

Group theory, Topology and Spin-1/2 Particles

From Dirac's belt to fermions

Louan Mol

Université Libre de Bruxelles

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1. Dirac's belt trick and the rotation group
2. Homotopy theory
3. Quantum spin and $SU(2)$
4. Spinors
5. Spinors in Physics
6. Conclusion

Dirac's belt trick and the rotation group

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Rules:

1. you can only move the end of the belt
2. you cannot twist or rotate it

Goal: untwist a 2π -twist.

\Rightarrow it turns out to be impossible ! One turn negates the twist: $2\pi \rightarrow -2\pi$.

Therefore, possible for a 4π twist ...

Why is that ?

Space of rotations: $\text{SO}(3)$ as a group

Rotations in 3-dimensional space: matrices that acts on \mathbb{R}^3 s.t.

1. preserve the **scalar product**: $O^T O = \mathbb{1}$ ($\Leftrightarrow O$ is orthogonal)
2. preserve the **orientation**: $\det O = 1$

Special uthogonal group

$\text{SO}(3)$ is the set of 3×3 real matrices such that $O^T O = \mathbb{1}$ and $\det O = 1$.

Three “fundamental” rotations:

$$x : \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \quad y : \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \quad z : \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

\Rightarrow It forms a **group**.

Space of rotations: $SO(3)$ as a topological space

Fundamental data that describes a rotation:

- an **axis** of rotation, i.e. a unit vector \vec{n} $\rightarrow 2$ parameters
- an **angle** of rotation $\theta \in [-\pi, \pi]$ (with $-\pi \sim \pi$) $\rightarrow 1$ parameter

The space of rotations can then alternately be defined as a **3-sphere of radius π and its antipodal points identified**:

$$SO(3) \cong B^3(\pi) / \sim$$

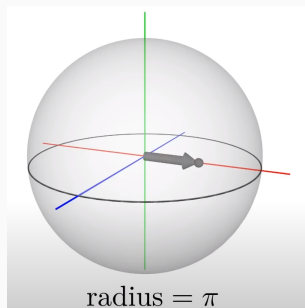
and for each point:

direction \leftrightarrow axis

norm \leftrightarrow angle

\Rightarrow It forms a **topological space**.

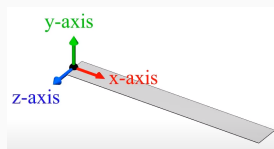
(group + topological space = Lie group)



Back to the belt

Mathematical description of the belt ?

- ▷ a belt is a strip, which is just a **path** + an **orientation**.
- ▷ given axis on the middle line along the belt, each set of axis is related by a rotation
- ▷ a belt configuration is equivalent to a continuous set of axis and therefore to a continuous set of translations, i.e. a **path in $SO(3)$**

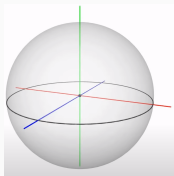


There is a bijection:

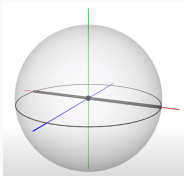
$$\text{belt configuration} \Leftrightarrow \text{path in } SO(3)$$

This gives us a new language to analyze the problem !

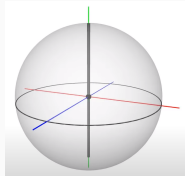
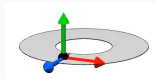
trivial rotation



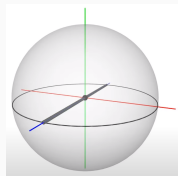
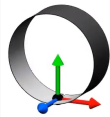
2π *x*-rotation



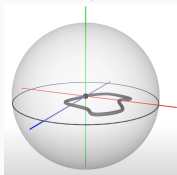
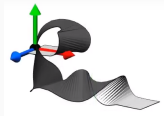
2π *y*-rotation



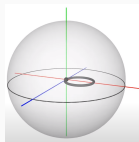
2π *z*-rotation



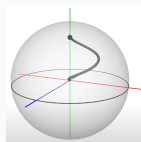
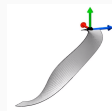
random rotation



closed



open path



<u>Belt</u>		<u>Path</u>
specific configuration	\longleftrightarrow	specific path
moving the ends	\longleftrightarrow	continuous deformation
ends have same orientation	\longleftrightarrow	closed path (loop)
can be flattened	\longleftrightarrow	contractible

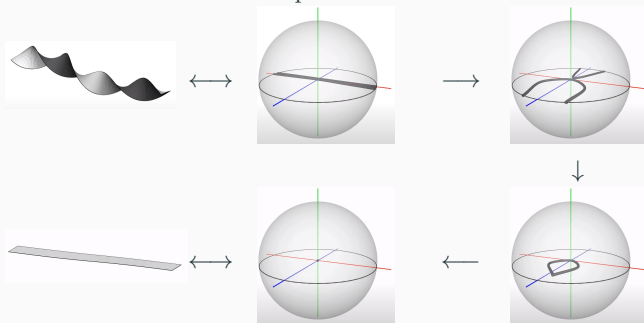
Back to Dirac's belt trick:

1. ends of the belt have same orientation \rightarrow we consider loops
(passing through the origin)
2. moving the ends of the belt \rightarrow continuous deformation
3. belt in original (flat) position \rightarrow trivial path

The question then becomes: **which loops are contractible ?**

Problem solved ?

