Group theory, Topology and Spin-$1/2$ Particles

From Dirac's belt to fermions

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Dirac's belt trick and the rotation group

Dirac's belt trick

You need:

- a belt (not necessarily Dirac's)
- a heavy book

Goal:

Deform the belt to untwist a 4π-twist. Possible ?

What about a 2π-twist ?

Dirac's belt trick

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- a belt (not necessarily Dirac's)
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Goal:

Deform the belt to untwist a 4π-twist. Possible ? Yes !

What about a 2π-twist ? No ! One turn negates the twist: $2\pi \to -2\pi$.

Result:

The torsion in the belt rotates two times faster than the ends of the belt.

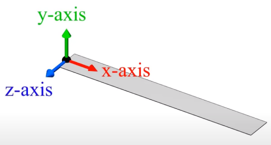
How can we explain that ?

Very interesting mathematics are hiding behind this simple demonstration.

Mathematicalizing the belt

Mathematical description of the belt ?

- a belt is a strip, which is just a path + an orientation.
- at each point along the middle the belt, we put a set of axis aligned with the belt. Each set of axis is a rotation of the initial set.
- this defines a continuous set of rotations or, more precisely, a path in SO(3)



There is a bijection:

$$\boxed{\text{belt configuration} \Leftrightarrow \text{path in SO(3)}}$$

It provides us a new language to analyze the problem !

Space of rotations

As a matrix group:

A rotation is a real 3×3 matrix R such that it

1. preserves the scalar product: $R^T R = \mathbb{1}$ ($\Leftrightarrow R$ is orthogonal)
2. preserves the orientation: $\det R = 1$

Special othogonal group

SO(3) is the set of 3×3 real matrices such that $R^T R = \mathbb{1}$ and $\det R = 1$.

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As a topological space:

Fundamentally, a rotation is a direction + an angle.

$\Rightarrow$ it's a vector where the direction is the axis and the norm is the angle.

The set of all such vectors is

$$\boxed{\mathrm{SO}(3) \cong B^3(\pi)/\sim}$$

Most famously known as $\mathbb{RP}^3$.

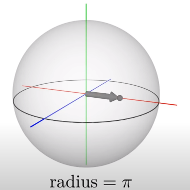
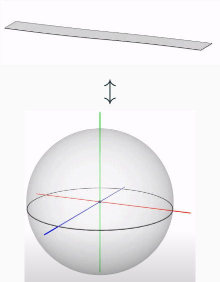


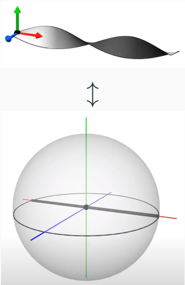
Figure 1: 3-sphere of radius π with its antipodal points identified on the boundary.

Examples

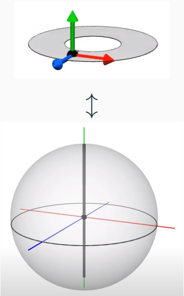
Trivial rotation



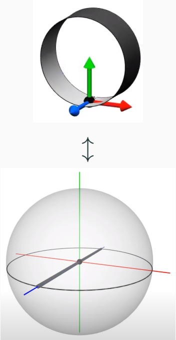
2π x-rotation



2π y-rotation

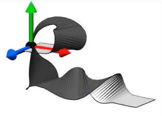


2π z-rotation



Examples

Random path



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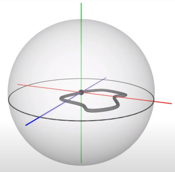

Closed path

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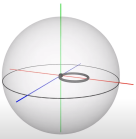
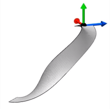


Open path



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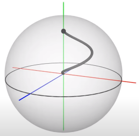


Dictionary

We see that:

Belt		**Path**
specific configuration	$\longleftrightarrow$	specific path
moving the ends	$\longleftrightarrow$	continuous deformation
ends have same orientation	$\longleftrightarrow$	loop
can be flattened	$\longleftrightarrow$	contractible

