Persistence for Circle Valued Maps

Dan Burghelea[*]   Tamal K. Dey[†]

Abstract

We study circle valued maps and consider the *persistence of the homology of their fibers*. The outcome is a finite collection of computable invariants (bar codes and Jordan cells) which answer the basic questions on persistence and in addition encode the topology of the source space and its relevant subspaces. We show how to recover the homology of the source space and of its relevant subspaces and how to compute the invariants. In particular, we reduce the computation of the bar codes to algorithms described for zigzag [4] and standard persistence [11, 16]. We show how persistence of circle valued maps can be extended to determine persistence for a class of 1-cocycles.

[*]Department of Mathematics, The Ohio State University, Columbus, OH 43210,USA. Email: `burghele@math.ohio-state.edu`

[†]Department of Computer Science and Engineering, The Ohio State University, Columbus, OH 43210, USA. Email: `tamaldey@cse.ohio-state.edu`

1 Introduction

Data analysis provides plenty of scenarios where one ends up with a nice space, most often a simplicial complex, a smooth manifold, or a stratified space equipped with a real valued or a circle valued map. The persistence theory, introduced in [11] and refined in [16], provides a great tool for analyzing real valued maps in terms of topological invariants. Similar theory for circle valued maps has not yet been developed in the literature. The work in [17] brings the concept of circle valued maps in the context of persistence by deriving a circle valued map for a given data using the existing persistence theory. In contrast, we develop a persistence theory for circle valued maps.

One place where circle valued maps appear naturally is the area of vector fields. The measurements in the dynamics described by a curl free vector field can be interpreted as 1-cocycles which are intimately connected to circle valued maps as we show later in this paper. Consequently, a notion of persistence for circle valued maps also provides a notion of persistence for 1-cocycles. In summary, persistence theory for circle valued maps promises to play the role for vector fields as does the standard persistence theory for the scalar fields.

One of the main concepts of the persistence theory is the notion of *bar codes* [16]–invariants that characterize a scalar map at the homology level. We show that circle valued maps, when characterized at homology level, require a new invariant called *Jordan cells* in addition to the bar codes. We describe these invariants using quiver representation theory which has been used in [4] to develop zigzag persistence. The standard persistence [11, 16] which we refer as *sublevel persistence* deals with sublevel sets. The notion when extended to *level sets* [4, 10] provides what we refer to as *level persistence*. The zigzag persistence provides complete invariants (bar codes) for level persistence using representation theory for linear quivers. It turns out that representation theory of cyclic quivers provides the complete invariants for persistence of the circle valued maps.

Our results include a derivation of the homology for the source space and its relevant subspaces in terms of the invariants (Theorem 3.1 and 3.2). The result also applies to real valued maps as they are special cases of the circle valued maps. This leads to a result (Corollary 3.3) which to our knowledge has not yet appeared in the literature.

After developing the results on invariants, we propose an algorithm to compute the bar codes. This involves lifting the circle valued map to a cyclic covering and then applying persistence on a relevant subset of this covering. For a simplicial complex, the entire computation can be done by manipulating the original matrix that encodes the input complex and the map even though the computation involves level sets and interval sets. Once the relevant matrix for a truncated covering is computed, one can use the algorithm of zigzag persistence [4] to compute the bar codes. We also indicate how one can reduce the problem to computing the bar codes for sublevel persistence and hence take advantage of the algorithms developed in [6, 11, 16]. The computation of Jordan cells is more difficult. We propose an approach to compute them and leave it open how to convert it into an efficient algorithm.

2 Definitions and background

We begin with the technical definition of tameness of a map which is essential for finite computations and elimination of pathological cases as recognized in earlier works as well [5, 12].

For a continuous map $f : X \to Y$ between two topological spaces X and Y, let $X_U = f^{-1}(U)$ for $U \subseteq Y$. When $U = y$ is a single point, the set X_y is called a *fiber* over y and is also commonly known as a level set. We call the continuous map $f : X \to Y$ *good* if every $y \in Y$ has a contractible neighborhood U so that the inclusion $X_y \to X_U$ is a homotopy equivalence. The continuous map $f : X \to Y$ is a *fibration* if each $y \in Y$ has a neighborhood U so that the maps $f : X_U \to U$ and $pr : X_y \times U \to U$ are

fiber wise homotopy equivalent. This implies that there exists continuous maps $l : X_U \to X_y \times U$ with $pr|_U \cdot l|_U = f|_U$ which, when restricted to the fiber for any $z \in U$, are homotopy equivalences. In particular, f is good.

Definition 2.1 *A proper continuous map $f : X \to Y$ is* tame *if it is good, and for some discrete closed subset $S \subset Y$, the restriction $f : X \setminus f^{-1}(S) \to Y \setminus S$ is a fibration. The points in $S \subset Y$ which prevent f to be a fibration are called* critical values.

If $Y = \mathbb{R}$ or $\mathbb{S}^1$ and X is compact, then the set of critical values is finite, say $s_1 < s_2 < \cdots s_k$. The fibers above them, X_{s_i}, are referred to as *singular fibers*. All other fibers are called *regular*. In the case of $\mathbb{S}^1$, s_i can be taken as angles and we can assume that $0 < s_i \leq 2\pi$. Clearly, for the open interval (s_{i-1}, s_i) the map $f : f^{-1}(s_{i-1}, s_i) \to (s_{i-1}, s_i)$ is a fibration which implies that all fibers over angles in (s_{i-1}, s_i) are homotopy equivalent with a fixed regular fiber, say X_{t_i}, with $t_i \in (s_{i-1}, s_i)$. In particular, there exist maps $a_i : X_{t_i} \to X_{s_i}$ and $b_i : X_{t_i} \to X_{s_{i-1}}$, unique up to homotopy, derived by restricting any inverse of the homotopy equivalence $X_u \subset X_U$ to the fibers of $f : X_U \to U$. These maps determine homotopically $f : X \to Y$, when $Y = \mathbb{R}$ or $\mathbb{S}^1$. For simplicity in writing, when $Y = \mathbb{R}$ we put $t_{k+1} \in (s_k, \infty)$ and $t_1 \in (-\infty, s_1)$ and when $Y = \mathbb{S}^1$ we put $t_{k+1} = t_1 \in (s_k, s_1 + 2\pi)$.

Note that all scalar or circle valued simplicial maps on a simplicial complex, and smooth maps with generic isolated critical points on a smooth manifold or stratified space are tame. In particular, Morse maps are tame. For the tame maps in this paper we will require an additional property that the space X is compact.

2.1 Persistence and invariants

Since our goal is to extend the notion of persistence from real valued maps to circle valued maps, we first summarize the questions that the persistence answers when applied to real valued maps, and then develop a notion of persistence for circle valued maps which can answer similar questions and more. We fix a field κ and write $H_r(X)$ to denote the homology vector space of X in dimension r with coefficients in a field κ.

Sublevel persistence. The persistent homology introduced in [11] and further developed in [16] is concerned with the following questions:

Q1. Does the class $h \in H_r(X_{(-\infty,t]})$ originate in $H_r(X_{(-\infty,t'']})$ for $t'' < t$? Does the class $h \in H_r(X_{(-\infty,t]})$ vanish in $H_r(X_{(-\infty,t']})$ for $t < t'$?

Q2. What are the smallest t' and t'' such that this happens?

This information is contained in the linear maps $H_r(X_{(-\infty,t]}) \to H_r(X_{(-\infty,t']})$ where $t' \geq t$ and is known as persistence. Since the involved subspaces are sublevel sets, we refer this persistence as *sublevel persistence*. When f is tame, the persistence for each $r = 0, 1, \cdots \dim X$ is determined by a finite collection of invariants referred to as **bar codes** [16]. For sublevel persistence the bar codes are a collection of *closed intervals* of the form $[s, s']$ or $[s, \infty)$ with s, s' being the critical values of f. From these bar codes one can derive the Betti numbers of $X_{(-\infty,a]}$, the dimension of $\text{img}(H_r(X_{(-\infty,t]}) \to H_r(X_{(-\infty,t']}))$ and get the answers to questions Q1 and Q2. For example, the number of r-bar codes which contain the interval $[a, b]$ is the dimension of $\text{img}(H_r(X_{(-\infty,a]}) \to H_r(X_{(-\infty,b]}))$. The number of r-bar codes corresponding to the interval $[a, b]$ is the maximal number of linearly independent homology classes born exactly in $H_r(X_{(-\infty,a]})$ but not before which also die exactly in $H_r(X_{(-\infty,b]})$ but not before.

Level persistence. Instead of sublevels, if we use levels, we obtain what we call level persistence. The level persistence was first considered in [10] and was completely characterized when the zigzag persistence was introduced in [4]. Level persistence is concerned with the homology of the fibers $H_r(X_t)$ and addresses questions of the following type.

Q1. Does the image of $h \in H_r(X_t)$ vanish in $H_r(X_{[t,t']})$, where $t' > t$ or in $H_r(X_{[t'',t]})$, where $t'' < t$?

Q2. Can h be detected in $H_r(X_{t'})$ where $t' > t$ or in $H_r(X_{t''})$ where $t'' < t$? The precise meaning of detection is explained below.

Q3. What are the smallest t' and t'' for the answers to Q1 and Q2 to be affirmative?

To answer such questions one has to record information about the following maps:

$$H_r(X_t) \rightarrow H_r(X_{[t,t']}) \leftarrow H_r(X_{t'}).$$

The **level persistence** is the information provided by this collection of vector spaces and linear maps for all t, t'. We say that $h \in H_r(X_t)$ is dead in $H_r(X_{[t,t']})$, $t' > t$, if its image by $H_r(X_t) \rightarrow H_r(X_{[t,t']})$ vanishes. Similarly, h is dead in $H_r(X_{[t'',t]})$, $t'' < t$, if its image by $H_r(X_t) \rightarrow H_r(X_{[t'',t]})$ vanishes.

We say that $h \in H_r(X_t)$ is detected in $H_r(X_{t'})$, $t' > t$, if its image in $H_r(X_{[t,t']})$ is contained in the image of $H_r(X_{t'}) \rightarrow H_r(X_{[t,t']})$. Similarly, the detection of h can be defined for $t'' < t$ also. In Figure 1, the class consisting of the sum of two circles at level t is not detected on the right, but is detected at all levels on the left up to (but not including) the level t'. In case of a tame map the collection of the vector spaces and linear maps is determined up to coherent isomorphisms by a collection of invariants called *bar codes* which are intervals of the form $[s, s'], (s, s'), (s, s'], [s, s')$ with s, s' critical values. These bar codes are called *invariants* because two tame maps $f : X \rightarrow \mathbb{R}$ and $g : Y \rightarrow \mathbb{R}$ which are fiber wise homotopy equivalent have the same associated bar codes. The open end of an interval signifies the death of a homology class at that end (left or right) whereas a closed end signifies that a homology class cannot be detected beyond this level (left or right). Level persistence provides considerably more information than the sub level persistence. The bar codes of the sub level persistence (for a tame map) can be recovered from the ones of level persistence [1]. It turns out that the bar codes of the level persistence can also be recovered from the bar codes of the sub level persistence of f and additional maps canonically associated to f as described in Appendix D.

