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NAME:	Luka Milic
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Entanglement of photons pairs generated in silicon ring resonators

Luka Milic

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Contents

1	Introduction	4
2	Detailed Background and Theory	5
2.1	Integrated silicon photonics	5
2.2	Marco Liscid - Why its okay to use classical to probe quantum and an introduction to four wavemixing	5
2.3	Ring Resonators	5
2.4	Bistability	6
2.5	Self phase modulation	7
2.6	Schmidt Rank and Purity	7
3	Method	7
3.1	Silicon Chips	8
3.1.1	Glassgow	8
3.1.2	Toshiba	9
3.1.3	a-Si	9
3.2	Coupling	9
3.3	Joint Spectrum	9
3.4	What experiments can be done?	10
3.5	$g^{(2)}(0)$	10
3.6	Analysing Data	10
4	Results	11
4.1	Glassgow	11
4.2	a-Si	14
4.3	Toshiba	14
4.3.1	Bistability Data	14
4.3.2	Pulse shaping	14
4.3.3	Power Scans	14
5	Discussion	15
6	Conclusion	16
A	Schmidt Number	18
A.1	Definition	18
A.2	Calculation from experimental data	18
A.2.1	Trace method	18
B	Equipment Specifications	20
C	Transfer matrix analysis of ring resonator cavities	20

1 Introduction

The endeavour to build a quantum computer holds the promise of solving computational problems which are currently intractable on classical computers. A particularly promising paradigm for this is the linear optical quantum computer (LOQC) model which in theory allows for scalable universal quantum computation. Work on LOQC can be done using bulk optics components but this quickly becomes impractical when the experiments need to be scaled up to more qubits. Integrated photonics is a solution to this problem and allows for experiments with more qubits in a much smaller space. Optical circuits can be implemented on such chips, popular materials are silicon-on-insulator (SOI), lithium niobate and glass materials. Here we focus on SOI chips as they have many promising properties for the implementation of complex quantum optical circuits.

A key requirement for the full implementation of LOQC is a scalable, bright, deterministic and indistinguishable single photon source. Single photon sources in the SOI platform are typically made from the waveguide itself and use the spontaneous four-wave mixing which occurs in silicon due to the third order non-linearity to create a single photon pair. This report aims to develop a method of measuring the indistinguishability of the produced photons with a classical technique, exploiting stimulated four-wave mixing. This method collects a joint spectrum which is an estimation of the spectral shape of the two photons produced by the source. For a full description the Joint Spectral Amplitude JSA is the desired quantity, this is a full description of the wavefunction of the single photons emitted by the silicon ring resonators. However it is only within the scope of this work to measure the Joint Spectral Intensity, which is the absolute value squared of the JSA. This gives an estimate of the JSA and allows to upper bound the purity of the source.

The mission is therefore to develop a methodology to reconstruct these wavefunctions and hence engineer indistinguishable (high purity) single photon sources. In this work we performed such measurements on three SOI chips. The experimental work started with an initial proof of concept that one can collect joint spectrum data in the way desired. This was done on a chip supplied by Marc Sorel from Glasgow University. Then due to the fragility of these chip at high powers the experiment progressed to a chip manufactured by Toshiba. Finally in order to investigate a promising new material amorphous silicon chip was used for experiments.

In parallel techniques of analysing the output data are developed. Filtering techniques which remove noise are developed in order to make the data usable. A general framework is set out which aims to quantify the certainty in the measurements.

Finally we conclude that there is still much to be done in this area, proposing an outline for how to carry out effective measurements in the future.

2 Detailed Background and Theory

2.1 Integrated silicon photonics

Integrated silicon photonics is a promising new platform on which to conduct quantum information experiments.

2.2 Marco Liscid - Why its okay to use classical to probe quantum and an introduction to four wavemixing

Marco [?]

2.3 Ring Resonators

Ring resonators are used as single photon sources. However to understand their behaviour to first order no quantum mechanics is needed. Here are the 3 governing equations:

SPLIT RESONANCES

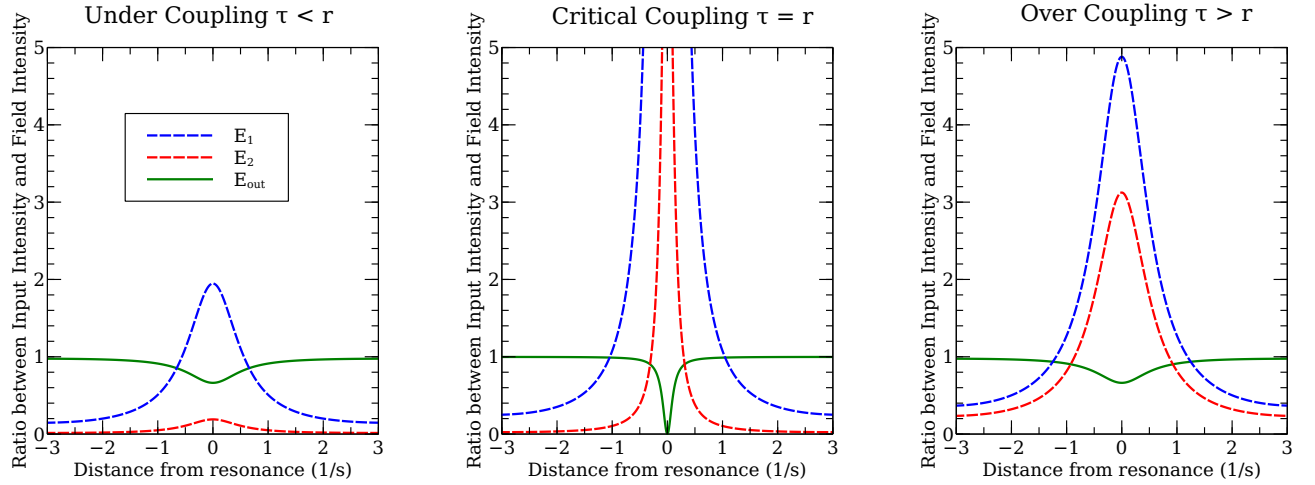


Figure 2.1: Notice how similar under and over coupling are to each other

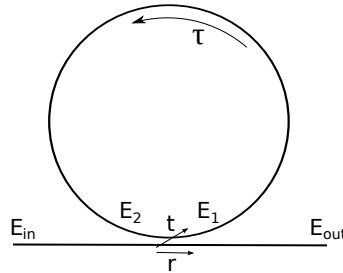


Figure 2.2: ahhh

$$\left| \frac{E_{out}}{E_0} \right|^2 = \frac{r^2 - 2r\tau \cos(\theta) + \tau^2}{1 + r^2\tau^2 - 2r\tau \cos(\theta)} \quad (2.1)$$

$$\left| \frac{E_1}{E_0} \right|^2 = \frac{t^2}{1 + r^2\tau^2 - 2r\tau \cos(\theta)} \quad (2.2)$$

$$\left| \frac{E_2}{E_0} \right|^2 = \tau^2 \left| \frac{E_1}{E_0} \right|^2 \quad (2.3)$$

