Group theory, Topology and Spin-1/2 Particles

From Dirac's belt to fermions

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Dirac's belt trick and the rotation group

Dirac's belt trick

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Goal: deform the belt to untwist a 4π-twist. Possible ? Yes !

Result: the belts rotates two times faster than its ends.

What about a 2π-twist ? It tuns out to be impossible ! One turn negates the twist: $2\pi \to -2\pi$.

Why is that ? How can we understand this ?

Space of rotations: SO(3) as a group

What is a rotation in $\mathbb{R}^3$? It is a real 3×3 matrix R that must

1. preserve the scalar product: $R^T R = \mathbb{1}$ ($\Leftrightarrow$ R is orthogonal)
2. preserve the orientation: $\det R = 1$

Special othogonal group

SO(3) is the set of 3×3 real matrices such that $R^T R = \mathbb{1}$ and $\det R = 1$.

Three "fundamental" rotations:

$$x : \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad y : \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad z : \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

any rotation can be obtained by composing those three rotation.

$\Rightarrow$ It forms a group.

Space of rotations: SO(3) as a manifold

Fundamental data that describes a rotation:

- an axis of rotation, i.e. a unit vector $\vec{n}$ $\rightarrow$ 2 parameters
- an angle of rotation $\theta \in [-\pi, \pi]$ (with $-\pi \sim \pi$) $\rightarrow$ 1 parameter

The space of rotations can then alternately be defined as a **3-sphere of radius π and its antipodal points identified**:

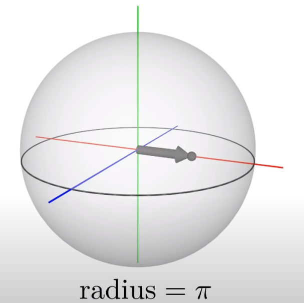
$$\boxed{\mathrm{SO}(3) \cong B^3(\pi)/\sim}$$

and for each point:

$$\text{direction} \leftrightarrow \text{axis}$$

$$\text{norm} \leftrightarrow \text{angle}$$

$\Rightarrow$ It is also a manifold , most famously known as $\mathbb{RP}^3$.



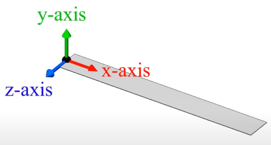
We have $\dim(\mathrm{SO}(3)) = 3$.

(Note that group + manifold = Lie group.)

Back to the belt

Mathematical description of the belt ?

- a belt is a strip, which is just a path + an orientation.
- given axis on the middle line along the belt, each set of axis is related by a rotation
- a belt configuration is equivalent to a continuous set of axis and therefore to a continuous set of translations, i.e. a path in SO(3)



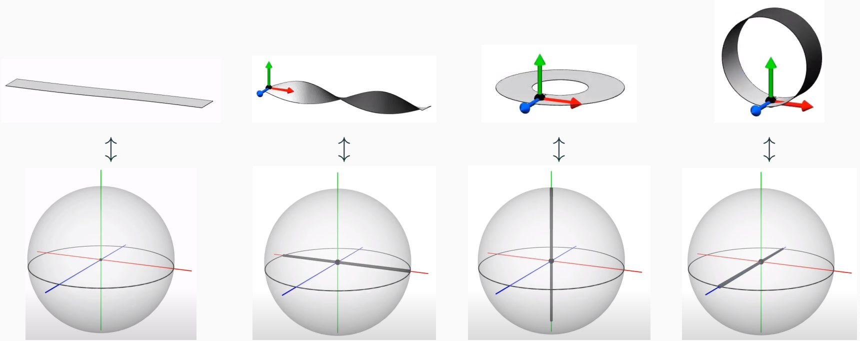
There is a bijection:

$$\boxed{\text{belt configuration} \Leftrightarrow \text{path in } SO(3)}$$

This gives us a new language to analyze the problem !