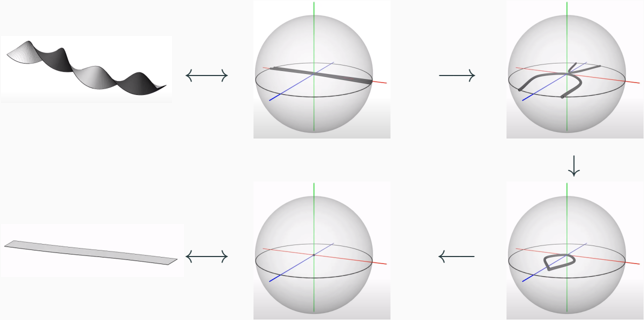
Back to Dirac's belt trick: the rules were

1. ends of the belt must keep the same orientation $\rightarrow$ we consider loops
2. moving the ends of the belt $\rightarrow$ continuous deformation
3. belt in original (flat) position $\rightarrow$ trivial constant path

The question "can the belt be flattened ?" then "**which loops are contractible ?**"

Problem solved ?

- <u>4π**-twist:**</u> we saw in the beginning the the 4π-twist can be flattened, how can we see this in terms of paths ?



$\Rightarrow$ the 4π-twist is contractible ! Great.

- <u>2π**-twist:**</u> we "clearly" see that is not contractible... no ?! Great..?..

Wierd aftertaste: our "proof" is good to show contractibility but bad to show non-contractibility and it only works for simple examples.
$\Rightarrow$ We want a consistent and general way of studying paths in topological spaces.

Homotopy theory

Homotopoy theory primer

Starting observation: depending on the topological space, all loops might not be contractible. Moreover, some loops are "fundamentally different" from each other, e.g. in $\mathbb{R}^3$, S^2, $\mathbb{T}^2$, etc.

Paths and homotopies

For a topological space X:

- *Path* in X: continuous map $\gamma : [0,1] \to X$,
- *Loop* : closed path, i.e. embedded circle,
- γ_1 and γ_2 are *homotopically equivalent* if one can be deformed into the other: there exists $f_t : [0,1] \to X$ with $t \in I$ such that

  $$f_0(s) = \gamma_1(s) \qquad \text{and} \qquad f_1(s) = \gamma_2(s).$$

  and the endpoints are fixed. This is an equivalence relation ($\sim$).

For each $x_0 \in X$, we define

$$\boxed{\pi_1(X, x_0) = \{\text{all loops based at } x_0\}/\sim,}$$

$\to$ set of "fundamentally different" loops passing through x_0.

Fundamental group

The elements of $\pi_1(X, x_0)$ are $[\gamma]$, called the *homotopy class* of γ.

Group structure on $\pi_1(X, x_0)$:

- **Product** of paths: $\gamma_1 \cdot \gamma_2 =$ "γ_1 then γ_2"
- **Inverse** path: $\gamma^{-1} =$ "γ traversed in the opposite direction"
- **Neutral** path: $e =$ "constant path at the identity"
- For homotopy classes: $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$ and $[\gamma]^{-1} = [\gamma^{-1}]$

Important fact: if X is path-connected, $\pi_1(X, x_0)$ does not depend on x_0, up to isomorphism.
$\Rightarrow$ we denote it as $\pi_1(X)$, it is called the fundamental group of X.

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Contractible loops are $\sim$ to a point, i.e. they are the elements of $[e]$.

Proposition (product of spaces)

If X and Y are path-connected, $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Proposition (maps between spaces)

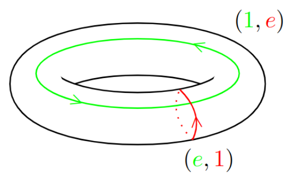
If $\varphi : X \to Y$ is a continuous map, it induces a homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ though $\varphi_*([\gamma]) = [\varphi \circ \gamma]$.

