Imperial College London

Advanced Research Project In Mathematics

Department of Mathematics

PERSPECTIVES ON ALGEBRAIC CURVES

Louis Bradshaw

A project submitted in partial fulfilment of the requirement for the award of MSci (Master of Science)

June 2022

Abstract

We give a survey on the theory compact Riemann surfaces. We aim to highlight the similarities between the algebraic and analytic theory, primarily as it is developed in [Mir95] and [Har77]. We provide an outline of the main result, that compact Riemann surfaces are non-singular projective varieties of dimension 1. We then explore this connection further, finishing with a statement of a version of Serre's GAGA Theorem for compact Riemann surfaces.

Acknowledgements

The main reference for Chapter 3 and 4 is [Mir95], in these chapters I follow the approach to proving the Riemann-Roch theorem found in Ch. 5, 6, 7 of [Mir95]. The main reference for Chapter 2 were my personal notes from the course 'Algebraic Geometry' taught by Dr. Travis Schedler at Imperial College London in Spring 2020, and Ch. 1 of [Har77]. The main reference (and where I learnt the material) for Chapter 5 is Ch. 9 in [Mir95]. I assert that this is my own work, unless specified otherwise. .

Table of Contents

1 Riemann Surfaces

1.1 Complex Manifolds and Holomorphic Maps

We will start our discussion by outlining the definition of a Riemann surface, the natural notion of maps between them, and some important examples. Some details will be omitted, complete descriptions can be found in Part II of [Don11]. We will also assume a familiarity with the basics of complex analysis and the theory of smooth manifolds, here we refer the reader to [Lee12] and Part I of [Don11].

Definition 1.1. A *complex chart* (ϕ, U) on a topological space X, is a homeomorphism $\phi : U \to V$ where $U \subseteq X$ is an open subset of X, and $V \subset \mathbb{C}^n$ is an open subset of $\mathbb{C}^n$. We say that two charts $\phi_1 : U_1 \to V_1$, $\phi_2 : U_2 \to V_2$ on X are *compatible* if given that $U_1 \cap U_2 \neq \emptyset$

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is holomorphic. $\phi_2 \circ \phi_1^{-1}$ is called a *transition function*.

Definition 1.2. A topological space X, along with collection of charts $\{(\phi_i, U_i)\}_{i \in I}$ is called a *complex manifold* if the following properties are satisfied:

1. X is a Hausdorff and second countable topological space.
2. $\{U_i\}_{i \in I}$ is an open cover for X, and all of the charts in $\{(\phi_i, U_i)\}_{i \in I}$ are pairwise compatible.

In the case that $n = 1$ and X is connected, we say that X is *Riemann Surface*.

By utilising the identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. we observe that any 'complex' chart ϕ on a Riemann surface X is also a real chart. Furthermore as all transition functions are necessarily holomorphic, they must be smooth. Every Riemann surface is therefore naturally a 2-dimensional real manifold. In particular, ideas inherited from the theory of smooth manifolds like *smooth functions* and *orientation* make sense. In fact, as a consequence of the Cauchy–Riemann equations, all Riemann surfaces are oriented when viewed as smooth surfaces. In the context of Riemann surfaces it is essential to refine the idea of smoothness of a function $f : X \to \mathbb{C}$. We do this in the following way.

Definition 1.3. Let X be a Riemann surface and let $f : X \to \mathbb{C}$ be a function. We say that f is *holomorphic* function on X if for every chart $\phi : U \to V \subseteq \mathbb{C}$, the composition $f \circ \phi^{-1} : V \to \mathbb{C}$ is a holomorphic. We denote the ring of holomorphic functions on X as $\mathcal{O}(X)$.

We can strengthen this definition to allow us to talk about holomorphic maps between Riemann surfaces.

Definition 1.4. Let X, Y be Riemann surfaces and let $F : X \to Y$ be a map. We say that F is a *holomorphic map* if for every point $x \in X$ there are charts (ϕ, U), (ψ, V) of X, Y such that $x \in U$, $F(x) \in V$ and $\psi \circ F \circ \phi^{-1}$ is holomorphic at $\phi(x)$.

If we endow $\mathbb{C}$ with its natural structure as a Riemann surface, these two definitions coincide for a function $f : X \to \mathbb{C}$. Given an holomorphic map $F : X \to Y$ between Riemann surfaces and a holomorphic function $f \in \mathcal{O}(Y)$, it can be shown that $f \circ F$ is a holomorphic function on X. This should not be surprising as we would expect the composition of holomorphic maps is again holomorphic. In this way, every holomorphic map $F : X \to Y$ induces a homomorphism of rings $F^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ called the *pullback* of F.

We can also extend the idea of meromorphic functions to Riemann surfaces. Recall that a meromorphic function $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is a function which is holomorphic outside a set of isolated points which are poles of f. Typically in complex analysis we loosely assign the value ∞ to f at its poles if it is meromorphic. Using [1.4] we can extend this idea concrete way. First we must introduce the complex manifold $\mathbb{P}^n$.

Definition 1.5. We define the topological space $\mathbb{P}^n$ as the quotient space of $\mathbb{C}^{n+1} - \{0\}$ by the equivalence relation $x \sim \lambda x, \ \forall \lambda \in \mathbb{C}$. We use the notation $[z_0 : \cdots : z_n] := [(z_0, \ldots z_n)]$.

Proposition 1.6. $\mathbb{P}^n$ along with the charts (ϕ_i, A_i) defined as

- $A_i = \{[z_0 : \ldots z_{i-1} : 1 : z_{i+1} : \ldots z_n] \in \mathbb{P}^n : z_j \in \mathbb{C} \ \forall i\}$
- $\phi_i : A_i \to \mathbb{C}^n$, $[z_0 : \ldots z_{i-1} : 1 : z_{i+1} : \ldots z_n] \mapsto (z_0, \ldots z_{i-1}, z_{i+1}, \ldots z_n)$

is a compact complex manifold of dimension n.

The proof of this statement can be found in any book on complex manifolds, for instance it is proved in Chapter 2.1 of [Huy05]. The crux of construction is that $\mathbb{C}^n$ lives inside of $\mathbb{P}^n$ via the chart ϕ_0 and outside of this chart we have a copy of $\mathbb{P}^{n-1}$. In the case of $\mathbb{P}^1$ we have $\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$. Since $\mathbb{P}^1$ has the structure of a complex manifold, we are now in a position to state what we mean by a meromorphic function on a Riemann surface.

Definition 1.7. Let X be a Riemann surface. A holomorphic function $f : X \to \mathbb{P}^1$ is called *meromorphic* if is not identically equal to $[0 : 1]$.

If $f : U \subseteq \mathbb{C} \to \mathbb{C}$ is a meromorphic function in the traditional sense, then the function

$$\tilde{f}(z) = \begin{cases} [0 : 1], & \text{if } z \text{ is a pole of } f \\ [1 : z], & \text{otherwise} \end{cases}$$

is the corresponding meromorphic function $\tilde{f} : U \to \mathbb{P}^1$ using our new more general definition. Conversely we can recover the original function as $f = \phi_0 \circ \tilde{f}$. This identification is 1-1 due to the fact that non-constant holomorphic maps have discrete preimages (Ch. 2, Prop. 3.12 in [Mir95]). We denote the set of meromorphic functions on a Riemann surface X as $\mathcal{M}(X)$. If we think locally, meromorphic functions $f : X \to \mathbb{P}^1$ are exactly those functions which meromorphic in the normal sense with respect to every chart (ϕ, U) of X. Following this logic, we conclude that $\mathcal{M}(X)$ is in fact a ring.

Now that we have extended the basic tools of complex analysis to Riemann surfaces, we will do the same to some relevant concepts from the theory of smooth manifolds. Of particular importance are the analogous versions of diffeomorphisms and sub-manifolds.

Definition 1.8. A holomorphic map $F : X \to Y$ is said to be a *biholomorphism* if it a bijection with holomorphic inverse.

In particular, if $F : X \to Y$ is a biholomorphism of Riemann surfaces then $F^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an isomorphism of rings with inverse $(F^{-1})^*$. If we put the appropriate complex structure on $\mathbb{C} \cup \{\infty\}$ then the natural map $\mathbb{P}^1 \to \mathbb{C} \cup \{\infty\}$ sending $[0 : 1] \mapsto \infty$ is a biholomorphism. Furthermore, the function sending $f \to \tilde{f}$ seen previously is exactly the pullback of this biholomorphism. In the category of Riemann surfaces biholomorphisms are the isomorphisms.

Definition 1.9. Let Y be a complex manifold of dimension n and suppose that $X \subset Y$ is a proper subset. We say that X is a k-dimensional *complex sub-manifold* of Y if for every point $x \in X$ there exists a chart of (ϕ, U) of Y such that $x \in U$ and

$$\phi(X \cap U) = \phi(U) \cap \mathbb{C}^k \subseteq \mathbb{C}^n$$

Just like for smooth sub-manifolds, complex sub-manifolds come with a unique topology and complex structure which is 'compatible' with the inclusion map $X \hookrightarrow Y$. Strictly speaking the inclusion map is what's called an holomorphic *embedding*. Furthermore, given an embedding $F : X \to Y$, $F(X)$ turns out to be a complex sub-manifold of Y which is biholomorphic to X. The details of this can be found in Chapter 4 of [FG02] however for now it suffices to think of complex sub-manifolds purely in terms of [1.9]. The type of complex manifolds which we will be primarily interested in are known as *projective* Riemann surfaces.

Definition 1.10. Let X be a Riemann surface. We say that X is *projective* if X is biholomorphic to a complex sub-manifold of $\mathbb{P}^n$ for some n.

By restating the definition of complex sub-manifold in this context, we have the following criterion for $X \subseteq \mathbb{P}^n$ to be a projective Riemann surface.

Proposition 1.11. Let $X \subseteq \mathbb{P}^n$ be a Riemann surface. X is a projective if for every $x = [x_0 : \cdots : x_n] \in X$ the following hold

1. There is an i such that $x_i \neq 0$.
2. For all $j \neq i$, the functions $[z_0 : \cdots : z_n] \mapsto z_j/z_i$ are holomorphic in a open neighbourhood of x.
3. There is an open neighbourhood U of x and some $k \neq i$ such that $\phi : U \to \mathbb{C}$ defined as $\phi : [z_0 : \cdots : z_n] \mapsto z_k/z_i$ is a chart of X.

The connection between these conditions and [1.9] is not immediately obvious. The crux of the proof is in an application of the implicit function theorem for functions in several complex variables. Essentially, conditions 2 and 3 allow for biholomorphic change of variables which gives us the chart required in [1.9]. More details can be found in Chapter 1.1 of [Huy05].

1.2 Meromorphic Functions

It is natural to expect that holomorphic functions will be of much importance in our discussion of Riemann surfaces. After all, when studying smooth manifolds there is emphasis put on the space of C^∞-functions. It is then surprising to find that meromorphic functions and the structure of the ring $\mathcal{M}(X)$ is of far more importance. The reason why will become clear later, for now we may use the following proposition to illustrate this.

Proposition 1.12. If X is a compact Riemann surface then the only holomorphic functions $f : X \to \mathbb{C}$ are the constant functions.

Proof. Let $f : X \to \mathbb{C}$ be a holomorphic. Since X is compact the function $|f| : X \to \mathbb{R}$ achieves a maximum in X say at p. Let (ϕ, U) be chart of p. By the maximum modulus principle the function $f \circ \phi^{-1} : \phi(U) \to \mathbb{C}$ is constant. After observing that the identity theorem in complex analysis also holds for Riemann surfaces (Ch. 4, Thm. 1.1 in [FG02]), we may conclude that f is constant as X is connected by definition.

$\square$

Since the structure of $\mathcal{O}(X)$ is so simple for compact Riemann surfaces, we will continue by developing the theory of their meromorphic functions.

Let X be a compact Riemann surface and let $f : X \to \mathbb{P}^1$ be a meromorphic function. For any point $x \in X$ and any chart (ϕ, U) of x we can realise $\tilde{f} = f \circ \phi^{-1} : \phi(U) \to \mathbb{C} \cup \{\infty\}$ as a meromorphic function. As such, $\tilde{f}$ has a Laurent series

$$\tilde{f} = \sum_{n=-\infty}^{\infty} a_n (z - \phi(x))^n$$

in a punctured neighbourhood of $\phi(x)$.

Definition 1.13. Let X and f be as above. We say that the order of f at the point x is the smallest exponent appearing in the Laurent series for $\tilde{f}$ at $\phi(x)$ which has non zero coefficient. We denote the order by $\mathrm{ord}_x(f)$.

The order f at x provides information about the nature of the f and the point x. For example if $\mathrm{ord}_x(f) \geq 1$ or $\mathrm{ord}_x(f) \leq -1$ then x is a zero or pole of f respectively. Our choice of chart may affect the Laurent series however the order is well defined in this respect.

Proposition 1.14. Let X be a Riemann surface and let $f : X \to \mathbb{P}^1$ be a meromorphic function. The function $\mathrm{ord}_x(f)$ is well defined.

Proof (p. 26 in [Mir95]). Let (ϕ, U) and (ψ, V) be two charts for $x \in X$. The transition function $\phi \circ \psi^{-1}$ is holomorphic with holomorphic inverse. Note that if $W, Z \subseteq \mathbb{C}$ are open and $F : W \to Z$ is any biholomorphism, then F has non zero first derivative at any point as

$$\frac{d}{dz}\big(F^{-1} \circ F\big)(z) = (F^{-1})'(F(z)) \cdot F'(z) = 1$$

by the complex chain rule. In particular this applies to $\phi \circ \psi^{-1}$. Furthermore, if we express $\phi \circ \psi^{-1}$ locally as a power series around $\psi(x)$

$$\phi \circ \psi^{-1}(z) = \phi(x) + \sum_{n=1}^{\infty} c_n (z - \psi(x))^n$$

then composing this with the Laurent series for $f \circ \phi^{-1}$ does not change the the term of lowest degree as $c_1 \neq 0$. Finally, as $(f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}) = f \circ \psi^{-1}$ the order calculations for both charts are the same and so $\mathrm{ord}_x(f)$ is well defined.

□

Thinking about meromorphic functions as holomorphic maps $F : X \to \mathbb{P}^1$ between Riemann surfaces is a powerful change of perspective. It allows us to apply theorems we prove about holomorphic maps equally to meromorphic functions. This is provides a more unified framework to think about 'well behaved' complex valued functions that we come across. A first example of such a theorem is the following.

Proposition 1.15. Let $F : X \to Y$ be a non constant holomorphic map between Riemann surfaces, let $x \in X$. There is a unique integer $k \geq 1$ such that for every chart $\psi : U_2 \to V_2$ for Y centred at $F(x)$ (i.e. $F(x) \in U_2$ and $\psi(F(x)) = 0$) there exists a chart $\phi : U_1 \to V_1$ centred at x such that

$$\psi \circ F \circ \phi^{-1}(z) = z^k$$

Proof (Ch. 2, Prop. 4.1 in [Mir95]). Let $\psi : U_2 \to V_2$ be a chart of Y centred at $F(x)$ and let $\phi : U_1 \to V_1$ be a chart of X centred at x. $\psi \circ F \circ \phi^{-1}$ is holomorphic and so by sufficiently shrinking V_1 we may express it as a convergent power series $\psi \circ F \circ \phi^{-1}(z) = \sum_{n=k}^{\infty} a_n z^n$. Furthermore we may write $\psi \circ F \circ \phi^{-1}(z) = z^k h(z)$ where h is holomorphic and $h(0) \neq 0$. Again by sufficiently shrinking V_1 we may assume that in this domain $h(z)$ has a holomorphic logarithm, i.e. $h(z) = \exp(g(z))$. Thus we have that $h(z) = \big(\exp(1/k \cdot g(z))\big)^k$ and so $h(z)$ has a holomorphic k-th root. If we write $h(z) = \alpha(z)^k$ then we have that $\psi \circ F \circ \phi^{-1} = (z \cdot \alpha(z))^k$.

Note that $(z \cdot \alpha(z))'(0) = \alpha(0) \neq 0$ and so by the inverse function theorem (Prop. 1.1.10 in [Huy05]) we may further restrict V_1 so that $z \cdot \alpha(z)$ biholomorphic. If we let $\beta : z \mapsto z \cdot \alpha(z)$ be the biholomorphic change of variables then $\beta \circ \phi$ is the chart that we require.

To show uniqueness we may use a topological argument which supposes that points close to $F(x)$ have a constant number of preimages close to x. If we except this then k is clearly unique as $\psi \circ F \circ \phi^{-1}(z) = z^k$ has exactly k preimages locally. We can alternatively postpone proving uniqueness for now and use [1.16] combined with [1.14] to give the result instead.

□

Given a non constant holomorphic map $F : X \to Y$ and a point $x \in X$ we call this local expression of F centred at x the *Local Normal Form* of F at x. The uniqueness of the exponent found in the local normal form allows us categorise the 'behaviour' of F at x in a compact way, we refer to the exponent as the *Multiplicity* of F at a point $x \in X$ and denote it by $\text{mult}_x(F)$.

As a quick example, consider the meromorphic function $f(z) = 1/z$ defined on the complex plane. Using our new language we can view this as a holomorphic map $f : \mathbb{C} \to \mathbb{P}^1$ and observe that the local normal form at 0 is given by the chart $\phi_1 : [z : 1] \mapsto z$ of $\mathbb{P}^1$. In particular in this function has the local form $z \mapsto [1 : 1/z] \sim [z : 1] \mapsto z$ and so the multiplicity of f at 0 is 1. Furthermore, it is immediately obvious that the order of f at 0 is -1. Although multiplicity can be applied to a wider class of functions than order, there is a pleasant relationship between the two notions when in the context of meromorphic functions.

Proposition 1.16. Let X be a Riemann surface and let $f : X \to \mathbb{P}^1$ be a non-constant meromorphic function, then

1. If $x \in X$ is a zero of f, then $\text{mult}_x(f) = \text{ord}_x(f)$
2. If $x \in X$ is a pole of f , then $\text{mult}_x(f) = -\text{ord}_x(f)$
3. If $x \in X$ is a neither a zero or a pole of f , then $\text{mult}_x(f) = \text{ord}_x(f - f(x))$

Proof (Ch. 2, Lem. 4.4 and Lem. 4.7 in [Mir95]). Let (ϕ, U) and (ψ, V) be a charts of $x \in X$ and $f(x) \in \mathbb{P}^1$ respectively. Consider the centred charts $\phi_c(z) = \phi(z) - \phi(x)$ and $\psi_c(z) = \psi(z) - \phi(f(x))$. Let $g = \phi \circ f \circ \psi^{-1}$ and consider the power series expansion $g(z) = g(\phi(x)) + \sum_{n=1}^{\infty} a_n(z - \phi(x))^n$ around $\phi(x)$. Let $g_c = \phi_c \circ f \circ \psi_c^{-1}$, we have that

$$g(z) - g(\phi(x)) = \sum_{n=1}^{\infty} a_n(z - \phi(x))^n$$

and therefore

$$g_c(z) = \sum_{n=1}^{\infty} a_n z^k$$

In particular, the coefficients are the same. As we have seen, $\text{mult}_x(f)$ is equal to the exponent of the lowest degree term in the power series expansion of g_c with non zero coefficient (since these charts are centred). Writing $g(z) = g(\phi(x)) + \sum_{n=k}^{\infty} a_n(z - \phi(x))^n$ with $a_k \neq 0$, we have that $k = \text{mult}_x(f)$. We can therefore find the multiplicity of a meromorphic function at any point by looking at any local power series expansion around that point.

Suppose that $x \in X$ is not a pole of f and let (ϕ, U) be any chart of x. The Laurent series $f \circ \phi^{-1}(z) = f(x) + \sum_{n=k}^{\infty} a_n(z - \phi(x))^n$ is a power series since x is not a pole. Furthermore, as we have seen $\text{mult}_x(f) = k = \text{ord}_x(f - f(x))$ which proves (1) and (3)

Now suppose that x is a pole of f. Consider the Laurent expansion $f \circ \phi^{-1}(z) = \sum_{n=k}^{\infty} a_n(z - \phi(x))^n$, then $f \circ \phi^{-1}(z) = (z - \phi(x))^k \cdot h(z)$ where h is holomorphic and non zero at $\phi(x)$ and therefore $\left(f \circ \phi^{-1}\right)^{-1} = (z - \phi(x))^{-k} \cdot h(z)^{-1}$. Since $f \circ \phi^{-1}$ has order $k < 0$ at $\phi(x)$, by expanding h^{-1} we observe that $(f \circ \phi^{-1})^{-1}$ has order $-k > 0$ at $\phi(x)$. It follows that $\mathrm{ord}_x(f) = -\mathrm{ord}_x(1/f)$. Now we have that $\mathrm{mult}_x(f) = \mathrm{mult}_{\phi(x)}(\phi_1 \circ f \circ \phi^{-1}) = \mathrm{ord}_x(1/f) = -\mathrm{ord}_x(f)$ proving (2).

$\square$

1.3 Riemann–Hurwitz Formula

In the previous section we saw how to best generalise 'order' in complex analysis to Riemann surfaces and holomorphic maps between them. Multiplicity as a concept is very powerful in distilling the information of holomorphic maps. To illustrate this, consider the holomorphic map $f : \mathbb{P}^1 \to \mathbb{P}^1$ which sends $[1 : z] \mapsto [1 : z^2]$ and $[0 : 1] \mapsto [0 : 1]$. By using [1.16], a quick calculation yields that the multiplicity of f at every point is 1 apart from at $[1 : 0]$ and $[0 : 1]$. These points are also the only points which uniquely map to their respective images. This is not a coincidence as we will see. We begin by defining the *degree* of a holomorphic map at a point.

Definition 1.17. Let X, Y be compact Riemann surfaces, and let $F : X \to Y$ be a non-constant holomorphic map. For every point $y \in Y$ we define the degree of F at y as

$$\deg_y(F) = \sum_{x \in F^{-1}(y)} \mathrm{mult}_x(F)$$

Going back to our example, we can see that the degree of f is 2 at every point $y \in \mathbb{P}^1$. This turns out to be true for any non-constant holomorphic map between compact Riemann surfaces.

Proposition 1.18. Let X, Y be compact Riemann surfaces, and let $F : X \to Y$ be a non-constant holomorphic map. Let $y_1, y_2 \in Y$ be any two points in the image of F, then $\deg_{y_1}(F) = \deg_{y_2}(F)$.

Proof (Ch. 2, Prop. 4.7 in [Mir95]). Let $y \in \mathrm{im}(F)$ and let $\{x_1, \dots, x_m\}$ be the preimages of y in F. Fix a chart (ψ, V) of Y centred at y, by [1.15] we have that there exists charts (ϕ_i, U_i) of X centred at x_i such that $\psi \circ F \circ \phi_i^{-1} = z^{k_i}$. We calculate the degree of F at y as $\deg_y(F) = \sum_{i=1}^m k_i$.

We now argue that there is a sufficiently small open neighbourhood $y \in V' \subseteq V$ such that for every $y' \in V'$, the pre-images $F^{-1}(y')$ all lie in $\cup_i U_i$. Suppose otherwise, we may form a sequence $\psi(y_1)$, $\psi(y_2), \dots$ which converges to $\psi(y)$ such that there are preimages $x_i \in X$ of y_i which do not lie in $\cup_i U_i$. X is compact and so there exists a subsequence $\{x_{l_j}\}_j$ of x_1, $x_2 \dots$ which converges to some $x \in X - \cup_i U_i$. The image of this subsequence in $\psi \circ F$ must also converge to $\psi(y)$ since $\psi \circ F$ is continuous, we therefore have that $\psi(F(x)) = \psi(y)$. This immediately yields a contradiction since $x \notin \cup_i U_i$ while simultaneously being a preimage of y.

Without loss of generality (by replacing V with V') we may therefore assume that if $F(x') \in V$ then $x' \in \cup_i U_i$. We may further shrink V and U_i such that the U_i are mutually disjoint and $\phi_i(U_i) = \psi(V) = \{z \in \mathbb{C} : |z| \leq \epsilon\}$ for some $\epsilon > 0$. Returning to our previous characterisation of $\psi \circ F \circ \phi_i^{-1}$ we therefore calculate that for every $y' \in V - y$, $\psi(y')$ has exactly k_i preimages in $\psi \circ F \circ \phi_i^{-1}$ which all have multiplicity 1. Furthermore, all preimages lie in some U_i and therefore

$$\deg_{y'}(F) = \sum_{x' \in F^{-1}(y')} \mathrm{mult}_{x'}(F) = \sum_{i=1}^{m} \sum_{x' \in F|_{U_i}^{-1}(y')} \mathrm{mult}_{x'}(F) = \sum_{i=1}^{m} k_i = \deg_y(F)$$

We have shown that the degree function is locally constant. Since Y is connected it follows that the degree function is globally constant, and we may unambiguously write $\deg(F)$ instead of $\deg_y(F)$.

□

Applying this result to meromorphic functions using [1.16] immediately yields an interesting corollary.

Corollary 1.19. Let X be a compact Riemann surface and let $f : X \to \mathbb{P}^1$ be a non-constant meromorphic function, then

$$\sum_{x \in X} \mathrm{ord}_x(f) = 0$$

Proof (Ch. 2, Prop. 4.12 in [Mir95]). Firstly, we observe that $x \in X$ is neither a zero or a pole of f if and only if $\mathrm{ord}_x(f) = 0$. This is because when we consider f as a function $f : X \to \mathbb{C} \cup \{\infty\}$, the Laurent series expansion around any such point $x \in X$ must have a constant term (as it it not a zero) and no terms with negative exponent (as it is not a pole).

Secondly, we observe that the zeros and poles of f must form discrete and therefore finite subset of X since it is compact. This follows from [1.16], in particular we saw that the multiplicity of f at a point $x \in X$ can be read from any local power series expansion of f at x. If $\psi \circ f \circ \phi^{-1}(z) = \phi(x) + \sum_{n \geq k} a_n (z - \phi(x))^n$ is such an expansion then

$$\mathrm{mult}_x(f) = k = 1 + \mathrm{ord}_{\phi(x)}(d(\psi \circ f \circ \phi^{-1})/dz)$$

The points where the multiplicity of f is greater than 1, or equivalently the zeros and poles of f, are the points locally given by the vanishing of a holomorphic function (i.e. $d(\psi \circ f \circ \phi^{-1})/dz$) which is discrete. We therefore finally calculate

$$\begin{aligned}\sum_{x \in X} \mathrm{ord}_x(f) &= \sum_{x \in f^{-1}(0)} \mathrm{ord}_x(f) + \sum_{x \in f^{-1}(\infty)} \mathrm{ord}_x(f) \\ &= \sum_{x \in f^{-1}(0)} \mathrm{mult}_x(f) - \sum_{x \in f^{-1}(\infty)} \mathrm{mult}_x(f) = \deg(f) - \deg(f) = 0\end{aligned}$$

□

Having developed the theory of holomorphic maps, we will now state an important theorem called the Riemann-Hurwitz formula. This theorem demonstrates the aforementioned connection between the topology of compact Riemann surfaces and the properties holomorphic maps between them. In order to do this we must first briefly talk about genus.