In Figure 1, we indicate the bar codes both for sub level and level persistence for some simple map in order to illustrate their differences. The reader can easily derive them by using the method described in Appendix D.

3   Persistence for circle valued maps

Let $f : X \rightarrow \mathbb{S}^1$ be a circle valued map. The sublevel persistence for such a map cannot be defined since circularity in values prevents defining sub-levels. Even level persistence cannot be defined as per say since the intervals may repeat over values. To overcome this difficulty we associate the infinite cyclic covering map $\tilde{f} : \tilde{X} \rightarrow \mathbb{R}$ for f. It is defined by the commutative diagram at left:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & \mathbb{R} \\
\psi \downarrow & & \downarrow p \\
X & \xrightarrow{f} & \mathbb{S}^1
\end{array}
$$

The map $p : \mathbb{R} \rightarrow \mathbb{S}^1$ is the universal covering of the circle (the map which assigns to the number $t \in \mathbb{R}$ the angle $\theta = t (mod\ 2\pi)$ and ψ is the pull back of p by the map f which is an infinite cyclic covering. Notice that $X_\theta = \tilde{X}_t$ if $p(t) = \theta$. If $h \in H_r(X_\theta) = H_r(\tilde{X}_t)$, $p(t) = \theta$, the questions Q1, Q2, Q3 for f and X can be formulated in terms of the level persistence for $\tilde{f}$ and $\tilde{X}$.
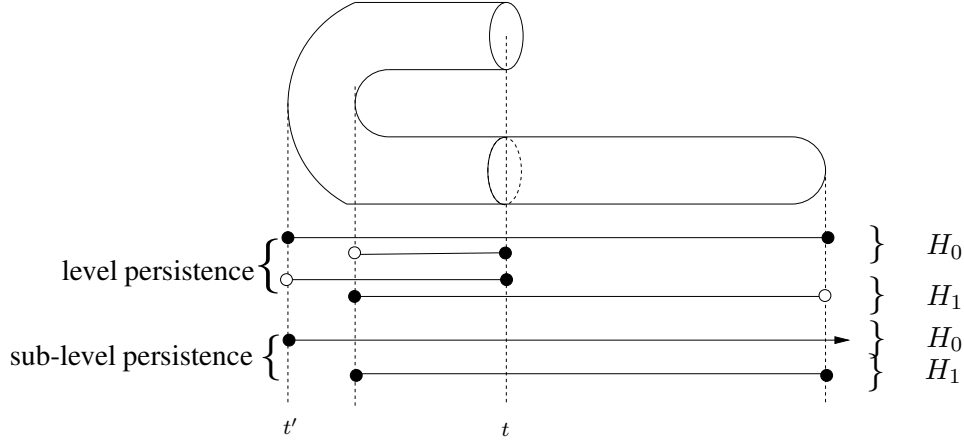
3

Figure 1: Bar codes for level and sub-level persistence.

Suppose that $h \in H_r(\tilde{X}_t) = H_r(X_\theta)$ is detected in $H_r(\tilde{X}_{t'})$ for some $t' > t$. If the interval $[t, t']$ contains a point t'' so that $p(t'') = \theta$, then, in some sense, h returns to $H_r(X_\theta)$ going along the circle $\mathbb{S}^1$ one or more times. When this happens, the class h may change in some respect . This gives rise to new questions that were not encountered in sublevel or level persistence.

Q4. When $h \in H_r(X_\theta)$ returns, how does the "returned class" compare with the original class h? It may disappear after going along the circle a number of times, or it might never disappear.

To answer Q1-Q4 one has to record information about $H_r(X_\theta) \to H_r(X_{[\theta, \theta']}) \leftarrow H_r(X_{\theta'})$ for any pair of angles θ and θ' which differ by at most 2π. This information is referred to as the **persistence for the circle valued map** f.

When f is tame, this is again completely determined up to coherent isomorphisms by a finite collection of invariants. However, unlike sublevel and level persistence, the invariants include structures other than bar codes called *Jordan cells*. Specifically, for any $r = 0, 1, \cdots, \dim(X)$ we have two types of invariants:

- *bar codes*: intervals of type $\{s, s'\}$, $0 < s \le 2\pi$, $s \le s' < \infty$, that are closed or open at s or s', precisely of the form $[s, s'], (s, s'], [s, s'), (s, s')$. These intervals can be geometerized as "spirals" with equations 1. For any interval $\{s, s'\}$ the spiral is the plane curve (see Figure 3 in section 4)

$$
\begin{aligned}
x(\theta) &= \left(\frac{1}{s' - s} \theta + \frac{s' - 2s}{s' - s} + 1 \right) cos\theta \\
y(\theta) &= \left(\frac{1}{s' - s} \theta + \frac{s' - 2s}{s' - s} + 1 \right) sin\theta \\
\theta &\in \{s, s'\}.
\end{aligned}
\tag{1}
$$

- *Jordan cells*. A Jordan cell is a pair (λ, k), $\lambda \in \overline{\kappa} \backslash 0$, $k \in \mathbb{Z}_{>0}$, where $\overline{\kappa}$ denotes the algebraic closure of the field κ. It corresponds to a $k \times k$ matrix of the form

$$
\begin{pmatrix}
\lambda & 1 & 0 \ldots & 0 \\
0 & \lambda & 1 \ldots & 0 \\
\vdots & & & \\
0 & \ldots & \lambda & 1 \\
0 & \ldots & 0 & \lambda
\end{pmatrix}.
$$

4

Notice that the bar codes for f can be inferred from $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$ with $[a,b]$ being any large enough interval. Specifically, the bar codes of $f : X \to \mathbb{S}^1$ are among the ones of $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$ for $(b-a)$ large enough. We explain in section 5 how to get a reasonable upper bound on $b - a$. Unfortunately, the Jordan cells can not be derived from $\tilde{f} : \tilde{X} \to \mathbb{R}$ or any of its truncations $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$. The collection of bar codes and Jordan cells for each $r = 1, 2, \cdots \dim X$ constitute the r-**invariants** of the map f which we define precisely in the next section.

The end points of any bar code correspond to critical angles, that is, s and $s'(mod\ 2\pi)$ of a bar code interval $\{s, s'\}$ are critical angles. One can recover the following information from the bar codes and Jordan cells:

1. The Betti numbers of each fiber,

2. The Betti numbers of the source space X, and

3. Betti numbers of $\tilde{X}_{[a,b]}$.

Theorems 3.1 and 3.2 make the above statement precise. Let B be a bar code described by a spiral (eqn. 1) and θ be any angle. Let $n_\theta(B)$ denote the cardinality of the intersection of the spiral with the ray originating at the origin and making an angle θ with the x-axis. For the Jordan cell $J = (\lambda, k)$, let $n(J) = k$. Furthermore, let $\mathcal{B}_r$ and $\mathcal{J}_r$ denote the set of bar codes and Jordan cells for H_r-homology. We have the following results.

Theorem 3.1 $\dim H_r(X_\theta) = \sum_{B \in \mathcal{B}_r} n_\theta(B) + \sum_{J \in \mathcal{J}_r} n(J)$.

Theorem 3.2 $\dim H_r(X) = \sharp\{B \in \mathcal{B}_r | \text{both ends closed}\} + \sharp\{B \in \mathcal{B}_{r-1} | \text{both ends open}\} + \sharp\{J \in \mathcal{J}_r | \lambda = 1\} + \sharp\{J \in \mathcal{J}_{r-1} | \lambda = 1\}$.

A real valued tame map $f : X \to \mathbb{R}$ can be regarded as a circle valued tame map $f' : X \to \mathbb{S}^1$ by identifying $\mathbb{R}$ to $(0, 2\pi)$ with critical values $t_1, \cdots, t_m$ becoming the critical angles $\theta_1, \cdots, \theta_m$ where $\theta_i = 2 \arctan t_i + \pi$. The map f' in this case does not have any Jordan cells. We have the following corollary:

Corollary 3.3 $\dim H_r(X_\theta) = \sum_{B \in \mathcal{B}_r} n_\theta(B)$ *and*
$\dim H_r(X) = \sharp\{B \in \mathcal{B}_r | \text{both ends closed}\} + \sharp\{B \in \mathcal{B}_{r-1} | \text{both ends open}\}$.

Theorem 3.1 is quite intuitive and is in analogy with the derived results for sublevel and level persistence [4, 16]. Theorem 3.2 is more subtle. Its counterpart for real valued function (Corollary 3.3) has not yet appeared in the literature. The proofs of these results require the definition of the bar codes and Jordan cells which appear in the next section. The proofs are sketched in Appendix A.

The Questions Q1-Q3 can be answered using the bar codes. The question Q4 about returned homology can be answered using the bar codes and Jordan cells.

Figure 2 indicates a tame map $f : X \to \mathbb{S}^1$ and the corresponding invariants, bar codes, and Jordan cells. The space X is obtained from Y in the figure by identifying its right end Y_1 (a union of three circles) to the left end Y_0 (again a union of three circles) following the map $\phi : Y_1 \to Y_0$. The map $f : X \to \mathbb{S}^1$ is induced by the projection of Y on the interval $[0, 2\pi]$. We have $H_1(Y_1) = H_1(Y_0) = \kappa \oplus \kappa \oplus \kappa$ and ϕ induces a linear map in H_1-homology represented by the matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix} \text{ whose } Jordan \text{ canonical form (see [7]) is the matrix } \begin{pmatrix} \mathbf{3} & 0 & 0 \\ 0 & \mathbf{1} & \mathbf{1} \\ 0 & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

with matrices (3) and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ at the "diagonal" which are Jordan cells with $k = 1$ and $k = 2$ respectively. Hence we have two Jordan cells $(3, 1)$ and $(1, 2)$ for $H_1(X)$.