2.4 Bistability

It can be experimentally observed that injecting power into a ring resonator will cause changes in the spectral position and shape of the resonance. Typically in silicon ring resonators the more power in the ring the more the resonance position is red-shifted by the thermo-optic effect [?]. A counter acting effect is carrier generation induced by two-photon absorption [1] which causes a blue-shift in the resonance position. This carrier generation process is much faster than the thermo-optic process so it more relevant to lasers with low repetition times.

The bi-stability effect is observed by changing the power injected into the ring resonator at a fixed wavelength. By steadily increasing the power of a monochromatic light source injected into the ring at a wavelength slightly higher than the resonance position λ_r of the ring, λ_r is increased (thermal effects dominate as the laser is a continuous wave and not pulsed). The shift in λ_r accelerates as more light is coupled into the ring and transmission falls as more light is coupled into the ring. The system is now in a different and stable state (assuming the injected light is not discontinued). With a low intensity probe it is now possible to map out the new position and shape of the resonance.

By doing the reverse experiment with the input power decreasing a similar phenomena is observed, however the sudden accelerating changing in resonance position is seen for a different power due to the ring coming from a different stable state.

Some knowledge of this effect is vital when planning experiments using high and variable powers, as one must take into account which state the ring is in. Further when automating equipment it may be vital to integrate knowledge of this into any scanning procedures.