As we previously mentioned, every Riemann surface has the structure of a orientable smooth manifold if we identify $\mathbb{C}$ with $\mathbb{R}^2$ in the natural way. In the case of compact Riemann surfaces we can apply the following well known classification of smooth surfaces in order understand their topological features.

Theorem 1.20. Let X be a connected compact oriented 2-dimensional smooth manifold, then X is diffeomorphic to one of the following

1. The sphere S^2
2. The connected sum of g tori for $g \geq 1$

The number g is called the *genus* of the surface, it can be thought of as the number of 'holes' that a surface has. Every compact Riemann surface is diffeomorphic (and therefore homeomorphic) to precisely one the surfaces in the above classification, it is in this way we may talk about the genus of a Riemann surface. Genus can be alternatively (perhaps more sensibly) defined in terms of a topological notion called *Euler characteristic*.

The Euler characteristic of a surface X, denoted $\chi(X)$, can be defined in many different ways. Most commonly we define it as the alternating sum $\sum_{n\geq 0}(-1)^n c_n$ where the $\{c_n\}$ are either the number of n-simplices in a simplicial structure (triangulation), the number n-cells in a CW-complex structure, or even the Betti numbers coming from the singular homology of X. For a compact orientable surface of genus g, it can be shown that $\chi(X) = 2 - 2g$. We may now finally state the Riemann-Hurwitz formula.

Theorem 1.21 (Riemann-Hurwitz formula)**.** Let X, Y be compact Riemann surfaces and let $F : X \to Y$ be a non-constant holomorphic map, then

$$\chi(X) = \deg(F) \cdot \chi(Y) - \sum_{x \in X} (\text{mult}_x(F) - 1)$$

1.4 Projective Algebraic Curves

In this section we turn our attention subsets of projective space which are cut out by homogeneous polynomial equations. Before studying these sets using the algebraic theory in the next chapter, we will first briefly talk about them in the context of Riemann Surfaces.

Suppose that $P \in \mathbb{C}[z_0, \dots, z_n]$ is a homogeneous polynomial of degree k. Since $F(\lambda z_0, \dots, \lambda z_n) = \lambda^k P(z_0, \dots, z_n)$ for every $\lambda \in \mathbb{C} - \{0\}$, the zero locus of P

$$\{P = 0\} = \{[z_0, \dots, z_n] \in \mathbb{P}^n : P(z_0, \dots, z_n) = 0\}$$

is a well defined subset of $\mathbb{P}^n$. In the charts (ϕ_i, A_i) of $\mathbb{P}^n$ we can interpret this zero set $\{P = 0\} \cap A_i$ as the zero locus of the polynomial

$$P_i := P(z_0, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n)$$

In order to answer the question of when $\{P = 0\}$ is a complex manifold, we need the aforementioned 'implicit function theorem' for functions in many complex variables.

Theorem 1.22 (Coro. 1.1.12 in [Huy05])**.** Let $U \subseteq \mathbb{C}^n$ be an open subset and let $f : U \to \mathbb{C}^m$ be a holomorphic map. Assume that $z \in U$ is such that the rank of the Jacobian matrix of f is maximal at z. If $n \leq m$ then there exists a biholomorphic map $g : V \to V'$ where V is an open subset of $\mathbb{C}^m$ containing $f(z)$ such that

$$g \circ f(z_1, \dots, z_n) = (z_1, \dots, z_n, 0, \dots, 0)$$

Using this theorem we can see that if P_i has Jacobian matrix of rank 1 at every point in the preimage of 0, then we obtain complex charts which give $\{P_i = 0\} = \{P = 0\} \cap A_i$ the structure of a $n - 1$ dimensional submanifold of A_i. This non-singularity condition on the polynomials P_i can be translated into a condition on P. In particular, it can be shown that all the polynomials P_i have Jacobian matrix of rank 1 at every point in the preimage of 0 if and only if there are no points $x \in P^{-1}(0) \subseteq \mathbb{C}^{n+1}$ such that $\partial/\partial z_i(P)(x) = 0$ for all i. The key idea here is to use something called *Eulers idenity*, the details of this argument can be found in Ch. 3, Prop. 3 of [Don11]. If P satisfies this property then we have complex submanifolds $\{P_i = 0\} \subseteq A_i$ which all agree on intersection, thus we have that the zero locus $\{P = 0\}$ is a complex submanifold of $\mathbb{P}^n$.

Definition 1.23. If $P \in \mathbb{C}[z_0, \dots, z_n]$ is a homogeneous polynomial such there are no solutions to the equation

$$\partial/\partial z_0(P)(x) = \cdots = \partial/\partial z_n(P)(x) = 0,\ P(x) = 0$$

Then the complex manifold $\{P = 0\} \subseteq \mathbb{P}^n$ is called a *projective hypersurface*.

Similarly, if we are given a collection of homogeneous polynomials $\{P_\alpha\}_\alpha$ we can consider the intersection of the zero sets $C = \cap_\alpha\{P_\alpha = 0\}$ and ask when this is a complex submanifold of $\mathbb{P}^n$. Specifically we are interested in when C is 1-dimensional and therefore a Riemann surface. Again, by using the implicit function theorem and following more or less the exact same logic as before, we are lead to the following characterisation for when this happens.

Proposition 1.24. Let $\{P_\alpha\}_\alpha \subseteq \mathbb{C}[z_0, \dots, z_n]$ be a collection of homogeneous polynomials and let $C = \cap_\alpha\{P_\alpha = 0\} \subseteq \mathbb{P}^n$. If for every point $x \in C$ there is an open neighbourhood $U \subseteq \mathbb{P}^n$ of x and polynomials $P_1, \dots, P_{n-1} \in \{P_\alpha\}_\alpha$ such that

1. $C \cap U = \{P_1 = \cdots = P_{n-1} = 0\}$
2. The Jacobian matrix $(\partial P_i/\partial z_j)_{ij}$ of the smooth function $(P_1, \dots, P_{n-1}) : \mathbb{C}^{n+1} \to \mathbb{C}^{n-1}$ has maximal rank at x.

then $C \subseteq \mathbb{P}^n$ is a submanifold of dimension 1.

Riemann surfaces of this form are called *local complete intersections* or alternatively *projective algebraic curves*.

We finish this chapter by commenting on the types of meromorphic functions on projective algebraic curves. To begin, let $P, Q \in \mathbb{C}[z_0, \dots, z_n]$ be homogeneous polynomials of the same degree. The so called rational expression P/Q gives a well defined function $P/Q : \mathbb{P}^n \to \mathbb{P}^1$. For any chart (ϕ_i, A_i) of $\mathbb{P}^n$, $P/Q \circ \phi_i^{-1}$ is the quotient of two polynomials and therefore P/Q is in some sense 'meromorphic'. For any projective algebraic curve $C \subseteq \mathbb{P}^n$, if C is not contained in $\{Q = 0\}$ then the function $(P/Q)|_C$ is truly a meromorphic function on C as it is a composition of holomorphic maps and not identically equally to $[0 : 1]$. The question remains: are these rational expressions are the only meromorphic functions on C? If so, which Riemann surfaces are biholomorphic to projective algebraic curves and therefore have $\{P/Q : Q \text{ not identically vanishing on } C\}$ as their field of meromorphic functions?

2 Algebraic Curves

2.1 Affine Algebraic Sets

As we have seen in the previous chapter, an interesting class of Riemann surfaces are those which can be described as the vanishing set of a collection of polynomials. In general such vanishing sets are not, of course, Riemann surfaces. In spite of this, they still have a rich algebraic structure which is surprisingly interconnected with complex-analytic theory that we have seen and will explore further in the next chapters. We will now proceed with outlining classical theory of algebraic geometry with eyes towards understanding complex projective algebraic curves.

We begin with the definition of affine space and an affine algebraic set. Throughout this chapter k will be an algebraically closed field, typically the field of complex numbers.

We define n-dimensional *affine space* as the set of all n-tuples of elements in k, i.e.

$$\mathbb{A}^n_k = \{(a_1, a_2, \ldots, a_n) : a_i \in k \ \forall i\}$$

If the base field k is clear we will simply write $\mathbb{A}^n$. We use the notation $\mathbb{A}^n$ to differentiate affine space from the vector space k^n. This is mainly because we don't typically think of $\mathbb{A}^n$ as having a vector space structure.

Definition 2.1. We say that a subset $S \subseteq \mathbb{A}^n$ is a *affine algebraic set* if there exists polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ such that

$$S = \{p \in \mathbb{A}^n : f_1(p) = f_2(p) = \cdots = f_m(p) = 0\}$$

An immediate observation we can make is that there is a connection between affine algebraic sets and the polynomial equations that vanish on them. Given an subset $I \subseteq k[x_1, \ldots, x_n]$, we can form the set $V(I)$ as

$$V(I) = \{p \in \mathbb{A}^n : f(p) = 0 \ \forall f \in I\} \subseteq \mathbb{A}^n$$

Due to Hilbert's Basis Theorem, the sets $V(I)$ are exactly the algebraic affine sets as $V(I) = V(\langle I \rangle) = V(\langle f_i, \ldots, f_m \rangle) = V(\{f_1, \ldots, f_m\})$. This allows us to form affine algebraic sets from ideals, or more general subsets of $k[x_1, \ldots x_m]$. Conversely, given an general subset $S \subseteq \mathbb{A}^n$, we define the set of polynomials which vanish on S as $I(S)$. More specifically

$$I(S) = \{f \in k[x_1, \ldots, x_n] : f(p) = 0 \ \forall p \in S\} \subseteq k[x_1, \ldots, x_n]$$

It is easy to check that $I(S)$ is a ideal of $k[x_1, \ldots, x_n]$ as if f, g vanish on S then so do hf, $f + g$ ect. We therefore have functions

$$V : \{\text{Subsets of } k[x_1, \ldots x_n]\} \to \{\text{Affine algebraic subsets of } \mathbb{A}^n\}$$

$$I : \{\text{Subsets of } \mathbb{A}^n\} \to \{\text{Ideals of } k[x_1, \ldots, x_n]\}$$

These two functions have a strong relationship, this is highlighted in the following theorem by Hilbert.

Theorem 2.2 (Hilbert's Nullstellensatz)**.** Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal, then $I(V(I)) = \mathrm{rad}(I)$.

As a corollary of this, when we restrict V and I to only the affine algebraic subsets of $\mathbb{A}^n$ and radical ideals of $k[x_1, \ldots, x_n]$, the two functions are inverses and furthermore order reversing. The proof of this theorem reasonably involved and mainly comprises of commutative algebra. It can be found in most texts on commutative algebra such as [AM69]. There are many geometric consequences of this theorem, the following are just a few which will be useful later.

Proposition 2.3. Let $R = k[x_1, \ldots, x_n]$ and let $I \subseteq R$ be an ideal.

1. If I is a maximal ideal then $V(I) = \{p\}$ for some $p \in \mathbb{A}^n$, moreover this correspondence is 1-1 and $I = \langle (x_1 - p_1), \ldots, (x_n - p_n) \rangle$. Additionally, $V(0) = \mathbb{A}^n$ and $V(R) = \emptyset$.

2. If I is a prime ideal then $V(I)$ is irreducible in the sense that if $V(I) = V(J) \cup V(K)$ then $V(J) \subseteq V(K)$ or $V(K) \subseteq V(J)$.

3. Every affine algebraic set S can be uniquely written as $V_1 \cap V_2 \cap \cdots \cap V_k$ where the V_i are all irreducible affine algebraic sets, no one containing another.

One issue that arises when studying affine algebraic sets is that the algebraically closed field k that we are working over does not come equipped with a topology by default. Topological tools are useful and we would like to have these tools available to us. In Chapter 1 the complex topology was an important part of the theory, allowing us to define things like genus. Since we are primarily interested in the case where $k = \mathbb{C}$ we could resort to using the complex topology again for affine algebraic sets. This approach, however, doesn't generalise and fails to capture the algebraic structure on $\mathbb{A}^n$. The correct topology to use is called the Zariski topology and is defined as follows.

Definition 2.4. The Zariski topology on $\mathbb{A}^n$ is defined as the topology which has open sets

$$\{\mathbb{A}^n - V(I) : I \subseteq k[x_1, \ldots, x_n] \text{ an ideal}\}$$

Alternatively, the Zariski topology is defined as the topology which has as its closed sets the affine algebraic subsets of $\mathbb{A}^n$. If $S \subseteq \mathbb{A}^n$ is an affine algebraic set then the Zariski topology on S is the subset topology induced by the Zariski topology on $\mathbb{A}^n$.

As $\langle 0 \rangle$ and $\langle 1 \rangle$ are ideals of $k[x_1, \ldots, x_n]$, both $\mathbb{A}^n$ and $\emptyset$ are open sets. Proving the finite union and arbitrary intersection axioms that the closed sets of a topology must satisfy comes down to the observation that $V(I) \cap V(J) = V(I + J)$ and $V(I) \cup V(J) = V(IJ)$.

The Zariski topology is unlike the complex topology in many ways. First of all, it is not Hausdorff. In fact if S is irreducible in the sense mentioned in [2.3] then all non-empty Zariski open subsets of S intersect. At first the Zariski topology may seem hard to work with. If we shift perspective, however, most of the topological notions turn out to be easy enough to visualise. Take for example the topological closure $\overline{U}$ of a subset $U \subseteq \mathbb{A}^n$. This is exactly the smallest affine algebraic set for which U is a subset.

In the case that $k = \mathbb{C}$, the Zariski and complex topology have a strong relationship. One example of this is that the notion of connectedness in both topologies agree. One direction is easy to show. First observe that the Zariski topology on $\mathbb{C}^n$ is contained in the complex topology. This is because complex valued polynomials are continuous in the complex topology and therefore all principle closed sets $V(p) = p^{-1}(0)$ are closed in both topologies ($\{0\}$ is a closed set in the complex topology). Therefore if $U \subseteq \mathbb{C}^n$ is both closed and open in the Zariski topology, then it is closed and open in the complex topology and so we have one side of the equivalence. As we can see, translating algebraic statements analytical ones isn't typically that difficult however as of yet we don't have any good tools for the other direction.

We can also translate the the notion of connectedness for an affine algebraic set to algebra. Assuming the full statement in the paragraph above, the following proposition provides us with three equivalent but philosophically different interpretations of connectedness.

Proposition 2.5. Let $S = V(I) \subseteq \mathbb{A}^n$ be an affine algebraic set, then S a disconnected (in the Zariski topology) if and only if there exist ideals $J, K \subseteq k[x_1, \ldots, x_n]$ such that $J \cap K = I$ and $J + K = R$.

Proof. Let S be disconnected. We have closed sets $V(J), V(K) \subseteq \mathbb{A}^n$ such that $V(J) \cup V(K) = S$ and $V(J) \cap V(K) = \emptyset$. Now $V(J + K) = V(J) \cap V(K) = \emptyset$ so by Hiblert's Nullstellensatz we have that $J + K = R$. We also have that $V(I) = V(J) \cup V(K) = V(J \cap K)$ and so by again applying Hilbert's Nullstellensatz it follows that $\mathrm{rad}(I) = \mathrm{rad}(J \cap K) = \mathrm{rad}(J) \cap \mathrm{rad}(K)$. Assuming that I, J, K are radical yields $I = J \cap K$.

Conversely suppose that there are radical ideals $J, K \subseteq k[x_1, \ldots, x_n]$ which satisfy the properties stated in the proposition. Applying Hilbert's Nullstellensatz immediately yields the that $V(J) \cap V(K) = V(J + K) = V(R) = \emptyset$ and $V(J) \cup V(K) = V(J \cap K) = V(I)$. Therefore $V(I)$ is disconnected.

□

For an affine algebraic set $S \subseteq \mathbb{A}^n$, the algebraic analogue of holomorphic functions are called *regular functions*. These functions carry information about the algebraic structure of S, they are defined as follows.

Definition 2.6. Let $S \subseteq \mathbb{A}^n$ be an affine algebraic set. We say that a function $f : S \to k$ is a *regular* if there exists $p \in k[x_1, \ldots, x_n]$ such that $f = p|_S$. We denote the set of regular functions $\mathcal{O}(S)$.

We can immediately make several observations about regular functions. First of all, they all come from polynomials defined on the affine space in which S is embedded. Secondly, the set $\mathcal{O}(S)$ comes equipped with a ring structure induced from multiplication and addition in $k[x_1, \dots, x_n]$. We therefore we have that there is an homomorphism of rings

$$\mathrm{res}_S : k[x_1, \dots, x_n] \to \mathcal{O}(S)$$

Thirdly, given two polynomials $p, q \in k[x_1, \dots, x_n]$, they will induce the same regular function if and only if $p|_S = q|_S$. If this is the case then $p - q$ must vanish on S, that is $p - q \in I(S)$. This leads us immediately to the observation res_S descends to an isomorphism

$$k[x_1, \dots, x_n]/I(S) \xrightarrow{\sim} \mathcal{O}(S)$$

Definition 2.7. Let $S \subseteq \mathbb{A}^n$ be an affine algebraic set. We define the coordinate ring of S as

$$k[S] = k[x_1, \dots, x_n]/I(S)$$

Even though they are isomorphic, we think of $\mathcal{O}(S)$ and $k[S]$ as being different objects. The reason for this will become clear when we generalise the theory in the next chapter.

Recall from Chapter 1 that we can extend the notation of a holomorphic function $f : X \to \mathbb{C}$ to a holomorphic map between Riemann surfaces. We can mimic this construction in the algebraic setting to define regular maps between affine algebraic sets.

Definition 2.8. Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be affine algebraic sets. We say that a map $F : X \to Y$ is a *regular map* if there exist polynomials $f_i \in k[x_1, \dots, x_n]$ such that

$$F(x_1, \dots, x_n) = (f_1(x), \dots, f_m(x))$$

An immediate observation about regular maps is that they are are Zariski continuous. If $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ are affine algebraic sets, $F : X \to Y$ is a regular map, and $V(f) \subseteq \mathbb{A}^m$ is any Zariski closed set, then $f \circ F \in \mathcal{O}(X)$. In particular $f \circ F$ is the restriction of a polynomial $p \in k[x_1, \dots, x_n]$, and $F^{-1}(V(f) \cap Y) = V(p) \cap X$ is Zariski closed as required. Note also that if $g \in \mathcal{O}(Y)$ and $\mathrm{res}_Y(q) = g$ then

$$g \circ F = q(f_1(x_1, \dots x_n), \dots, f_m(x_1, \dots, x_n)) \in k[X]$$

and so $g \circ F \in \mathcal{O}(X)$.

Regular maps turn the set of affine algebraic sets into a category. Indeed it is easy to show that the constant map $\mathrm{Id}_X : X \to X$ is regular, and that the composition of two regular maps is a regular map. This allows us to define the notation of *isomorphism* for affine algebraic sets. X and Y are isomorphic if there exist regular maps $F : X \to Y$, $G : Y \to X$ such that $G \circ F = \mathrm{Id}_X$ and $F \circ G = \mathrm{Id}_Y$.

We end this section with a useful proposition about regular maps. This should remind you of the identity theorem in complex analysis. Roughly the identity theorem states that a holomorphic function is determined by its behaviour on a suitable open subset.

Proposition 2.9. Let $X \subseteq \mathbb{A}^n$ $Y \subseteq \mathbb{A}^m$ be affine algebraic sets, and let $F, G : X \to Y$ be regular maps. Suppose that there exists an dense open subset $U \subseteq X$ such that $F|_U = G|_U$, then $F = G$.

Proof. Let F', G' be the functions $(f_1, \ldots, f_m)$, $(g_1, \ldots, g_m) : \mathbb{A}^n \to \mathbb{A}^m$ such that $F'|_X = F$ and $G'|_X = G$. We have that $\{F = G\} = \{x \in X : F(x) = G(x)\} = V(f_1 - g_1, \ldots, f_m - g_m) \cap X$ which is closed. Since $U \subseteq \{F = G\}$ is a dense in X it immediately follows that $\{F = G\} = X$ as required.

□

2.2 Quasi-Projective Varieties

In order study projective algebraic curves using algebraic tools, we need to generalise the theory in the previous section to vanishing sets of homogeneous polynomials in $\mathbb{P}^n$. We begin by defining the Zariski topology on $\mathbb{P}^n$ where

$$\mathbb{P}^n = k^{n+1} - \{0\} \,/\, x \sim \lambda x, \forall \lambda \in k$$

Definition 2.10. We say that a set $S \subseteq \mathbb{P}^n$ is a *projective algebraic set* (or just projective for short) if there exists homogeneous polynomials $f_1, \ldots, f_m \in k[x_0, \ldots, x_n]$ such that

$$S = \{x \in \mathbb{P}^n : f_1(x) = \cdots = f_m(x) = 0\}$$

The Zariski topology on $\mathbb{P}^n$ is defined as the topology which has the projective algebraic subsets of $\mathbb{P}^n$ as its closed sets.

Like in the affine case, the proof that this is in fact a topology isn't too difficult. $\mathbb{P}^n$ and $\emptyset$ are projective algebraic sets, as are finite unions and intersections of projective sets. The only non-trivial axiom to check is that arbitrary intersections of projective algebraic sets are projective. We will approach this using the idea of a homogeneous ideal.

Definition 2.11. An ideal $I \subseteq k[x_0, \ldots, x_n]$ is *homogeneous* if it can be generated by a finite number of homogeneous polynomials. If $I \subseteq k[x_0, \ldots, x_n]$ is a homogeneous ideal and $\{f_1, \ldots, f_m\}$ is a homogeneous generating set, we define the vanishing locus of I as

$$V(I) = \{x \in \mathbb{P}^n : f_1(x) = \cdots = f_m(x) = 0\}$$

Similarly, if $S \subseteq \mathbb{P}^n$ is a projective algebraic set then we define $I(S)$ as the ideal generated by the homogeneous polynomials which vanish on S, that is

$$I(S) = \langle \{f \in k[x_0, \ldots, x_n] : f \text{ homogeneous and } f(x) = 0 \ \forall x \in S\} \rangle$$

If I and J are homogeneous ideals then so is $I + J$ and IJ. As in the affine case, once we observe that $V(I) \cap V(J) = V(I + J)$ and $V(I) \cup V(J) = V(IJ)$ it is clear that the Zariski topology on $\mathbb{P}^n$ is in fact a topology.

Definition 2.12. Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set with radical homogeneous ideal $I(X)$. We define the *homogeneous coordinate ring* of X as

$$S[X] = k[x_0, \ldots, x_n]/I(X)$$

Since every projective subset of $\mathbb{P}^n$ is defined by a finite collection of polynomials in $k[x_0, \ldots, x_n]$, we can associate it with an affine algebraic subset of $\mathbb{A}^{n+1}$. This allows us to formulate Hilbert's Nullstellensatz in the projective setting.

Theorem 2.13 (Projective Nullstellensatz)**.** Let $I \subseteq k[x_0, \ldots, x_n]$ be a homogeneous ideal not containing $\langle x_0, \ldots, x_n \rangle$, then $\text{rad}(I)$ is also homogeneous and $I(V(I)) = \text{rad}(I)$.

As we have seen in Chapter 1, $\mathbb{P}^n$ has *affine charts* (ϕ_i, A_i) defined by fixing the i'th coordinate

$$A_i = \{[x_0, \ldots, x_n] \in \mathbb{P}^n : x_i = 1\}$$

These affine charts are open sets of $\mathbb{P}^n$ in Zariski topology. We can write them as

$$A_i = \mathbb{P}^n - V(x_i)$$

If we conciser the natural association $A_i \xrightarrow{\sim} \mathbb{A}^n$ we can view the affine algebraic subsets of $\mathbb{A}^n$ as subsets of $\mathbb{P}^n$ and vice versa. Let $I \subseteq k[x_1, \ldots, x_n]$ be the ideal generated by $f_1, \ldots f_m$ and let $S = V(I) \subseteq \mathbb{A}^n$. Once we homogenise the f_i by setting

$$\tilde{f}_i = x_0^d \, f_i\Big(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\Big)$$

we can form the *projective closure* of S, $\widetilde{S} \subseteq V(\tilde{f}_1, \ldots, \tilde{f}_m) \subseteq \mathbb{P}^n$, defined as the Zariski closure of $S \subset A_0$ in $\mathbb{P}^n$. $\widetilde{S}$ is a closed subset of $\mathbb{P}^n$ which satisfies $\phi_0(\widetilde{S} \cap A_0) = S \subseteq \mathbb{A}^n$. Therefore we may view S as the intersection of an open and closed subset in $\mathbb{P}^n$. This leads us to the following definition.

Definition 2.14. Let $X \subseteq \mathbb{P}^n$. We say that X is a *quasi-projective set* if we can express X as $U \cap C$ where $U, C \subseteq \mathbb{P}^n$ are open and closed respectively. If $X \subseteq \mathbb{P}^n$ is a quasi-projective variety then say X is irreducible if it is irreducible with respect to the subspace topology induced by $\mathbb{P}^n$. If $X \subseteq \mathbb{P}^n$ is an irreducible quasi-projective (or projective) set, we will call it a quasi-projective (respectfully, projective) variety.

In the previous section we defined regular maps between affine algebraic sets. We would like to generalise this notation to projective and quasi-projective varieties. We have already made progress for the former. Given a projective variety $X = V(f_1, \ldots f_m) \subseteq \mathbb{P}^n$ we have an covering of irreducible affine algebraic sets $A_i \cap X$ given by

$$\mathbb{A}^n \supseteq V(f_1(x_0, \ldots x_{i-1}, 1, x_{i+1}, \ldots, x_n), \ldots) = A_i \cap V(f_1, \ldots f_m) \subseteq A_i$$

Given a function $f : X \to k$ and a point $x \in X$, one way we could define f being regular is by requiring that $f|_{A_i \cap X}$ be regular for every i. This turns out to be a special case of the general definition for a quasi-projective variety.

Definition 2.15. Let $X \subseteq \mathbb{P}^n$ be quasi-projective. We say that a function $f : X \to k$ is *regular* at $p \in X$ if there exists an open neighbourhood U of p and homogeneous polynomials $h, g \in k[x_0, \ldots, x_n]$ of the same degree such that $V(g) \cap U = \emptyset$ and $f|_U(x) = h(x)/g(x)$. We say that f is regular if it is regular at every point in X. We denote the set of all regular functions on X by $\mathcal{O}(X)$.