5

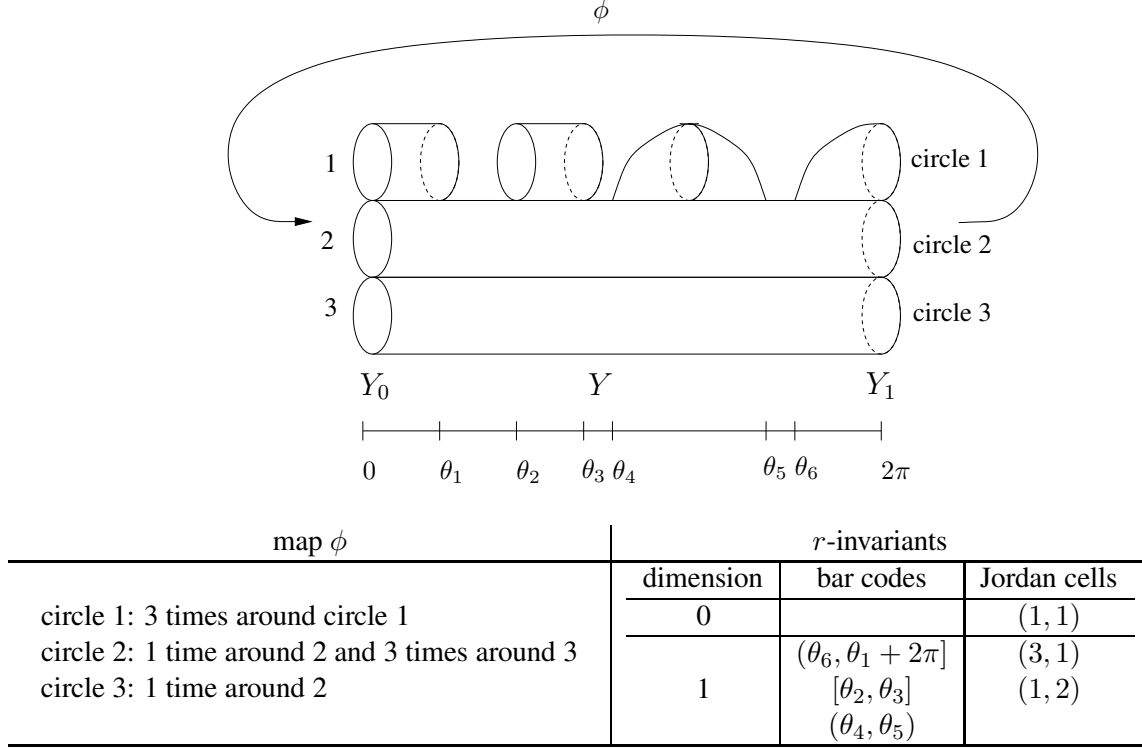| map ϕ	r-invariants		
	dimension	bar codes	Jordan cells
circle 1: 3 times around circle 1	0		$(1,1)$
circle 2: 1 time around 2 and 3 times around 3		$(\theta_6, \theta_1 + 2\pi]$	$(3,1)$
circle 3: 1 time around 2	1	$[\theta_2, \theta_3]$	$(1,2)$
		(θ_4, θ_5)	

Figure 2: Example of r-invariants for a circle valued map

4 Representation theory and r-invariants

The invariants for the circle valued map are derived from the representation theory of quivers. The quivers are directed graphs. The representation theory of simple quivers such as paths with directed edges was described by Gabriel [8] and is at the heart of the derivation of the invariants for zigzag and then level persistence in [4]. For circle valued maps, one needs representation theory for circle graphs with directed edges. This theory appears in the work of Nazarova [13], and Donovan and Ruth-Freislich [9]. The reader can find a refined treatment in Kac [14].

Let G_{2m} be a directed graph with $2m$ vertices, $x_1, x_2, \cdots x_{2m}$. Its underlying undirected graph is a simple cycle. The directed edges in G_{2m} are of two types: *forward* $a_i : x_{2i-1} \to x_{2i}$, $1 \le i \le m$, and *backward* $b_i : x_{2i+1} \to x_{2i}$, $1 \le i \le m-1$, $b_m : x_1 \to x_{2m}$.
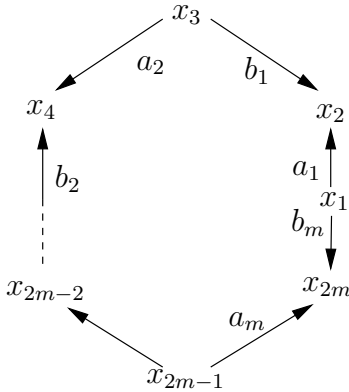
We think of this graph as being residing on the unit circle centered at the origin o in the plane.

A representation ρ on G_{2m} is an assignment of a vector space V_x to each vertex x and a linear map $V_e : V_x \to V_y$ for each oriented edge $e = \{x, y\}$. Two representations ρ and ρ' are isomorphic if for each vertex x there exists an isomorphism from the vector space V_x of ρ to the vector space V_x' of ρ', and these isomorphisms intertwine the linear maps $V_x \to V_y$ and $V_x' \to V_y'$. A *non-trivial* representation assigns at least one vector space which is not zero-dimensional. A representation is *indecomposable* if it is not isomorphic to the sum of two nontrivial representations. It is not hard to observe that each representation has a decomposition as a sum of indecomposable representations unique up to isomorphisms.

6

We provide a description of indecomposable representations of the quiver G_{2m}. For any triple of integers $\{i, j, k\}$, $1 \le i, j \le m$, $k \ge 0$, one may have any of the four representations, $\rho^I([i,j];k)$, $\rho^I((i,j];k)$, $\rho^I([i,j);k)$, and $\rho^I((i,j);k)$ defined below.

Suppose that the evenly indexed vertices $\{x_2, x_4, \cdots x_{2m}\}$ of G_{2m} which are the targets of the directed arrows correspond to the angles $0 < s_1 < s_2 < \cdots < s_m \le 2\pi$. Draw the spiral curve given by equation (1) for the interval $\{s_i, s_j + 2k\pi\}$; refer to Figure 3.

For each x_i, let $\{e_i^1, e_i^2, \cdots\}$ denote the ordered set (possibly empty) of intersection points of the ray ox_i with the spiral. While considering these intersections, it is important to realize that the point $(x(s_i), y(s_i))$ (resp. $(x(s_j + 2k\pi), y(s_j + 2k\pi))$) does not belong to the spiral if the interval in $\rho^I(\{i,j\}, k)$ is open at i (resp. j). For example, in Figure 3, the last circle on the ray ox_{2j} is not on the spiral since $\rho^I([i,j), 2)$ is open at right.

Let V_{x_i} denote the vector space generated by the base $\{e_i^1, e_i^2, \cdots\}$. Furthermore, let $\alpha_i : V_{x_{2i-1}} \to V_{x_{2i}}$ and $\beta_i : V_{x_{2i+1}} \to V_{x_{2i}}$ be the linear maps defined on bases and extended by linearity as follows: assign to $e_{2i\pm1}^\ell$ the vector $e_{2i}^d \in V_{x_i}$ if e_{2i}^d is an adjacent intersection point to the points $e_{2i\pm1}^\ell$ on the spiral. If e_{2i}^d does not exist, assign zero to $e_{2i\pm1}^\ell$. If zero is not assigned to $e_{2i\pm1}^\ell$, d must be l, $l-1$, or $l+1$.

The construction above provides a representation on G_{2m} which is indecomposable. One can also think these representations as the bar codes $[s_i, s_j + 2k\pi]$, $(s_i, s_j + 2k\pi]$, $[s_i, s_j + 2k\pi)$, and $(s_i, s_j + 2k\pi)$.
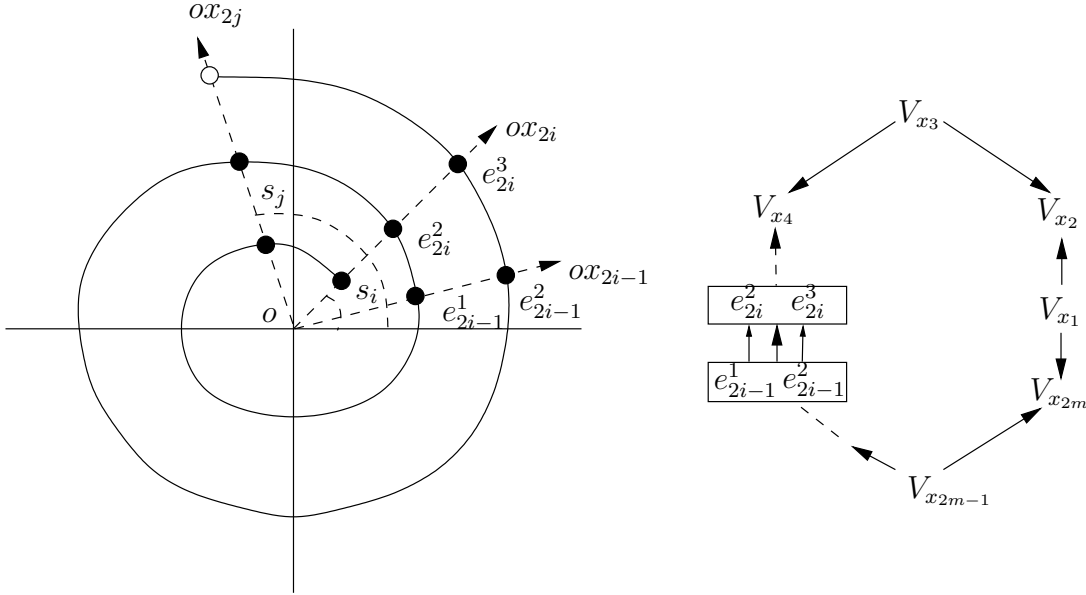

Figure 3: The spiral for $[s_i, s_j + 4\pi)$.

For any Jordan cell (λ, k) we associate a representation $\rho^J(\lambda, k)$ defined as follows. Assign the vector space with base $e_1, e_2, \cdots, e_k$ to each x_i and take all linear maps α_i but one (say α_1) and β_i the identity. The map α_1 is given by the Jordan cell matrix (λ, k). Again this representation is indecomposable.

It follows from the work of [9, 13, 14] that bar codes and Jordan cells as constructed above are all and only indecomposable representations of the quiver G_{2m}.

Observation 4.1 *If a representation ρ does not contain any indecomposable representations of type ρ^I in its decomposition, all linear maps $\alpha_i s$ and $\beta_i s$ are isomorphisms. For such a representation, starting with an index i, consider the linear isomorphism*

$$T_i = \beta_i^{-1} \cdot \alpha_i \cdot \beta_{i-1}^{-1} \cdot \alpha_{i-1} \cdots \beta_2^{-1} \cdot \alpha_2 \cdot \beta_1^{-1} \cdot \alpha_1 \cdot \beta_m^{-1} \cdot \alpha_m \cdot \beta_{m-1}^{-1} \cdot \alpha_{m-1} \cdots \beta_{i+1}^{-1} \cdot \alpha_{i+1}.$$

The Jordan canonical form of the isomorphism T_i is independent of i and is a block diagonal matrix with the diagonal consisting of Jordan cells (λ, k)s. Clearly, ρ is the direct sum of $\rho^J(\lambda, k)$s with each (λ, k) being a Jordan cell of T_i.