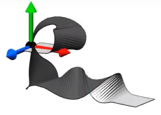
Examples

trivial rotation | **2π x-rotation** | **2π y-rotation** | **2π z-rotation**



Examples

random rotation



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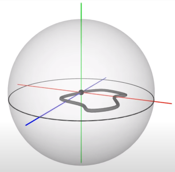

closed

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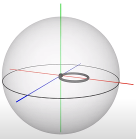
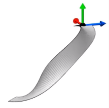


open path



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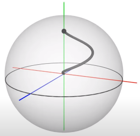


Dictionary

We see that:

Belt		Path
specific configuration	$\longleftrightarrow$	specific path
moving the ends	$\longleftrightarrow$	continuous deformation
ends have same orientation	$\longleftrightarrow$	loop
can be flattened	$\longleftrightarrow$	contractible

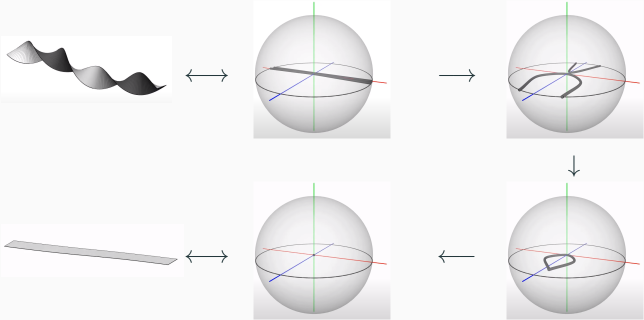
Back to Dirac's belt trick: the rules were

1. ends of the belt must keep the same orientation $\rightarrow$ we consider loops
2. moving the ends of the belt $\rightarrow$ continuous deformation
3. belt in original (flat) position $\rightarrow$ trivial constant path

The question "can the belt be flattened ?" then becomes "**which loops are contractible ?**"

Problem solved ?

- **4π-twist:** We saw in the beginning the the 4π-twist can be flattened, how can we see this in terms of paths ?



$\Rightarrow$ the 4π-twist is contractible ! Great.

- **2π-twist:** we "clearly" see that is not contractible... no ?! Great..?..

Wierd aftertaste: our "proof" is good to show contractibility but bad to show non-contractibility and it only works for simple examples.
$\Rightarrow$ We want a consistent and general way of studying paths in topological spaces.

Homotopy theory

Starting observation: depending on the topological space, all loops might not be contractible. Moreover, some loops are "fundamentally different" from each other, e.g. in $\mathbb{R}^3$, S^2, $\mathbb{T}^2$, etc.

Paths and homotopies

For a topological space X:

- *Path* in X: continuous map $\gamma : [0,1] \to X$,
- *Loop* : closed path, i.e. embedded circle,
- γ_1 and γ_2 are *homotopically equivalent* if one can be deformed into the other: there exists $f_t : [0,1] \to X$ with $t \in I$ such that

$$f_0(s) = \gamma_1(s) \qquad \text{and} \qquad f_1(s) = \gamma_2(s).$$

and the endpoints are fixed. This is an equivalence relation ($\sim$).

For each $x_0 \in X$, we define

$$\boxed{\pi_1(X, x_0) = \{\text{all loops based at } x_0\}/\sim,}$$

$\to$ set of "fundamentally different" loops passing through x_0.

Fundamental group

The elements of $\pi_1(X, x_0)$ are $[\gamma]$, called the *homotopy class* of γ.

Group structure on $\pi_1(X, x_0)$:

- **Product** of paths: $\gamma_1 \cdot \gamma_2 =$ "γ_1 then γ_2"
- **Inverse** path: $\gamma^{-1} =$ "γ traversed in the opposite direction"
- **Neutral** path: $e =$ "constant path at the identity"
- For homotopy classes: $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$ and $[\gamma]^{-1} = [\gamma^{-1}]$

Important fact: if X is path-connected, $\pi_1(X, x_0)$ does not depend on x_0, up to isomorphism.
$\Rightarrow$ we denote it as $\pi_1(X)$, it is called the fundamental group of X.

Contractible loops are $\sim$ to a point, i.e. they are the element of $[e]$.

Proposition (product of spaces)

If X and Y are path-connected, $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Proposition (maps between spaces)

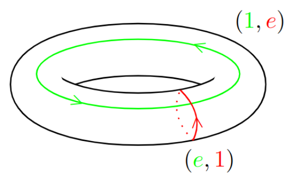
If $\varphi : X \to Y$ is a continuous map, is induces a homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ though $\varphi_*([\gamma]) = [\varphi \circ \gamma]$.

