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Hausdorff Stability of Persistence Spaces

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Abstract Multidimensional persistence modules do not admit a concise representation analogous to that provided by persistence diagrams for real-valued functions. However, there is no obstruction for multidimensional persistent Betti numbers to admit one. Therefore, it is reasonable to look for a generalization of persistence diagrams concerning those properties that are related only to persistent Betti numbers. In this paper, the *persistence space* of a vector-valued continuous function is introduced to generalize the concept of persistence diagram in this sense. The main result is its stability under function perturbations: Any change in vector-valued functions implies a not greater change in the Hausdorff distance between their persistence spaces.

Keywords Multidimensional persistence · Persistent Betti numbers · Multiplicity · Homological critical value

Mathematics Subject Classification Primary 55N99 · 68U05

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1 Introduction

Topological data analysis deals with the study of global features of data to extract information about the phenomena that data represent. The persistent homology approach to topological data analysis is based on computing homology groups at different scales to see which features are long-lived and which are short-lived. The basic assumption is that relevant features and structures are the ones that persist longer.

In classical persistence, a topological space X is explored through the evolution of the sublevel sets of a real-valued continuous function f defined on X. The role of X is to represent the data set, while f is a descriptor of some property which is considered relevant for the analysis. These sublevel sets, being nested by inclusion, produce a filtration of X. Focusing on the occurrence of important topological events along this filtration—such as the birth and death of connected components, tunnels and voids – it is possible to obtain a global description of data, which can be formalized via an algebraic structure called a *persistence module* [10]. Such information can be encoded in a parameterized version of the Betti numbers, known in the literature as *persistent Betti numbers* [13], a *rank invariant* [4] and—for the 0th homology—a *size function* [15]. The key point is that these descriptors can be represented in a very simple and concise way, by means of multisets of points called *persistence diagrams*. Moreover, they are stable with respect to the bottleneck and Hausdorff distances, thus implying resistance to noise [11]. Thanks to this property, persistence is a viable option for analyzing data from the topological perspective, as shown, for example, in a number of concrete problems concerning shape comparison and retrieval [1,2,5,12].

A common scenario in applications is to deal with multi-parameter information. The use of vector-valued functions enables the study of multi-parameter filtrations, whereas a scalar-valued function only gives a one-parameter filtration. Therefore, Frosini and Mulazzani [16] and Carlsson and Zomorodian [4] proposed *multidimensional persistence* to analyze richer and more complex data.

A major issue in multidimensional persistence is that, when filtrations depend on multiple parameters, it is not possible to provide a complete and discrete representation for multidimensional persistence modules analogous to that provided by persistence diagrams for one-dimensional persistence modules [4]. This theoretical obstruction discouraged so far the introduction of a multidimensional analogue of the persistence diagram.

One can immediately see that the lack of such an analogue is a severe drawback for the actual application of multidimensional persistence to the analysis of data. Therefore, a natural question we may ask ourselves is the following one: In which other sense may we hope to construct a generalization of a persistence diagram for the multidimensional setting?

Cohen-Steiner et al. [11] showed that the persistence diagram satisfies the following important properties (see also [7] for the generalization from tame to arbitrary continuous functions):

- It can be defined via *multiplicities* obtained from persistent Betti numbers;
- It allows to completely reconstruct persistent Betti numbers;

- It is stable with respect to function perturbations; and
- The coordinates of its off-diagonal points are homological critical values.

Therefore, it is reasonable to require that a generalization of a persistence diagram for the multidimensional setting satisfies all these properties. We underline that, because of the aforementioned impossibility result in [4], no generalization of a persistence diagram exists that can achieve the goal of representing completely a persistence module, but only its persistent Betti numbers. For this reason, in this paper, we will only study persistent Betti numbers and not persistence modules.

In the present work, we introduce a *persistence space* to generalize the notion of a persistence diagram in the aforementioned sense. More precisely, we define a persistence space as a multiset of points defined via multiplicities. In the one-dimensional case, it coincides with a persistence diagram. Moreover, it allows for a complete reconstruction of multidimensional persistent Betti numbers (Multidimensional Representation Theorem 3.12). As a further contribution, we show that the coordinates of the off-diagonal points of a persistence space are multidimensional homological critical values (Theorem 4.3). These ideas were anticipated in [9].

Our main result is the stability of persistence spaces under function perturbations (Stability Theorem 5.1): The Hausdorff distance between the persistence spaces of two functions $f, g : X \to \mathbb{R}^n$ is never greater than $\max_{x \in X} \max_{1 \le i \le n} |f_i(x) - g_i(x)|$.

Outline In Sect. 2, we review the basics on multidimensional persistent Betti numbers functions and we fix notations. In Sect. 3, we look at discontinuity points of persistent Betti numbers functions in order to define multiplicity of points. Then, persistence spaces are introduced and are proven to characterize persistent Betti numbers. In Sect. 4, we show that points of a persistence space have coordinates that are homological critical values. We establish the stability result in Sect. 5. Section 6 is about the relation between a persistence space and the persistence diagrams corresponding to certain one-parameter filtrations. Section 7 concludes the paper.

2 Background on Persistence

The main reference about multidimensional persistence modules is [4]. As for multidimensional persistence Betti numbers, we refer the reader to [7]. In accordance with the main topic of this paper, in what follows we will stick to the notations and working assumptions adopted in the latter.

Hereafter, X is a topological space which is assumed to be compact and triangulable, and any function from X to $\mathbb{R}^n$ is supposed to be continuous. When $\mathbb{R}^n$ is viewed as a vector space, its elements are denoted using overarrows. Moreover, in this case, we endow $\mathbb{R}^n$ with the max-norm defined by $\|\vec{v}\|_\infty = \max_i |v_i|$.

For every $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, we write $u \preceq v$ (resp. $u \prec v$, $u \succ v$, $u \succeq v$) if and only if $u_i \le v_i$ (resp. $u_i < v_i$, $u_i > v_i$, $u_i \ge v_i$) for all $i = 1, \ldots, n$. Note that $u \succ v$ is not the negation of $u \preceq v$.

We also use the following notations: D_n^+ will be the open set $\{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : u \prec v\}$, while $D_n = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : u \preceq v \wedge \exists j \text{ s.t.} u_j = v_j\}$. D_n^* will denote the set $D_n^+ \cup \{(u, \infty) : u \in \mathbb{R}^n\}$. Finally, $\overline{D_n^*} = D_n^* \cup D_n$. Points of D_n^+ are called *proper points*, those of $D_n^* \setminus D_n^+$ are *points at infinity*.

For every function $f : X \to \mathbb{R}^n$, we denote by $X\langle f \preceq u \rangle$ the sublevel set $\{x \in X : f(x) \preceq u\}$. For $u \preceq v$, we can consider the inclusion of $X\langle f \preceq u \rangle$ into $X\langle f \preceq v \rangle$. This inclusion induces a homomorphism $\iota_k^{u,v} : \check{H}_k(X\langle f \preceq u\rangle \to \check{H}_k(X\langle f \preceq v\rangle)$, where $\check{H}_k$ denotes the kth Čech homology group for every $k \in \mathbb{Z}$. The image of $\iota_k^{u,v}$ consists of the k-homology classes of cycles "born" no later than u and "still alive" at v. The use of Čech homology will shortly be motivated.

Definition 2.1 (Multidimensional persistent homology group) For $u \prec v$, the image of $\iota_k^{u,v}$ is called the *multidimensional kth persistent homology group of* (X, f) *at* (u, v).

We assume to work with coefficients in a field $\mathbb{K}$. Hence, homology groups are vector spaces, and homomorphisms induced in homology by continuous maps are linear maps. As usual, by the rank of a linear map we mean the dimension of its image. Thus, the rank of $\iota_k^{u,v}$ completely determines persistent homology groups, leading to the notion of *persistent Betti numbers*.

Definition 2.2 (Persistent Betti Numbers) The *persistent Betti numbers function of* $f : X \to \mathbb{R}^n$ (briefly PBNs) is the function $\beta_f : D_n^+ \to \mathbb{N}$ defined, for $(u, v) \in D_n^+$, by

$$\beta_f(u, v) = \operatorname{rk} \iota_k^{u,v}.$$

Obviously, for each $k \in \mathbb{Z}$, we have different PBNs for f (which should be denoted $\beta_{f,k}$, say) but, for the sake of notational simplicity, we omit adding any reference to k. This will also apply to the notations used for other concepts in this paper, such as multiplicities.

Among the properties of PBNs, it is worth mentioning those useful in the rest of the paper.

Proposition 2.3 *(Finiteness) For every* $(u, v) \in D_n^+$, $\beta_f(u, v) < +\infty$.

We remark that the above Proposition 2.3, whose formal proof can be found in [7], holds without any tameness assumption for the continuous function f, only requiring the triangulability of the topological space X.

Proposition 2.4 *(Monotonicity) As an integer-valued function in* $(u, v) \in D_n^+$, β_f *is non-decreasing in u and non-increasing in v with respect to* $\preceq$.

Proposition 2.5 *(Right-Continuity) As a function in* $(u, v) \in D_n^+$, β_f *is right-continuous with respect to both u and v, that is,* $\lim_{u \to \bar{u}, u \succeq \bar{u}} \beta_f(u, v) = \beta_f(\bar{u}, v)$ *and* $\lim_{v \to \bar{v}, v \succeq \bar{v}} \beta_f(u, v) = \beta_f(u, \bar{v})$.

The latter property, proved in [7] for $n = 1$ but valid also for $n > 1$, justifies the use of Čech theory. The proof is based of the continuity axiom of Čech homology (the reader can refer to [14] for details). Using the right-continuity property, in [7] it has been proved that, for $n = 1$, a multiset of points of D_n^*, called a *persistence diagram*, completely describes persistent Betti numbers, without requiring tameness of functions.

3 Persistence Space

The aim of this section is to introduce persistence spaces by analogy with persistence diagrams. In order to do this, we preliminarily study the behavior of discontinuity points of PBNs. In particular, we will prove some results about the propagation of discontinuities of PBNs (Proposition 3.4) and about local constancy of PBNs (Propositions 3.5 and 3.6). These facts will be used to introduce the notion of multiplicity of a point (either proper or at infinity). Points of a persistence space will be exactly those with a positive multiplicity.

The main result of this section is that a persistence space is sufficient to reconstruct the underlying PBNs (Representation Theorem 3.12), in analogy with the one-dimensional framework (cf. the k-Triangle Lemma in [11] and the Representation Theorem 3.11 in [7]).

3.1 PBNs and Discontinuities

We recall that PBNs are functions from D_n^+ to $\mathbb{N}$. Being integer-valued functions, PBNs have jump discontinuities (unless they are identically zero). Precisely, discontinuity points are points (u, v) of D_n^+ such that in every neighborhood of (u, v) in D_n^+, there is a point (u', v') with $\beta_f(u, v) \neq \beta_f(u', v')$. We now study the behavior of discontinuities of PBNs. We start with some lemmas. Lemma 3.1 is analogous to [15, Lemma 1], Lemma 3.2 is analogous to [15, Lemma 2]. Proposition 3.4 is analogous to [15, Cor.1].

It is convenient to introduce the following notations. For $v \in \mathbb{R}^n$, $\beta_f(\cdot, v) : \mathbb{R}^n \to \mathbb{N}$ denotes the function taking each n-tuple u with $u \prec v$ to the number $\beta_f(u, v)$. Analogous meaning will be given to $\beta_f(u, \cdot)$. For every $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n) \in \mathbb{R}^n$, we denote by $\mathbb{R}^n_{\pm}(\bar{u})$ the subset of $\mathbb{R}^n$ given by $\{u \in \mathbb{R}^n : u \prec \bar{u} \vee u \succ \bar{u}\}$. In particular, for a point $u = (u_1, \ldots, u_n)$ in $\mathbb{R}^n_{\pm}(\bar{u})$ it holds that $u_i \neq \bar{u}_i$ for every $i = 1, \ldots, n$.