How to compute $\pi_1(X)$?

Can be difficult, there are different methods (e.g. Van Kampen theorem, Hopf fibrations, Hurewicz theorems, etc), not discussed here. A lot of homotopy groups are still unknown !

Examples:

1. $\pi_1(\mathbb{R}^2) = 0$
2. $\pi_1(S^2) = 0$
3. $\pi_1(S^1) = \mathbb{Z}$
4. $\pi_1(\mathbb{T}^2) = \pi_1(S^1) \times \pi_1(S^1) = \mathbb{Z} \times \mathbb{Z}$
5. $\pi_1(\mathbb{R}^2 \backslash \{p\}) = \mathbb{Z}$



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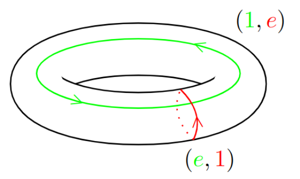

Remarks:

- $\pi_1(S^1) = \mathbb{Z}$ implies various famous theorems: fundamental theorem of algebra, Brouwer's fixed point theorem, Borusk-Ulam theorem etc.
- $\pi_1(\mathbb{R}^2 \backslash \{p\}) = \mathbb{Z}$ but $\pi_1(\mathbb{R}^3 \backslash \{p\}) = 0$, higher homotopy groups for higher-dimensional holes ? Yes, *nth homotopy group*:

$$\pi_n(X, x_0) = \{S^n \text{ based at } x_0\}/\sim .$$

Homotopy groups of spheres

Good example of the complexity of homotpy groups:

- ▷ embedding a sphere in a higher-dimensional one: always trivial
- ▷ embedding a sphere in itself: always $\mathbb{Z}$ ways
- ▷ embedding a sphere in lower-dimensional one: much more complicated, periodic for a bit, then completely chaotic

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	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

Figure 2: Homotopy groups of spheres.

Back to $\mathrm{SO}(3)$

Question we had: are all loops in SO(3) contractible ?

In homotopy language: is $\pi_1(\mathrm{SO}(3))$ trivial ?

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Answer: NO, one can show that

$$\boxed{\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2}$$

$\Rightarrow$ There only two “fundamentally different” loops in SO(3) !
$\Rightarrow$ there is only one kind of non-contractible loop !

Indeed, there only two different initial configurations (i.e. two possible loops in SO(3)):

- $4\pi k$-twists which are all equivalent
- $(4\pi k + 2\pi)$-twists which are all equivalent

with $k \in \mathbb{Z}$.

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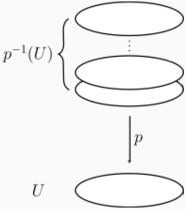
We have a better understanding Dirac's belt trick. **But still no proof !**
Homotopy theory allowed us to understand the ways of embedding loops in some spaces, we now need a tool to lift this ambiguity: covering spaces !

Covering spaces

Covering space

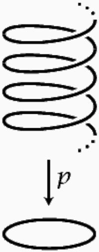
For a topological space X, a *covering space* is a topological space $\widetilde{X}$ with a *projection map* $p : \widetilde{X} \to X$ such that there exists an open cover $\{U_\alpha\}$ for which $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\widetilde{X}$, each of which is mapped by p homeomorphically on U_α.
If X is connected, $|p^{-1}(x)|$ is constant and called the number of *sheets*.



Examples

There are many possibilities to cover the circle:

- $\mathbb{R}$ covers S^1 with $p_1(t) = (\cos(2\pi t), \sin(2\pi t))$,
- $\mathbb{R}$ covers S^1 with $p_2(t) = (\cos(5t), \sin(5t))$,
- S^1 covers S^1 in several ways, with $p(z) = z^n$, $n \in \mathbb{N}$.



