Group theory, Topology and Spin-$1/2$ Particles

From Dirac's belt to fermions

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Dirac's belt trick and the rotation group

Dirac's belt trick

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Rules:

1. you can only move the end of the belt
2. you cannot twist or rotate it

Goal: untwist a 2π-twist.

$\Rightarrow$ it tuns out to be impossible ! One turn negates the twist: $2\pi \to -2\pi$.

Therefore, possible for a 4π twist ...

Why is that ?

Space of rotations: SO(3) as a group

Rotations in 3-dimensional space: matrices that acts on $\mathbb{R}^3$ s.t.

1. preserve the scalar product: $O^T O = \mathbb{1}$ ($\Leftrightarrow$ O is orthogonal)
2. preserve the orientation: $\det O = 1$

Special othogonal group

SO(3) is the set of 3×3 real matrices such that $O^T O = \mathbb{1}$ and $\det O = 1$.

Three "fundamental" rotations:

$$x : \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad y : \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad z : \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ It forms a group.

Space of rotations: SO(3) as a topological space

Fundamental data that describes a rotation:

- an axis of rotation, i.e. a unit vector $\vec{n}$ $\rightarrow$ 2 parameters
- an angle of rotation $\theta \in [-\pi, \pi]$ (with $-\pi \sim \pi$) $\rightarrow$ 1 parameter

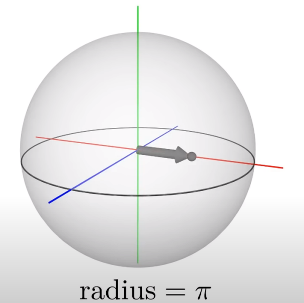
The space of rotations can then alternately be defined as a **3-sphere of radius $\boldsymbol{\pi}$ and its antipodal points identified**:

$$\boxed{\mathrm{SO}(3) \cong B^3(\pi)/\sim}$$

and for each point:

$$\text{direction} \leftrightarrow \text{axis}$$
$$\text{norm} \leftrightarrow \text{angle}$$

$\Rightarrow$ It forms a topological space.

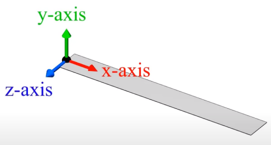


(group + topological space = Lie group)

Back to the belt

Mathematical description of the belt ?

- a belt is a strip, which is just a path + an orientation.
- given axis on the middle line along the belt, each set of axis is related by a rotation
- a belt configuration is equivalent to a continuous set of axis and therefore to a continuous set of translations, i.e. a path in SO(3)



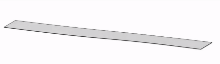
There is a bijection:

$$\boxed{\text{belt configuration} \Leftrightarrow \text{path in } SO(3)}$$

This gives us a new language to analyze the problem !

Dictionary

trivial rotation



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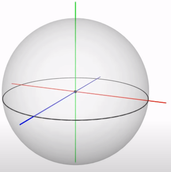
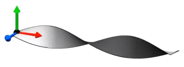


2π x**-rotation**



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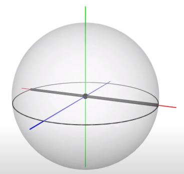
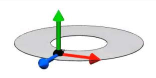


2π y**-rotation**



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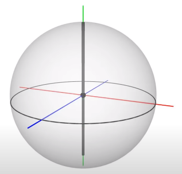
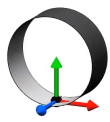


2π z**-rotation**



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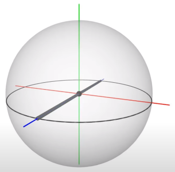
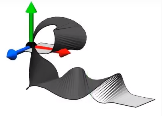


Dictionary

random rotation



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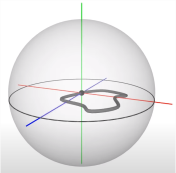
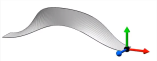


closed



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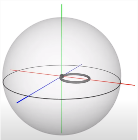
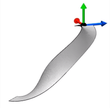


open path



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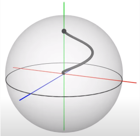