The following Lemma 3.1 formalizes the observation that, for $u^1 \preceq u^2 \prec v^1 \preceq v^2$, the number of linearly independent homology classes "born" between u^1 and u^2 and "still alive" at v^1 is not smaller than the number of those still alive at v^2.

Lemma 3.1 *(Multidimensional Jump Monotonicity) Let $u^1, u^2, v^1, v^2 \in \mathbb{R}^n$. If $u^1 \preceq u^2 \prec v^1 \preceq v^2$, then*

$$\beta_f(u^2, v^1) - \beta_f(u^1, v^1) \geq \beta_f(u^2, v^2) - \beta_f(u^1, v^2).$$

Proof For every $u \in \mathbb{R}^n$ with $u \prec v^1$, the map $\iota^{v^1,v^2} : \check{H}(X\langle f \preceq v^1\rangle \to \check{H}(X\langle f \preceq v^2\rangle)$ induces a map $\iota_u^{v^1,v^2} : \mathrm{im}\, \iota^{u,v^1} \to \mathrm{im}\, \iota^{u,v^2}$ that turns out to be surjective, implying $\beta_f(u, v^1) - \beta_f(u, v^2) = \dim \ker \iota_u^{v^1,v^2}$. From $u^1 \preceq u^2$, it follows that $\ker \iota_{u^1}^{v^1,v^2} \subseteq \ker \iota_{u^2}^{v^1,v^2}$. Hence $\beta_f(u^1, v^1) - \beta_f(u^1, v^2) \leq \beta_f(u^2, v^1) - \beta_f(u^2, v^2)$, proving the claim. □

Discontinuities points of PBNs exhibit a peculiar behavior. For $u \prec v$, discontinuities in the variable u propagate toward D_n being confined in the "lower" component of $u \times \mathbb{R}^n_{\pm}(v)$; quite conversely, discontinuities in the variable v propagate toward D_n

never escaping from the "upper" component of $v \times \mathbb{R}^n_{\pm}(u)$. Also, being characterized by integer jumps, it is possible to show that discontinuity points of PBNs are precluded from large portions of D^+_n. To formally prove these facts (Propositions 3.4, 3.5 and 3.6), we need the following two results.

Lemma 3.2 is about the fact that a discontinuity point of β_f corresponds to either the "birth" or the "death" of a homology class.

Lemma 3.2 *In D^+_n, any neighborhood of a discontinuity point of β_f contains a point (u, v) with $\beta_f(\cdot, v)$ discontinuous at u, or $\beta_f(u, \cdot)$ discontinuous at v.*

Proof Every neighborhood of p in D^+_n contains an open hypercube Q centered at p. If p is a discontinuity point of β_f, there is a point $q \in Q$ with $\beta_f(p) \neq \beta_f(q)$. We can connect p and q by a path entirely contained in Q made of segments such that either the n-tuple u or the n-tuple v is constant for all points (u, v) of each such segment. β_f cannot be constant along this path. This proves the claim. □

Lemma 3.3 states that if a homology class is "born" at $\bar{u}$, then the linearly independent homology classes immediately before and after $\bar{u}$ are different in number; the case when a class "dies" at $\bar{v}$ is analogous.

Lemma 3.3 *For every $(\bar{u}, \bar{v}) \in D^+_n$, the following statements hold:*

(i) If $\bar{u}$ is a discontinuity point of $\beta_f(\cdot, \bar{v})$, then, for every real number $\varepsilon > 0$, there is a point $u \in \mathbb{R}^n_{\pm}(\bar{u})$ such that $\|u - \bar{u}\|_\infty < \varepsilon$ and $\beta_f(u, \bar{v}) \neq \beta_f(\bar{u}, \bar{v})$;
(ii) If $\bar{v}$ is a discontinuity point of $\beta_f(\bar{u}, \cdot)$, then, for every real number $\varepsilon > 0$, there is a point $v \in \mathbb{R}^n_{\pm}(\bar{v})$ such that $\|v - \bar{v}\|_\infty < \varepsilon$ and $\beta_f(\bar{u}, v) \neq \beta_f(\bar{u}, \bar{v})$.

Proof (i) If $\bar{u}$ is a discontinuity point of $\beta_f(\cdot, \bar{v})$, then, for every $\varepsilon > 0$, there is a point $u' \in \mathbb{R}^n$ such that $\|u' - \bar{u}\|_\infty < \varepsilon$ and $\beta_f(u', \bar{v}) \neq \beta_f(\bar{u}, \bar{v})$. Let us consider the case when $\beta_f(u', \bar{v}) < \beta_f(\bar{u}, \bar{v})$. If $u' \notin \mathbb{R}^n_{\pm}(\bar{u})$, we take a point $u \in \mathbb{R}^n_{\pm}(\bar{u})$ such that $u \preceq u'$ and $\|u - \bar{u}\|_\infty < \varepsilon$. By the monotonicity of PBNs (Proposition 2.4), $\beta_f(u, \bar{v}) \leq \beta_f(u', \bar{v})$. Hence, $\beta_f(u, \bar{v}) < \beta_f(\bar{u}, \bar{v})$, yielding the claim. The case when $\beta_f(u', \bar{v}) > \beta_f(\bar{u}, \bar{v})$ can be handled in much the same way.
(ii) The proof is analogous. □

Proposition 3.4 *For every $(\bar{u}, \bar{v}) \in D^+_n$, the following statements hold:*

(i) If $\bar{u}$ is a discontinuity point of $\beta_f(\cdot, \bar{v})$, then it is a discontinuity point of $\beta_f(\cdot, v)$ for every $\bar{u} \prec v \preceq \bar{v}$;
(ii) If $\bar{v}$ is a discontinuity point of $\beta_f(\bar{u}, \cdot)$, then it is a discontinuity point of $\beta_f(u, \cdot)$ for every $\bar{u} \preceq u \prec \bar{v}$.

Proof We shall confine ourselves to prove only statement (i). Indeed, proving statement (ii) is completely analogous.

Contrary to our claim assume that, for some v with $\bar{u} \prec v \preceq \bar{v}$, $\beta_f(\cdot, v)$ is continuous at $\bar{u}$. Then $\lim_{u \to \bar{u}, u \succeq \bar{u}} \beta_f(u, v) - \beta_f(\bar{u}, v) = 0$. Hence, Lemma 3.1 together with the fact that PBNs are non-decreasing in u (Proposition 2.4) implies that $\lim_{u \to \bar{u}, u \succeq \bar{u}} \beta_f(u, \bar{v}) - \beta_f(\bar{u}, \bar{v}) = 0$. Analogously, $\lim_{u \to \bar{u}, u \preceq \bar{u}} \left(\beta_f(\bar{u}, \bar{v}) - \beta_f(u, \bar{v})\right) = 0$. Hence, for some sufficiently small $\varepsilon > 0$, and for every $u \in \mathbb{R}^n_{\pm}(\bar{u})$ such that $\|u - \bar{u}\|_\infty < \varepsilon$, recalling that β_f is integer-valued, we have $\beta_f(u, \bar{v}) = \beta_f(\bar{u}, \bar{v})$. By Lemma 3.3(i) this implies that $\bar{u}$ cannot be a discontinuity point of $\beta_f(\cdot, \bar{v})$. □

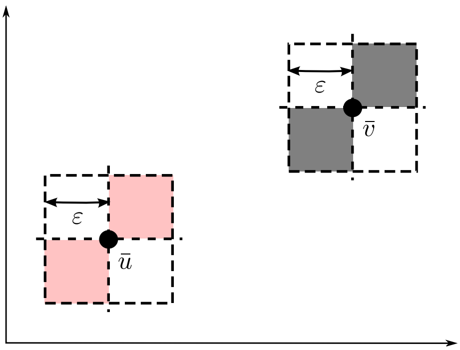

Fig. 1 Graphical representation of $W_\varepsilon(\bar{p}) \subseteq D_n^+$, with $n = 2$, corresponding to the Cartesian product of the *pink area* with the *gray one* (Color figure online)

The next two propositions (analogous to [15, Proposition 6] and [15, Proposition 7], respectively) give some constraints on the presence of discontinuity points of PBNs.

Proposition 3.5 *Let $\bar{p} = (\bar{u}, \bar{v})$ be a proper point of D_n^+. Then, a real number $\varepsilon > 0$ exists, such that the open set*

$$W_\varepsilon(\bar{p}) = \{(u, v) \in \mathbb{R}^n_\pm(\bar{u}) \times \mathbb{R}^n_\pm(\bar{v}) : \|u - \bar{u}\|_\infty < \varepsilon,\ \|v - \bar{v}\|_\infty < \varepsilon\}$$

is a subset of D_n^+, and does not contain any discontinuity point of β_f.

Proof Obviously, there always exists a sufficiently small $\varepsilon > 0$ such that $W_\varepsilon(\bar{p}) \subseteq D_n^+$ (see Fig. 1 to visualize this). Let us now fix $N \in \mathbb{N}$ such that $1/N < \varepsilon$, and suppose, contrary to our assertion, that for every $j \geq N$ a discontinuity point p^j of β_f exists in $W_{1/j}(\bar{p})$. We want to prove that this contradicts the finiteness of β_f (cf. Proposition 2.3).

We set $\bar{u}_{+1/N} = (\bar{u}_1 + 1/N, \ldots, \bar{u}_n + 1/N)$, $\bar{v}_{-1/N} = (\bar{v}_1 - 1/N, \ldots, \bar{v}_n - 1/N)$.

By the previous Lemma 3.2, we know that arbitrarily close to each p^j there is a point $q^j = (u^j, v^j)$ such that either u^j is a discontinuity point of $\beta_f(\cdot, v^j)$, or v^j is a discontinuity point of $\beta_f(u^j, \cdot)$. Therefore, possibly by extracting a subsequence from $(q^j)_{j \geq N}$, we can assume that either each u^j is a discontinuity point of $\beta_f(\cdot, v^j)$ or each v^j is a discontinuity point of $\beta_f(u^j, \cdot)$. We treat in detail only the first case because the other one can be treated similarly.

Since q^j can be taken arbitrarily close to p^j, we can also assume that $u^j \preceq \bar{u}_{+1/N}$ and, possibly by considering again a subsequence, that either $u^j \preceq u^{j+1} \preceq \bar{u}$ for every $j \geq N$, or $\bar{u} \preceq u^{j+1} \preceq u^j$ for every $j \geq N$.

By Proposition 3.4(i) it holds that u^j is a discontinuity point of $\beta_f(\cdot, v)$, for every $u^j \prec v \preceq v^j$. In particular, for $v = \bar{v}_{-1/N}$, $\beta_f(\cdot, v)$ has an integer jump at each u^j. Moreover, we have also that $u^j \preceq \bar{u}_{+1/N}$ for every $j \geq N$. Therefore, since $\beta_f(\cdot, \bar{v}_{-1/N})$ is non-decreasing (cf. Proposition 2.4), and recalling that either $u^j \preceq u^{j+1}$ for every $j \geq N$, or $u^{j+1} \preceq u^j$ for every $j \geq N$, we deduce that $\beta_f(\bar{u}_{+1/N}, \bar{v}_{-1/N}) = +\infty$, thus contradicting the finiteness of β_f. □

Proposition 3.6 *Let $\bar{p} = (\bar{u}, \infty)$ be a point at infinity of D_n^*. Then, a real number $\varepsilon > 0$ exists, such that the open set*

$$V_\varepsilon(\bar{p}) = \left\{(u, v) \in \mathbb{R}^n_\pm(\bar{u}) \times \mathbb{R}^n : \|u - \bar{u}\|_\infty < \varepsilon,\ v_i > \frac{1}{\varepsilon},\ i = 1, \ldots, n\right\}$$

is a subset of D_n^+, and does not contain any discontinuity point of β_f.

