

Group theory, Topology and Spin-1/2 Particles

From Dirac's belt to fermions

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2. Homotopy theory
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Dirac's belt trick and the rotation group

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Rules:

1. you can only move the end of the belt
2. you cannot twist or rotate it

Goal: untwist a 2π -twist.

\Rightarrow it turns out to be impossible ! One turn negates the twist: $2\pi \rightarrow -2\pi$.

Therefore, possible for a 4π twist ...

Why is that ?

Space of rotations: $\text{SO}(3)$ as a group

Rotations in 3-dimensional space: matrices that acts on \mathbb{R}^3 s.t.

1. preserve the **scalar product**: $O^T O = \mathbb{1}$ ($\Leftrightarrow O$ is orthogonal)
2. preserve the **orientation**: $\det O = 1$

Special uthogonal group

$\text{SO}(3)$ is the set of 3×3 real matrices such that $O^T O = \mathbb{1}$ and $\det O = 1$.

Three “fundamental” rotations:

$$x : \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \quad y : \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \quad z : \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

\Rightarrow It forms a **group**.

Space of rotations: $\text{SO}(3)$ as a topological space

Fundamental data that describes a rotation:

- an **axis** of rotation, i.e. a unit vector \vec{n} $\rightarrow 2$ parameters
- an **angle** of rotation $\theta \in [-\pi, \pi]$ (with $-\pi \sim \pi$) $\rightarrow 1$ parameter

The space of rotations can then alternately be defined as a **3-sphere of radius π and its antipodal points identified**:

$$\boxed{\text{SO}(3) \cong B^3(\pi) / \sim}$$

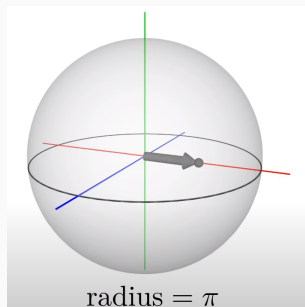
and for each point:

direction \leftrightarrow axis

norm \leftrightarrow angle

\Rightarrow It forms a **topological space**.

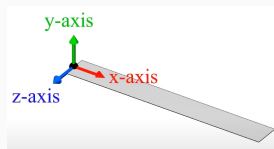
(group + topological space = Lie group)



Back to the belt

Mathematical description of the belt ?

- ▷ a belt is a strip, which is just a **path** + an **orientation**.
- ▷ given axis on the middle line along the belt, each set of axis is related by a rotation
- ▷ a belt configuration is equivalent to a continuous set of axis and therefore to a continuous set of translations, i.e. a **path in $SO(3)$**

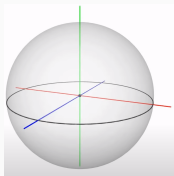


There is a bijection:

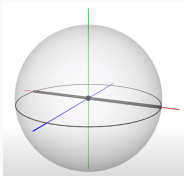
$$\text{belt configuration} \Leftrightarrow \text{path in } SO(3)$$

This gives us a new language to analyze the problem !

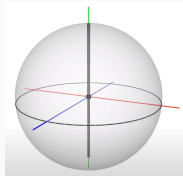
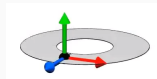
trivial rotation



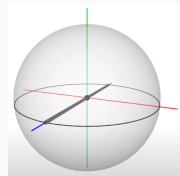
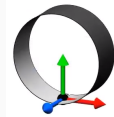
2π *x*-rotation



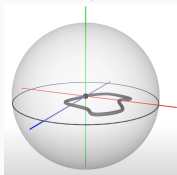
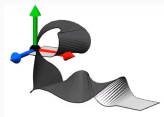
2π *y*-rotation



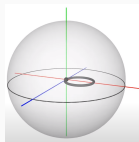
2π *z*-rotation



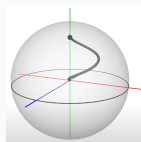
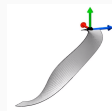
random rotation



closed



open path



<u>Belt</u>		<u>Path</u>
specific configuration	\longleftrightarrow	specific path
moving the ends	\longleftrightarrow	continuous deformation
ends have same orientation	\longleftrightarrow	closed path (loop)
can be flattened	\longleftrightarrow	contractible

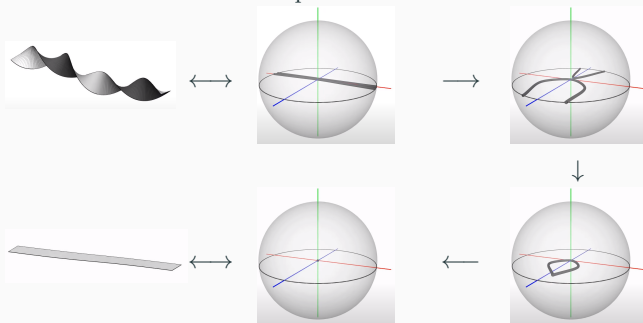
Back to Dirac's belt trick:

1. ends of the belt have same orientation \rightarrow we consider loops
(passing through the origin)
2. moving the ends of the belt \rightarrow continuous deformation
3. belt in original (flat) position \rightarrow trivial path

The question then becomes: **which loops are contractible ?**

Problem solved ?

