Group theory, Topology and Spin-$1/2$ Particles

From Dirac's belt to fermions

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Brussels Summer School of Mathematics 2022

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Dirac’s belt trick and rotations

Dirac's belt trick

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Rules:

1. you can only move the end of the belt
2. you cannot twist or rotate it

Goal: untwist a 2π-twist.

$\Rightarrow$ it tuns out to be impossible ! One turn negates the twist: $2\pi \to -2\pi$.

Therefore, possible for a 4π twist ...

Why is that ?

Rotations in 3-dimensional space: matrices that acts on $\mathbb{R}^3$ s.t.

1. preserve the scalar product: $O^T O = \mathbb{1}$ ($\Leftrightarrow O$ is orthogonal)
2. preserve the orientation: $\det O = 1$

Special othogonal group

SO(3) is the set of 3×3 real matrices such that $O^T O = \mathbb{1}$ and $\det O = 1$.

Three "fundamental" rotations:

$$x : \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad y : \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad z : \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ It forms a group.

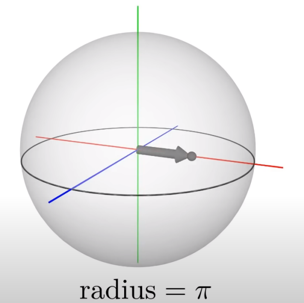
Space of rotations: SO(3) as a topological space

Fundamental data that describes a rotation:

- an axis of rotation, i.e. a unit vector $\overrightarrow{n}$ $\rightarrow$ 2 parameters
- an angle of rotation $\theta \in [-\pi, \pi]$ (with $-\pi \sim \pi$) $\rightarrow$ 1 parameter

The space of rotations can then alternately be defined as a **3-sphere of radius π and its antipodal points identified**:

$$\boxed{\mathrm{SO}(3) \cong B^3(\pi)/\sim}$$



and for each point:

$$\text{direction} \leftrightarrow \text{axis}$$
$$\text{norm} \leftrightarrow \text{angle}$$

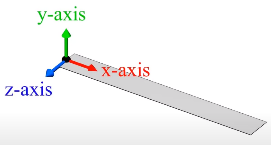
$\Rightarrow$ It forms a topological space.

(group + topological space = Lie group)

Back to the belt

Mathematical description of the belt ?

- a belt is a strip, which is just a path + an orientation.
- given axis on the middle line along the belt, each set of axis is related by a rotation
- a belt configuration is equivalent to a continuous set of axis and therefore to a continuous set of translations, i.e. a path in SO(3)



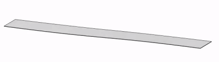
There is a bijection:

$$\boxed{\text{belt configuration} \Leftrightarrow \text{path in SO(3)}}$$

This gives us a new language to analyze the problem !

Dictionary

trivial rotation



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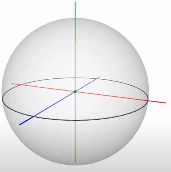
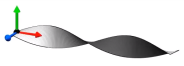


2π x-rotation



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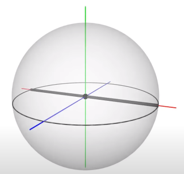
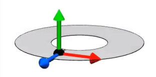


2π y-rotation



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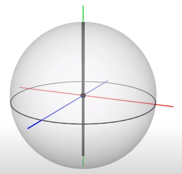
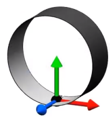


2π z-rotation



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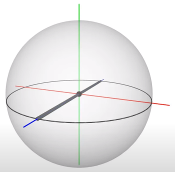
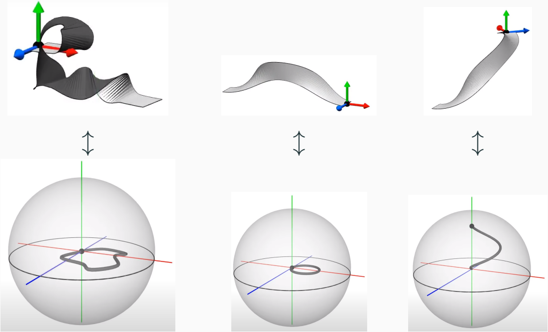


Dictionary

random rotation | **closed** | **open path**



Dictionary

Belt		**Path**
specific configuration	$\longleftrightarrow$	specific path
moving the ends	$\longleftrightarrow$	continuous deformation
ends have same orientation	$\longleftrightarrow$	closed path (loop)
can be flattened	$\longleftrightarrow$	contractible