Proposition 2.16. Let $S \subseteq A_0 \subset \mathbb{P}^n$ is an affine algebraic subset. If a function $f : S \to k$ is regular in the above sense, then it satisfies the regularity condition outlined in [2.6].

Proof. Suppose that $X \subseteq A_0$ is an affine algebraic set and that $f : X \to \mathbb{P}^n$ is regular in the above sense. For every point $p \in X$ let $U_p \subseteq X$ be an open neighbourhood of p such that $f|_{U_p}(x) = h_p(x)/g_p(x)$ on U_p. As $h_p(x)/g_p(x)$ is well defined on $X - V(g_p)$, applying [2.9] to the function $f_p(x)g_p(x)$ defined on U_p and $X - V(g_p)$ separately, allows us to set $U_p = X - V(g_p)$ without problem. Note that as a consequence of Hilbert's Baisis Theorem, there is a finite subset $\{V(g_1), \ldots, V(g_m)\}$ of $\{V(g_p)\}_{p \in X}$ with empty intersection. By Hilbert's Nullstellensatz, $\cap_{i=1}^m V(g_i) = \emptyset$ implies that $\langle g_1, \ldots, g_m \rangle = k[x_1, \ldots, x_n]$ and so there are $l_i \in k[x_1, \ldots, x_n]$ such that $l_1 g_1 + \cdots + l_m g_m = 1$. Finally we observe that

$$F := l_1 g_1 \left(\frac{h_1}{g_1}\right) + \cdots + l_m g_m \left(\frac{h_m}{g_m}\right) = l_1 h_1 + \cdots + l_m h_m \in k[x_1, \ldots, x_n]$$

Conversely, $F(x) = (1)f(x)$ and so f satisfies [2.6].

□

Just as we did in the affine case, we extend the idea regular functions to regular maps between quasi-projective varieties.

Definition 2.17. Let $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ be quasi-projective varieties. A map $F : X \to Y$ is *regular* if it is continuous, and if for every open subset $U \subseteq Y$ and regular function $f \in \mathcal{O}(U)$ the composition $f \circ F$ is a regular function on $F^{-1}(U)$. This notion makes the set of quasi-projective varieties into a category and thus defines what it means for $F : X \to Y$ to be an isomorphism.

It can be checked that in the affine case, this definition of regular map reduces to the one that we are familiar with from the previous section. Let X, Y be affine algebraic sets and let $F : X \to Y$ be regular in the above sense. The coordinate functions y_i of Y are clearly regular and so $y_i \circ F$ are also regular functions on X. By Prop [2.16] the components of F are therefore polynomials and so F fulfils the requirements of [2.8]. This shows that the two definitions of regular map are equivalent in the affine case.

One downside of working with projective algebraic varieties instead of affine ones is that the homogeneous coordinate ring S[X] does not consist of regular functions on X. To ratify this, we will now introduce possibly the most important idea in projective algebraic geometry: rational functions, the rational function field, and rational maps between algebraic varieties.

Definition 2.18. Let $X \subseteq \mathbb{P}^n$ be a quasi-projective variety. A *rational function* $f : X \to k$ is a pair (U, f) where $U \subseteq X$ is an non-empty open subset and $f : U \to k$ is a regular function. We say that two rational functions $(U_1, f_1), (U_2, f_2)$ are equivalent if they agree on their intersection. Note that since X is a quasi-projective variety, it is irreducible and so all such non-empty open subsets U_1, U_2 must intersect.

Requiring X to be a quasi-projective variety turns the above into an equivalence relation, as such we think of a rational function as an equivalence class of this equivalence relation. The main difference between regular and rational functions is that regular functions must be defined in an open neighbourhood of every point. On the other hand, a rational function $f : X \to k$ may have points for which there is no representative $(U_x, f_x) \sim f$ defining the value of f at $x \in X$. The regular locus (the points where $f : X \to k$ can be prescribed a value) can be shown to be an open subset of X and so in some sense f must be defined almost everywhere. Furthermore, a regular function on X is just a rational function with regular locus equal to X.

We denote the set of all rational functions on $X \subseteq \mathbb{P}^n$ by $k(X)$. Given two rational functions f, g with representatives (U, f), (V, g) we can add and multiply them. Furthermore, in the case that g is not identically zero we can also divide f by g. Writing f, g as $f|_U(x) = a(x)/b(x)$, $g|_V(x) = c(x)/d(x)$ we define these operations in the natural way on the open subsets $U \cap V$ and $U \cap V \cap (\mathbb{P}^n - V(c))$. This gives $k(X)$ the structure of a field.

Rational functions can be extended to more general maps between quasi-projective varieties.

Definition 2.19. Let $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ be quasi-projective varieties. A *rational map* $F : X \to Y$ is a pair (U, F) where $U \subseteq X$ is a non-empty open subset, and $F : U \to Y$ is a regular map. Just as before, we say that two rational maps (U_1, F_1), (U_2, F_2) are equivalent if they agree on their intersection.

Just as in [2.9], we can show that if two regular maps agree on a dense open subset (since X is irreducible, any non-empty open subset will suffice) then they must be equal. We therefore also have a well defined notion of equivalence for rational maps.

We say that quasi-projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ are *birationally equivalent* if there is a rational map $F :\to Y$ which has a rational inverse. More precisely: X and Y are birationally equivalent if there are non-empty open subsets $U \subseteq X$, $V \subseteq Y$ such that $F : U \xrightarrow{\sim} V$ is an isomorphism.

We finish this section with an important proposition which illustrates a direct connection between the rational function field $k(X)$ and the homogeneous coordinate ring $S[X]$ in the case that $X \subset \mathbb{P}^n$ is a projective variety.

Proposition 2.20. Let $X \subseteq \mathbb{P}^n$ be a projective variety. Let $T \subset S[X]$ be the multiplicative subset defined as

$$T = \{[f] \in S[X] - \langle 0 \rangle : f \in k[x_0, \dots, x_n] \text{ homogeneous}\}$$

We have an isomorphism of fields

$$k(X) \cong S(X)$$

where $S(X)$ is the subring of $T^{-1}S[X]$ defined as

$$S(X) = \{f/g \in T^{-1}S[X] : f, g \text{ homogeneous s.t. } \deg(f) = \deg(g)\}$$

There is a lot to verify in this statement. The full proof can be found on p. 18 of [Har77]. The basic idea is to form an isomorphism of $k[X \cap A_0]$ with the subring of $S[X]_{x_0}$ consisting of elements of the form f/x_0^k where $\deg f = k$. This isomorphism is defined by sending $f(x_1, \dots, x_n)$ to the polynomial $f(x_1/x_0, \dots, x_n/x_0)$. We then prove that for any irreducible affine variety S, the rational function field is precisely the field of fractions $k[S]_{\langle 0 \rangle}$. Finally, after proving that $k(X) = k(X \cap A_0)$ the result quickly follows. This proposition provides us with a very algebraic way to think about the rational function field of a projective variety.

If we let $k = \mathbb{C}$ there is clearly a lot of similarities between rational and meromorphic functions on projective varieties. For example, in Chapter 1 saw that for a projective algebraic curve C, expressions of the form P/Q (P, Q homogeneous of the same degree) are rational functions if $C \not\subseteq \{Q = 0\}$. Using [2.20] we can see that these are exactly the rational functions on C as a projective variety instead. This should be somewhat surprising as a priori meromorphic functions have no reason to be algebraic in nature.

2.3 More on Regular and Rational maps

One of the powerful ideas in algebraic geometry is the relationship between varieties and their rings/fields of regular/rational functions. We begin by discussing this in the context of affine algebraic sets. We want to turn the assignment $X \mapsto \mathcal{O}(X)$ into a functor from the category of affine algebraic sets to the category of rings. We need a way of translating regular map $F : X \to Y$ to a homomorphism of their coordinate rings.

Definition 2.21. Let $X \subseteq \mathbb{A}^n$, $Y, \subseteq \mathbb{A}^m$ be affine algebraic sets, and let $F : X \to Y$ be a regular map. Let $g \in \mathcal{O}(Y)$ then we define the *pullback* of g by F, which we will denote by $F^*(g)$, as

$$F^*(g) = g \circ F$$

Note that $F^*(g) \in \mathcal{O}(X)$ and so F^* is a map $F^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Furthermore, it is easy to see that this map is a homomorphism of rings. For instance $F^*(f+g)(x) = (f+g) \circ (F)(x) = f(F(x)) + g(F(x)) = F^*(f)(x) + F^*(g)(x)$.

Proposition 2.22. The assignment $X \mapsto \mathcal{O}(X)$, $F \mapsto F^*$ is a contravariant functor between the category of affine algebraic sets (over k) and the category of finitely generated k-algebras with no nilpotents.

Proof. The functoriality of this assignment is easy to check. The interesting part is that $\mathcal{O}(X)$ is always a finitely generated k-algebra with no nilpotent elements. For an affine algebraic set $X \subseteq \mathbb{A}^n$, we have that $\mathcal{O}(X) = k[X] = k[x_1, \dots, x_n]/I(X)$ and therefore $\mathcal{O}(X)$ is a k-algebra. Additionally, the constant function $\lambda \in k \subseteq \mathcal{O}(Y)$ is pulled back to the the same constant function on X and so F^* is a k-algebra homomorphism.

Since $\{x_1, \dots x_n\}$ generate $k[x_1, \dots, x_n]$ over k, this descends to a finite generating set of $k[X]$ over k and so $\mathcal{O}(X)$ is finitely generated. If $[f] \in \mathcal{O}(X)$ is such that $[f]^m = 0$, then $f^m \in I(X)$ however since $I(X)$ is radical we have that $f \in I(X)$ and so $[f] = 0$. Therefore $\mathcal{O}(X)$ contains no nilpotent elements.

□

Suppose now that we have a finitely generated k-algebra R, with no nilpotent elements. This can be interpreted as saying that there is a surjective homomorphism of k-algebras $\phi : k[x_1, \dots, x_n] \to R$ for some $n \in \mathbb{Z}_{\geq 1}$. Therefore we have by the first isomorphism theorem that $R \cong k[x_1, \dots, x_n]/\ker(\phi)$. Also, by the same argument as above it follows that $\ker(\phi)$ is a radical ideal. By the Nullstellensatz $\ker(\phi) = I(X)$ for some affine algebraic set $X \subseteq k[x_1, \dots, x_n]$ and therefore $R \cong k[X]$. We have shown that every finitely generated k-algebra corresponds to an affine algebraic however this is far from canonical. Indeed the correspondence depends on ϕ, i.e. on the finite generating set of R that we choose. We address this issue and complete the equivalence between the two categories with the following proposition.

Proposition 2.23. Let R, S be two finitely generated k-algebras with no nilpotent elements and let $\psi : R \to S$ be a homomorphism of k-algebras. Suppose that $\phi_R : R \xrightarrow{\sim} k[Y]$ and $\phi_S : S \xrightarrow{\sim} k[X]$ where $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ are affine algebraic sets. Then there exists a unique regular map $F : X \to Y$ such that $F^* = \phi_S \circ \psi \circ \phi_R^{-1}$.

Proof. I will only present a sketch of the proof. The full details can be found in Ch. 1, Prop. 3.5 of [Har77]. Let ψ be as above and assume w.l.o.g. that $R = k[Y]$ and $S = k[X]$. Let $\{\tilde{y}_i\}$ be the coordinate functions in $k[Y]$. $\psi(\tilde{y}_i)$ are all regular functions on X and therefore $F := (\psi(\tilde{y}_1), \dots, \psi(\tilde{y}_m))$ is a regular map $F : X \to \mathbb{A}^m$. Note that for $y_i \in k[\mathbb{A}^m]$ we have $F^*(y_i)(x) = y_i \circ F(x) = y_i(\psi(\tilde{y}_1)(x), \dots, \psi(\tilde{y}_m)(x)) = \psi(\tilde{y}_i)(x)$. Therefore if $f \in I(Y) \subseteq k[x_1, \dots, x_m]$ then as ψ and F^* are k-algebra homomorphisms we have that

$$f \circ F(x) = F^*(f(y_1, \dots y_m))(x) = \psi(f(\tilde{y}_1, \dots, \tilde{y}_m))(x) = 0(x) = 0$$

and thus $\mathrm{Im}(F) \subseteq Y$. We have shown that $F : X \to Y$ is a regular map and $F^* = \psi$.

□

Corollary 2.24. Two affine algebraic sets X, Y are isomorphic if and only if their rings of regular functions are k-isomorphic.

This proposition completes the equivalence between affine algebraic sets and finitely generated k-algebras without nilpotent elements. We may therefore think of an isomorphism class of affine algebraic sets $[X]$ simply as a isomorphism class of k-algebras $[R]$, and in embedding $X \subseteq \mathbb{A}^n$ as a choice of surjective k-homomorphism $k[x_1, \dots, x_n] \to R$. In this language, prime ideals $\mathfrak{p} \subseteq R$ correspond to affine varieties $Y \subseteq X \subseteq \mathbb{A}^n$ by considering the k-algebra homomorphism $R \to R/\mathfrak{p}$.

We now move onto the case of quasi-projective varieties. It may be tempting to take the same approach as we did for affine algebraic sets, however these ideas either fail to generalise or are not useful. We can demonstrate this in two ways. Firstly, although regular maps between quasi-projective varieties induce homomorphisms on their rings of regular functions, most of the time these rings doesn't carry useful information about X. For instance, it can be shown that if X is a projective variety then the only regular functions on it are the constant functions (Ch. 1, Thm. 3.4 in [Har77]). Secondly, while the coordinate ring of a quasi-projective variety X is a useful (as we have seen in [2.20]), it is dependant on the embedding and as such there are isomorphic quasi-projective varieties which have different coordinate rings. Because of these reasons, the approach we will take for quasi-projective varieties is different.

Definition 2.25. Let X and Y be quasi-projective varieties and let $F : X \dashrightarrow Y$ be a rational map. We say that F is *dominant* if for every representative (f, U) of F, $f(U)$ is dense in Y.

Note that since X is irreducible it is sufficient for a single representative to have dense image. A rational map $F : X \dashrightarrow Y$ being dominant is precisely the condition which allows us take 'pullbacks' of rational functions. Take for instance a rational function $g \in k(Y)$ with regular locus $U_g \subseteq Y$. Let (f, U) be a representative of F. If F is dominant then $U_g \cap F(U)$ must intersect as otherwise $F(U) \subseteq \mathbb{P}^n - U_g$ which contradicts the fact that $F(U)$ is dense. We can therefore define a rational map

$$F^*(g) := g \circ F : U \cap F^{-1}(U_g) \to k$$

We formalise this in the following definition.

Definition 2.26. Let X, Y be quasi-projective varieties and let $F : X \dashrightarrow Y$ be a dominant rational map. The map F^* described above is well defined homomorphism of fields $F^* : k(Y) \to k(X)$ called the *pullback* of F.

The proof that this is a homomorphism of fields is exactly the same as the comments proceeding [2.21].

Now that we have a notion of pullback for dominant rational maps, we can state and prove the correspondence we are after.

Proposition 2.27. Let X, Y be quasi-projective varieties. There is a 1-1 correspondence

$$\{\text{Dominant rational maps } X \dashrightarrow Y\} \longleftrightarrow \{\text{Field homomorphisms } k(Y) \to k(X)\}$$

Furthermore, if $Z \subseteq \mathbb{P}^n$ is a quasi-projective variety then $k(Z)$ is a finitely generated field extension over k. If K/k is any finitely generated field extension then there exists a quasi-projective variety Z such that $K \cong k(Z)$.

Proof (Ch. 1, Coro. 4.5 in [Har77]). Let K/k be a finitely generated field extension and let $a_1, \ldots, a_m \in K$ be a finite generating set, i.e. $K = k(a_1, \ldots a_m)$. Let $R = k[a_1, \ldots, a_m] \subseteq K$ be the k-algebra generated by $a_1, \ldots, a_m$ over k. We have a surjective homomorphism of k-algebras $\phi : k[x_1, \ldots, x_m] \to R$ and therefore we have that $R \cong k[x_1, \ldots, x_m]/\ker(\phi)$ by the 1st isomorphism theorem. Since K is an integral domain, R has no nilpotents and $\ker(\phi)$ is a prime ideal. R is therefore coordinate ring of the affine algebraic set $Z = V(\ker(\phi)) \subseteq \mathbb{A}^m$. Furthermore,

$k[Z] \cong R$ and so $k(Z) \cong R_{\langle 0 \rangle} = k[a_1, \dots, a_m]_{\langle 0 \rangle} = K$. Therefore K is isomorphic to the rational function field of some quasi-projective variety.

Let $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ be quasi-projective varieties and let $\psi : k(X) \to k(Y)$ be a homomorphism of fields. Without loss of generality we may assume that there exists a dense open affine algebraic subset $S \subseteq Y \cap A_i$. Let $y_1, \dots y_m$ be a generating set of $k[S] \subseteq k(Y)$. The rational functions $\psi(y_1), \dots, \psi(y_m) \in k(X)$ are all defined on some open subset $U \subseteq X$. As in the proof of [2.23] they therefore define a regular map $F : U \to S \subseteq Y$ with $F^* = \psi|_{\mathcal{O}(S)} : \mathcal{O}(S) \to \mathcal{O}(U)$. Note that as ψ is a field homomorphism, $\psi|_{\mathcal{O}(S)}$ is injective. $I(\mathrm{im}(F)) = \ker(\psi|_{\mathcal{O}(S)})$ and so $\mathrm{im}(F) = S$ which is dense in Y. F thus determines a dominant rational map $F : X \to Y$ with $F^* = \psi$.

□

Corollary 2.28. Two quasi-projective varieties X, Y are birational if and only if their function fields are isomorphic as fields.

We have accomplished our goal of firmly connecting the invariants of quasi-projective and affine varieties with their appropriate notions of 'morphism'. If we are only interested in the properties of quasi-projective varieties (over k) up to birational equivalence, we can use our dictionary to translate into questions about finitely generated field extensions of k. On the other hand, given such a field K we don't yet have a tool to find a corresponding quasi-projective variety X which is 'nice as possible'. In what remains of this chapter we will answer this in the case where K has 'dimension' 1.

2.4 Dimension and Smoothness

In this section we will explore 'dimension' and 'smoothness' in the context of quasi-projective varieties. As we may expect, there are equivalent geometric and algebraic formulations for these concepts in terms of the set $X \subseteq \mathbb{P}^n$ and its ring of rational functions $k(X)$.

Definition 2.29. Let X be a quasi-projective variety. Conciser a chain of irreducible non-empty closed subsets of X

$$X_0 \subset X_1 \cdots \subset X_n = X$$

We define the *dimension* of X as the length of the longest such chain i.e. if the chain above is of maximal length then $\dim(X) = n$.

This notion of dimension is different from how it is defined for manifolds. Since the singleton $\{p\}$ is always a irreducible closed set, if X has dimension 1 then the only 'sub-varieties' of X are points. Dimension provides us a good geometric invariant of quasi-projective varieties. Suppose now that $X \subseteq \mathbb{A}^n \subseteq \mathbb{P}^n$. Irreducible closed subsets $Z \subset X$ correspond prime ideals $\mathfrak{p} \supset I(X)$. By the Nullstellensatz and the 3rd isomorphism theorem, we may interpret the dimension of an irreducible affine algebraic set in the following way.

Proposition 2.30. Let $X \subseteq \mathbb{A}^n$ be an irreducible affine algebraic set, then $\dim(X)$ is equal to the Krull dimension of $k[X]$.

An equivalent definition of dimension for quasi-projective varieties can given in term of the rational function field. Recall that if $X \subseteq \mathbb{P}^n$ is quasi-projective, then $k(X)$ is a finitely generated field extension of k. This can be broken down into $k \subseteq L \subseteq K$ where L/k is a pure transcendental extension and K/L is algebraic. Since L is finitely generated over k we may write $L = k(a_1, \dots, a_m)$. If $\{a_1, \dots, a_m\}$ is algebraically independent we say that the transcendence degree of K/k is m.

Proposition 2.31. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. Then the transcendence degree of $k(X)$ is equal to the Krull dimension of $k[X]$. Furthermore if R is any finitely generated k-algebra without nilpotents then $\mathrm{tr}_{\mathrm{deg}}(R_{\langle 0 \rangle}) = \dim(R)$.

Proof. See Ch. 13, Thm. A of [Eis95]. $\square$

We generalise this result to projective varieties in the following proposition.

Proposition 2.32. Let $X \subseteq \mathbb{P}^n$ be a projective variety. The dimension of X is equal to the transcendence degree of $k(X)$ and is one less than the Krull dimension of $S[X]$.

Proof. Note that $\{A_i \cap X\}_{i=0}^n$ is an open cover for X and as such, atleast one $A_i \cap X$ must have the same dimension as X. As in the comments proceeding [2.20] we have that $k(A_i \cap X) = k(X)$ and so $\dim(X) = \dim(A_i \cap X) = \mathrm{tr}_{\mathrm{deg}}(k(X))$ as required. Also by [2.20] we have that $k[A_i \cap X]$ is isomorphic to $R := \{f/x_i^k \in S[X]_{x_i} : \deg(f) = k\}$. Using this we may conclude that $S[X]_{x_i} = k[A_i \cap X][x_i, x_i^{-1}]$. It follows that $\dim(S[X]) = \mathrm{tr}_{\mathrm{deg}}(S[X]_{\langle 0 \rangle}) = \mathrm{tr}_{\mathrm{deg}}((S[X]_{x_i})_{\langle 0 \rangle})) = \mathrm{tr}_{\mathrm{deg}}(k(A_i \cap X)) + 1 = \dim(X) + 1$.

$\square$

The text topic to tackle is 'smoothness'. Recall from Chapter 1 that given homogeneous polynomials $f_1, \dots, f_m \in k[x_0, \dots, x_n]$, the Jacobian matrix $(\partial f_i / \partial x_j)_{ij}$ gave a condition for when $\{f_1 = \cdots = f_m = 0\}$ is a complex submanifold of $\mathbb{P}^n$. The tactic, as demonstrated in the case of a hypersurface, was to find a conditions on the polynomials $f_1, \dots, f_m$ such that $A_i \cap \{f_1 = \cdots = f_m = 0\}$ was submanifold of A_i. Similarly in the algebraic case, we will define smoothness first for affine algebraic sets as smoothness should be a local property. As $\partial/\partial x_j$ are algebraic operators we may directly transplant the original definition to the algebraic setting.

Definition 2.33. Let $X \subseteq \mathbb{A}^n$ be an irreducible affine algebraic set and let $f_1, \dots, f_m$ be a set of generators for $I(X)$. We say that X is a *smooth* or *non-singular* at a point $x \in X$ if the rank of the Jacobian matrix $(\partial f_i / \partial x_j)_{ij}$ has rank $n - r$ where r is the dimension of X.

In the analytic case this condition on the Jacobian matrix insured that we had a chart (ϕ, U) of the point which was compatible with the complex structure of $\mathbb{P}^n$. In the algebraic case we don't have access to charts, instead we have access to algebraic (rational) functions defined on Zariski open subsets of X. Given a point $x \in X$ we can gather these together into what is called the local ring of a point.

Definition 2.34. Let $X \subseteq \mathbb{P}^n$ be a quasi-projective variety and let $x \in X$. We define the *local ring* of x as the subring $\mathcal{O}_x \subseteq k(X)$ defined as

$$\mathcal{O}_x = \{(U, f) \in k(X) : x \in U\}$$

The key here is that rational functions $f \in \mathcal{O}_x$ have to be defined in a neighbourhood of x. The following proposition highlights some key properties of local rings.

Proposition 2.35. Let $X \subseteq \mathbb{P}^n$ be a projective variety and let $x \in X$ with local ring $\mathcal{O}_x$, then

1. $\mathcal{O}_x$ is a Noetherian local ring with unique maximal ideal $\mathfrak{m}_x$.
2. $\mathcal{O}_x$ has residue field isomorphic to k and $\mathfrak{m}_x/\mathfrak{m}_x^2$ has the structure of k-vector space.
3. Let $\mathfrak{m}_x \in S[X]$ be the ideal $\{f \in S[X] : f(x) = 0\}$. Then $\mathfrak{m}_x$ is maximal and the localisation $S[X]_{\mathfrak{m}_x}$ is isomorphic to $\mathcal{O}_x$.

Proof (Ch. 1, Thm. 3.4 in [Har77]). Let $\mathfrak{m}_x = \{f \in \mathcal{O}_x : f(x) = 0\}$. It is clear that $\mathfrak{m}_x$ is an ideal of $\mathcal{O}_x$ as it is additive and if $f(x) = 0$ then $g(x)f(x) = 0$ for every $g \in \mathcal{O}_x$. Suppose now that $f, g \in \mathcal{O}_x$ such that $f(x) = g(x)$, then $f - g(x) = 0$ and so $[f] = [g]$ in $\mathcal{O}_x/\mathfrak{m}_x$. Since the constant functions are in $\mathcal{O}_x$ we conclude that $\mathcal{O}_x/\mathfrak{m}_x \cong k$ and so $\mathfrak{m}_x$ is a maximal ideal. To see that $\mathcal{O}_x$ is indeed a local ring, we observe that every element $f \notin \mathfrak{m}_x$ has $f(x) \neq 0$ and thus $1/f$ is regular in some neighbourhood of x. The group of units is exactly $\mathcal{O}_x - \mathfrak{m}_x$ and so $\mathcal{O}_x$ is a local ring.

To see that $\mathcal{O}_x$ is Noetherian, suppose that $x \in A_i$ and consider the affine algebraic set $X_i := X \cap A_i$. The inclusion map $\iota : k[X_i] \to \mathcal{O}_x$ is injective but not surjective. If we localise $k[X_i]$ by the prime ideal $\mathfrak{m}_x \cap k[X_i]$ we may consider the ring homomorphism $\mathcal{O}_x \hookrightarrow k[X_i]_{\mathfrak{m}_x} \subseteq k(X_i)$. This map is both injective and surjective as every $f \in \mathcal{O}_x$ takes the form g/h where $g \in k[X_i]$ and $h \in k[X_i] - \mathfrak{m}_x$. This proves that $\mathcal{O}_x$ is Noetherian as it is the localisation of a Noetherian ring at a prime ideal. Furthermore, if we recall the isomorphism between $k[X_i]$ and the sub ring $\{f/x_i^k \in S[X]_{x_i} : k \in \mathbb{Z}_{\geq 0}, \deg(f) = k\}$ of $S[X]_{x_i}$ from [2.20], we can see that the prime ideal $k[X_i] \cap \mathfrak{m}_x$ corresponds to the ideal $\{f/x_i^k : fx = 0\}$ and since $x_i(x) \neq 0$ (since $x \in A_i$) it follows that

$$\mathcal{O}_x \cong k[X_i]_{\mathfrak{m}_x} \cong S[X]_{\mathfrak{m}'_x}$$

where $\mathfrak{m}'_x = \{f \in S[X] : f(x) = 0\}$. This proves (1) and (3). Finally, (2) is a well known algebraic fact about Noetherian locals rings, it is a direct consequence of Prop. 2.8 in [AM69].