Dictionary

Belt		Path
specific configuration	$\longleftrightarrow$	specific path
moving the ends	$\longleftrightarrow$	continuous deformation
ends have same orientation	$\longleftrightarrow$	closed path (loop)
can be flattened	$\longleftrightarrow$	contractible

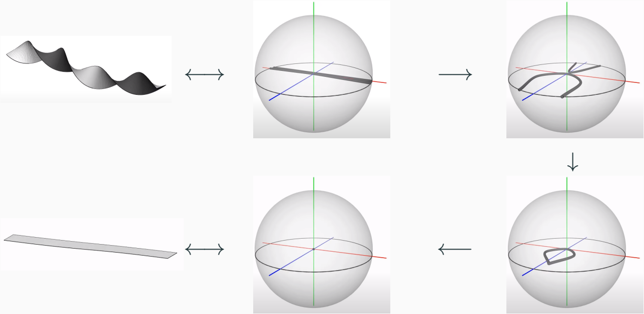
Back to Dirac's belt trick:

1. ends of the belt have same orientation $\rightarrow$ we consider loops (passing through the origin)
2. moving the ends of the belt $\rightarrow$ continuous deformation
3. belt in original (flat) position $\rightarrow$ trivial path

The question then becomes: **which loops are contractible ?**

Problem solved ?

- **4π-twist:** We saw in the beginning the the 4π-twist can be flattened, how can we see this in terms of paths ?



$\Rightarrow$ the 4π-twist is contractible ! Great.

- **2π-twist:** we "clearly" see that is not contractible... no ?! Great..?..

Wierd aftertaste: our "proof" is good to show contractibility but bad to show non-contractibility and it only works for simple examples.
$\Rightarrow$ We want a consistent and general way of studying paths in topological spaces.

Homotopy theory

Homotopoy theory primer

Starting observation: depending on the topological space, all loops might not be contractible. Moreover, some loops are "fundamentally different" from each other.
Examples: $\mathbb{R}^3$, S^2, $\mathbb{T}^2$, etc.

Paths and homotopies

For a topological space X:

- *Path* in X: continuous map $\gamma : [0,1] \to X$, *loop* if closed
- γ_1 and γ_2 are *homotopically equivalent* if one can be deformed into the other: there exists $H : [0,1] \times [0,1] \to X$ such that

  $$H(0,t) = \gamma_1(t) \qquad \text{and} \qquad H(1,t) = \gamma_2(t).$$

  This is an equivalence relation ($\sim$).

For each $x_0 \in X$, we define

$$\pi_1(X, x_0) = \{\text{all loops based at } x_0\} / \sim,$$

it is the set of "fundamentally different" loops passing through x_0. $[\gamma]$ is called the *homotopy class* of γ.

Fundamental group

Group structure on $\pi_1(X, x_0)$:

- **Product** of paths: $\gamma_1 \cdot \gamma_2 =$ "γ_1 then γ_2"
- **Inverse** path: $\gamma^{-1} =$ "γ traversed in the opposite direction"
- **Neutral** path: $e =$ "constant path at the identity"
- For homotopy classes: $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$ and $[\gamma]^{-1} = [\gamma^{-1}]$

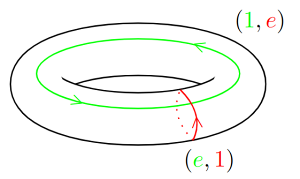
Important fact: up to isomorphism, $\pi_1(X, x_0)$ does not depend on x_0
$\Rightarrow$ we denote it as $\pi_1(X)$, it is called the fundamental group of X.

Contractible loops are $\sim$ to a point, i.e. they are the element of $[e]$.

How to compute $\pi_1(X)$? Can be difficult, not discussed here.

Examples:

- $\pi_1(\mathbb{R}^3) = \{e\}$
- $\pi_1(S^2) = \{e\}$
- $\pi_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$
- $\pi_1(\mathbb{R}^2 \backslash \{p\}) = \mathbb{Z}$



Remark: $\pi_1(\mathbb{R}^3 \backslash \{p\}) = \{e\}$, higher homotopy groups for higher-dimensional holes ?

