Group theory, Topology and Spin-$1/2$ Particles

From Dirac's belt to fermions

Louan Mol

Unversité Libre de Bruxelles

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Table of contents

Dirac's belt trick and the rotation group

Dirac's belt trick

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Rules:

1. you can only move the end of the belt
2. you cannot twist or rotate it

Goal: untwist a 2π-twist.

$\Rightarrow$ it tuns out to be impossible ! One turn negates the twist: $2\pi \to -2\pi$.

Therefore, possible for a 4π twist ...

Why is that ?

Space of rotations: SO(3) as a group

Rotations in 3-dimensional space: matrices that acts on $\mathbb{R}^3$ s.t.

1. preserve the scalar product: $O^T O = \mathbb{1}$ ($\Leftrightarrow O$ is orthogonal)
2. preserve the orientation: $\det O = 1$

Special othogonal group

SO(3) is the set of 3×3 real matrices such that $O^T O = \mathbb{1}$ and $\det O = 1$.

Three "fundamental" rotations:

$$x : \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad y : \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad z : \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ It forms a group.

Space of rotations: SO(3) as a topological space

Fundamental data that describes a rotation:

- an axis of rotation, i.e. a unit vector $\vec{n}$ $\rightarrow$ 2 parameters
- an angle of rotation $\theta \in [-\pi, \pi]$ (with $-\pi \sim \pi$) $\rightarrow$ 1 parameter

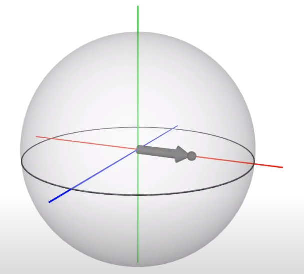
The space of rotations can then alternately be defined as a **3-sphere of radius $\boldsymbol{\pi}$ and its antipodal points identified**:

$$\boxed{\mathrm{SO}(3) \cong B^3(\pi)/\sim}$$

and for each point:

$$\text{direction} \leftrightarrow \text{axis}$$
$$\text{norm} \leftrightarrow \text{angle}$$

$\Rightarrow$ It forms a topological space.

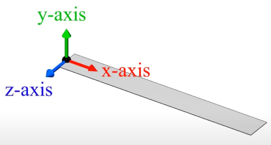


(group + topological space = Lie group)

Back to the belt

Mathematical description of the belt ?

- a belt is a strip, which is just a path + an orientation.
- given axis on the middle line along the belt, each set of axis is related by a rotation
- a belt configuration is equivalent to a continuous set of axis and therefore to a continuous set of translations, i.e. a path in SO(3)



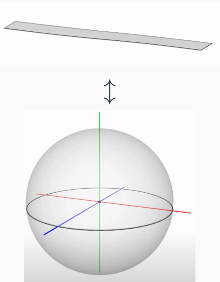
There is a bijection:

$$\boxed{\text{belt configuration} \Leftrightarrow \text{path in SO(3)}}$$

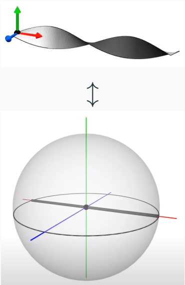
This gives us a new language to analyze the problem !

Dictionary

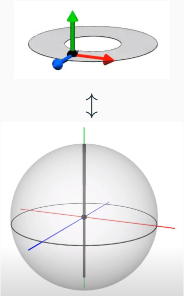
trivial rotation



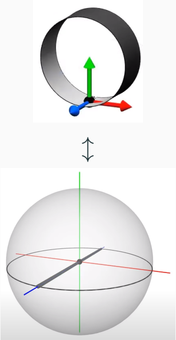
2π x**-rotation**



2π y**-rotation**

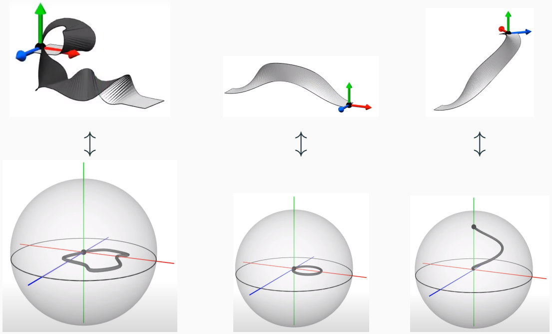


2π z**-rotation**



Dictionary

random rotation **closed** **open path**



Dictionary

Belt		**Path**
specific configuration	$\longleftrightarrow$	specific path
moving the ends	$\longleftrightarrow$	continuous deformation
ends have same orientation	$\longleftrightarrow$	closed path (loop)
can be flattened	$\longleftrightarrow$	contractible