How to compute $\pi_1(X)$?

Can be difficult, there a different methods (e.g. Van Kampen theorem, Hopf fibrations, Hurewicz theorems, etc), not discussed here. A lot of homotopy groups are still unknown !

Examples: ✏️

- $\pi_1(\mathbb{R}^2) = 0$
- $\pi_1(S^2) = 0$
- $\pi_1(S^1) = \mathbb{Z}$
- $\pi_1(\mathbb{T}^2) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$
- $\pi_1(\mathbb{R}^2 \backslash \{p\}) = \mathbb{Z}$



Just $\pi_1(S^1) = \mathbb{Z}$ implies various famous theorems: fundamental theorem of algebra, Brouwer's fixed point theorem, Borusk-Ulam theorem etc.

Remark: $\pi_1(\mathbb{R}^2 \backslash \{p\}) = \mathbb{Z}$ but $\pi_1(\mathbb{R}^3 \backslash \{p\}) = 0$, higher homotopy groups for higher-dimensional holes ? Yes, higher-homotopy groups:

$$\pi_n(X, x_0) = \{S^n \text{ based at } x_0\} / \sim .$$

Homotopy groups of spheres

Good example of the complexity of homotpy groups:

- ▷ embedding a sphere in a higher-dimensional one is always trivial
- ▷ embedding a sphere in itself always works in the same way, regardless of the dimension
- ▷ embedding a sphere in lower-dimensional one is much more complicated: periodic for a bit, then completely chaotic

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

Back to SO(3)

Question we had: are all loops in SO(3) contractible ?
In homotopy language: is $\pi_1(\mathrm{SO}(3))$ trivial ?

Answer: NO, one can show that

$$\boxed{\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2}$$

$\Rightarrow$ There only two "fundamentally different" loops in SO(3) !
$\Rightarrow$ there is only one kind of non-contractible loop !

Indeed, there only two different initial configurations (i.e. two possible loops in SO(3)):

- $4\pi k$-twists which are all equivalent
- $(4\pi k + 2\pi)$-twists which are all equivalent

with $k \in \mathbb{Z}$.

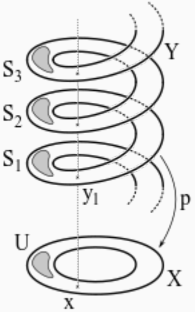
We have a better understanding Dirac's belt trick. **But still no proof !**
Homotopy theory allowed us to understand the ways of embedding loops in some spaces, we now need a tool to lift this ambiguity: covering space !

Covering spaces

Covering spaces

Covering space

For a topological space X, a *covering space* is a topological space $\widetilde{X}$ with a *projection map* $p : \widetilde{X} \to X$ such that there exists an open cover $\{\mathrm{U}_\alpha\}$ for which $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped by p homeomorphically on U_α.
If X is connected, $|p^{-1}(x)|$ is constant and called the number of *sheets*.



Example: covering the circle. Many possibilities:✏️

- $\mathbb{R}$ covers S^1 with $p_1(t) = (\cos(2\pi t), \sin(2\pi t))$,
- $\mathbb{R}$ covers S^1 with $p_2(t) = (\cos(5t), \sin(5t))$,
- S^1 covers S^1 in several ways, with $p(z) = z^n$, $n \in \mathbb{N}$,
- $S^1 \sqcup S^1$ covers S^1 with $p(k, t) = t$, $k = 0, 1$.

General properties

1. Some covering spaces are "equivalent": isomorphisms between covering spaces.
   For S^1, p_1 and p_2 are isomorphic.
2. The lifting of point can, by definition, be ambiguous: related by Deck transformations, which form a group $G(\widetilde{X})$.
   For S^1, $G(\mathbb{R}) = \mathbb{Z}$ and $G(S^1) = \mathbb{Z}_n$.
3. Some covering spaces are "included" in others: we restrict ourselves to connected covering spaces.
4. There can exists many covering spaces for the same base space. Under some assumtions, there exists a unique, maximal covering space, called *universal covering space* (UCS).
   $\mathbb{R}$ is the UCS of S^1.

