# Group theory, Topology and Spin-$1/2$ Particles

From Dirac's belt to fermions

Louan Mol

Unversité Libre de Bruxelles

Brussels Summer School of Mathematics 2022

# Table of contents

# Dirac’s belt trick and rotations

# Dirac's belt trick

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Rules:

1. you can only move the end of the belt
2. you cannot twist or rotate it

**Goal:** untwist a $2\pi$-twist.

$\Rightarrow$ it tuns out to be impossible ! One turn negates the twist: $2\pi \to -2\pi$.

Therefore, possible for a $4\pi$ twist ...

Why is that ?

# Space of rotations: SO(3) as a group

Rotations in 3-dimensional space: matrices that acts on $\mathbb{R}^3$ s.t.

1. preserve the scalar product: $O^T O = \mathbb{1}$ ($\Leftrightarrow O$ is orthogonal)
2. preserve the orientation: $\det O = 1$

**Special othogonal group**

SO(3) is the set of $3 \times 3$ real matrices such that $O^T O = \mathbb{1}$ and $\det O = 1$.

Three "fundamental" rotations:

$$x : \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad y : \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad z : \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ It forms a group.

# Space of rotations: SO(3) as a topological space

Fundamental data that describes a rotation:

- an axis of rotation, i.e. a unit vector $\vec{n}$ $\rightarrow$ 2 parameters
- an angle of rotation $\theta \in [-\pi, \pi]$ (with $-\pi \sim \pi$) $\rightarrow$ 1 parameter

The space of rotations can then alternately be defined as a **3-sphere of radius $\pi$ and its antipodal points identified**:

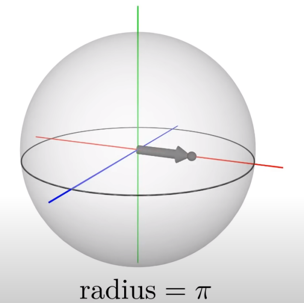
$$\boxed{\mathrm{SO}(3) \cong B^3(\pi)/\sim}$$

and for each point:

$$\text{direction} \leftrightarrow \text{axis}$$

$$\text{norm} \leftrightarrow \text{angle}$$

$\Rightarrow$ It forms a topological space.

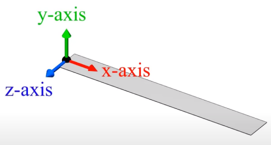



(group + topological space = Lie group)

# Back to the belt

Mathematical description of the belt ?

- a belt is a strip, which is just a path + an orientation.
- given axis on the middle line along the belt, each set of axis is related by a rotation
- a belt configuration is equivalent to a continuous set of axis and therefore to a continuous set of translations, i.e. a path in SO(3)



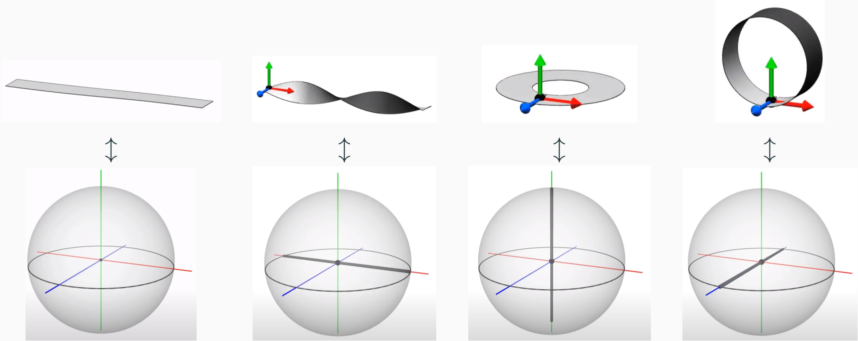

There is a bijection:

$$\boxed{\text{belt configuration} \Leftrightarrow \text{path in SO(3)}}$$

This gives us a new language to analyze the problem !

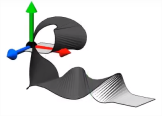
# Dictionary

**trivial rotation** | **$2\pi$ $x$-rotation** | **$2\pi$ $y$-rotation** | **$2\pi$ $z$-rotation**

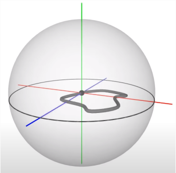


# Dictionary

**random rotation**

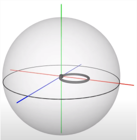


$\updownarrow$

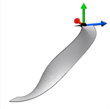


**closed**

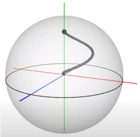
$\updownarrow$



**open path**



$\updownarrow$



# Dictionary

| **Belt** | | **Path** |
|---|---|---|
| specific configuration | $\longleftrightarrow$ | specific path |
| moving the ends | $\longleftrightarrow$ | continuous deformation |
| ends have same orientation | $\longleftrightarrow$ | closed path (loop) |
| can be flattened | $\longleftrightarrow$ | contractible |