So, for a general representation ρ, once we separate out all indecomposables of type ρ^I, we can obtain the ones corresponding to the Jordan cells from the Jordan canonical form of the linear isomorphism T_i. A procedure to decompose a representation ρ into indecomposable representations is provided in Appendix E.

The r-invariants. Let f be a circle valued tame map defined on a topological space X. For f with m critical angles $0 < s_1 < s_2, \cdots s_m \leq 2\pi$, consider the quiver G_{2m} with the vertices x_{2i} specified by the angles s_i and the vertices x_{2i-1} by the angles t_i that satisfy $0 < t_1 < s_1 < t_2 < s_2, \cdots t_m < s_m$.

For any r, consider the representation ρ_r of G_{2m} with $V_{x_i} = H_r(X_{x_i})$ and the linear maps α_is and β_is induced in the H_r-homology by the maps $a_i : X_{x_{2i-1}} \to X_{x_{2i}}$ and $b_i : X_{x_{2i+1}} \to X_{x_{2i}}$ described in section 2. The bar codes and the Jordan cells of this representation, also referred as r-bar codes and r-Jordan cells, are independent of the choice of t_is and are the r-**invariants** of f.

5 Algorithm

Given a circle valued tame map $f : X \to \mathbb{S}^1$, we now present an algorithm to compute the bar codes when X is a finite simplicial complex, and f is generic and linear. Genericity means that f is injective on vertices. The genericity condition is not necessary if one starts with an angle θ which is not the image of any vertex. To explain linearity we recall that, for any simplex $\sigma \in X$, the restriction $f|_\sigma$ admits liftings $\hat{f} : \sigma \to \mathbb{R}$, i.e. $\hat{f}$ is a continuous map which satisfies $p \cdot \hat{f} = f|_\sigma$. The map $f : X \to \mathbb{S}^1$ is called *linear* if for any simplex σ, at least one of the liftings (and then any other) is linear.

Most of tame circle valued maps of practical interest are fiber wise equivalent to linear circle valued maps defined on a simplicial complex.

Given a tame circle valued map $f : X \to \mathbb{S}^1$, we consider the infinite cyclic covering map $\tilde{f} : \tilde{X} \to \mathbb{R}$ of $f : X \to \mathbb{S}^1$ as described in section 4. Precisely, one has the infinite cyclic coverings $\psi : \tilde{X} \to X$, and $p : \mathbb{R} \to \mathbb{S}^1$ with $p(t) = t(mod\, 2\pi)$, which satisfies $p \cdot \tilde{f} = f \cdot \psi$. Recall that $p(t) = \theta$ implies $\tilde{X}_t = X_\theta$.

Since $\tilde{X}$ is not compact, we truncate $\tilde{X}$ to $\tilde{X}_{[t,t+2\pi k]}$. The restriction $\tilde{f} : \tilde{X}_{[t,t+2\pi k]} \to \mathbb{R}$ is tame. In view of Theorem 3.1, if k is an integer larger than $(\inf_\theta \dim H_r(X_\theta)) + 1$, the r-bar codes of f are among the (level persistence) r-bar codes of $\tilde{f} : \tilde{X}_{[t,t+2\pi k]} \to \mathbb{R}$. One can verify that if $t \in [0, 2\pi]$, they are the same as the r-bar codes which begin in the interval $(2\pi, 4\pi]$ for the truncation $\tilde{f} : \tilde{X}_{[t,t+2\pi k]} \to \mathbb{R}$. To compute the r-bar codes for f we proceed as follows.

Step 1. Choose a $\theta \in [0, 2\pi]$, compute the rank d of $H_r(X_\theta)$, and take $k = d + 2$. Since X_θ is a cell complex we can represent its incidence structure with a matrix. One may apply the standard persistence algorithm to this matrix for computing the rank of $H_r(X_\theta)$. Computing the cell complex X_θ from X explicitly is cumbersome. We propose a simple method to compute the incidence matrix of X_θ from that of X. Let $M(C)$ denote the incidence matrix of any finite cell (in particular simplicial) complex C, that is, $M(C)[i, j] = 1$ if the cell $\sigma_i \in C$ is a face of $\sigma_j \in C$ with codimenison 1. All other entries in $M(C)$ are zero. For any θ, consider a matrix $\hat{M}$ as the minor of $M(X)$ which consist of the rows and columns i with $\theta \in f(\text{int}\,\sigma_i)$ and regard the simplex σ_i as the cell $\hat{\sigma}_i$ with $\dim(\hat{\sigma}_i) = \dim(\sigma_i) - 1$. If there are $\ell < n$ such simplices, $\hat{M}$ is an $\ell \times \ell$ matrix.

- If no vertex takes the value θ, then $M(X_\theta) = \hat{M}$ is the incidence matrix of the cell complex X_θ.

- If there exists a vertex v (unique since f is generic) so that $f(v) = \theta$, then the incidence matrix $M(X_\theta)$ is an $(\ell+1) \times (\ell+1)$ matrix with one additional row and column corresponding to the vertex v viewed as a cell of dimension 0 which should be indexed before all other cells. The entry for the pair $(v, \hat{\sigma}_j)$ in $M(X_\theta)$ is 1 if σ_j is a 2−simplex which has v as a vertex and 0 otherwise.

The second situation can be avoided by choosing θ different from the value of f on vertices.

Step 2. After determining k in the previous step, we construct $\tilde{X}_{[t,t+2\pi k]}$ by first cutting open X at $X_{\theta=t}$ for some $t \in [0, 2\pi]$ and then putting copies of this dissected X one after another joining along copies of X_θ. See Figure 4. It turns out that we do not need to compute explicitly the space $\tilde{X}_{[t,t+2\pi k]}$, but instead can compute its incidence matrix from the matrix $M(X)$. For a more formal description of the process, see Appendix C.
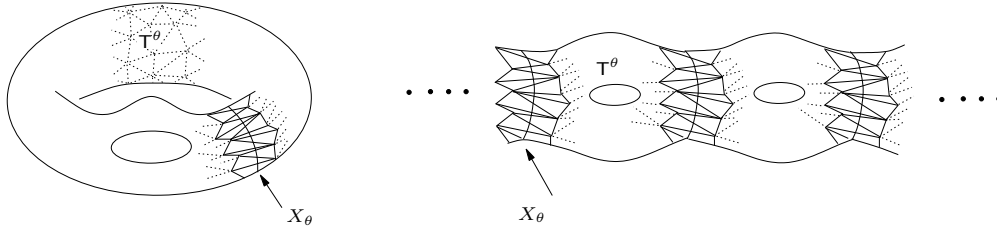

Figure 4: Complex X with level X_θ on left. Part of the complex $\tilde{X}$ on right.

Step 3. The algorithm for zigzag persistence [4] can compute the bar codes for a real-valued function defined on a finite simplicial complex. Once we compute the incidence matrix of $\tilde{X}_{[t,t+2\pi k]}$ that is compatible with $\tilde{f}$, we can apply the zigzag persistence algorithm to compute the bar codes. In Appendix D we provide an alternative algorithm to compute the bar codes using the standard persistence algorithm [11, 16].

The computation of Jordan cells is harder. We describe an approach to compute them though we do not yet have an efficient algorithm to implement it. Consider the quiver representation ρ_r defined by $f : X \to \mathbb{S}^1$ in H_r-homology. Suppose that ρ_r is described by the set of matrices that represent maps α_is and β_is. These matrices can be computed by standard persistence algorithm. We split off all indecomposables of type ρ^I from ρ_r first using an iterative procedure. Then, we apply Observation 4.1 to get the Jordan cells. The procedure is described in details in Appendix E.

6   1-cocycles

In this section we discuss how one may leverage the definition and computation of persistence for circle valued maps to derive the same for 1-cocycles. These 1-cocycles appear in scientific studies and engineering applications. For example, a smooth curl free vector field on a Riemannian manifold provides a 1-cocycle when integrated along the edges of any triangulation of the manifold. Similarly, a ranking problem which is locally consistent [18, 19] defines such a 1-cocycle. The persistence invariants that we define for such 1-cocycles carry information about the dynamics of the vector field in the first case (at least as much as Novikov theory provides [3]) and the quality of the global inconsistency of the ranking in the second.

Let X be a simplicial complex with $\mathcal{X}_0$ being the set of vertices. Denote by $\mathcal{X}_1 \subseteq \mathcal{X}_0 \times \mathcal{X}_0$ the collection of pairs (x,y) with $x,y \in \mathcal{X}_0$ so that x,y are the end points of a 1-simplex in X. Note that if $(x,y) \in \mathcal{X}_1$ then $(y,x) \in \mathcal{X}_1$.

A 0-*cochain* is a function $f : \mathcal{X}_0 \to \mathbb{R}$. A 0-cochain can be identified with a continuous map $f : X \to \mathbb{R}$ whose restriction to each simplex is linear. The 0-cochain f is *generic* if $f : \mathcal{X}_0 \to \mathbb{R}$ is injective. The

1-cochains are a generalization of 0-cochains in that their domain is the set $\mathcal{X}_1$ of oriented edges of X. Let $\mathbf{f} : \mathcal{X}_1 \to \mathbb{R}$ be a 1-cochain defined on X. The map $\mathbf{f}$ is a 1-*cocycle* if it satisfies:

1. $\mathbf{f}(x, y) = -\mathbf{f}(y, x)$ for any ordered pair $(x, y) \in \mathcal{X}_1$, and

2. if $(x, y, z) \in \mathcal{X}_2$ then $\mathbf{f}(x, y) + \mathbf{f}(y, z) + \mathbf{f}(z, x) = 0$; equivalently $\mathbf{f}(x, y) + \mathbf{f}(y, z) = \mathbf{f}(x, z)$.

If $St(x)$ denotes the star of the vertex $x \in \mathcal{X}_0$ (the star of any simplex is a sub complex), a 1-cocycle $\mathbf{f}$ defines a unique function $f_x : St(x) \to \mathbb{R}$ by the formulae $f_x(x) = 0$ and $f_x(y) = \mathbf{f}(x, y)$ for any vertex $y \neq x$ in $St(x)$. Clearly $(f_x - f_y)(z)$ is constant in z for any z in a connected component of $St(x) \cap St(y)$.