- **Some covering spaces are equivalent:**

Isomorphisms

Two covering space $\widetilde{X}$ and $\widetilde{X}'$ of X are *isomorphic* if there exists a homeomorphism $h : \widetilde{X} \to \widetilde{X}'$ such that $p_2 \circ h = p_1$.

For S^1, p_1 and p_2 are isomorphic.

General properties

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- **The lifting of point can, by definition, be ambiguous:**

Deck transformations

A *Deck transformation* is a homeomorphism $d : \widetilde{X} \to \widetilde{X}$ such that $p \circ d = p$. With composition, they form a group $G(\widetilde{X})$.

For S^1, $G(\mathbb{R}) = \mathbb{Z}$ and $G(S^1) = \mathbb{Z}_n$.

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- **Many covering spaces for the same base space:**

Universal covering space

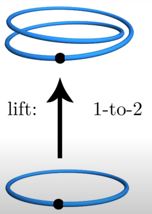
If $\widetilde{X}$ is simply connected and X is (locally) path-connected, there exists covering space of any other covering space. It is maximal, unique and called *universal covering space* (UCS).

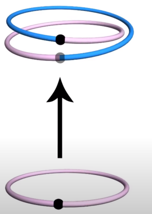
$\mathbb{R}$ is the UCS of S^1.

Lifting properties

Observation:

1. Lifting points is ambiguous.
2. Lifting path is not ambiguous if the starting point is fixed.
3. Constant paths are lifted to constant paths.
4. The projection of a homotopy is a homotopy for the projected paths.
5. The lifts of homotopy-equivalent paths are homotopically equivalent! $\rightarrow$ relation between $\pi_1(X)$ and $\widetilde{X}$?





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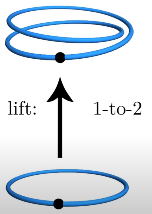

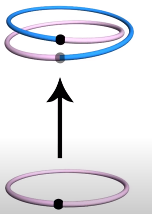

If $\widetilde{X}$ is the UCS of X, we actually have

$$\boxed{\pi_1(X) = G(\widetilde{X}).}$$

$\Rightarrow$ algebraic features of $\pi_1(X)$ can be seen as geometric features of $\widetilde{X}$.

Covering space of SO(3)

One can show that

$$\boxed{\text{The universal covering space of SO(3) is SU(2).}}$$

Special unitary group

SU(2) is the set of 2×2 complex matrices such that $U^\dagger U = \mathbb{1}$ and $\det U = 1$.

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Properties of SU(2):

What is the most general element of SU(2) ? Imposing $U^\dagger = U^{-1}$ and $\det U = 1$, we find

$$U = \begin{bmatrix} X + iY & Z + iW \\ -Z + iY & X - iY \end{bmatrix} \tag{1}$$

with $X^2 + Y^2 + Z^2 + W^2 = 1 \Rightarrow \mathrm{SU}(2) \cong S^3$.

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SU(2) and SO(3):

1. both of dimension three
2. both are connected
3. both isometry groups
4. $-\mathbb{1} \in \mathrm{SU}(2)$ but $-\mathbb{1} \notin \mathrm{SO}(3)$

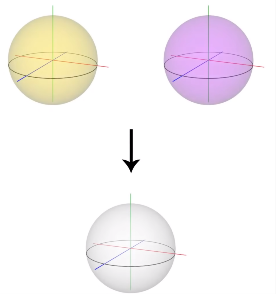
Representating SU(2)

How could we represent $\mathrm{SU}(2) \cong S^3$ in $3d$?

Observation: S^2 is equivalent to two disks glued along their boundary.

Similarly, S^3 is equivalent to balls glued along their boundary.

Question: are those balls related to SO(3) ?



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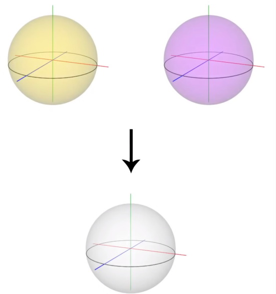
They are the sheets !