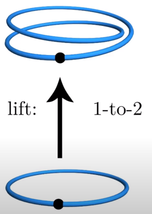
Lifting properties

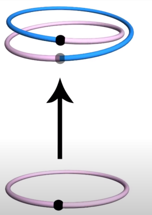
- lifting points is ambiguous
- lifting path is not ambiguous if the starting point is fixed
- constant paths are lifted to constant paths
- the projection of a homotopy is a homotopy for the projected paths
- the lifts of homotopy-equivalent paths are homotopically equivalent! $\rightarrow$ relation between $\pi_1(X)$ and $\widetilde{X}$?

If $\widetilde{X}$ is the UCS of X, we actually have

$$\boxed{\pi_1(X) = G(\widetilde{X}).}$$

$\Rightarrow$ algebraic features of $\pi_1(X)$ can be seen as geometric features of $\widetilde{X}$.





Covering space of SO(3)

One can show that

$$\boxed{\text{The universal covering space of SO(3) is SU(2).}}$$

Special unitary group

SU(2) is the set of 2×2 complex matrices such that $U^\dagger U = \mathbb{1}$ and $\det U = 1$.

Properties of SU(2):

What is the most general for of $U \in \mathrm{SU}(2)$? Imposing $U^\dagger = U^{-1}$ and $\det U = 1$, we find

$$U = \begin{bmatrix} X + iY & Z + iW \\ -Z + iY & X - iY \end{bmatrix} \tag{1}$$

with $X^2 + Y^2 + Z^2 + W^2 = 1 \Rightarrow \mathrm{SU}(2) \cong S^3$

$\Rightarrow$ SU(2) can be viewed as a group and a manifold, i.e. it is a Lie group.

SU(2) and SO(3):

1. both Lie groups of dimension three
2. both are connected
3. $-\mathbb{1} \in \mathrm{SU}(2)$ but $-\mathbb{1} \notin \mathrm{SO}(3)$

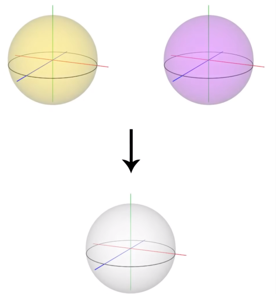
SU(2) and SO(3)

How could we represent $\mathrm{SU}(2) \cong S^3$ in $3d$?

Observation: S^2 is equivalent to two disks glued along their boundary.

Similarly, S^3 is equivalent to balls glued along their boundary.

Question: are those balls related to SO(3) ? In other words: how are the two notions of rotations related ? They are the sheets!



The projection is

$$p\left(\begin{bmatrix} x & y \\ -\overline{y} & \overline{x} \end{bmatrix}\right) = \begin{bmatrix} \mathrm{Re}(x^2 - y^2) & \mathrm{Im}(x^2 + y^2) & -2\mathrm{Re}(xy) \\ -\mathrm{Im}(x^2 - y^2) & \mathrm{Re}(x^2 + y^2) & 2\mathrm{Im}(xy) \\ 2\mathrm{Re}(x\overline{y}) & 2\mathrm{Im}(x\overline{y}) & |x|^2 - |y|^2 \end{bmatrix}$$

with $|x|^2 + |y|^2 = 1$.

SU(2) and SO(3)

Group relation:

The fact that SU(2) is a double-cover of SO(3) can can see in practice with

$$p(U) = p(-U).$$

Intuitively, we should be able to recover SO(3) from SU(2) if $U \sim -U$.
And, indeed,

$$\boxed{\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2,}$$

where the quotient means exactly that we identity U with $-U$.

Other formulation:

We saw that S^3 is a universal double-sheeted cover of $\mathbb{RP}^3$, $\pi_1(S^3) = \{e\}$, and $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$. This makes sense since

$$\mathbb{RP}^3 = S^3/\{(x, y, z) \sim (-x, -y, -z)\} = S^3/\mathbb{Z}_2 = \mathrm{SU}(2)/\mathbb{Z}_2,$$

we get the previous group relation.