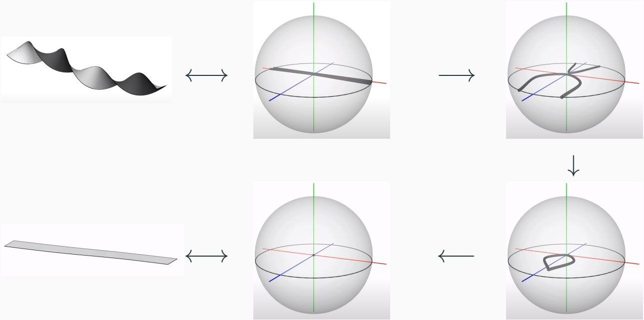
**Back to Dirac's belt trick:**

1. ends of the belt have same orientation $\rightarrow$ we consider loops
   (passing through the origin)
2. moving the ends of the belt $\rightarrow$ continuous deformation
3. belt in original (flat) position $\rightarrow$ trivial path

The question then becomes: **which loops are contractible ?**

# Problem solved ?

- <u>$4\pi$**-twist:**</u> We saw in the beginning the the $4\pi$-twist can be flattened, how can we see this in terms of paths ?



  $\Rightarrow$ the $4\pi$-twist is contractible ! Great.
- <u>$2\pi$**-twist:**</u> we "clearly" see that is not contractible... no ?! Great..?..

**Wierd aftertaste:** our "proof" is good to show contractibility but bad to show non-contractibility and it only works for simple examples.
$\Rightarrow$ We want a consistent and general way of studying paths in topological spaces.

# Homotopy theory

# Homotopoy theory primer

**Starting observation:** depending on the topological space, all loops might not be contractible. Moreover, some loops are "fundamentally different" from each other.
Examples: $\mathbb{R}^3$, $S^2$, $\mathbb{T}^2$, etc.

**Paths and homotopies**

For a topological space $X$:

- *Path* in $X$: continuous map $\gamma : [0,1] \to X$, *loop* if closed
- $\gamma_1$ and $\gamma_2$ are *homotopically equivalent* if one can be deformed into the other: there exists $H : [0,1] \times [0,1] \to X$ such that

  $$H(0,t) = \gamma_1(t) \qquad \text{and} \qquad H(1,t) = \gamma_2(t).$$

  This is an equivalence relation ($\sim$).

For each $x_0 \in X$, we define

$$\pi_1(X, x_0) = \{\text{all loops based at } x_0\}/\sim,$$

it is the set of "fundamentally different" loops passing through $x_0$.

# Fundamental group

Group structure on $\pi_1(X, x_0)$:

- **Product** of paths: $\gamma_1 \cdot \gamma_2 =$ "$\gamma_1$ then $\gamma_2$"
- **Inverse** path: $\gamma^{-1} =$ "$\gamma$ traversed in the opposite direction"
- **Neutral** path: $e =$ "constant path at the identity"
- For equivalence classes: $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$ and $[\gamma]^{-1} = [\gamma^{-1}]$

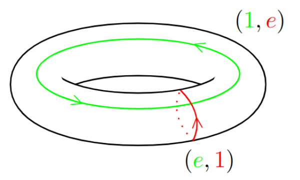
Important fact: up to isomorphism, $\pi_1(X, x_0)$ does not depend on $x_0$
$\Rightarrow$ we denote it as $\pi_1(X)$, it is called the fundamental group of $X$.

Contractible loops are $\sim$ to a point, i.e. they are the element of $[e]$.

How to compute $\pi_1(X)$ ? Can be difficult, not discussed here.

**Examples:**

- $\pi_1(\mathbb{R}^3) = \{e\}$
- $\pi_1(S^2) = \{e\}$
- $\pi_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$
- $\pi_1(\mathbb{R}^2 \backslash \{p\}) = \mathbb{Z}$




Remark: $\pi_1(\mathbb{R}^3 \backslash \{p\}) = \{e\}$, higher homotopy groups for higher-dimensional holes ?

# Back to SO(3)

**Question we had:** are all loops in SO(3) contractible ?
In homotopy language: is $\pi_1(\mathrm{SO}(3))$ trivial ?

**Answer:** NO, one can compute that

$$\boxed{\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2}$$

$\Rightarrow$ There only two "fundamentally different" loops in SO(3) !
$\Rightarrow$ all non-contractible loops are deformations of the $2\pi$-twist !