The projection map is

$$p\left(\begin{bmatrix} x & y \\ -\overline{y} & \overline{x} \end{bmatrix}\right) = \begin{bmatrix} \mathrm{Re}(x^2 - y^2) & \mathrm{Im}(x^2 + y^2) & -2\mathrm{Re}(xy) \\ -\mathrm{Im}(x^2 - y^2) & \mathrm{Re}(x^2 + y^2) & 2\mathrm{Im}(xy) \\ 2\mathrm{Re}(x\overline{y}) & 2\mathrm{Im}(x\overline{y}) & |x|^2 - |y|^2 \end{bmatrix}$$

with $|x|^2 + |y|^2 = 1$.

Group relation:

The fact that SU(2) is a double-cover of SO(3) can can see in practice with

$$p(U) = p(-U).$$

Intuitively, we should be able to recover SO(3) from SU(2) if $U \sim -U$.
And, indeed,

$$\boxed{\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2,}$$

where the quotient means exactly that we identity U with $-U$.

SU(2) and SO(3)

Group relation:

The fact that SU(2) is a double-cover of SO(3) can can see in practice with

$$p(U) = p(-U).$$

Intuitively, we should be able to recover SO(3) from SU(2) if $U \sim -U$. And, indeed,

$$\boxed{\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2,}$$

where the quotient means exactly that we identity U with $-U$.

Other formulation:

We saw that S^3 is a universal double-sheeted cover of $\mathbb{RP}^3$, $\pi_1(S^3) = \{e\}$, and $\pi_1(\mathbb{RP}^3) = \mathbb{Z}_2$. This makes sense since

$$\mathbb{RP}^3 = S^3/\{(x,y,z) \sim (-x,-y,-z)\} = S^3/\mathbb{Z}_2 = \mathrm{SU}(2)/\mathbb{Z}_2,$$

we get the previous group relation.

Bringing it all together

What are the lifts of the 2π-twist and the 4π-twist ?

$$2\pi\text{-twist} \to \text{path from } I \text{ to } -I$$

$$4\pi\text{-twist} \to \text{path from } I \text{ to } -I \text{ to } I \text{ again}$$

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Proof that 2π-twist is non-contractible in SO(3):

Let us suppose that the 2π-twist is contractible. At each step of its contraction, we can lift the path to SU(2). This provides us with a contraction of the lifted 2π-twist. However, the lifted 2π-twist does not have the same start and endpoint, therefore it is not contractible, and so is the non-lifted path.

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Proof that 4π-twist is contractible in SO(3):

The lift of the 4π-twist is a loop. Since $\pi_1(\mathrm{SU}(2)) = \pi_1(S^3) = 0$, this loop is necessarily contractible. Projecting each step of its contraction provides us with a contraction of the 4π-twist path in SO(3).

Summary of the analysis of Dirac's belt trick

- Belt configurations are equivalent to paths in SO(3). New question: we want to classify the contractable and non-contractable loops.
- Fundamental groups and covering spaces: $\pi_1(X)$ and $\widetilde{X}$ are two pictures of the same thing. $\widetilde{X}$ is the space that contains the same information plus the topological information of non-equivalent paths, i.e. it "solves" the homotopy ambiguity.
- The UCS of SO(3) is SU(2), and it is a double cover.
- There only two kinds topologically-distinguishable loops: [$4\pi k$-twists] is contractible and the [$(4\pi k + 2\pi)$-twists] is not contractible.

The belt trick is a way of physically demonstrating that the fundamental group of SO(3) is $\mathbb{Z}_2$.

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The belt trick is a way of physically demonstrating that the fundamental group of SO(3) is $\mathbb{Z}_2$.

Are there other manifestations of homotopy in our practical world ?

Yes: the spin ! (You don't need a belt, but you need an electron.)