Bringing it all together

What are the lifts of the 2π-twist and the 4π-twist ?✏️

$$2\pi\text{-twist} \to \text{path from } I \text{ to } -I$$

$$4\pi\text{-twist} \to \text{path from } I \text{ to } -I \text{ to } I \text{ again}$$

Proof that 2π-twist is non-contractible in SO(3):

Let us suppose that the 2π-twist is contractible. At each step of its contraction, we can lift the path to SU(2). This provides us with a contraction of the lifted 2π-twist. However, the lifted 2π-twist does not have the same start and endpoint, therefore it is not contractible, and so is the non-lifted path.

Proof that 4π-twist is contractible in SO(3):

The lift of the 4π-twist is a loop. Since $\pi_1(\mathrm{SU}(2)) = \pi_1(S^3) = 0$, this loop is necessarily contractible. Projecting each step of its contraction provides us with a contraction of the 4π-twist path in SO(3).

Summary of the analysis of Dirac's belt trick

- Belt configurations are equivalent to paths in SO(3). New question: we want to classify the contractable and non-contractable loop.
- Fundamental groups and covering spaces: $\pi_1(X)$ and $\widetilde{X}$ are two pictures of the same thing. $\widetilde{X}$ is the space that contains the same information plus the topological information of non-equivalent paths, i.e. it "solves" the homotopy ambiguity.
- The UCS of SO(3) is SU(2), and it is a double cover.
- There only two kinds topologically-distinguishable loops: $[4\pi k\text{-twists}]$ is contractible and the $[(4\pi k + 2\pi)\text{-twists}]$ is not contractible.

The belt trick is a way of physically demonstrating that the fundamental group of SO(3) is $\mathbb{Z}_2$.

Are there other manifestations of homotopy in our practical world ?

Yes: the spin ! (You don't need a belt, but you need an electron.)
Initially, this trick was a demonstration invented by P. Dirac (1902-1984) to explain the notion of spin to his students.

Quantum spin

What is the spin ?

Skipping most of the physics background:

Spin in quantum mechanics

1. The *spin* is an inherent property of any "particle":
   - it's a number $s \in \frac{1}{2}\mathbb{N}$, in our case $s = 1/2$
   - does not change (like the mass, charge, etc)
   - classifies particles
2. A particle of spin s is, at a given moment, in a certain state described by the *spin vector*:
   - unit vector of $v \in \mathbb{C}^{2s+1}$, in our case $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2$
   - this state can evolve over time
3. What we can measure yet another quantity, called *observed spin*:
   - discrete value $s_{\text{obs.}} \in \{s, s-1, \dots, 0, \dots, -s+1, -s\}$
     in our case, $s_{\text{obs.}} \in \{1/2, -1/2\}$ that we denote $\uparrow$ and $\downarrow$
   - given a direction, e.g. $i = x, y, z$
   - outcome is random, we can only compute de probabilities of the different outcomes

How do measures happen ?

Let us introduce

$$v_{x,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, v_{x,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, v_{y,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}, v_{y,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}, v_{z,\uparrow} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_{z,\downarrow} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The probability of measuring $s_{\text{obs.}}$ in the direction i is given by the projection

$$P(i, s_{\text{obs.}}) = |\langle v_{i,s_{\text{obs.}}}, v \rangle_{\mathbb{C}^2}|^2 \tag{2}$$

where v is the spin vector of the particle and $\langle v, w \rangle = v^\dagger w$ is a scalar product on $\mathbb{C}^2$.

Example: in the direction z,

$$P(z, \uparrow) = |\alpha|^2 , \qquad P(z, \downarrow) = |\beta|^2 . \tag{3}$$

Consequently:

- we must have $\langle v, v \rangle_{\mathbb{C}^2} = |\alpha|^2 + |\beta|^2 = 1$
- to “measure” the spin state, we must repeat the experience many times
- there are states that are always spin $\uparrow$ or always spin $\downarrow$

How to rotate a spin vector ?

We need to leave the probabilities $\Rightarrow$ scalar product invariant:

$$\boxed{\text{Spin vectors transform under the isometry group of } \mathbb{C}^2\text{, i.e. SU(2) !}}$$

How do space rotations act on spin vectors ?

The rotation group of euclidean space is still SO(3), so we need a way of doing an SO(3) rotation through SU(2) transformations.