Thus, a 1-cocycle can be thought of as a collection of linear maps $\{f_x : St(x) \to \mathbb{R}\}$ for each vertex x, such that the difference $f_x - f_y$ is constant on each connected component of $St(x) \cap St(y)$. A 1-cocycle $\mathbf{f}$ is *generic* if all linear maps f_x are generic, i.e., injective when restricted to vertices of $St(x)$.

Any 1-cocycle $\mathbf{f}$ represents a cohomology class $< \mathbf{f} > \in H^1(X; \mathbb{R})$ and any such cohomology class is represented by a 1-cocycle. Two 1-cocycles $\mathbf{f}_1$ and $\mathbf{f}_2$ represent the same cohomology class iff $\mathbf{f}_1 - \mathbf{f}_2 = \delta f$ for some 0-cochain f.

An *almost integral* 1-cocycle is a pair $(\mathbf{f}, \alpha)$ where $\mathbf{f}$ is a 1-cocycle whose values on integral 1-cycles are integer multiple of a fixed positive real α.[1] These cocycles include the class of *rational* 1-cocycles whose values on integral 1-cycles are rational numbers. In particular, a 1-cocycle $\mathbf{f} : \mathcal{X}_1 \to \mathbb{R}$ with rational numbers as values is a rational 1-cocycle and therefore an almost integral 1-cocycle for some rational number α. One can show that an almost integral 1-cocycle corresponds to a circle valued map and vice versa.

Proposition 6.1 *Any circle valued map $f : X \to \mathbb{S}^1$ defines an almost integral 1-cocycle $(\mathbf{f}, \alpha)$ and any almost integral 1-cocycle $(\mathbf{f}, \alpha)$ defines a circle valued map $f : X \to \mathbb{S}^1 = \mathbb{R}/\alpha$ whose associated 1-cocycle is $(\mathbf{f}, \alpha)$.*

A proof is given in Appendix B.

Persistence of an (almost integral) 1-cocycle: We define the persistence of an almost integral 1-cocycle and therefore of any rational 1-cocycle (for α the largest positive rational number which makes the rational 1-cocycle almost integral) as the persistence of the associated circle valued map.

7   Conclusions

We have analyzed circle valued maps from the perspective of topological persistence. We show that the notion of persistence for such maps incorporate an invariant that is not encountered in persistence studied erstwhile. Our results also shed lights on computing ranks of homological vector spaces from bar codes (Theorems 3.1 and 3.2). We have given algorithms to compute the bar codes of the invariants, one uses the zigzag persistence algorithm and the other uses standard persistence algorithm. We have also proposed an approach to compute the Jordan cells.

Some open questions ensue from this research. What can be said about the stability of the invariants as was established for standard persistence [5]? How can we convert the approach sketched in Appendix E for computing bar codes and Jordan cells into an efficient algorithm? For 1-cocyles, we define and compute persistence via a circle valued map. Is it possible to skip this intermediate map and define its invariants directly?

---

[1]in the language of algebraic topology the cohomology class $< \mathbf{f} > \in H^1(X; \mathbb{R})$ has degree of rationality 1, or equivalently the image of the induced homomorphism $H_1(X; \mathbb{Z}) \to \mathbb{R}$ is $\alpha\mathbb{Z} \subset \mathbb{R}$.

The standard persistence is related to Morse theory. In a similar vein, the persistence for circle valued map is related to Morse Novikov theory [15]. The work of Burghelea and Haller applies Morse Novikov theory to instantons and closed trajectories for vector field with Lyapunov closed one form [3]. The results in this paper will very likely provide additional insight on the dynamics of these vector fields [2].

References

[1] D. Burghelea, T. K. Dey, and D. Dong. Defining and computing topological persistence for 1-cocycles (arXiv:1012.3763), 2010.

[2] D. Burghelea. New topological invariants for Novikov theory. in preparation.

[3] D. Burghelea and S. Haller. Dynamics, Laplace transform and spectral geometry. *J.Topol.* **1** (2008), 115-151.

[4] G. Carlsson, V. de Silva, and D. Morozov. Zigzag persistent homology and real-valued functions. *Proc. 25th Annu. Sympos. Comput. Geom.* (2009), 247–256.

[5] D. Cohen-Steiner, H. Edelsbrunner, and J. Harer. Stability of persistence diagrams. *Discrete Comput. Geom.* **37** (2007), 103-120.

[6] D. Cohen-Steiner, H. Edelsbrunner, and D. Morozov. Vines and vineyards by updating persistence in linear time. *Proc. 22nd Annu. Sympos. Comput. Geom.* (2006), 119–134.

[7] N. Dunford and J.T. Schwartz. Linear Operators, Part I: General Theory, Interscience, 1958.

[8] P. Gabriel. Unzerlegbare Darstellungen I. *Manuscr. Math.* **6** (1972), 71–103.

[9] P. Donovan and M. R. Freislich. Representation theory of finite graphs and associated algebras *Carleton Math. Lecture Notes.* **5** (1973), Carleton University, Ottawa.

[10] T. K. Dey and R. Wenger. Stability of critical points with interval persistence. *Discrete Comput. Geom.* **38** (2007), 479–512.

[11] H. Edelsbrunner, D. Letscher, and A. Zomorodian. Topological persistence and simplification. *Discrete Comput. Geom.* **28** (2002), 511–533.

[12] H. Edelsbrunner and J. L. Harer. Computational Topology, An Introduction. *AMS*, Providence, Rhode Island, 2009.

[13] L. A. Nazarova. Representations of quivers of infinite type (Russian). *IZV.Akad.Nauk SSSR Ser. Mat.* **37** (1973), 752-791.

[14] V Kac. Infinite root systems, representation of graphs and invariant theory. *Invent. Math.* **56** (1980), 57-92.

[15] S. P. Novikov. Quasiperiodic structures in topology. *In Topological methods in modern mathematics, Proc. Sympos. in honor of John Milnor's sixtieth birthday*, New York, 1991. eds L. R. Goldberg and A. V. Phillips, *Publish or Perish*, Houston, TX, 1993, 223–233.

[16] A. Zomorodian and G. Carlsson. Computing persistent homology. *Discr. Comput. Geom.* **33** (2005), 249–274.

[17] V. de Silva and M. Vejdemo-Johansson. Persistent cohomology and circular coordinates. *Proc. 25th Annu. Sympos. Comput. Geom.* (2009), 227–236.

[18] Xiaoye Jiang, Lek-Heng Lim, Yuan Yao and Yinyu Ye. Statistical Ranking and Combinatorial Hodge Theory. *arxiv.org/abs/0811.1067.*

[19] Yuan Yao. Combinatorial Laplacians and Rank Aggregation. *the 6th International Congress of Industrial and Applied Mathematics (ICIAM), mini symposium: Novel Matrix Methods for Internet Data Mining. Zurich, Switzerland, July 16-20, 2007*

Appendix

A Proofs

A careful look at Figure 2 and the bar codes indicate why a semi-closed (one end open, the other closed) bar code does not contribute to the homology of the total space X while a closed r-bar code (both ends closed) and an open (both end open) $(r-1)$-bar code contributes one unit each to the H_r-homology of the total space. Informally, the lack of contribution of a Jordan cell with $\lambda \neq 1$ and the contribution of one unit of a r-Jordan cell with $\lambda = 1$ to both r and $r+1$ dimensional homology of the total space can be explained with the homology calculation for a " mapping torus". Below we will provide rigorous but schematic arguments for the proofs of Theorems 3.1, 3.2 and Corollary 3.3.

First recall that a representation ρ of the graph G_{2m} is indicated by the vector spaces $V_{x_{2i-1}}, V_{x_{2i}}$ and the linear maps α_i and β_i. To such representation ρ we associate the block matrix $M_\rho : \bigoplus_{1 \leq i \leq m} V_{x_{2i-1}} \to \bigoplus_{1 \leq i \leq m} V_{x_{2i}}$ defined by:

$$
\begin{pmatrix}
\alpha_1 & -\beta_1 & 0 & \dots & & \dots & & 0 \\
0 & \alpha_2 & -\beta_2 & \dots & & \dots & & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & & \\
0 & \dots & \dots & \dots & & \dots \alpha_{m-1} & -\beta_{m-1} & \\
-\beta_m & \dots & \dots & \dots & & \dots & & \alpha_m
\end{pmatrix}
$$

We define the "dimension" of ρ as the $2m$-tuple of positive integers $\dim(\rho) = (n_1, d_1 \cdots n_m, d_m)$ with $n_i = \dim V_{x_{2i-1}}$ and $d_i = \dim V_{x_{2i}}$ and the numbers $\mathrm{K}(\rho) = \dim \ker M_\rho$ and $\mathrm{C}K(\rho) = \dim coker M_\rho$. For the sum of two such representations $\rho = \rho_1 \oplus \rho_2$ we have:

Proposition A.1

1. $\dim(\rho_1 \oplus \rho_2) = \dim(\rho_1) + \dim(\rho_2)$,

2. $\mathrm{K}(\rho_1 \oplus \rho_2) = \mathrm{K}(\rho_1) + \mathrm{K}(\rho_2)$,

3. $\mathrm{C}K(\rho_1 \oplus \rho_2) = \mathrm{C}K(\rho_1) + \mathrm{C}K(\rho_2)$.

The explicit description of the representations corresponding to the bar codes permits explicit calculations.

Proposition A.2

1. *If $i \leq j$ then*

   (a) *$\dim \rho^I([i, j], k)$ is given by:*
      $n_l = k + 1$ if $(i + 1) \leq l \leq j$ and k otherwise,
      $d_l = k + 1$ if $i \leq l \leq j$ and k otherwise
   (b) *$\dim \rho^I((i, j], k)$ is given by:*
      $n_l = k + 1$ if $(i + 1) \leq l \leq j$ and k otherwise,
      $d_l = k + 1$ if $(i + 1) \leq l \leq j$ and k otherwise,
   (c) *$\dim \rho^I([i, j), k)$ is given by:*
      $n_l = k + 1$ if $(i + 1) \leq l \leq j$ and k otherwise,
      $d_l = k + 1$ if $i \leq l \leq (j - 1)$ and k otherwise,

(d) $\dim \rho^I((i,j),k)$ is given by:

$n_l = k+1$ if $(i+1) \le l \le j$ and k otherwise,

$d_l = k+1$ if $(i+1) \le l \le (j-1)$ and k otherwise

2. *If $i > j$ then similar statements hold.*

(a) $\dim \rho^I([i,j],k)$ is given by:

$n_l = k$ if $(j+1) \le l \le i$ and $k+1$ otherwise;

$d_l = k$ if $(j+1) \le l \le (i-1)$ and $k+1$ otherwise

(b) $\dim \rho^I((i,j],k)$ is given by:

$n_l = k$ if $(j+1) \le l \le i$ and $k+1$ otherwise.