Proof First of all, let us observe that $V_\varepsilon(\bar{p}) \subseteq D_n^+$ for ε sufficiently close to 0. Next, we fix $N \in \mathbb{N}$ such that $1/N < \varepsilon$, and suppose, contrary to our assertion, that for every $j \geq N$ a discontinuity point p^j of β_f exists in $V_{1/j}(\bar{p})$. We want to prove that such an assumption leads to contradict the finiteness of β_f (cf. Proposition 2.3).

By the previous Lemma 3.2, we know that arbitrarily close to each p^j there is a point $q^j = (u^j, v^j)$ such that either u^j is a discontinuity point of $\beta_f(\cdot, v^j)$, or v^j is a discontinuity point of $\beta_f(u^j, \cdot)$. It is not restrictive to assume N sufficiently large such that $\max_{x \in X} \|f(x)\|_\infty \leq N$. Since q^j can be taken arbitrarily close to p^j, we can assume that $q^j \in V_{1/j}(\bar{p})$, implying that $v_i^j > j$ for $i = 1, \ldots, n$. Hence, $X\langle f \preceq v^j\rangle = X$ for every $j \geq N$. Therefore, v^j is not a discontinuity point of $\beta_f(u^j, \cdot)$, and hence, u^j is a discontinuity point of $\beta_f(\cdot, v^j)$. Thus, by Proposition 3.4(i), we have that, for every $j \geq N$, u^j is a discontinuity point of $\beta_f(\cdot, v)$, with $u^j \prec v \preceq v^j$. Before going on note that, possibly by considering a subsequence of $(q^j)_{j \geq N}$, we can assume that either $u^j \preceq u^{j+1}$ for every $j \geq N$, or $u^{j+1} \preceq u^j$ for every $j \geq N$.

We now set $\bar{u}_{+1/N} = (\bar{u}_1 + 1/N, \ldots, \bar{u}_n + 1/N)$, $\bar{v} = (N, \ldots, N)$, and consider the function $\beta_f(\cdot, \bar{v})$. According to the previous considerations, such a function should have an infinite number of integer jumps. Indeed, for every $j \geq N$, we have $\bar{v} \prec v^j$, and hence, u^j is a discontinuity point of $\beta_f(\cdot, \bar{v})$. Moreover, we have also that $u^j \preceq \bar{u}_{+1/N}$ for every $j \geq N$. Therefore, since $\beta_f(u, v)$ is non-decreasing in the variable u (cf. Proposition 2.4), and recalling that either $u^j \preceq u^{j+1}$ for every $j \geq N$, or $u^{j+1} \preceq u^j$ for every $j \geq N$, it should be that $\beta_f(\bar{u}_{+1/N}, \bar{v}) = +\infty$, thus contradicting the finiteness of β_f. $\square$

3.2 Persistence Space and Representation Theorem

In this section, we introduce persistence spaces of vector-valued functions, in analogy to persistence diagrams of scalar-valued functions. To do this, we first generalize the notion of multiplicity of a point to the multidimensional setting. Then we introduce persistence spaces as multisets of points with a strictly positive multiplicity. Finally, we show that the PBNs of a continuous function can be reconstructed from multiplicities of points.

We begin with defining the multiplicity of a proper point. For every $(u, v) \in D_n^+$ and $\vec{e} \in \mathbb{R}^n$ with $\vec{e} \succ 0$ and $u + \vec{e} \prec v - \vec{e}$, we consider the number

$$\begin{aligned}\mu_{f,\vec{e}}(u, v) = {} & \beta_f(u + \vec{e}, v - \vec{e}) - \beta_f(u - \vec{e}, v - \vec{e}) \\ & -\beta_f(u + \vec{e}, v + \vec{e}) + \beta_f(u - \vec{e}, v + \vec{e}).\end{aligned} \tag{3.1}$$

The computation of $\mu_{f,\vec{e}}(u, v)$ is illustrated in Fig. 2.

PBNs being integer-valued functions (Proposition 2.3), we have that $\mu_{f,\vec{e}}(u, v)$ is an integer number, and by Lemma 3.1, it is non-negative. Once again by Lemma 3.1, if $0 \prec \vec{e} \preceq \vec{\eta}$, then

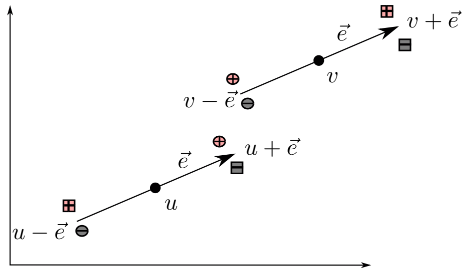

Fig. 2 Computation of $\mu_{f,\vec{e}}(u,v)$ involves the algebraic sum of the values that β_f takes at the four points $(u+\vec{e}, v-\vec{e})$, $(u-\vec{e}, v-\vec{e})$, $(u+\vec{e}, v+\vec{e})$, $(u-\vec{e}, v+\vec{e})$. In this picture, the pairs of coordinates of each point are *shape*- and *color-coded*, and the *plus sign* in the sum is represented by $\oplus$ or $\boxplus$, the *minus sign* by $\ominus$ or $\boxminus$

$$\begin{aligned}
\beta_f(u+\vec{\eta}, v-\vec{\eta}) - \beta_f(u-\vec{\eta}, v-\vec{\eta}) &\geq \beta_f(u+\vec{\eta}, v-\vec{e}\,) - \beta_f(u-\vec{\eta}, v-\vec{e}\,),\\
\beta_f(u+\vec{\eta}, v+\vec{\eta}) - \beta_f(u-\vec{\eta}, v+\vec{\eta}) &\leq \beta_f(u+\vec{\eta}, v+\vec{e}\,) - \beta_f(u-\vec{\eta}, v+\vec{e}\,),\\
\beta_f(u+\vec{\eta}, v-\vec{e}\,) - \beta_f(u+\vec{\eta}, v+\vec{e}\,) &\geq \beta_f(u+\vec{e}, v-\vec{e}\,) - \beta_f(u+\vec{e}, v+\vec{e}\,),\\
\beta_f(u-\vec{\eta}, v-\vec{e}\,) - \beta_f(u-\vec{\eta}, v+\vec{e}\,) &\leq \beta_f(u-\vec{e}, v-\vec{e}\,) - \beta_f(u-\vec{e}, v+\vec{e}\,).
\end{aligned}$$

These inequalities easily imply that the sum defining $\mu_{f,\vec{e}}(u,v)$ is non-decreasing in $\vec{e}$ (with respect to $\preceq$). Moreover, by Proposition 3.5, each term in that sum is constant on the set of those $\vec{e} \in \mathbb{R}^n$ for which $\vec{e} \succ 0$ and $\|\vec{e}\|_\infty$ is sufficiently close to 0. These remarks justify the following definition.

Definition 3.7 For every $p = (u,v) \in D_n^+$, the *multiplicity* $\mu_f(p)$ is the nonnegative integer number defined by setting

$$\mu_f(p) = \min_{\substack{\vec{e}\succ 0\\ u+\vec{e}\prec v-\vec{e}}} \mu_{f,\vec{e}}(u,v)$$

with $\vec{e} \in \mathbb{R}^n$.

In plain words, a proper point with a strictly positive multiplicity captures a topological feature with bounded persistence. Persistence can be defined as follows.

Definition 3.8 The *persistence* of a point $p = (u,v) \in D_n^+$ with multiplicity $\mu_f(p) > 0$ is given by

$$\mathrm{pers}(p) = \min_{i=1,\ldots,n} v_i - u_i .$$

The motivation behind this definition of persistence is that $\min_{i=1,\ldots,n} v_i - u_i$ is directly proportional to the distance of (u,v) to D_n, as explained at the beginning of Sect. 5. Therefore, it gives a measure of the amount of perturbation needed to move a proper point to D_n. Interestingly, $\min_{i=1,\ldots,n} v_i - u_i$ is also strictly related to the one-dimensional persistence in the filtration along the line passing through u and v; for more details on these facts, we refer the reader to Sect. 6.

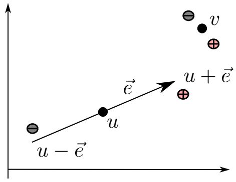

Fig. 3 Computation of $\mu^{\infty}_{f,\vec{e}}(u, v)$ involves the algebraic sum of the values that β_f takes at the points $(u + \vec{e}, v)$, $(u - \vec{e}, v)$. In this picture, the pairs of coordinates of each point are *color-coded*, and the sign in the sum is represented by ⊕ or ⊖

We now similarly define the multiplicity of a point at infinity. For every $(u, v) \in D_n^+$ and $\vec{e} \in \mathbb{R}^n$ with $\vec{e} \succ 0$ and $u + \vec{e} \prec v$, we consider the number

$$\mu^{\infty}_{f,\vec{e}}(u, v) = \beta_f(u + \vec{e}, v) - \beta_f(u - \vec{e}, v). \tag{3.2}$$

The computation of $\mu^{\infty}_{f,\vec{e}}(u, v)$ is illustrated in Fig. 3.

By Proposition 2.3, $\mu^{\infty}_{f,\vec{e}}(u, v)$ is an integer number, and by Proposition 2.4, we know that it is nonnegative. Lemma 3.1 easily implies that it is non-decreasing in $\vec{e}$ and non-increasing in v (with respect to $\preceq$). Moreover, by Proposition 3.6, each term of the sum defining $\mu^{\infty}_{f,\vec{e}}(u, v)$ is constant on the set of those $\vec{e}, v \in \mathbb{R}^n$ for which $\vec{e} \succ 0$, $\|\vec{e}\|_\infty$ is sufficiently close to 0, and $v_i > 1/\|\vec{e}\|_\infty$, for $i = 1, \ldots, n$. These remarks justify the following definition.

Definition 3.9 For every $p = (u, \infty) \in D_n^*$, the *multiplicity* $\mu_f(p)$ is the nonnegative integer number defined by setting

$$\mu_f(p) = \min_{\substack{\vec{e} \succ 0 \\ v : u + \vec{e} \prec v}} \mu^{\infty}_{f,\vec{e}}(u, v).$$

with $\vec{e} \in \mathbb{R}^n$.

As we will see in Corollary 3.13, points at infinity with a nonzero multiplicity capture essential topological features of X. In other words, they correspond to features with unbounded persistence.

Remark 3.10 For $n = 1$, Definitions 3.7 and 3.9 coincide with the definitions of multiplicity of proper points and points at infinity, respectively, used to define persistence diagrams.

Having extended the notion of multiplicity to the multidimensional setting, the definition of a persistence space is now completely analogous to the one of a persistence diagram for real-valued functions.

Definition 3.11 (Persistence Space) The persistence space $\mathrm{Spc}(f)$ is the multiset of all points $p \in D_n^*$ such that $\mu_f(p) > 0$, counted with their multiplicity, union the points of D_n, counted with infinite multiplicity.

Persistence spaces can be reasonably thought as the analogue, in the case of a vector-valued function, of persistence diagrams. Indeed, similarly to the one-dimensional case, a persistence space is completely and uniquely determined by the corresponding persistent Betti numbers. Moreover, even in the multi-parameter situation, the converse is true as well, since it is possible to prove the following Multidimensional Representation Theorem. In what follows, $\langle \vec{e}\,\rangle$ denotes the line in $\mathbb{R}^n$ spanned by $\vec{e}$.

