Notes on Quiver Gauge Theories

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Abstract

In these notes, we present some basic ideas around the large topic of quiver gauge theories. The goal is to reproduce and regroup the basics of quiver gauge theories. Note that this is a draft, it may contain a lot of typos, errors and imprecisions. It is only meant as a work support.

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Contents

1 | Physical setup

Brane-world paradigm

We consider our world to be a slice in the ten-dimensional spacetime of type II superstring theory, i.e. the worldvolume of a D3-brane. More precisely, we consider a stack of N D3-branes in order to have U(N) Chan-Paton factors resulting in a U(N) gauge group. To have D3-branes, we need to consider type IIB superstring theory. The spacetime is therefore not necessarily $\mathbb{R}^{1,9}$ but of the more general form

$$M = \mathbb{R}^{1,3} \times M^{(6)}.$$

This is the so-called *brane-world paradigm*.

Supersymmetry on the worldvolume: holonomy and Calabi-Yau compactification

Independently from string theory, we can ask for the wolrdvolume theory to be supersymmetric. Preserving any degree of supersymmetry constrains the transverse space $M^{(6)}$ to be compact, complex, Kähler and to have $G \subset$ SU(3) holonomy. Namely, $M^{(6)}$ must be a Calabi-Yau threefold.

Let us discuss the the holonomy part of the constrain in more details as it will play an important role in the following. We start from type IIB superstring theory which is 10-dimensional and has $\mathcal{N} = 2$ supersymmetry so it possesses 32 supercharges. As usual, they transform under the minimal spinor representation of the bulk Lorentz group, here SO$(1, 9)$. In ten dimensions this representation is 8-dimensional (complex) which is why there are $2(8 + 8) = 32$ supercharges: 8 transforming in the 8-dimensional minimal spinor representation and 8 transforming in the 8-dimensional conjugate minimal representation and whole thing times two since $\mathcal{N} = 2$. Let us now consider a four-dimensional field theory resulting from compactification of the transverse six-dimensional space. The number of supercharges that generate supersymmetries for this theory is the number of Killing spinor (covariantly constant spinor): each Killing spinor contracted with the local supersymmetry current generates a residual supersymmetry(*). Now the link with holonomy: a generic curved six-dimensional manifold has O(6) holonomy and SO(6) if it is orientable. Since SO$(6) \cong$ SU(4), minimal spinors can be viewed as having four complex component and as transforming under SU(4). Indeed, minimal spinors in six dimensions have four complex components. In order to have one covariantly conserved spinor, we look for the biggest subgroup of SU(4) that leaves a component of the spinor invariant. This is clearly $\{e\} \times$ SU$(3) \subset$ SU(4) that acts trivially on the first component. The spinor $(1, 0, 0, 0)$ is then covariantly conserved. Our transverse space must therefore have SU(3) holonomy such that the parallel transport of the spinor $(1, 0, 0, 0)$ under any closed loop is a lower SU(3) rotation. We conclude that if the transverse Calabi-Yau has SU(3) holonomy, the worldvolume theory has $\mathcal{N} = 1$ supersymmetry. The same reasoning can be used to obtain that SU(2) holonomy implies $\mathcal{N} = 2$ supersymmetry.

Non-compact transverse space

If we let the worldvolume of the D3-branes carry the requisite gauge theory while the bulk contains gravity, we can relax the compactness condition and study non-compact threefolds. In other words, $M^{(6)}$ is taken to be Calabi-Yau variety, instead of a Calabi-Yau manifold. A Calabi-Yau varitey is an affine variety that locally models a Calabi-Yau manifold, therefore allowing for singularities.

Using a non-compact transverse space can intuitively be understood as a Kaluza-Klein compactification where we take the size of the compact dimensions to infinity. The four-dimensional gravity coupling constant being inversely proportional to this quantity, there is no gravity in this limit. This makes the analysis much simpler and therefore also serves as an argument to ignore gravity in the worldvolume theory. Consequently, we will mostly ignore gravity and not care about the metric of the spacetime, see appendix K for more details.

Singular transverse space

The only smooth Calabi-Yau threefold is $\mathbb{C}^3$ so we are lead to consider singular Calabi-Yau varieties. We usually denote $S \equiv M^{(6)}$ to remind us of the singular aspect. String theory being a theory of extended objects, it is well-defined on such singularities. In a sense that will be clarified later, this singular structure of the geometry requires to “project” the theory. As a result, the gauge group $\mathrm{U}(N)$ will be broken down into products of smaller gauge groups. The simplest examples of singular Calabi-Yau varieties are Calabi-Yau orbifolds. We will mainly be interested these examples.

From the point of view of the orbifold, the D3-brane is a point. Consequently, the D3-branes parametrize the transverse space. This is the first clue of the crucial relationship between the worldvolume theory and the Calabi-Yau singularity. Eventually, we will see that the classical vacuum of the gauge theory should be, in explicit coordinates, the defining equation of S. This is precisely the opposite of the projection manipulation we mentioned above: recovering the transverse space from the gauge theory.

Projecting and computing the classical vacua are therefore inverse operations with respect to each other. This suggest a bijection between the singular transverse space and the gauge theory: the former can be computed from the latter and vice-versa. This is called “forward algorithm” and “inverse algorithm”. We will of course discuss this in more details.

Mathematical formulation

Mathematically, this brane-world paradigm is the realization of branes as supports of vector bundles (sheaf). Gauge theories on branes are intimately related to algebraic constructions of stable bundles, i.e. holomorphic or algebraic vector bundles that are stable in the sense of geometric invariant theory. In particular, D-brane gauge theories manifest as a natural description of symplectic quotients and their resolutions in geometric invariant theory.

To summarize in more mathematical terms, our D-branes, together with the stable vector bundle (sheaf) supported thereupon, resolves the transverse Calabi-Yau orbifold which is the vacuum for the gauge theory on the worldvolume as a GIT quotient (*).

Summary

We consider N D3-branes in type IIB superstring theory carrying a $\mathrm{U}(N)$ gauge group. The transverse space S is taken to be a non-compact singular Calabi-Yau variety.

2 | Smooth transverse space: the simplest case

2.1 | Generalities

Let us start by considering the simplest configuration were the transverse Calabi-Yau space is non-singular, i.e. it is a proper smooth Calabi-Yau threefold. As mentioned above, the only smooth Calabi-Yau threefold is $S = \mathbb{C}^3$. In this case, the spacetime is simply flat space $\mathbb{R}^{1,9} = \mathbb{R}^{1,3} \times \mathbb{R}^6$ with a choice a complex structure on $\mathbb{R}^6$. From the $\mathrm{U}(N)$ Chan-Paton factors, the worldvolume theory inherits from a $\mathrm{U}(N)$ gauge group. Type IIB superstring theory is a ten-dimensional $\mathcal{N} = 2$ theory so it has 32 supercharges. The presence of the branes breaks the Lorentz symmetry of $\mathbb{R}^{1,9}$ as

$$\mathrm{SO}(1,9) \to \mathrm{SO}(1,3) \times \mathrm{SO}(6), \tag{2.1}$$

whereby breaking half of the supersymmetries, as we explained in the previous section. We are thus left with 16 supercharges. In four dimensions, this corresponds to $\mathcal{N} = 4$. The worldvolume theory for $S = \mathbb{C}^3$ is therefore $D = 4, \mathcal{N} = 4$ $\mathrm{U}(N)$ SCFT gauge theory. This worldvolume theory, obtained in the non-singular case $S = \mathbb{C}^3$, is called the *parent theory*.

Note that the D3-brane will warp the flat space metric to that of $AdS_5 \times S^5$ and the bulk geometry is not strictly $\mathbb{C}^3$. However, as stated above, we are only concerned with the local gauge theory and not with gravitational back-reaction, therefore it suffices to consider S as $\mathbb{C}^3$.

2.2 | Matter content

As discussed in appendix B, there is only one $D = 4, \mathcal{N} = 4$ SCFT theory, up to a choice of gauge group G. In our case, $G = \mathrm{U}(N)$. The isometry group of the transverse space $\mathbb{R}^6$ is $\mathrm{SO}(6) \cong \mathrm{SU}(4)$. Since the scalar fields living on the branes are interpreted as its tranverse oscillations, SO(6) is a global symmetry of the field theory. These global symmetries of worldvolume theory lead to the R-symmetry group SU$(4)_R$. The only $\mathcal{N}=4$ supermutliplet can be rewritten in terms of $\mathcal{N}=1$ supermultiplets as follows:

$$[\mathcal{N}=4 \text{ vector multiplet}] : V=(\lambda_\alpha, A_\mu, D) \oplus \Phi_A=(\phi^A, \psi^A_\alpha, F^A). \tag{2.2}$$

with $A=1,2,3$. In other words, after removing the auxiliary fields D and F^A, the matter content is

- a U(N) gauge field A_μ which transforms as a singlets under SU$(4)_R$:

$$\text{Gauge transformation} : A_\mu \mapsto U A_\mu U^{-1} + U\partial_\mu U^{-1}, \qquad U \in \text{U}(N) \tag{2.3}$$
$$\text{R-symmetry} : A_\mu \mapsto A_\mu. \tag{2.4}$$

  Note that the usual term

- 4 Weyl fermions $\psi^a_\alpha \equiv (\lambda_\alpha, \psi^1_\alpha, \psi^2_\alpha, \psi^3_\alpha,)$ that transform under the adjoint of U(N) and are mixed together under the representation **4** of SU$(4)_R$. This means that each fermion ψ^a takes values in $\mathfrak{u}(N)$. We denote the components by ψ^a_{IJ} $(I,J=1,\ldots,N)$. Explicitely:

$$\text{Gauge transformation} : \psi^a \mapsto U\psi^a U^\dagger, \qquad U \in \text{U}(N), \tag{2.5}$$
$$\text{R-symmetry} : \psi^a \mapsto R^a{}_b \psi^b, \qquad R \in \text{SU}(4)_R. \tag{2.6}$$

  Note that this gives us $4N^2$ Weyl fermions in total.