2.5 Self phase modulation

2.6 Schmidt Rank and Purity

3 Method

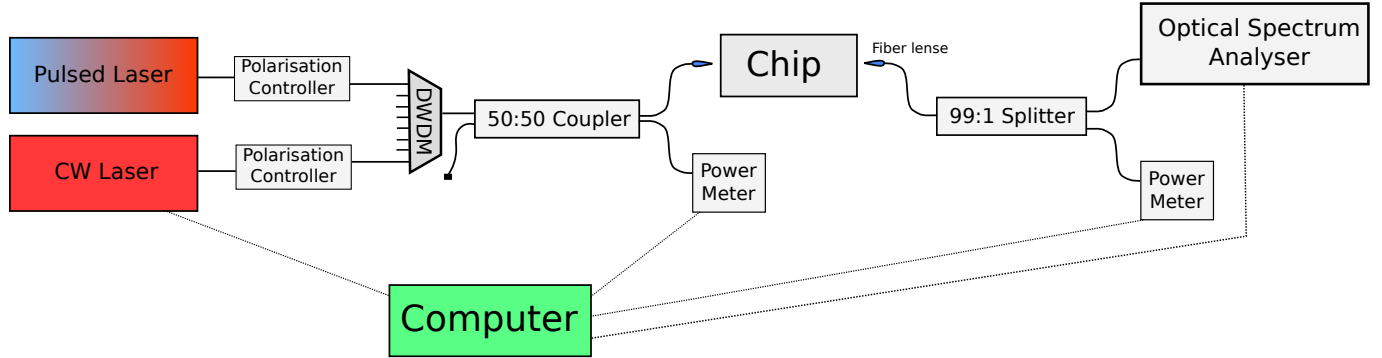


Figure 3.1: Glasgow test structure chip

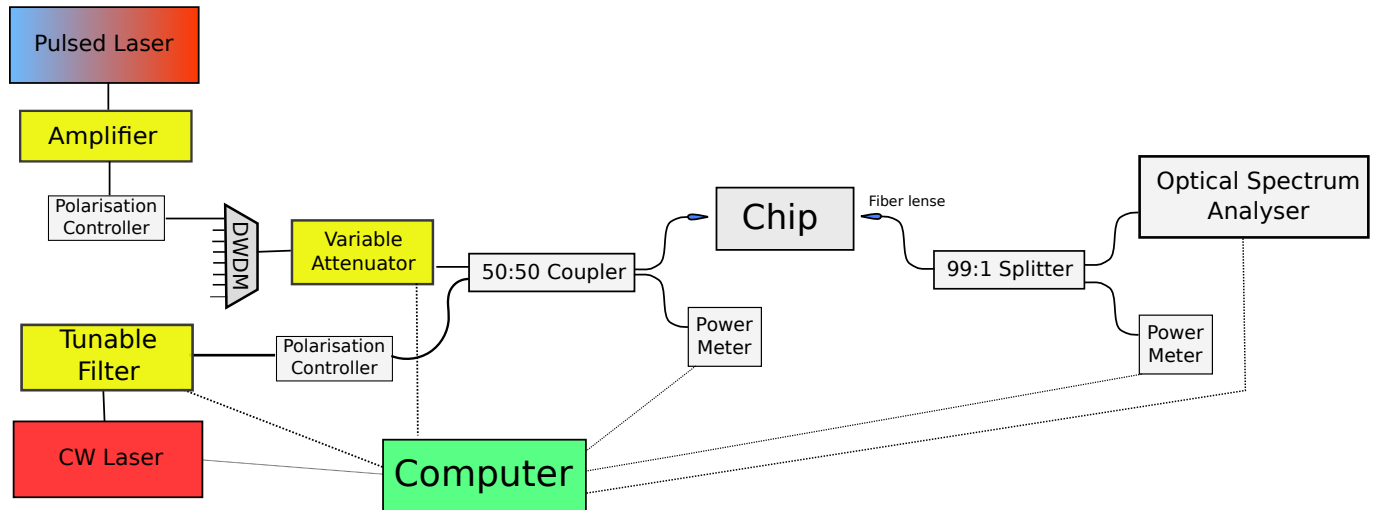


Figure 3.2: Glasgow test structure chip

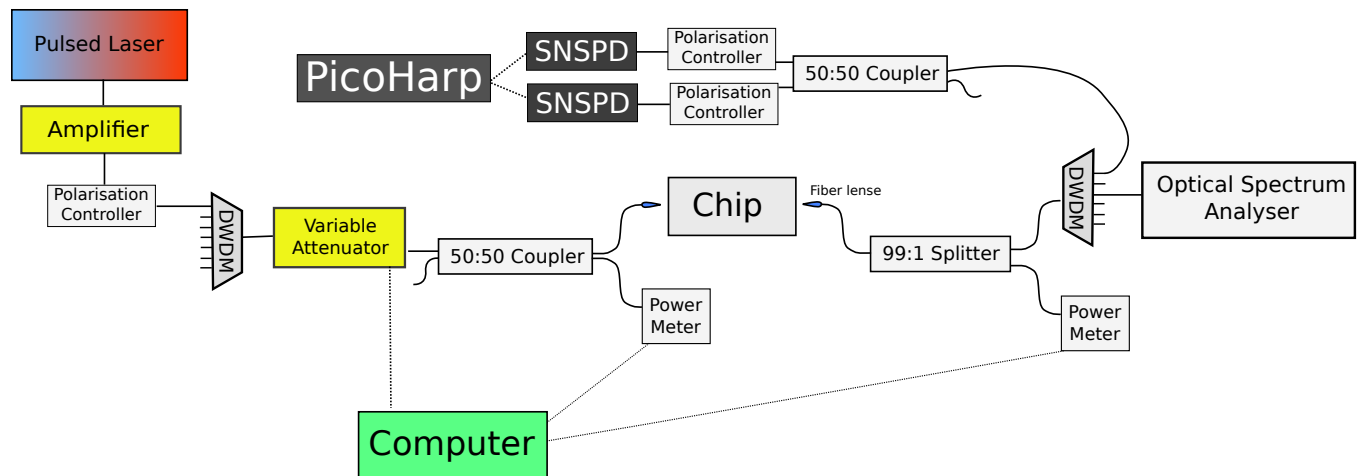


Figure 3.3: Glasgow test structure chip

3.1 Silicon Chips

3.1.1 Glassgow

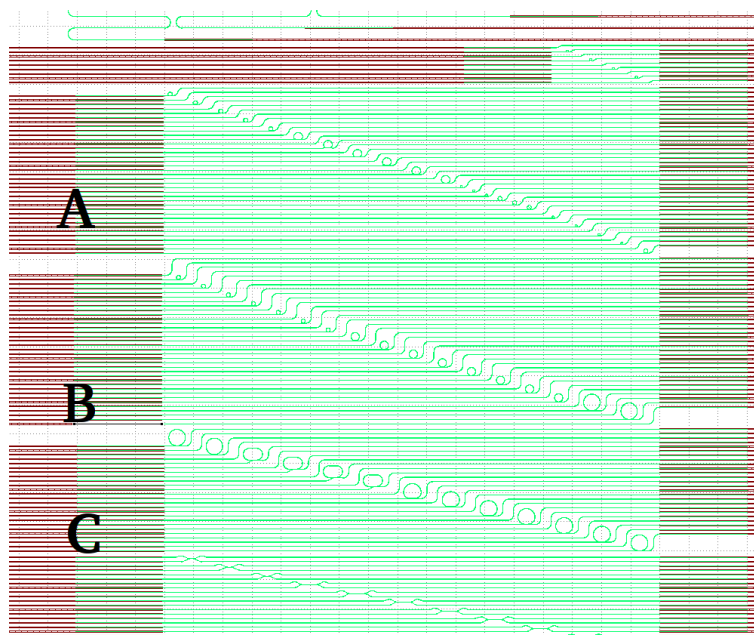


Figure 3.4: Glasgow test structure chip

3.1.2 Toshiba

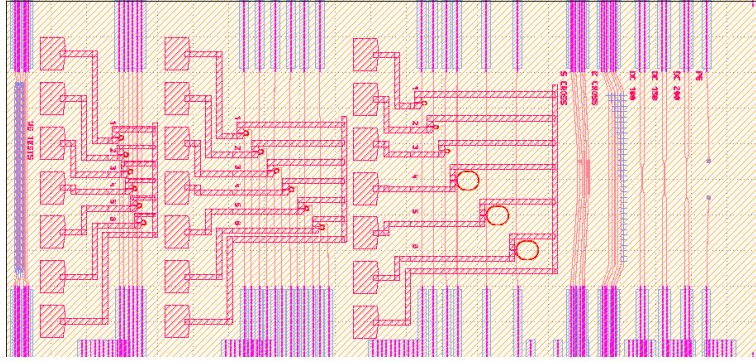


Figure 3.5: Glassgow test structure chip

3.1.3 a-Si

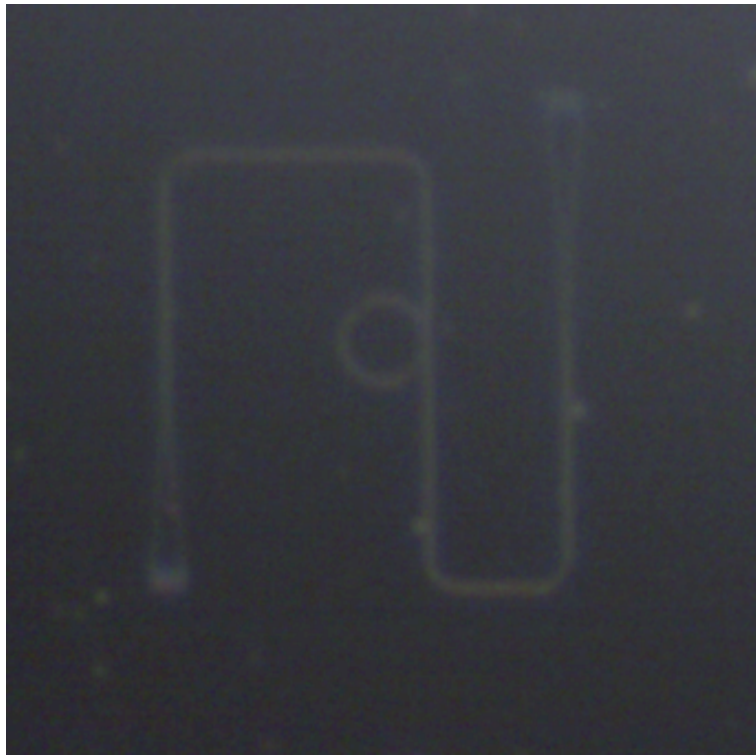


Figure 3.6: Glassgow test structure chip

3.2 Coupling

3.3 Joint Spectrum

Take the convention that the **signal** photon is the one we measure and the **idler** is the one we stimulate.