$d_l = k$ if $(j+1) \le l \le i$ and $k+1$ otherwise,

(c) $\dim \rho^I([i,j),k)$ is given by:

$n_l = k$ if $(j+1) \le l \le i$ and $k+1$ otherwise;

$d_l = k$ if $j \le l \le (i-1)$ and $k+1$ otherwise,

(d) $\dim \rho^I((i,j),k)$ is given by:

$n_l = k$ if $(j+1) \le l \le i$ and $k+1$ otherwise;

$d_l = k$ if $j \le l \le i$ and $k+1$ otherwise.

3. $\dim \rho^J(\lambda, k)$ is given by $n_i = d_i = k$

Proposition A.3

1. $\mathrm{K}(\rho^I([i,j],k)) = 0$, $\mathrm{C}K(\rho^I([i,j],k)) = 1$,

2. $\mathrm{K}(\rho^I([i,j),k)) = 0$, $\mathrm{C}K(\rho^I([i,j),k)) = 0$,

3. $\mathrm{K}(\rho^I((i,j],k)) = 0$, $\mathrm{C}K(\rho^I((i,j],k)) = 0$,

4. $\mathrm{K}(\rho^I((i,j),k)) = 1$, $\mathrm{C}K(\rho^I((i,j),k)) = 0$,

5. $\mathrm{K}(\rho^J(\lambda, k)) = 0$ (resp. 1) if $\lambda \ne 1$ (resp. 1),

6. $\mathrm{C}K(\rho^J(\lambda, k)) = 0$ (resp. 1) if $\lambda \ne 1$ (resp. 1).

The proof of Theorem 3.1 is a consequence of Propositions A.1 and A.2. The proof of Theorem 3.2 goes on the following lines. First observe that, up to homotopy, the space X can be regarded as the iterated mapping torus $\mathcal{T}$ described below. Consider the collection of spaces and continuous maps:

$$X_m = X_0 \overset{b_0=b_m}{\longleftarrow} R_1 \xrightarrow{a_1} X_1 \overset{b_1}{\longleftarrow} R_2 \xrightarrow{a_2} X_2 \cdots X_{m-1} \overset{b_{m-1}}{\longleftarrow} R_m \xrightarrow{a_m} X_m.$$

Denote by $\mathcal{T} = T(a_1 \cdots a_m; b_1 \cdots b_m)$ the space obtained from the disjoint union

$$(\bigsqcup_{1 \le i \le m} R_i \times [0,1]) \sqcup (\bigsqcup_{1 \le i \le m} X_i)$$

by identifying $R_i \times \{1\}$ to X_i by a_i and $R_i \times \{0\}$ to X_{i-1} by b_{i-1}. Denote by $f^{\mathcal{T}} : \mathcal{T} \to \mathbb{R}/m\mathbb{Z}$ where $f^{\mathcal{T}} : R_i \times [0,1] \to [i, i+1]$ is the projection on $[0,1]$ followed by the translation of $[0,1]$ to $[i, i+1]$. This map is a homotopical reconstruction of $f : X \to \mathbb{S}^1$ with the choice of angles t_i, s_i as described in section 2 where $X_i := f^{-1}(s_i)$, $R_i := f^{-1}(t_i)$.

Let $\mathcal{P}'$ denote the space obtained from the disjoint union

$$(\bigsqcup_{1 \le i \le m} R_i \times (\epsilon, 1]) \sqcup (\bigsqcup_{1 \le i \le m} X_i)$$

by identifying $R_i \times \{1\}$ to X_i by a_i, and $\mathcal{P}''$ denote the space obtained from the disjoint union

$$(\bigsqcup_{1 \le i \le m} R_i \times [0, 1 - \epsilon) \sqcup (\bigsqcup_{1 \le i \le m} X_i)$$

by identifying $R_i \times \{0\}$ to X_{i-1} by b_{i-1}.

Let $\mathcal{R} = \bigsqcup_{1 \le i \le m} R_i$ and $\mathcal{X} = \bigsqcup_{1 \le i \le m} X_i$. Then, one has:

1. $\mathcal{T} = \mathcal{P}' \cup \mathcal{P}''$,

2. $\mathcal{P}' \cap \mathcal{P}'' = (\bigsqcup_{1 \le i \le m} R_i \times (\epsilon, 1 - \epsilon)) \sqcup \mathcal{X}$, and

3. the inclusions $(\bigsqcup_{1 \le i \le m} R_i \times \{1/2\}) \sqcup \mathcal{X} \subset \mathcal{P}' \cap \mathcal{P}''$ as well as the obvious inclusions $\mathcal{X} \subset \mathcal{P}'$ and $\mathcal{X} \subset \mathcal{P}''$ are homotopy equivalences.

The Mayer-Vietoris long exact sequence leads to the diagram

$$\begin{array}{ccc}
H_r(\mathcal{R}) & \xrightarrow{M_r(\alpha,\beta)} & H_r(\mathcal{X}) \\
\end{array}$$

$$\cdots \longrightarrow H_{r+1}(\mathcal{T}) \xrightarrow{\partial_{r+1}} H_r(\mathcal{R}) \oplus H_r(\mathcal{X}) \xrightarrow{N} H_r(\mathcal{X}) \oplus H_r(\mathcal{X}) \xrightarrow{(i^r, -i^r)} H_r(\mathcal{T}) \longrightarrow$$

$$\begin{array}{ccc}
H_r(\mathcal{X}) & \xrightarrow{Id} & H_r(\mathcal{X}) \\
\end{array}$$

with maps pr_1, $(Id, -Id)$, in_2, Δ.

Here Δ denotes the diagonal, in_2 the inclusion on the second component, pr_1 the projection on the first component, i^r the linear map induced in homology by the inclusion $\mathcal{X} \subset \mathcal{T}$, and $M_r(\alpha, \beta)$ the linear map given by the matrix

$$\begin{pmatrix}
\alpha_1^r & -\beta_1^r & 0 & \dots & \dots & 0 \\
0 & \alpha_2^r & -\beta_2^r & \dots & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & \dots & \dots & \dots & \dots \alpha_{m-1}^r & -\beta_{m-1}^r \\
-\beta_m^r & \dots & \dots & \dots & \dots & \alpha_m^r
\end{pmatrix}$$

with $\alpha_i^r : H_r(R_i) \to H_r(X_i)$ and $\beta_i^r : H_r(R_{i+1}) \to H_r(X_i)$ induced by the maps α_i and β_i, and N defined by

$$\begin{pmatrix} \alpha^r & Id \\ \beta^r & Id \end{pmatrix}$$

where α^r and β^r are the matrices

$$\begin{pmatrix}
\alpha_1^r & 0 & \dots & \dots & 0 \\
0 & \alpha_2^r & \dots & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \dots & 0 & \alpha_m^r
\end{pmatrix}$$

15

$$\begin{pmatrix} 0 & \beta_1^r & 0 & \ldots & 0 \\ 0 & 0 & \beta_2^r & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & 0 & \beta_{m-1}^r \\ \beta_m^r & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

As a consequence of the long exact sequence in the diagram above we obtain

$$H_r(\mathcal{T}) = \mathrm{coker}\,\mathrm{M_r}(\alpha, \beta) \oplus \ker \mathrm{M_{r-1}}(\alpha, \beta). \tag{2}$$

Theorem 3.2 follows from Propositions A.1, A.3 and the equation 2 above.

B Proof of Proposition 6.1

From circle valued map to 1-cocycle. Consider a continuous *circle valued* map $f : X \to \mathbb{S}^1$. Let $p : \mathbb{R} \to \mathbb{S}^1 = \mathbb{R}/\alpha\mathbb{Z}$ be the map defined by $p(t) = t(\mathrm{mod}\ \alpha)$, α a positive real number. For any simplex $\sigma \in X$, the restriction $f|_\sigma$ admits liftings $\hat{f} : \sigma \to \mathbb{R}$, i.e., $\hat{f}$ is a continuous map which satisfies $p \cdot \hat{f} = f|_\sigma$. Assign to each pair $(x, y) \in \mathcal{X}_1$, $\mathbf{f}(x, y) = \hat{f}(y) - \hat{f}(x)$ where $\hat{f}$ is a lift of f. The assignment is independent of the lifting $\hat{f}$. We obtain an almost integral 1-cocycle $(\mathbf{f}, \alpha)$.

Construction of the covering $\psi : \tilde{X} \to X$**.** Regard X as a topological space. Choose a base point $x \in X$ and consider the space of continuous paths $\gamma : [0, 1] \to X$ with $\gamma(0) = x$, equipped with compact open topology. Make two continuous paths γ_1 and γ_2 equivalent iff the $\gamma_1(1) = \gamma_2(1)$ and the closed path $\gamma_1 \star \gamma_2^{-1}$ satisfies $< \mathbf{f} > ([\gamma_1 \star \gamma^{-1}]) = 0$. Here $\star$ denotes the concatenation of the paths γ_1 and γ_2^{-1} defined by $\gamma_2^{-1}(t) = \gamma_2(1-t)$, and $[\gamma_1 \star \gamma_2^{-1}]$ denotes the homology class of $\gamma_1 \star \gamma_2^{-1}$. The quotient space $\tilde{X}$, whose underlying set is the set of equivalence classes of paths, is equipped with the canonical map $\psi : \tilde{X} \to X$ induced by assigning to γ the point $\gamma(1) \in X$. The map ψ is a local homeomorphism and $\tilde{X}$ is the total space of a principal covering with group $G = \mathrm{img}(< \mathbf{f} >: H_1(X; \mathbb{Z}) \to \mathbb{R})$. When $\mathbf{f}$ is almost integral G is isomorphic to $\mathbb{Z}$. If X is equipped with a triangulation, then $\tilde{X}$ gets a triangulation whose simplices, when viewed as subsets of $\tilde{X}$, are homeomorphic by ψ to simplices of X.

Construction of $\tilde{f}$ **from an almost integral 1-cocycle** $(\mathbf{f}, \alpha)$**.**

Step 1. Consider $\psi : \tilde{X} \to X$ the principal $\mathbb{Z}-$covering associated with the cohomology class $< \mathbf{f} >$ defined by $\mathbf{f}$. This means that $\tilde{X}$ is a simplicial complex equipped with a free simplicial action $\mu : \mathbb{Z} \times \tilde{X} \to \tilde{X}$ whose quotient space, $\tilde{X}/\mu$, is the simplicial complex X. The map ψ identifies to $\tilde{X} \to \tilde{X}/\mu = X$ and satisfies $\psi^*(< \mathbf{f} >) = 0$.