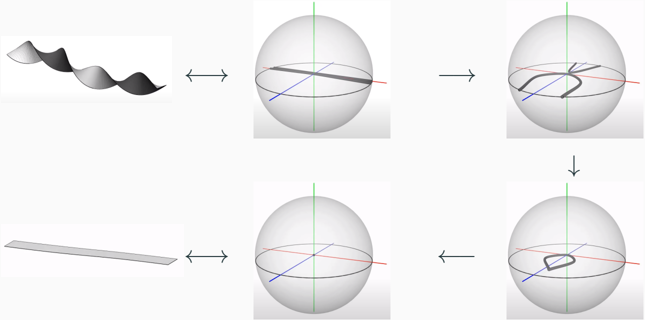
Back to Dirac's belt trick:

1. ends of the belt have same orientation $\rightarrow$ we consider loops
   (passing through the origin)
2. moving the ends of the belt $\rightarrow$ continuous deformation
3. belt in original (flat) position $\rightarrow$ trivial path

The question then becomes: **which loops are contractible ?**

Problem solved ?

- <u>4π**-twist:**</u> We saw in the beginning the the 4π-twist can be flattened, how can we see this in terms of paths ?



$\Rightarrow$ the 4π-twist is contractible ! Great.

- <u>2π**-twist:**</u> we "clearly" see that is not contractible... no ?! Great..?..

Wierd aftertaste: our "proof" is good to show contractibility but bad to show non-contractibility and it only works for simple examples.
$\Rightarrow$ We want a consistent and general way of studying paths in topological spaces.

Homotopy theory

Homotopoy theory primer

Starting observation: depending on the topological space, all loops might not be contractible. Moreover, some loops are "fundamentally different" from each other.
Examples: $\mathbb{R}^3$, S^2, $\mathbb{T}^2$, etc.

Paths and homotopies

For a topological space X:

- *Path* in X: continuous map $\gamma : [0,1] \to X$, *loop* if closed
- γ_1 and γ_2 are *homotopically equivalent* if one can be deformed into the other: there exists $H : [0,1] \times [0,1] \to X$ such that

$$H(0,t) = \gamma_1(t) \qquad \text{and} \qquad H(1,t) = \gamma_2(t).$$

  This is an equivalence relation ($\sim$).

For each $x_0 \in X$, we define

$$\pi_1(X, x_0) = \{\text{all loops based at } x_0\}/\sim,$$

it is the set of "fundamentally different" loops passing through x_0. $[\gamma]$ is called the *homotopy class* of γ.

Fundamental group

Group structure on $\pi_1(X, x_0)$:

- **Product** of paths: $\gamma_1 \cdot \gamma_2 =$ "γ_1 then γ_2"
- **Inverse** path: $\gamma^{-1} =$ "γ traversed in the opposite direction"
- **Neutral** path: $e =$ "constant path at the identity"
- For homotopy classes: $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$ and $[\gamma]^{-1} = [\gamma^{-1}]$

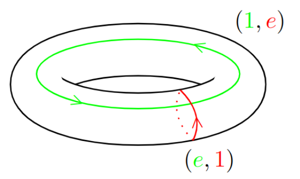
Important fact: up to isomorphism, $\pi_1(X, x_0)$ does not depend on x_0
$\Rightarrow$ we denote it as $\pi_1(X)$, it is called the fundamental group of X.

Contractible loops are $\sim$ to a point, i.e. they are the element of $[e]$.

How to compute $\pi_1(X)$? Can be difficult, not discussed here.

Examples:

- $\pi_1(\mathbb{R}^3) = \{e\}$
- $\pi_1(S^2) = \{e\}$
- $\pi_1(\mathbb{T}^2) = \mathbb{Z} \times \mathbb{Z}$
- $\pi_1(\mathbb{R}^2 \backslash \{p\}) = \mathbb{Z}$



Remark: $\pi_1(\mathbb{R}^3 \backslash \{p\}) = \{e\}$, higher homotopy groups for higher-dimensional holes ?

Back to SO(3)

Question we had: are all loops in SO(3) contractible ?
In homotopy language: is $\pi_1(\mathrm{SO}(3))$ trivial ?