- 3 complex scalar fields ϕ^A transforming under the adjoint representation of U(N) and under the two-times anti-symmettric representation of SU$(4)_R$. This means that each ϕ^A takes values in $\mathfrak{u}(N)$ and we denote the components by ϕ^A_{IJ}. Recall that SU$(4) \cong$ SO(6) so the action of the R-symmetry can be seen as the **3** of SU$(3) \subset$ SU$(4)_R$ acting on three complex scalars ϕ^A or equivalently as the **6** of SO$(6)_R$ acting on 6 real scalars X^m, the real and imaginary parts of the ϕ^A. They are interpreted as the oscillations of the branes in the transverse space. Explicitely:

$$\text{Gauge transformation} : X^m \mapsto U X^m U^\dagger, \qquad U \in \text{U}(N), \tag{2.7}$$
$$\text{R-symmetry} : X^m \mapsto R^m{}_n X^n, \qquad R \in \text{SO}(6)_R. \tag{2.8}$$

  Note that this gives $6N^2$ real scalars in total. They are the superpartners of the fermions(*).

Note that the gauge group U(N) can also be seen as the group of isometries of the metric space $\mathbb{C}^N$, i.e. Hom$(\mathbb{C}^N, C^N)$. From this point of view, the transformations (2.3)-(2.8)can be summarized as

$$A_\mu \in \text{Hom}(\mathbb{C}^N, \mathbb{C}^N), \tag{2.9}$$
$$\psi \in \mathbf{4} \otimes \text{Hom}(\mathbb{C}^N, \mathbb{C}^N), \tag{2.10}$$
$$X \in \mathbf{6} \otimes \text{Hom}(\mathbb{C}^N, \mathbb{C}^N). \tag{2.11}$$

> If the transverse space is non-singular, the only possibility is $S=\mathbb{C}^3$. In this case, the worldvolume theory is therefore $D=4, \mathcal{N}=4$ U(N) SCFT gauge theory. It is called the *parent theory*.

3 | Singular transverse space: orbifold singularities

When the transverse space is singular, the worldvolume theory corresponds to a specific projection of the parent theory that we found in the smooth case $S=\mathbb{C}^3$. We call it the *daughter theory*. This projections depends on the type of singularity that one considers. The simplest case is the the case of orbifold singularity, i.e. when the transverse space is a quotient space with a non-free action.

3.1 | Generalities

We now wish to pick a discrete group Γ and which acts non-trivially on $\mathbb{R}^6$. There are several possibilities:

- $\Gamma \subset \mathrm{SU}(4)$ naturally acts on $\mathbb{R}^6$. This does not require a choice of complex structure. We get an $\mathcal{N} = 0$ theory.
- $\Gamma \subset \mathrm{SU}(3)$ naturally acts on $\mathbb{C}^3$, this also requires a choice of complex structure on $\mathbb{R}^6$. We get an $\mathcal{N} = 1$ theory.
- $\Gamma \subset \mathrm{SU}(2)$ naturally acts on the second factor of $\mathbb{C} \times \mathbb{C}^2$, so this requires a choice of complex structure on $\mathbb{R}^6$. We get an $\mathcal{N} = 2$ theory.

We are interested in supersymmetric theories so we take $\Gamma \subset \mathrm{SU}(3)$ with the action

$$\cdot : \left(\begin{array}{ccc} \Gamma \times \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3 \\ (\gamma, z) & \longmapsto & \gamma \cdot z \end{array} \right) \tag{3.1}$$

is the representation of Γ coming from the fundamental representation of $\mathrm{SU}(3)$, so $\cdot$ is just the matrix product. We can see that the origin is always a fixed point so this action is never free. Since $\mathbb{C}^3$ is a smooth manifold, this makes $\mathbb{C}^3/\Gamma$ an orbifold. Note this case naturally includes the case $\Gamma \subset \mathrm{SU}(2)$ too (as $\mathrm{SU}(2) \subset \mathrm{SU}(3)$), just not the case $\Gamma \in \mathrm{SO}(6)$. When $\Gamma \subset \mathrm{SU}(2 \subset \mathrm{SU}(3)$, it acts trivially on one component so we write $S = \mathbb{C} \times \mathbb{C}^2/\Gamma$.

If Γ is a general finite group the condition that $\mathbb{C}^3/\Gamma$ is an Calabi-Yau orbifold means that there must exist a resolution of this orbifold such that the corresponding smooth space is Calabi-Yau, i.e. a crepant resolution. Existence of such a resolution constrains Γ (*), see appendix H.

If the transverse space is $S = \mathbb{C}^3/\Gamma$, the field theory must be projected to a theory which also invariant under Γ, seen as a subgroup of the R-symmetry group. The prescription is straihgt-forward: we can use the elements $\gamma \in \Gamma$ to project out that states that are not Γ-invariant. That is, if ρ is an embedding of Γ in the gauge group $\mathrm{U}(N)$, only the the fields such that

$$\rho(\gamma) A_\mu \rho(\gamma)^{-1} = A_\mu, \tag{3.2}$$

$$R(\gamma)\rho(\gamma)\psi_{IJ}\rho(\gamma)^{-1} = \psi_{IJ}, \tag{3.3}$$

$$R(\gamma)\rho(\gamma)X_{IJ}\rho(\gamma)^{-1} = X_{IJ} \tag{3.4}$$

are kept in the spectrum, where ρ is a unitary representation of Γ on $\mathbb{C}^N$ and $R = \mathbf{4}, \mathbf{6}$. Let us make two remarks:

- the term $U\partial_\mu U^{-1}$ is absent from 3.6. This comes from the fact that since Γ is a finite group, the only smooth functions $x \mapsto \Gamma$ are the constant ones, so transformations of the gauge field under a finite subgroup of its gauge group cannot depend on x.
- the fields that transform non-trivially under R-symmetry also have an extra induced action of Γ, in agreement with (2.9)-(2.11). The R-symmetry untouched by Γ will be the resulting R-symmetry of daughter theory.

3.2 | Representation theory realization

Let $\{(\rho_i, V_i)\}_{i\in I}$ be a complete set of irreducible representations of Γ. Since Γ is finite, it is particular compact and those representation can be taken to be unitary. Moreover, i takes a finite number of values. Let us consider a representation of Γ on $\mathbb{C}^N$, we denote it $(\rho, \mathbb{C}^N)$ and also take it to be unitary. In that case, $\rho(\gamma) \in \mathrm{U}(N)$. This is what we mean by "the embedding of Γ in $\mathrm{U}(N)$". The adjoint representation of $\mathrm{U}(N)$ defined as[1]

$$\mathrm{Ad} : \left(\begin{array}{ccc} \mathrm{U}(n) & \longrightarrow & \mathrm{GL}(\mathfrak{u}(N)) \\ U & \longmapsto & \mathrm{Ad}_U \end{array} \right), \qquad \mathrm{Ad}_U : \left(\begin{array}{ccc} \mathfrak{u}(N) & \longrightarrow & \mathfrak{u}(N) \\ \omega & \longmapsto & \mathrm{Ad}_U(\omega) \equiv U\omega U^{-1} \end{array} \right), \tag{3.5}$$

[1] this is well-defined since for all $\omega \in \mathfrak{u}(N)$ and $U \in \mathrm{U}(N)$, $U\omega U^{-1} \in \mathfrak{u}(N)$.

now allows us to act with Γ on $\mathfrak{u}(N)$. We use this representation in the expression (3.6)-(3.8). More formally, these relations can be rewritten as

$$\mathrm{Ad}_{\rho(\gamma)} A_\mu = A_\mu, \tag{3.6}$$
$$R(\gamma)\mathrm{Ad}_{\rho(\gamma)} \psi_{IJ} = \psi_{IJ}, \tag{3.7}$$
$$R(\gamma)\mathrm{Ad}_{\rho(\gamma)} X_{IJ} = X_{IJ} \tag{3.8}$$

We can decompose $(\rho, \mathbb{C}^N)$ as follows:

$$(\rho, \mathbb{C}^N) = \bigoplus_{i \in I} (\rho_i, V_i)^{N_i} \tag{3.9}$$
$$= \bigoplus_{i \in I} (\mathbf{1}^{N_i} \otimes \rho_i, \mathbb{C}^{N_i} \otimes V_i) \tag{3.10}$$

were N_i are integer multiplicities ($(\rho_i, V_i)^{N_i} \equiv (\rho_i, V_i)^{\oplus N_i}$) and $\mathbf{1}$ is the trivial representation, so Γ acts trivially on the $\mathbb{C}^{N_i}$. We have $\sum_i N_i \dim(\rho_i) = N$. The rewritting (3.10) will be useful later on.