- **4π -twist**: We saw in the beginning the the 4π -twist can be flattened, how can we see this in terms of paths ?



\Rightarrow the 4π -twist is **contractible** ! Great.

- **2π -twist**: we “clearly” see that is not contractible... no ?! Great..?..

Wierd aftertaste: our “proof” is good to show contractibility but bad to show non-contractibility and it only works for simple examples.

\Rightarrow We want a consistent and general way of studying paths in topological spaces.

Homotopy theory

Starting observation: depending on the topological space, all loops might not be contractible. Moreover, some loops are “fundamentally different” from each other.

Examples: \mathbb{R}^3 , S^2 , \mathbb{T}^2 , etc.

Paths and homotopies

For a topological space X :

- *Path* in X : continuous map $\gamma : [0, 1] \rightarrow X$, *loop* if closed
- γ_1 and γ_2 are *homotopically equivalent* if one can be deformed into the other: there exists $H : [0, 1] \times [0, 1] \rightarrow X$ such that

$$H(0, t) = \gamma_1(t) \quad \text{and} \quad H(1, t) = \gamma_2(t).$$

This is an equivalence relation (\sim).

For each $x_0 \in X$, we define

$$\pi_1(X, x_0) = \{\text{all loops based at } x_0\} / \sim,$$

it is the set of “fundamentally different” loops passing through x_0 . $[\gamma]$ is called the *homotopy class* of γ .

Fundamental group

Group structure on $\pi_1(X, x_0)$:

- **Product** of paths: $\gamma_1 \cdot \gamma_2 = \text{“}\gamma_1 \text{ then } \gamma_2\text{”}$
- **Inverse** path: $\gamma^{-1} = \text{“}\gamma \text{ traversed in the opposite direction”}$
- **Neutral** path: $e = \text{“constant path at the identity”}$
- For homotopy classes: $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$ and $[\gamma]^{-1} = [\gamma^{-1}]$

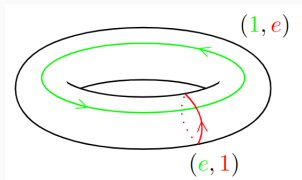
Important fact: up to isomorphism, $\pi_1(X, x_0)$ does not depend on x_0
 \Rightarrow we denote it as $\pi_1(X)$, it is called the **fundamental group** of X .

Contractible loops are \sim to a point, i.e. they are the element of $[e]$.

How to compute $\pi_1(X)$? Can be difficult, not discussed here.

Examples:

- $\pi_1(\mathbb{R}^3) = \{e\}$
- $\pi_1(S^2) = \{e\}$
- $\pi_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$
- $\pi_1(\mathbb{R}^2 \setminus \{p\}) = \mathbb{Z}$



Remark: $\pi_1(\mathbb{R}^3 \setminus \{p\}) = \{e\}$, higher homotopy groups for higher-dimensional holes ?

Question we had: are all loops in $SO(3)$ contractible ?

In homotopy language: is $\pi_1(SO(3))$ trivial ?

Answer: NO, one can compute that

$$\pi_1(SO(3)) = \mathbb{Z}_2$$

\Rightarrow There only two “fundamentally different” loops in $SO(3)$!

\Rightarrow all non-contractible loops are deformations of the 2π -twist !

The belt trick is a way of physically demonstrating that the fundamental group of $SO(3)$ is \mathbb{Z}_2 .

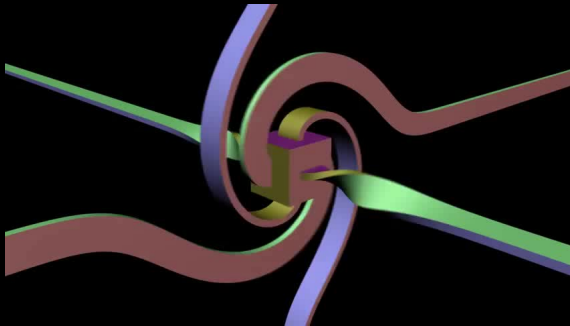
We can now say, with more confidence, that we understood Dirac's belt trick.