Theorem 3.12 *(Multidimensional Representation Theorem) For every $(\bar{u}, \bar{v}) \in D_n^+$ and every $\vec{e} \succ 0$, it holds that*

$$\beta_f(\bar{u}, \bar{v}) = \sum_{\substack{u \preceq \bar{u},\, v \succ \bar{v} \\ \bar{u}-u, v-\bar{v} \in \langle \vec{e}\,\rangle}} \mu_f(u, v) + \sum_{\substack{u \preceq \bar{u} \\ \bar{u}-u \in \langle \vec{e}\,\rangle}} \mu_f(u, \infty). \tag{3.3}$$

Proof We have seen that, for every $(u, v) \in D_n^+$, and for every $\vec{e} \succ 0$, a positive real number $\varepsilon = \varepsilon(u, v, \vec{e})$ sufficiently small exists, for which

$$\mu_f(u, v) = \mu_{f,\varepsilon\vec{e}}(u, v). \tag{3.4}$$

As for points at infinity, for every $(u, \infty) \in D_n^*$, and for every $\vec{e} \succ 0$, we can choose a positive real number $\varepsilon = \varepsilon(u, \vec{e})$ sufficiently small such that, setting $v^\varepsilon = (\varepsilon^{-1}, \ldots, \varepsilon^{-1})$, we have

$$\mu_f(u, \infty) = \mu^{\infty}_{f,\varepsilon\vec{e}}(u, v^\varepsilon). \tag{3.5}$$

Thus, by (3.4) and (3.5), we get

$$\sum_{\substack{u \preceq \bar{u},\, v \succ \bar{v} \\ \bar{u}-u, v-\bar{v} \in \langle \vec{e}\,\rangle}} \mu_f(u, v) = \sum_{\substack{u \preceq \bar{u},\, v \succ \bar{v} \\ \bar{u}-u, v-\bar{v} \in \langle \vec{e}\,\rangle}} \mu_{f,\varepsilon\vec{e}}(u, v),$$
$$\sum_{\substack{u \preceq \bar{u} \\ \bar{u}-u \in \langle \vec{e}\,\rangle}} \mu_f(u, \infty) = \sum_{\substack{u \preceq \bar{u} \\ \bar{u}-u \in \langle \vec{e}\,\rangle}} \mu^{\infty}_{f,\varepsilon\vec{e}}(u, v^\varepsilon).$$

Now, by the finiteness and the monotonicity of PBNs, at most finitely many discontinuity points of $\beta_f(\cdot, \bar{v})$ exist, say $u^1, \ldots, u^p \in \mathbb{R}^n$, such that $u^i \preceq \bar{u}$ and $\bar{u} - u^i \in \langle \vec{e}\,\rangle$ for all $i = 1, \ldots, p$. Without loss of generality, we can assume that $u^1 \prec \cdots \prec u^p$. Analogously, let $v^1 \prec \cdots \prec v^q$ be the discontinuity points of $\beta_f(\bar{u}, \cdot)$ for which $v^j \succ \bar{v}$ and $v^j - \bar{v} \in \langle \vec{e}\,\rangle$ for all $j = 1, \ldots, q$. In conclusion, we have $u^1 \prec \cdots \prec u^p \preceq \bar{u} \prec \bar{v} \prec v^1 \prec \cdots \prec v^q$.

Note that, for every $v \succ \bar{v}$ with $v - \bar{v} \in \langle \vec{e}\,\rangle$, the restriction of $\beta_f(\cdot, v)$ to the set $U = \{u \in \mathbb{R}^n \mid u \preceq \bar{u},\ \bar{u} - u \in \langle \vec{e}\,\rangle\}$ is continuous at all $u \neq u^i$. Indeed, suppose to the contrary that, for a given $\hat{v} \succ \bar{v}$ with $\hat{v} - \bar{v} \in \langle \vec{e}\,\rangle$, the function $\beta_f(\cdot, \hat{v})$ restricted to the above set is discontinuous at a point $\hat{u} \neq u^i$. Then, by Proposition 3.4, we would have that $\beta_f(\cdot, v)$ is discontinuous at $\hat{u}$ for every v with $\hat{u} \prec v \preceq \hat{v}$. In particular, $\hat{u}$ would be a discontinuity point of $\beta_f(\cdot, \bar{v})$, against the assumption that $u^1, \ldots, u^p$ are the only discontinuity points.

In much the same way we can show that, for any $u \preceq \bar{u}$ with $\bar{u} - u \in \langle \vec{e} \rangle$, the restriction of $\beta_f(u, \cdot)$ to $V = \{v \in \mathbb{R}^n \mid v \succ \bar{v},\ v - \bar{v} \in \langle \vec{e} \rangle\}$ is continuous for all $v \neq v^j$.

These remarks imply that, for $u \in U \setminus \{u^1, \ldots, u^p\}$, $v \in V \setminus \{v^1, \ldots, v^q\}$, and for any real $\varepsilon > 0$ sufficiently small, $\mu_{f,\varepsilon\vec{e}}(u, v) = 0$. Indeed, β_f is integer-valued so that, by recalling equality (3.1), the sum defining $\mu_{f,\varepsilon\vec{e}}(u, v)$ for any such u and v is over constant terms. Similarly, by equality (3.2) it holds that $\mu^{\infty}_{f,\varepsilon\vec{e}}(u, v^{\varepsilon}) = 0$. Therefore, we have

$$\sum_{\substack{u \preceq \bar{u},\ v \succ \bar{v} \\ \bar{u}-u, v-\bar{v} \in \langle \vec{e} \rangle}} \mu_{f,\varepsilon\vec{e}}(u, v) = \sum_{\substack{i=1,\ldots,p \\ j=1,\ldots,q}} \mu_{f,\varepsilon\vec{e}}(u^i, v^j),$$

$$\sum_{\substack{u \preceq \bar{u} \\ \bar{u}-u \in \langle \vec{e} \rangle}} \mu^{\infty}_{f,\varepsilon\vec{e}}(u, v^{\varepsilon}) = \sum_{i=1,\ldots,p} \mu^{\infty}_{f,\varepsilon\vec{e}}(u^i, v^{\varepsilon}).$$

Before going on, note that the following facts hold for every $u \in U$ and $v \in V$:

(F1) If $u^i \preceq u \prec u^{i+1}$ and $v^j \preceq v \prec v^{j+1}$, then $\beta_f(u, v) = \beta_f(u^i, v^j)$. Indeed, β_f is integer-valued and right-continuous (Propositions 2.3 and 2.5). Moreover, $\beta_f(\cdot, v)$ and $\beta_f(u, \cdot)$ are continuous in $U \setminus \{u^1, \ldots, u^p\}$ and $V \setminus \{v^1, \ldots, v^q\}$, respectively;

(F2) If $u \prec u^1$ then $\beta_f(u, v) = 0$;

(F3) If $v \succeq v^q$ then $\beta_f(u, v) = \beta_f(u, v^q)$.

Thus, by applying (F1) and (F2), we get

$$\begin{aligned}\sum_{\substack{i=1,\ldots,p \\ j=1,\ldots,q}} \mu_{f,\varepsilon\vec{e}}(u^i, v^j) &= \beta_f(u^p + \varepsilon\vec{e}, v^1 - \varepsilon\vec{e}) - \beta_f(u^1 - \varepsilon\vec{e}, v^1 - \varepsilon\vec{e}) \\ &\quad + \beta_f(u^1 - \varepsilon\vec{e}, v^q + \varepsilon\vec{e}) - \beta_f(u^p + \varepsilon\vec{e}, v^q + \varepsilon\vec{e}) \\ &= \beta_f(u^p + \varepsilon\vec{e}, v^1 - \varepsilon\vec{e}) - \beta_f(u^p + \varepsilon\vec{e}, v^q + \varepsilon\vec{e}),\end{aligned}$$

ε being a sufficiently small positive real number. Analogously, by applying (F1), (F2) and (F3), we get

$$\begin{aligned}\sum_{i=1,\ldots,p} \mu^{\infty}_{f,\varepsilon\vec{e}}(u^i, v^{\varepsilon}) &= \beta_f(u^p + \varepsilon\vec{e}, v^{\varepsilon}) - \beta_f(u^1 - \varepsilon\vec{e}, v^{\varepsilon}) \\ &= \beta_f(u^p + \varepsilon\vec{e}, v^{\varepsilon}) = \beta_f(u^p + \varepsilon\vec{e}, v^q + \varepsilon\vec{e}).\end{aligned}$$

Indeed, all other terms cancel out, as shown in Fig. 4. So we can write

$$\sum_{\substack{i=1,\ldots,p \\ j=1,\ldots,q}} \mu_{f,\varepsilon\vec{e}}(u^i, v^j) + \sum_{i=1,\ldots,p} \mu^{\infty}_{f,\varepsilon\vec{e}}(u^i, v^{\varepsilon}) = \beta_f(u^p + \varepsilon\vec{e}, v^1 - \varepsilon\vec{e}).$$

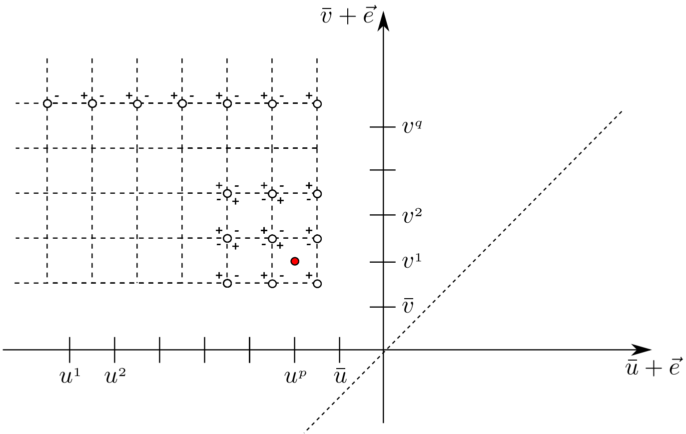

Fig. 4 Idea behind the cancellation process used in the proof of Theorem 3.12

It is not restrictive to assume ε sufficiently small such that, by the right-continuity of $\beta_f(u, v)$ in u and v, $\beta_f(u^p + \varepsilon\vec{e}, v^1 - \varepsilon\vec{e}) = \beta_f(\bar{u}, \bar{v})$ thus getting the claim. $\square$

As a corollary of the Multidimensional Representation Theorem, we get that multiplicities of points at infinity contain the information necessary to compute the Betti numbers $\beta(X)$ of the space X.

Corollary 3.13 *For every $f : X \to \mathbb{R}^n$, there exists $\bar{u} \in \mathbb{R}^n$ for which $X\langle f \preceq \bar{u}\rangle = X$. Then, for every $\vec{e} \succ 0$, it holds that*

$$\beta(X) = \sum_{\bar{u}-u\in\langle\vec{e}\,\rangle} \mu_f(u, \infty).$$

Proof The existence of some point $\bar{u} \in \mathbb{R}^n$ such that $X\langle f \preceq \bar{u}\rangle = X$ is a consequence of the compactness of X and the continuity of f.

Taking $\bar{u}$ such that $X\langle f \preceq \bar{u}\rangle = X$, we get $\beta(X) = \beta_f(\bar{u}, \bar{v})$ for any $\bar{v} \succ \bar{u}$. By the Multidimensional Representation Theorem 3.12,

$$\beta_f(\bar{u}, \bar{v}) = \sum_{\substack{u\preceq\bar{u},\, v\succ\bar{v} \\ \bar{u}-u,\, v-\bar{v}\in\langle\vec{e}\,\rangle}} \mu_f(u, v) + \sum_{\substack{u\preceq\bar{u} \\ \bar{u}-u\in\langle\vec{e}\,\rangle}} \mu_f(u, \infty).$$

We observe that $\mu_f(u, v) = 0$ for every $v \succ \bar{v}$. Indeed, taking $v \succ \bar{v}$, for any $\vec{e} \succ 0$ whose norm is sufficiently close to 0, we have $\beta_f(u+\vec{e}, v-\vec{e}) - \beta_f(u+\vec{e}, v+\vec{e}) = 0$ and $\beta_f(u-\vec{e}, v+\vec{e}) - \beta_f(u-\vec{e}, v-\vec{e}) = 0$, implying that $\mu_{f,\vec{e}}(u, v) = 0$. Analogously, $\mu_f(u, \infty) = 0$ for every $u \succ \bar{u}$. Hence, the claim. $\square$

As a consequence of Corollary 3.13, we have that the existence of points at infinity of $\mathrm{Spc}(f)$ is actually independent from f. This leads to the following remark.