After the projection, the resulting gauge group is given by the Γ-invariant part of the gauge group, that is $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)^\Gamma$. We use the superscript Γ to indicate that we only keep the trivial representations in the decomposition, that is, we only keep that subspaces that transform trivially. First, we can see that by Schur's lemma (*)

$$(V_i \otimes V_j^*)^\Gamma = \delta_{ij} \tag{3.11}$$

Now since $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N) \cong \mathbb{C}^N \otimes (\mathbb{C}^N)^*$, we get

$$\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)^\Gamma = (\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^\Gamma \tag{3.12}$$
$$= \bigoplus_{i,j \in I} \left((\mathbb{C}^{N_i} \otimes V_i) \otimes (\mathbb{C}^{N_j} \otimes V_j)^*\right)^\Gamma \tag{3.13}$$
$$= \bigoplus_{i,j \in I} \left(\mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^* \otimes V_i \otimes V_j^*\right)^\Gamma \tag{3.14}$$
$$= \bigoplus_{i,j \in I} \left(\mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^*\right)^\Gamma \otimes \left(V_i \otimes V_j^*\right)^\Gamma \tag{3.15}$$
$$= \bigoplus_{i \in I} \mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_i})^* \tag{3.16}$$

so the daughter gauge group is

$$G_{\text{proj}} = \bigotimes_{i \in I} \mathrm{U}(N_i). \tag{3.17}$$

Now it turns out that in the low energy effective field theory the $\mathrm{U}(1)$ factor of every $\mathrm{U}(N_i)$ decouples (*)so the resulting gauge group is in fact

$$G_{\text{proj}} = \bigotimes_{i \in I} \mathrm{SU}(N_i). \tag{3.18}$$

For the matter fields, the reasoning is similar but we now have to take into account the R-symmetry. Let $\mathbf{4} \equiv (\rho_\mathbf{4}, V_\mathbf{4})$ be the fundamental representation of $\mathrm{SU}(4)_R$ and $\mathbf{6} \equiv (\rho_\mathbf{6}, V_\mathbf{6})$ be the fundamental representation of $\mathrm{SO}(6)_R$. Wish to compute $(V_\mathcal{R} \otimes \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N))^\Gamma$ with $\mathcal{R} = \mathbf{4}, \mathbf{6}$:

$$(V_\mathcal{R} \otimes \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N))^\Gamma = \bigoplus_{i,j \in I} \left(V_\mathcal{R} \otimes (\mathbb{C}^{N_i} \otimes V_i) \otimes (\mathbb{C}^{N_j} \otimes V_j)^*\right)^\Gamma \tag{3.19}$$
$$= \bigoplus_{i,j \in I} \left(V_\mathcal{R} \otimes \mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^*\right)^\Gamma \otimes \left(V_i \otimes V_j^*\right)^\Gamma \tag{3.20}$$
$$= \bigoplus_{i,j \in I} a_{ij}^\mathcal{R} \left(\mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^*\right) \tag{3.21}$$

with

$$(\rho_{\mathcal{R}}, V_{\mathcal{R}}) \otimes (\rho_i, V_i) = \bigoplus_{j \in I} a_{ij}^{\mathcal{R}} (\rho_j, V_j). \tag{3.22}$$

This expression makes sense because $(\rho_{\mathcal{R}}, V_{\mathcal{R}})$ is a representation of $\mathrm{SU}(4)$ so it is in particular a representation of $\mathrm{SU}(3)$ and therefore also in particular a representation of Γ.

Using the orthogonality of characters of irreducible non-equivalent representations, we obtain the explicit expression of the coefficient $a_{ij}^{\mathcal{R}}$:

$$a_{ij}^{\mathcal{R}} = \frac{1}{|\Gamma|} \sum_{\gamma=1}^{r} r_\gamma \chi^{\mathcal{R}}(\gamma) \chi^i(\gamma) \chi^j(\gamma)^* \tag{3.23}$$

where r_γ is the order of the conjugacy class containing γ and χ^i is the character of ρ_i.

In the end, we can see that:

> In the daughter theory, the matter fields become a total of $a_{ij}^{\mathbf{4}}$ bi-fundamental fermions and $a_{ij}^{\mathbf{6}}$ bi-fundamental bosons transforming as the $(\mathbf{N_i}, \overline{\mathbf{N}}_{\boldsymbol{j}})$ of $\mathrm{SU}(N_i) \times \mathrm{SU}(N_j)$ under the products of gauge groups.

3.3 | Field content, quivers and McKay graphs

A convenient way the represent the matter content of a daughter theory is to use *quiver diagrams*. A quiver is a finite oriented graph such that each node i represents a gauge factor $\mathrm{SU}(N_i)$ and each arrow $i \to j$ represents a bi-fundamental field transforming under $(\mathbf{N_i}, \overline{\mathbf{N}}_{\boldsymbol{j}})$. The *adjacency matrix* A of the graph is a $k \times k$ with k being the number of nodes (gauge factors) whose elements A_{ij} are the number of arrows (bi-fundamental fields) from i to j. In other words, from (3.21), the adjacency matrix of the fermions has elements $A_{ij} = a_{ij}^{\mathbf{4}}$ and the one of the scalars has elements $A_{ij} = a_{ij}^{\mathbf{6}}$.

On the other hand, given finite group Γ, a representation (ρ_W, W) and a complete set of irreducible representations $\{(\rho_i, V_i)\}_{i \in I}$ of the latter, one can construct a McKay graph (or quiver) as follows:

1. Draw a vertex for every representation (ρ_i, V_i).
2. For every representation (ρ_i, V_i), compute the decomposition

$$(\rho_W, V_W) \otimes (\rho_i, V_i) = \bigoplus_j (\rho_j, V_j)^{\oplus n_{ij}}$$

   where n_{ij} is the multiplicity of (ρ_j, V_j) in the decomposition of $(\rho_W, V_W) \otimes (\rho_i, V_i)$.
3. For every $n_{ij} > 0$, draw n_{ij} arrows from the vertex of (ρ_i, V_i) to the one of (ρ_j, V_j).

When $\Gamma \subset \mathrm{SU}(2)$ and that (ρ_W, W) is its defining representation, the McKay graphs are in one-to-one correspondence with the extended Dynkin diagrams of the simply laced Lie algebras. This is the classical McKay correspondence, see appendix F.

From (3.22) we see that $n_{ij} = a_{ij}^{\mathcal{R}}$, i.e. the matter quivers that we defined previously are exactly the McKay graph associated to the matter representation in question. Put differently, the matter content of the daughter theory is encapsulated in the McKay graphs of Γ and with respect to $\mathcal{R}$ with $\mathcal{R} = \mathbf{4}$ for spinors and $\mathcal{R} = \mathbf{6}$ for scalars. This very important point allows us to use known results on McKay graphs such as the McKay correspondence for example.

3.4 | A simple example: $S = \mathbb{C}^3/\mathbb{Z}_3$

We illustrate the previous discussion by the simple case where $\Gamma = \mathbb{Z}_3$ acts on $\mathbb{C}^2$ as

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{3.24}$$

i.e. the transverse space is the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. This simple example is a good first approach in which we will to explain in details each step so that we can go faster afterwards.

3.4.1 | Projection

Let us consider a representation $(\rho, \mathbb{C}^N)$ of $\mathbb{Z}_3$. A complete set of irreducible representations of $\mathbb{Z}_3$ is given by $\{(\rho_k, V_k)\}_{k=0,1,2}$ with $V_k = \mathbb{C}$ and

$$\rho_i(g) = \zeta_3^k \tag{3.25}$$

where g is the generator of $\mathbb{Z}_3$. The representation (ρ, V) can be decomposed as

$$(\rho, V) = \bigoplus_{i=0}^{2} N_i(\rho_i, V_i). \tag{3.26}$$

In other words, it is equivalent to the representation

$$\rho(g) = \begin{array}{l} \left[\begin{array}{ccccccccc} 1 & & & & \cdots & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \zeta_3 & & & & & \\ \vdots & & & & \ddots & & & & \vdots \\ & & & & & \zeta_3 & & & \\ & & & & & & \zeta_3^2 & & \\ & & & & & & & \ddots & \\ 0 & & & & \cdots & & & & \zeta_3^2 \end{array}\right] \end{array} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} N_0 \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} N_1 \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} N_2 \end{array} . \tag{3.27}$$