Answer: NO, one can compute that

$$\boxed{\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2}$$

$\Rightarrow$ There only two "fundamentally different" loops in SO(3) !
$\Rightarrow$ all non-contractible loops are deformations of the 2π-twist !

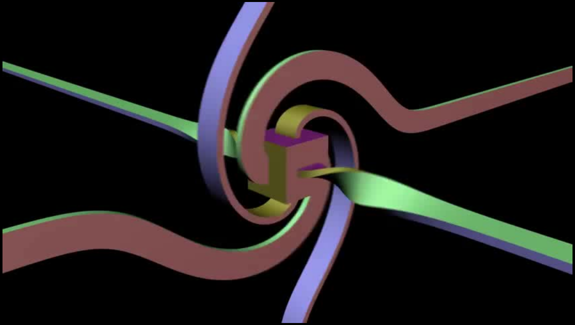
The belt trick is a way of physically demonstrating that the fundamental group of SO(3) is $\mathbb{Z}_2$.

We can now say, with more confidence, that we understood Dirac's belt trick.

Are there other manifestation of homotopy in our practical world ?

Yes: the spin ! (you don't need a belt, but you need an electron)
Initially, this trick was a demonstration invented by P. Dirac (1902-1984) to explain the notion of spin to his students.

Animation 1



An object attached to belts or strings can spin continuously without becoming tangled. Notice that after the cube completes a 360° rotation, the spiral is reversed from its initial configuration. The belts return to their original configuration after spinning a full 720°.

Animation 2

A more extreme example demonstrating that this works with any number of strings. In the limit, a piece of solid continuous space can rotate in place like this without tearing or intersecting itself.

Quantum spin and SU(2)

What is the spin ?

Skipping most of the physics background:

Spin in quantum mechanics

1. the spin is an inherent property of any "particle":
   - number $s \in \frac{1}{2}\mathbb{N}$, in our case $s = 1/2$
   - does not change, like the mass, charge, etc
   - classifies particles
2. a particle of spin s is, at a given moment, in a certain state described by the spin vector:
   - unit vector of $v \in \mathbb{C}^{2s+1}$, in our case $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2$
   - can evolve over time
3. what we can measure yet another quantity, called observed spin:
   - discrete value $s_{\text{obs.}} \in \{s, s-1, \ldots, 0, \ldots, -s+1, -s\}$
     In our case, $s_{\text{obs.}} \in \{1/2, -1/2\}$ that we denote $\uparrow$ and $\downarrow$
   - given a direction, e.g. $i = x, y, z$
   - outcome is random, we can only compute de probabilities of the different outcomes

How do measures happen ?

Let us introduce

$$v_{x,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, v_{x,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, v_{y,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}, v_{y,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}, v_{z,\uparrow} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_{z,\downarrow} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The probability of measuring $s_{\text{obs.}}$ in the direction i is given by the projection

$$P(i, s_{\text{obs.}}) = \left|\langle v_{i,k}, v\rangle_{\mathbb{C}^2}\right|^2 \tag{1}$$

where v is the spin vector of the particle.

Example: in the direction z,

$$P(z, \uparrow) = |\alpha|^2, \qquad P(z, \downarrow) = |\beta|^2. \tag{2}$$

Consequently:

- we must have $\langle v, v\rangle_{\mathbb{C}^2} = |\alpha|^2 + |\beta|^2 = 1$
- to “measure” the spin state, we must repeat the experience many times
- there are states that are always spin $\uparrow$ or always spin $\downarrow$

Rotating a spin vector

For vectors: recall that the scalar product on $\mathbb{R}^3$ is $\langle v_1, v_2 \rangle_{\mathbb{R}^3} = (v_1)^T v_2$ and

$$\langle Rv_1, Rv_2 \rangle_{\mathbb{R}^3} = \langle v_1, v_2 \rangle_{\mathbb{R}^3} \qquad \Leftrightarrow \qquad R^T R = \mathbb{1}$$

so SO(3) is the isometry group of $\mathbb{R}^3$ (+ orientation preserving).