Remark 3.14 For any continuous $f, g : X \to \mathbb{R}^n$, it holds that

$$\mathrm{Spc}(f) \cap \{(u, \infty) : u \in \mathbb{R}^n\} \neq \emptyset \text{ iff } \mathrm{Spc}(g) \cap \{(u, \infty) : u \in \mathbb{R}^n\} \neq \emptyset.$$

We end this section with two results, showing that discontinuity points of β_f propagate from points of a persistence space, both proper and at infinity, toward D_n. For what follows, it is convenient to observe that equality (3.3) can be reformulated as

$$\beta_f(\bar{u}, \bar{v}) = \sum_{s \geq 0, t > 0} \mu_f(\bar{u} - s\vec{e}, \bar{v} + t\vec{e}\,) + \sum_{s \geq 0} \mu_f(\bar{u} - s\vec{e}, \infty). \tag{3.6}$$

Proposition 3.15 *If $(\bar{u}, \bar{v}) \in D_n^+$ is a point with multiplicity $\mu_f(\bar{u}, \bar{v}) > 0$, then the following statements hold:*

(i) $\bar{u}$ is a discontinuity point of $\beta_f(\cdot, \hat{v})$, for every $\hat{v}$ with $\bar{u} \prec \hat{v} \prec \bar{v}$;
(ii) $\bar{v}$ is a discontinuity point of $\beta_f(\hat{u}, \cdot)$, for every $\hat{u}$ with $\bar{u} \preceq \hat{u} \prec \bar{v}$.

Proof Let us prove assertion (i). Fix $\hat{v}$ such that $\bar{u} \prec \hat{v} \prec \bar{v}$. Let $\vec{e} \in \mathbb{R}^n$, with $\vec{e} \succ 0$ and $\|\vec{e}\,\|_\infty$ sufficiently small so that $\bar{u} + \vec{e} \prec \hat{v} \prec \bar{v} - \vec{e}$. By applying the Multidimensional Representation Theorem 3.12, and recalling equality (3.6), we get

$$\begin{aligned}\beta_f(\bar{u} + \vec{e}, \bar{v} - \vec{e}\,) - \beta_f(\bar{u} - \vec{e}, \bar{v} - \vec{e}\,) &= \sum_{\substack{-1 \leq s < 1 \\ t > -1}} \mu_f(\bar{u} - s\vec{e}, \bar{v} + t\vec{e}\,) \\ &\quad + \sum_{-1 \leq s < 1} \mu_f(\bar{u} - s\vec{e}, \infty).\end{aligned}$$

Now, $\mu_f(\bar{u}, \bar{v})$ is an addend of the first sum in the above equality. Since $\mu_f(\bar{u}, \bar{v}) > 0$, we have $\beta_f(\bar{u} + \vec{e}, \bar{v} - \vec{e}) - \beta_f(\bar{u} - \vec{e}, \bar{v} - \vec{e}) > 0$. The Multidimensional Jump Monotonicity Lemma 3.1 implies that $\beta_f(\bar{u} + \vec{e}, \hat{v}) - \beta_f(\bar{u} - \vec{e}, \hat{v}) \geq \beta_f(\bar{u} + \vec{e}, \bar{v} - \vec{e}) - \beta_f(\bar{u} - \vec{e}, \bar{v} - \vec{e})$. By the arbitrariness of $\vec{e}$, we deduce that $\bar{u}$ is a discontinuity point of $\beta_f(\cdot, \hat{v})$. The proof of assertion (ii) is analogous. □

Figure 5 provides a pictorial representation of Proposition 3.15.

Proposition 3.16 *If $(\bar{u}, \infty) \in D_n^*$ is a point at infinity with multiplicity $\mu_f(\bar{u}, \infty) > 0$, then $\bar{u}$ is a discontinuity point of $\beta_f(\cdot, \hat{v})$ for every $\bar{u}$ with $\bar{u} \prec \hat{v}$.*

Proof Analogous to that of Proposition 3.15(i). □

4 The Points of the Persistence Space are Pairs of Homological Critical Values

In this section, we review the concept of homological critical value for a vector-valued continuous function from [6], and we show that the points of a persistence space are pairs of homological critical values.

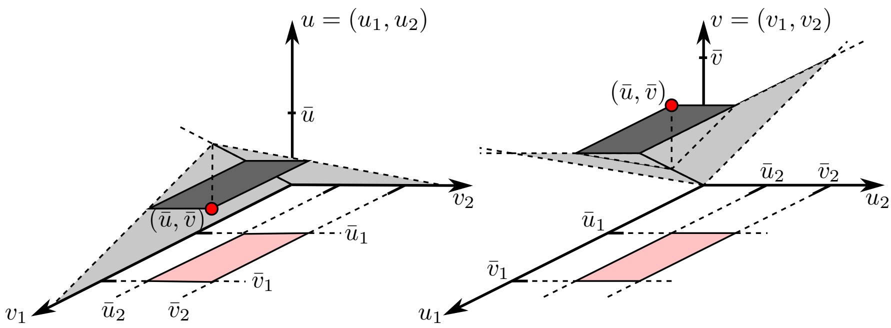

Fig. 5 Graphical description, for $n = 2$, of Proposition 3.15(i) on the *left*, and Proposition 3.15(ii) on the *right*: from a point $(\bar{u}, \bar{v}) \in D_n^+$ with nonzero multiplicity (*red circles*), the discontinuity points of β_f (*dark gray areas*) propagate toward D_n (*light gray areas*); the *pink areas* are the projections of the *dark gray* ones on the planes (v_1, v_2) (*left*) and (u_1, u_2) (*right*) (Color figure online)

Definition 4.1 We shall say that $u \in \mathbb{R}^n$ is a *homological critical value* for $f : X \to \mathbb{R}^n$ if there exists an integer number k such that, for all sufficiently small real values $\varepsilon > 0$, two values $u', u'' \in \mathbb{R}^n$ can be found with $u' \preceq u \preceq u''$, $\|u' - u\|_\infty < \varepsilon$, $\|u'' - u\|_\infty < \varepsilon$, such that the homomorphism $\iota_k^{u',u''} : \check{H}_k(X\langle f \preceq u'\rangle) \to \check{H}_k(X\langle f \preceq u''\rangle)$ induced by inclusion is not an isomorphism.

Let us observe that homological critical values of a vector-valued function f do not necessarily belong to the image of f.

Proposition 4.2 *Let* $(u, v) \in D_n^+$. *The following statements hold:*

(i) If u *is a discontinuity point of* $\beta_f(\cdot, v)$*, then* u *is a homological critical value of* f*;*
(ii) If v *is a discontinuity point of* $\beta_f(u, \cdot)$*, then* v *is a homological critical value of* f.

Proof Let us begin by proving (i). If u is a discontinuity point of $\beta_f(\cdot, v)$ in the homology degree k, then, by Lemma 3.3(i), for every real number $\varepsilon > 0$, there is a point $u' \in \mathbb{R}^n_\pm(u)$ such that $\|u - u'\|_\infty < \varepsilon$ and $\beta_f(u', v) \neq \beta_f(u, v)$.

In the case when $\beta_f(u', v) < \beta_f(u, v)$, we have $u' \prec u$ by the monotonicity of β_f, and hence, it does make sense to consider the homomorphism $\iota_k^{u',u} : \check{H}_k(X\langle f \preceq u'\rangle) \to \check{H}_k(X\langle f \preceq u\rangle)$ induced by inclusion. Let us now prove that $\iota_k^{u',u}$ is not an isomorphism. By contradiction, suppose that $\iota_k^{u',u}$ is an isomorphism. Then, by the commutativity of the diagram

$$\begin{tikzcd}
\check{H}_k(X\langle f \preceq u'\rangle) \arrow[rr, "\iota_k^{u',u}"] \arrow[dr, "\iota_k^{u',v}"'] & & \check{H}_k(X\langle f \preceq u\rangle) \arrow[dl, "\iota_k^{u,v}"] \\
& \check{H}_k(X\langle f \preceq v\rangle) &
\end{tikzcd}$$

it would follow that $\operatorname{im}\iota_k^{u',v}$ and $\operatorname{im}\iota_k^{u,v}$ are isomorphic, thus contradicting the assumption that $\beta_f(u', v) = \operatorname{rk}\iota_k^{u',v} \neq \operatorname{rk}\iota_k^{u,v} = \beta_f(u, v)$. The case $\beta_f(u', v) > \beta_f(u, v)$ can be analogously handled.

A similar proof works for (ii). If v is a discontinuity point of $\beta_f(u, \cdot)$ in the homology degree k, then, by Lemma 3.3(ii), for every real number $\varepsilon > 0$, there is a point $v' \in \mathbb{R}^n_{\pm}(v)$ such that $\|v - v'\|_\infty < \varepsilon$ and $\beta_f(u, v') \neq \beta_f(u, v)$. In the case when $\beta_f(u, v) < \beta_f(u, v')$, we have $v \prec v'$ again by the monotonicity of β_f. By considering the commutative diagram

$$\begin{tikzcd} & \check{H}_k(X\langle f \preceq u\rangle) \arrow[dl, "\iota_k^{u,v}"'] \arrow[dr, "\iota_k^{u,v'}"] & \\ \check{H}_k(X\langle f \preceq v\rangle) \arrow[rr, "\iota_k^{v,v'}"] & & \check{H}_k(X\langle f \preceq v'\rangle) \end{tikzcd}$$

we can prove, again by contradiction, that $\iota_k^{v,v'}$ is not an isomorphism. The case when $\beta_f(u, v) > \beta_f(u, v')$ is similar. □

Theorem 4.3 *If $(\bar{u}, \bar{v}) \in D_n^+$ is a point with multiplicity $\mu_f(\bar{u}, \bar{v}) > 0$, then both $\bar{u}$ and $\bar{v}$ are homological critical values for f. Moreover, if $(\bar{u}, \infty)$, with $\bar{u} \in \mathbb{R}^n$, is a point at infinity with multiplicity $\mu_f(\bar{u}, \infty) > 0$, then $\bar{u}$ is a homological critical value for f.*

Proof The first claim follows immediately from Propositions 3.15 and 4.2, applied in this order. Analogously, the second claim follows from Propositions 3.16 and 4.2, applied in this order. □

We end this section with a further result about homological critical values, for which it is crucial to use a homology theory of Čech type.