Choose a vertex x of X and call it a *base point*. Notice that the vertices $\tilde{\mathcal{X}}_0$ of $\tilde{X}$ can be also described as equivalence classes of sequences $\{x = x_0, x_1, \cdots x_{N-1}, x_N\}$ with x_is being consecutive vertices of X (i.e. x_i, x_{i+1} are vertices of an edge). Two such sequences, $\{x = x_0, x_1, \cdots, x_{N-1}, x_N\}$ and $\{x = y_0, y_1, \cdots y_{L-1}, y_L\}$ are equivalent if $x_N = y_L$ and the sequence $\{x = z_0, \cdots, z_{N+L} = x\}$ with $z_i = x_i$ if $i \leq N$ and $z_{j+N} = y_{L-j}$ if $j \leq L$, satisfies

$$\sum_{0 \leq i \leq L+N-1} \mathbf{f}(z_i, z_{i+1}) = 0.$$

Step 2. Define the map $\tilde{f} : \tilde{\mathcal{X}}_0 \to \mathbb{R}$ by $\tilde{f}(\tilde{y}) := \sum_{0 \leq i \leq L-1} \mathbf{f}(y_i, y_{i+1})$ where $\tilde{y} \in \tilde{\mathcal{X}}_0$ is the vertex corresponding to the equivalent class of $\{x = y_0, \cdots, y_L\}$. The description of $\tilde{X}$ given above guarantees that $\tilde{f}$ is well defined. Extend $\tilde{f}$ to a linear map $\tilde{f} : \tilde{X} \to \mathbb{R}$. Observe that if $\tilde{y}_1$ and $\tilde{y}_2$ satisfy $\psi(\tilde{y}_1) = \psi(\tilde{y}_2)$ then $\tilde{f}(\tilde{y}_1) - \tilde{f}(\tilde{y}_2) \in \alpha\mathbb{Z}$. In addition, if $\tilde{e}_1$ and $\tilde{e}_2$ are two edges of $\tilde{X}$ from $\tilde{y}_1$ to $\tilde{y}_1'$ and $\tilde{y}_2$ to $\tilde{y}_2'$ respectively with $\psi(\tilde{e}_1) = \psi(\tilde{e}_2)$, then $\tilde{f}(\tilde{y}_1') - \tilde{f}(\tilde{y}_2') = \tilde{f}(\tilde{y}_1) - \tilde{f}(\tilde{y}_2)$. This implies that if $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are two simplices with $\psi(\tilde{\sigma}_1) = \psi(\tilde{\sigma}_2) = \sigma$, and ψ_1 and ψ_2 are the homeomorphisms defined by the restriction of ψ to $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ respectively, then the map $\tilde{f} \cdot \psi_1^{-1} - \tilde{f} \cdot \psi_2^{-1} : \sigma \to \mathbb{R}$ is constant and an integer multiple of the real number α.

From cocycles to circle valued maps. Assume that $(\mathbf{f}, \alpha)$, an almost integral 1-cocycle, has been given. Observe that the map $p \cdot \tilde{f} : \tilde{X} \to \mathbb{R}$ (with $p : \mathbb{R} \to \mathbb{S}^1 = \mathbb{R}/\alpha\mathbb{Z}$) factors through $\tilde{X}/\mu = X$ inducing a circle valued map from X to $\mathbb{S}^1$. This is our circle valued map whose associated 1-cocycle is $\mathbf{f}$.

C Construction of $\tilde{X}_{[t,t+2\pi k]}$

We show how to construct a simplicial complex from X that contains $\tilde{X}_{[t,t+2\pi k]}$ Before we describe the construction, we need some properties of ordering of simplices which we apply later.

Suppose that X is any complex equipped with a filtration $\mathcal{F} = X_0 \subseteq \cdots X_i \subseteq X_{i+1} \cdots \subseteq X_m = X$ with X_is subcomplexes of X. Let $\mathcal{X}$ denote the set of simplices in X.

Definition C.1 *A total order on $\mathcal{X} = \{\sigma_1, \ldots, \sigma_n\}$ is called topologically consistent if the condition A below is satisfied and filtration compatible if the condition B below is satisfied.*

- **Condition A.** σ_i *is a face of* σ_j *implies* $i < j$.

- **Condition B.** $\sigma_i \in X_k$ *and* $\sigma_j \in X_{k'} \setminus X_k$ *with* $k < k'$ *implies* $i < j$.

Given a filtration $\mathcal{F}$, one can canonically modify any total order which satisfies Condition A into one which satisfies both conditions A and B.

Now consider the input complex X on which the circle valued map f is defined. For any $\theta \in \mathbb{S}^1$, decompose $\mathcal{X}$ as a disjoint union $\mathcal{X} = \mathcal{T}^\theta \sqcup \mathcal{L}^\theta \sqcup \partial_- \mathcal{L}^\theta \sqcup \partial_+ \mathcal{L}^\theta$ where (see Figure 5)

- $\mathcal{L}^\theta$ consists of the set of all simplices whose closure do intersect the level X_θ. Let L^θ be the simplicial complex generated by simplices in $\mathcal{L}^\theta$,

- $\mathcal{T}^\theta$ is the set of simplices which do not belong to L^θ. Let T^θ denote the simplicial complex generated by the the simplices in $\mathcal{T}^\theta$ and consider $\mathsf{T}^\theta \cap \mathsf{L}^\theta$. This simplicial complex is the disjoint union of two simplicial complexes $\partial_- \mathsf{L}^\theta$ and $\partial_+ \mathsf{L}^\theta$ characterized by $f(\sigma) < \theta$ for $\sigma \in \partial_- \mathsf{L}^\theta$ and $f(\sigma) > \theta$ for $\sigma \in \partial_+ \mathsf{L}^\theta$.

- $\partial_\pm \mathcal{L}^\theta$ represent the simplices in $\partial_\pm \mathsf{L}^\theta$.

Our purpose is to build a collection of simplicial complexes which contain $\tilde{X}_{[t,t+2\pi k]}$ where $p(t) = \theta$ and calculate the bar codes contained in $[t, t + 2\pi k]$.

Introduce a nested sequence of simplices $\tilde{\mathcal{X}}^\theta(0) \subseteq \tilde{\mathcal{X}}^\theta(1) \subseteq \cdots \tilde{\mathcal{X}}^\theta(k)$ as follows. Since we will repeat copies of each of the sets $\mathcal{T}^\theta$, $\mathcal{L}^\theta$, $\partial_- \mathcal{L}^\theta$, and $\partial_+ \mathcal{L}^\theta$, let $\mathcal{T}^\theta(n)$, $\mathcal{L}^\theta(n)$, $\partial_- \mathcal{L}^\theta(n)$, and $\partial_+ \mathcal{L}^\theta(n)$ denote their nth. copies respectively. Taking $\mathcal{L}^\theta(0) = \mathcal{L}^\theta$, $\mathcal{T}^\theta(0) = \mathcal{T}^\theta$, $\partial_+ \mathcal{L}^\theta(0) = \partial_+ \mathcal{L}^\theta$, define inductively,

$$
\begin{aligned}
\tilde{\mathcal{X}}^\theta(0) &= \partial_- \mathcal{L}^\theta \\
\tilde{\mathcal{X}}^\theta(n+1) &= \tilde{\mathcal{X}}^\theta(n) \sqcup \mathcal{L}^\theta(n) \sqcup \partial_+ \mathcal{L}^\theta(n) \sqcup \mathcal{T}^\theta(n) \sqcup \partial_- \mathcal{L}^\theta(n+1)
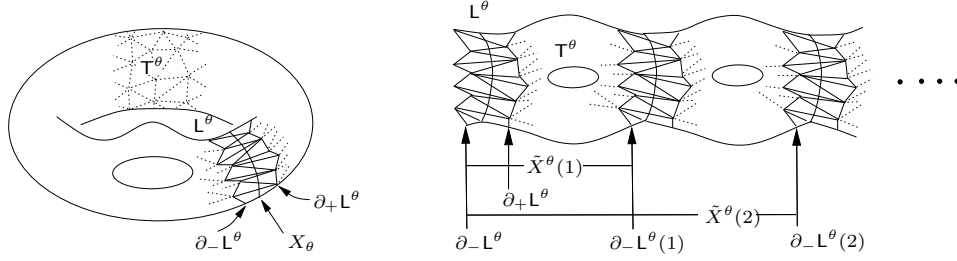\end{aligned}
$$

Figure 5: Complex X with level X_θ on left. Complexes $\tilde{X}^\theta(1)$, $\tilde{X}^\theta(2)$, $\cdots$ providing a filtration of the total space $\tilde{X}$ on right.

Let $I(\sigma, \tau)$ denote the incidence between σ and τ, that is, $I(\sigma, \tau) = 1$ if τ is a face of σ of codimenison 1 and 0 otherwise. Taking $I_0(\sigma, \tau) = I(\sigma, \tau)$, the incidences among the simplices are described by

$$
\begin{aligned}
I_{n+1}(\sigma, \tau) &= I_n(\sigma, \tau) \text{ if } \sigma \in \tilde{\mathcal{X}}^\theta(n) \text{ and } \tau \text{ is a face of } \sigma \text{ of codimension } 1 \\
I_{n+1}(\sigma, \tau) &= I(\sigma, \tau) \text{ if } \sigma \in \mathcal{L}(n) \sqcup \partial_+\mathcal{L}^\theta(n) \subset \mathsf{L}^\theta \text{ and } \tau \text{ is a face of } \sigma \text{ of codimension } 1 \\
I_{n+1}(\sigma, \tau) &= I(\sigma, \tau) \text{ if } \sigma \in \mathcal{T}^\theta(n) \sqcup \partial_-\mathcal{L}^\theta(n+1) \subset \mathsf{T}^\theta \text{ and } \tau \text{ is a face of } \sigma \text{ of codimension } 1.
\end{aligned}
$$

In all other cases $I_{n+1}(\sigma, \tau) = 0$.

Notice that each $\tilde{\mathcal{X}}^\theta(i)$ forms a simplicial complex $\tilde{X}^\theta(i)$ (Figure 5). To describe $\tilde{f}$ it suffices to provide its values on vertices. We write

$$
\mathcal{P} = \partial_-\mathcal{L}^\theta \sqcup \mathcal{L}^\theta \sqcup \partial_+\mathcal{L}^\theta \sqcup \mathcal{T}^\theta
$$

and $\mathcal{P}_0$ for the subset of vertices in $\mathcal{P}$. We write $\mathcal{P}_0(n)$ for the $n-$th copy of $\mathcal{P}_0$ and define $\tilde{f}(n) : \mathcal{P}_0(n) \to \mathbb{R}$ by $\tilde{f}(n) := \tilde{f} + 2\pi n$ where $\tilde{f} = p^{-1} \cdot f$ with $p : (t - \pi, t + \pi] \to \mathbb{S}^1$ which [2] sends t to θ. Once defined on vertices, $\tilde{f}$ is extended by linearity to each simplex of the simplicial complex $\tilde{X}^\theta(n)$.