Initially, this trick was a demonstration invented by Paul Dirac (1902-1984) to explain the notion of spin to his students.

Quantum spin

What is the spin ?

Skipping most of the physics background:

Spin in quantum mechanics

1. The *spin* is an inherent property of any "particle":
   - it's a number $s \in \frac{1}{2}\mathbb{N}$, in our case $s = 1/2$
   - does not change (like the mass, charge, etc)
   - classifies particles
2. A particle of spin s is, at a given moment, in a certain state described by the *spin vector*:
   - unit vector of $v \in \mathbb{C}^{2s+1}$, in our case $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \in \mathbb{C}^2$
   - this state can evolve over time
3. What we can measure yet another quantity, called *observed spin*:
   - discrete value $s_{\text{obs.}} \in \{s, s-1, \dots, 0, \dots, -s+1, -s\}$
     in our case, $s_{\text{obs.}} \in \{1/2, -1/2\}$ that we denote $\uparrow$ and $\downarrow$
   - given a direction, e.g. $i = x, y, z$
   - outcome is random, QM predicts the probability of each outcome

How do measures happen ?

Let us introduce

$$v_{x,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, v_{x,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, v_{y,\uparrow} = \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}, v_{y,\downarrow} = \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}, v_{z,\uparrow} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_{z,\downarrow} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The probability of measuring $s_{\text{obs.}}$ in the direction i is given by the projection

$$P(i, s_{\text{obs.}}) = \left|\langle v_{i,s_{\text{obs.}}}, v\rangle_{\mathbb{C}^2}\right|^2 \tag{2}$$

where v is the spin vector of the particle and $\langle v, w\rangle = v^\dagger w$ is a scalar product on $\mathbb{C}^2$.

Example: in the direction z,

$$P(z,\uparrow) = |\alpha|^2, \qquad P(z,\downarrow) = |\beta|^2. \tag{3}$$

Consequently:

- we must have $\langle v, v\rangle_{\mathbb{C}^2} = |\alpha|^2 + |\beta|^2 = 1$
- to “measure” the spin state, we must repeat the experience many times
- there are states that are always spin $\uparrow$ or always spin $\downarrow$

How to rotate a spin vector ?

We need to leave the probabilities conserved $\Rightarrow$ scalar product invariant:

$$\boxed{\text{Spin vectors transform under SU(2) !}}$$

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How do space rotations act on spin vectors ?

The rotation group of euclidean space is still SO(3), so we need a way of doing an SO(3) rotation through SU(2) transformations.

This is exactly what the covering technology provides us: a unique way to lift a rotation from SO(3) to SU(2).

The bizarre nature of fermions

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Interpretation ?

The 2π-twist is not closed $\Rightarrow$ walking around such a particle would not give back the particle in the same states, it would negate the spin state. Very odd property ... Could such exotic particles exist ? Error of interpretation ?

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Yes, they do exist! Out of the 18 elementary particles, 12 of them have spin 1/2 ! And this have been observed experimentally.

The bizarre nature of fermions

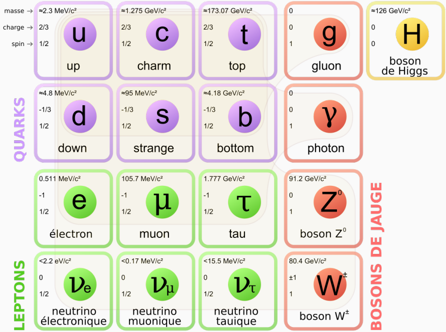

Figure 3: Standard Model of particle physics.

Practical details:

- Instead of walking around the particle, we rotate it using a magnetic field (Lamor procession).
- We cannot detect the "“−” sign if only one particle, at least two are necessary.
- We do not actually use electrons but neutrons (see neutron interferometry).

Generalizations

Other spins: $s \in \frac{1}{2}\mathbb{N}$.