Proposition 4.4 *Let X be a compact space having a triangulation of dimension d, and let $f : X \to \mathbb{R}^n$ be a continuous function. Then f has no homological critical values for the homology degrees $k > d$. In particular, β_f is identically zero for $k > d$.*

Proof It is well known (cf. [14, p.320]) that, for any compact pair (Y, A) in X, the Čech homology theory with coefficients in a field ensures that $H_k(Y, A) = 0$ for $k > d$. Applying this result with $Y = X\langle f \preceq u\rangle$ and $A = \emptyset$, we obtain that $H_k(X\langle f \preceq u\rangle)$ is trivial for every $u \in \mathbb{R}^n$ and every $k > d$. □

5 Stability of Persistence Spaces

In this section, we prove that small changes in the considered functions induce not greater changes in the corresponding persistence spaces. In particular, the distance between two functions $f, g : X \to \mathbb{R}^n$ is measured by $\max_{x\in X} \|f(x) - g(x)\|_\infty$, while the distance between $\operatorname{Spc}(f)$ and $\operatorname{Spc}(g)$ is measured according to the Hausdorff distance induced on $\overline{D_n^*}$ by the max-norm:

$$
\begin{aligned}
& d_H(\mathrm{Spc}(f), \mathrm{Spc}(g)) \\
& \quad = \max\left\{ \sup_{p \in \mathrm{Spc}(f)} \inf_{q \in \mathrm{Spc}(g)} \|p - q\|_\infty, \sup_{q \in \mathrm{Spc}(g)} \inf_{p \in \mathrm{Spc}(f)} \|p - q\|_\infty \right\},
\end{aligned}
$$

where, for $p = (u, v), q = (u', v') \in \overline{D_n^*}$, we set

$$
\|p - q\|_\infty = \max\left\{\|u - u'\|_\infty, \|v - v'\|_\infty\right\}, \tag{5.1}
$$

with the convention that $\infty - \infty = 0$ and $v - \infty = \infty - v = \infty$ for every $v \in \mathbb{R}^n$. In this way, in $d_H(\mathrm{Spc}(f), \mathrm{Spc}(g))$, also recalling Remark 3.14, points at infinity are more conveniently compared with other points at infinity. Moreover, a direct computation yields the following formula for the distance of a point $p = (u, v) \in D_n^+$ to D_n:

$$
\inf_{q \in D_n} \|p - q\|_\infty = \min_{i=1,\ldots,n} \frac{v_i - u_i}{2}.
$$

In this setting, our stability result can be stated as follows.

Theorem 5.1 *(Stability Theorem) Let $f, g : X \to \mathbb{R}^n$ be continuous functions. Then*

$$
d_H(\mathrm{Spc}(f), \mathrm{Spc}(g)) \le \max_{x \in X} \|f(x) - g(x)\|_\infty.
$$

The proof of this result is based on the next propositions holding for any continuous function $f : X \to \mathbb{R}^n$. For every $p = (u, v) \in D_n^+$ and every $\vec{e} \in \mathbb{R}^n$ with $\vec{e} \succeq 0$, we set

$$
\mathcal{L}_{\vec{e}}(p) = \{(u - s\vec{e}, v + t\vec{e}) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, s, t \in \mathbb{R}, -1 \le s < 1, -1 < t \le 1\}.
$$

Note that, if $u + \vec{e} \prec v - \vec{e}$, then $\mathcal{L}_{\vec{e}}(p) \subseteq D_n^+$.

Proposition 5.2 *Let $\bar{p} = (\bar{u}, \bar{v}) \in D_n^+$ and $\vec{e} \in \mathbb{R}^n$, with $\vec{e} \succ 0$ and $\bar{u} + \vec{e} \prec \bar{v} - \vec{e}$. Then*

$$
\begin{aligned}
& \beta_f(\bar{u} + \vec{e}, \bar{v} - \vec{e}) - \beta_f(\bar{u} - \vec{e}, \bar{v} - \vec{e}) + \\
& \quad -\beta_f(\bar{u} + \vec{e}, \bar{v} + \vec{e}) + \beta_f(\bar{u} - \vec{e}, \bar{v} + \vec{e})
\end{aligned} \tag{5.2}
$$

is equal to the cardinality of the set $\mathcal{L}_{\vec{e}}(\bar{p}) \cap \mathrm{Spc}(f)$, where proper points of $\mathrm{Spc}(f)$ *are counted with multiplicity.*

Proof It is sufficient to apply the Multidimensional Representation Theorem 3.12, and recall equality (3.6). □

Note that, by the finiteness of β_f and the Multidimensional Jump Monotonicity Lemma 3.1, the sum in (5.2) returns necessarily a nonnegative, integer number. Thus, the cardinality of the set $\mathcal{L}_{\vec{e}}(\bar{p}) \cap \mathrm{Spc}(f)$ is finite, and corresponds to the number of proper points of $\mathrm{Spc}(f)$, counted with their multiplicity, in $\mathcal{L}_{\vec{e}}(\bar{p})$.

Proposition 5.3 *Let $\bar{p} = (\bar{u}, \bar{v}) \in D_n^+$. A real value $\bar{\eta} > 0$ exists such that, for every $\eta \in \mathbb{R}$ with $0 \le \eta \le \bar{\eta}$, the following statements hold:*

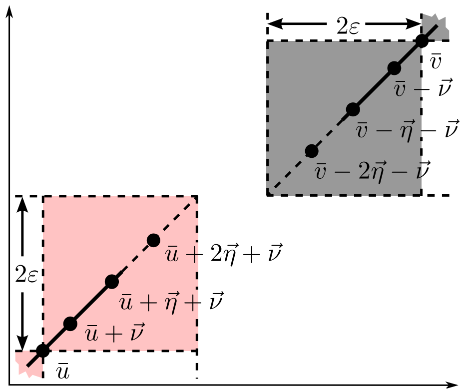

Fig. 6 Points relevant to the proof of Proposition 5.3. *Bold segments* represent part of the set $\mathcal{L}_{\vec{\varepsilon}}(\bar{p}) \subseteq W_{2\varepsilon}(\bar{p})$ used in that proof (Color figure online)

(i) For $\vec{\eta} = (\eta, \eta, \ldots, \eta) \in \mathbb{R}^n$, *the set*

$$\overline{\mathcal{L}_{\vec{\eta}}(\bar{p})} = \left\{(\bar{u} - s\vec{\eta}, \bar{v} + t\vec{\eta}\,) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, s, t \in \mathbb{R}, -1 \le s \le 1, -1 \le t \le 1\right\}$$

is contained in D_n^+*;*

(ii) For every continuous function $g : X \to \mathbb{R}^n$ *with* $\max_{x \in X} \|f(x) - g(x)\|_\infty \le \eta$*, the persistence space* $\mathrm{Spc}(g)$ *has exactly* $\mu_f(\bar{p})$ *proper points in* $\overline{\mathcal{L}_{\vec{\eta}}(\bar{p})}$*, counted with multiplicity.*

Proof By Proposition 3.5, we know that a sufficiently small real number $\varepsilon > 0$ exists such that the set $W_{2\varepsilon}(\bar{p})$ is entirely contained in D_n^+ and does not contain any discontinuity point of β_f. For every η with $0 \le \eta \le \varepsilon$, the set $\overline{\mathcal{L}_{\vec{\eta}}(\bar{p})}$ is contained in D_n^+ thus proving (i).

In order to prove (ii), let $\bar{\eta}$ be any real number such that $0 < \bar{\eta} < \frac{\varepsilon}{2}$, and let $\eta \in \mathbb{R}$ be such that $0 \le \eta \le \bar{\eta}$.

If $\eta = 0$, then $g = f$, $\overline{\mathcal{L}_{\vec{\eta}}(\bar{p})} = \{\bar{p}\}$, and hence, the claim follows. Otherwise, if $\eta > 0$, let us take a sufficiently small real number ν with $0 < \nu < \eta$. We have $\eta + \nu < \varepsilon$ and $2\eta + \nu < 2\varepsilon$. Moreover, setting $\vec{\nu} = (\nu, \nu, \ldots, \nu) \in \mathbb{R}^n$, it holds that $\bar{u} + 2\vec{\eta} + \vec{\nu} \prec \bar{v} - 2\vec{\eta} - \vec{\nu}$. Now, if $g : X \to \mathbb{R}^n$ is a continuous function for which $\max_{x \in X} \|f(x) - g(x)\|_\infty \le \eta$, by [7, Lemma 2.5] we get

$$\beta_f(\bar{u} + \vec{\nu}, \bar{v} - \vec{\nu}\,) \le \beta_g(\bar{u} + \vec{\eta} + \vec{\nu}, \bar{v} - \vec{\eta} - \vec{\nu}\,) \le \beta_f(\bar{u} + 2\vec{\eta} + \vec{\nu}, \bar{v} - 2\vec{\eta} - \vec{\nu}\,).$$

Since β_f is constant on each connected component of $W_{2\varepsilon}(\bar{p})$, and $(\bar{u} + \vec{\nu}, \bar{v} - \vec{\nu}\,)$ and $(\bar{u} + 2\vec{\eta} + \vec{\nu}, \bar{v} - 2\vec{\eta} - \vec{\nu}\,)$ belong to the same connected component of $W_{2\varepsilon}(\bar{p})$, as illustrated in Fig. 6, we have

$$\beta_f(\bar{u} + \vec{\nu}, \bar{v} - \vec{\nu}\,) = \beta_f(\bar{u} + 2\vec{\eta} + \vec{\nu}, \bar{v} - 2\vec{\eta} - \vec{\nu}\,),$$

thus implying that $\beta_f(\bar{u}+\vec{\eta}+\vec{\nu}, \bar{v}-\vec{\eta}-\vec{\nu}\,) = \beta_g(\bar{u}+\vec{\eta}+\vec{\nu}, \bar{v}-\vec{\eta}-\vec{\nu}\,)$. Analogously, $\beta_f(\bar{u} - \vec{\eta} - \vec{\nu}, \bar{v} - \vec{\eta} - \vec{\nu}\,) = \beta_g(\bar{u} - \vec{\eta} - \vec{\nu}, \bar{v} - \vec{\eta} - \vec{\nu}\,)$, $\beta_f(\bar{u} + \vec{\eta} + \vec{\nu}, \bar{v} + \vec{\eta} + \vec{\nu}\,) = \beta_g(\bar{u} + \vec{\eta} + \vec{\nu}, \bar{v} + \vec{\eta} + \vec{\nu}\,)$ and $\beta_f(\bar{u} - \vec{\eta} - \vec{\nu}, \bar{v} + \vec{\eta} + \vec{\nu}\,) = \beta_g(\bar{u} - \vec{\eta} - \vec{\nu}, \bar{v} + \vec{\eta} + \vec{\nu}\,)$.

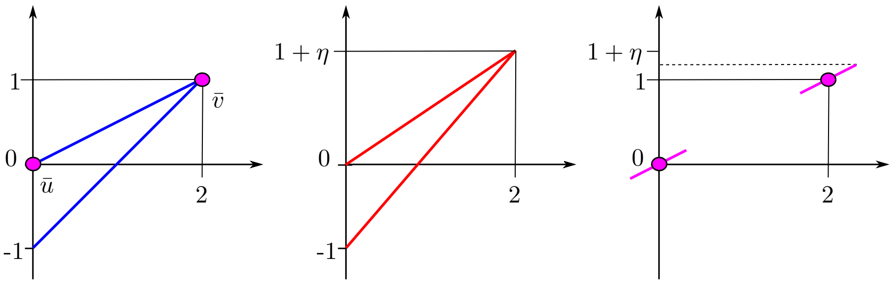

Fig. 7 im f (*bold blue line*, *left*), im g (*bold red line*, *center*) and a representation of the set $\overline{\mathcal{L}_{\vec{\eta}}(\bar{p})}$ (*bold purple segments*, *right*) as defined in Example 5.4 (Color figure online)

Therefore, by Proposition 5.2 applied both to g and f, the number of proper points of $\mathrm{Spc}(g)$ in $\mathcal{L}_{\vec{\eta}+\vec{\nu}}(\bar{p})$, counted with multiplicity, is equal to that of $\mathrm{Spc}(f)$. On the other hand, $\mathcal{L}_{\vec{\eta}+\vec{\nu}}(\bar{p}) \subseteq \mathcal{L}_{\vec{\varepsilon}}(\bar{p})$. Hence, by Proposition 3.15, recalling that $W_{2\varepsilon}(\bar{p})$ does not contain any discontinuity point of β_f, no proper point of $\mathrm{Spc}(f)$ is in $\mathcal{L}_{\vec{\eta}+\vec{\nu}}(\bar{p})$, except possibly for $\bar{p}$. In conclusion, $\mu_f(\bar{p})$ equals the number of proper points of $\mathrm{Spc}(g)$, counted with multiplicity, contained in the set $\mathcal{L}_{\vec{\eta}+\vec{\nu}}(\bar{p})$.