Note that an order of the simplices of X satisfying condition A induces an order on the simplices of $\mathcal{T}^\theta(n)$, $\partial_\pm\mathcal{L}^\theta(n)$ and $\mathcal{L}^\theta(n)$ and by juxtaposition an order on the simplices of $\tilde{X}^\theta(n)$ which continue to satisfy condition A. It implies that one can build a matrix $M(\tilde{X}^\theta(n))$ which satisfies condition A by juxtaposing the minors of $M(X)$ that represent $\mathcal{L}^\theta$, $\partial_\pm L^\theta$, $\mathcal{T}^\theta$, and their copies in an appropriate order. Note also that $\tilde{X}^\theta(n)$ is a sub complex of $\tilde{X}$ and therefore the restriction of $\tilde{f}$ provides tame maps on each of these spaces. The columns and rows of $M(\tilde{X}^\theta(n))$ can be reordered so that they become filtration compatible with $\tilde{f}$.

D   Computing bar codes with standard persistence

In what follows we propose an alternative method to derive the bar codes using the standard persistence algorithm which computes the sub level persistence applied to various subspaces Y of X canonically derived from X and f as indicated in [1].

For this purpose we need also a minor extension of sublevel persistence, which we refer as simultaneous persistence. To describe it, we consider two maps $f^\pm : W^\pm \to [0, \infty)$ with the condition that $(f^\pm)^{-1}(0) = A \subset W^\pm$. Denote by $\omega^{f^-, f^+}(s, t)$ the maximal number of linearly independent elements in $H_r(A)$ which die with respect to f^- exactly at s and with respect to f^+ exactly at t. These numbers are analogues of the numbers $\mu_r(\cdot)$ considered for the sublevel persistence and can be computed by running the standard persistence algorithms for W^+ and W^-, see [12].

Let $f : X \to \mathbb{R}$ be a tame map with critical values $\cdots s_{i-1} < s_i < s_{i+1} < s_{i+2} \cdots$. We apply the discussion below to $\tilde{f} : \tilde{X} \to \mathbb{R}$.

---

[2]p is bijective and continuous, but p^{-1} is not continuous

For $s_i < s_j$, let

$N\{s_i, s_j) =$ the number of bar codes which intersect the level X_{s_i} with open end at s_j,

$N(s_i, s_j\} =$ the number of bar codes which intersect the level X_{s_j} with open end at s_i,

$N(s_i, s_j) =$ the number of bar codes which have both ends s_i and s_j open, and

These numbers can be computed using the sublevel persistence and the simultaneous persistence as follows. First we introduce some notations.

For $s \in \mathbb{R}$ consider the maps $g_s^{\pm} : Y^{\pm}(s) \to [0, \infty)$ defined by:

$$Y^+(s) = X_{[s,\infty)}, Y^-(s) = X_{(-\infty,s]}$$
$$g_s^+ = f|_{X_{[s,\infty)}} - s, g_s^- = -f|_{X_{(-\infty,s]}} + s.$$

Letting $\mu_r^{g_{s_i}^+}$ and $\mu_r^{g_{s_j}^-}$ denote the numbers $\mu(\cdot)$ as defined in [12] for maps $g_{s_i}^+$ and $g_{s_j}^-$ respectively we have:

$$N\{s_i, s_j) = \mu_r^{g_{s_i}^+}(0, s_j - s_i)$$
$$N(s_i, s_j\} = \mu_r^{g_{s_j}^-}(0, s_j - s_i)$$
$$N(s_i, s_j) = \omega^{g_s^-, g_s^+}(s - s_i, s_j - s), s \in (s_i, s_j).$$

Hence, the cardinality of $[s_i, s_j)$ is

$$N[s_i, s_j) := N\{s_i, s_j) - N\{s_{i-1}, s_j) - N(s_i, s_j).$$

Similarly, the cardinality of $(s_i, s_j]$ is

$$N(s_i, s_j] = N(s_i, s_j\} - N(s_i, s_{j+1}\} - N(s_i, s_{j+1}).$$

and the cardinality of (s_i, s_j) equals $N(s_i, s_j)$. We are only left with computing the number of bar codes of type $[s_i, s_j]$. Let

$N\{s_i, s_j] =$ the number of bar codes which intersect the level X_{s_i} and with closed end at s_j,

$N[s_i, s_j\} =$ the number of bar codes which intersect the levels X_{s_j} and with closed end at s_i,

$N[s_i, s_j] =$ the number of bar codes which have both ends s_i and s_j closed, and

$N\{s_i, s_j\} =$ the number of bar codes which intersect both levels X_{s_i} and X_{s_j}.

Then we have
$$N\{s_i, s_j\} = \Sigma_{j=i}^{\infty} \mu_r^{g_{s_i}^+}(0, s_j - s_i),$$
$$N\{s_i, s_j] = N\{s_i, s_j\} - N\{s_i, s_{j+1}\} - N\{s_i, s_{j+1}),$$
$$N[s_i, s_j\} = N\{s_i, s_j\} - N\{s_{i-1}, s_j\} - N(s_{i-1}, s_j\}$$

with the last terms $N\{s_i, s_{j+1})$ and $N(s_{i-1}, s_j\}$ being computed as discussed above and finally

$$N[s_i, s_j] = N\{s_i, s_j] - N\{s_{i-1}, s_j] - N(s_{i-1}, s_j].$$

E   Decomposing a quiver representation

We present a procedure to decompose a representation ρ into a sum of indecomposable representations, i.e, find its bar codes and Jordan cells.

For notational purpose we introduce the infinite sequences $\{x_1, x_2, \cdots\}$, $\{V_{x_1}, V_{x_2}, \cdots\}$, $\{\alpha_1, \alpha_2, \cdots\}$, and $\{\beta_1, \beta_2, \cdots\}$ defined by: $x_{i+2km} := x_i$, $V_{x_{i+2km}} := V_{x_i}$, $\alpha_{i+km} := \alpha_i$, and $\beta_{i+km} := \beta_i$.

An *l-chain* denoted by c starting at the index p, $1 \le p \le 2m$, is a collection of successive l elements $c = \{h_i \in V_{x_i}, h_i \ne 0, p \le i \le p + l - 1\}$, which satisfy:

P1: $\alpha_i(h_{2i-1}) = h_{2i}, \beta_i(h_{2i+1}) = h_{2i}$ provided $2i - 1 \ge p$, and $2i + 1 \le p + l - 1$.

P2: If p is even, then h_p is not in the image of $\alpha_{p/2}$, and if p is odd, then $\beta_{(p-1)/2}(h_p) = 0$.

P3: If $(p+l-1)$ is even, then h_{p+l-1} is not in the image of $\beta_{(p+l-1)/2}$, and if odd, then $\alpha_{(p+l)/2}(h_{p+l-1}) = 0$.

P4: For any i, $1 \le i \le 2m$, all elements in the collection which belong to V_{x_i} are linearly independent.

Any such chain generates an indecomposable representation ρ^c, a sub-representation of ρ, whose vector space $V_{x_i}^c \subseteq V_{x_i}$ is the linear span of the elements in the chain which lie in V_{x_i}. Property P4 ensures that this sub-representation can be split off, i.e., one can find a representation ρ' so that $\rho = \rho^c \oplus \rho'$.

Indeed, assume that we record the representation ρ by the matrices $\alpha_i \in \mathcal{M}^{n_i \times d_i}$ and $\beta_i \in \mathcal{M}^{d_{i+1} \times n_i}$, where $n_1, n_2, \cdots n_m, d_1, d_2, \cdots d_m$ are the integers defining the dimension of ρ as described in Appendix A. Then, we can find a new base in each V_{x_i} which contains the elements of the chain lying in V_{x_i} as the first elements of the base. With respect to this new base, the linear maps α_i and β_i become $\begin{pmatrix} \alpha_i^c & \alpha_i'' \\ 0 & \alpha_i' \end{pmatrix}$ and $\begin{pmatrix} \beta_i^c & \beta_i'' \\ 0 & \beta_i' \end{pmatrix}$. The representation ρ' is then given by the matrices α_i' and β_i'.

Our procedure splits off indecomposable representations corresponding to chains of length 1 first, then of length 2, and so on until no more chain exists. This happens when all α_is and β_is become isomorphisms. At this stage Observation 4.1 permits us to find the indecomposable representations which give the Jordan cells. The following proposition allows us to find all chains iteratively.

Proposition E.1

1. *If $h \in V_{x_{2i+1}}$ satisfies $\beta_i(h) = 0$, then h is the beginning of a collection of successive elements which satisfy P1,P2,P3.*

2. *If $h \in V_{x_{2i}}$ is not in the image of α_i, then h is the beginning of a collection of successive elements which satisfy P1,P2,P3.*

3. *If a representation ρ does not contain l-chains for $0 \le l \le k$, and we have a collection of $(k+1)$ successive elements which satisfy P1, P2, P3, then they do satisfy P4; hence the collection is a $(k+1)$-chain.*

Note that a complete collection of the 1-chains is given by a maximal set of linearly independent elements of $\ker \beta_i \cap \ker \alpha_{i+1} \subset V_{x_{2i+1}}$ which gives bar codes of the form $(i, i+1)$ and also a maximal set of linearly independent elements in $V_{x_{2i}} \setminus (\alpha_i(V_{x_{2i-1}}) + \beta_i(V_{x_{2i+1}}))$ which give bar codes $[i, i]$. At any generic step of the algorithm, we search for a l-chain assuming that all k-chains where $k < l$ have been eliminated. We start at a vertex satisfying either 1 or 2 of Proposition E.1, and then extend the chain as far as we can using the current matrices α_i and β_i. If the extension fails to provide an l-chain, we search for another vertex

to begin a new l-chain. If there is no l-chain in the current representation, we will exhaust all vertices, or will find one otherwise. Once an l-chain is found, we split it off the current representation and continue.

At the end of the above procedure, we will eliminate all indecomposable representations of type ρ^I from ρ and then can find the Jordan cells from the remaining representation using Jordan decomposition as indicated by Observation 4.1.

The original maps α_i and β_i for the quiver representation ρ_r are the linear maps induced in homology by the inclusions $X_{t_i} \subset X_{[t_i, s_i]}$ and $X_{t_{i+1}} \subset X_{[s_i, t_{i+1}]}$. The matrix representations of these maps can be computed using the standard persistence algorithm.