3.4 What experiments can be done?

Assuming from the above that the procedure for collecting the JSI is fixed and fully understood the question that now needs to be answered is: what parameters can be reliably varied to change the JSI? One that we pursue and that forms the main part of this work is varying the power of the pump laser injected into the ring. This is of interest as it may

3.5 $g^{(2)}(0)$

3.6 Analysing Data

4 Results

4.1 Glasgow

This chip was used to do an initial proof of concept that the JSI of a ring resonator could be measured at high resolution. Here the aim was to explore different ring geometries and develop an intuition on how to do the experiment. The Pritel pulsed laser was used with a pulse duration of 2 ps, a FWHM of 1.0 nm with wavelength range 1530 nm to 1530 nm and a peak power of 100 W. Due to the pulsed laser sometimes destroying the spot size converter on the chip a 3 dB attenuator was added just after the laser output, this resulted in roughly about 5 dB m of power coming out of the lense fiber.

As each device on the chip has unique characteristics due to fabrication errors only some devices produced good data meaning that a full characterisation of the chip was not possible and only a few devices were used. Typically it was imperfections in the spot size converters on the chips being imperfect which prevented good JSI's from being collected but another issue encountered was under coupling of the rings.

Here we present three data sets collected, each has high SNR of at least 20 allowing for the maximum purity P_{max} to be calculated with high accuracy. The relevant spectral scan is presented with the JSIs for reference, these are routine scans performed with the tunable CW laser typically at 1 m W of which roughly -14 dB m gets into the chip.

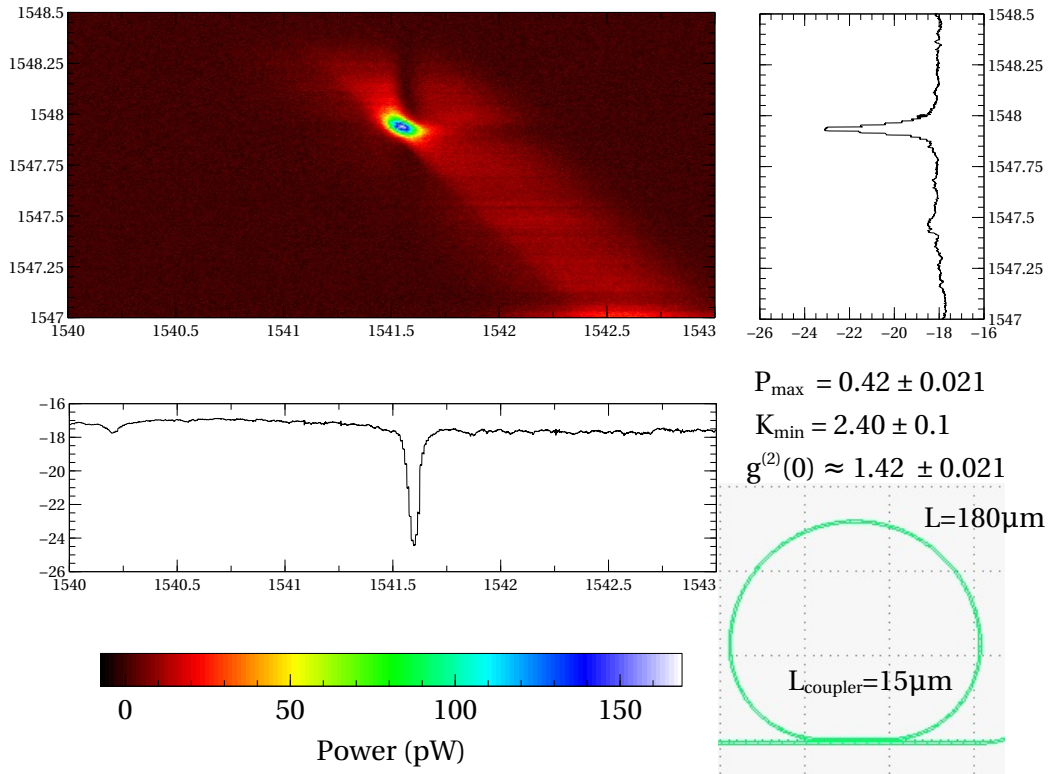


Figure 4.1: JSI Ring C17 We calculate a quality factor of $Q \approx 25000$. In this instance the coupling loss was on average 20 dB and the input power of the pulsed laser was 5.1 dB giving a estimated power in waveguide of -5 dB. The probe laser operated at 1 m W. Fitting the spectral scans of the ring resonators gives coupling parameters of the order $r = 0.946$ $\tau = 0.985$ $n_{eff} = 4.136$, note these are only estimates.

Figure 4.1 shows a clear response from the ring resonator at the resonant frequencies, superimposed on this there is also straight waveguide stimulated four wave mixing observed at much lower intensities. It can be deduced that much of this straight wave guide contribution happens before the ring as it is filtered at the resonance wavelength. The asymmetry is an experimental artefact of imperfect alignment of the resonances with the AWG channels and the pump laser. This is P_{max} value.