Since $\dim \rho_i = 1$, we have

$$N_0 + N_1 + N_2 = N. \tag{3.28}$$

The gauge field configurations that are left invariant under the action of $\mathbb{Z}_n$ are therefore the ones that satisfy

$$\rho(g) A_\mu \rho(g)^{-1} = A_\mu. \tag{3.29}$$

We actually want this relation to be true for any element of $\mathbb{Z}_n$ but in this case it is invariant under any element of $\mathbb{Z}_n$ if and only if it is invariant under the generator g of $\mathbb{Z}_3$, so we only need to impose (3.29). The constrained is easily solved by using the bi-index notation $A_{\mu;i\alpha_i,j\beta_j}$ $(i, j = 0, 1, 2, \alpha_i, \beta_i = 1, \ldots, N_i)$ for the component of the gauge fields. From (4.2), we can see that

$$A_{\mu;i\alpha_i,i\beta_j} \mapsto \rho_i(g) A_{\mu;i\alpha_i,j\beta_j} \rho_j(g)^{-1} = \zeta_n^{i-j} A_{\mu;i\alpha_i,j\beta_j}. \tag{3.30}$$

thus only the configurations with $A_{\mu;i\alpha_i,j\beta_j} = 0$ if $i \neq j$ are invariant. The gauge field has therefore a block diagonal form:

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & \\ & A_{\mu;11} & \\ & & A_{\mu;2,2} \end{bmatrix} \tag{3.31}$$

with $A_{\mu;ij} \equiv (A_{\mu;i\alpha_i,j\beta_j})_{\alpha_i=0,\ldots,N_i,\beta_j=0,\ldots,N_j}$. The block A_{ii} transforms under $\mathbb{Z}_3$ under $(\rho_i, V_i)^{N_i}$. Consequently, the gauge group is now broken to

$$G_{\text{proj}} = \mathrm{U}(N_0) \times \mathrm{U}(N_1) \times \mathrm{U}(N_1), \tag{3.32}$$

the biggest subgroup of $\mathrm{U}(N)$ that preserves those form of configurations.

Let us study the scalars. $\mathbb{Z}_3$ acts on the three complex scalars through

$$R(g) = \rho_1^{\oplus 3}(g) = \zeta_3 \mathbb{1}_3 = \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} \tag{3.33}$$

or, equivalently, on the real scalars as $R(g) = \zeta_3 \mathbb{1}_6$. According to (2.7)-(2.8), the scalar field configurations that are left invariant satisfy

$$R(g)^m{}_n \rho(g) X^n \rho(g)^{-1} = X^m \tag{3.34}$$

for all $g \in \mathbb{Z}_n$. Using the bi-index notations, this becomes

$$X^m_{i\alpha_i,j\beta_j} \mapsto \zeta_n \delta^m{}_n \rho_i(g) X^n_{i\alpha_i,j\beta_j} \rho_j(g)^{-1} = \zeta_3^{i-j+1} X^m_{i\alpha_i,j\beta_j} \tag{3.35}$$

$$\overline{X}^m_{i\alpha_i,j\beta_j} \mapsto \zeta_3^{-1} \delta^m{}_n \rho_i(g) \overline{X}^n_{i\alpha_i,j\beta_j} \rho_j(g)^{-1} = \zeta_3^{i-j-1} \overline{X}^m_{i\alpha_i,j\beta_j} \tag{3.36}$$

thus only the configurations with $X^n_{i\alpha_i,j\beta_j} = 0$ if $i-j+1 \neq 0$ are left invariant and only the configurations with $\overline{X}^n_{i\alpha_i,j\beta_j} = 0$ if $i-j-1 \neq 0$ are left invariant. The scalar fields X have a block off-diagonal form:

$$X^m = \begin{bmatrix} 0 & X^m_{01} & 0 \\ 0 & 0 & X^m_{12} \\ X^m_{21} & & 0 \end{bmatrix}, \qquad \overline{X}^m = \begin{bmatrix} 0 & 0 & \overline{X}^m_{02} \\ \overline{X}^m_{10} & 0 & 0 \\ 0 & \overline{X}^m_{21} & 0 \end{bmatrix} \tag{3.37}$$

with the block notations

$$X^m_{ij} \equiv (X^m_{i\alpha_i,j\beta_j})_{\alpha_i=0,\ldots,N_i,\beta_j=0,\ldots,N_j}$$
$$\overline{X}^m_{ij} \equiv (\overline{X}^m_{i\alpha_i,j\beta_j})_{\alpha_i=0,\ldots,N_i,\beta_j=0,\ldots,N_j}.$$

The block X^m_{ij} is an $N_i \times N_j$ block and transforms under the representation $(\mathbf{N_i}, \overline{\mathbf{N}}_{\boldsymbol{j}})$ of $\mathrm{U}(N_i) \times \mathrm{U}(N_j)$:

$$X^m_{i,i+1} \in \mathbf{N_{i+1}} \otimes \overline{\mathbf{N}}_{\boldsymbol{i}} \cong \mathrm{Hom}(V_{i+1}, V_i), \tag{3.38}$$

$$\overline{X}^m_{i+1,i} \in \mathbf{N_i} \otimes \overline{\mathbf{N}}_{\boldsymbol{i+1}} \cong \mathrm{Hom}(V_i, V_{i+1}). \tag{3.39}$$

Let us make an important observation: the form of the scalar fields are the same for every $m = 0, \ldots, 5$. This can be traced back to the fact that $R(g) = \zeta_3 \mathbb{1}_6$ so the R-symmetry action of $\mathbb{Z}_n$ is the same for all m.

Let us now study the four Weyl fermions ψ^a.(*)

3.4.2 | Quiver

We can draw the quiver of this daughter theory. We have three types of bi-fundamental scalar fields:

$$X^m_{01} \in (\mathbf{N_1}, \overline{\mathbf{N}}_{\mathbf{0}}), \quad X^m_{12} \in (\mathbf{N_2}, \overline{\mathbf{N}}_{\mathbf{1}}), \quad X^m_{20} \in (\mathbf{N_0}, \overline{\mathbf{N}}_{\mathbf{2}}). \tag{3.40}$$

In each representation bi-fundamental representation there are six real scalars, i.e. 3 complex scalars. They are each represented by an arrow between the right representations, see fig. 1.

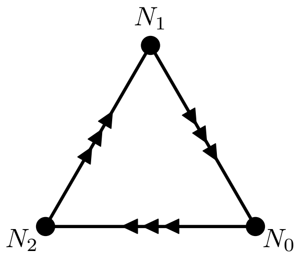


Figure 1: Quiver of the $\mathbb{C}^3/\mathbb{Z}_3$ daughter theory.

The adjacency matrix of this quiver is

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{bmatrix} \tag{3.41}$$

which is coherent with the fact that the coefficients of the McKay decomposition of $\rho_1 \oplus \rho_1 \oplus \rho_1$ are

$$\begin{aligned} n_{00} &= 0, \quad n_{01} = 3 \quad\quad n_{02} = 0, \\ n_{10} &= 0, \quad n_{11} = 0, \quad\quad n_{12} = 3, \\ n_{20} &= 3, \quad n_{21} = 0, \quad\quad n_{22} = 0. \end{aligned} \tag{3.42}$$

3.4.3 | Gauge anomaly cancellation

Finally, let us discuss the gauge anomaly cancellation. Our fields transform under the adjoint representation of SU(N), under the fundamentals of SU(N_i) and under the anti-fundamentals of SU(N_i). The adjoint representation being reel, it is self-conjugate and therefore does not contribute to the anomaly. Fundamentals of SU(N_i) have a $+1$ contribution to the anomaly and anti-fundamentals of SU(N_i) have a -1 contribution to the anomaly. Anomaly cancellation therefore imposes that the contribution of the fundamental and of the anti-fundamental of SU(N_i) cancel each other for each $i = 0, 1, 2$, see appendix (C). The bi-fundamental representation $(\mathbf{N_i}, \overline{\mathbf{N}}_{\boldsymbol{j}})$ counts as N_j fundamentals of SU(N_i) and as N_i anti-fundamentals SU(N_j). We get the three following conditions:

$$\text{SU}(N_0) : N_2 - N_1 = 0, \tag{3.43}$$

$$\text{SU}(N_1) : N_0 - N_2 = 0, \tag{3.44}$$

$$\text{SU}(N_2) : N_1 - N_0 = 0. \tag{3.45}$$

which immediately imply

$$N_0 = N_1 = N_2. \tag{3.46}$$

From (3.28), we get that $N_0 = N_1 = N_2 = N/3$, meaning that that the daughter theory has quantum gauge symmetry if and only if the parent theory with has gauge group SU(N) where N is a multiple of 3. Or, in other words, if the number of D-branes is a multiple of 3.

3.5 | $(p+1)$-dimensional quiver gauge theories

Let us mention that we can generalize our initial brane-world paradigm and consider Dp-branes in type II string theory (type IIA if p is even and type IIB if p is odd) instead of just D3-branes. The spacetime is then of the form

$$M = \mathbb{R}^{1,p} \times \mathbb{R}^{9-p}/\Gamma \tag{3.47}$$

where Γ is a discrete subgroup of Spin$(9-p)$. If Γ is a subgroup of a special holonomy group, we recover a somewhat generalized version of the paradigm that we discussed above. In this case the transverse space is a Calabi-Yau orbifold and some degree of supersymmetry is preserved. Note that the fermionic and bosonic quivers that coincides. If Γ is not a subgroup of a special holonomy group, then the $(p+1)$-dimensional quiver gauge theory that we obtain in the low-energy limit is not supersymmetric. We then have different quivers for the fermions and the bosons, although with the same vertices, by definition.