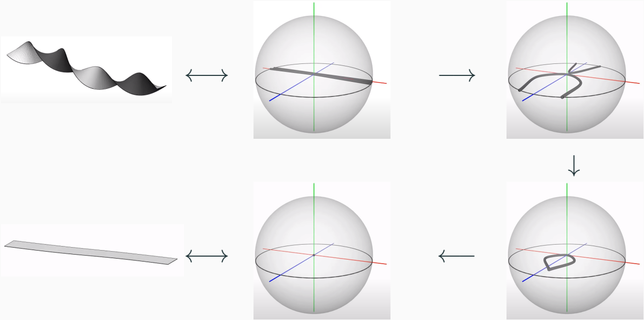
Back to Dirac's belt trick:

1. ends of the belt have same orientation $\rightarrow$ we consider loops
   (passing through the origin)
2. moving the ends of the belt $\rightarrow$ continuous deformation
3. belt in original (flat) position $\rightarrow$ trivial path

The question then becomes: **which loops are contractible ?**

Problem solved ?

- **4π-twist:** We saw in the beginning the the 4π-twist can be flattened, how can we see this in terms of paths ?



$\Rightarrow$ the 4π-twist is contractible ! Great.

- **2π-twist:** we “clearly” see that is not contractible... no ?! Great..?..

Wierd aftertaste: our “proof” is good to show contractibility but bad to show non-contractibility and it only works for simple examples.
$\Rightarrow$ We want a consistent and general way of studying paths in topological spaces.

Homotopy theory

Homotopoy theory primer

Starting observation: depending on the topological space, all loops might not be contractible. Moreover, some loops are "fundamentally different" from each other.
Examples: $\mathbb{R}^3$, S^2, $\mathbb{T}^2$, etc.

Paths and homotopies

For a topological space X:

- *Path* in X: continuous map $\gamma : [0,1] \to X$, *loop* if closed
- γ_1 and γ_2 are *homotopically equivalent* if one can be deformed into the other: there exists $H : [0,1] \times [0,1] \to X$ such that

  $$H(0,t) = \gamma_1(t) \qquad \text{and} \qquad H(1,t) = \gamma_2(t).$$

  This is an equivalence relation ($\sim$).

For each $x_0 \in X$, we define

$$\pi_1(X, x_0) = \{\text{all loops based at } x_0\}/\sim,$$

it is the set of "fundamentally different" loops passing through x_0.

Fundamental group

Group structure on $\pi_1(X, x_0)$:

- **Product** of paths: $\gamma_1 \cdot \gamma_2 =$ "γ_1 then γ_2"
- **Inverse** path: $\gamma^{-1} =$ "γ traversed in the opposite direction"
- **Neutral** path: $e =$ "constant path at the identity"
- For equivalence classes: $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$ and $[\gamma]^{-1} = [\gamma^{-1}]$

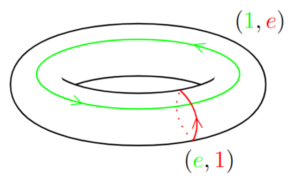
Important fact: up to isomorphism, $\pi_1(X, x_0)$ does not depend on x_0
$\Rightarrow$ we denote it as $\pi_1(X)$, it is called the fundamental group of X.

Contractible loops are $\sim$ to a point, i.e. they are the element of $[e]$.

How to compute $\pi_1(X)$? Can be difficult, not discussed here.

Examples:

- $\pi_1(\mathbb{R}^3) = \{e\}$
- $\pi_1(S^2) = \{e\}$
- $\pi_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$
- $\pi_1(\mathbb{R}^2 \backslash \{p\}) = \mathbb{Z}$



Remark: $\pi_1(\mathbb{R}^3 \backslash \{p\}) = \{e\}$, higher homotopy groups for higher-dimensional holes ?

Back to SO(3)

Question we had: are all loops in SO(3) contractible ?
In homotopy language: is $\pi_1(\mathrm{SO}(3))$ trivial ?

Answer: NO, one can compute that

$$\boxed{\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2}$$

$\Rightarrow$ There only two "fundamentally different" loops in SO(3) !
$\Rightarrow$ all non-contractible loops are deformations of the 2π-twist !

> The belt trick is a way of physically demonstrating that the fundamental group of SO(3) is $\mathbb{Z}_2$.

We can now say, with more confidence, that we understood Dirac's belt trick.

Are there other manifestation of homotopy in our practical world ?

Yes: the spin ! (Pro: you don't need a belt, Con: you need an electron)
Initially, this trick was a demonstration invented by P. Dirac (1902-1984) to explain the notion of spin to his students.

Quantum spin and SU(2)

Conclusion

Blocks

Three different block environments are pre-defined and may be styled with an optional background color.

Some text.

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Default

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Example

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Backup slides

Sometimes, it is useful to add slides at the end of your presentation to refer to during audience questions.

The best way to do this is to include the `appendixnumberbeamer` package in your preamble and call `\appendix` before your backup slides.

metropolis will automatically turn off slide numbering and progress bars for slides in the appendix. [**?**]

References i

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