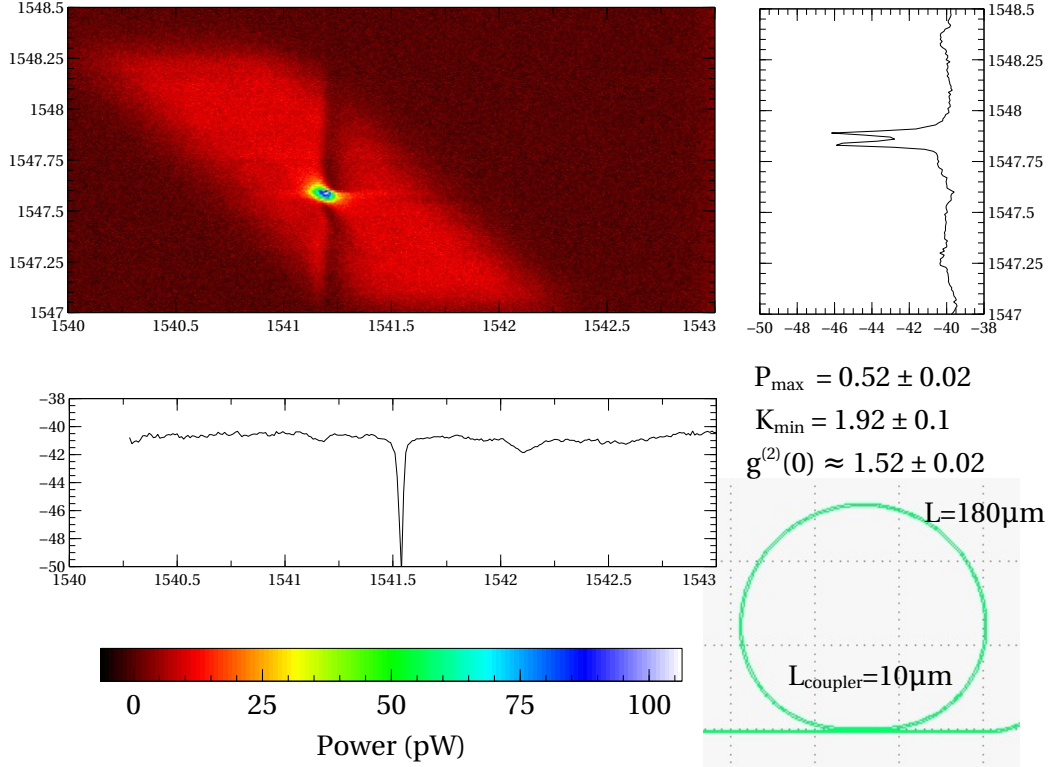


Figure 4.2: JSI Ring C21 We calculate a quality factor of $Q \approx 28000$. In this instance the coupling loss was on average 21 dB and the input power of the pulsed laser was 5.5 dB m giving a estimated power in waveguide of -5 dB m. The probe laser operated at 1 m W. Fitting the spectral scans of the ring resonators gives coupling parameters of the order $r = 0.957$ $\tau = 0.977$ $n_{eff} = 4.144$, note these are only estimates.

In figure 4.2 we see ring C21 which has many physical similarities with ring C17 shown in figure 4.1 however we observe a few key differences. Firstly the resonance that the CW laser probes is split but this is not reflected in the JSI because the split feature is on the order of 0.01 nm which under the 0.03 nm resolution of the OSA. This inability to resolve finer features which are almost certainly encoded in the real JSA is an important topic discussed later on in this work.

Secondly the coupling distance of the egg shaped resonator is slightly shorter here which is reflected in a slightly smaller value for r , however this difference isn't significant enough to use for a comparison as there are many other factors between the two experiments that were not held

constant, in particular the alignment of the AWG channels.

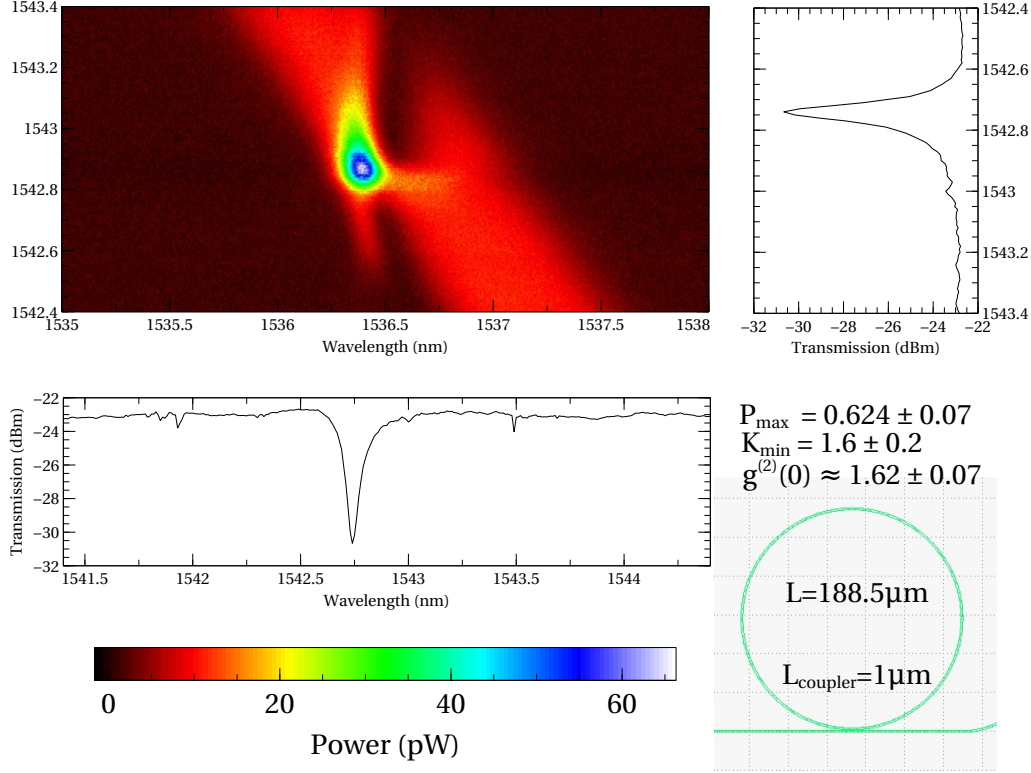


Figure 4.3: JSI Ring B32 We calculate a quality factor of $Q \approx 14000$. In this instance the coupling loss was on average 22 dBm and the input power of the pulsed laser was 5.1 dBm giving a estimated power in waveguide of 0 dBm. The probe laser operated at 1 mW. Fitting the spectral scans of the ring resonators gives coupling parameters of the order $r = 0.985$ $\tau = 0.947$ $n_{eff} = 4.136$, note these are only estimates.

The final JSI collected for this chip is shown in figure 4.3, here the ring geometry is of the more traditional circular shape. It is observed that the resonance has some asymmetry due to more prominent non-linear effects. We observe the intensities of the ring resonator and straight wave guide FWM processes to be more similar in this scan which can be accounted for by the lower quality factory. Again it is notable to see that the ring seems to be collecting some of the straight wave guide FWM, however at a shifted frequency to the peak of the ring FWM, this is contradictory to the two previous examples and may be due to the more non-linear response of this ring.

Summary The three purities observed (0.42, 0.52, 0.62) coincide with previous characterisations of the chip using the single photon detectors which measured a purity of on the order 0.45 [2]. A advantage to these measurements is they allow one to easier develop ways of increasing the purity.

