Group theory, Topology and Spin-1/2 Particles

From Dirac's belt to fermions

Louan Mol

Unversité Libre de Bruxelles

Brussels Summer School of Mathematics 2022

Table of contents

Dirac's belt trick and rotations

Dirac's belt trick

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Rules:

1. you can only move the end of the belt
2. you cannot twist or rotate it

Goal: untwist a 2π-twist.

Dirac's belt trick

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Rules:

1. you can only move the end of the belt
2. you cannot twist or rotate it

Goal: untwist a 2π-twist.

$\Rightarrow$ it tuns out to be impossible ! One turn negates the twist: $2\pi \to -2\pi$.

Dirac's belt trick

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Rules:

1. you can only move the end of the belt
2. you cannot twist or rotate it

Goal: untwist a 2π-twist.

$\Rightarrow$ it tuns out to be impossible ! One turn negates the twist: $2\pi \to -2\pi$.

Therefore, possible for a 4π twist ...

Why is that ?

Space of rotations: SO(3) as a group

Rotations in 3-dimensional space: matrices that acts on $\mathbb{R}^3$ s.t.

1. preserve the scalar product: $O^T O = \mathbb{1}$ ($\Leftrightarrow O$ is orthogonal)
2. preserve the orientation: $\det O = 1$

Special othogonal group

SO(3) is the set of 3×3 real matrices such that $O^T O = \mathbb{1}$ and $\det O = 1$.

Three "fundamental" rotations:

$$x : \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad y : \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad z : \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ It forms a group.

Space of rotations: SO(3) as a topological space

Fundamental data that describes a rotation:

- an axis of rotation, i.e. a unit vector $\vec{n}$ $\rightarrow$ 2 parameters
- an angle of rotation $\theta \in [-\pi, \pi]$ (with $-\pi \sim \pi$) $\rightarrow$ 1 parameter

The space of rotations can then alternately be defined as a **3-sphere of radius $\boldsymbol{\pi}$ and its antipodal points identified**:

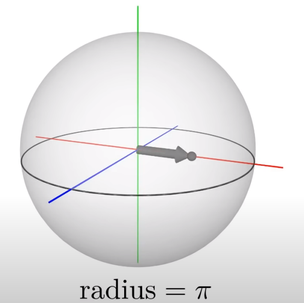
$$\boxed{SO(3) \cong B^3(\pi)/\sim}$$

and for each point:

$$\text{direction} \leftrightarrow \text{axis}$$

$$\text{norm} \leftrightarrow \text{angle}$$

$\Rightarrow$ It forms a topological space.



(group + topological space = Lie group)

Back to the belt

Lets "mathematicalize" the belt:

A belt (strip) is just a path + an orientation.

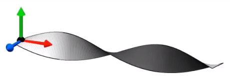
There is a bijection:

$$\boxed{\text{belt configuration} \Leftrightarrow \text{path in SO(3)}}$$

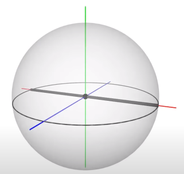
This gives us a new language to analyze the problem !

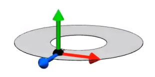
Dictionary

Belt | **Path**

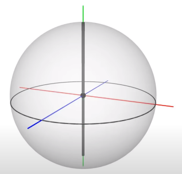


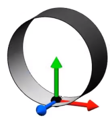
$\longleftrightarrow$



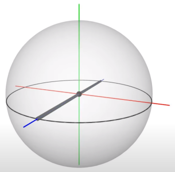


$\longleftrightarrow$



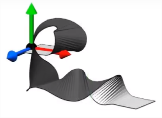


$\longleftrightarrow$

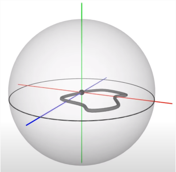


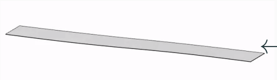
(The color code is consistent.)

Belt **Path**

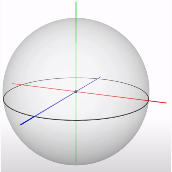


$\longleftrightarrow$





$\longleftrightarrow$

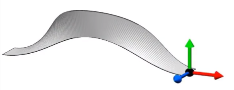


(The color code is consistent.)

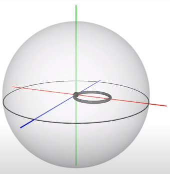
Back to Dirac's belt trick

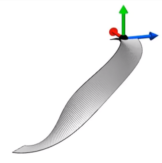
Recall the rules of the game:

1. translating the end of the belt
2. not turning the end of the belt

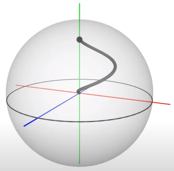


$\longleftrightarrow$





$\longleftrightarrow$



Back to Dirac's belt trick

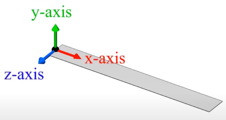
Back to Dirac's belt trick

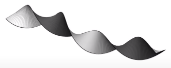
Paths in topological spaces (homotopy theory)

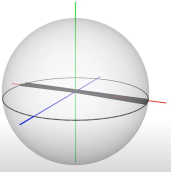
The belt trick is a way of physically demonstrating the the fundamental group of SO(3) is $\mathbb{Z}_2$.

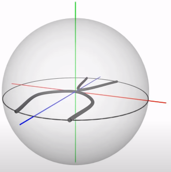
Electrons

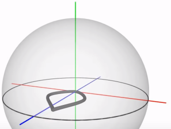
In quantum mechanics


y-axis
x-axis
z-axis









Section 2

Section 3

Section 4

Section 5

Blocks

Three different block environments are pre-defined and may be styled with an optional background color.

Some text.

aaaaa

Default

Block content.

Alert

Block content.

Example

Block content.

Backup slides

Sometimes, it is useful to add slides at the end of your presentation to refer to during audience questions.

The best way to do this is to include the `appendixnumberbeamer` package in your preamble and call `\appendix` before your backup slides.

metropolis will automatically turn off slide numbering and progress bars for slides in the appendix. [1]

R. Graham, D. Knuth, and O. Patashnik.
Concrete mathematics.
Addison-Wesley, Reading, MA, 1989.