Wuhan to SF infection model

Will Stedden

February 2020

1 Introduction

This is an exploration of a simple model to see how much time there is after the start of an epidemic to stop transportation between two cities in order to prevent the spread from one to another. The model looks similar to [1], but I made a simpler model of spreading, quarantine, and transit between two cities, and I made lots of other simplifying assumptions too. This model is not intended to make quantitative predictions. It's just meant to illustrate that there is a limited amount of time to stop transit before it's too late to contain the spread.

2 Model Definition

I adapted a standard SIR model in the limit where there are lots of people in the susceptible (S) stage and no one in the recovered (R) stage. Therefore this model only tries to measure the number of infected, $I(t)$, in two populations. I also assumed that there is a spreading rate, α, and a quarantine rate that gets smaller when there are more people who are sick, $\beta + \kappa/I$, (assuming that we are using up limited healthcare resources). Finally, there is some small factor μ that defines the rate of people moving from the infected city (eg Wuhan) to the non-infected city (eg San Francisco). Combining these assumptions into a set of equations gives

$$\frac{\mathrm{d}I_{\mathrm{wu}}}{\mathrm{d}t} = \alpha I_{\mathrm{wu}} - (\beta + \frac{\kappa}{I_{\mathrm{wu}}})I_{\mathrm{wu}} \tag{1}$$

$$\frac{\mathrm{d}I_{\mathrm{sf}}}{\mathrm{d}t} = \alpha I_{\mathrm{sf}} - (\beta + \frac{\kappa}{I_{\mathrm{sf}}})I_{\mathrm{sf}} + \mu I_{\mathrm{wu}} \tag{2}$$

We can simplify this by carrying out the products and setting $\gamma = \alpha - \beta$ to get the primary model equations.

$$\frac{\mathrm{d}I_{\mathrm{wu}}}{\mathrm{d}t} = \gamma I_{\mathrm{wu}} - \kappa \tag{3}$$

$$\frac{\mathrm{d}I_{\mathrm{sf}}}{\mathrm{d}t} = \gamma I_{\mathrm{sf}} - \kappa + \mu I_{\mathrm{wu}} \tag{4}$$

I should point out that the equation for I_{sf} is only valid when it doesn't send I_{sf} negative. Whenever $I_{\text{sf}} = 0$ and the equation for $dI_{\text{sf}}/dt < 0$ we just set $dI_{\text{sf}}/dt = 0$.

This equation still has 3 free parameters, which makes it cumbersome to simulate all possible outcomes. In the next section I renormalize variables to transform this into a single parameter differential equation.

3 Simplifying and setting initial conditions

In this section, I perform the nondimensionalization to simplify the model and I introduce the parameter ε to set the models initial conditions.

3.1 Nondimensionalization

Following, [2], I perform a normalization of our variables, t, I_{wu}, and I_{sf} by the constants t_0, I_0, giving us

$$w = \frac{I_{\text{wu}}}{I_0}, x = \frac{I_{\text{sf}}}{I_0}, \tau = \frac{t}{t_0} \tag{5}$$

and

$$\frac{\mathrm{d}w}{\mathrm{d}\tau} = \frac{\mathrm{d}I_{\text{wu}}}{\mathrm{d}t}\frac{t_0}{I_0} \tag{6}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}I_{\text{sf}}}{\mathrm{d}t}\frac{t_0}{I_0} \tag{7}$$

I sub these into equations 3 to get

$$\frac{\mathrm{d}w}{\mathrm{d}\tau}\frac{I_0}{t_0} = \gamma w I_0 - \kappa \tag{8}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau}\frac{I_0}{t_0} = \gamma x I_0 - \kappa + \mu w I_0 \tag{9}$$

and rearranging to get

$$\frac{\mathrm{d}w}{\mathrm{d}\tau} = \gamma w t_0 - \kappa\frac{t_0}{I_0} \tag{10}$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \gamma x t_0 - \kappa\frac{t_0}{I_0} + \mu w t_0 \tag{11}$$

We can set the first term in each equation to 1 by choosing $t_0 = 1/\gamma$. After that substitution, we can use $I_0 = \kappa/\gamma$ to set the second term to 1 as well, leaving

$$\frac{\mathrm{d}w}{\mathrm{d}\tau} = w - 1 \tag{12}$$
$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x - 1 + \frac{\mu}{\gamma} w \tag{13}$$

Finally, we rename the variable μ/γ to ρ for simplicity leaving our final dimensionless dynamical equation

$$\frac{\mathrm{d}w}{\mathrm{d}\tau} = w - 1 \tag{14}$$
$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x - 1 + \rho w \tag{15}$$

which we can use to simulate the dynamics without worrying about so many parameters.

It's important to remember that for this equation to hold, we must assume that ρ is much less than 1. Otherwise, we would need to account for the $-\rho w$ in our equation for w. This is reasonable though because we expect $\mu << \gamma$, since the probability of infecting another person is much higher than the probability of flying to San Francisco.

3.2 Initial infected population

The solution for $w(\tau)$ with $w(0) = n$ (from wolfram alpha) is

$$w(\tau) = (n-1)e^{\tau} + 1 \tag{16}$$

To make our model interesting we assume that the infected population in Wuhan is going to be growing. This amounts to assuming that the initial value of w at $\tau = 0$ (ie n) is such that $dw/dt > 0$ or $n > 1$.