4.2 a-Si

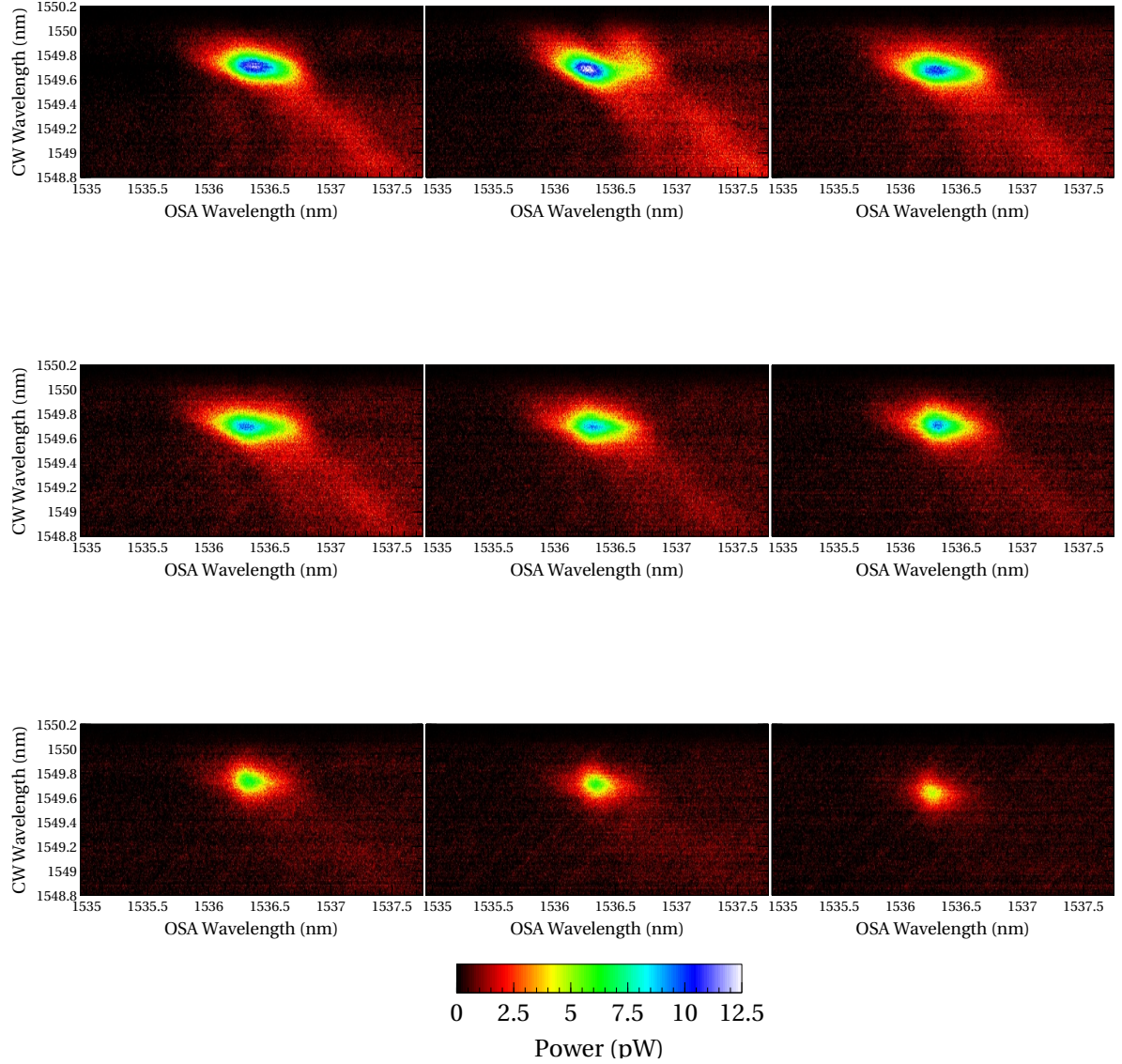


Figure 4.4: JSI Ring C21 We calculate a quality factor of $Q \approx 28000$. In this instance the coupling loss was on average 21 dB and the input power of the pulsed laser was 5.5 dB m giving a estimated power in waveguide of -5 dB m. The probe laser operated at 1 m W. Fitting the spectral scans of the ring resonators gives coupling parameters of the order $r = 0.957$ $\tau = 0.977$ $n_{eff} = 4.144$, note these are only estimates.

4.3 Toshiba

4.3.1 Bistability Data

4.3.2 Pulse shaping

4.3.3 Power Scans

5 Discussion

6 Conclusion

References

- [1] Qianfan Xu and Michal Lipson. Carrier-induced optical bistability in silicon ring resonators. *Opt. Lett.*, 31(3):341–343, February 2006.
- [2] Eric Scammell. Indistinguishable heralded photons from silicon ring resonators, 2014.
- [3] Andreas Eckstein, Guillaume Boucher, Aristide Lematre, Pascal Filloux, Ivan Favero, Giuseppe Leo, John E. Sipe, Marco Liscidini, and Sara Ducci. High-resolution spectral characterization of two photon states via classical measurements. *Laser & Photonics Reviews*, 8(5):L76–L80, September 2014.

A Schmidt Number

A.1 Definition

Starting with some arbitrary state ψ :

$$|\psi\rangle = \sum_{i,j} \alpha(i,j) |i\rangle_A \otimes |j\rangle_B \quad (\text{A.1})$$

The schmidt number K of this state measures the degree of entanglement. If $K = 1$ then you can find $|\psi\rangle = |\xi\rangle \otimes |\eta\rangle$ and for $K > 1$ you can find:

$$|\psi\rangle = \sum_i^K r_i |\xi_i\rangle_A \otimes |\eta_i\rangle_B \quad (\text{A.2})$$