$\square$

As we may expect, smoothness of a projective variety at a point $x \in X$ can be interpreted equivalently as a statement about the local ring $\mathcal{O}_x$. This is not too surprising as in the complex-analytic case the corresponding notion of 'local ring of a point', the so called 'ring of germs of holomorphic functions at a point', gives information about the 'tangent plane' at that point. More specifically, this ring is also a local ring and the vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ can be shown to be isomorphic to the cotangent space of X at x. By naively translating this over to the algebraic setting, we might expect that if the dimension of $\mathfrak{m}_x/\mathfrak{m}_x^2$ is equal to the dimension of X then x is not a singular point as the 'tangent plane' at x has the correct dimension.

Proposition 2.36. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic set. Let $x \in X$ and let $\mathfrak{m}_x$ be the unique maximal ideal of $\mathcal{O}_x$. Then X is non-singular at x if and only if the k-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ has dimension $\dim(X)$.

Proof (Ch. 1, Thm. 5.1 in [Har77]). Let $\theta : k[x_1, \dots, x_n] \to k^n$ be the k-linear map defined as

$$\theta(f) = (\partial f/\partial x_1(x), \dots, \partial f/\partial x_n(x))$$

If $I = \langle f_1, \dots, f_m \rangle \subseteq k[x_1, \dots, x_n]$ is an ideal then the rank of the matrix $J = ((\partial f_i/\partial x_j)(x))_{ij}$ is exactly the dimension of $\theta(I) \subseteq k^n$. Conversely, if we write $x = (a_1, \dots, a_n)$, by directly calculating we find that θ vanishes on $\mathfrak{a}^2$ where $\mathfrak{a} = \langle x_1 - a_1, \dots, x_n - a_n \rangle$. In particular, θ descends to an isomorphism $\theta : \mathfrak{a}/\mathfrak{a}^2 \to k^n$. This observation allows us to also interpret the rank of J as the dimension of $I/\mathfrak{a}^2$ in $\mathfrak{a}/\mathfrak{a}^2$. In the proof of [2.35] we saw that $\mathcal{O}_x \cong k[X]_{\mathfrak{m}_x}$ where $\mathfrak{m}_x$ is the ideal of regular functions which vanish at x, in case that $X = V(I)$ we observe that $k[X] = k[x_1, \dots, x_n]/I$ and $\mathfrak{m}_x = \mathfrak{a}$. This shows us that $\mathfrak{m}_x \cong \mathfrak{a}\,(k[x_1, \dots, x_n]/I)_{\mathfrak{a}}$ and so $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong \mathfrak{a}/(I + \mathfrak{a}^2)$. Therefore we have

$$\dim(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim(\mathfrak{a}/\mathfrak{a}^2) - \dim(I/\mathfrak{a}^2) = n - \mathrm{rk}(J)$$

and it follows that $\dim(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim(X)$ if and only if $\mathrm{rk}(J) = n - \dim(X)$ as required.

□

Since these two definitions of smoothness are equivalent, it makes sense to define non-singular projective varieties in the following way.

Definition 2.37. Let $X \subseteq \mathbb{P}^n$ be a projective variety. Let $\mathcal{O}_x$ be the local ring at a point x with unique maximal ideal $\mathfrak{m}_x$. We say that $x \in X$ is a *non-singular* point if the k-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ has the same dimension as X. We say that X is non-singular if it is non-singular at every point.

We finish this section with two propositions about local rings which we will need later.

Proposition 2.38. Suppose that $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ are quasi projective varieties and $F : X \to Y$ is a regular map, then F induces a homomorphism $F_x^* : \mathcal{O}_{F(x)} \to \mathcal{O}_x$ for every $x \in X$. Furthermore, F is an isomorphism if and only F is a homeomorphism and F_x^* is an isomorphism for every $x \in X$.

Proof. The induced map $F_x^* : \mathcal{O}_{F(x)} \to \mathcal{O}_x$ is defined as $F_x^* : (g, V) \mapsto (g \circ F, F^{-1}(V))$. This is clearly a well defined homomorphism, it is in fact the restriction of F^* to the local ring $\mathcal{O}_{F(x)} \subseteq k(Y)$. If F is an isomorphism then F_x^* is a isomorphism as F^* is isomorphic and $F_x^* = F^*|_{\mathcal{O}_{F(x)}}$. Conversely, suppose that F is a homeomorphism and F_x^* is an isomorphism for every $x \in X$. For every open set $U \subset X$ and every regular function $f \in \mathcal{O}(U)$ we have that $f \in \mathcal{O}_x$ for any $x \in U$. Fix an $x \in U$ and let $g : \mathcal{O}_{F(x)}$ be $g = (F_x^*)^{-1}(f)$, then $f = g \circ F$ and therefore $f \circ F^{-1} = g$ is regular. Therefore F^{-1} is a regular map and so F is an isomorphism as it is a regular homeomorphism with regular inverse.

□

Proposition 2.39. Let $X \subseteq \mathbb{P}^n$ be quasi-projective and suppose $x, y \in X$. Then, $\mathcal{O}_x \subseteq \mathcal{O}_y$ if and only if $x = y$.

Proof (Ch. 1, Lem. 6.4 in [Har77]). By possibly performing a projective transformation of $\mathbb{P}^n$, without loss of generality assume that $x, y \in A_i$ for some i. Consider the affine algebraic set $X_i = X \cap A_i$. As we have seen in the proof of [2.35] we have that $\mathcal{O}_x = k[X_i]_{\mathfrak{m}_x} \subseteq k(X_i)$. By Prop [2.3] we know that the ideals $\mathfrak{m}_x$, $\mathfrak{m}_y$ are not only prime but also maximal in $k[X_i]$. Therefore, if $\mathcal{O}_x \subseteq \mathcal{O}_y$ then $k[X_i]_{\mathfrak{m}_x} \subseteq k[X_i]_{\mathfrak{m}_y}$ and so $\mathfrak{m}_y \subseteq \mathfrak{m}_x$. Since these ideals are both maximal, $\mathfrak{m}_x = \mathfrak{m}_y$ and therefore $x = y$.

$\square$

2.5 Smooth Projective Curves

As we have seen, projective varieties can be studied in terms of their field of rational functions $k(X)$. If X is a projective algebraic curve (as defined in [1.21]) then the interpretation of the Jacobian matrix in [2.36] together with [2.31] show that the dimension of X is the same whether we regard it as a Riemann surface or projective variety over $k = \mathbb{C}$. In terms of the field of rational functions, this translates to $k(X)$ having transcendence degree 1 and the local ring of every point $\mathcal{O}_x \subseteq X$ being 'regular' (p. 123 in [AM69]).

We approach this problem from the other direction. That is, under what conditions is a finitely generated field extension $k \subseteq K$ isomorphic to $k(X)$ for a projective algebraic curve X? We begin by introducing discrete valuation rings, the prototype for local rings of points on curves.

Definition 2.40. Let K be a field. A discrete valuation on K is a homomorphism of abelian groups $\nu : (K - \{0\}, \times) \to \mathbb{Z}$ such that

$$\nu(x + y) \geq \min(\nu(x), \nu(y))$$

The set $R = \{x \in K : \nu(x) \geq 0\} \cup \{0\} \subseteq K$ is a ring, we call R the *valuation ring* of the discrete valuation ν. Any integral domain R which arises as the valuation ring of a discrete valuation of its field of fractions, is called a *discrete valuation ring*.

Let $X \in \mathbb{P}^n$ be a non-singular curve, and let $x \in X$. So far we know the following things about the local ring $\mathcal{O}_x$

- $\mathcal{O}_x$is a Noetherian integral domain
- $\mathcal{O}_x$ is a local ring with and residue field k and field of fractions $k(X)$.
- $\dim(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim(X) = 1$

It turns out that these conditions are equivalent to $\mathcal{O}_x$ being a discrete valuation ring. In particular, they imply that the maximal ideal $\mathfrak{m}_x$ is principle and that every other non-zero ideal takes the form $\langle f^n \rangle$ where $\mathfrak{m} = \langle f \rangle$. We form the discrete valuation by defining $\nu(a) = n$ where $\langle a \rangle = \langle f^n \rangle$ and extending this to the field of fractions as $\nu(a/b) = \nu(a) - \nu(b)$. A full proof of this statement can be found on p. 94-95 of [AM69]. This reference also gives another equivalent characterisation of discrete valuations rings, instead of requiring $\dim(\mathfrak{m}_x/\mathfrak{m}_x^2) = 1$ we may require that $\mathcal{O}_x$ is integrally closed.

Since the field of fractions of $\mathcal{O}_x$ is $k(X)$, $\mathcal{O}_x$ is a discrete valuation ring of $k(X)$. Note also that if ν is the discrete valuation and $\lambda \in k - \{0\} \subseteq k(X)$ then $\nu(\lambda a) = \nu(\lambda) + \nu(a)$. Since $\langle \lambda a \rangle = \langle a \rangle$ we have that $\nu(\lambda a) = \nu(a)$ and therefore $\nu(\lambda) = 0$ for every $\lambda \in k \subseteq k(X)$. If R is any discrete valuation ring of $k(X)$ which has this property then we will call R a discrete valuation ring of $k(X)/k$.

If X is a smooth projective variety of dimension 1 and $x, y \in X$ are two distinct points, then by [2.39] x and y correspond to different discrete valuations rings $\mathcal{O}_x$, $\mathcal{O}_y \subset k(X)$ of $k(X)/k$. The following proposition helps with the other direction of this correspondence.

Proposition 2.41. Let K/k be a function field of transcendence degree 1 and let $R \subseteq K$ be a discrete valuation ring of K/k. Then there is a non-singular affine curve $S \subseteq \mathbb{A}^n$ and a point $x \in S$ such that $R \cong \mathcal{O}_x$.

Proof. The idea of the proof is to show that a every discrete valuation ring of K/k is the localisation of a finitely generated k-algebra at a maximal ideal. The proof in more detail can be found on p. 41 in [Har77]. We need some results from commutative algebra (p. 40 in [Har77]) :

(a) R is a discrete valuation ring of K if and only if for every local ring S of K such that such $R \subseteq S$ and $\mathfrak{m}_s \cap R = \mathfrak{m}_r$, $R = S$.

(b) The localisation of a Dedekind domain at a non-zero prime ideal is a discrete valuation ring.

(c) Let R be an integral domain with field of fractions K, let L/K be an algebraic field extension and let R' denote the integral closure of $R \subseteq L$. The field of fractions of R' is L. If R is a finitely generated k-algebra then so is R'. If R is a Dedekind domain then so is R'.

Let R be as above, let $\mathfrak{m} \subseteq R$ be the unique maximal ideal, and let $r \in \mathfrak{m}$ be a generator of $\mathfrak{m}$. We know that $r \in K - k$ since it generates $\mathfrak{m}$. As k is algebraically closed, we have that $k \subset k(r)$ and $k(r) \subseteq K$ are transcendental and algebraic field extensions respectfully. In particular the ring $k[r]$ is the polynomial ring in 1 variable over k. Let S be the integral closure of $k[r] \subseteq K$ then by (c) it is finitely generated k-algebra and furthermore Dedekind (as $k[r]$ is a PID).

$R \subseteq K$ is discrete valuation ring of K/k and so is integrally closed in K. Therefore as $k[r] \subseteq R$ we have that $S \subseteq R$ and in particular $\mathfrak{a} = \mathfrak{m} \cap S$ is a maximal ideal of S (S is Dedekind and so non-zero prime ideals are maximal). S is Dedekind and so by (b) and (c), $S_{\mathfrak{a}}$ is a discrete valuation ring of K/k. Finally we have that by (a), $S_{\mathfrak{a}} = R$. Therefore R is isomorphic to a the localisation of a finitely generated k-algebra at a maximal ideal. Using the ideas from [2.24] and [2.35] we may view R as a the local ring at a point x (corresponding to the maximal ideal $\mathfrak{a}$) on an affine curve with coordinate ring S. Because S is Dedekind the localisation at every maximal ideal is a discrete valuation ring, or equivalently integrally closed.

Thus we observe that the curve S is nonsingular.

Note additionally that since S is a coordinate ring of an affine variety Y and $r \in \mathfrak{a} \subseteq S$, r vanishes at the point x. Because $r \neq 0$ it can only be in finitely many maximal ideals of S since otherwise r would vanish at a infinite subset of Y. This is impossible as r is a regular function on Y and Y is a curve.

□

We will now construct a non-singular projective curve from a function field K/k of transcendence degree 1. [2.41] suggests that there is a connection between the discrete valuation rings $R \subseteq K$ and local rings at a point on non-singular affine curves. A priori, different discrete valuation rings do not to represent different points (that is, local rings of points) on the same curve X. Our strategy will be to show that the space of discrete valuation rings of K/k is 'isomorphic' to the points of a non-singular projective curve X. The method we will use is found in Ch. 1.6 of [Har77]. The first step is to put structure on the set of discrete valuation rings of K/k, Hartshorne calls the finished product an *abstract non-singular curve*.

Recall that when thinking about the dimension of a projective variety $X \subseteq \mathbb{P}^n$ we had two options: the transcendence degree of $k(X)$ or the topological dimension of X. If we interpret the dimension of a projective curve using the topological definition we observe that the only strict chains of irreducible closed subsets are $\{pt\} \subset X$. The Zariski topology on any projective curve is therefore just the co-finite topology. Of course we can still distinguish between projective curves by restricting our morphisms from general Zariski-continuous functions to just the regular maps, however the fact that they are all topologically identical is useful and motivates the following definition.

Definition 2.42. Let K be a function field of transcendence degree 1 over k. We define D_K as the set of all discrete valuation rings of K/k, i.e.

$$D_K = \{R \subseteq K : R \text{ a discrete valuation ring of } K/k\}$$

We make D_K into a topological space by giving it the co-finite topology. We call an open subset $U \subseteq D_K$ an *abstract non-singular curve*.

Definition 2.43. Let $U \subseteq D_K$ be an open set. We define the ring of regular functions on U as

$$\mathcal{O}(U) = \bigcap_{R \in U} R$$

Let $U \subseteq X \subseteq \mathbb{P}^n$ be an open subset of a non-singular projective curve. An element $f \in \cap_{x \in U} \mathcal{O}_x$ is a rational function $f \in k(X)$ for which there are representatives $\{(U_x, f_x)\}_{x \in U}$ such that $x \in U_x$. Clearly f is then a regular function on U. If we instead have a regular function $f \in \mathcal{O}(U)$ where $U \subseteq D_K$ is an open set, then f also defines a unique function $f : U \to k$ in the following way: for $R \in D_K$ let $f(R) := \pi_{\mathfrak{m}_R}(f)$ where $\pi_{\mathfrak{m}_R} : R \to R/\mathfrak{m}_R$ is quotient map and $\mathfrak{m}_R$ is the maximal ideal of R.

Definition 2.44. Let X, Y be abstract non-singular curves. We say that a map $F : X \to Y$ is regular if it continuous and if for every $f \in \mathcal{O}(Y)$, the composition $f \circ F$

is a regular function on X (i.e. there is a $g \in \mathcal{O}(X)$ such that $\forall R \in X$, $f \circ F(R) = \phi_R(g)$).

The definition of regular map is the same for non-singular curves and abstract ones. We can also define a regular map between the two in the same way. If X is an abstract non-singular curve and Y is any quasi-projective variety (or vice versa), then a regular map $F : X \to Y$ is any function for which $(\cdot \circ F) : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a homomorphism of rings. This enlarges the category of quasi-projective varieties, and it is in this sense that we will consider a quasi-projective variety isomorphic to an abstract one. We are now in a position to state the main theorem of this chapter.

Theorem 2.45. Let K/k be a finitely generated field extension of transcendence degree 1 over k. The abstract non-singular curve D_K is isomorphic to a non-singular projective curve.

In order to prove this theorem we need two lemmas. The first lemma states that every non-singular quasi-projective curve is isomorphic the abstract non-singular curve comprising of it's local rings.

Lemma 2.46. If $X \subseteq \mathbb{P}^n$ is a non-singular quasi-projective curve then $U = \{\mathcal{O}_x \subseteq k(X) : x \in X\} \subseteq D_{k(X)}$ is an open set and $X \cong U$.

Proof (Ch. 1, Prop. 6.7 in [Har77]. Let $Y = A_0 \cap X$. By [2.35] we have that $k(Y) = k(X)$ and $\mathcal{O}_{y,Y} = \mathcal{O}_{y,X}$ (denoting the local ring of $y \in Y$ in Y and X respectively) as A_0 is an open subset of $\mathbb{P}^n$. Let $V = \{\mathcal{O}_y \subseteq k(X) : y \in Y\} \subseteq U$, and let $\{x_1, \dots x_n\}$ be a generating set for $k[Y]$. If $R \in D_{k(X)}$ is such that $k[Y] \subseteq R \subseteq k(X)$ then $R \in V$ by the same reasoning we used in the proof of [2.41]. Therefore V is comprised of exactly those discrete valuation rings which contain $x_1, \dots, x_n$. This must be a co-finite set of $D_{k(X)}$ as $\{R \in D_{k(X)} : x_1 \notin R\} = \{R \in D_{k(X)} : x_1 \in \mathfrak{m}_R\}$ is a finite set. $V \subseteq U$ implies that U is also co-finite and therefore an open subset of $D_{k(X)}$.

We argue that the function $F : X \to U$ defined by $F(x) = \mathcal{O}_x$ is an isomorphism. Since F is 1-1 (by [2.39]), it is continuous. Suppose $f \in \mathcal{O}_x$, then it is clear that $\pi_{\mathfrak{m}_x}(f) = f(x)$. Therefore if $f \in \mathcal{O}(U)$ then the map $f \circ F(x) = \pi_{\mathfrak{m}_x}(f) = f(x)$ is regular on X. Conversely if $f \in \mathcal{O}(X)$ then $f \circ F^{-1} = f \in \mathcal{O}(U)$ is also regular. Therefore F is an isomorphism.

□

The second lemma states that we can extend regular maps $X - \{p\} \to Y$ where X is a non-singular projective curve.

Lemma 2.47. Let X be a non-singular projective curve and let Y be a projective variety. If $F : X - \{p\} \to Y$ is a regular map then we can extend this uniquely to a regular map $\tilde{F} : X \to Y$.

Proof (Outline). Because $X \subset \mathbb{P}^n$ is a non-singular projective curve, the local ring $\mathcal{O}_x$ is a discrete valuation ring of $k(X)/k$ and elements take the form g/h where $g, h \in k[x_0, \dots, x_n]$ are homogeneous of the same degree and $h(p) \neq 0$. The unique maximal ideal $\mathfrak{m}_p \subseteq \mathcal{O}_p$ consists of precisely the functions which vanish at p and as we have seen this ideal is principle and generated by some $l \in \mathcal{O}_p$. As

$Y \subseteq \mathbb{P}^m$ we may consider $F : X - \{p\} \to \mathbb{P}^n$. As a consequence of [2.20], locally we can write $F(x) = [f_0 : \cdots : f_m]$ where $f_i \in k(X)$. If we let $\nu : k(X) \to k(X)$ be the valuation defining $\mathcal{O}_x$, we can consider $r = \min\{\nu(f_i)\}_{i=1,\ldots m}$. We then have that $\nu(l^{-r}f_i) \geq 0\ \forall i$, and $\nu(l^{-r}f_i) = 0$ for atleast one i. Therefore $l^{-r}f_i \in \mathcal{O}_p\ \forall i$ and so the function

$$\tilde{F}(x) = [l^{-r}f_0 : \cdots : l^{-r}f_m]$$

is a well defined local extension of F to the point p as every homogeneous component is defined at p and they do not all identically vanish at p. Defining $\tilde{F}(x) = F(x)$ everywhere else yields a regular function $\tilde{F} : X \to \mathbb{P}^m$ extending F. Furthermore $\tilde{F}(X) \subseteq Y$ as otherwise $\tilde{F}^{-1}(Y) = X - \{p\}$ yields a contradiction since $\tilde{F}$ is regular and therefore continuous.

□

By a very similar argument, the lemma also holds when we replace X with an abstract non-singular curve. The full proof of this generalised statement can be found in Ch. 1, Prop. 6.8 in [Har77].

The next thing that we need is the ability to take 'products' of non-singular projective curves. Defining the product of two projective varieties is more subtle than just using the set theoretical product. Take for example the set theoretic product $\mathbb{P}^1 \times \mathbb{P}^1$. Breaking this down into its components we observe that $\mathbb{P}^1 \times \mathbb{P}^1 = (\mathbb{A}^1 \sqcup \{pt\}) \times (\mathbb{A}^1 \sqcup \{pt\}) = \mathbb{A}^2 \sqcup \mathbb{A}^1 \sqcup \mathbb{A}^1 \sqcup \{pt\}$ whereas $\mathbb{P}^2 = \mathbb{A}^2 \sqcup \mathbb{P}^1 = \mathbb{A}^2 \sqcup \mathbb{A}^1 \sqcup \{pt\}$. To define the product we first introduce the Segre embedding.

Definition 2.48. Let $n, m \in \mathbb{Z}_{\geq 1}$ and let $N = mn + m + n$. The Segre embedding is the map $\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ defined as

$$\sigma_{n,m} : ([x_0, \ldots x_n], [y_0, \ldots, y_m]) \mapsto [\ldots, x_i y_j \ldots]$$

in lexicographic order. Given two quasi-projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ we define their product $X \times Y$ as image of the set theoretic product in the Segre embedding, i.e. $X \times Y = \sigma_{n,m}(X \times_{\text{Set}} Y) \subseteq \text{Im}(\sigma_{n,m})$.

The following are a list of facts about the Segre embedding that we need. I will not prove them however Chapter 5 of [Sha13] is a good reference for this topic.

Proposition 2.49. Let $X \subseteq \mathbb{P}^n$, $Y \subseteq \mathbb{P}^m$ be projective varieties and let $\Sigma_{n,m} = \text{Im}(\sigma_{n,m})$, then

- $\sigma_{n,m}$ is a bijection onto it's image.
- $\Sigma_{n,m} \subseteq \mathbb{P}^N$ and $X \times Y \subseteq \Sigma_{n,m}$ are projective varieties.
- The projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are regular. Furthermore, if $Z \subseteq \mathbb{P}^k$ is a projective variety and $\phi_X : Z \to X$ and $\phi_X : Z \to Y$ are regular maps then there is a unique regular map $F : Z \to X \times Y$ such that $\pi_X \circ F = \phi_X$ and $\pi_Y \circ F = \phi_Y$.

We now have all the tool required to prove [2.45]. We will prove this theorem in several steps.

Proof of Theorem 2.45 (Chap. 1, Thm. 6.9 in [Har77]). Let K be finitely generated function field of transcendence degree 1 over k.

Step 1: Let $R \in D_K$. By [2.41] there is an non-singular affine curve Y and a point $y \in Y$ such that $R \cong \mathcal{O}_y$. Note that as in [2.41], Y has rational function field $k(Y) = K$ and furthermore $R = \mathcal{O}_y$ when we view $\mathcal{O}_y \subseteq k[Y] \subseteq K$. Let $Y \subseteq A_0 \subseteq \mathbb{P}^n$ then Y is a quasi-projective variety and so by [2.46] Y is isomorphic to a open set $U \subseteq D_K$. Furthermore, since $\mathcal{O}_y = R$ we have that $R \in U$. Therefore every point $R \in D_K$ has an open neighbourhood which is isomorphic to a non-singular affine curve.

Step 2: As every open set of D_K contains all but finitely many points, every open cover must have a finite sub-cover. Let $U_R \subseteq D_K$ be the open neighbourhood of D_K which is isomorphic to an non-singular affine curve. Since $\{U_R : R \in D_K\}$ is an open cover of D_K, there is a finite sub-cover which we will denote $\{U_1, \ldots U_k\}$. Let X_i be the projective closure of $Y_i \subseteq \mathbb{P}_{n_i}$. We can write the isomorphisms $\phi_i : U_i \to Y_i$ as regular maps $\phi_i : U_i \to X_i$. As U_i is a co-finite set, we can apply the generalisation of [2.46] recursively to extend ϕ_i to regular maps $\phi_i : D_K \to X_i$. By Prop [2.49] we have a unique regular map $\Psi : D_K \to \prod_{j=1}^k X_j$. Let X be the projective closure of $\Psi(D_k)$, and let $\pi_i = \pi_{X_i} \circ \iota_X$ where π_{X_i}, ι_X are projection $\pi_{X_i} : \prod_{j=1}^k X_j \to X_i$ and the inclusion $\iota_X : X \to \prod_{j=1}^k X_j$ respectfully. The following diagram of regular maps commutes:

$$\begin{tikzcd} U_i \arrow[r, "\phi_i"] \arrow[d, hook, "\iota_i"] & X_i \\ D_K \arrow[r, "\Psi"] & X \arrow[u, two heads, "\pi_i"'] \end{tikzcd}$$

Step 3: First we note that all of the maps are dominant by construction. As Ψ is dominant we have that $\dim(X) \leq \dim(D_K) = 1$. As X is not finite it therefore has dimension 1 (i.e. X is a projective curve). The sequence of dominant regular maps $D_K \to X \to X_i$ induces injective homomorphisms $\mathcal{O}_{\phi_i(R)} \hookrightarrow \mathcal{O}_{\Psi(R)} \hookrightarrow R$. Because $R = \mathcal{O}_{\phi_i(R)}$ we have that $\mathcal{O}_{\Psi(R)} \cong R$ and so Ψ_R^* is an isomorphism for every $R \in D_K$.

Step 4: Because ψ_i is injective, and π_i is surjective, the composition $\Psi \circ \iota_i$ must be injective and therefore Ψ must be injective. Conversely let $x \in X$. The dominant regular map gives us a field embedding $\Psi^* : k(X) \hookrightarrow K$, and so the local ring $\Psi^*(\mathcal{O}_x)$ is a local subring of K. Just as we did in the proof of [2.41], we may construct a discrete valuation ring R of K/k such that $\Psi^*(\mathcal{O}_x) \subseteq R$ and $\Psi^*(\mathfrak{m}_x) = \mathfrak{m}_R \cap \Psi^*(\mathcal{O}_x)$. Let $y = \Psi(R)$, then $\Psi^*(\mathcal{O}_y) = R$ and therefore $\mathcal{O}_x \subseteq \mathcal{O}_y$. By Prop [2.39] we therefore have that $y = x$ and so Ψ is surjective. We have therefore shown that Ψ is a bijective regular map which induces isomorphisms on local rings. By [2.38] it follows that Ψ is an isomorphism.