Back to SO(3)

Question we had: are all loops in SO(3) contractible ?
In homotopy language: is $\pi_1(\mathrm{SO}(3))$ trivial ?

Answer: NO, one can compute that

$$\boxed{\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2}$$

$\Rightarrow$ There only two "fundamentally different" loops in SO(3) !
$\Rightarrow$ all non-contractible loops are deformations of the 2π-twist !

The belt trick is a way of physically demonstrating that the fundamental group of SO(3) is $\mathbb{Z}_2$.

We can now say, with more confidence, that we understood Dirac's belt trick.

Are there other manifestation of homotopy in our practical world ?

Yes: the spin ! (you don't need a belt, but you need an electron)
Initially, this trick was a demonstration invented by P. Dirac (1902-1984) to explain the notion of spin to his students.

Quantum spin and SU(2)

Rotating a spin vector

For vectors: recall that the scalar product on $\mathbb{R}^3$ is $\langle v_1, v_2 \rangle_{\mathbb{R}^3} = (v_1)^T v_2$ and

$$\langle Rv_1, Rv_2 \rangle_{\mathbb{R}^3} = \langle v_1, v_2 \rangle_{\mathbb{R}^3} \qquad \Leftrightarrow \qquad R^T R = \mathbb{1}$$

so SO(3) is the isometry group of $\mathbb{R}^3$ (+ orientation preserving).

For spin vectors: the scalar product on $\mathbb{C}^2$ is $\langle v_1, v_2 \rangle_{\mathbb{C}^2} = (v_1)^\dagger v_2$ and

$$\langle Uv_1, Uv_2 \rangle_{\mathbb{C}^2} = \langle v_1, v_2 \rangle_{\mathbb{C}^2} \qquad \Leftrightarrow \qquad U^\dagger U = \mathbb{1}$$

so, similarly,

Special unitary group

SU(2) is the set of 2×2 complex matrices such that $U^\dagger U = \mathbb{1}$ and $\det U = 1$.

and SU(2) is the isometry group of $\mathbb{C}^2$ (+ orientation preserving).

Like SO(3) it is a Lie group so it can be viewed

SU(2) and SO(3)

What is the most general for of $U \in \mathrm{SU}(2)$? Imposing $U^\dagger = U^{-1}$ and $\det U = 1$, we find

$$U = \begin{bmatrix} X + iY & Z + iW \\ -Z + iY & X - iY \end{bmatrix} \tag{1}$$

with $X^2 + Y^2 + Z^2 + W^2 = 1 \Rightarrow \mathrm{SU}(2) \cong S^3$
so SU(2) can be viewed as a group and a manifold, it is a Lie group

SU(2) and SO(3):

1. both Lie groups of dimension three
2. both are connected
3. $-\mathbb{1} \in \mathrm{SU}(2)$ but $-\mathbb{1} \notin \mathrm{SO}(3)$

How could we represent $\mathrm{SU}(2) \cong S^3$ in $3d$?

Observation: S^2 is equivalent to two disks glued along their boundary

Similarly: S^3 is equivalent to balls glued along their boundary

BUT, are those spheres related to SO(3) ?
In other words: how are the two notions of rotations related ?

$\Rightarrow$ covering spaces !

Covering spaces

Covering spaces

Covering space

For a topological space X, a *covering space* is a topological space E with a *projection map* $p = E \to X$ such that

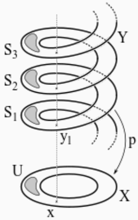
- p is continuous
- there exists a discrete set D and U and open neighborhood of $x \in X$ such that

$$p^{-1}(U) = \bigsqcup_{d \in D} V_d$$

  and $p|_{V_d} = V_d \to U$ is a homeomorphism. V_d are called the *sheets*.