Note that $1 \leq K \leq D$ where D is the dimension of the system. The purity is the inverse of K so:

$$P = 1/K \quad (\text{A.3})$$

An expression for K can be found using the density matrix for ψ :

$$\rho_{AB} = |\psi\rangle\langle\psi| = \sum_{i,j,k,l} \alpha(i,j) \alpha^*(k,l) |i\rangle\langle k| \otimes |j\rangle\langle l| \quad (\text{A.4})$$

$$\rho_A = \text{Tr}_B(\rho_{AB}) = \sum_{i,j,k} \alpha(i,j) \alpha^*(k,j) |i\rangle\langle k| \quad (\text{A.5})$$

$$\rho_A^2 = \sum_{i',j',k'} \sum_{i,j,k} \alpha(i,j) \alpha(k,j) \alpha^*(i',j') \alpha^*(k',j') |i\rangle\langle k| |i'\rangle\langle k'| \quad (\text{A.6})$$

$$= \sum_{j',k'} \sum_{i,j,k} \alpha(i,j) \alpha^*(k,j) \alpha(k,j') \alpha^*(k',j') |i\rangle\langle k'| \quad (\text{A.7})$$

$$\text{Tr}_A(\rho_A^2) = \sum_{i,j,k,j'} \alpha(i,j) \alpha^*(k,j) \alpha(k,j') \alpha^*(i,j') \quad (\text{A.8})$$

$$(\text{A.9})$$

For a unentangled ψ we know that $\text{Tr}_A(\rho_A^2) = 1$ For ψ entangled this will be smaller than 1 (proof comes from the property of the density operator that its eigenvalues are all smaller than 1). This fits the definition of the purity of a quantum state hence we can write:

$$P = \frac{1}{K} = \sum_{i,j,k,l} \alpha(i,j) \alpha^*(k,j) \alpha(k,l) \alpha^*(i,l) \quad (\text{A.10})$$

A.2 Calculation from experimental data

A.2.1 Trace method

In the lab we can measure $|\phi(\omega_1, \omega_2)|^2$, here I outline how to extract the schmidt number from this set of values. Taking the positive square root of the matrix of values obtained from the lab you have a matrix \mathbf{f} given by:

$$\mathbf{f} = \sum_{\omega_1, \omega_2} \phi(\omega_1, \omega_2) |\omega_1\rangle\langle\omega_2| \quad (\text{A.11})$$

(This seems to be some weird way of writing the wavefunction as a matrix, bare with me it turns out to be useful)

$$\mathbf{f}^\dagger \mathbf{f} = \sum_{\omega_1, \omega_2, \omega_3} \phi(\omega_1, \omega_2) \phi(\omega_3, \omega_2) |\omega_1\rangle \langle \omega_3| \quad (\text{A.12})$$

$$(\mathbf{f}^\dagger \mathbf{f})^2 = \sum_{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6} \phi(\omega_1, \omega_2) \phi(\omega_3, \omega_2) \phi(\omega_4, \omega_5) \phi(\omega_6, \omega_5) |\omega_1\rangle \langle \omega_3| \omega_4\rangle \langle \omega_6| \quad (\text{A.13})$$

$$(\mathbf{f}^\dagger \mathbf{f})^2 = \sum_{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6} \phi(\omega_1, \omega_2) \phi(\omega_3, \omega_2) \phi(\omega_3, \omega_5) \phi(\omega_6, \omega_5) |\omega_1\rangle \langle \omega_6| \quad (\text{A.14})$$

$$\text{Tr} [(\mathbf{f}^\dagger \mathbf{f})^2] = \sum_{\omega_1, \omega_2, \omega_3, \omega_4} \phi(\omega_1, \omega_2) \phi(\omega_3, \omega_2) \phi(\omega_3, \omega_4) \phi(\omega_1, \omega_4) \quad (\text{A.15})$$

I've done it this way because I wanted to figure out where the equation in [3] comes from. You can now see that equation A.10 is of exactly the same form as $\text{Tr} [(\mathbf{f}^\dagger \mathbf{f})^2]$ (barring the conjugates but this is okay since ϕ is real.) Taking the parallel further it can be seen that equation A.12 is of the form of a reduced density matrix. Here we must make sure to normalise to make sure this is a valid reduced density matrix. The normalisation is:

$$N = \text{Tr} [\mathbf{f}^\dagger \mathbf{f}] = \sum_{\omega_1, \omega_2} \phi(\omega_1, \omega_2)^2 \quad (\text{A.16})$$

Giving:

$$\rho_A = \frac{\mathbf{f}^\dagger \mathbf{f}}{N} \quad (\text{A.17})$$

We can then write:

$$\frac{1}{K} = \frac{\text{Tr} [(\mathbf{f}^\dagger \mathbf{f})^2]}{\text{Tr} [\mathbf{f}^\dagger \mathbf{f}]^2} \quad (\text{A.18})$$

B Equipment Specifications

C Transfer matrix analysis of ring resonator cavities

A useful way to describe the spectral response of a ring resonator is by defining a transfer matrix. This 2×2 matrix relates the electric fields of the inputs and outputs of the waveguide and resonator. In our analysis we define the electric fields to be complex valued and also normalised to conserve energy, this condition implies the transfer matrix must be unitary.

First write down the obvious relations

$$E_{out} = r_1 E_{in} + t_1 E_2 \quad (C.1)$$

$$E_1 = r_2 E_2 + t_2 E_{in} \quad (C.2)$$

Define a matrix.

$$M = \begin{pmatrix} r_1 & t_1 \\ t_2 & r_2 \end{pmatrix} \quad (C.3)$$

Enforce unitarity

$$\begin{pmatrix} r_1 & t_1 \\ t_2 & r_2 \end{pmatrix} \begin{pmatrix} r_1^* & t_2^* \\ t_1^* & r_2^* \end{pmatrix} = \begin{pmatrix} |r_1|^2 + |t_1|^2 & r_1 t_2^* + r_2^* t_1 \\ r_2 t_1^* + r_1^* t_2 & |r_2|^2 + |t_2|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (C.4)$$

Furthermore $\det M = 1$

$$r_1 r_2 - t_1 t_2 = 1 \quad (C.5)$$

so

$$r_2 = \frac{1 + t_1 t_2}{r_1} \quad (C.6)$$

Also $M^{-1} = M^*$

$$\begin{pmatrix} r_2 & -t_1 \\ -t_2 & r_1 \end{pmatrix} = \begin{pmatrix} r_1^* & t_1^* \\ t_2^* & r_2^* \end{pmatrix} \quad (C.7)$$

This tells us r_1 and r_2 must be real. So we can simplify the unitarity equation. So

$$t_1 = iT_1 \quad (C.8)$$

$$t_2 = iT_2 \quad (C.9)$$

Rewrite the interesting equations: Use also $r_2 = r_1^*$

$$\begin{pmatrix} |r_1|^2 + T_1^2 & -ir_1 T_2 + ir_1 T_1 \\ -ir_1^* T_1 + ir_1^* T_2 & |r_1|^2 + T_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (C.10)$$

Ok so $T_2 = T_1 = t$

$$\begin{pmatrix} |r_1|^2 + t^2 & 0 \\ 0 & |r_1|^2 + t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (C.11)$$

$$|r_1|^2 = 1 - t^2 \quad (C.12)$$

$$|r_2|^2 = 1 - t^2 \quad (C.13)$$

d Basically I have a degree of freedom. $r_1 = e^{i\theta}r$ and $r_2 = e^{-i\theta}r$. Choose $\theta = 0$ for simplicity.