Recall that the fraction of supercharges that is preserved by compactifying on a Calabi-Yau n-fold (with SU(n) holonomy) is 2^{1-n}. Starting from the $\mathcal{N} = 2$ 10-dimensional type IIB string theory with 32 supercharges, this means that

- if we compactify on a 1-fold, we get 32 supercharges in 8 dimensions so $\mathcal{N} = 2$,
- if we compactify on a 2-fold, we get 16 supercharges in 6 dimensions so $\mathcal{N} = 2$,
- if we compactify on a 3-fold, we get 8 supercharges in 4 dimensions so $\mathcal{N} = 2$,
- is we compactify on a 4-fold, we get 4 supercharges in 2 dimensions so $\mathcal{N} = 4$.

We will however mostly mostly consider 4-dimensional quiver gauge theories, i.e. living on D3-branes.

4 | Orbifold constructions of $\mathcal{N}=2$ daughter theories

4.1 | $S=\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_n$

We consider a representation $(\rho,\mathbb{C}^N)$ of $\mathbb{Z}_n$. We decompose it on the set of irreducible representations of $\mathbb{Z}_n$ as

$$(\rho,V)=\bigoplus_{i=0}^{n-1}N_i(\rho_i,V_i). \tag{4.1}$$

In other words, it is equivalent to the representation

$$\rho(g)=\begin{bmatrix}1 & & & \cdots & & & 0\\ & \ddots & & & & & \\ & & 1 & & & & \\ \vdots & & & \ddots & & & \vdots\\ & & & & \zeta_n^{n-1} & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & \zeta_n^{n-1}\end{bmatrix}\begin{matrix}\left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\}N_0\\ \vdots\\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\}N_{n-1}\end{matrix}. \tag{4.2}$$

Since $\dim\rho_i=1$, $\sum_i N_i=N$.

The gauge field configurations that are left invariant under the action of $\mathbb{Z}_n$ are therefore the ones that satisfy

$$\rho(g)A_\mu\rho(g)^{-1}=A_\mu. \tag{4.3}$$

The constrained is easily solved by using the bi-index notation:

$$A_{\mu;i\alpha_i,i\beta_j}\mapsto\rho_i(g)A_{\mu;i\alpha_i,j\beta_j}\rho_j(g)^{-1}=\zeta_n^{i-j}A_{\mu;i\alpha_i,j\beta_j}. \tag{4.4}$$

thus only the configurations with $A_{\mu;i\alpha_i,j\beta_j}=0$ if $i\neq j$ are invariant. The gauge field has therefore a block diagonal form:

$$A_\mu=\begin{bmatrix}A_{\mu;00} & & & \\ & A_{\mu;11} & & \\ & & \ddots & \\ & & & A_{\mu;n-1,n-1}\end{bmatrix} \tag{4.5}$$

with $A_{\mu;ij}\equiv(A_{\mu;i\alpha_i,j\beta_j})_{\alpha_i=0,\ldots,N_i,\beta_j=0,\ldots,N_j}$. The block A_{ii} transforms under $\mathbb{Z}_n$ as $(\rho_i,V_i)^{N_i}$. For now it is only a simple generalization of the case $\mathbb{C}^3/\mathbb{Z}_3$. This makes sense: projection of the gauge field only depends the discrete group Γ, not on the way it acts on $\mathbb{C}^3$ because it does not transform under R-symmetry.

The gauge group is now broken to

$$G_{\text{proj}}=\prod_{i=0}^{n-1}\mathrm{U}(N_i). \tag{4.6}$$

Now for the scalar fields. The action of $\mathbb{Z}_n$ that we consider leaves the first component of $\mathbb{C}^3$ untouched so we take the action $\mathbf{1}\oplus\mathbf{2}$ where $\mathbf{2}$ is the usual action of $\mathbb{Z}_n$ on $\mathbb{C}^2$. In other words,

$$R(g)=\begin{bmatrix}1 & 0 & 0\\ 0 & \zeta_n & 0\\ 0 & 0 & \zeta_n^{-1}\end{bmatrix}. \tag{4.7}$$

Or, equivalently, $R(g)^m{}_n=\delta^m{}_nA_n$ with $A_m=(1,1,\zeta_n,\zeta_n,\zeta_n^{-1},\zeta_n^{-1})$. The scalar field configurations that are left invariant satisfy

$$R(g)^m{}_n\rho(g)X^n\rho(g)^{-1}=X^m \tag{4.8}$$

for all $g \in \mathbb{Z}_n$. Using the bi-index notations, this becomes

$$X^m_{i\alpha_i,j\beta_j} \mapsto \delta^m{}_n A_n \rho_i(g) X^m_{i\alpha_i,j\beta_j} \rho_j(g)^{-1} = \delta^m{}_n A_n \zeta_n^{i-j} X^m_{i\alpha_i,j\beta_j} = \begin{cases} \zeta_n^{i-j} X^m_{i\alpha_i,j\beta_j}, & m = 0,1 \\ \zeta_n^{i-j+1} X^m_{i\alpha_i,j\beta_j}, & m = 2,3 \\ \zeta_n^{i-j-1} X^m_{i\alpha_i,j\beta_j}, & m = 4,5 \end{cases} \tag{4.9}$$

$$\overline{X}^m_{i\alpha_i,j\beta_j} \mapsto \delta^m{}_n \overline{A_n} \rho_i(g) \overline{X}^m_{i\alpha_i,j\beta_j} \rho_j(g)^{-1} = \delta^m{}_n \overline{A_n} \zeta_n^{i-j} X^m_{i\alpha_i,j\beta_j} = \begin{cases} \zeta_n^{i-j} X^m_{i\alpha_i,j\beta_j}, & m = 0,1 \\ \zeta_n^{i-j-1} X^m_{i\alpha_i,j\beta_j}, & m = 2,3 \\ \zeta_n^{i-j+1} X^m_{i\alpha_i,j\beta_j}, & m = 4,5 \end{cases} \tag{4.10}$$

thus only the configurations with $X^{0,1}_{i\alpha_i,j\beta_j} = 0$ if $i - j \neq 0$, $X^{2,3}_{i\alpha_i,j\beta_j} = 0$ if $i - j + 1 \neq 0$ and $X^{4,5}_{i\alpha_i,j\beta_j} = 0$ if $i - j - 1 \neq 0$ are left invariant (and similarly for the conjugated fields). The scalar fields X have a the following forms:

$$X^{0,1} = \begin{bmatrix} X^{0,1}_{00} & & 0 \\ & \ddots & \\ 0 & & X^{0,1}_{n-1,n-1} \end{bmatrix}, \tag{4.11}$$

$$X^{2,3} = \begin{bmatrix} 0 & X^{2,3}_{01} & & 0 \\ & \ddots & \ddots & \\ & & 0 & X^{2,3}_{n-2,n-1} \\ X^{2,3}_{n-1,0} & & & 0 \end{bmatrix}, \quad X^{4,5} = \begin{bmatrix} 0 & & & X^{4,5}_{0,n-1} \\ X^{4,5}_{10} & 0 & & \\ & \ddots & \ddots & \\ 0 & & X^{4,5}_{n-1,n-2} & 0 \end{bmatrix}, \tag{4.12}$$

$$\overline{X}^{0,1} = \begin{bmatrix} \overline{X}^{0,1}_{00} & & 0 \\ & \ddots & \\ 0 & & \overline{X}^{0,1}_{n-1,n-1} \end{bmatrix}, \tag{4.13}$$

$$\overline{X}^{2,3} = \begin{bmatrix} 0 & & & \overline{X}^{2,3}_{0,n-1} \\ \overline{X}^{2,3}_{10} & 0 & & \\ & \ddots & \ddots & \\ 0 & & \overline{X}^{2,3}_{n-1,n-2} & 0 \end{bmatrix}, \quad \overline{X}^{4,5} = \begin{bmatrix} 0 & \overline{X}^{4,5}_{01} & & 0 \\ & \ddots & \ddots & \\ & & 0 & \overline{X}^{4,5}_{n-2,n-1} \\ \overline{X}^{4,5}_{n-1,0} & & & 0 \end{bmatrix} \tag{4.14}$$

so X^m_{ij} is an $N_i \times N_j$ block and transforms under the representation $(\mathbf{N_i}, \overline{\mathbf{N}}_\mathbf{j})$ of $\mathrm{U}(N_i) \times \mathrm{U}(N_j)$:

$$X^{0,1}_{i,i} \in \mathbf{N_i} \otimes \overline{\mathbf{N}}_\mathbf{i} \cong \mathrm{Hom}(V_i, V_i), \tag{4.15}$$

$$X^{2,3}_{i,i+1} \in \mathbf{N_{i+1}} \otimes \overline{\mathbf{N}}_\mathbf{i} \cong \mathrm{Hom}(V_{i+1}, V_i), \tag{4.16}$$

$$X^{4,5}_{i+1,i} \in \mathbf{N_i} \otimes \overline{\mathbf{N}}_\mathbf{i+1} \cong \mathrm{Hom}(V_i, V_{i+1}). \tag{4.17}$$

So the scalar fields are split up in three families depending on the way they transform under R-symmetry. We now see a big difference with the case $\mathbb{C}^3/\mathbb{Z}_3$: since the R-symmetry does not act the same way on each directions in $\mathbb{C}^3$, the invariant scalar field configurations are not the same in each direction either.