Examples:

- $\mathbb{R}$ can cover S^1 with $p(t) = (\cos(2\pi t), \sin(2\pi t))$, $G(E) = \mathbb{Z}$
- S^1 can cover S^1 in several ways, with $p(z) = z^n$, $n \in \mathbb{N}$, $G(E) = \mathbb{Z}_n$ ✏️
- other S^n ? S^1 is a special case



Properties of the covering space

Important remarks:

1. **Some covering spaces are "equivalent".**

   Isomorphisms

   Two covering space E_1 and E_2 of X are *isomorphic* if there exists a homeomorphism $h : E_1 \to E_2$ such that $p_2 \circ h = p_1$.

2. **The lifting of point can, by definition, be ambiguous.**

   Deck transformations

   A *Deck transformation* is a homeomorphism $d : E \to E$ such that $p \circ d = p$. With composition, they form a group $G(E)$.

3. **There can exists many covering spaces for the same base space.** In some cases there exists a unique, maximal covering space:

Back to our examples:

- $\mathbb{R}$ can cover S^1: $G(E) = \mathbb{Z}$
- S^1 can cover S^1 with n sheets: $G(E) = \mathbb{Z}_n$ ✏️
- other S^n ? S^1 is a special case

Covering spaces and paths

Back to $\mathbf{SU}(2)$

From SU(2) to SO(3)

What is the projection map ?

We introduce the *Pauli matrices*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2}$$

Then, for $\overrightarrow{r} = (x, y, z) \in \mathrm{SO}(3)$, the matrix

$$\overrightarrow{r} \cdot \overrightarrow{\sigma} = r^i \sigma_i = \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix}$$

is traceless and self-adjoint, i.e. $\overrightarrow{r} \cdot \overrightarrow{\sigma} \in \mathfrak{su}(2)$. More precisely, σ_i are the generators of $\mathfrak{su}(2)$. Moreover $\det(\overrightarrow{r} \cdot \overrightarrow{\sigma}) = -(x^2 + y^2 + z^2)$.

We can then shown that, the new matrix $U(\overrightarrow{r} \cdot \overrightarrow{\sigma})U^\dagger$, with $U \in \mathrm{SU}(2)$ is

- still traceless and self-adjoint
  $\Rightarrow \exists \overrightarrow{r}_U \in \mathrm{SO}(3)$ such that $U(\overrightarrow{r} \cdot \overrightarrow{\sigma})U^\dagger = \overrightarrow{r}_U \cdot \overrightarrow{\sigma}$
- has same determinant, i.e $|\overrightarrow{r}| = |\overrightarrow{r}_U|$
  $\Rightarrow \exists R_U \in \mathrm{SO}(3)$ such that $\overrightarrow{r}_U = R\overrightarrow{r}$

In then end, for each $U \in \mathrm{SU}(2)$, we have $p(U) \equiv R_U \in \mathrm{SO}(3)$. This maps can be shown to be locally a homeomorphism.

Last comments on $SU(2)$ and $\mathbf{SO}(3)$

Group relation:

There is recurring theme: two things SU(2) correspond to one in SO(3). This comes from the fact that SU(2) is a double-cover of SO(3), which can can see in practice with

$$p(U) = p(-U).$$

Intuitively, we should be able to recover SO(3) from SU(2) if $U \sim -U$.

And, indeed,

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2,$$

where the quotient means exactly that we identity U with $-U$.

Other formulation:

We saw that $\mathrm{SU}(2) \cong S^2$ and $\mathrm{SO}(3) \cong B^2(\pi)/\sim$ but $B^2(\pi)/\sim \cong \mathbb{RP}^3$ so, in more convenient language: S^3 is a double cover of $\mathbb{RP}^3$, $\pi_1(S^3) = \{e\}$, and $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$.

Back to $\mathrm{SU}(2)$

Back to SU(2)

What is the lift of the 2π-twist ? The path going from I to $-I$

<u>Proof that 2π-twist is non-contractible in SO(3):</u>

> Let us suppose that the 2π-twist is contractible. At each step of its contraction, we can lift the path to SU(2). This provides us with a contraction of the lifted 2π-twist. However, the lifted 2π-twist does not have the same start and endpoint, which does not change during the contraction, therefore it is non-contractible. And so is hte non-lifted path.