For spin vectors: the scalar product on $\mathbb{C}^2$ is $\langle v_1, v_2 \rangle_{\mathbb{C}^2} = (v_1)^\dagger v_2$ and

$$\langle Uv_1, Uv_2 \rangle_{\mathbb{C}^2} = \langle v_1, v_2 \rangle_{\mathbb{C}^2} \qquad \Leftrightarrow \qquad U^\dagger U = \mathbb{1}$$

so, similarly,

Special unitary group

SU(2) is the set of 2×2 complex matrices such that $U^\dagger U = \mathbb{1}$ and $\det U = 1$.

and SU(2) is the isometry group of $\mathbb{C}^2$ (+ orientation preserving).

Like SO(3) it is a Lie group so it can be viewed

SU(2) and $\mathcal{S}0(3)$

What is the most general for of $U \in \mathrm{SU}(2)$? Imposing $U^\dagger = U^{-1}$ and $\det U = 1$, we find

$$U = \begin{bmatrix} X + iY & Z + iW \\ -Z + iY & X - iY \end{bmatrix} \tag{3}$$

with $X^2 + Y^2 + Z^2 + W^2 = 1 \Rightarrow \mathrm{SU}(2) \cong S^3$
so SU(2) can be viewed as a group and a manifold, it is a Lie group

SU(2) and SO(3):

1. both Lie groups of dimension three
2. both are connected
3. $-\mathbb{1} \in \mathrm{SU}(2)$ but $-\mathbb{1} \notin \mathrm{SO}(3)$

How could we represent $\mathrm{SU}(2) \cong S^3$ in $3d$?

Observation: S^2 is equivalent to two disks glued along their boundary

Similarly: S^3 is equivalent to balls glued along their boundary

BUT, are those spheres related to SO(3) ?
In other words: how are the two notions of rotations related ?

$\Rightarrow$ covering spaces !

Covering spaces

Covering spaces

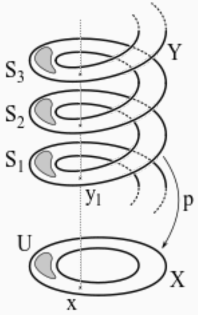
Covering space

For a topological space X, a *covering space* is a topological space E with a *projection map* $\pi = E \to X$ such that

- π is continuous
- there exists a discrete set D and U and open neighborhood of $x \in X$ such that

$$\pi^{-1}(U) = \bigsqcup_{d \in D} V_d$$

  and $\pi|_{V_d} = V_d \to U$ is a homeomorphism. V_d are called the *sheets*.



Examples:

- $\mathbb{R}$ can cover S^1 with $\pi(t) = (\cos(2\pi t), \sin(2\pi t))$
- S^1 can cover S^1 in several ways, with $\pi(z) = z^n$, $n \in \mathbb{N}$ (1 turn in covering space is n turns in S^1) ✏️

Properties of the coverin space

Important remarks:

1.

Proof that 2π-twist is non-contractible in SO(3)

Spinors

Short detour: spinors in Physics

Conclusion

1. There are two topologically distinguishable classes (homotopy classes) of paths through rotations that result in the same overall rotation, as illustrated by the Dirac's belt trick. (True in any dimension)
2. Spinors change in different ways depending not just on the overall final rotation, but the details of how that rotation was achieved (by a continuous path in the rotation group).
3. The spin group is the group of all rotations keeping track of the class. It doubly covers the rotation group, since each rotation can be obtained in two inequivalent ways as the endpoint of a path.
4. The space of spinors by definition is equipped with a (complex) linear representation of the spin group.

Blocks

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Some text.

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Default

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Alert

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Example

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Introduction to the spin

Spin in quantum mechanics

The spin of an "particle" is a number $s \in \frac{1}{2}\mathbb{N}$.
The spin state of a particle of spin s is a unit vector in $\mathbb{C}^{2s+1}$.

The spin is a property, it cannot change (e.g. mass, charge)
The spin state is a characteristic, it evolves

How to interpret it ?

1. **directions:** we choose the direction in which we want to measure it
2. **probabilistic theory:** the outcome of the measure, we can only compute the probabilities of the different outcomes
3. **discrete quantity:** in the chosen direction, the spin will either appear to up or down ($\uparrow$ or $\downarrow$)

The probability of measuring the spin $k = \uparrow, \downarrow$ in the direction $i = x, y, z$ is given by

$$P(i,k) = |\langle v_{i,k}, v \rangle|^2 \tag{4}$$

where v is the spin state of the particle, for some given vectors $v_{i,k}$.

Spin in quantum mechanics

The Lie algebra $\mathfrak{su}(2)$ is generated by the Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{5}$$

References i