Let us draw the quiver for the case $n = 3$ so that we can compare to 1. We have $2 \cdot 9 = 18$ real scalar fields in 9 different representations:

$$X^0_{00}, X^1_{00} \in (\mathbf{N_0}, \overline{\mathbf{N}}_\mathbf{0}), \quad X^0_{11}, X^1_{11} \in (\mathbf{N_1}, \overline{\mathbf{N}}_\mathbf{1}), \quad X^0_{22}, X^1_{22} \in (\mathbf{N_2}, \overline{\mathbf{N}}_\mathbf{2}), \tag{4.18}$$

$$X^2_{01}, X^3_{01} \in (\mathbf{N_1}, \overline{\mathbf{N}}_\mathbf{0}), \quad X^2_{12}, X^3_{12} \in (\mathbf{N_2}, \overline{\mathbf{N}}_\mathbf{1}), \quad X^2_{20}, X^3_{20} \in (\mathbf{N_0}, \overline{\mathbf{N}}_\mathbf{2}), \tag{4.19}$$

$$X^4_{10}, X^5_{10} \in (\mathbf{N_0}, \overline{\mathbf{N}}_\mathbf{1}), \quad X^4_{21}, X^5_{21} \in (\mathbf{N_1}, \overline{\mathbf{N}}_\mathbf{2}), \quad X^4_{02}, X^5_{02} \in (\mathbf{N_2}, \overline{\mathbf{N}}_\mathbf{0}), \tag{4.20}$$

We now only have 1 complex scalar in each representation and the quiver is given by 2.

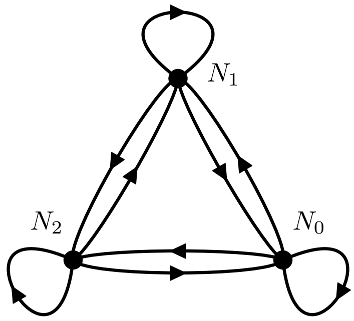


Figure 2: Quiver of the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_3$ daughter theory.

It is easy to see how the construction of the the quiver generalizes for arbitrary n. The adjacency matrix is

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \tag{4.21}$$

which is, as it should, coincides with the McKay decomposition of $\mathbf{1} \oplus \mathbf{2}$:

$$\begin{aligned} n_{00} = 1, \quad \ldots, \qquad & n_{0,n-1} = 1, \\ \vdots \qquad\qquad\qquad & \quad \vdots \\ n_{n-1,0} = 1, \quad \ldots, \qquad & n_{n-1,n-1} = 1. \end{aligned} \tag{4.22}$$

Gauge anomaly cancellation now imposes that

$$N_{i-1} - N_{i-1} + N_{i+1} + N_{i+1} = 0 \tag{4.23}$$

for $i = 0, \ldots, n-1$. Those constrains are always satisfied so the the factors N_i are arbitrary, as long as $\sum_i N_i = N$ of course.

This quiver is easy to scale up for any n. We therefore implement that into Mathematica and get the quivers for any n.

4.2 | $S = \mathbb{C} \times \mathbb{C}^2/\mathcal{D}_n$

We quotient $\mathbb{C}^3$ by the binary dihedral group $\mathcal{D}_n$ that acts on the last two components. A set of irreducible representations of $\mathcal{D}_n$ is given by

$$\{(\rho_i, V_i)\}_{i=0,\ldots,n+2} \tag{4.24}$$

with $V_i = \mathbb{C}$ for $i = 0, \ldots, 3$ and $V_i = \mathbb{C}^2$ for $i = 4, \ldots, n+2$, so there are 4 one-dimensional representations and $n-1$ two-dimensional representations. They are explicitely given in section E.1.2. Using the bi-index notations $A_{\mu;i\alpha_i,j\beta_j}$ with $i, j = 0, \ldots, n+2$ and $\alpha_i, \beta_j = 0, \ldots, \dim \rho_i \cdot N_i - 1$, the invariant configurations must have $A_{\mu;ij} = 0$ if $i \neq j$, i.e. it must have a diagonal block-form, wiht block of size $\mathbb{N}_i \times N_i$ for $i = 0, \ldots, 3$ and of size $2N_i \times 2N_i$ for $i = 4, \ldots, n+1$. The blocks transforming under the 2-dimensional representations must have the form

$$A_{\mu;ii} = \begin{bmatrix} A_{\mu;i0,i0} & A_{\mu;i0,i1} & \cdots & A_{\mu;i,0,i,2N_i-2} & A_{\mu;i,0,i,2N_i-1} \\ \mp A_{\mu;i0,i1} & A_{\mu;i0,i0} & \cdots & \mp A_{\mu;i,0,i,2N_i-1} & A_{\mu;i,0,i,2N_i-2} \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ A_{\mu;i,2N_i-2,i,0} & A_{\mu;i,2N_i-2,i1} & \cdots & A_{\mu;i,2N_i-2,i,2N_i-2} & A_{\mu;i,2N_i-2,i,2N_i-1} \\ \mp A_{\mu;i,2N_i-2,i1} & A_{\mu;i,N_i-2,i,0} & \cdots & \mp A_{\mu;i,2N_i-2,i,2N_i-1} & A_{\mu;i,2N_i-2,i,2N_i-2} \end{bmatrix} \tag{4.25}$$

so each block is composed of N_i^2 2×2 matrices of the form

$$\begin{bmatrix} A & B \\ \mp B & A \end{bmatrix}. \tag{4.26}$$

We take the "$-$" signs when i is even and the "$+$" signs when i is odd. This comes from the fact that the 2-dimensional representations involve alternating signs.

R-symmetry acts on the three complex scalar fields as

$$R(A) = A \tag{4.27}$$

(*)

4.3 | $S = \mathbb{C} \times \mathbb{C}^2/\mathcal{T}, \mathcal{O}, \mathcal{I}$

(*)

5 | Orbifold constructions of $\mathcal{N} = 1$ daughter theories

5.1 | $S = \mathbb{C}^3/\mathbb{Z}_n$

Let us now consider the general case of $\mathbb{Z}_n$ acting on $\mathbb{C}^3$, i.e. we want to generalize the case that we treated in section 3.4.

5.1.1 | Actions of $\mathbb{Z}_n$ on $\mathbb{C}^3$

The first thing to do is to specify a representation of $\mathbb{Z}_n$ on $\mathbb{C}^3$. Any representation $(R, \mathbb{C}^3)$ can be decomposed as

$$R = \bigoplus_{k=0}^{n-1} \rho_k^{\oplus n_k} \tag{5.1}$$

and is therefore equivalent to a block-diagonal representation. This implies that $\sum_k n_k = 3$ so R can always be written as

$$R(g) = (\rho_a \oplus \rho_b \oplus \rho_c)(g) = \begin{bmatrix} \zeta_n^a & 0 & 0 \\ 0 & \zeta_n^b & 0 \\ 0 & 0 & \zeta_n^c \end{bmatrix} \tag{5.2}$$

with a, b, c some arbitrary exponents. We denote the representation (5.2) by (a, b, c). Let us make a few remarks on these representations:

- Since we consider $\mathbb{Z}_n$ as a subgroup of SU(3), we must have

  $$(a + b + c) \mod n = 0 \tag{5.3}$$

  such that $\det(R(g)) = 1$.

- Taking a or $a + n$ gives the same representation and the same is true for b and c, so we we can actually restrict ourselves to $0 < a, b, c < n$. We can trade the strict inequalities and allow the values 0 or n if we want to consider trivial representations as well. In this case, we will find that at least one direction of $\mathbb{C}^3$ is left untouched, i.e. we are in the situation where the orbifold is actually $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$, which we will not consider here.

- Permuting a, b, c gives equivalent representations so we can take a, b, c to be ordered, so $0 < a \leq b \leq c < n$.