□

If we analyse this theorem, we can see that it implies a categorical equivalence between finitely generated field extensions K/k of transcendence degree 1 and non-singular projective varieties of dimension 1. In the case that $k = \mathbb{C}$ we can showcase the relationships we have demonstrated in the following diagram.

$$\begin{tikzcd}
\text{Function fields } K/\mathbb{C} \text{ of transcendence degree 1} \arrow[r, leftrightarrow, "\sim"] \arrow[rr, bend left, dashed, leftrightarrow] & \text{Smooth projective varieties of dimension 1} \arrow[r, bend left] & \text{Compact Riemann surfaces} \arrow[l, bend left, dashed]
\end{tikzcd}$$

In the next two chapters our aim will be to solidify the dotted arrows.

3 Divisors and the Riemann-Roch Theorem

3.1 Differential Forms and Divisors

We want to find conditions for a Riemann surface X to be biholomorphic to an non-singular projective curve. Surprisingly, this is possible whenever X is compact. The key step in proving this is to show that all compact Riemann surfaces are projective. This result builds off a major result originally attributed to Riemann himself called the Riemann Existence Theorem (see [Har15]). This asserts that all compact Riemann surfaces have a non-constant meromorphic function. In this chapter we will use Riemann's theorem to prove our result by way of the Riemann-Roch theorem.

In order to discuss the Riemann-Roch theorem and its consequences, we must first discuss differential forms and integration in the context of Riemann surfaces. Recall that for a smooth surface X, a differential 1-form is a smooth section of the cotangent bundle T^*X. Given a smooth coordinate chart (ϕ, U) for X, the induced coordinates $\{x, y\}$ allow us to express any differential 1-form ω locally as $\omega = f_1 dx + f_2 dy$ for some smooth functions f_1, f_1. If we have a smooth map $F : X \to Y$ then there is a pullback $F^* : \Omega^1(Y) \to \Omega^1(X)$ defined by

$$(F^*\eta)_x(\nu) = \eta_{F(x)}(F_*(\nu))$$

where F_* is the derivative (pushforward) of F. In particular, we can transform the local expressions of ω to a different coordinate chart (ψ, V) via the smooth function $\phi \circ \psi^{-1} : \psi(V \cap U) \to \phi(V \cap U)$.

In the context of complex manifolds, we may formulate the theory of differential forms in a similar way. This is done by enriching tangent / cotangent vector bundles of a complex manifold X with a 'complex structure' which in turn allows us to define when sections of T^*X (1-forms) and T_*X (vector fields) are holomorphic. For our purposes however, it is sufficient think about 1-forms in the following less rigorous way.

Definition 3.1. Let $U \subseteq \mathbb{C}^2$ be an open set. A *holomorphic 1-form* on U is an expression of the form $f(z)dz$ where f is a holomorphic function on U. Let $U, V \subseteq \mathbb{C}$ and let $f(z)dz$ be a holomorphic 1-form on U, given a holomorphic function $\psi : V \to U$ we say that the holomorphic 1-form $f \circ \psi(w) \cdot \psi'(w)dw$ on V is the transformation of $f(z)dz$ via ψ.

Definition 3.2. Let X be a Riemann surface. A *holomorphic 1-form* ω on X is an assignment of a holomorphic 1-form $f(z)dz$ on $\phi(U)$ to every chart (ϕ, U) of X such that if any two charts (ϕ, U), (ψ, V) overlap then in this intersection their assigned holomorphic 1-forms are transformations of one and other via $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$.

Similarly, we can define a meromorphic 1-form $f(z)dz$ on an open set $U \subseteq \mathbb{C}$ by relaxing the requirement on $f(z)$ to just being meromorphic. Extending this in an analogous way to [3.2] allows us to define meromorphic 1-forms on Riemann surfaces. We denote the space of holomorphic/meromorphic 1-forms on X by $\Omega^1(X)$

and $\mathcal{M}^1(X)$ respectively.

Just as in the real case, we can define what it means to pullback holomorphic and meromorphic 1-forms by a holomorphic map.

Definition 3.3. Let $F : X \to Y$ be a non constant holomorphic map between Riemann surfaces and let ω be a holomorphic 1-form on Y. Let (ϕ, U) and (ψ, V) be a charts of X, Y respectively such that $F(U) \subseteq V$. Suppose with respect to (ψ, V), ω locally takes the form $\omega = f(w)dw$. We define the *pullback* of ω locally in $(\phi, F^{-1}(V))$ as

$$F^*\omega = f \circ (\psi \circ F \circ \phi^{-1})(z) \cdot (\psi \circ F \circ \phi^{-1})'(z)dz$$

By choosing a suitable basis of open sets $\{U_i\}_{i \in I}$ for X and repeating this procedure for all of them, we may form a holomorphic 1-form on X which can be checked to be consistent with [3.2]. This 1-form is called the pullback of ω through F and is denoted it $F^*\omega$. We can pullback meromorphic 1-forms using the same construction with the addendum that we must exclude the poles of ω.

Meromorphic 1-forms are the objects which we can integrate along paths on Riemann surfaces. Let ω be a meromorphic 1-form on X and let $\gamma : [a, b] \to X$ be a smooth path contained in a single chart (ϕ, U). We define the integral of ω along γ as

$$\int_\gamma \omega = \int_{\phi \circ \gamma} f(z)dz = \int_a^b f(\phi \circ \gamma(t)) \cdot (\phi \circ \gamma)'(t)dt$$

where $\omega = f(z)dz$ in (ϕ, U). The way 1-forms transform insure this is well defined. If $\gamma = \gamma_1 \cdot \ldots \cdot \gamma_k$ is any path not necessarily contained in a single chart, then if (ϕ_i, U_i) are charts such that $\gamma_i \in U_i$ we define

$$\int_\gamma \omega = \sum_i \int_{\phi_i \circ \gamma_i} f_i(z)dz$$

where $(\cdot)$ denotes path concatenation and $\omega = f_i(z)dz$ in the chart (ϕ_i, U_i).

As holomorphic/meromorphic 1-forms are locally expressions of the form $f(z)dz$, they have a notion of 'order at a point' in a similar way to holomorphic/meromorphic functions.

Definition 3.4. Let X be a Riemann surface, let $x \in X$ be a point, and let ω be a meromorphic 1-form. Suppose (ϕ, U) is a chart for X centred at $x \in U$ and suppose in this chart ω takes the form $\omega = f(z)dz$. We define the *order* of ω at x as

$$\mathrm{ord}_x(\omega) = \mathrm{ord}_0(f)$$

This definition is independent of our choice of the chart (ϕ, U). To see this consider that in any other coordinate chart centred at x, ω will take the form $(f \circ T)(w) \cdot T'(w)dw$ where T is biholomorphic. Locally around 0, $T(w) = cw + O(w^2)$ and $T'(w) = c + O(w)$. By expanding we can see that the order calculations must be the same.

We now turn to object of primary study in this chapter, divisors. We begin with the definition.

Definition 3.5. Let X be a Riemann surface. A *divisor* on X is a function $f : X \to \mathbb{Z}$ which takes the value 0 at all but a discrete subset of X. The set of divisors on X is denoted $\mathrm{Div}(X)$, and under the operation $(f+g)(x) = f(x) + g(x)$ has the structure of a subgroup of $\mathbb{Z}^X$. In the case that X is compact, divisors necessarily have finite support. In this case we define may the *degree* of D as

$$\deg(D) = \sum_{x \in X} D(x)$$

The degree function $\deg : \mathrm{Div}(X) \to \mathbb{Z}$ is a group homomorphism.

Divisors enable us to talk rigorously about 'finite sums' of codimension-1 complex manifolds inside Riemann surfaces. This language efficiently packages and allows us to manipulate data about functions, forms, and other constructions built on top of Riemann surfaces. Two illustrative and important examples are the following.

Definition 3.6. Let X be a Riemann surface and let f be a meromorphic function on X. We define the *divisor of f* as

$$\mathrm{div}(f) = \sum_{x \in X} \mathrm{ord}_x(f) \cdot x$$

Since the the poles and zeros of meromorphic functions are discrete, the divisor of f is a divisor. We say that a *principle divisor* is any divisor which is the divisor of some meromorphic function. The set of principle divisors of X is denoted $\mathrm{PDiv}(X)$ and forms a subgroup of $\mathrm{Div}(X)$. Similarly, if ω is a meromorphic 1-form then we define the divisor of ω as

$$\mathrm{div}(\omega) = \sum_{x \in X} \mathrm{ord}_x(\omega) \cdot x$$

Divisors which arise in this way are called *canonical divisors* and the set canonical divisors on X is denoted $\mathrm{KDiv}(X)$.

Similarly to 1-forms, non-constant holomorphic maps allow us to 'pullback' divisors in a natural way.

Definition 3.7. Let $F : X \to Y$ be a non-constant holomorphic map between Riemann surfaces. Let $D \in \mathrm{Div}(Y)$, we define the *pullback* $F^*(D)$ of D as

$$F^*(D)(x) = \mathrm{mult}_x(F) \cdot D(F(x))$$

Proposition 3.8. Let $F : X \to Y$ be a non-constant holomorphic map between Riemann surfaces and let $f : Y \to \mathbb{C}$ be a meromorphic function, then

$$F^*(\mathrm{div}(f)) = \mathrm{div}(f \circ F)$$

Proof (Ch. 5, Lem. 1.17 in [Mir95]). We first calculate $\mathrm{ord}_x(f \circ F)$. Fix a point $x \in X$, consider centred local coordinates such that F is in local normal form, i.e. $F(z) =$

z^m where $m = \mathrm{mult}_x(F)$. In these coordinates we may also write $f = \sum_{k \geq n} a_k z^k$ where $n = \mathrm{ord}_{F(x)}(f)$. It follows that in these coordinates $f \circ F = \sum_{k \geq n} a_k z^{mk}$ and so $\mathrm{ord}_x(f \circ F) = \mathrm{mult}_x(F) \cdot \mathrm{ord}_{F(x)}(f)$.

By definition we have that $F^*(\mathrm{div}(f))(x) = \mathrm{mult}_x(F) \cdot \mathrm{div}(f)(F(x)) = \mathrm{mult}_x(F) \cdot \mathrm{ord}_{F(x)}(f)$. Since $\mathrm{ord}_x(f \circ F) = \mathrm{mult}_x(F) \cdot \mathrm{ord}_{F(x)}(f)$, we have that $F^*(\mathrm{div}(f))(x) = \mathrm{ord}_x(f \circ F) = \mathrm{div}(f \circ F)(x)$ as required. In particular, the pullback of a principle divisor is also principle

$\square$

3.2 The space L(D) and Embeddings into Projective Space

Our main motivation for developing divisors is so we can translate statements about meromorphic functions into statements about spaces of divisors and vice versa. We begin with the following observation (Ch. 5, Lem. 1.12 in [Mir95]). Let X be a Riemann surface and let ω, η be meromorphic 1-forms on X which are not identically zero. Locally in a chart (ϕ, U) we can write $\omega = f_1(z)dz$ and $\eta = f_2(z)dz$ where f_1, f_2 are meromorphic. If we let $h = f_1/f_2$ then locally in U we have $\omega = h \cdot \eta$. Repeating this process for every chart gives us a unique globally defined meromorphic function h such that $\omega = h \cdot \eta$. Furthermore, locally we have that $\mathrm{ord}_x(f_1) = \mathrm{ord}_x(f_2) + \mathrm{ord}_x(h)$ and so this statement about 1-forms is equivalent to the following statement about divisors.

Proposition 3.9. Let X be a Riemann surface C and C' be canonical divisors, then there is a principle divisor P such that $C = C' + P$.

The property two divisors differing by a principle divisor is called being *linearly equivalent*, we will denote this as $D \sim D'$. As we would expect, linear equivalence is an equivalence relation and so we may talk about the equivalence class of a divisor. Another way that we can compare divisors is by introducing a partial order $\geq$ on $\mathrm{Div}(X)$, this is defined as follows.

Definition 3.10. Let X be a Riemann surface and let $D, D' \in \mathrm{Div}(X)$. We say that $D \geq D'$ if $D(x) \geq D'(x)$ for every $x \in X$. Likewise we say that $D > D'$ if $D(x) \geq D'(x)$ for every $x \in X$ and $D \neq D'$.

Using linear equivalence and our partial order, we may better organise the space of divisors $\mathrm{Div}(X)$ into what are called 'complete linear systems'.

Definition 3.11. Let X be a Riemann surface and let $D \in \mathrm{Div}(X)$. We define the *complete linear system* of D, denoted $|D|$ as

$$|D| = \{Q \in \mathrm{Div}(X) : Q \sim D \text{ and } Q \geq 0\}$$

Note that while any divisor D can define a complete linear system, if $D < 0$ then $D \notin |D|$.

Finally, we have two more constructions regarding divisors which finally concretely link the world of divisors with meromorphic functions and 1-forms.

Definition 3.12. Let $D \in \mathrm{Div}(X)$ where X is a Riemann surface. We define the space of *meromorphic functions with poles bounded by D*, denoted $L(D)$, as

$$L(D) = \{f \in \mathcal{M}(X) : \mathrm{div}(f) \geq -D\} \cup \{0\}$$

Abusing notation by setting $\mathrm{ord}_x(0) = \infty$ we me omit $\cup\{0\}$ from our definition. Similarly we define the space of *meromorphic 1-forms with poles bounded by D*, denoted $L^1(D)$, as

$$L^1(D) = \{\omega \in \mathcal{M}^1(X) : \mathrm{div}(\omega) \geq -D\}$$

$L(D)$ and $L^1(D)$ are complex vector spaces. They can be interpreted as subspaces of $\mathcal{M}(X)$ which have required zeros and permitted poles as described by the divisor D. As an example consider the case $X = \mathbb{C}$, the space $L(-n \cdot 0)$ consists of functions which have Laurent series of the form $f(z) = \sum_{k \geq n} c_k z^k$ at 0. As the 'behaviour' of the meromorphic functions $f(z)$ and $\lambda \cdot f(z)$ are essentially the same, it is sometimes useful to think about 1-dimensional linear subspaces of $L(D)$ instead of the individual elements. The projectivization

$$\mathbb{P}(L(D)) = \big(L(D) - \{0\}\big)/f \sim \lambda f,\ \lambda \in \mathbb{C}$$

gives us exactly this perspective. Using this we can compare complete linear systems with their corresponding space of meromorphic functions.

Proposition 3.13. Let D be a divisor on a compact Riemann surface X. The function $\mathcal{S} : L(D) \to |D|$ defined as $\mathcal{S} : f \mapsto D + \mathrm{div}(f)$ descends to a well defined bijection $\mathcal{S} : \mathbb{P}(L(D)) \to |D|$.

Proof (Ch. 5, Lem. 3.7 in [Mir95]). Let $f \in L(D)$. $\mathrm{div}(f) \geq -D$ implies that $D + \mathrm{div}(f) \geq 0$ and so $\mathcal{S}(f) \in |D|$. Let $g = \lambda f$ where $\lambda \neq 0$, clearly $\mathrm{div}(f) = \mathrm{div}(g)$ and so $\mathcal{S}$ descends to a well defined map $\mathcal{S} : \mathbb{P}(L(D)) \to |D|$. Suppose now that $g \in L(D)$ is a meromorphic function such that $\mathcal{S}(f) = F(g)$, it follows that $\mathrm{div}(f/g) = 0$ and so f/g has no zeros or poles in X. As X is compact, by [1.12] f/g must be a constant and so $f = \lambda g$. Conversely let $Q \in |D|$, by definition there exists $f \in \mathcal{M}(X)$ such that $Q = D + \mathrm{div}(f)$ and so $\mathcal{S}(f) = Q$. Therefore $\mathcal{S}$ is a bijection on $\mathbb{P}(L(D))$.

□

Going back to our example $X = \mathbb{C}$, we may observe that the spaces $L(D)$ are often infinite dimensional. For instance, consider the divisor $D = 1 \cdot 1$. For every $n \in \mathbb{Z}_{\geq 1}$ the function z^n is in $L(D)$. If we instead consider the corresponding meromorphic functions on $\mathbb{P}^1$ the situation is different. A quick calculation yields that $\mathrm{ord}_\infty(z^n) = -n$ and so this family of functions cannot all lie in any space of the form $L(m \cdot 1 + k \cdot \infty)$ for any $m, k \in \mathbb{Z}$. This difference is precisely because $\mathbb{P}^1$ is compact whereas $\mathbb{C}$ is not.

Proposition 3.14. Let X be a compact Riemann surface and let D a divisor on X, then $L(D)$ is finite dimensional. In particular, if we write $D = P - N$ where P, N are disjoint divisors such that $P, N \geq 0$, then $\dim(L(D)) \leq 1 + \deg(P)$.

Proof (Ch. 5, Lem. 3.15 & Prop. 3.16 in [Mir95]). Write $D = P - N$ as described. We proceed by doing induction on the degree of the positive part P. If $\deg(P) = 0$

then $D \leq 0$ and so $L(D) \subseteq \mathcal{O}(X) = \mathbb{C}$ since X is compact. Therefore $\dim(L(D)) \leq 1 + \deg(P)$ as required.

Suppose for induction that the inequality holds whenever $\deg P \leq k$. Let D be a divisor with positive part P such that $\deg(P) = k + 1$ and let $x \in X$ be any point such that $P(x) \geq 1$. Consider the divisor $D - x$, clearly this has positive part $P - x$ which has degree k, by applying the induction hypothesis we have that $\dim(L(D - x)) \leq 1 + \deg(P - x) = \deg(P)$.

Fix a coordinate chart (ϕ, U) centred at x and let $n = -D(x)$. Consider the the function $\mu : L(D) \to \mathbb{C}$ which is defined by sending $f \in L(D)$ to the coefficient c_n in the Laurent series expansion $f \circ \phi^{-1}(z) = \sum_{k \geq n} c_k z^k$ at the point $\phi(x) = 0$. μ is clearly a linear map with kernel $L(D - x)$. Since μ is a linear it is either the zero map in which case $L(D - x) = L(D)$, or it is surjective in which case $L(D - x)$ has codimension 1 in $L(D)$. In either case we have that since $L(D - x)$ is finite dimensional, $\dim(L(D)) \leq L(D - x) + 1$ and so $\dim(L(D)) \leq 1 + \deg(P)$ which completes the induction argument.

□

As of yet we do not know much about structure of the space $L(D)$ apart from the fact that it is finite dimensional. Even the fact that $\mathcal{M}(X)$ is non-empty is highly non-trivial. Proving that we can embed every compact Riemann surface into a projective space $\mathbb{P}^n$ will require facts about $L(D)$ which will come as a consequence of the Riemann-Roch theorem. Instead immediately proving this theorem, we will first investigate what exactly these conditions are. We begin with the following proposition which links collections of meromorphic functions with holomorphic maps into projective space.

Proposition 3.15. Let X be a Riemann surface and let $f_0, \ldots, f_n$ be a collection of meromorphic functions which are not all identically zero. There is a holomorphic map $F : X \to \mathbb{P}^n$ such that if $x \in X$ is not a pole of any f_i and not a zero of every f_i then

$$F(x) = [f_0(x) : \cdots : f_n(x)]$$

Furthermore, if $g_0, \ldots, g_n$ is another collection of meromorphic functions which posses this property, i.e.

$$F(x) = [g_0(x) : \cdots : g_n(x)]$$

whenever defined, then there is a meromorphic function λ such that $f_i = \lambda g_i$ for every i.

Proof (Ch. 5, Lem. 4.2 & Prop. 4.3 in [Mir95]). Let $f_0, \ldots f_n$ be as described, let $x \in X$ and let (ϕ, U) be a chart of X centred at x. Without loss of generality we may shrink U such that every $f_i|_U$ can only possibly have a zero or pole at x. Let $k = \min_i\{\mathrm{ord}_x(f_i)\}$ and consider the map $\tilde{F} : \phi(U) \to \mathbb{C}^{n+1}$ defined as $\tilde{F}(z) = (\tilde{f}_0(z), \ldots, \tilde{f}_n(z))$, where $\tilde{f}_i(z)$ is the unique holomorphic function such that $f_i \circ \phi^{-1}(z) = z^k \tilde{f}_i(z)$. Note that the functions $\tilde{f}_i(z)$ are not all identically zero at any point in $\phi(U)$. We may therefore define the map function $F := \pi \circ \tilde{F} \circ \phi : U \to \mathbb{P}^n$, where $\pi : \mathbb{C}^{n+1} \to \mathbb{P}^n$ is the quotient map. This map is clearly holomorphic and if $y \in U - \{x\}$ then

$$F(y) = [\tilde{f}_0(\phi(y)) : \cdots : \tilde{f}_n(\phi(y))] = [\phi(y)^{-k} f_0(y) : \cdots : \phi(y)^{-k} f_n(y)]$$
$$= [f_0(y) : \cdots : f_n(y)]$$

By repeating this process for every point $x \in X$ we may extend F to a holomorphic map $F : X \to \mathbb{P}^n$ which has the desired property.

Suppose as in the proposition that $g_0, \ldots g_n$ is another collection of meromorphic functions such that $F(x) = [g_0(x) : \cdots : g_n(x)]$ whenever defined. If any of the functions f_i are identically zero then g_i must also be. In this case any meromorphic function λ has the property that $f_i = \lambda g_i$. For every point $x \in X$ that is not a zero or pole of any of the f_i or g_i (excluding those which are identically zero), we have that

$$[f_0(x) : \cdots : f_n(x)] = [g_0(x) : \cdots : g_n(x)]$$

In particular there is a $\lambda_x \in \mathbb{C} - \{0\}$ such that $f_i(x) = \lambda_x g_i(x)$ for every $i = 0, \ldots, n$. The function $\lambda(x) = \lambda_x$ is clearly holomorphic as it is the ratio of non-vanishing holomorphic functions. Furthermore, as it is a ratio of meromorphic functions on X it is meromorphic as a function $\lambda : X \to \mathbb{C}$. We therefore have a meromorphic function λ such that $f_i = \lambda g_i$ for every $i = 0, \ldots, n$ as required.

□

If $F : X \to \mathbb{P}^n$ is a holomorphic map and $f = (f_0, \ldots, f_n)$ is a collection of meromorphic functions which together satisfy the conditions of [3.15], we denote this by writing $F = \phi_f$.

Corollary 3.16. For every holomorphic map $F : X \to \mathbb{P}^n$, there is a collection of meromorphic functions $f = (f_0, \ldots, f_n)$ such that $F = \phi_f$.

Proof. Let $F : X \to \mathbb{P}^n$ be a holomorphic map and let (ϕ_i, A_i) be the charts of $\mathbb{P}^n$. Let k be such that $F(X) \cap A_k \neq \emptyset$, i.e. $F(X) \not\subset \{z_k = 0\} \subset \mathbb{P}^n$. Consider the function $\pi_i : \mathbb{P}^n \to \mathbb{C} \cup \{\infty\}$ defined by sending the point $[z_0 : \cdots : z_n] \mapsto z_i / z_k$ and let $f_i := \pi_i \circ F : X \to \mathbb{C} \cup \{\infty\}$. Let $x \in X$. Since F is a holomorphic map, there exists a chart (ψ, U) of x such that with respect to these coordinates we may write $F \circ \psi^{-1}(z) = [g_0(z) : \cdots : g_n(z)]$ where the $g_i : \psi(U) \to \mathbb{C}$ are holomorphic. We therefore have that $f_i \circ \psi^{-1}(z) = g_i(z)/g_k(z)$ is meromorphic in $\psi(U)$ as it is the ratio of holomorphic functions. It follows that the functions f_i are all meromorphic on X. Finally, if we follow the procedure outlined in [3.15] with the meromorphic functions $f = (f_0, \ldots, 1, \ldots, f_n)$ we observe that $F = \phi_f$ as required.

□

Since every holomorphic map $F : X \to \mathbb{P}^n$ is described completely by a tuple of meromorphic functions, it is natural to expect that properties of F will correspond to properties of these meromorphic functions. Suppose that the meromorphic functions $f_0, \ldots, f_n$ define the holomorphic map F and suppose for simplicity that the image $F(X)$ is not contained in a hyperplane $H \subset \mathbb{P}^n$. This property is equivalent to the functions $f_0, \ldots, f_n \in \mathcal{M}(X)$ being linearly independent. This is because any linear dependence $f_0 = \sum_{i=1}^n a_i f_i$ implies that $F(X)$ is contained in the hyperplane $z_0 - \sum_{i=1}^n a_i z_i = 0$ and vice versa. Consider the divisor D defined by $D(x) = -\min_i\{\mathrm{ord}_x(f_i)\}$, clearly we have that $f_i \in L(D)$ and therefore we may more appropriately think of $V_f := \langle f_0, \dots, f_n \rangle$ as a $n+1$ dimensional subspace of $L(D)$.

We can connect $F : X \to \mathbb{P}^n$ directly to the group of divisors by utilising the bijection $\mathcal{S} : \mathbb{P}(L(D)) \to |D|$. Since $V_f \subseteq L(D)$ is a subspace, $\mathbb{P}(V_f)$ can be thought of as a 'linear subspace' of $\mathbb{P}(L(D))$. Using the map $\mathcal{S}$ we can consider the set $S(\mathbb{P}(V_f)) \subseteq |D|$. Unwinding this definition we find that

$$S(\mathbb{P}(V_f)) = \{\mathrm{div}(f) + D : f \in V_f\}$$

Definition 3.17. Let $F : X \to \mathbb{P}^n$ be a holomorphic map from a compact Riemann surface X such that the image $F(X)$ is not contained in any hyperplane of $\mathbb{P}^n$. Suppose that $F = \phi_f$ where $f = (f_0, \dots, f_n)$ is a collection of meromorphic functions. We define the *linear system* of the map F as $|F| = S(\mathbb{P}(V_f))$.

Generally, if V is a finite dimensional vector space and $W \subseteq V$ is a subspace, then we call sets of the form $\mathbb{P}(W) \subseteq \mathbb{P}(V)$ *linear subspaces* of $\mathbb{P}(V)$. For a complete linear system $|D|$, we call a subset of $|D|$ a *general linear system* if it corresponds to a linear subspace of $\mathbb{P}(L(D))$ through the map $\mathcal{S}$. If $Q \subseteq |D|$ is a general linear system and $W \subseteq L(D)$ is the vector space such that $\mathcal{S}(\mathbb{P}(W)) = Q$, then we define the *dimension* of Q as $\dim(W) - 1$.