Now the matrix is:

$$\begin{pmatrix} r & it \\ it & r \end{pmatrix} \quad (\text{C.14})$$

Now we add that

$$E_2 = \tau e^{i\theta} E_1$$

Recall the initial equations.

$$E_{out} = rE_{in} + itE_2 \quad (\text{C.15})$$

$$E_1 = rE_2 + itE_{in} \quad (\text{C.16})$$

$$E_1 = r\tau e^{i\theta} E_1 + itE_{in} \quad (\text{C.17})$$

$$\frac{E_1}{E_0} = \frac{it}{1 - r\tau e^{i\theta}} \quad (\text{C.18})$$

Okay so that's E_1 done.

$$\frac{E_1}{E_0} = \frac{it}{1 - r\tau e^{i\theta}} \quad (\text{C.19})$$

$$\left| \frac{E_1}{E_0} \right|^2 = \frac{t^2}{1 + r^2\tau^2 - 2r\tau\cos(\theta)} \quad (\text{C.20})$$

For E_2 we have:

$$\frac{E_2}{E_0} = \frac{it\tau e^{i\theta}}{1 - r\tau e^{i\theta}} \quad (\text{C.21})$$

$$\left| \frac{E_2}{E_0} \right|^2 = \frac{\tau^2 - \tau^2 r^2}{1 + r^2\tau^2 - 2r\tau\cos(\theta)} \quad (\text{C.22})$$

For E_{out} we have:

$$\frac{E_{out}}{E_0} = r + it \frac{it\tau e^{i\theta}}{1 - r\tau e^{i\theta}} \quad (\text{C.23})$$

$$\frac{E_{out}}{E_0} = \frac{r - r^2\tau e^{i\theta} - t^2\tau e^{i\theta}}{1 - r\tau e^{i\theta}} \quad (\text{C.24})$$

$$\left| \frac{E_{out}}{E_0} \right|^2 = \frac{(r - r^2\tau e^{i\theta} - t^2\tau e^{i\theta})(r - r^2\tau e^{-i\theta} - t^2\tau e^{-i\theta})}{(1 - r\tau e^{i\theta})(1 - r\tau e^{-i\theta})} \quad (\text{C.25})$$

$$\left| \frac{E_{out}}{E_0} \right|^2 = \frac{(r(r - r^2\tau e^{-i\theta} - t^2\tau e^{-i\theta}) - r^2\tau e^{i\theta}(r - r^2\tau e^{-i\theta} - t^2\tau e^{-i\theta}) - t^2\tau e^{i\theta}(r - r^2\tau e^{-i\theta} - t^2\tau e^{-i\theta}))}{1 + r^2\tau^2 - 2r\tau\cos(\theta)} \quad (\text{C.26})$$

$$\left| \frac{E_{out}}{E_0} \right|^2 = \frac{r^2 - r^3\tau e^{-i\theta} - rt^2\tau e^{-i\theta} - r^3\tau e^{i\theta} + r^4\tau^2 + 2r^2\tau^2 t^2 - rt^2\tau e^{i\theta} + t^4\tau^2}{1 + r^2\tau^2 - 2r\tau\cos(\theta)} \quad (\text{C.27})$$

$$\left| \frac{E_{out}}{E_0} \right|^2 = \frac{r^2 - e^{-i\theta}(r^3\tau + rt^2\tau) - e^{i\theta}(r^3\tau + rt^2\tau) + r^4\tau^2 + t^4\tau^2 + 2r^2\tau^2 t^2}{1 + r^2\tau^2 - 2r\tau\cos(\theta)} \quad (\text{C.28})$$

$$\left| \frac{E_{out}}{E_0} \right|^2 = \frac{r^2 + 2(r^3\tau + rt^2\tau)\cos(\theta) + r^4\tau^2 + t^4\tau^2 + 2r^2\tau^2 t^2}{1 + r^2\tau^2 - 2r\tau\cos(\theta)} \quad (\text{C.29})$$

$$\left| \frac{E_{out}}{E_0} \right|^2 = \frac{r^2 2(r^3 \tau + r(1 - r^2) \tau) \cos(\theta) + r^4 \tau^2 + (1 - r^2)^2 \tau^2 + 2r^2 \tau^2 (1 - r^2)}{1 + r^2 \tau^2 - 2r\tau \cos(\theta)} \quad (C.30)$$

$$\left| \frac{E_{out}}{E_0} \right|^2 = \frac{r^2 - 2(r^3 \tau + r(1 - r^2) \tau) \cos(\theta) + r^4 \tau^2 + (1 - 2r^2 + r^4) \tau^2 + 2r^2 \tau^2 - 2r^4 \tau^2}{1 + r^2 \tau^2 - 2r\tau \cos(\theta)} \quad (C.31)$$

$$\left| \frac{E_{out}}{E_0} \right|^2 = \frac{r^2 - 2r\tau \cos(\theta) + r^4 \tau^2 + \tau^2 - 2r^2 \tau^2 + r^4 \tau^2 + 2r^2 \tau^2 - 2r^4 \tau^2}{1 + r^2 \tau^2 - 2r\tau \cos(\theta)} \quad (C.32)$$

FINAL RESULTS

$$\left| \frac{E_{out}}{E_0} \right|^2 = \frac{r^2 - 2r\tau \cos(\theta) + \tau^2}{1 + r^2 \tau^2 - 2r\tau \cos(\theta)} \quad (C.33)$$

$$\left| \frac{E_2}{E_0} \right|^2 = \frac{\tau^2 - \tau^2 r^2}{1 + r^2 \tau^2 - 2r\tau \cos(\theta)} = \tau^2 \left| \frac{E_1}{E_0} \right|^2 \quad (C.34)$$

$$\left| \frac{E_1}{E_0} \right|^2 = \frac{t^2}{1 + r^2 \tau^2 - 2r\tau \cos(\theta)} \quad (C.35)$$