The problem has now been reformulated as follows: we are looking for all possible ordered triplets (a, b, c) such that $0 < a \leq b \leq c < n$ and (5.3). For $n = 1$ and $n = 2$, it is clear that this is not possible, the possibilities involve at least one trivial representation. For $n = 3$, the only possibility is $(1, 1, 1)$, the one we used in 3.4. What about bigger values of n? First, note that $0 < a \leq b \leq c < n$ implies that the maximum value of $a+b+c$ is $3n-1$. From equation (5.3), the only two possibilities are then $a+b+c = n$ or $a+b+c = 2n$. What simplifies the analysis is that the second case can actually be ignored because it is already being taken care of by the first case. Let us explain that in more details: (*)

Now that we have seen that the only possibility is $a+b+c = n$, we find that the maximum value that a can take is $\lfloor n/3 \rfloor$, so $a = 1, \ldots, \lfloor n/3 \rfloor$. c can then be expressed in terms of a and b as $c = n - a - b$. The constraint $b \leq c$ then implies $b \leq (n-a)/2$, i.e. $b = a, \ldots, \lfloor (n-a)/2 \rfloor$. To summarize, the representations are all the representations of the form

$$(a, b, n-a-b) \tag{5.4}$$

with $a = 1, \ldots, \lfloor n/3 \rfloor$ and $b = a, \ldots, \lfloor (n-a)/2 \rfloor$. For small values, we get

$$\begin{aligned}
&\mathbb{Z}_1 : \text{necessarily involves a trivial representation}\\
&\mathbb{Z}_2 : \text{necessarily involves a trivial representation}\\
&\mathbb{Z}_3 : (1,1,1)\\
&\mathbb{Z}_4 : (1,1,2)\\
&\mathbb{Z}_5 : (1,1,3), (1,2,2)(*)\\
&\mathbb{Z}_6 : (1,1,4), (1,2,3), (2,2,2)\\
&\mathbb{Z}_7 : (1,1,5), (1,2,4), (1,3,3), (2,2,3)(*)\\
&\quad\vdots
\end{aligned}$$

For an arbitrary n, the total number of different representation is

$$\sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} + 1 \right\rfloor = \begin{cases} 3k^2, & \text{if } n = 6k \\ 3k^2 + k, & \text{if } n = 6k+1, \\ 3k^2 + 2k, & \text{if } n = 6k+2, \\ 3k^2 + 3k + 1, & \text{if } n = 6k+3, \\ 3k^2 + 4k + 1, & \text{if } n = 6k+4, \\ 3k^2 + 5k + 2, & \text{if } n = 6k+5 \end{cases} \tag{5.5}$$

The details are presented in appendix L.1.

5.1.2 | Example: $\mathbb{Z}_5$

For $\mathbb{Z}_5$, we saw that there are two nonequivalent ways of acting on $\mathbb{C}^3$, see (**??**). Let us first consider the action $(1,1,3)$

$$R(g) = \begin{bmatrix} \zeta_5 & 0 & 0 \\ 0 & \zeta_5 & 0 \\ 0 & 0 & \zeta_5^3 \end{bmatrix} \tag{5.6}$$

i.e. $R = \rho_1 \oplus \rho_1 \oplus \rho_3$.

For the gauge field, the reasoning is exactly the same than for $\mathbb{Z}_3$ and we get

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & \\ & \ddots & \\ & & A_{\mu;44} \end{bmatrix} \tag{5.7}$$

where each block $A_{\mu;ii}$ is of size $N_i \times N_i$. The projected gauge group is

$$G_{\text{proj}} = \mathrm{U}(N_0) \times \mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \mathrm{U}(N_3) \times \mathrm{U}(N_4). \tag{5.8}$$

The scalar fields transform as

$$X^m_{i\alpha_i, j\beta_j} \to R(g)^m{}_n \rho_i(g) X^n_{i\alpha_i, j\beta_j} \rho_j(g)^{-1} \tag{5.9}$$

so the invariant configurations must satisfy

$$X^m_{i\alpha_i,j\beta_j} = R(g)^m{}_n \zeta_4^{i-j} \rho_i(g) X^n_{i\alpha_i,j\beta_j} = \begin{cases} \zeta^{i-j+1} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 0,1,2,3 \\ \zeta^{i-j+3} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 4,5. \end{cases} \tag{5.10}$$

and are therefore of the form

$$X^m = \begin{bmatrix} 0 & X^m_{01} & 0 & 0 & 0 \\ 0 & 0 & X^m_{12} & 0 & 0 \\ 0 & 0 & 0 & X^m_{23} & 0 \\ 0 & 0 & 0 & 0 & X^m_{34} \\ X^m_{40} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.11}$$

for $m = 0,1,2,3$ and

$$X^m = \begin{bmatrix} 0 & 0 & 0 & X^m_{03} & 0 \\ 0 & 0 & 0 & 0 & X^m_{14} \\ X^m_{20} & 0 & 0 & 0 & 0 \\ 0 & X^m_{31} & 0 & 0 & 0 \\ 0 & 0 & X^m_{42} & 0 & 0 \end{bmatrix} \tag{5.12}$$

for $m = 4,5$. Gauge anomaly cancellation imposes

$$-N_{i-2} + N_{i+2} - 2N_{i+1} + 2N_{i-1} = 0 \tag{5.13}$$

for $i = 0,1,2,3,4$ which implies that

$$N_0 = N_1 = N_2 = N_3 = N_4. \tag{5.14}$$

Now if we choose the representation $(1,2,2)$, i.e.

$$R(g) = \begin{bmatrix} \zeta_5 & 0 & 0 \\ 0 & \zeta_5^2 & 0 \\ 0 & 0 & \zeta_5^2 \end{bmatrix} \tag{5.15}$$

we get instead that the invariant field configurations must satisfy

$$X^m_{i\alpha_i,j\beta_j} = R(g)^m{}_n \zeta_4^{i-j} \rho_i(g) X^n_{i\alpha_i,j\beta_j} = \begin{cases} \zeta^{i-j+1} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 0,1 \\ \zeta^{i-j+2} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 2,3,4,5. \end{cases} \tag{5.16}$$

and are therefore of the form

$$X^m = \begin{bmatrix} 0 & X^m_{01} & 0 & 0 & 0 \\ 0 & 0 & X^m_{12} & 0 & 0 \\ 0 & 0 & 0 & X^m_{23} & 0 \\ 0 & 0 & 0 & 0 & X^m_{34} \\ X^m_{40} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.17}$$

for $m = 0,1$ and

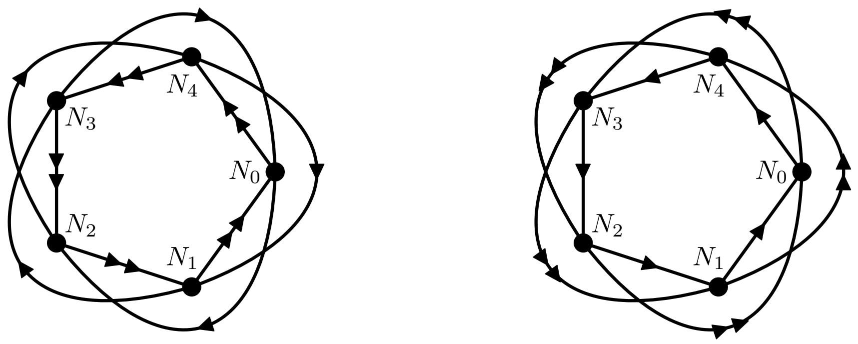
$$X^m = \begin{bmatrix} 0 & 0 & X^m_{02} & 0 & 0 \\ 0 & 0 & 0 & X^m_{13} & 0 \\ 0 & 0 & 0 & 0 & X^m_{24} \\ X^m_{30} & 0 & 0 & 0 & 0 \\ 0 & X^m_{41} & 0 & 0 & 0 \end{bmatrix} \tag{5.18}$$

for $m = 2,3,4,5$. Gauge anomaly cancellation imposes

$$-2N_{i+2} + 2N_{i-2} - N_{i+1} + N_{i-1} = 0(*) \tag{5.19}$$

for $i = 0,1,2,3,4$ which implies that

$$N_0 = N_1 = N_2 = N_3 = N_4. \tag{5.20}$$



(a) $(1,1,3)$ *(b)* $(1,2,2)$

Figure 3: Quivers of the $\mathbb{C}^3/\mathbb{Z}_5$ daughter theories.

Note that, even we thought that $(1,1,3)$ and $(1,2,2)$ were different representation, that are actually equivalent in the sense that $(1,1,3)+(1,1,3)=(2,2,1)$. The two quivers in fig. 3 should therefore be the same. And indeed, upon further inspection, the two are the same. To see this, we can rename the vertices in the second graph as $N_1 \to N_3, N_2 \to N_1, N_3 \to N_4, N_4 \to N_2$. This renaming defines the bijection between the two graphs. A more pragmatic way to see this is to look at the adjacency matrices:

$$a_{(1,1,3)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 2 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \qquad a_{(1,2,2)} = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}. \tag{5.21}$$

Changing the names of the vertices is equivalent to swapping line and columns. For example,

5.1.3 | General $\mathbb{Z}_n$

Let us now consider the general action

$$R(g) = \begin{bmatrix} \zeta_n^a & 0 & 0 \\ 0 & \zeta_n^b & 0 \\ 0 & 0 & \zeta_n^c \end{bmatrix} \tag{5.22}$$

of $\mathbb{Z}_n$ on $\mathbb{C}^3$, where (a,b,c) one of the representation that we studied before. In particular, recall that $a+b+c=n$. Following the same reasoning than before, we get that the gauge field of of the form

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & \\ & \ddots & \\ & & A_{\mu;n-1,n-1} \end{bmatrix}. \tag{5.23}$$

Invariant scalar field configurations transform as

$$X^m_{i\alpha_i,j\beta_j} = \begin{cases} \zeta_n^{i-j+a} X^m_{i\alpha_i,j\beta_j}, & \text{if } m=0,1 \\ \zeta_n^{i-j+b} X^m_{i\alpha_i,j\beta_j}, & \text{if } m=2,3 \\ \zeta_n^{i-j+c} X^m_{i\alpha_i,j\beta_j}, & \text{if } m=4,5 \end{cases} \tag{5.24}$$

so

$$X^{0,1}_{j-a,j} \in (\mathbf{N_j}, \overline{\mathbf{N}}_{\boldsymbol{j-a}}), \tag{5.25}$$

$$X^{2,3}_{j-b,j} \in (\mathbf{N_j}, \overline{\mathbf{N}}_{\boldsymbol{j-b}}), \tag{5.26}$$
$$X^{4,5}_{j-c,j} \in (\mathbf{N_j}, \overline{\mathbf{N}}_{\boldsymbol{j-c}}) \tag{5.27}$$

are the only possible non-vanishing components. This allows us to quickly draw all the possible quivers for a given n. Once again, the difficulty is only computational, not conceptual. This can therefore easily be implemented into Mathematica and we get the quiver for any n and any representation (a, b, c).