As we have seen, holomorphic maps of the form $F : X \to \mathbb{P}^n$ may be expressed as ϕ_f for different collections of functions $f = (f_0, \dots, f_n)$. Suppose $g = (g_0, \dots, g_n)$ is another collection and let λ be the meromorphic function such that $f_i = \lambda g_i$. This does not affect the linear system $|F|$. If D' is the divisor corresponding to $g = (g_0, \dots, g_n)$ then we observe that $D = D' + \mathrm{div}(\lambda)$ and so $D \sim D'$. Since $\mathrm{div}(f_i) + D = \mathrm{div}(g_i) + \mathrm{div}(\lambda) + D = \mathrm{div}(g_i) + D'$, by extending $\mathbb{Z}$-linearly we have equality of sets

$$\{\mathrm{div}(f) + D : f \in V_f\} = \{\mathrm{div}(g) + D' : g \in V_g\}$$

If the image of a holomorphic map $F : X \to \mathbb{P}^n$ is not contained in any hyperplane of $\mathbb{P}^n$, it follows that the associated general linear system $|F|$ has dimension n. Furthermore, since X is compact, $\deg(\mathrm{div}(f)) = 0$ for every meromorphic function f and therefore every element of $|F|$ has degree $d = \mathrm{Deg}(D)$. For simplicity, if a general linear system Q has these two properties we will call it a 'g_d^n'. We are now in a position to state and prove the first theorem connecting general linear systems and holomorphic maps into projective space.

Theorem 3.18. Let X be a compact Riemann surface. There is a 1-1 correspondence between holomorphic maps $F : X \to \mathbb{P}^n$ (up to 'linear change of coordinates') whose image is not contained in any hyperplane of $\mathbb{P}^n$, and general linear systems g_d^n which have the property that for every point $x \in X$ there is a divisor $C \in g_d^n$ such that $C(x) = 0$.

Proof (Ch. 5, Lem. 4.6 & Prop. 4.15 in [Mir95]). Given a such a holomorphic map $F : X \to \mathbb{P}^n$, we have already described the process in which we obtain the divisor D and the general linear system $|F| \subseteq |D|$. Let $F = [f_0 : \cdots : f_n]$ and let $V = \langle f_0, \ldots, f_n \rangle \subseteq L(D)$ be the vector space such that $\mathcal{S}(\mathbb{P}(V)) = |F|$. Let $x \in X$ be any point and let f_k be the meromorphic function which has minimal order at x among the f_i. Recall that $D(x) = -\min_i\{\mathrm{ord}_x(f_i)\}$ and therefore $(\mathrm{div}(f_k) + D)(x) = 0$. Clearly $f_k \in V$ and so $C := \mathrm{div}(f_i) + D \in |F|$. Since $x \in X$ was arbitrary, the general linear system $|F|$ has the desired property.

Suppose that $G = [g_0 : \cdots : g_n]$ is another holomorphic map such that $|G| = |F|$. Let D', V' be such that $V' \subseteq L(D')$ and $\mathcal{S}(\mathbb{P}(V')) = |G|$. Since $|G| = |F|$ we may choose a basis $\{g_0', \ldots g_n'\}$ for V' such that $\mathrm{div}(f_i) + D = \mathrm{div}(g_i') + D'$. The meromorphic functions $\lambda_i = f_i/g_i'$ all have the property that $\mathrm{div}(\lambda_i) = D' - D$, in particular $\mathrm{div}(\lambda_i) = \mathrm{div}(\lambda_j)$ and so $\mathrm{div}(\lambda_i/\lambda_j) = 0$ which implies that $\lambda_i = \lambda_j$ for all i, j since X is compact. By appropriately rescaling $\{g_0', \ldots g_n'\}$ to a new basis $\{g_0^*, \ldots g_n^*\}$ for V', there is therefore a meromorphic function λ such that $f_i = \lambda \cdot g_i^*$ for all i. Let $A : V' \to V'$ be the linear change of basis sending $g_i \mapsto g_i^*$. This descends to a well defined linear change of coordinates $A_{\mathbb{P}} : \mathbb{P}^n \xrightarrow{\sim} \mathbb{P}^n$ such that $A_{\mathbb{P}} \circ G = [g_0^* : \cdots : g_n^*] = [\lambda \cdot g_0^* : \cdots : \lambda \cdot g_n^*] = F$,

Conversely, let D be a divisor and let $Q \subseteq |D|$ be a g^n_d satisfying the conditions of the theorem. By definition there is n+1-dimensional subspace $V \subseteq L(D)$ such that $\mathcal{S}(\mathbb{P}(V)) = C$. Let $\{g_0, \ldots, g_n\}$ be a basis for V and consider the function $\phi_g : X \to \mathbb{P}^n$, i.e. $\phi_g(x) = [g_0(x) : \cdots : g_n(x)]$. Let $D' = -\min_i\{\mathrm{ord}(g_i)\}$. A general element of $|\phi_g|$ and Q have the forms $\mathrm{div}(g) + D'$ and $\mathrm{div}(g) + D$ respectively where $g \in V$. We argue now that $D = D'$. Let $x \in X$ be a point and let $f \in V$ be a meromorphic function which has minimal order at x. As $\mathrm{div}(a_i g_i + a_j g_j)(x) = \min\{\mathrm{ord}_x(g_i), \mathrm{ord}_x(g_j)\}$ we have that w.l.o.g. we may assume that $f = g_i$ for some i. It follows that $(\mathrm{div}(f) + D)(x) = 0$ as otherwise there would be no divisor $C \in Q$ such that $C(x) = 0$. Since we also know that $(\mathrm{div}(f) + D')(x) = 0$ we therefore have shown that $D(x) = D'(x)$. Since x was arbitrary, we have shown that $D' = D$ and therefore that $Q = |\phi_g|$ as required.

□

Suppose from now on that X is a compact Riemann surface. If we have a general linear system $Q \subseteq |D|$ for some divisor $D \in \mathrm{Div}(X)$ then every divisor $C \in |D|$ has the same degree as $C \sim D$ by definition. Furthermore since X is compact, $L(D)$ is finite dimensional. It follows that every general linear system is a g^n_d. In order to apply [3.18] to Q, we need Q to also have property that for every point $x \in X$ there is a divisor $C \in Q$ such that $C(x) = 0$. We call this property *freeness* of the general linear system Q. To construct a map into projective space, it is useful to narrow our focus to complete linear systems which are *free*. In fact, the following proposition allows us construct a map from any divisor.

Proposition 3.19. Let D be a divisor. There is a divisor $D' \leq D$ such that $|D'|$ is free and $L(D') = L(D)$.

Proof (Ch. 5, Lem. 4.6 in [Mir95]). By definition of $|D|$, every $C \in |D|$ is such that $C \geq 0$. Let F be the largest non-negative divisor such that $F \leq C$ for all $C \in |D|$. Note that if $|D|$ is free then $F = 0$. The divisor $D' = D - F$ obviously satisfies $D' \leq D$ and furthermore it is clear that the complete linear system $|D'|$ is free. $D' \leq D$ also implies that $L(D') \subseteq L(D)$. Conversely, if $f \in L(D)$ then $\mathrm{div}(f) + D \in$

$|D|$. Since $F \in |D|$ is minimal we have that $\mathrm{div}(f) + D \geq F$ which implies that $\mathrm{div}(f) + (D - F) \geq 0$ and therefore that $\mathrm{div}(f) + D' \geq 0$. This is exactly the condition for $\mathrm{div}(f) + D' \in |D|$. As $\mathcal{S} : \mathbb{P}(L(D')) \to |D'|$ is a 1-1 map it follows that $f \in L(D')$ and therefore that $L(D) = L(D')$ as required.

□

For divisor D, by replacing it by $D - F$ we can still have access to [3.18] without changing the underlying space of meromorphic functions $L(D)$. For notational simplicity, if D is any divisor then we denote any map defined as $[f_0 : \cdots : f_n]$ where $f_0, \ldots, f_n \in L(D)$ is a basis for $L(D)$, by ϕ_D. As we have seen, maps of the form ϕ_D are unique up to linear change of coordinates and if F is as above we have that $|D - F| = |\phi_D|$.

Given a holomorphic map $F : X \to \mathbb{P}^n$, as in Chapter 1 we say that F is an embedding if $F(X) \subseteq \mathbb{P}^n$ is a complex sub-manifold and F is a biholomorphism onto its image $F(X)$. In this case we can distil this into two properties of the map F.

1. $F : X \to \mathbb{P}^n$ is a bijection into its image.
2. For every point $x \in X$ there are charts (ψ, U) and (ϕ_i, A_i) of the points x and $F(x)$ such that the map $\phi_i \circ F \circ \psi^{-1}$ has non-vanishing Jacobian matrix at $\psi^{-1}(x)$.

The key point is that if the map $\phi_i \circ F \circ \psi^{-1}$ has non-zero derivative at $\psi^{-1}(x)$ then the implicit function theorem guarantees that there is a biholomorphic change of coordinates which gives a chart $\{\beta, V\}$ for $F(x)$ such that $\beta(F(X) \cap V) = \beta(V) \cap \mathbb{C} \subseteq \mathbb{C}^n$. Furthermore, the inverse function theorem implies that if this property is true for every point $x \in X$ then F is a biholomorphism onto its image. Given a map of the form $F = \phi_D$, we can translate (1) and (2) into conditions on the space $L(D)$.

Proposition 3.20. Let D be a divisor on X such that $|D|$ is free and of dimension $n > 0$. Every map of the form $\phi_D : X \to \mathbb{P}^n$ satisfies property (1) if and only if for every pair of distinct points x, $y \in X$, $\dim(L(D - x - y)) = \dim(L(D)) - 2$.

Proof (Ch. 5, Lem. 4.17 & Prop. 4.18 in [Mir95]). Let D be as in the proposition and fix a point $x \in X$. Since $|D|$ is free, there is a function $f \in L(D)$ such that $\mathrm{ord}_x(f) = -D(x)$, in particular $L(D - x) \subseteq L(D)$ has codimension 1 by the same reasoning as in the proof of [3.14]. Let $\{f_1, \ldots, f_n\}$ be a basis for $L(D - x)$ and extend this to a basis $\{f_0, f_1, \ldots, f_n\}$ for $L(D)$. Let $F : X \to \mathbb{P}^n$ be the holomorphic map defined by $F(z) = [f_0(z) : \cdots : f_n(z)]$. Since $\mathrm{ord}_x(f_0) = -D(x)$ and $\mathrm{ord}_x(f_i) > -D(x)$ for $i \geq 1$, by the process outlined in [3.15] we can determine that $F(x) = [1 : 0 \cdots : 0]$. Since every map of the form ϕ_D differs by a linear change of coordinates, it is sufficient to prove that if $y \in X - \{x\}$ then $F(x) = F(y)$ if and only if $\dim(L(D - x - y)) \neq \dim(L(D)) - 2$.

.

Let $y \in X - \{x\}$ and suppose that $F(x) = F(y)$, i.e. $F(y) = [1 : 0 \cdots : 0]$. This occurs if and only if $\mathrm{ord}_y(f_0) > \mathrm{ord}_y(f_i)$ for every $i \geq 1$. Since $\mathrm{ord}_y(f_0) = -D(y)$, this is equivalent to $\{f_1, \ldots, f_n\}$ being a basis for $L(D - y)$ and therefore that

$L(D-y) = L(D-x)$. Finally we observe that $L(D-x) = L(D-y)$ if and only if $L(D-x-y) = L(D-x) \subseteq L(D)$ if and only if $\dim(L(D-x-y)) = \dim(L(D)) - 1$. By [3.14] we know that either $L(D-x-y)$ either has codimension 1 or 2 in $L(D)$ and so $\dim(L(D-x-y)) = \dim(L(D)) - 1$ if and only if $\dim(L(D-x-y)) \neq \dim(L(D)) - 2$, completing the proof.

□

Proposition 3.21. Let D be a divisor on X such that $|D|$ is free and of dimension $n > 0$. Every holomorphic map of the form $\phi_D : X \to \mathbb{P}^n$ which satisfies property (1) also satisfies (2) if and only if for every point $L(D-2x) \neq L(D-x)$ for every point $x \in X$.

Proof (Ch. 5, Lem. 4.19 in [Mir95]). Let D be as in the proposition. A linear change of coordinates is a biholomorphism $\mathbb{P}^n \to \mathbb{P}^n$ and so it suffices to prove the proposition for a singe holomorphic map of the form ϕ_D. Let $\{f_0, \dots, f_n\}$ be a basis for $L(D)$ and let $F = [f_0 : \cdots : f_n]$. Fix a point $x \in X$. $|D|$ is free and so we know that there is a f_i satisfying $\mathrm{ord}_x(f_i) = -D(x)$. Let (ψ, U) be a chart of X centred at x and recall that (ϕ_i, A_i) is that chart of $\mathbb{P}^n$ sending $[z_0 : \cdots : z_n] \mapsto (z_0/z_i, \dots, z_{i-1}/z_i, z_{i+1}/z_i, \dots, z_n/z_i)$. By sufficiently shrinking U and using [3.15], we have that

$$F \circ \psi^{-1} : z \mapsto [z^{D(x)}(f_0 \circ \psi^{-1})(z) : \cdots : z^{D(x)}(f_n \circ \psi^{-1})(z)]$$

If we write $g_j(z) = (f_j \circ \psi^{-1}(z))/(f_i \circ \psi^{-1}(z))$ then

$$\phi_i \circ F \circ \psi^{-1} : z \mapsto (g_0(z), \dots, g_{i-1}(z), g_{i+1}(z), \dots, g_n(z))$$

Let $m_j = \mathrm{ord}_0(g_j) = \mathrm{ord}_x(f_i) - \mathrm{ord}_x(f_j)$. It we see that

$$\phi_i \circ F \circ \psi^{-1}(z) = (c_0 z^{m_0} + \dots, \dots, c_n z^{m_n} + \dots)$$

and therefore the Jacobian matrix of $\phi_i \circ F \circ \psi^{-1}(z)$ at 0 doesn't vanish if and only if $m_j = 1$ for atleast one j. This happens if and only if there is a function f_j such that $\mathrm{ord}_x(f_j) = -D(x) + 1$ which as $|D|$ is free, occurs if and only if there is a function $f \in L(D-x)$ such that $\mathrm{ord}_x(f) = -D(x) + 1$. Finally, by Prop [3.14] this is equivalent to $L(D-2x) \neq L(D-x)$.

□

By combining the results of this section, we have proved the following theorem.

Theorem 3.22. Let X be a compact Riemann surface and let D be a divisor such that $|D|$ is free and has dimension $n > 0$. If for every pair of (not necessarily distinct) points $x, y \in X$ we have that $\dim(L(D-x-y)) = \dim(L(D)) - 2$ then any map of the form $F = \phi_D$ is an embedding of X as a complex submanifold of $\mathbb{P}^n$.

3.3 Laurent Tails and $H^1(D)$

Before we continue, one issue we need to comment on is this existence of meromorphic functions. As we have mentioned before, proving that $\mathcal{M}(X)$ is non-empty for a general compact Riemann surface X is a non-trivial but old theorem of Riemann. To proceed we will assume the following version of the statement. Proofs of the following statement can be found at Ch. 9, Thm. 6.1 in [NN01] or in [Har15].

Theorem 3.23 (Riemann Existence Theorem)**.** Let X be a compact Riemann surface. For every point $x \in X$ there exists a meromorphic function f which has a simple pole at x (i.e. $\mathrm{ord}_x(f) = -1$). Furthermore, for every pair of distinct points $x, y \in X$ there exists a meromorphic function g such that $f(x) \neq g(x)$.

In essence, the Riemann-Roch theorem is a statement about the dimension of spaces of the form $L(D)$. The existence of a divisor D such that $\dim(L(D - x - y)) = \dim(L(D)) - 2$ will follow as a consequence. Our strategy going forward split up into two parts. Firstly, in this section, we will develop our understanding of $L(D)$ using a concept called Laurent tail divisors.

Definition 3.24. A Laurent polynomial of the form $r(z) = \sum_{i=n}^{m} c_i z^i$ is called a *Laurent tail* of a Laurent series f if the Laurent series $f - r$ only has terms of degree larger than z^m, i.e.

$$f(z) = \sum_{i=n}^{m} c_i z^i + \sum_{j>m} c_j z^j$$

Laurent tails provide more information about the behaviour of a Laurent series than just the term of lowest degree. As such, they allow us to give a more precise approximation of a Laurent series than just the order alone. Using this concept, we can extend the Riemann Existence Theorem in the following way.

Theorem 3.25 (Laurent Approximation Theorem)**.** Let X be a compact Riemann surface. Let $x_1, \ldots, x_m \in X$ be distinct points and for each $i = 1, \ldots m$ let (ϕ_i, U_i) be a chart of X centred at x_i. For any collection of Laurent tails $r_1, \ldots, r_m$ there exists a meromorphic function f such that the Laurent series of $f \circ \phi_i^{-1}$ at 0 has r_i as a Laurent tail.

Proof (Ch. 6, Lem. 1.15 in [Mir95]). We prove this in three steps. First, we prove the theorem for a single point $x \in X$ with centred chart (ϕ, U). Suppose r is a Laurent tail of the form $r(z) = c_n z^n$ where $n \in \mathbb{Z}$. By [3.23] we know there is a function f such that $f \circ \phi^{-1}$ has Laurent series $f \circ \phi^{-1}(z) = az^{-1} + \sum_{i \geq 0} a_i z^i$. Clearly the function $c_n(f/a)^{-n}$ is a meromorphic function with r as a Laurent tail when composed with ϕ^{-1}. We continue by induction on the number of terms in the Laurent tail: Suppose that whenever $r(z) = \sum_{i=n}^{m} c_i z^i$ is such that $n - m = k$ then there exists a meromorphic function f such that $f \circ \phi^{-1}$ has r has a Laurent tail. Now suppose that $n - m = k + 1$. Applying the induction hypothesis to $r_1(z) = c^n z^n$ and we have a meromorphic function f_1 such that $f_1 \circ \phi^{-1}$ has Laurent series expansion $f_1 \circ \phi^{-1}(z) = c_n z^n + \sum_{j>n} b_j z^j$. Let $r_2(z) = \sum_{i=n+1}^{m} (b_i - c_i) z^i$ then we can again apply to induction hypothesis to get a meromorphic function f_2 with Laurent expansion $f_2 \circ \phi^{-1}(z) = \sum_{i=n+1}^{m} (b_i - c_i) z^i + \sum_{j>m} a_j z^j$. In particular we have that $(f_1 - f_2) \circ \phi^{-1}$ has Laurent expansion $(f_1 - f_2) \circ \phi^{-1}(z) = \sum_{i=n}^{m} c_i z^i + \ldots$ and therefore has $r(z)$ as a Laurent tail. This completes the induction argument.

The next step is to prove that for any collection of distinct points $x_0, \dots, x_m \in X$ there is a meromorphic function with a zero at x_0 and a pole at $x_1, \dots, x_m$. Given just two points x_0, x_1, let f be a meromorphic function such that $f(x_0) \neq f(x_1)$. Without loss of generality we may assume that x_0 is a zero of f by possibly replacing f by $1/f$ or $f - f(x_0)$. Since $f(x_0) \neq f(x_1)$, the meromorphic function $f/(f - f(x_1))$ has a zero at x_0 and a pole at x_1 as required. Assume for induction that the statement holds whenever $m = k$. Let $x_0, \dots x_{k+1}$ be any collection of distinct points, using the induction hypothesis there is a function f which has a zero at x_0 and a pole at $x_1, \dots x_k$. Also, as we have shown, there is a function g which has a zero at x_0 and a pole at x_{k+1}. The function $h = f + g$ has a zero at x_0, by replacing g with g^N for some sufficiently large N we can also insure that h has a pole at $x_1, \dots x_{k+1}$ as required, competing the induction argument.

We now combine these results to prove the proposition. Let $x_1, \dots, x_m \in X$ be distinct point with corresponding centred charts (ϕ_i, U_i), and let $r_1, \dots, r_m$ be any collection of Laurent tails. Write $r_i(z) = \sum_{j=k_i}^{l_i} c_{i,j} z^j$ and let N, M be integers such that $N > \max_i\{l_i\}$ and $M = \min_i\{k_i\}$. By adding sufficient terms with zero coefficient, consider w_i as the Laurent tail of r_i with terms up to degree N, i.e. $w_i(z) = r_i(z) + \sum_{j>l_i}^{N} 0 \cdot z^j$. Let $f_1, \dots, f_m$ be meromorphic functions such that $f_i \circ \psi^{-1}$ has w_i as a Laurent tail at x_i. Let g_i be the function which has a zero at x_i and a pole at x_j for every $j \neq i$. By replacing g_i with $1/(1 + g_i^{N-M})$ we can assume that $\operatorname{ord}_{x_i}(g_i) = 0$ and $\operatorname{ord}_{x_j}(g_i) \geq N - M$ whenever $j \neq i$. Consider the function

$$h = \sum_{j=1}^{m} f_j \cdot g_j$$

For any point x_i, we calculate that $(f_i \cdot g_i) \circ \phi_i^{-1}$ and $(f_j \cdot g_j) \circ \phi_i^{-1}$ have Laurent expansions

$$(f_i \cdot g_i) \circ \phi_i^{-1}(z) = w_i(z) + \mathcal{O}(z^N) \quad \text{and} \quad (f_j \cdot g_j) \circ \phi_i^{-1}(z) = \mathcal{O}(z^N)$$

whenever $j \neq i$. Therefore h has w_i, and thus r_i, as a Laurent tail at x_i. Since this holds for every $i = 1, \dots m$ and h is meromorphic, it satisfies the conditions of the proposition.

$\square$

This theorem is called Laurent *approximation* for good reason. While it guarantees the existence of a meromorphic function with specified behaviours at certain points, it does not say anything about how this these functions behave outside of them. Crucially, it does not allow us to construct meromorphic functions which are strictly holomorphic outside of a collection prescribed zeros and poles. This theorem therefore says nothing about $L(D)$. We now define divisors of Laurent tails in order to state this investigate this issue further.

From now on we will fix a compact Riemann surface X and centred coordinate charts (ϕ_x, U_x) for every point $x \in X$. For simplicity, if we say that a meromorphic function f has Laurent series $\sum_{i \geq n} c_i z^i$ at x, rigorously this should be interpreted as $f \circ \phi_x^{-1}$ having Laurent series $f \circ \phi_x^{-1}(z) = \sum_{i \geq n} c_i z^i$ at 0.

Definition 3.26. Let T be the group Laurent polynomials of the form $r(z) = \sum_{i=n}^{m} c_i z^i$. The group of *Laurent tail divisors*, denoted $T(X)$ is defined as the direct sum of abelian groups $T(X) := T^X$. Given a divisor D we also define the truncation homomorphism $t_D : T(X) \to T(X)$ which is defined by sending

$$t_D(z^n \cdot x) = \begin{cases} 0, & \text{if } n \geq -D(x) \\ z^n \cdot x, & \text{otherwise} \end{cases}$$

and extending $\mathbb{C}$-linearly. We denote the image of t_D by $T(X)[D]$ and if we are given another divisors D' such that $D' \leq D$ then we can extend t_D to a map $t_D^{D'} =: t_D|_{T(X)[D']}$.

Definition 3.27. Fix a divisor D. Define the map $\alpha_D : \mathcal{M}(X) \to T(X)[D]$ which sends a meromorphic function f to the Laurent tail divisor $\sum_{x \in X} r_x(z) \cdot x$ where $r_x(z)$ is the Laurent series of f up to (but not including) the $-D(x)$'th term.

The map α_D is well defined as D is non-zero at only a finite number of points. At all other points f has only a finite number of poles and so $\sum_{x \in X} r_x(z) \cdot x$ is a finite sum for any meromorphic function f and any divisor D.

As an initial observation, by unwinding this definition we see that $\ker(\alpha_D) = L(D)$. This is because if $\alpha_D(f) = 0$ then for any point $x \in X$, the Laurent series of f at x has no terms of degree greater than $-D(x)$ and so in particular $f \in L(D)$. Furthermore, a Laurent tail divisor $R = \sum_i r_i(z) \cdot x_i$ is in the image of α_D if and only if there is a function f which has Laurent tails $r_i(z)$ at x_i and is holomorphic everywhere else. Combining these results, we have an exact sequence of vector spaces

$$0 \to L(D) \to \mathcal{M}(X) \xrightarrow{\alpha_D} T(X)[D] \to H^1(D) \to 0$$

where $H^1(D)$ is defined as the cokernel of α_D, $H^1(D) := T(X)[D]/\mathrm{im}\alpha_D$. $H^1(D)$ measures the degree to which we cannot find meromorphic functions with specified Laurent tails (up to terms of degree $-D(x_i)$) at a collection of points $x_1, \ldots, x_m$. Although not immediately obvious, $H^1(D)$ is directly related to $L(D)$ as showcased in the following result.

Theorem 3.28. Let D be a divisor on a compact Riemann surface X, then

$$\dim(L(D)) - \dim(H^1(D)) = \deg(D) + 1 - \dim(H^1(0))$$

Proof (p. 181-182 & p. 185-186 in [Mir95]. The key result needed to prove this is that $H^1(D)$ is a finite dimensional space. Let us first prove the result assuming this fact. Consider that for every pair of divisors $D_1 \leq D_2$ we have the following digram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{M}(X)/L(D_1) \arrow[r, "\alpha_{D_1}"] \arrow[d, "\pi"] & T(X)[D_1] \arrow[r] \arrow[d, "t"] & H^1(D_1) \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{M}(X)/L(D_2) \arrow[r, "\alpha_{D_2}"] & T(X)[D_2] \arrow[r] & H^1(D_2) \arrow[r] & 0 \end{tikzcd}$$

where t denotes the map $t_{D_2}^{D_1}$. Since the left square commutes and the rows are exact, we can complete this to a commutative diagram by adding a map

$l : H^1(D_1) \to H^1(D_2)$ defined by sending $R + \mathrm{im}\alpha_{D_1} \mapsto t^{D_1}_{D_2}(R) + \mathrm{im}\alpha_{D_2}$. Noting that the map π is surjective as $L(D_1) \subseteq L(D_2)$, we can use the snake lemma to obtain a short exact sequence

$$0 \to \ker(\pi) \to \ker(t) \to \ker(l) \to 0$$

Analysis of the map t reveals that $\sum_{x\in X} r_x(z) \cdot x \in \ker(t)$ if and only if $r_x(z)$ only has terms of degree between $-D_2(x)$ and $-D_1(x)$. It follows that $\ker(t)$ is spanned by the the set $\{z^k \cdot x : x \in X,\ -D_2(x) \le k \le -D_1(x)\}$. Therefore $\ker(t)$ has dimension $\deg(D_2) - \deg(D_1)$. The dimension of $\ker(\pi)$ is also easy to calculate as $\ker(\pi) \cong L(D_2)/L(D_2)$ and so it has dimension $\dim(L(D_2)) - \dim(L(D_1))$. Using the short exactly sequence, we compare dimensions to find that

$$\dim(\ker(l)) = \big(\deg(D_2) - \dim(L(D_2))\big) - \big(\deg(D_1) - \dim(L(D_1))\big)$$

As the map t is surjective, we observe that by design l is also. It follows that $0 \to \ker(t) \to H^1(D_1) \to H^1(D_2) \to 0$ is a short exact sequence. If we crucially assume that spaces of the form $H^1(D)$ are finite dimensional, again by comparing dimensions we have

$$\dim(\ker(l)) = \dim(H^1(D_1)) - \dim(H^1(D_2))$$

and combining these two results yields

$$\dim(H^1(D_1)) + \deg(D_1) - \dim(L(D_1)) = \dim(H^1(D_2)) + \deg(D_2) - \dim(L(D_2))$$

Now let D be any divisor and let $P \ge 0$ be any strictly positive divisor which greater than D. Since either side of the previous equation only relies on a single divisor, is it clear that we can replace D_1 with D and D_2 with 0 as $P \ge D_2$ and $P \ge 0$. Since $\deg(D_2) = 0$ and $\dim(L(0)) = 1$ it finally follows that

$$\dim(L(D)) - \dim(H^1(D)) = \deg(D) + 1 - \dim(H^1(0))$$

as required.

