Group theory, Topology and Spin-$1/2$ Particles

From Dirac's belt to fermions

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Dirac’s belt trick and rotations

Dirac's belt trick

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Rules:

1. you can only move the end of the belt
2. you cannot twist or rotate it

Goal: untwist a 2π-twist.

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Therefore, possible for a 4π twist ...

Why is that ?

Space of rotations: SO(3) as a group

Rotations in 3-dimensional space: matrices that acts on $\mathbb{R}^3$ and that preserve the scalar product, and in particular lengths.

In other words: matrix O such that $O^T O = \mathbb{1}$ ($\Leftrightarrow O$ is orthogonal)

Additional requirement: orientation preserving $\Leftrightarrow \det O = 1$

Special othogonal group

SO(3) is the set of 3×3 real matrices such that $O^T O = \mathbb{1}$ and $\det O = 1$.

Three “fundamental” rotations:

$$x : \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad y : \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad z : \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

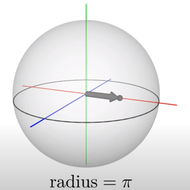
$\Rightarrow$ It forms a group.

Space of rotations: SO(3) as a topological space

The belt trick is a way of physically demonstrating the the fundamental group of SO(3) is $\mathbb{Z}_2$.

Electrons

In quantum mechanics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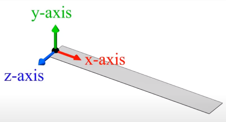
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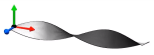

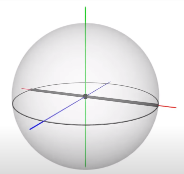

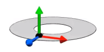

Section 2

Section 3

Section 4

Section 5

Blocks

Three different block environments are pre-defined and may be styled with an optional background color.

Some text.

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Default

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Alert

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Example

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Backup slides

Sometimes, it is useful to add slides at the end of your presentation to refer to during audience questions.

The best way to do this is to include the `appendixnumberbeamer` package in your preamble and call `\appendix` before your backup slides.

metropolis will automatically turn off slide numbering and progress bars for slides in the appendix. [1]

References i

R. Graham, D. Knuth, and O. Patashnik.
Concrete mathematics.
Addison-Wesley, Reading, MA, 1989.