- **4π -twist**: We saw in the beginning the the 4π -twist can be flattened, how can we see this in terms of paths ?



\Rightarrow the 4π -twist is **contractible** ! Great.

- **2π -twist**: we “clearly” see that is not contractible... no ?! Great..?..

Wierd aftertaste: our “proof” is good to show contractibility but bad to show non-contractibility and it only works for simple examples.

\Rightarrow We want a consistent and general way of studying paths in topological spaces.

Homotopy theory

Starting observation: depending on the topological space, all loops might not be contractible. Moreover, some loops are “fundamentally different” from each other.

Examples: \mathbb{R}^3 , S^2 , \mathbb{T}^2 , etc.

Paths and homotopies

For a topological space X :

- *Path* in X : continuous map $\gamma : [0, 1] \rightarrow X$, *loop* if closed
- γ_1 and γ_2 are *homotopically equivalent* if one can be deformed into the other: there exists $H : [0, 1] \times [0, 1] \rightarrow X$ such that

$$H(0, t) = \gamma_1(t) \quad \text{and} \quad H(1, t) = \gamma_2(t).$$

This is an equivalence relation (\sim).

For each $x_0 \in X$, we define

$$\pi_1(X, x_0) = \{\text{all loops based at } x_0\} / \sim,$$

it is the set of “fundamentally different” loops passing through x_0 . $[\gamma]$ is called the *homotopy class* of γ .

Fundamental group

Group structure on $\pi_1(X, x_0)$:

- **Product** of paths: $\gamma_1 \cdot \gamma_2 = \text{“}\gamma_1 \text{ then } \gamma_2\text{”}$
- **Inverse** path: $\gamma^{-1} = \text{“}\gamma \text{ traversed in the opposite direction”}$
- **Neutral** path: $e = \text{“constant path at the identity”}$
- For homotopy classes: $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$ and $[\gamma]^{-1} = [\gamma^{-1}]$

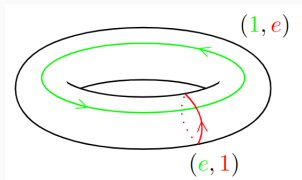
Important fact: up to isomorphism, $\pi_1(X, x_0)$ does not depend on x_0
 \Rightarrow we denote it as $\pi_1(X)$, it is called the **fundamental group** of X .

Contractible loops are \sim to a point, i.e. they are the element of $[e]$.

How to compute $\pi_1(X)$? Can be difficult, not discussed here.

Examples:

- $\pi_1(\mathbb{R}^3) = \{e\}$
- $\pi_1(S^2) = \{e\}$
- $\pi_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$
- $\pi_1(\mathbb{R}^2 \setminus \{p\}) = \mathbb{Z}$



Remark: $\pi_1(\mathbb{R}^3 \setminus \{p\}) = \{e\}$, higher homotopy groups for higher-dimensional holes ?

Question we had: are all loops in $\text{SO}(3)$ contractible ?

In homotopy language: is $\pi_1(\text{SO}(3))$ trivial ?

Answer: NO, one can compute that

$$\pi_1(\text{SO}(3)) = \mathbb{Z}_2$$

\Rightarrow There only two “fundamentally different” loops in $\text{SO}(3)$!

\Rightarrow all non-contractible loops are deformations of the 2π -twist !

The belt trick is a way of physically demonstrating that the fundamental group of $\text{SO}(3)$ is \mathbb{Z}_2 .

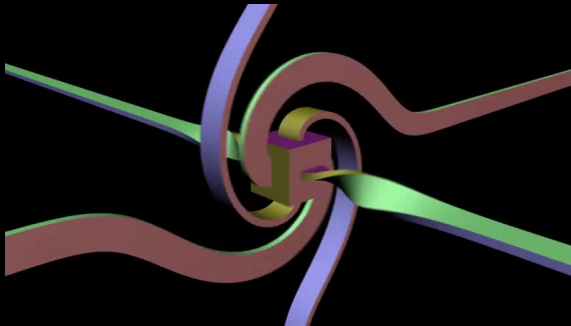
We can now say, with more confidence, that we understood Dirac's belt trick.

Are there other manifestation of homotopy in our practical world ?

Yes: the **spin** ! (you don't need a belt, but you need an electron)

Initially, this trick was a demonstration invented by P. Dirac (1902-1984) to explain the notion of spin to his students.