□

Proving that $H^1(D)$ is finite dimensional is a vital component of our proof of the Riemann-Roch theorem. The approach that Miranda takes in [Mir95] is to use [3.23] analyse the function field $\mathcal{M}(X)$. He demonstrates that $M(X)$ is a finitely generated field extension of $\mathbb{C}$ with transcendence degree 1 (see Ch. 6, Prop. 1.17 in [Mir95]). Furthermore, he shows that any set of meromorphic functions which include those described in [3.23] generate $\mathcal{M}(X)$ over $\mathbb{C}$ (see Ch. 6, Prop. 1.22 in [Mir95]). This observation, which fundamentally is a consequence of [3.23], completes the long dotted arrow in the figure at the end of Chapter 2. It also answers the question we posed at the end of Chapter 1: the meromorphic function field of a projective algebraic curve is exactly the rational function field coming from algebraic theory.

Due to this observation about $\mathcal{M}(X)$, it can be shown with a little effort that $H^1(D)$ is always finite dimensional (see p. 182-184 in [Mir95]). Going forward we will assume this result. This is not the only approach to the Riemann-Roch theorem. For example, in [For81] the author bypasses $H^1(D)$ (and Serre Duality) instead focusing a 'cohomology group' $H^1(X, \mathcal{O}_D)$ which measures exactly when there fails to global meromorphic functions with prescribed poles. These approaches turn out to be essentially same according to something called the GAGA principle, we will discuss this briefly in the next chapter.

3.4 Serre Duaility and the Riemann-Roch Theorem

Given that $H^1(D)$ is finite dimensional, the crux of the proof of the Riemann-Roch theorem is in finding an expression for its dimension. This will allow us to simplify the expression for $\dim(L(D))$ in [3.28]. The key result we will use is called the Serre Duality theorem. In particular, we will use the following corollary of it.

Corollary 3.29. Let X be a compact Riemann surface and let D be a divisor, then

$$\dim(H^1(D)) = \dim(L^1(-D))$$

Let us first see how this result gives us our desired form of the Riemann-Roch theorem. From now on we assume that X is a compact Riemann surface. We begin with some lemmas.

Lemma 3.30. Let K be a canonical divisor (i.e. $K = \operatorname{div}(\omega)$), then for any other divisor D we have $L^1(D) \cong L(K + D)$.

Proof (Ch. 5, Lem. 3.11 in [Mir95]. Let ω be the meromorphic function such that $\operatorname{div}(\omega) = K$. Define the function $\nu : L(K + D) \to L^1(D)$ defined by $\nu(f) = f\omega$. This is well defined as if $\operatorname{div}(f) \geq -D - K$ then $\operatorname{ord}_x(f\omega) = \operatorname{ord}_x(f) + \operatorname{ord}_x(\omega) \geq -D(x)$ and so $f\omega \in L^1(D)$. ν is injective since $f\omega = g\omega$ then $f = g$. For surjectivity, let $\eta \in L^1(D)$ then by [3.9] there is meromorphic function h such that $h\omega = \eta$. As $\operatorname{div}(h) + \operatorname{div}(\omega) = \operatorname{div}(\eta) \geq -D$ we have $\operatorname{div}(h) \geq -K - D$ and so $h \in L(K + D)$ as required.

□

Lemma 3.31. There exists a canonical divisor K of X such that $\operatorname{Deg}(K) = 2g - 2$.

Proof (Ch. 5, Prop. 1.14 & Ch. 4, Lem 2.6 in [Mir95]. Let f be any non-constant meromorphic function on X and let ω be the meromorphic 1-form on $\mathbb{P}^1$ defined as dz on A_0 and $-1/z^2 dz$ on A_2. This is a meromorphic 1-form as the transition function is $\phi_1 \circ \phi_0^{-1}(z) = 1/z$. By considering f as a holomorphic map $f : X \to \mathbb{P}^1$, if we fix a point $x \in X$ we may apply [1.15] to get a charts (ψ, U) (ϕ, V) centred at x, $f(x)$ respectively such that $\psi \circ F \circ \phi^{-1}(z) = z^k$ where $k = \operatorname{mult}_x(f)$. Let $\eta \in M^1(X)$ be the pullback $\eta = f^*(\omega)$. If $\omega = g(z)dz$ with respect to (ϕ, V) then $\eta = g(z^k) \cdot kz^{k-1}dz$ with respect to (ψ, U). It follows that $\operatorname{ord}_x(\eta) = \operatorname{ord}_{f(x)}(\omega) \cdot \operatorname{mult}_x(f) + \operatorname{mult}_x(f) - 1$.

Since ω has order -2 at $[0 : 1]$ and order 0 at every other point, it follows that $\operatorname{ord}_x(\eta) = -\operatorname{mult}_x(f) - 1$ whenever $f(x) = [0 : 1]$ and $\operatorname{ord}_x(\eta) = \operatorname{mult}_x(f) - 1$ otherwise. We therefore calculate that

$$\begin{aligned}\operatorname{Deg}(\operatorname{div}(\eta)) = \sum_{x\in X} \operatorname{ord}_x(\eta) &= \sum_{f(x)\neq[0:1]} (\operatorname{mult}_x(f)-1) + \sum_{f(x)=[0:1]} (-\operatorname{mult}_x(f)-1) \\ &= \sum_{x\in X} (\operatorname{mult}_x(f)-1) - 2\sum_{x\in f^{-1}([0:1])} \operatorname{mult}_x(f)\end{aligned}$$

By definition $\deg(f) = \sum_{x\in f^{-1}([0:1])} \operatorname{mult}_x(f)$ and by the Riemann-Hurwitz formula $\sum_{x\in X}(\operatorname{mult}_x(f)-1) = \chi(X) - \deg(f)\cdot\chi(\mathbb{P}^1)$. Since $\chi(\mathbb{P}^1) = 2$ it follows that $\operatorname{Deg}(\operatorname{div}(\eta)) = \chi(X) = 2g-2$ where X has genus g.

□

Assuming Corollary [3.29] we can prove our desired version of the Riemann-Roch Theorem.

Theorem 3.32 (Riemann-Roch Theorem)**.** Let X be a compact Riemann surface and let D be any divisor, then

$$\dim(L(D)) - \dim(L(D-K)) = \deg(D) + 1 - g$$

Proof (Ch. 6, Thm. 3.11 in [Mir95]). Firstly, for any canonical divisor K we have $\dim(H^1(K)) = \dim(L^1(-K)) = \dim(L(0)) = 1$ and $\dim(H^1(0)) = \dim(L^1(0)) = \dim(L(K))$ by the first lemma. Applying [3.28] to the canonical divisor $K = \operatorname{div}(\eta)$ from the second lemma yields

$$\dim(H^1(0)) - 1 = 2g - 2 + 1 - \dim(H^1(0))$$

and so we calculate $\dim(H^1(0)) = g$. Secondly, we notice that $\dim(H^1(D)) = \dim(L^1(-D)) = \dim(L(K-D))$ by the first lemma. Finally, applying [3.28] and combining these two results gives us

$$\dim(L(D)) - \dim(L(K-D)) = \deg(D) + 1 - g$$

as required.

□

The crux of proving the Serre Duality Theorem, and therefore the corollary that we need, lies in generalising the Residue theorem from complex analysis to integrals on compact Riemann surfaces. Recall that if $U \subseteq \mathbb{C}$ is an open set and $f : U \to \mathbb{C}$ is a meromorphic function, then we define the residue of f at $x \in U$ as $\operatorname{Res}_x(f) = c_{-1}$ where c_{-1} is the coefficient of the term $(z-x)^{-1}$ in the Laurent expansion of f in an annulus of the point x. The residue theorem then states the following

Theorem 3.33. Let $U \subseteq \mathbb{C}$ be an open subset and let $f : U \to \mathbb{C}$ be a meromorphic function with poles $\{a_0, \ldots, a_k\}$. If γ is a positively oriented simple closed curve in U which contains all poles of f in its interior, then

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_j \operatorname{Res}_{a_j}(f)$$

To generalise this to Riemann surfaces, instead of thinking about the residues of functions we should instead think about residues of forms. The reason we need to do this is because while the order of a meromorphic function is independent of coordinate chart, the coefficients of the Laurent series can differ. For 1-forms however the transformation rule for changing coordinates charts fixes this problem for the coefficient c_{-1}.

Definition 3.34. Let ω be a meromorphic 1-form on X. We define the residue of ω at a pole $x \in X$ as

$$\mathrm{Res}_x(\omega) = \frac{1}{2\pi i}\oint_\gamma \omega$$

where γ is a positively oriented simple closed curve in X such that x is the only pole of ω in its interior which is in turn contained in a single chart of X.

Proposition 3.35. Let ω be as above and let x be a pole of ω, then $\mathrm{Res}_x(\omega)$ is well defined. Let (ϕ, U) be any chart of X centred at x, and suppose that with respect to this chart $\omega = f(z)dz$. Then $\mathrm{Res}_x(\omega) = c_{-1}$, the coefficient of the z^{-1} term in the Laurent expansion of f at 0.

Proof (Ch. 4, Lem. 3.12 in [Mir95]). By working backwards, a simple application of Theorem [3.33] shows that $\mathrm{Res}_x(\omega) = c_{-1}$. Furthermore, it follows that $\mathrm{Res}_x(\omega)$ is well defined as the value of an integral is independent of the coordinate charts we use.

□

Theorem 3.36 (Residue Theorem)**.** Let X be a compact Riemann surface and let ω be a meromorphic 1-form on X. Suppose that $\{a_1, \dots, a_k\}$ are the poles (points with $\mathrm{ord}_x(\omega) < 0$) of ω, then

$$\sum_{j=1}^{k} \mathrm{Res}_{a_j}(\omega) = 0$$

Proof (Ch. 4, Thm. 3.17 in [Mir95]). We will provide an outline of the proof. Let ω and $\{a_1, \dots, a_k\}$ be as in the proposition. Let $\gamma_1, \dots, \gamma_k$ be a simple closed curves such that

$$\mathrm{Res}_{a_j}(\omega) = \frac{1}{2\pi i}\oint_{\gamma_j} \omega$$

We have that

$$\sum_j \mathrm{Res}_{a_j}(\omega) = \frac{-1}{2\pi i}\sum_j \oint_{-\gamma_j} \omega = \frac{-1}{2\pi i}\oint_{-\sum_j \gamma_j} \omega$$

Where $-\gamma_j$ denotes γ_j traversed backwards. By translating this expression into an integral on X as a compact smooth surface, we can apply stokes theorem for manifolds by considering that $\partial(X - \cup_j U_j) = -\sum_j \gamma_j$ where U_j is the interior of γ_j. In particular, if $\omega = f(z)dz$ with respect to a chart, then if we translate we can express this as $\omega = f(x,y)(dx + idy)$ where coordinate $z = x + iy$. It follows that

$$d\omega = (\partial_y f(x,y) - i\partial_x f(x,y))dx \wedge dy$$

This expression is therefore locally 0 whenever f is locally holomorphic by the Cauchy-Riemann equations. It follows that since ω is strictly holomorphic inside $X - \cup_j U_j$ we finally have that

$$\oint_{-\sum_j \gamma_j} \omega = \int_{X - \cup_j U_j} d\omega = 0$$

and therefore that $\sum_j \mathrm{Res}_{a_j}(\omega) = 0$.

$\square$

It is not immediately obvious how the residue theorem is linked to $H^1(D)$. Like in the last section, we fix compact Riemann surface X and centred coordinate chart for every point $x \in X$. To start, let D be a divisor and assume that ω is a meromorphic 1-form such that $\omega \in L^1(-D)$. If f is a meromorphic function then as we have demonstrated before, we may calculate the Laurent expansion of $f\omega$ at a point $x \in X$ as

$$f\omega = \big(\sum_j c_j z^j\big) \cdot \big(\sum_{k \geq D(x)} a_k z^k\big) dz$$

where the first term and second terms are the Laurent expansions of f and g at x ($\omega = g(z)dz$ at x). In particular, we calculate using Prop [3.35] that $\mathrm{Res}_x(f\omega) = \sum_{j \geq D(x)} c_{-j-1} \cdot a_j$. It follows that $\mathrm{Res}_x(f\omega)$ only depends on c_j for $i < -D(x)$ and therefore if $\alpha_D(f) = \sum_{x \in X} r_x(z) \cdot x$ then it only depends on the Laurent tail $r_x(z)$. The map $R_{x,\omega} : T(X)[D] \to \mathbb{C}$ defined as

$$R_{x,\omega} : \big(\sum_j c_j z^j\big) \cdot x + \sum_{y \neq x} r_y(z) \cdot y \mapsto \sum_{j \geq D(x)} c_{-j-1} \cdot a_j$$

therefore satisfies $\mathrm{Res}_x(f\omega) = R_{x,\omega} \circ \alpha_D(f)$. The residue theorem then motivates the following definition.

Definition 3.37. Given a divisor D and a meromorphic 1-form $\omega \in L^1(-D)$, we define the residue function $\mathrm{Res}_\omega : T(X)[D] \to \mathbb{C}$ as

$$\mathrm{Res}_\omega : L \mapsto \sum_{x \in X} R_{x,\omega}(L)$$

Proposition 3.38. Res_ω is a linear map and for every meromorphic function f we have that $\mathrm{Res}_\omega \circ \alpha_D(f) = 0$.

Proof. Firstly, Res_ω is linear as it is a sum of linear maps. Secondly, by unwinding the definition we see that $\mathrm{Res}_\omega \circ \alpha_D(f) = \sum_{x \in X} \mathrm{Res}_x(\omega)$ which is equal to 0 by the residue theorem.

$\square$

The major consequence of this proposition, the main work of which is done by the residue theorem, is that Res_ω descends to a linear functional on $H^1(D)$. We therefore have an association $L^1(-D) \to H^1(D)^*$. The statement of Serre Duality is that this association is an isomorphism. In plain language this says that any basis of the meromorphic 1-forms with poles bounded by $-D$ provide independent linear conditions for elements of $T(X)[D]$ to be in the image of α_D.

Theorem 3.39 (Serre Duality Theorem)**.** Let X be a compact Riemann surface and let D be a divisor on X. The function $\mathrm{Res} : L^1(-D) \to H^1(D)^*$ defined by $\mathrm{Res} : \omega \mapsto \mathrm{Res}_\omega$ is an isomorphism of vector spaces.

Proof (Ch. 6, Thm. 3.3 in [Mir95]). Injectivity is easy to prove. Let $\omega \in L^1(-D)$ be non-zero, then there is a point $x \in X$ such that ω has Laurent expansion $(\sum_{j\geq n} a_j z^j) \cdot dz$ with $a_n \neq 0$ and $n \geq D(x)$. Consider the Laurent tail divisor $z^{-n-1} \cdot x$, this is clearly in $T(X)[D] - \mathrm{im}(\alpha_D)$ and $\mathrm{Res}_\omega(z^{-n-1} \cdot x) = a_n \neq 0$. Therefore Res is injective as we have shown that if $\omega \neq 0$ then Res_ω is not the zero functional.

The proof of surjectivity is more involved. Fix $\phi \in H^1(D)^*$ and for simplicity we will consider ϕ as a linear functional $\phi : T(X)[D] \to \mathbb{C}$ which vanishes on $\mathrm{im}(\alpha_D)$ as this uniquely identifies ϕ. Let $\omega \in M^1(X)$ be a non zero meromorphic 1-form. Since every meromorphic 1-form takes the form $\eta = g\omega$ for some meromorphic function g, the problem reduces to finding such a g such that $\mathrm{Res}_{g\omega} = \phi$. Let $K = \mathrm{div}(\omega)$ and fix a divisor A such that $A \leq K, D$. By definition, since $A \leq K, D$ we may define $\phi_A = \phi \circ t_D^A : T(X)[A] \to \mathbb{C}$ and furthermore as as $L^1(-A) \subseteq L^1(-K)$ we can consider Res_ω as a functional $\mathrm{Res}_\omega : T(X)[A] \to \mathbb{C}$ which vanishes on $\mathrm{im}\alpha_A$. The maps, ϕ_A and Res_ω are now of the same type, to proceed we will use the following lemma.

Lemma 3.40. For any two linear functionals $\psi_1, \psi_2 \in H^1(A)^*$, there is a positive divisor C and meromorphic functions $f_1, f_2 \in L(C)$ such that

$$\psi_1 \circ t_A^{A-C-\mathrm{div}(f_1)} \circ \mu_{f_1} = \psi_2 \circ t_A^{A-C-\mathrm{div}(f_2)} \circ \mu_{f_2}$$

where μ_f is the unique map $\mu_f : T(X)[A - C] \to T(X)[A - C - \mathrm{div}(f)]$ which satisfies $\mu_f \circ \alpha_{A-C}(g) = \alpha_{A-C-\mathrm{div}(f)}(fg)$.

Proof (Ch. 6, Lem. 3.4 in [Mir95]). The map μ_f simply sends a Laurent tail divisor $r(z) \cdot x$ to the Laurent tail divisor $((\sum_{j\geq k} c_i z^i) \cdot r(z)) \cdot x$ where $\sum_{j\geq k} c_i z^i$ is the Laurent expansion of f at x, this map is linear and well defined as the image is surely contained in $T(X)[A - C - \mathrm{div}(f)]$. It is also an isomorphism with inverse $\mu_{1/f}$.

Suppose that no so such triple C, f_1, f_2 exist. Then for all divisors C the map $h : L(C) \bigoplus L(C) \to H^1(A - C)^*$ defined as

$$h(f, g) = (\psi_1 \circ t_A^{A-C-\mathrm{div}(f)} \circ \mu_f) - (\psi_2 \circ t_A^{A-C-\mathrm{div}(g)} \circ \mu_g)$$

is injective and therefore $\dim(H^1(A - C)^*) \geq 2\dim(L(C))$. We now apply Theorem 4.28 to $A - C$ to see that

$$\begin{aligned}\dim(H^1(A - C)) &= \dim(L(A - C)) - \deg(A - C) - 1 + \dim(H^1(0)) \\ &\leq \dim(L(A)) - \deg(A) + \deg(C) - 1 + \dim(H^1(0))\end{aligned}$$

as $C > 0$. If we also apply Theorem [3.28] to C to find that $L(C) \geq \deg(C) + 1 - \dim(H^1(0))$. By choosing C such that $\deg(C)$ is sufficiently large, these inequalities are inconsistent and so we have a contradiction. □

If we apply this lemma to ϕ_A and Res_ω we have a positive divisor C and meromorphic functions f_1, $f_2 \in L(C)$ such that

$$\phi_A \circ t_A^{A-C-\mathrm{div}(f_1)} \circ \mu_{f_1} = \mathrm{Res}_\omega \circ t_A^{A-C-\mathrm{div}(f_2)} \circ \mu_{f_2}$$

Since $A - C - \mathrm{div}(f_2) < A$, we have $L^1(A - C - \mathrm{div}(f_2)) \subseteq L^1(A)$ and so we can interpret $\mathrm{Res}_\omega \circ t_A^{A-C-\mathrm{div}(f_2)} = \mathrm{Res}_\omega \in T(X)[A - C - \mathrm{div}(f_2)]$. Note also that for any divisor E and any differential form $\eta \in L^1(-E)$ we have that $\mathrm{Res}_\eta \circ \mu_g = \mathrm{Res}_{g\eta} : T(X)[E + \mathrm{div}(g)] \to \mathbb{C}$ for any meromorphic function g. Combining this with our previous result yields

$$\mathrm{Res}_{(f_2/f_1)\omega} = \phi_A \circ t_A^{A-C-\mathrm{div}(f_1)} = \phi \circ t_D^{A-C-\mathrm{div}(f_1)}$$

as linear functionals on $T(X)[A - C - \mathrm{div}(f_1)]$. It follows that $(f_2/f_1)\omega \in L^1(-D)$ as otherwise there would be a point $x \in X$ such that $n = \mathrm{ord}_x((f_2/f_1)\omega) < D(x)$ and so we would have a Laurent tail divisor $z^{-n-1} \cdot x \in \ker(t_D^{A-C-\mathrm{div}(f_1)})$ such that $\mathrm{Res}_{(f_2/f_1)\omega}(z^{-n-1} \cdot x) \neq 0$. We therefore have that $\mathrm{Res}_{(f_2/f_1)\omega} = \phi$ as functionals on $T(X)[D]$ which vanish on α_D and therefore as elements of $H^1(D)^*$, competing the proof of surjectivity.

□

3.5 Compact Riemann Surfaces are Projective Algebraic Curves

Recall that if we have a divisor D on a compact Riemann X such that $|D|$ is free and $\dim(L(D - x - y) = L(D) - 2$ for all $x, y \in X$, then $\phi_D : X \to \mathbb{P}^n$ is an embedding of X into projective space. The Riemann-Roch Theorem makes such a divisor easy to construct. Consider the term $\dim(L(K - D))$ in the statement of the Riemann-Roch theorem. If D is such that $\deg(D) > 2g - 2$ then $L(K - D)$ must have dimension 0 as any non-zero meromorphic function $f \in L(K - D)$ would contradict [1.19]. It follows that if $\deg(D) > 2g$ then by the Riemann-Roch theorem we have that

$$\dim(L(D)) = \deg(D) + 1 - g$$

and furthermore, this also holds when we replace D with $D - x - y$ for any points $x, y \in X$. The Riemann-Roch theorem thus directly gives us the result that we have been trying to prove.

Proposition 3.41. Let X be a compact Riemann surface, there exists a divisor D on X such that $\phi_D : X \to \mathbb{P}^n$ is an embedding of X as a complex submanifold of $\mathbb{P}^n$.

Proof. Let D be such that $\deg(D) > 2g$ where g is the genus of X. For any two points $x, y \in X$, by the Riemann-Roch theorem we have that

$$\begin{aligned}\dim(L(D - x - y)) = \deg(D - x - y) + 1 - g &= (\deg(D) + 1 - g) - 2 \\ &= \dim(L(D)) - 2\end{aligned}$$

Therefore by [3.22], ϕ_D is an embedding. $\square$

Corollary 3.42. Every compact Riemann surface is projective.

This is the principle result that we have been striving towards. The final step is to prove that all projective Riemann surfaces, that is all compact Riemann surfaces, are projective algebraic curves. This final step is reasonably complicated and there are many approaches depending on the machinery one has at their disposal. One approach is to directly invoke the following general and powerful theorem.

Theorem 3.43 (Chow's theorem)**.** If $X \subseteq \mathbb{P}^n$ is an complex sub-manifold, or more generally any analytic set, then there are homogeneous polynomials $f_1, f_2, \dots, f_m \in \mathbb{C}[x_0, \dots, x_n]$ such that $X = V(f_1, f_2, \dots, f_m) \subseteq \mathbb{P}^n$.

Chow's theorem cements the connection between the analytic and algebraic structure on $\mathbb{P}^n$. It completes the short dotted arrow in the figure at the end of Chapter 2. One can prove Chow's theorem using analytic methods or by using the GAGA principle. [War] is a good survey on different approaches to proving Chow's theorem, the proof can also be found in Ch. 5, Thm. 5.13 in [FG02]. In either case, the proofs are very involved and too complicated to prove with the machinery we have developed. We will instead spend the rest of this chapter describing a method for finding polynomial equations satisfied by projective Riemann surfaces. We begin with a definition.

Definition 3.44. Let X be a compact Riemann surface of genus g, and let K be a canonical divisor. We call the linear system $|K| = \{K + \mathrm{div}(f) : f \in \mathcal{M}(X)\} = \{\mathrm{div}\omega : \omega \in \mathcal{M}^1(X)\}$ the canonical linear system of X.

Proposition 3.45. If X has genus $g = 0$ then it is biholomorphic to the Riemann sphere $\mathbb{P}^1$. If instead $g > 0$ then the canonical linear system $|K|$ of X is free and therefore corresponds to a map $\phi_K : X \to \mathbb{P}^n$ where $n = g - 1$.

Proof (Ch. 7, Prop. 1.7 & Lem. 1.14 in [Mir95]). In the case that X has genus $g = 0$, by applying Riemann-Roch theorem to the divisor $D = 1 \cdot x$ we see that $\dim(L(x)) = \deg(x) + 1 - g = 2$ as $\deg(x) = 1 > 2g$. It follows that X has a non-constant meromorphic function f which has a unique pole at x of order -1. It follows by [1.18] that f must have degree 1 and therefore must be a 1-1 biholomorphism $f : X \to \mathbb{P}^1$.

Suppose now that X has genus $g > 0$. Applying the Riemann-Roch theorem to K reveals that $\dim(L(K)) = \dim(L(0)) + \deg(K) + 1 - g = g$. Also, by applying the Riemann-Roch theorem to $D = 1 \cdot x$ we find

$$\dim(L(x)) = \dim(L(K - x)) + \deg(x) + 1 - g$$

Note that $\dim(L(x)) = 1$ as otherwise X would have a meromorphic function of degree 1 and therefore by the previous analysis it would be biholomorphic to $\mathbb{P}^1$. It follows that

$$\dim(L(K - x)) = 1 - 1 - 1 + g = g - 1$$

and so $\dim(L(K - x)) = \dim(L(K)) - 1$ which as we have previously seen is equivalent to $|K|$ being a free linear system.