Other dimensions: $\mathrm{SO}(3) \to \mathrm{SO}(n)$ and $\mathrm{SU}(2) \to \mathrm{Spin}(n)$.

Spinor

A spinor of spin s in dimension n is an a element of $\mathbb{C}^{2s+1}$ transforming under a (complex) linear representation of $\mathrm{Spin}(n)$.

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Summary on spinors:

1. There are two topologically distinguishable classes of paths through rotations that result in the same overall rotation. (True in any dimension, $\mathrm{Spin}(n)$ is always double-sheeted.)
2. The most general object should take that difference into account: spinors.
3. A spinor is characterized by its spin. Depending on the dimension of the space, the dimensions of the spin representations vary but all spins are always possible.
4. Other approach: Clifford algebra !

Ending remarks on spinors

Spin in nature:

- only spins 0 (Higgs boson), 1/2 (electrons, quarks, etc), 1 (photons, gluons, etc) and 2 (graviton) are found in nature
- spins higher than 2 are technically very problematic, and not well-understood yet. Current topic of research (U Mons !)
- spin-1/3 particles ? No, impossible, because $\pi_1(SO(3)) = \mathbb{Z}_2$. Example of mathematical constraint on physical models.

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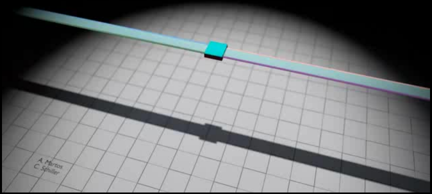
Behind quantum mechanics:

The spinors we encountered previously are spinors in Quantum Mechanics, which is non-relativistic. Modern Physics is relativistic therefore we mainly care about the indefinite rotation groups rather than the usual rotation groups, because of special relativity. The whole theory can be generalized accordingly:

	Non-relativistic	**Relativistic**
rotation group	$\mathrm{SO}(3)$	$\mathrm{SO}(1,3)$
UCS	$\mathrm{Spin}(3) = \mathrm{SU}(2)$	$\mathrm{Spin}(1,3) = \mathrm{SL}(2,\mathbb{C})$

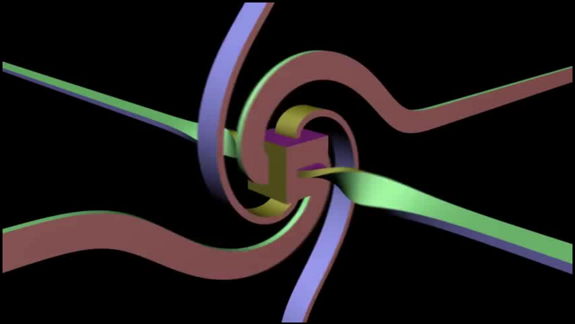
More belts, more fun

Anti-twister mechanisms



Expanding the Dirac's belt trick setup, one can attach two belts to an object and rotate it by 720° without it getting tangled. Combining the two movements, the object can spin continuously without becoming tangled.

Anti-twister mechanisms



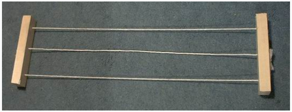
Increasing the number of belts does not change this behavior. Notice that after the cube completed a 360° rotation, the spiral is reversed from its initial configuration. It only returns to its original configuration after spinning a full 720°.

Anti-twister mechanisms

A more extreme example demonstrating that this works with any number of strings. In the limit, a piece of solid continuous space can rotate in place like this without tearing or intersecting itself.

Fun facts

- Anti-twister mechanisms are used in engineering to supply electric power to rotating devices.
- The cup on the hand trick (balinese candle dance or Philippine wine dance).
- Tangloids is mathematical gamed base on the same principles.
- Link with quaternions.



(a) Tangloids.

(b) Balinese candle dance.