Animation 1



An object attached to belts or strings can spin continuously without becoming tangled. Notice that after the cube completes a 360° rotation, the spiral is reversed from its initial configuration. The belts return to their original configuration after spinning a full 720° .



A more extreme example demonstrating that this works with any number of strings. In the limit, a piece of solid continuous space can rotate in place like this without tearing or intersecting itself.

Quantum spin and $SU(2)$

What is the spin ?

Skipping most of the physics background:

Spin in quantum mechanics

1. the **spin** is an inherent property of any “particle”:
 - number $s \in \frac{1}{2}\mathbb{N}$, in our case $s = 1/2$
 - does not change, like the mass, charge, etc
 - classifies particles into different classes
2. a particle of spin s is, at a given moment, in a certain **spin state**:
 - unit vector of $v \in \mathbb{C}^{2s+1}$, in our case $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2$
 - can evolve over time
3. what we can measure is the **observed spin**:
 - discrete value $s_{\text{obs.}} \in \{s, s-1, \dots, 0, \dots, -s+1, -s\}$ ($2s+1$ possibilities).
 - In our case, $s_{\text{obs.}} = 1/2, -1/2$ that we denote \uparrow and \downarrow
 - given a direction, e.g. $i = x, y, z$
 - outcome is random, we can only compute the probabilities of the different outcome (repeat experience)

What is the spin ?

Let us introduce

$$v_{x,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, v_{x,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, v_{y,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}, v_{y,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}, v_{z,\uparrow} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_{z,\downarrow} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The probability of measuring $s_{\text{obs.}}$ in the direction i is given by

$$P(i, s_{\text{obs.}}) = |\langle v_{i,k}, v \rangle|^2 \quad (1)$$

where v is the spin state of the particle.

Example: in the direction z ,

$$P(z, \uparrow) = |\alpha|^2, \quad P(z, \downarrow) = |\beta|^2. \quad (2)$$

Note: we must have $|\alpha|^2 + |\beta|^2 = 1$.

Special unitary group

$SU(2)$ is the set of 2×2 complex matrices such that $U^\dagger U = \mathbb{1}$ and $\det U = 1$.

Like $SO(3)$ it is a Lie group so it can be viewed

- as a group
- as a topological space $SU(2) \cong S^3$

If we take a step back:

Scalar product on \mathbb{R}^3 :

$$\langle v_1, v_2 \rangle = (v_1)^T v_2$$

is such that

$$\langle Rv_1, Rv_2 \rangle = \langle v_1, v_2 \rangle$$

if and only if $R^T R = \mathbb{1}$.

\Rightarrow SO(3) is the (orientation preserving) isometry group

Scalar product on \mathbb{C}^2 :

$$\langle v_1, v_2 \rangle = (v_1)^\dagger v_2$$

is such that

$$\langle Uv_1, Uv_2 \rangle = \langle v_1, v_2 \rangle$$

if and only if $U^\dagger U = \mathbb{1}$.

\Rightarrow SU(2) is the (orientation preserving) isometry group

Spinors

Spinors in Physics

Conclusion

1. There are two topologically distinguishable classes (homotopy classes) of paths through rotations that result in the same overall rotation, as illustrated by the Dirac's belt trick. (True in any dimension)
2. Spinors change in different ways depending not just on the overall final rotation, but the details of how that rotation was achieved (by a continuous path in the rotation group).
3. The spin group is the group of all rotations keeping track of the class. It doubly covers the rotation group, since each rotation can be obtained in two inequivalent ways as the endpoint of a path.
4. The space of spinors by definition is equipped with a (complex) linear representation of the spin group.

Three different block environments are pre-defined and may be styled with an optional background color.

Some text.

Default

Block content.

Alert

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Example

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aaaaa

Introduction to the spin

Spin in quantum mechanics

The **spin** of an “particle” is a number $s \in \frac{1}{2}\mathbb{N}$.

The **spin state** of a particle of spin s is a unit vector in \mathbb{C}^{2s+1} .

The spin is a **property**, it cannot change (e.g. mass, charge)

The spin state is a **characteristic**, it evolves

How to interpret it ?

1. **directions:** we choose the direction in which we want to measure it
2. **probabilistic theory:** the outcome of the measure, we can only compute the probabilities of the different outcomes
3. **discrete quantity:** in the chosen direction, the spin will either appear to up or down (\uparrow or \downarrow)

The probability of measuring the spin $k = \uparrow, \downarrow$ in the direction $i = x, y, z$ is given by

$$P(i, k) = |\langle v_{i,k}, v \rangle|^2 \quad (3)$$

where v is the spin state of the particle, for some given vectors $v_{i,k}$.

The Lie algebra $\mathfrak{su}(2)$ is generated by the **Pauli matrices**

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (4)$$



A. Hatcher.

Algebraic Topology.

Algebraic Topology. Cambridge University Press, 2002.



N. Miller.

Representation theory and quantum mechanics, 2018.