This is exactly what the covering technology provides us: a unique way to lift a rotation from SO(3) to SU(2).

Interpretation !?

The 2π-twist is not closed $\Rightarrow$ walking around such a particle would not give back the particle in the same states, it would negate the spin state. Very odd property ... Could such exotic particles exist ? Error of interpretation ?

Yes, they do exist! Out of the 18 elementary particles, 12 of them have spin 1/2 ! And this have been observed experimentally.

The bizarre natre of fermions

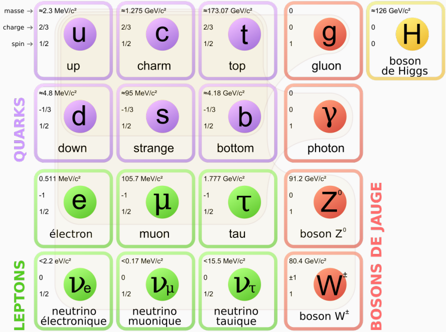


Figure 1: Standard Model of particle physics.

Practical details:

- Instead of walking around the particle, we rotate it using a magnetic field (Lamor procession).
- We cannot detect the "'−" sign if only one particle, at least two are necessary.
- We do not actually use electrons but neutrons (see neutron interferometry).

Generalizations

Other spins:

Other dimensions:

Ending remarks on spinors

Spin in nature:

- only spins 0 (Higgs boson), 1/2 (electrons, quarks, etc), 1 (photons, gluons, etc) and 2 (graviton) are found in nature
- spins higher than 2 are technically very problematic, and not well-understood yet. Current topic of research (U Mons !)
- spin-1/3 particles ? No, impossible, because $\pi_1(SO(3)) = \mathbb{Z}_2$. Example of mathematical constraint on physical models.

Behind quantum mechanics:

The spinors we encountered previously are spinors in Quantum Mechanics, which is non-relativistic. Modern Physics is relativistic therefore we mainly care about the indefinite rotation groups rather than the usual rotation groups, because of special relativity. The whole theory can be generalized accordingly:

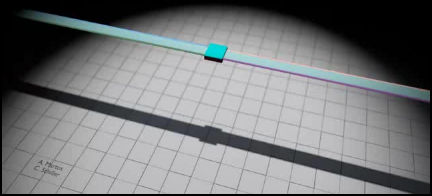
	Non-relativistic	**Relativistic**
rotation group	$\mathrm{SO}(3)$	$\mathrm{SO}(1,3)$
UCS	$\mathrm{Spin}(3) = \mathrm{SU}(2)$	$\mathrm{Spin}(1,3) = \mathrm{SL}(2,\mathbb{C})$

Summary on spinors

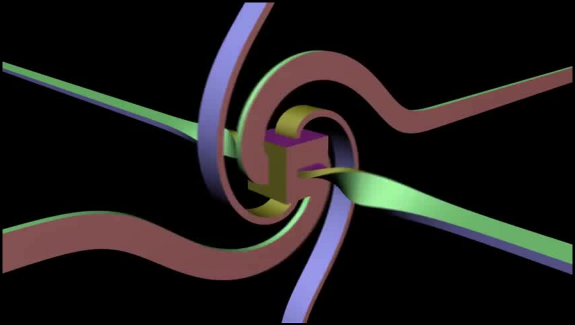
1. There are two topologically distinguishable classes (homotopy classes) of paths through rotations that result in the same overall rotation, as illustrated by the Dirac's belt trick. (True in any dimension.)
2. Nothing forces us to consider only objects that do not make this difference (vectors). We should consider the most general case, this is what spinors are.
3. Spinors change in different ways depending not just on the overall final rotation, but the details of how that rotation was achieved (by a continuous path in the rotation group).
4. The spin group is the group of all rotations keeping track of the class. It doubly covers the rotation group, since each rotation can be obtained in two in-equivalent ways as the endpoint of a path.
5. The space of spinors by definition is equipped with a (complex) linear representation of the spin group.
6. A spinor is characterized by its spin. Depending on the dimension of the space, the dimensions of the spin representations vary but all spins are always possible.

More belts and fun facts

Anti-twister mechanisms



Expanding the Dirac's belt trick setup, one can attach two belts to an object and rotate it by 720° without it getting tangled. Combining the two movements,the object can spin continuously without becoming tangled.