This is true for every sufficiently small $\nu > 0$. Therefore, $\mu_f(\bar{p})$ is equal to the number of points of $\mathrm{Spc}(g)$ contained in the intersection $\bigcap_{\nu>0} \mathcal{L}_{\vec{\eta}+\vec{\nu}}(\bar{p})$, thus proving the claim. □

Before going on, we remark that the proposed proof of Proposition 5.3 cannot work without assuming that $\vec{\varepsilon}$, $\vec{\eta}$ and $\vec{\nu}$ are multiples of $\vec{1} = (1, 1, \dots, 1) \in \mathbb{R}^n$. The obstruction is in the application of [7, Lemma 2.5]. Also, such requirement ensures that the intersection $\bigcap_{\nu>0} \mathcal{L}_{\vec{\eta}+\vec{\nu}}(\bar{p})$ in the end of the proof is over a sequence of nested sets, thus implying the claim. These remarks actually mirror the fact that Proposition 5.3 does not hold without our assumptions on $\vec{\eta}$, as shown by the following example where we take $\vec{\eta} \in \mathbb{R}^2$ with $\vec{\eta} = \left(\eta, \frac{\eta}{2}\right)$. Similar examples for which Proposition 5.3 does not hold can be built for every $\vec{\eta} \neq (\eta, \eta)$.

Example 5.4 Let X be the closed interval $[0, 1]$, and let $f, g : X \to \mathbb{R}^2$ be two functions linearly interpolating the following values: $f(0) = g(0) = (0, 0)$, $f(1) = g(1) = (0, -1)$, $f\left(\frac{1}{2}\right) = (2, 1)$ and $g\left(\frac{1}{2}\right) = (2, 1 + \eta)$, with $\eta > 0$, as depicted in Fig. 7. We have $\max_{x\in X} \|f(x) - g(x)\|_\infty = \eta$.

Set $\bar{u} = (0, 0)$ and $\bar{v} = (2, 1)$. According to Definition 3.7, the point $\bar{p} = (\bar{u}, \bar{v})$ has multiplicity $\mu_f(\bar{p})$ equal to 1 (we are considering $0th$ homology). Hence, $\bar{p} \in \mathrm{Spc}(f)$. Suppose now η sufficiently small so that, taking $\vec{\eta} = \left(\eta, \frac{\eta}{2}\right)$, the set $\overline{\mathcal{L}_{\vec{\eta}}(\bar{p})}$ is entirely contained in D_n^+. In this case, it is easy to check that, in contrast with Proposition 5.3, $\overline{\mathcal{L}_{\vec{\eta}}(\bar{p})}$ does not contain points of $\mathrm{Spc}(g)$.

Proposition 5.5 *Let $f, g : X \to \mathbb{R}^n$ be two continuous functions such that $\max_{x\in X} \|f(x) - g(x)\|_\infty \leq \varepsilon$. Then, for every proper point $p \in \mathrm{Spc}(f)$, a point $q \in \mathrm{Spc}(g)$ exists such that $\|p - q\|_\infty \leq \varepsilon$.*

Proof Let $p \in \mathrm{Spc}(f)$. If a point $q \in D_n$ exists, for which $\|p-q\|_\infty \le \varepsilon$, then there is nothing to prove because, by definition, $q \in \mathrm{Spc}(g)$. Hence, let us assume that $\|p-q\|_\infty > \varepsilon$ for all $q \in D_n$. For every $\tau \in [0,\varepsilon]$, let h_τ be the function defined as $h_\tau = \frac{\varepsilon-\tau}{\varepsilon} f + \frac{\tau}{\varepsilon} g$. Note that, for every $\tau, \tau' \in [0,\varepsilon]$, we have $\max_{x\in X} \|h_\tau(x) - h_{\tau'}(x)\|_\infty \le |\tau - \tau'|$.

For $\tau \in [0,\varepsilon]$, let $\vec{\tau} = (\tau, \tau, \ldots, \tau) \in \mathbb{R}^n$. Since $\|p-q\|_\infty > \varepsilon$ for all $q \in D_n$, we have that $\overline{\mathcal{L}_{\vec{\tau}}(p)} \subset D_n^+$ for every $\tau \in [0,\varepsilon]$. Now, consider the set

$$A = \left\{\tau \in [0,\varepsilon] : \exists\, q_\tau \in \mathrm{Spc}(h_\tau) \cap \overline{\mathcal{L}_{\vec{\tau}}(p)}\right\}.$$

A is non-empty, since $0 \in A$. Let us set $\tau_* = \sup A$ and show that $\tau_* \in A$. Indeed, let (τ_j) be a sequence in A converging to τ_*. Since $\tau_j \in A$, for each j there is a proper point $q_j \in \mathrm{Spc}(h_{\tau_j})$ such that $q_j \in \overline{\mathcal{L}_{\vec{\tau}_j}(p)} \subseteq \overline{\mathcal{L}_{\vec{\varepsilon}}(p)}$. By the compactness of $\overline{\mathcal{L}_{\vec{\varepsilon}}(p)}$, possibly by extracting a convergent subsequence, we can define $q = \lim_j q_j$.

We have that $q \in \overline{\mathcal{L}_{\vec{\tau}_*}(p)}$. Indeed, if $q \notin \overline{\mathcal{L}_{\vec{\tau}_*}(p)}$, then for every sufficiently large index j, we have $q_j \notin \overline{\mathcal{L}_{\vec{\tau}_*}(p)}$. On the other hand, since $\tau_j \le \tau_*$ for all j, it holds that $\overline{\mathcal{L}_{\vec{\tau}_j}(p)} \subseteq \overline{\mathcal{L}_{\vec{\tau}_*}(p)}$, thus giving a contradiction.

Moreover, the multiplicity $\mu_{h_{\tau_*}}(q)$ of q for $\beta_{h_{\tau_*}}$ is strictly positive. Indeed, since $\tau_j \to \tau_*$ and $q \in \overline{\mathcal{L}_{\vec{\tau}_*}(p)}$, for every arbitrarily small $\eta > 0$ and any sufficiently large j, the set $\overline{\mathcal{L}_{\vec{\eta}}(q)}$, with $\vec{\eta} = (\eta, \eta, \ldots, \eta) \in \mathbb{R}^n$, contains at least one proper point $q_j \in \mathrm{Spc}(h_{\tau_j})$. But Proposition 5.3 implies that, for each sufficiently small $\eta > 0$, the set $\overline{\mathcal{L}_{\vec{\eta}}(q)}$ contains exactly as many proper points of $\mathrm{Spc}(h_{\tau_j})$ as $\mu_{h_{\tau_*}}(q)$, provided that $|\tau_j - \tau_*| \le \eta$. Therefore, the multiplicity $\mu_{h_{\tau_*}}(q)$ is strictly positive, thus implying that $\tau_* \in A$.

To conclude the proof, we have to show that $\max A = \varepsilon$. If $\tau_* < \varepsilon$, by using Proposition 5.3 once again we see that there exist a real value $\eta > 0$ with $\tau_* + \eta < \varepsilon$, and a point $q_{\tau_*+\eta} \in \mathrm{Spc}(h_{\tau_*+\eta})$ for which $q_{\tau_*+\eta} \in \overline{\mathcal{L}_{\vec{\eta}}(q)}$. Consequently, $\|q - q_{\tau_*+\eta}\|_\infty \le \eta$. Thus, by the triangle inequality we would have $\|p - q_{\tau_*+\eta}\|_\infty \le \tau_* + \eta$ and hence $q_{\tau_*+\eta} \in \overline{\mathcal{L}_{\vec{\tau}_*+\vec{\eta}}(p)}$, implying that $\tau_* + \eta \in A$. Obviously, this would contradict the fact that $\tau_* = \max A$. Therefore, $\varepsilon = \max A$, so that $\varepsilon \in A$. Clearly, this proves the claim. □

In analogy to proper points, we prove the equivalent of Propositions 5.2, 5.3 and 5.5 for points at infinity. For every $p = (u, \infty) \in D_n^*$ and every $\vec{e} \in \mathbb{R}^n$ with $\vec{e} \succeq 0$, we set

$$\mathcal{N}_{\vec{e}}(p) = \{(u - s\vec{e}, \infty) \in D_n^* \mid -1 \le s < 1\}.$$

Further, for every real value $\varepsilon > 0$ we denote by v^ε the n-tuple $(\varepsilon^{-1}, \varepsilon^{-1}, \ldots, \varepsilon^{-1})$.

Proposition 5.6 *Let $\bar{p} = (\bar{u}, \infty) \in D_n^*$. A sufficiently small real value $\varepsilon > 0$ exists such that $\bar{u} + \vec{e} \prec v^\varepsilon$ for all $\vec{e} \in \mathbb{R}^n$ with $\vec{e} \succ 0$ and $\|\vec{e}\|_\infty < \varepsilon$. Moreover, for every $v \succ v^\varepsilon$,*

$$\beta_f(\bar{u} + \vec{e}, v) - \beta_f(\bar{u} - \vec{e}, v) \tag{5.3}$$

is equal to the cardinality of the set $\mathcal{N}_{\vec{e}}(\bar{p}) \cap \mathrm{Spc}(f)$*, where points at infinity of* $\mathrm{Spc}(f)$ *are counted with multiplicity.*

Proof Clearly, a sufficiently small real value $\varepsilon > 0$ exists for which $\bar{u} + \vec{e} \prec v^{\varepsilon}$ whenever $\vec{e} \succ 0$ and $\|\vec{e}\|_{\infty} < \varepsilon$. For every $v \succ v^{\varepsilon}$, by applying the Multidimensional Representation Theorem and using (3.6) we obtain

$$\beta_f(\bar{u} + \vec{e}, v) - \beta_f(\bar{u} - \vec{e}, v) = \sum_{\substack{-1 \le s < 1 \\ t > 0}} \mu_f(\bar{u} - s\vec{e}, v + t\vec{e}) + \sum_{-1 \le s < 1} \mu_f(\bar{u} - s\vec{e}, \infty). \tag{5.4}$$

Now, suppose ε is small enough so that $\varepsilon^{-1} \ge \max_{x \in X} \|f(x)\|_{\infty}$. It follows that $X\langle f \preceq v^{\varepsilon} \rangle = X$. Hence, if $v \succ v^{\varepsilon}$, then v cannot be a discontinuity point of $\beta_f(u, \cdot)$, for any $u \prec v$. By Proposition 3.15, this implies that the first sum in (5.4) runs over proper points with null multiplicity, and hence, all its terms vanish. This proves the claim. □

The finiteness and the monotonicity of β_f imply that the sum in (5.3) results in a nonnegative integer number. Hence, the cardinality of the set $\mathcal{N}_{\vec{e}}(\bar{p}) \cap \mathrm{Spc}(f)$ is finite, and corresponds to the number of points at infinity of $\mathrm{Spc}(f)$, counted with multiplicity, in $\mathcal{N}_{\vec{e}}(\bar{p})$.

Proposition 5.7 *Let* $\bar{p} = (\bar{u}, \infty) \in D_n^*$. *A real value* $\bar{\eta} > 0$ *exists such that, for every* $\eta \in \mathbb{R}$ *with* $0 \le \eta \le \bar{\eta}$ *and every continuous function* $g : X \to \mathbb{R}^n$ *with* $\max_{x \in X} \|f(x) - g(x)\|_{\infty} \le \eta$*, the persistence space* $\mathrm{Spc}(g)$ *has exactly* $\mu_f(\bar{p})$ *points at infinity, counted with multiplicity, in the set*

$$\overline{\mathcal{N}_{\vec{\eta}}(\bar{p})} = \{(\bar{u} - s\vec{\eta}, \infty) \in D_n^* | -1 \le s \le 1\},$$

with $\vec{\eta} = (\eta, \eta \dots, \eta) \in \mathbb{R}^n$.