Conclusion

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1. Dirac's belt trick can be understood by studying the fundamental group of SO(3).
2. The universal cover of SO(3) is SU(2), in which the homotopy ambiguity is solved. Spin vectors transform under SU(2) and covering space technology then allows us to better understand the nature of the spin in quantum mechanics.
3. Spinors can be defined in any dimension and for any spin. Leading to a generalization of usual vectors that take into account the topological difference between some rotations that, a priori, could look equivalent.
4. Spinors are fundamental in modern theories of fundamental interactions. Spinors model most of elementary particles. In particular, exactly like Dirac's belt, electrons rotate through the lift in SU(2) thus taking into account the homotopy class of the rotation, how cool ?!

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Thank you !

More on rotations

Three “fundamental” rotations:

$$x : \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad y : \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \qquad z : \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

any rotation can be obtained by composing those three rotation.

Bijection between rotations and unit quaternions:

$$(\overrightarrow{n}, \theta) \qquad \Leftrightarrow \qquad e^{\frac{\theta}{2}(n_x\mathbf{i}+n_y\mathbf{j}+n_z\mathbf{k})} = \cos\frac{\theta}{2} + (n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k})\sin\frac{\theta}{2}$$

using an extension of Euler’s formula. Then,

- rotation of a vector $\overrightarrow{r}$: $\mathbf{p}' \to \mathbf{q}\mathbf{p}\mathbf{q}^{-1}$ with $\mathbf{p} = r_x\mathbf{i} + r_y\mathbf{j} + r_z\mathbf{k}$,
- composition of rotations: $\mathbf{q} = \mathbf{q}_1\mathbf{q}_2$.

Construction of the projection map

We introduce the *Pauli matrices*(generators of $\mathfrak{su}(2)$):

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

Then, for $\overrightarrow{r} = (x, y, z) \in \mathbb{R}^3$, the matrix

$$\overrightarrow{r} \cdot \overrightarrow{\sigma} = r^i \sigma_i = \begin{bmatrix} z & x - iy \\ x + iy & -z \end{bmatrix}$$

is traceless and self-adjoint, i.e. $\overrightarrow{r} \cdot \overrightarrow{\sigma} \in \mathfrak{su}(2)$. More precisely, σ_i are the generators of $\mathfrak{su}(2)$. Moreover $\det(\overrightarrow{r} \cdot \overrightarrow{\sigma}) = -(x^2 + y^2 + z^2)$.

We can then shown that, the new matrix $U(\overrightarrow{r} \cdot \overrightarrow{\sigma})U^\dagger$, with $U \in \mathrm{SU}(2)$ is

- still traceless and self-adjoint
  $\Rightarrow \exists \overrightarrow{r}_U \in \mathrm{SO}(3)$ such that $U(\overrightarrow{r} \cdot \overrightarrow{\sigma})U^\dagger = \overrightarrow{r}_U \cdot \overrightarrow{\sigma}$
- has same determinant, i.e $|\overrightarrow{r}| = |\overrightarrow{r}_U|$
  $\Rightarrow \exists R_U \in \mathrm{SO}(3)$ such that $\overrightarrow{r}_U = R\overrightarrow{r}$

In then end, for each $U \in \mathrm{SU}(2)$, we have $p(U) \equiv R_U \in \mathrm{SO}(3)$. This maps can be shown to be locally a homeomorphism.

More on homotpy

Proposition

If $\varphi : X \to Y$ are is a homotopy equivalence, then the induced homomorphism $\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism for all $x_0 \in X$.

Theorem (fundamental theorem of algebra)

Every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

Theorem (Brouwer fixed point)

Every continuous map $h : D^2 \to D^2$ has a fixed point.

Theorem (Borusk-Ulam)

For every continuous map $f = S^2 \to \mathbb{R}^2$ there exist a pair of antipodal points x and $-x$ in S^2 such that $f(x) = f(-x)$.

One can generalize and show that $\pi_n(S^n) = \mathbb{Z}$. These theorems are then generalized accordingly.

More belts

We can see that

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