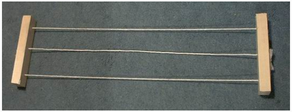
Increasing the number of belts does not change this behavior. Notice that after the cube completed a 360° rotation, the spiral is reversed from its initial configuration. It only returns to its original configuration after spinning a full 720°.

Anti-twister mechanisms

A more extreme example demonstrating that this works with any number of strings. In the limit, a piece of solid continuous space can rotate in place like this without tearing or intersecting itself.

Fun facts

- Anti-twister mechanisms are used in engineering to supply electric power to rotating devices.
- The cup on the hand trick (balinese candle dance or Philippine wine dance).
- Tangloids is mathematical gamed base on the same principles.
- Link with quaternions.



(a) Tangloids.

(b) Balinese candle dance.

Conclusion

1. Dirac's belt trick can be understood by studying the fundamental group of SO(3).
2. The universal cover of SO(3) is SU(2), in which the homotopy ambiguity is solved. Spin vectors transform under SU(2) and covering space technology then allows us to better understand the nature of the spin in quantum mechanics.
3. Spinors can be defined in any dimension and for any spin. Leading to a generalization of usual vectors that take into account the topological difference between some rotations that, a priori, could look equivalent.
4. Spinors are fundamental in Physics and in particular all modern theories of fundamental interactions. Spinors model most of elementary particles. In particular, exactly like Dirac's belt, electrons rotate through the lift in SU(2) thus taking into account the homotopy class of the rotation, how cool !

Thank you

Spin in quantum mechanics

The spin of an "particle" is a number $s \in \frac{1}{2}\mathbb{N}$.

The spin state of a particle of spin s is a unit vector in $\mathbb{C}^{2s+1}$.

The spin is a property, it cannot change (e.g. mass, charge)

The spin state is a characteristic, it evolves

How to interpret it ?

1. **directions:** we choose the direction in which we want to measure it
2. **probabilistic theory:** the outcome of the measure, we can only compute the probabilities of the different outcomes
3. **discrete quantity:** in the chosen direction, the spin will either appear to up or down ($\uparrow$ or $\downarrow$)

The probability of measuring the spin $k = \uparrow, \downarrow$ in the direction $i = x, y, z$ is given by

$$P(i,k) = |\langle v_{i,k}, v \rangle|^2 \tag{4}$$

where v is the spin state of the particle, for some given vectors $v_{i,k}$.

Spin in quantum mechanics

The Lie algebra $\mathfrak{su}(2)$ is generated by the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5}$$

From SU(2) to SO(3)

(maybe put the appendix if no time)

What is the projection map ?

We introduce the *Pauli matrices*

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{6}$$

Then, for $\vec{r} = (x, y, z) \in \mathbb{R}^3$, the matrix

$$\vec{r} \cdot \vec{\sigma} = r^i \sigma_i = \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix}$$

is traceless and self-adjoint, i.e. $\vec{r} \cdot \vec{\sigma} \in \mathfrak{su}(2)$. More precisely, σ_i are the generators of $\mathfrak{su}(2)$. Moreover $\det(\vec{r} \cdot \vec{\sigma}) = -(x^2 + y^2 + z^2)$.

We can then shown that, the new matrix $U(\vec{r} \cdot \vec{\sigma})U^\dagger$, with $U \in \mathrm{SU}(2)$ is

- still traceless and self-adjoint
  $\Rightarrow \exists \vec{r}_U \in \mathrm{SO}(3)$ such that $U(\vec{r} \cdot \vec{\sigma})U^\dagger = \vec{r}_U \cdot \vec{\sigma}$
- has same determinant, i.e $|\vec{r}| = |\vec{r}_U|$
  $\Rightarrow \exists R_U \in \mathrm{SO}(3)$ such that $\vec{r}_U = R\vec{r}$

In then end, for each $U \in \mathrm{SU}(2)$, we have $p(U) \equiv R_U \in \mathrm{SO}(3)$. This maps

For vectors: recall that the scalar product on $\mathbb{R}^3$ is $\langle v_1, v_2 \rangle_{\mathbb{R}^3} = (v_1)^T v_2$ and

$$\langle Rv_1, Rv_2 \rangle_{\mathbb{R}^3} = \langle v_1, v_2 \rangle_{\mathbb{R}^3} \qquad \Leftrightarrow \qquad R^T R = \mathbb{1}$$

so SO(3) is the isometry group of $\mathbb{R}^3$ (+ orientation preserving).