Are there other manifestation of homotopy in our practical world ?

Yes: the **spin** ! (you don't need a belt, but you need an electron)

Initially, this trick was a demonstration invented by P. Dirac (1902-1984) to explain the notion of spin to his students.

Animation 1



An object attached to belts or strings can spin continuously without becoming tangled. Notice that after the cube completes a 360° rotation, the spiral is reversed from its initial configuration. The belts return to their original configuration after spinning a full 720° .

Animation 2



A more extreme example demonstrating that this works with any number of strings. In the limit, a piece of solid continuous space can rotate in place like this without tearing or intersecting itself.

Quantum spin and $SU(2)$

What is the spin ?

Skipping most of the physics background:

Spin in quantum mechanics

1. the **spin** is an inherent property of any “particle”:
 - number $s \in \frac{1}{2}\mathbb{N}$, in our case $s = 1/2$
 - does not change, like the mass, charge, etc
 - classifies particles
2. a particle of spin s is, at a given moment, in a certain state described by the **spin vector**:
 - unit vector of $v \in \mathbb{C}^{2s+1}$, in our case $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2$
 - can evolve over time
3. what we can measure yet another quantity, called **observed spin**:
 - discrete value $s_{\text{obs.}} \in \{s, s-1, \dots, 0, \dots, -s+1, -s\}$
In our case, $s_{\text{obs.}} \in \{1/2, -1/2\}$ that we denote \uparrow and \downarrow
 - given a direction, e.g. $i = x, y, z$
 - outcome is random, we can only compute the probabilities of the different outcomes

What is the spin ?

How do measures happen ?

Let us introduce

$$v_{x,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, v_{x,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, v_{y,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}, v_{y,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}, v_{z,\uparrow} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_{z,\downarrow} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The probability of measuring $s_{\text{obs.}}$ in the direction i is given by the projection

$$P(i, s_{\text{obs.}}) = |\langle v_{i,k}, v \rangle_{\mathbb{C}^2}|^2 \quad (1)$$

where v is the spin vector of the particle.

Example: in the direction z ,

$$P(z, \uparrow) = |\alpha|^2, \quad P(z, \downarrow) = |\beta|^2. \quad (2)$$

Consequently:

- we must have $\langle v, v \rangle_{\mathbb{C}^2} = |\alpha|^2 + |\beta|^2 = 1$
- to “measure” the spin state, we must repeat the experience many times
- there are states that are always spin \uparrow or always spin \downarrow

Rotating a spin vector

For vectors: recall that the scalar product on \mathbb{R}^3 is $\langle v_1, v_2 \rangle_{\mathbb{R}^3} = (v_1)^T v_2$ and

$$\langle Rv_1, Rv_2 \rangle_{\mathbb{R}^3} = \langle v_1, v_2 \rangle_{\mathbb{R}^3} \quad \Leftrightarrow \quad R^T R = \mathbb{1}$$

so $\text{SO}(3)$ is the isometry group of \mathbb{R}^3 (+ orientation preserving).

For spin vectors: the scalar product on \mathbb{C}^2 is $\langle v_1, v_2 \rangle_{\mathbb{C}^2} = (v_1)^\dagger v_2$ and

$$\langle Uv_1, Uv_2 \rangle_{\mathbb{C}^2} = \langle v_1, v_2 \rangle_{\mathbb{C}^2} \quad \Leftrightarrow \quad U^\dagger U = \mathbb{1}$$

so, similarly,

Special unitary group

$\text{SU}(2)$ is the set of 2×2 complex matrices such that $U^\dagger U = \mathbb{1}$ and $\det U = 1$.

and $\text{SU}(2)$ is the isometry group of \mathbb{C}^2 (+ orientation preserving).

Like $\text{SO}(3)$ it is a Lie group so it can be viewed

SU(2) and SO(3)

What is the most general form of $U \in \text{SU}(2)$? Imposing $U^\dagger = U^{-1}$ and $\det U = 1$, we find

$$U = \begin{bmatrix} X + iY & Z + iW \\ -Z + iY & X - iY \end{bmatrix} \quad (3)$$

with $X^2 + Y^2 + Z^2 + W^2 = 1 \Rightarrow \text{SU}(2) \cong S^3$

so $\text{SU}(2)$ can be viewed as a group and a manifold, it is a Lie group

$\text{SU}(2)$ and $\text{SO}(3)$:

1. both Lie groups of dimension three
2. both are connected
3. $-\mathbb{1} \in \text{SU}(2)$ but $-\mathbb{1} \notin \text{SO}(3)$

How could we represent $\text{SU}(2) \cong S^3$ in $3d$?

