MEASURE CONCENTRATION ON SUBSPHERES

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1. The Problem

Set S_α^{k-1} denote the angular α-neighbourhood of a $(k-1)$-dimensional subsphere of the $(d-1)$-sphere S^{d-1}. For what follows, in order to avoid having to deal with special cases, assume $3 \leq k \leq d-2$. The problem we are concerned with is a measure concentration bound

$$\sigma_{d-1}(\mathsf{S}_\alpha^{k-1}) \begin{cases} \leq C \cdot e^{-c\lambda^2 d} & \text{if } \sin^2(\alpha) = (1-k/n) - \lambda, \\ \geq 1 - C \cdot e^{-c\lambda^2 d} & \text{if } \sin^2(\alpha) = (1-k/n) + \lambda, \end{cases}$$

where σ_{d-1} denotes the uniform measure on the sphere and c, C are some constants. Three approaches to this problem are described.

(1) The first approach is based on a projection $\mathsf{S}^{d-1} \to \mathsf{B}_k$ to a k-dimensional ball, and describes the set S_α^{k-1} as the set of those point in S^{d-1} that project to a point of length greater than $\cos\alpha$.
(2) The second approach uses the representation of $\sigma_{d-1}(\mathsf{S}_\alpha^{k-1})$ as normalized incomplete beta function, i.e., the cumulative distribution function of a beta distribution. An asymptotic analysis of this function reveals Gaussian-like behaviour.
(3) The third approach is based on the binomial interpretation of the beta distribution.

Almost unavoidable is the representation of the neighorhood measure in terms of the normalized incomplete beta function

$$\sigma_{d-1}(\mathsf{S}_\alpha^{k-1}) = I_{\sin^2(\alpha)}((d-k)/2, k/2). \tag{1.1}$$

This function is defined as $I_x(a,b) = B_x(a,b)/B_1(a,b)$, where

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt. \tag{1.2}$$

is the incomplete beta function.

2. Length of projection

We have the representation

$$\sigma_{d-1}(\mathsf{S}_\alpha^{k-1}) = \mathbb{P}\{\|\mathbf{\Pi}_k(\boldsymbol{p})\|^2 \geq \cos^2(\alpha)\},$$

where $\boldsymbol{p} \sim \text{Uniform}(\mathsf{S}^{d-1})$ and $\mathbf{\Pi}_k$ denotes the projection on the first k coordinates. The mean of $\|\mathbf{\Pi}_k(\boldsymbol{p})\|^2$ is readily established:

$$m = \mathbb{E}[\|\mathbf{\Pi}_k(\boldsymbol{p})\|^2] = k/d.$$

From this point, there are different directions one can go. One argument uses concentration around the median and the fact that the median is not

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far off [5, 15.2.2], while another argument is based on looking at Gaussian projections [3]. We describe the first one.

First note that the square of the projected length is a 2-Lipshitz function on the sphere. By Lévy's lemma on the concentration of Lipshitz functions [5, 14.3.2], we have concentration around the median μ ($\lambda > 0$):

$$\mathbb{P}\{\|\mathbf{\Pi}_k(\boldsymbol{p})\|^2 \geq \mu + \lambda\} \leq 2e^{-\lambda^2 d/8} \quad \text{and} \quad \mathbb{P}\{\|\mathbf{\Pi}_k(\boldsymbol{p})\|^2 \leq \mu - \lambda\} \leq 2e^{-\lambda^2 d/8}$$

In order to get conentration around the mean, we could use the generic argument that the mean and median are within distance of order $c/\sqrt{d}$ for a constant c [5, 14.3.3], or follow the trick used in the proof of [5, 15.2.2]. One can also use known facts about the beta distribution (1.1). From a known inequality relating the mode, median and mean of the beta distribution [4], we get for $5 < d < 2k$:

$$\frac{k-2}{d-4} \geq \mu \geq \frac{k}{d},$$

and the reverse inequality for $5 < 2k < d$, provided the parameters are all in a range where this makes sense. For $2 \leq k \leq d-4$, the left-hand side is bounded from above by $k/d + 2/d$. Similarly, in the case $2k < d$, one also gets within an additive factor of $2/d$ to the mean. The concentration inequalities then rewrite as

$$\mathbb{P}\{\|\mathbf{\Pi}_k(\boldsymbol{p})\|^2 \geq k/n + \lambda\} \leq 2e^{-(\lambda-2/d)^2 d/8} \leq 4e^{-\lambda^2 d/8}$$

and similar for the other direction. This readily translates into a concentration inequality for the measure around subspheres as

$$\sigma_{d-1}(\mathsf{S}_\alpha^{k-1}) \begin{cases} \leq 4e^{-\lambda^2 d/8} & \text{if } \sin^2(\alpha) = (1-k/n) - \lambda, \\ \geq 1 - 4e^{-\lambda^2 d/8} & \text{if } \sin^2(\alpha) = (1-k/n) + \lambda, \end{cases}$$

where $\lambda > 0$.