For spin vectors: the scalar product on $\mathbb{C}^2$ is $\langle v_1, v_2 \rangle_{\mathbb{C}^2} = (v_1)^\dagger v_2$ and

$$\langle Uv_1, Uv_2 \rangle_{\mathbb{C}^2} = \langle v_1, v_2 \rangle_{\mathbb{C}^2} \qquad \Leftrightarrow \qquad U^\dagger U = \mathbb{1}$$

so SU(2) is the isometry group of $\mathbb{C}^2$ (+ orientation preserving).

Homotopy

Which space have the same fundamental groups ? Evidently, homeomorphic ones, but we can be less restrictive:

Proposition

If $\varphi : X \to Y$ are is a homotopy equivalence, then the induced homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism for all $x_0 \in X$.

Fundamental group of the circle

Using the covering space technology, we can show that

$$\boxed{\pi_1(S^1) = \mathbb{Z},}$$

which intuitively make sense. This results alone implies many famous theorems of topology:

Theorem (fundamental theorem of algebra)

Every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

Theorem (Brouwer fixed point)

Every continuous map $h : D^2 \to D^2$ has a fixed point.

And, using the same techniques:

Theorem (Borusk-Ulam)

For every continuous map $f = S^2 \to \mathbb{R}^2$ there exist a pair of antipodal points x and $-x$ in S^2 such that $f(x) = f(-x)$.

One can generalize and show that $\pi_n(S^n) = \mathbb{Z}$. These theorems are then generalized accordingly.

Important observations

1. **Some covering spaces are "equivalent":**

   Isomorphisms

   Two covering space $\widetilde{X}$ and $\widetilde{X}'$ of X are *isomorphic* if there exists a homeomorphism $h : \widetilde{X} \to \widetilde{X}'$ such that $p_2 \circ h = p_1$.

   For S^1, p_1 and p_2 are isomorphic.

2. **The lifting of point can, by definition, be ambiguous.**

   Deck transformations

   A *Deck transformation* is a homeomorphism $d : \widetilde{X} \to \widetilde{X}$ such that $p \circ d = p$. With composition, they form a group $G(\widetilde{X})$.

   For S^1, $G(\mathbb{R}) = \mathbb{Z}$ and $G(S^1) = \mathbb{Z}_n$.

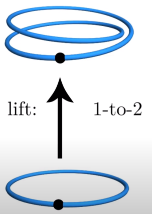
3. **Some covering spaces are "included" in others:** we restrict ourselves to connected covering spaces.

4. **There can exists many covering spaces for the same base space.** If $\widetilde{X}$ is simply connected and X is (locally) path-connected, then it is a covering space of any other covering space. It is maximal, unique and called *universal covering space* (UCS). $\mathbb{R}$ is the UCS of S^1.

Covering spaces and paths

Path lifting property

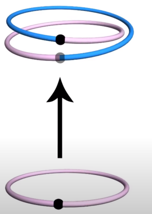
Any path can be lifted. If we fix the lift of the starting point, the lifted path is unique.



$\Rightarrow$ The lift of a constant path is still constant.

$\Rightarrow$ Loop are not necessarily lifted to loops.

Clearly, the projection of a homotopy is a homotopy for the projected paths. What about they other way around ?



Homotopy lifting property

Any homotopy can be lifted. If we fix the lift of the starting path, the lifted homotopy is unique.

$\Rightarrow$ The lifts of homotopy-equivalent paths are homotopically equivalent! $\rightarrow$ relation between $\pi_1(X)$ and $\widetilde{X}$?

If $\widetilde{X}$ is the UCS of X, we actually have

$$\boxed{\pi_1(X) = G(\widetilde{X}).}$$

$\Rightarrow$ algebraic features of $\pi_1(X)$ can be seen as geometric features of $\widetilde{X}$.

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