Observation: S^2 is equivalent to two disks glued along their boundary

Similarly: S^3 is equivalent to balls glued along their boundary

BUT, are those spheres related to $\text{SO}(3)$?

In other words: how are the two notions of rotations related ?

\Rightarrow covering spaces !

Covering spaces

Covering space


For a topological space X , a *covering space* is a topological space E with a *projection map* $p = E \rightarrow X$ such that

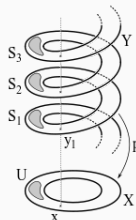
- p is continuous
- there exists a discrete set D and U and open neighborhood of $x \in X$ such that

$$p^{-1}(U) = \bigsqcup_{d \in D} V_d$$

and $p|_{V_d} = V_d \rightarrow U$ is a homeomorphism. V_d are called the *sheets*.

Examples:

- \mathbb{R} can cover S^1 with
 $p(t) = (\cos(2\pi t), \sin(2\pi t))$, $G(E) = \mathbb{Z}$
- S^1 can cover S^1 in several ways, with
 $p(z) = z^n$, $n \in \mathbb{N}$, $G(E) = \mathbb{Z}_n$ 
- other S^n ? S^1 is a special case



Properties of the covering space

Important remarks:

1. Some covering spaces are “equivalent”.

Isomorphisms

Two covering space E_1 and E_2 of X are *isomorphic* if there exists a homeomorphism $h : E_1 \rightarrow E_2$ such that $p_2 \circ h = p_1$.


2. The lifting of point can, by definition, be ambiguous.

Deck transformations

A *Deck transformation* is a homeomorphism $d : E \rightarrow E$ such that $p \circ d = p$. With composition, they form a group $G(E)$.

3. There can exist many covering spaces for the same base space. In some cases there exists a unique, maximal covering space:

Back to our examples:

- \mathbb{R} can cover S^1 : $G(E) = \mathbb{Z}$
- S^1 can cover S^1 with n sheets: $G(E) = \mathbb{Z}_n$ 
- other S^n ? S^1 is a special case

How to find the SO(3) matrix corresponding to an SU(2) matrix ?

We introduce the *Pauli matrices*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (4)$$

Then, for $\vec{r} = (x, y, z) \in \text{SO}(3)$, the matrix

$$\vec{r} \cdot \vec{\sigma} = r^i \sigma_i = \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix}$$

is **traceless** and **self-adjoint**, i.e. $\vec{r} \cdot \vec{\sigma} \in \mathfrak{su}(2)$. More precisely, σ_i are the generators of $\mathfrak{su}(2)$. Moreover $\det(\vec{r} \cdot \vec{\sigma}) = -(x^2 + y^2 + z^2)$.

We can then shown that, the new matrix $U(\vec{r} \cdot \vec{\sigma})U^\dagger$, with $U \in \text{SU}(2)$ is

- still **traceless** and **self-adjoint**
 $\Rightarrow \exists \vec{r}_U \in \text{SO}(3)$ such that $U(\vec{r} \cdot \vec{\sigma})U^\dagger = \vec{r}_U \cdot \vec{\sigma}$
- has **same determinant**, i.e. $|\vec{r}| = |\vec{r}_U|$
 $\Rightarrow \exists R_U \in \text{SO}(3)$ such that $\vec{r}_U = R_U \vec{r}$

In then end, for each $U \in \text{SU}(2)$, we have $p(U) \equiv R_U \in \text{SO}(3)$. This maps can be shown to be locally a **homeomorphism**.

Group relation:

There is recurring theme: two things $SU(2)$ correspond to one in $SO(3)$. This comes from the fact that $SU(2)$ is a **double**-cover of $SO(3)$, which can see in practice with

$$p(U) = p(-U).$$

Intuitively, we should be able to recover $SO(3)$ from $SU(2)$ if $U \sim -U$.

And, indeed,

$$SO(3) \cong SU(2)/\mathbb{Z}_2,$$

where the quotient means exactly that we identify U with $-U$.


Other formulation:

We saw that $SU(2) \cong S^3$ and $SO(3) \cong B^2(\pi)/\sim$ but $B^2(\pi)/\sim \cong \mathbb{RP}^3$ so, in more convenient language: S^3 is a double cover of \mathbb{RP}^3 , $\pi_1(S^3) = \{e\}$, and $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$.