If we make this assumption it becomes cleaner to describe 16 in terms of a new variable $\varepsilon = n - 1$, which is how much the starting infected population in Wuhan exceeds the minimum needed for an outbreak in Wuhan, 1. So we can rewrite 16 as

$$w(\tau) = \varepsilon e^{\tau} + 1 \tag{17}$$

4 Outbreak Critical Time

Now we can get to the interesting part, where we can estimate the critical time where we reach an outbreak in San Francisco, τ_x.

There are actually two distinct phases in this critical time. First, there is the time until the infected population in SF becomes greater than 0, meaning the number of infected people arriving exceeds the capacity to quarantine them fast enough. I'll label this time as τ_+.

Second, there is the time from when SF's infected population starts to grow up until the point where the infection becomes self-sustaining even if the influx from Wuhan goes to 0. This is the time that we've defined as τ_x. However, the dynamics are different once $\tau > \tau_x$ so let's assume that we will renormalize time after τ_+ as $\tau' = \tau - \tau_+$. So in these new time call this time when San Francisco's infection becomes self sustaining τ'_x.

So in the end we have to calculate both τ_+ and τ'_x separately and combine of these terms to get an estimate of the time of outbreak, $\tau_x = \tau_+ + \tau'_{x'}$.

4.1 Calculating time to SF spread

I can calculate τ_+ directly from the solution to the differential equation. In the beginning, $dx/d\tau < 0$, and the infection only starts spreading in SF once it reach $dx/d\tau = 0$. We can sub the equation for $w(\tau)$ 17 into the equation for $dx/d\tau$ to get

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = x - 1 + \rho \left(\varepsilon e^{\tau}_{+} + 1\right) = 0 \tag{18}$$

Since x is by definition 0 up until this point, we can set it to 0 in the equation and rearrange to get

$$\rho\left(\varepsilon e^{\tau} + 1\right) = 1 \tag{19}$$

and rearranging further we get to

$$\tau_+ = \ln\left(\frac{1-\rho}{\rho\varepsilon}\right) \tag{20}$$

4.2 Estimating time to self-sustaining spread in SF

If at some point we stop further input from Wuhan, (ie set $\rho = 0$), the infection in SF will continue to spread anyway if $x > 1$ at that point. We can evaluate $x(\tau)$ in the time after τ_+ by subbing in equation 17, shifting our time variable to $\tau' = \tau - \tau_+$, and solving equation 15 with $x(\tau') = 0$ at $\tau' = 0$ (using wolfram alpha).

$$x(\tau') = \rho\varepsilon e^{\tau'}\tau' + (1-\rho)(1-e^{\tau'}) \tag{21}$$

Unfortunately, finding the time τ'_x when this is 1 isn't possible because of the mixture of exponential and linear τ'. However we can estimate it in two different limiting cases, one where τ'_x is really short and another where τ'_x is really long.

4.2.1 Limit where $\tau'_x << 1$

If we assume that we're in the limit where the time to outbreak (τ'_x) will be small then we can expand $e^{\tau'}$ as $1 + \tau'$, approximate 21, and find when $x(\tau'_x) = 1$.

$$x(\tau_x') \approx \tau_x'(\rho\varepsilon - (1-\rho)) = 1 \tag{22}$$

and rearranging for τ_x'

$$\tau_x = \ln\left(\frac{1-\rho}{\rho\varepsilon}\right) = \frac{1}{\rho(\varepsilon+1)-1} \tag{23}$$

In order for τ_x' to be $<< 1$, we need the denominator to be large. But since, $\rho << 1$ we know that what we really need is $\varepsilon >> 1/\rho$. In this limit we can say that $\tau_x' \approx 0$ or in other words, $\tau_x \approx \tau_+$

4.2.2 Limit where $\tau_x' >> 1$

In the opposite limit, I just assume that the first term in 21 dominates to we have

$$x(\tau_x') \approx \rho\varepsilon\tau_x' e^{\tau_x'} = 1 \tag{24}$$

According to wolfram alpha this is solved by $\tau_x' = W(1/\rho\varepsilon)$, where W is the product log function. This is some crazy function that I don't know how to think about, but it seems to be mostly just something like $0.8 \cdot \ln(1/\rho\varepsilon)$ for values of $1/(\rho\varepsilon)$ from 10 to 100000 and asymptotically approaching $\ln(1/\rho\varepsilon)$ as $\rho\varepsilon \to 0$.

So I'll just approximate it as $0.8 \cdot \ln(1/\rho\varepsilon)$ and give our final result for τ_x in this limit as

$$\tau_x = \ln\left(\frac{1-\rho}{\rho\varepsilon}\right) + 0.8 \cdot \ln(1/\rho\varepsilon) \tag{25}$$

Finally, since we assume that $\rho << 1$, we can drop the $\ln(1-\rho)$ and simplifiy the logarithms to

$$\tau_x = 1.7 * \ln\frac{1}{\rho\varepsilon} \tag{26}$$

5 Conclusion

To see how this compares to simulated results, check out this jupyter notebook.

References

[1] Aleksa Zlojutro, David Rey, and Lauren Gardner. A decision-support framework to optimize border control for global outbreak mitigation. *Scientific Reports*, 9(1):2216, 2019.

[2] Steven H. Strogatz. *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering.* Westview Press, 2000.