> The belt trick is a way of physically demonstrating that the fundamental group of SO(3) is $\mathbb{Z}_2$.

We can now say, with more confidence, that we understood Dirac's belt trick.

Are there other manifestation of homotopy in our practical world ?

Yes: the spin ! (you don't need a belt, but you need an electron)
Initially, this trick was a demonstration invented by P. Dirac (1902-1984) to explain the notion of spin to his students.

# Quantum spin and SU(2)

# What is the spin ?

Skipping most of the physics background:

## Spin in quantum mechanics

1. the spin is an inherent property of any "particle":
   - number $s \in \frac{1}{2}\mathbb{N}$, in our case $s = 1/2$
   - does not change, like the mass, charge, etc
   - classifies particles into different classes
2. a particle of spin $s$ is, at a given moment, in a certain spin state:
   - unit vector of $v \in \mathbb{C}^{2s+1}$, in our case $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2$
   - can evolve over time
3. what we can measure is the observed spin:
   - discrete value $s_{\text{obs.}} \in \{s, s-1, \ldots, 0, \ldots, -s+1, -s\}$ ($2s+1$ possibilities).
     In our case, $s_{\text{obs.}} = 1/2, -1/2$ that we denote $\uparrow$ and $\downarrow$
   - given a direction, e.g. $i = x, y, z$
   - outcome is random, we can only compute de probabilities of the different outcome (repeat experience)

# What is the spin ?

Let us introduce

$$v_{x,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, v_{x,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, v_{y,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}, v_{y,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}, v_{z,\uparrow} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_{z,\downarrow} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The probability of measuring $s_{\text{obs.}}$ in the direction $i$ is given by

$$P(i, s_{\text{obs.}}) = |\langle v_{i,k}, v \rangle|^2 \tag{1}$$

where $v$ is the spin state of the particle.

**Example**: in the direction $z$,

$$P(z,\uparrow) = |\alpha|^2, \qquad P(z,\downarrow) = |\beta|^2. \tag{2}$$

Note: we must have $|\alpha|^2 + |\beta|^2 = 1$.

# Special unitary group

**Special unitary group**

SU(2) is the set of $2 \times 2$ complex matrices such that $U^\dagger U = \mathbb{1}$ and $\det U = 1$.

Like SO(3) it is a Lie group so it can be viewed

- as a group
- as a topological space $\mathrm{SU}(2) \cong S^3$

# SU(2) and SO(3): isometries

If we take a step back:

Scalar product on $\mathbb{R}^3$:

$$\langle v_1, v_2 \rangle = (v_1)^T v_2$$

is such that

$$\langle R v_1, R v_2 \rangle = \langle v_1, v_2 \rangle$$

if and only if $R^T R = \mathbb{1}$.

$\Rightarrow$ SO(3) is the (orientation preserving) isometry group

Scalar product on $\mathbb{C}^2$:

$$\langle v_1, v_2 \rangle = (v_1)^\dagger v_2$$

is such that

$$\langle U v_1, U v_2 \rangle = \langle v_1, v_2 \rangle$$

if and only if $U^\dagger U = \mathbb{1}$.

$\Rightarrow$ SU(2) is the (orientation preserving) isometry group

# SU(2) and SO(3: covering space

# Conclusion

# Blocks

Three different block environments are pre-defined and may be styled with an optional background color.

Some text.

aaaaa

**Default**

Block content.

**Alert**

Block content.

**Example**

Block content.

# Introduction to the spin

**Spin in quantum mechanics**

The spin of an "particle" is a number $s \in \frac{1}{2}\mathbb{N}$.
The spin state of a particle of spin $s$ is a unit vector in $\mathbb{C}^{2s+1}$.

The spin is a property, it cannot change (e.g. mass, charge)
The spin state is a characteristic, it evolves

How to interpret it ?

1. **directions:** we choose the direction in which we want to measure it
2. **probabilistic theory:** the outcome of the measure, we can only compute the probabilities of the different outcomes
3. **discrete quantity:** in the chosen direction, the spin will either appear to up or down ($\uparrow$ or $\downarrow$)

The probability of measuring the spin $k = \uparrow, \downarrow$ in the direction $i = x, y, z$ is given by

$$P(i,k) = |\langle v_{i,k}, v \rangle|^2 \tag{3}$$

where $v$ is the spin state of the particle, for some given vectors $v_{i,k}$.

# Spin in quantum mechanics

The Lie algebra $\mathfrak{su}(2)$ is generated by the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

A. Hatcher.
***Algebraic Topology.***
Algebraic Topology. Cambridge University Press, 2002.

N. Miller.
**Representation theory and quantum mechanics, 2018.**