3. Analysis of Beta distribution

The relationship (1.1) suggests the use of asymptotic methods on the beta function. Set $p = k/n$. In [2], the following bound was shown (among other estimates, including a precise asymptotic analysis)

$$\sigma_{d-1}(\mathsf{S}_\alpha^{k-1}) \begin{cases} \leq \frac{c_1}{\sqrt{-u}} e^u & \text{if } \sin^2(\alpha) = 1 - p - \lambda \\ \geq 1 - \frac{c_2}{\sqrt{-u}} e^u & \text{if } \sin^2(\alpha) = 1 - p + \lambda, \end{cases}$$

where $u = (d/2) \cdot (1-p)\ln(1 + \lambda/(1-p)) + p\ln(1 - \lambda/p)$ (we use different notation and a slight reformulation). The constants c_1, c_2 are assumed to depend on absolute lower and upper bounds of $1 + \lambda/(1-p)$ and $1 - \lambda/p$. Using an upper bound on the second derivative and integrating, one finds the upper bound:

$$(1-p)\ln(1 + \lambda/(1-p)) + p\ln(1 - \lambda/p) \leq -\lambda^2.$$

From this, we get the bounds

$$\sigma_{d-1}(\mathsf{S}_\alpha^{k-1}) \begin{cases} \leq \frac{c_1}{\lambda\sqrt{d/2}} e^{-\lambda^2 d/2} & \text{if } \sin^2(\alpha) = 1 - p - \lambda \\ \geq 1 - \frac{c_2}{\lambda\sqrt{d/2}} e^{-\lambda^2 d/2} & \text{if } \sin^2(\alpha) = 1 - p + \lambda. \end{cases}$$

The asymptotic result of [2] can also be obtain by direct means, by an argument related to the Laplace method, I can include this if needed.

4. Relation to binomial distribution

Setting $x = \sin^2(\alpha)$ and $p = k/d$, $q = 1 - p$, we have the following expansion of the beta function (recall identity 1.1):

$$\sigma^{d-1}(\mathsf{S}^{k-1}_\alpha) = \sum_{\substack{i=qd/2 \\ i\equiv d-1(2)}}^{d-1} \binom{\frac{d-2}{2}}{\frac{i-1}{2}} x^{\frac{i-1}{2}}(1-x)^{\frac{d-i-1}{2}} + \begin{cases} R(x) & k \text{ odd}, \\ 0 & k \text{ even}, \end{cases} \tag{4.1}$$

with remainder $R(x) = I_x((d-1)/2, 1/2)$. Alternatively [1, A.1.1], identity (4.1) can be derived by noting that

$$\sigma^{d-1}(\mathsf{S}^{k-1}_\alpha) = \mathbb{P}\{\boldsymbol{Q}\mathsf{S}^{k-1} \cap \mathrm{cap}^{d-1}(\alpha)\},$$

where $\boldsymbol{Q}$ is a random rotation, chosen uniformly from the orthogonal group. The kinematic formula gives the expression

$$\sigma^{d-1}(\mathsf{S}^{k-1}_\alpha) = \sum_{i=0}^{k}(1-(-1)^i)\ v_{d-k+i}(\mathrm{cap}^{d-1}(\alpha)).$$

Plugging in the spherical intrinsic volumes $v_k(\mathrm{cap}^{d-1}(\alpha))$ gives rise to identity (4.1).

Note that, compared with approach 3, we reduced the beta integral to a binomial distribution plus one simpler beta integral. Set $x = q - \lambda$, $\lambda > 0$. The remainder in (4.1) can be bounded by $R(q-\lambda) \le 1.24 \cdot e^{-(d-2)\lambda^2/2}$, I omit the details of this calculation. For the other terms, note that if d and k are even, we have a binomial distribution with mean $x(d-2)/2$, for which we have the standard Chernoff bounds (recall $x = \sin^2(\alpha)$)

$$\begin{aligned} \sigma^{d-1}(\mathsf{S}^{k-1}_\alpha) &= \mathbb{P}\{X \ge q(d-2)/2\} \\ &= \mathbb{P}\{X \ge (x+\lambda)(d-2)/2\} \\ &\le e^{-\lambda^2(d-2)/2}, \end{aligned}$$

where X denotes the binomial random variable. The bound in the other direction is similar. If d is odd, we might have to use known identities for $\Gamma(j+1/2)$, it seems that the most we lose are factors of $\sqrt{\pi}$ and from the estimate of the remainder.

References

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