On the other hand, the 4π-twist lifts to a path going from I to -1, to I again (). So its a loop and the argument does not hold anymore. Make sens, since we already “showed” its contractibility.

Summary on SU(2) and SO(3)

1. SU(2) is the universal covering space of SO(3), it has two sheets
2. we constructed an explicit projection map
3.

Spinors

Summary on spinors

1. There are two topologically distinguishable classes (homotopy classes) of paths through rotations that result in the same overall rotation, as illustrated by the Dirac's belt trick. (True in any dimension.)
2. Spinors change in different ways depending not just on the overall final rotation, but the details of how that rotation was achieved (by a continuous path in the rotation group).
3. The spin group is the group of all rotations keeping track of the class. It doubly covers the rotation group, since each rotation can be obtained in two in-equivalent ways as the endpoint of a path.
4. The space of spinors by definition is equipped with a (complex) linear representation of the spin group.

Spinors in Physics

What is the spin ?

Skipping most of the physics background:

Spin in quantum mechanics

1. the *spin* is an inherent property of any "particle":
   - number $s \in \frac{1}{2}\mathbb{N}$, in our case $s = 1/2$
   - does not change, like the mass, charge, etc
   - classifies particles
2. a particle of spin s is, at a given moment, in a certain state described by the *spin vector*:
   - unit vector of $v \in \mathbb{C}^{2s+1}$, in our case $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2$
   - can evolve over time
3. what we can measure yet another quantity, called *observed spin*:
   - discrete value $s_{\text{obs.}} \in \{s, s-1, \dots, 0, \dots, -s+1, -s\}$
     In our case, $s_{\text{obs.}} \in \{1/2, -1/2\}$ that we denote $\uparrow$ and $\downarrow$
   - given a direction, e.g. $i = x, y, z$
   - outcome is random, we can only compute de probabilities of the different outcomes

What is the spin ?

How do measures happen ?

Let us introduce

$$v_{x,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, v_{x,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, v_{y,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}, v_{y,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}, v_{z,\uparrow} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_{z,\downarrow} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The probability of measuring $s_{\text{obs.}}$ in the direction i is given by the projection

$$P(i, s_{\text{obs.}}) = \left|\langle v_{i,k}, v\rangle_{\mathbb{C}^2}\right|^2 \tag{3}$$

where v is the spin vector of the particle.

Example: in the direction z,

$$P(z, \uparrow) = |\alpha|^2, \qquad P(z, \downarrow) = |\beta|^2. \tag{4}$$

Consequently:

- we must have $\langle v, v\rangle_{\mathbb{C}^2} = |\alpha|^2 + |\beta|^2 = 1$
- to “measure” the spin state, we must repeat the experience many times
- there are states that are always spin $\uparrow$ or always spin $\downarrow$

Rotating a spin vector

The group which acts on spin vectors is SU(2).

Question: how do rotations act on spin vectors ? The rotation group of euclidean space is still SO(3), so we need a way of doing an SO(3) rotation through SU(2) transformations.

This is exactly what the covering technology provides us: a unique way to lift a rotation from SO(3) to SU(2).

We saw that the 2π-twist is not closed $\Rightarrow$ walking around such a particle would not give back the particle in the same states, it would inverts the spin state.

But, very odd property ... could such exotic particles exist ? Or is it an error of interpretation ?

Yes, they do exist. Out of the 18 elementary particles, 12 of them have spin 1/2 ! E.g. electrons.

Remarks

Technical details:

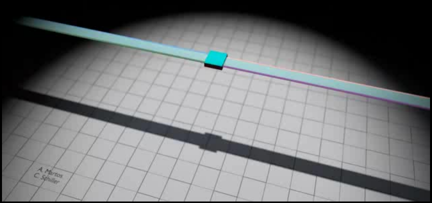
- instead of walking around the particle, we rotate it using a magnetic field (Lamor procession)
- we cannot detect the effect if there only one particle
- we do not actually use electrons but neutrons (see neutron interferometry).