Proof By Proposition 3.6, a sufficiently small $\varepsilon > 0$ exists such that the set $V_{2\varepsilon}(\bar{p})$ is entirely contained in D_n^+, it does not contain any discontinuity point of β_f and, setting $\vec{\varepsilon} = (\varepsilon, \varepsilon, \dots, \varepsilon)$, $\bar{u} + 2\vec{\varepsilon} \prec v^{\varepsilon}$. Proposition 3.16 implies that $\mathrm{Spc}(f)$ has no points at infinity in $\mathcal{N}_{\vec{\varepsilon}}(\bar{p})$, except possibly for $\bar{p}$.

Take $\bar{v} \in \mathbb{R}^n$ such that $\bar{v} - \frac{\vec{\varepsilon}}{2} \succ v^{\varepsilon}$. Let $\bar{\eta}$ be any real number for which $0 < \bar{\eta} < \frac{\varepsilon}{2}$, and let $\eta \in \mathbb{R}$ be such that $0 \le \eta \le \bar{\eta}$.

If $\eta = 0$ then $g = f$, $\overline{\mathcal{N}_{\vec{\eta}}(\bar{p})} = \{\bar{p}\}$, and hence, the claim follows.

Otherwise, if $\eta > 0$, let us consider a sufficiently small $\nu \in \mathbb{R}$ with $0 < \nu < \eta$. We have $\eta + \nu < \varepsilon$ and $2\eta + \nu < 2\varepsilon$. Moreover, setting $\vec{\nu} = (\nu, \nu, \dots, \nu) \in \mathbb{R}^n$, it holds that $\bar{u} + 2\vec{\eta} + \vec{\nu} \prec \bar{v} - \vec{\eta}$. Now, if $g : X \to \mathbb{R}^n$ is a continuous function for which $\max_{x \in X} \|f(x) - g(x)\|_{\infty} \le \eta$, by [7, Lemma 2.5] we get

$$\beta_f(\bar{u} + \vec{\nu}, \bar{v} + \vec{\eta}) \le \beta_g(\bar{u} + \vec{\eta} + \vec{\nu}, \bar{v}) \le \beta_f(\bar{u} + 2\vec{\eta} + \vec{\nu}, \bar{v} - \vec{\eta}).$$

Note that $(\bar{u}+\vec{\nu},\bar{v}+\vec{\eta})$ and $(\bar{u}+2\vec{\eta}+\vec{\nu},\bar{v}-\vec{\eta})$ belong to the same connected component of $V_{2\varepsilon}(\bar{p})$. Since β_f is constant on each connected component of $V_{2\varepsilon}(\bar{p})$, we have

$$\beta_f(\bar{u}+\vec{\nu},\bar{v}+\vec{\eta})=\beta_f(\bar{u}+2\vec{\eta}+\vec{\nu},\bar{v}-\vec{\eta}),$$

thus implying that $\beta_f(\bar{u}+\vec{\eta}+\vec{\nu},\bar{v})=\beta_g(\bar{u}+\vec{\eta}+\vec{\nu},\bar{v})$. Analogously, $\beta_f(\bar{u}-\vec{\eta}-\vec{\nu},\bar{v})=\beta_g(\bar{u}-\vec{\eta}-\vec{\nu},\bar{v})$.

Since $\mathrm{Spc}(f)$ has no points at infinity in $\mathcal{N}_{\vec{\varepsilon}}(\bar{p})$ except possibly for $\bar{p}$, by Proposition 5.6 and the previous equalities we get that $\mu_f(\bar{p})$ equals the number of points at infinity of $\mathrm{Spc}(g)$ contained in the set $\mathcal{N}_{\vec{\eta}+\vec{\nu}}(\bar{p})$. This is true for every sufficiently small $\nu>0$. Therefore, $\mu_f(\bar{p})$ is equal to the number of points at infinity of $\mathrm{Spc}(g)$ contained in the intersection $\bigcap_{\nu>0}\mathcal{N}_{\vec{\eta}+\vec{\nu}}(\bar{p})$, thus proving the claim. □

Proposition 5.8 *Let $f,g:X\to\mathbb{R}^n$ be two continuous functions such that $\max_{x\in X}\|f(x)-g(x)\|_\infty\le\varepsilon$. Then, for every point at infinity $p=(u,\infty)\in\mathrm{Spc}(f)$, a point $q=(u',\infty)\in\mathrm{Spc}(g)$ exists such that $\|p-q\|_\infty\le\varepsilon$.*

Proof The proof is analogous to that of Proposition 5.5, after noting that the norm $\|\cdot\|_\infty$ introduced in (5.1) naturally induces a topology on D_n^*. □

We are now ready to prove the stability of persistence spaces.

Proof of Therorem 5.1 Let $\max_{x\in X}\|f(x)-g(x)\|_\infty=\varepsilon$. Propositions 5.5 and 5.8 imply that $\sup_{p\in\mathrm{Spc}(f)}\inf_{q\in\mathrm{Spc}(g)}\|p-q\|_\infty\le\varepsilon$. Moreover, by exchanging the roles of f and g, once more by Propositions 5.5 and 5.8 we also get $\sup_{q\in\mathrm{Spc}(g)}\inf_{p\in\mathrm{Spc}(f)}\|p-q\|_\infty\le\varepsilon$. Thus, $d_H(\mathrm{Spc}(f),\mathrm{Spc}(g))\le\varepsilon$, and the claim follows. □

6 Persistence Spaces and Reduction to One-Parameter Filtrations

In this section, we study the relation between the persistence space of a vector-valued function and the persistence diagrams of scalar functions corresponding to one-parameter filtrations along monotonically increasing lines.

We recall from Sect. 3.2 that the multiplicity of a point, which if strictly positive signals points of the persistence space, is initially based on minimizing a certain quantity varying a vector $\vec{e}\succ 0$ of $\mathbb{R}^n$ in length and direction. As a consequence of Propositions 3.5 and 3.6, we have seen that the direction of $\vec{e}$ is ultimately irrelevant to compute the multiplicity of a point. On the other hand, this arbitrariness in the choice of the direction of $\vec{e}$ can be deployed to deduce specific properties of persistence spaces. For instance, in order to prove the stability of persistence spaces in the Hausdorff sense, it is crucial to take $\vec{e}=(1,\dots,1)$ as shown in Example 5.4. In this way, we are led to study multiciplicities along parallel lines.

On the other hand, a natural question is what happens when the multiplicity of a point (u,v) is instead computed along the direction $v-u$, thus allowing us to use a single line as in previous papers [2,3,7]. The next result answers this question by characterizing points (u, v) of the persistence space $\mathrm{Spc}(f)$ of a function $f : X \to \mathbb{R}^n$ as points of the persistence diagram $\mathrm{Dgm}(F_{(u,v)})$ of a scalar function $F_{(u,v)} : X \to \mathbb{R}$.

We need some premises. For a point $(u, v) \in D_n^+$, the line $L \subseteq \mathbb{R}^n$ passing trough u and v can be parameterized by a parameter $s \in \mathbb{R}$ as $L : u = s\vec{e} + b$ with $\vec{e} \succ 0$ and $b \in \mathbb{R}^n$ a point belonging to L. It is worth mentioning that further constraints on $\vec{e}$ and b ensure that they can be uniquely chosen. This setting allows us to define

$$F_{(u,v)}(x) = \max_i \left\{ \frac{f_i(x) - b_i}{e_i} \right\}.$$

Proposition 6.1 *The following statements hold:*

(i) For every $(u, v) \in D_n^+$ with $(u, v) = (s\vec{e}+b, t\vec{e}+b)$, it holds that $(u, v) \in \mathrm{Spc}(f)$ if and only if $(s, t) \in \mathrm{Dgm}(F_{(u,v)})$;
(ii) For every $(u, \infty) \in D_n^*$, with $u = s\vec{e} + b$, it holds that $(u, \infty) \in \mathrm{Spc}(f)$ if and only if $(s, \infty) \in \mathrm{Dgm}(F_{(u,v)})$.

Proof By [3, Lemma1], the one-parameter filtration of X obtained by sweeping the line $L : u = s\vec{e} + b$ corresponds to the sublevel sets of $F_{(u,v)}$. Next, by [2, Prop. 1], if $u = s\vec{e} + b$ and $v = t\vec{e} + b$ then $\mu_f(u, v) = \mu_{F_{(u,v)}}(s, t)$, and $\mu_f(u, \infty) = \mu_{F_{(u,v)}}(s, \infty)$, yielding the claim. □

As a consequence of Proposition 6.1, it is now possible to establish a relation between the persistence of a proper point $(u, v) \in \mathrm{Spc}(f)$ and that of its corresponding point $(s, t) \in \mathrm{Dgm}(F_{(u,v)})$.

Corollary 6.2 *If $(u, v) \in \mathrm{Spc}(f) \cap D_n^+$ and $(u, v) = (s\vec{e} + b, t\vec{e} + b)$, then*

$$\frac{pers(u, v)}{pers(s, t)} = \min_i e_i. \tag{6.1}$$

Proof It is sufficient to observe that, for all $i = 1, \ldots, n$, we can write

$$v_i - u_i = te_i + b_i - (se_i + b_i) = e_i(t - s),$$

which leads to the claim after recalling Definition 3.8. □

Hence, $\mathrm{pers}(u, v)$ and $\mathrm{pers}(s, t)$ are equal up to the scaling factor $\min_i e_i$. This coefficient $\min_i e_i$ is inversely proportional to how much the stability degrades in the reduction to scalar functions for lines that are nearly parallel to coordinate hyperplanes [7].

7 Discussion

We have presented a complete representation of multidimensional persistent Betti numbers via persistence spaces, which is stable in the Hausdorff sense under function perturbations.

In our present treatment of persistence spaces, we have focused on data belonging to the topological category. We briefly discuss here a different setting which has been treated in [8], namely the case when the considered space X and the function $f : X \to \mathbb{R}^n$ belong to the smooth category. The ideas contained in that work can be used to establish a link between persistence spaces and the concept of *Pareto criticality*.

More in detail, assume X is a smooth, closed (i.e. compact without boundary), connected Riemannian manifold, and $f : X \to \mathbb{R}^n$ is a smooth function. A point $x \in X$ is a *Pareto critical point* of f if the convex hull of $\nabla f_1(x), \ldots, \nabla f_n(x)$ contains the null vector. Moreover, $u \in \mathbb{R}^n$ is a *Pareto critical value* of f if $u = f(x)$ for some Pareto critical point $x \in X$.

In this setting, following [8], it is possible to show that the coordinates of the proper points of $\mathrm{Spc}(f)$ can be projected, through a suitable map $\rho : \mathbb{R}^n \to \mathbb{R}^h$, with $h \leq n$, to Pareto critical values of $\rho \circ f$.

From the application viewpoint, an interesting case is when data come in the form of triangulated compact spaces endowed with interpolated functions.

The discrete case of multidimensional persistent Betti numbers has been treated in [6] so we refer the interested reader to that paper for further details. Here we confine ourselves to report that in that case the set of homological critical values of f is a nowhere dense set in $\mathbb{R}^n$. Moreover its n-dimensional Lebesgue measure is zero. Finally, it is worth mentioning that, although the set of homological critical values may be an uncountable set even in the discrete setting, it admits a finite representative set as stated in [6, Prop. 4.6].

These results suggest the following open question: in the discrete case, is it possible to determine a finite representative for the corresponding persistence spaces? Clearly, a positive answer to this question would open the way to the definition of a bottleneck distance between these representative points.

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