What is the lift of the 2π -twist ? The path going from I to $-I$ .

Proof that 2π -twist is non-contractible in $SO(3)$:

Let us suppose that the 2π -twist is contractible. At each step of its contraction, we can lift the path to $SU(2)$. This provides us with a contraction of the lifted 2π -twist. However, the lifted 2π -twist does not have the same start and endpoint, which does not change during the contraction, therefore it is non-contractible. And so is the non-lifted path.

On the other hand, the 4π -twist lifts to a path going from I to -1 , to I again (). So it's a loop and the argument does not hold anymore. Make sense, since we already “showed” its contractibility.

1. $SU(2)$ is the universal covering space of $SO(3)$, it has two sheets
2. we constructed an explicit projection map
- 3.

Spinors

1. There are two topologically distinguishable classes (homotopy classes) of paths through rotations that result in the same overall rotation, as illustrated by the Dirac's belt trick. (True in any dimension)
2. Spinors change in different ways depending not just on the overall final rotation, but the details of how that rotation was achieved (by a continuous path in the rotation group).
3. The spin group is the group of all rotations keeping track of the class. It doubly covers the rotation group, since each rotation can be obtained in two inequivalent ways as the endpoint of a path.
4. The space of spinors by definition is equipped with a (complex) linear representation of the spin group.

Spinors in Physics

The group which acts on spin vectors is $SU(2)$.

Question: how do rotations act on spin vectors ? The rotation group of euclidean space is still $SO(3)$, so we need a way of doing an $SO(3)$ rotation through $SU(2)$ transformations.

This is exactly what the covering technology provides us: a unique way to lift a rotation from $SO(3)$ to $SU(2)$.

We saw that the 2π -twist is not closed \Rightarrow walking around such a particle would not give back the particle in the same states, it would invert the spin state.

But, very odd property ... could such exotic particles exist ? Or is it an error of interpretation ?

Yes, they do exist. Out of the 18 elementary particles, 12 of them have spin 1/2 ! E.g. electrons.

Technical details:

- instead of walking around the particle, we rotate it using a magnetic field (Lamor procession)
- we cannot detect the effect if there only one particle
- we do not actually use electrons but neutrons (see neutron interferometry).

Spin in nature:

- in nature, only spins 0 (Higgs boson), $1/2$ (electrons, quarks, etc) 1 (photons, gluinos, etc) and 2 (graviton)
- spins higher than 2 are very problematic, and not well-understood yet. Current topic of research (UMons !)
- spin- $1/3$ particles ? No, impossible, because $\pi_1(SO(3)) = \mathbb{Z}_2$. Example of mathematical constraint on physical models.

Conclusion

1. Dirac's belt trick can be understood by studying the fundamental group of $SO(3)$.
2. The universal cover of $SO(3)$ is $SU(2)$, in which the homotopy ambiguity is solved. Spin vectors transform under $SU(2)$ and covering space technology then allows us to better understand the nature of the spin in quantum mechanics.
3. Spinors can be defined in any dimension and for any spin. Leading to a generalization of usual vectors that take into account the topological difference between some rotations that, a priori, could look equivalent.
4. Spinors are fundamental in Physics and in particular all modern theories of fundamental interactions. Spinors model most of elementary particles. In particular, exactly like Dirac's belt, electrons rotate through the lift in $SU(2)$ thus taking into account the homotopy class of the rotation, how cool !

Thank you

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Example

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Introduction to the spin

Spin in quantum mechanics

The **spin** of an “particle” is a number $s \in \frac{1}{2}\mathbb{N}$.

The **spin state** of a particle of spin s is a unit vector in \mathbb{C}^{2s+1} .

The spin is a **property**, it cannot change (e.g. mass, charge)

The spin state is a **characteristic**, it evolves

How to interpret it ?

1. **directions:** we choose the direction in which we want to measure it
2. **probabilistic theory:** the outcome of the measure, we can only compute the probabilities of the different outcomes
3. **discrete quantity:** in the chosen direction, the spin will either appear to up or down (\uparrow or \downarrow)

The probability of measuring the spin $k = \uparrow, \downarrow$ in the direction $i = x, y, z$ is given by

$$P(i, k) = |\langle v_{i,k}, v \rangle|^2 \quad (5)$$

where v is the spin state of the particle, for some given vectors $v_{i,k}$.

The Lie algebra $\mathfrak{su}(2)$ is generated by the **Pauli matrices**

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \quad (6)$$



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