Spin in nature:

- in nature, only spins 0 (Higgs boson), 1/2 (electrons, quarks, etc), 1 (photons, gluons, etc) and 2 (graviton)
- spins higher than 2 are very problematic, and not well-understood yet. Current topic of research (UMons !)
- spin-1/3 particles ? No, impossible, because $\pi_1(SO(3)) = \mathbb{Z}_2$. Example of mathematical constraint on physical models.

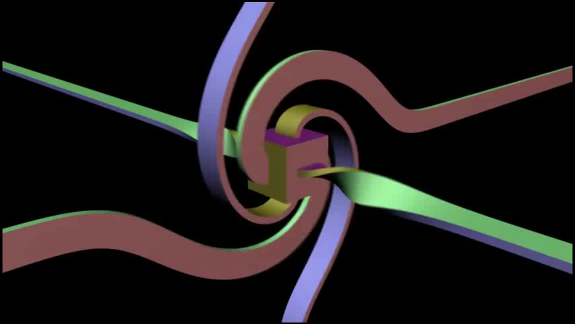
Behind quantum mechanics: modern fundamental physics is relativistic so we use a pseudo-scalar product. The isometry group becomes $\mathrm{SO}(1,3)$. The spinor theory we depicted can be generalized to $\mathrm{SO}(p,q)$.

Fun facts



Expanding the Dirac's belt trick setup, one can attach two belts to an object and rotate it by 360° without getting tangled. So it can spin continuously without becoming tangled

Anti-twister mechanisms



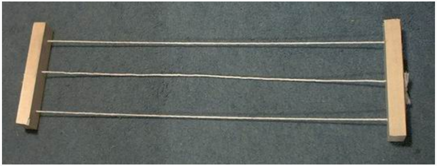
Increasing the number of belts does not change the behavior; the cube completes a 360° rotation, the spiral is reversed from its initial configuration. The belts return to their original configuration after spinning a full 720°.

Anti-twister mechanisms

A more extreme example demonstrating that this works with any number of strings. In the limit, a piece of solid continuous space can rotate in place like this without tearing or intersecting itself.

Fun facts

- anti-twister machnism is used in engineering to supply electric power to rotating devices
- cup on the hand trick (balinese candle dance or Philippine wine dance)
- tangloids



Conclusion

1. Dirac's belt trick can be understood by studying the fundamental group of SO(3).
2. The universal cover of SO(3) is SU(2), in which the homotopy ambiguity is solved. Spin vectors transform under SU(2) and covering space technology then allows us to better understand the nature of the spin in quantum mechanics.
3. Spinors can be defined in any dimension and for any spin. Leading to a generalization of usual vectors that take into account the topological difference between some rotations that, a priori, could look equivalent.
4. Spinors are fundamental in Physics and in particular all modern theories of fundamental interactions. Spinors model most of elementary particles. In particular, exactly like Dirac's belt, electrons rotate through the lift in SU(2) thus taking into account the homotopy class of the rotation, how cool !

Thank you

Blocks

Three different block environments are pre-defined and may be styled with an optional background color.

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Example

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Introduction to the spin

Spin in quantum mechanics

The spin of an "particle" is a number $s \in \frac{1}{2}\mathbb{N}$.
The spin state of a particle of spin s is a unit vector in $\mathbb{C}^{2s+1}$.

The spin is a property, it cannot change (e.g. mass, charge)
The spin state is a characteristic, it evolves

How to interpret it ?

1. **directions:** we choose the direction in which we want to measure it
2. **probabilistic theory:** the outcome of the measure, we can only compute the probabilities of the different outcomes
3. **discrete quantity:** in the chosen direction, the spin will either appear to up or down ($\uparrow$ or $\downarrow$)

The probability of measuring the spin $k = \uparrow, \downarrow$ in the direction $i = x, y, z$ is given by

$$P(i,k) = \left|\langle v_{i,k}, v\rangle\right|^2 \tag{5}$$

where v is the spin state of the particle, for some given vectors $v_{i,k}$.

Spin in quantum mechanics

The Lie algebra $\mathfrak{su}(2)$ is generated by the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{6}$$

References i

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