$\square$

In the case that X has genus $g > 0$, the map $\phi_K : X \to \mathbb{P}^{g-1}$ is called the *canonical map* of X. For Riemann surfaces of low genus the canonical map fails to be an embedding, in the case $g \geq 2$ however it can be shown that ϕ_k fails to be an embedding if and only if X is *hyperelliptic* (see Ch. 7, Prop. 2.1 in [Mir95]), that is X has a meromorphic function of degree 2 (see Ch. 3, Prop. 4.11 in [Mir95]). In any case, given any divisor D such that ϕ_D is an embedding, we can use the following insight to find homogeneous polynomial equations for the image.

Definition 3.46. Let X be a compact Riemann surface and let D be a divisor such that $\phi_D : X \to \mathbb{P}^n$ is an embedding. Let $F \in \mathbb{C}[x_0, \dots x_n]$ be homogeneous and let $H = V(F)$. Given a point $\phi_D(x) \in \mathbb{P}^n$ we note that if G is another homogenuous polynomial of the same degree such that $\phi_D(x) \notin V(G)$ then the ratio F/G is a meromorphic function on $\mathrm{im}(\phi_D) \subseteq \mathbb{P}^n$ which is holomorphic at $\phi_D(x)$. We define the intersection divisor $\mathrm{div}(F)$ on X as

$$\mathrm{div}(F) = \mathrm{ord}_x((F/G) \circ \phi_D)$$

where G is any homogeneous polynomial such that $\phi_D(x) \notin V(G)$.

This definition is independent of the choice of the homogeneous polynomials G as we are only concerned about the order of F/G at $\phi_D(x)$. Suppose that H is a hyperplane, i.e. $H = V(F)$ where $F = \sum_i a_i z_i$ is linear and homogeneous. In this case we can calculate $\mathrm{div}(F)$ as follows. Let $\phi_D = [f_0 : \cdots : f_n]$ and suppose that $x \in X$ is a point. Suppose that the j'th coordinate of $\phi_D(x)$ is non-zero, then we calculate

$$\mathrm{div}(F)(x) = \mathrm{ord}_x((\sum_i a_i f_i)/f_j) = \mathrm{ord}_x(\sum_i a_i f_i) - \mathrm{ord}_x(f_j)$$

Letting $H = V(F)$ vary over the hyperplanes of $\mathbb{P}^n$, we notice that $\mathrm{div}(F)$ are exactly the divisors in the general linear system $|D|$, this is because j was selected such that f_j had minimal order among the f_i at x and thus $\mathrm{ord}_x(f_j) = -D(x)$.

If we let F, G be any two homogeneous polynomials of degree k, from the definition it immediately follows that $\mathrm{div}(F) - \mathrm{div}(F) = \mathrm{div}(F/G)$ where $\mathrm{div}(F/G)$ is divisor of the meromorphic function F/G on X. Using this, it follows that $\mathrm{div}(F) \sim \mathrm{div}(L^k)$ for any linear homogeneous polynomial L. Again, unwinding the definition shows that $\mathrm{div}(L^k) = k \cdot \mathrm{div}(L) \sim k \cdot D$ as $\mathrm{div}(L) \in |D|$. We have therefore shown that $\mathrm{div}(F) \sim k \cdot D$ where k is the degree of F.

Let $\mathcal{P}(n, k)$ be the vector space of homogeneous polynomials of degree k in $n+1$ variables. Fix a homogeneous polynomial F of degree k and write $\mathrm{div}(F) = k \cdot D + \mathrm{div}(f)$ where f is some meromorphic function on X. We can define the linear map

$$R_k : \mathcal{P}(n, k) \to L(kD)$$

where $R_k : G \to f \cdot (G/F)$. This is well defined as $f \cdot (G/F)$ has poles bounded by $\mathrm{div}(F) - \mathrm{div}(f) = k \cdot D$. Polynomials in the kernel of R_k are exactly the homogeneous polynomials of degree k which vanish on $X \subseteq \mathbb{P}^n$. This is easy to see as if $f \cdot (G/F) = 0$ identically on X then since $f \neq 0$, G must be also. Finally, analysing the dimensions of $\mathcal{P}(n, k)$ and $L(kD)$ by using a combinatorial argument and applying the Riemann-Roch theorem respectively, we arrive at the following inequality.

Proposition 3.47. Suppose that D is a divisor such that $\phi_D : X \to \mathbb{P}^n$ is an embedding. If k is such that $\deg(kD) \geq 2g - 1$ then

$$\dim(\ker(R_k)) \geq \binom{n+k}{k} - k \cdot \deg(D) - 1 + g$$

Analysing the kernels of R_k for various divisors D gives us homogeneous polynomial equations for X if we consider $X \subseteq \mathbb{P}^n$. Of particular interest is the case where $g \geq 2$ and X is not hyperelliptic as then we may set $D = K$ which simplifies things.

4 Sheaves and the GAGA Principle

4.1 Sheaves and Sheaf Maps

To finish, we will briefly explore 'sheaves' in order to state a result called the GAGA principle. We begin by defining sheaves and their notion of morphism.

Definition 4.1. A *pre-sheaf* $\mathcal{F}$ on a topological space X is the assignment of a set $\mathcal{F}(X)$ to every open set of X, and a collection of *restriction maps* $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ whenever $V \subseteq U$ which satisfy the following properties:

1. For every open set $U \subseteq X$, $\rho^U_U = \mathcal{F}(U) \to \mathcal{F}(U)$ is the identity map.
2. If $W \subseteq V \subseteq U$ are all open sets of X, then $\rho^U_W = \rho^U_V \circ \rho^V_W$.

The elements of $\mathcal{F}(U)$ are called the sections of $\mathcal{F}$ over U. The sections over the entire space X are called the global sections of $\mathcal{F}$. By further requiring that the sets $\mathcal{F}$ are groups or rings, and that the restriction maps are morphisms in their respective context, we call $\mathcal{F}$ a pre-sheaf of groups or rings.

Let X be a Riemann surface. Given a group G we can consider the pre-sheaf of groups G^X defined as

$$G^X(U) = G^U = \{f : U \to G\}\,,\ G^X(\emptyset) = \{e\}$$

where the group structure is defined as $(fg)(x) = f(x)g(x)$. Let $V \subseteq U$, we define the restriction maps naturally to be $\rho^U_V : f \to f|_V$. If we let $G = (\mathbb{C}, +)$ then this pre-sheaf associates to each open set $U \subseteq X$ the group of functions $\{f : U \to \mathbb{C}\}$. This pre-sheaf doesn't carry much information on its own. Notice that if we require that the sections be smooth the restriction maps are well defined and axiom (2) is still satisfied. A function $f : U \to \mathbb{C}$ is smooth if it is smooth at every point $p \in U$, so if we restrict to a smaller open set $V \subseteq U$ the smoothness is preserved. This works for any property of functions which is local and encapsulates what the data of a pre-sheaf is, in essence this is data which is invariant under restricting to a smaller open set. When we are in such a situation we can often do the inverse of this procedure too, for instance if $f : U \to G$ and $g : V \to G$ agree on the intersection of their domains then we have a third function $h : U \cup V \to G$ which restricts to f and g respectively on the domains that they are defined on. This motivates the following:

Definition 4.2. Let $\mathcal{F}$ be pre-sheaf on X and $U \subseteq X$ be an open subset with open cover $\{U_i\}$. We say that $\mathcal{F}$ satisfies the *sheaf axioms* for U and $\{U_i\}_i$ if the following are satisfied:

3. If $f_1, f_2 \in \mathcal{F}(U)$ are such that $\rho^U_{U_i}(f_1) = \rho^U_{U_i}(f_2)\ \forall i$, then $f_1 = f_2$.
4. If there are sections $f_i \in \mathcal{F}(U_i)$ such that $\rho^{U_i}_{U_i \cap U_j}(f_i) = \rho^{U_j}_{U_i \cap U_j}(f_j)\ \forall i, j$, then there is some $f \in \mathcal{F}(U)$ such that $\rho^U_{U_i}(f) = f_i\ \forall i$.

Given that the sections 'agree on intersection', Axiom (3) says that there is at most one way to glue them, and axiom (4) says there is atleast one way to glue. If $\mathcal{F}$ is a pre-sheaf on a topological space X which satisfies the sheaf axioms for every open set U and open cover $\{U_i\}$ of U, then we say that $\mathcal{F}$ is a *sheaf*. Note that it can be inferred from the above axioms that if $\mathcal{F}$ is a pre-sheaf of groups then $\mathcal{F}(\emptyset) = \{e\}$. More generally, if one views a (pre)sheaf as a contravariant functor $\mathcal{F} : \mathrm{Op}_X \to \mathcal{C}$ (Op_X is the category of open sets of X, with morphisms given by inclusion maps) then it may be inferred that $\mathcal{F}$ is the terminal object in $\mathcal{C}$ (see p. 75 of [Vak17]). For rings with unity the terminal object is the zero ring $\{0\}$.

In the context of Riemann surfaces, as the topological spaces are Hausdorff and the functions under consideration are defined according to a local property, they define sheaves of rings. Some examples of these for these are as follows:

· $\mathcal{O}_X(U) = \{f : U \to \mathbb{C} : f \text{ holomorphic}\}$

· $\mathcal{C}^\infty_X(U) = \{f : U \to \mathbb{C} : f \text{ smooth}\}$

· $\mathcal{M}_X(U) = \{f : U \to \mathbb{C} : f \text{ meromorphic}\}$

Another sheaf worth mentioning is motivated by divisors on Riemann surfaces. Recall a divisor D on an Riemann surface X is a function $D : X \to \mathbb{Z}$ whose support is a discrete subset of X. We may think about divisors as a formal sum $\sum D(p) \cdot p$ where the points with $D(p) \neq 0$ form a discrete subset of X. We want to define the sheaf of divisors Div_X however if we naively define $\mathrm{Div}_X(U) = \{D \in \mathbb{Z}^U : D(p) \text{ has discrete support}\}$ we encounter a problem as this does not define a sheaf. Indeed if X is compact then we could have $D : U \to \mathbb{Z}$ with $|\mathrm{supp}(D)| = \infty$ as U is open. This could not be lifted to a global section as Div_X consists of the finitely supported functions $D : X \to \mathbb{Z}$. We instead give the following definition:

$$\mathrm{Div}_X(U) = \{D \in \mathbb{Z}^U : \mathrm{supp}(D) \text{ has discrete support in } X\}$$

If $\mathcal{F}$, $\mathcal{G}$ are sheaves on a topological space X then there is a natural notion of 'morphism' $\mathcal{F} \to \mathcal{G}$.

Definition 4.3. Let $\mathcal{F}$ and $\mathcal{G}$ be two pre-sheaves on a topological space X, a map (or morphism) of pre-sheaves $\phi : \mathcal{F} \to \mathcal{G}$ is a collection of morphisms

$$\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$$

for every open set $U \subseteq X$, under the condition that whenever $V \subseteq U$ are open subsets of X the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \big\downarrow \rho^U_V & & \big\downarrow \rho^U_V \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \end{array}$$

Essentially this definition is saying that a map of pre-sheaves on X is a collection of functions between the sections on every open subset, which respects both the algebraic structure (depending on the type of pre-sheaf) and the restriction maps. If $\mathcal{F}, \mathcal{G}$ are sheaves then we say that ϕ is a map (or morphism) of sheaves.

Morphisms turn the collection of sheaves (of Abelian groups) on a topological space X into a category. This category has analogues of the constructions found in linear algebra. One example is the kernel of a morphism.

Definition 4.4. Let $\mathcal{F}$ and $\mathcal{G}$ be pre-sheaves of abelian groups and let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of pre-sheaves, the kernel of ϕ denoted $\ker\phi$ is a sub pre-sheaf of $\mathcal{F}$ defined by:

$$\ker\phi(U) = \ker(\phi_U) \leq \mathcal{F}(U)$$

We note that the kernel $\ker\phi$ is a pre-sheaf since if $V \subseteq U$ and $g \in \ker\phi(U)$ then $\rho^U_V(g) \in \ker\phi(V)$ as $\phi_V(\rho^U_V(g)) = \rho^U_V(\phi_U(g)) = \rho^U_V(e) = e$. Therefore $\ker\phi$ is indeed a sub pre-sheaf of $\mathcal{F}$ as the inclusion $\ker\phi(U) \to \mathcal{F}(U)$ is a well defined pre-sheaf map. It turns out that if ϕ is a map of sheaves, then the kernel is actually a sub sheaf too.

Proposition 4.5. Let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups and let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves, then the sub pre-sheaf $\ker\phi$ is a sub sheaf of $\mathcal{F}$.

Proof. We need to verify axioms (3) and (4). Let $U \subseteq X$ be an open subset with open cover $\{U_i\}_i$. Axiom (3) is immediately satisfied as $\ker\phi$ is a sub pre-sheaf of $\mathcal{F}$ which is itself a sheaf. Surely if $f_1, f_2 \in \ker\phi(U)$ such that $\rho^U_{U_i}(f_1) = \rho^U_{U_i}(f_2)\ \forall i$, then this is also true when we view $f_1, f_2 \in \mathcal{F}(U)$ and thus $f_1 = f_2$ as required.

To verify axiom (4) we use a similar argument. Let $f_i \in \ker\phi(U_i)$ be as in axiom (4). Again we have that there exists some unique $f \in \mathcal{F}(U)$ such that $\rho^U_{U_i}(f) = f_i$ since $\ker\phi$ is a sub pre-sheaf. Now we note that

$$\rho^U_{U_i}(\phi_U(f)) = \phi_{U_i}(\rho^U_{U_i}(f)) = \phi_{U_i}(f_i)) = e \in \mathcal{G}(U_i)$$

as ϕ is a sheaf map and $f_i \in \ker\phi(U_i)$. Now by axiom 3 we know that $\phi_U(f) \in \mathcal{G}(U)$ is the unique section which restricts to the identity element on each $U_i \in U$ and thus is must be itself the identity element $e \in \mathcal{G}(U)$. Therefore $\phi_U(f) = e$ and so $f \in \ker\phi(U)$ as required. It follows that $\ker\phi$ is a sheaf and thus a sub sheaf of $\mathcal{F}$.

□

We finish this section by defining exact sequences of sheaves.

Definition 4.6. Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be sheaves of abelian groups on X, we say that a sequence

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

is exact at $\mathcal{G}$ if the composition of the maps is zero, and if for every $x \in X$ and every open set $x \in U \subseteq X$ the following holds: for every $f \in \ker(\psi_U) \subseteq \mathcal{G}(U)$, there is an open set $x \in V \subset U$ such that $\rho^U_V(f) \in \mathrm{im}(\phi_V)$.

This allows us to define short/long exact sequences for sheaves in the normal way. In fact, for a short exact sequences can be check that they are characterised in the following way.

Definition 4.7. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be sheaves of abelian groups on X, then the sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$$

is exact if any only if $\mathcal{F} = \ker(\phi)$ and if for every point $x \in X$ and every open set $x \in U$, there is an open set $x \in V \subseteq U$ such that ψ_V is surjective.

4.2 Cohomology and the GAGA Principle

Sheaves are perfect for talking about objects that are defined on all open sets of a topological space. Sheaf cohomology is a tool built on top of this in order to allow us to ask 'local to global' questions about sheaves. As an example, let D be a divisor on a compact Riemann surface X and consider the sheaf $\mathcal{O}_D$ defined as

$$\mathcal{O}_D(U) = \{f \in \mathcal{M}(U) : \mathrm{ord}_x(f) \geq -D(x) \text{ for every } x \in U\}$$

The Riemann-Roch theorem is a statement about the dimension of $L(D)$, i.e. the space of global sections $\mathcal{O}_D(X)$. It turns out that is exactly a question about the dimension of the cohomology group $H^1(X, \mathcal{O}_D)$.

Definition 4.8. Let X be topological space, let $\mathcal{F}$ be a sheaf of abelian groups and let $U = \{U_i\}_{i \in I}$ be an open cover of X. We define the *q'th cochain group* of $\mathcal{F}$ as

$$\mathcal{C}^q(U, \mathcal{F}) = \prod_{(i_0, \ldots, i_q) \in I^{q+1}} \mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q})$$

Elements of $\mathcal{C}^q(U, \mathcal{F})$ are called q-cochains. For clarity, a 0-cochain is just a collection of sections $\{f_i \in \mathcal{F}(U_i)\}_{i \in I}$, i.e. a section for ever U_i in the open cover. A 1-chain is then a collection of sections $\{f_{ij} \in \mathcal{F}(U_i \cap U_j)\}_{i,j \in I}$, i.e. one for every intersection $U_i \cap U_j$ in the open cover. For simplicity we denote 0, 1 and 2 cochains as $(f_i)_i$, $(f_{ij})_{ij}$ and $(f_{ijk})_{ijk}$ respectively.

Definition 4.9. We define the *coboundry operator*, $\delta : \mathcal{C}^0(U, \mathcal{F}) \to \mathcal{C}^1(U, \mathcal{F})$ as

$$\delta((f_i)_i) = (f_j - f_i)_{ij}$$

We can also define coboundry operator $\delta : \mathcal{C}^1(U, \mathcal{F}) \to \mathcal{C}^2(U, \mathcal{F})$ as

$$\delta((f_{ij})_{ij}) = (f_{jk} - f_{ik} + f_{ij})_{ijk}$$

where f_{jk}, f_{ik}, f_{ij} are considered as elements in $\mathcal{F}(U_i \cap U_j \cap U_k)$.

The coboundary operators can be checked to be group homomorphisms. Furthermore it is clear that $\delta^2(f_i)_i = \delta((f_j - f_i)_{ij}) = ((f_k - f_j) - (f_k - f_i) + (f_j - f_i)_{ijk} = 0$. We therefore have a chain complex

$$\mathcal{C}^0(U, \mathcal{F}) \xrightarrow{\delta} \mathcal{C}^1(U, \mathcal{F}) \xrightarrow{\delta} \mathcal{C}^2(U, \mathcal{F})$$

and following these lines we can extend δ in a way to get a chain complex

$$\mathcal{C}^0(\mathfrak{U},\mathcal{F}) \xrightarrow{\delta} \mathcal{C}^1(\mathfrak{U},\mathcal{F}) \xrightarrow{\delta} \mathcal{C}^2(\mathfrak{U},\mathcal{F}) \xrightarrow{\delta} \mathcal{C}^3(\mathfrak{U},\mathcal{F}) \xrightarrow{\delta} \dots$$

Definition 4.10. We define the cohomology groups of $\mathcal{F}$ with respect to the open cover $\mathfrak{U}$, denoted $H^k(\mathfrak{U},\mathcal{F})$, as the cohomology groups of this chain complex.

The cohomology groups are dependant on the open cover we use. The way be fix this to take the 'direct limit' of the cohomology groups for every open cover $\mathfrak{U} = \{U_i\}_{i\in I}$. For the full detail of this construction see p. 98-100 of [For81]. For our purposes all we need to know is that if $\mathfrak{U}, \mathfrak{V}$ are open covers of X such that for all for every $U_i \in \mathfrak{U}$ there is $V_j \in \mathfrak{V}$ such that $V_j \subseteq U_i$, then there is a 'refinement' map $t^{\mathfrak{U}}_{\mathfrak{V}} : \mathcal{C}^q(\mathfrak{U},\mathcal{F}) \to \mathcal{C}^q(\mathfrak{V},\mathcal{F})$ sending

$$(f_{i_1\dots i_q})_{i_1\dots i_q} \mapsto (g_{j_1\dots j_q})_{j_1\dots j_q}$$

where $(g_{j_1\dots j_q})_{j_1\dots j_q} = (f_{j_1\dots j_q})_{j_1\dots j_q}$ and $f_{j_1\dots j_q} \in \mathcal{F}(V_{j_1} \cap \dots \cap V_{j_q}) \subseteq \mathcal{F}(U_{i_1} \cap \dots \cap V_{i_q})$. Furthermore, this map descends to a map on cohomology groups, and so we have a map $H^q(\mathfrak{U},\mathcal{F}) \to H^q(\mathfrak{V},\mathcal{F})$. If we take the 'direct limit' of these cohomology groups (see Sect. 1.4 in [Vak17]), we can define the cohomology groups of $\mathcal{F}$ without the need for a open cover.

Definition 4.11. The cohomology groups of $\mathcal{F}$, denoted $H^q(X,\mathcal{F})$ are defined as

$$H^q(X,\mathcal{F}) = \varinjlim_{\mathfrak{U}} H^q(\mathfrak{U},\mathcal{F})$$

The key idea here is for every open cover $\mathfrak{U}$ we have a homomorphism $f_{\mathfrak{U}} : H^q(\mathfrak{U},\mathcal{F}) \to H^q(X,\mathcal{F})$. Furthermore, if for every open cover $\mathfrak{U}$ we have homomorphism $\phi_{\mathfrak{U}} : H^q(\mathfrak{U},\mathcal{F}) \to G$ that commutes with the refinement maps t, then there is a unique homomorphism $\psi : H^q(X,\mathcal{F}) \to G$ such that $\psi \circ f_{\mathfrak{U}} = \phi_{\mathfrak{U}}$ for all $\mathfrak{U}$.

The main reason why sheaf cohomology is useful for answering questions about global sections is because of the following theorem (see Ch. 9, Thm. 3.18 of [Mir95]).

Theorem 4.12. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves of abelian groups on X, then there is a exact sequence in cohomology groups

$$\begin{tikzcd} H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{G}) \arrow[r] & H^0(X,\mathcal{H}) \arrow[dll] \\ H^1(X,\mathcal{F}) \arrow[r] & H^1(X,\mathcal{G}) \arrow[r] & H^1(X,\mathcal{H}) \end{tikzcd}$$

We will now use these observations to shed light on the topics we have previously discussed. Let X be a compact Riemann surface. As we have seen, we may regard X equally as a Riemann surface or a projective algebraic curve $X = V(I) \subseteq \mathbb{P}^n$. As such X comes equipped with two natural topologies, the complex topology (denoted $T_{\mathbb{C}}$) and the Zariski topology (denoted T_Z). As we have seen, the Zariski topology on a curve is cofinite. Since cofinite sets are open in the complex topology in X, if we are given a sheaf $\mathcal{F}$ on $(X, T_{\mathbb{C}})$ then we may define a sheaf $\mathcal{F}'$ on (X, T_Z). The amazing fact about Riemann surfaces is that when we do this, although restrict to a sub topology, if we pick our sheaves wisely then we don't lose any information. To see an example of this, consider the following sheaves defined on $(X, T_{\mathbb{C}})$

- $\mathcal{O}_{R,\mathbb{C}}(U) = \{f \in \mathcal{M}(X) : f \text{ holomorphic in } U\}$
- $\mathcal{M}_{R,\mathbb{C}}(U) = \{f \in \mathcal{M}(U)\}$

These sheaves are very close to how we thought about regular and rational functions in Chapter 2. If we define $\mathcal{O}_{R,Z}(U)$ as the corresponding sheaf on (X, T_Z), we observe that if $f \in \mathcal{O}_{R,\mathbb{C}}(U)$ then since meromorphic functions have a finite number of poles, there is a cofinite set $V = X - \{x \in X : \mathrm{ord}_x(f) < 0\}$ such that $f \in \mathcal{O}_{R,\mathbb{C}}(V)$ (i.e. $\exists g$ s.t $\rho^V_U(g) = f$). Therefore $f \in \mathcal{O}_{R,Z}(V)$ and we have really lost any sections. The advantage of this is that since the Zariski topology is so simple, it is often easier to calculations about sheaf cohomology here. As an quick example, if we consider the sheaf of Laurent tail divisors and sheaf of meromorphic functions with poles bounded by D, defined as

- $T_{\mathbb{C}}[D](U) = \{R \in T^U : R \text{ has discrete support }\}$
- $\mathcal{O}_{\mathbb{C}}[D](U) = \{f \in \mathcal{M}(U) : \mathrm{ord}_x(f) \geq -D(x) \text{ for all } x \in U\}$

Then again we don't lose any sections when moving to $T_Z[D]$ and $\mathcal{O}_Z[D]$ on (X, T_Z). Moreover, in this case it turns out we have an exact sequence

$$0 \to \mathcal{O}_Z[D] \to \mathcal{M}_{R,Z} \to T_Z[D] \to 0$$

and because of the simplicity of $\mathcal{M}_{R,Z}$ the long exact sequence in cohomology is easy to calculate and shows that

$$0 \to L(D) \to M(X) \to T[D](X) \to H^1(X, \mathcal{O}_Z[D]) \to 0$$

showing that $H^1(D) = H^1(X, \mathcal{O}_Z[D])$ (see p. 311-315 in [Mir95]). We finish by stating the GAGA principle, which makes this discussion rigorous.

Theorem 4.13 (GAGA Principle)**.** Let $X \subset \mathbb{P}^n$ be a projective algebraic curve, then for every $n \geq 0$ there is an isomorphism

$$H^n(X, \mathcal{O}_Z[D]) \xrightarrow{\sim} H^n(X, \mathcal{O}_{\mathbb{C}}[D])$$

References

[AM69] M. Atiyah and I. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley, 1969.

[Don11] S. Donaldson. *Riemann Surfaces*. Oxford Graduate Texts in Mathematics. Oxford University Press, 2011.

[Eis95] D. Eisenbud. *Commutative Algebra*. Graduate Texts in Mathematics. Springer, 1995.

[FG02] K. Fritzsche and H. Grauert. *From Holomorphic Functions to Complex Manifolds*. Graduate Texts in Mathematics. Springer, 2002.

[For81] O. Forster. *Lectures on Riemann Surfaces*. Graduate Texts in Mathematics. Springer, 1981.

[Har15] D. Harbater. *Riemann's Existence Theorem*. 2015. URL: `https://www2.math.upenn.edu/~harbater/RETppr.pdf`.

[Har77] R. Hartshorne. *Algebraic Geometry*. Springer, 1977.

[Huy05] D. Huybrechts. *Complex Geometry*. Universitext. Springer, 2005.

[Lee12] J. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer, 2012.

[Mir95] R. Miranda. *Algebraic Curves and Riemann Surfaces*. Graduate Studies in Mathematics. American Mathematical Society, 1995.

[NN01] R. Narasimhan and Y. Nievergelt. *Complex Analysis in One Variable*. Birkhäuser, 2001.

[Sha13] I. Shafarevich. *Basic Algebraic Geometry 1*. Springer, 2013.

[Vak17] R. Vakil. *The Rising Sea: Foundations of Algebraic Geometry*. 2017. URL: `http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf`.

[War] E. Warner. *Three Approaches To Chow's Theorem*. URL: `https://www.math.columbia.edu/~warner/notes/AlgGeomSemTalk.pdf`.