5.2 | $S = \mathbb{C}^3/\Delta(3n^2), \Delta(6n^2)$

5.3 | $S = \mathbb{C}^3/\Sigma_{36\times 3}, \Sigma_{60\times 3}, \Sigma_{168\times 3}, \Sigma_{216\times 3}, \Sigma_{360\times 3}$

5.4 | $S = \mathbb{C}^3/(\mathbb{Z}_m \times \mathbb{Z}_n)$

5.4.1 | Representations of $\mathbb{Z}_m \times \mathbb{Z}_n$

We consider the group $\mathbb{Z}_m \times \mathbb{Z}_n$. Let us denote by $\{\mu_i\}_{i=0,\dots,m-1}$ and $\{\sigma_j\}_{j=0,\dots,n-1}$ two complete set of irreducible representation of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively, with

$$\mu(g_1) = \zeta_m^i \tag{5.28}$$
$$\sigma_j(g_2) = \zeta_n^j \tag{5.29}$$

where g_1 and g_2 are the generating elements of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively. $\mathbb{Z}_m \times \mathbb{Z}_n$ is of order mn and possesses the same number of equivalency classes (abelian). It has therefore mn irreducible representations. Since the group is abelian, they are all of dimension 1. Let us denote by $\{T_k\}_{k=0,\dots,m+n-1}$ a complete set of irreducible representations. Since we have a product group, for any k there exists indices $i(k)$ and $j(k)$ such that $T_k = \mu_{i(k)} \otimes \sigma_{j(k)}$. We choose the indices $i(k)$ and $j(k)$ such that

$$\begin{aligned}
T_0 &= \mu_0 \otimes \mu_0, \\
T_1 &= \mu_0 \otimes \mu_1, \\
T_2 &= \mu_0 \otimes \mu_2, \\
&\vdots \\
T_{n-1} &= \mu_0 \otimes \mu_{n-1}, \\
T_n &= \mu_1 \otimes \mu_0, \\
&\vdots \\
T_{2n-1} &= \mu_1 \otimes \mu_{n-1}, \\
&\vdots \\
T_{mn-1} &= \mu_{m-1} \otimes \mu_{n-1}.
\end{aligned}$$

That is, we take

$$\begin{cases} i(k) &= \lfloor k/n \rfloor \\ j(k) &= k \mod n \end{cases} \qquad \Leftrightarrow \qquad k = i(k)n + j(k). \tag{5.30}$$

Note that this simply a dictionary between a line notation and a matrix notation and that it is indeed a bijection $k \Leftrightarrow i, j$. In this way, we can proceed to similar manipulations than before, were we used line notations.

Any representation R of $\mathbb{Z}_m \times \mathbb{Z}_n$ can be decomposed as

$$R = \bigoplus_{i,j} N_k T_k = \bigoplus_{i,j} N_{ij} (\mu_i \otimes \sigma_j) \tag{5.31}$$

with $N_k = N_{i(k)j(k)}$. We must have $\sum_k N_k = 3$. In other words, any representation of $\mathbb{Z}_m \times \mathbb{Z}_n$ on $\mathbb{C}^3$ is equivalent to

$$R(g_1, g_2) = [(\mu_a \otimes \sigma_{a'}) \oplus (\mu_{b'} \otimes \sigma_{b'}) \oplus (\mu_c \otimes \sigma_{c'})](g_1, g_2) = \begin{bmatrix} \xi_m^a \xi_n^{a'} & 0 & 0 \\ 0 & \xi_m^b \xi_n^{b'} & 0 \\ 0 & 0 & \xi_m^c \xi_n^{c'} \end{bmatrix}. \tag{5.32}$$

The determinant condition is

$$\xi_m^{a+b+c} = \xi_n^{-a'-b'-c'} \tag{5.33}$$
$$\Leftrightarrow (a+b+c) \mod m = (a'+b'+c') \mod n \tag{5.34}$$

$$R(g_1, g_2) = \begin{bmatrix} \xi_m & 0 & 0 \\ 0 & \xi_n & 0 \\ 0 & 0 & \xi_m^{-1}\xi_n^{-1} \end{bmatrix}. \tag{5.35}$$

5.4.2 | Projection

Let us start by the gauge field. We consider a unitary representation $(\rho, \mathbb{C}^N)$ of $\mathbb{Z}_m \times \mathbb{Z}_n$ on $\mathbb{C}^n$ and decompose it as

$$\rho = \bigoplus_{i,j} N_k T_k \tag{5.36}$$

such that

$$\rho(g) = \begin{bmatrix} T_0(g) & & \cdots & & 0 \\ & \ddots & & & \\ & & T_0(g) & & \\ \vdots & & \ddots & & \vdots \\ & & & T_{mn-1}(g) & \\ & & & \ddots & \\ 0 & & \cdots & & T_{mn-1}(g) \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} N_0 \\ \vdots \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} N_{mn-1} \end{matrix} . \tag{5.37}$$

We can now use our usual bi-index notation $A_{\mu;k\alpha_k,l\beta_l}$ with $k,l = 0,\dots mn-1$ and $\alpha_k, \beta_k = 0, \dots, N_k$ but instead it is more convenient to come back to our matrix notation by writing the block $A_{\mu;k,l}$ as $A_{\mu;i(k)j(k),i(l)j(l)}$ that we simply denote by $A_{\mu;ij,i'j'}$ with $i, i' \in \{0, \dots, n-1\}$ and $j, j' \in \{0, \dots, n-1\}$. So for $m = 2, n = 3$ for example, the link between the two notations is

$$\begin{bmatrix} A_{\mu;00} & A_{\mu;01} & A_{\mu;02} & A_{\mu;03} & A_{\mu;04} & A_{\mu;05} \\ A_{\mu;10} & A_{\mu;11} & A_{\mu;12} & A_{\mu;13} & A_{\mu;14} & A_{\mu;05} \\ A_{\mu;20} & A_{\mu;21} & A_{\mu;22} & A_{\mu;23} & A_{\mu;24} & A_{\mu;05} \\ A_{\mu;30} & A_{\mu;31} & A_{\mu;32} & A_{\mu;33} & A_{\mu;34} & A_{\mu;05} \\ A_{\mu;40} & A_{\mu;41} & A_{\mu;42} & A_{\mu;43} & A_{\mu;44} & A_{\mu;05} \\ A_{\mu;50} & A_{\mu;51} & A_{\mu;52} & A_{\mu;53} & A_{\mu;54} & A_{\mu;05} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{\mu;00,00} & A_{\mu;00,01} & A_{\mu;00,02} \\ A_{\mu;01,00} & A_{\mu;01,01} & A_{\mu;01,02} \\ A_{\mu;02,00} & A_{\mu;02,01} & A_{\mu;02,02} \end{bmatrix} & \begin{bmatrix} A_{\mu;00,10} & A_{\mu;00,11} & A_{\mu;00,12} \\ A_{\mu;01,10} & A_{\mu;01,11} & A_{\mu;01,12} \\ A_{\mu;02,10} & A_{\mu;02,11} & A_{\mu;02,12} \end{bmatrix} \\ \begin{bmatrix} A_{\mu;10,00} & A_{\mu;10,01} & A_{\mu;10,02} \\ A_{\mu;11,00} & A_{\mu;11,01} & A_{\mu;11,02} \\ A_{\mu;12,00} & A_{\mu;12,01} & A_{\mu;12,02} \end{bmatrix} & \begin{bmatrix} A_{\mu;10,10} & A_{\mu;10,11} & A_{\mu;10,12} \\ A_{\mu;11,10} & A_{\mu;11,11} & A_{\mu;11,12} \\ A_{\mu;12,10} & A_{\mu;12,11} & A_{\mu;12,12} \end{bmatrix} \end{bmatrix} \tag{5.38}$$

So instead of considering A_μ to be a single $mn \times mn$ matrix of element $A_{\mu;kl}$, where $A_{\mu;kl}$ are $N_k \times N_l$ matrices, we consider it as an $m \times m$ where each element $A_{\mu,ii'}$ (line i column i') is itself an $n \times n$ matrices with elements $A_{\mu;ij,i'j'}$ (line j column j'), as shown above.

Using these notations, the gauge field transforms as

$$A_{\mu;ij,i'j'} \mapsto (\mu_i(g) \otimes \sigma_j(g)) A_{\mu;ij,i'j'} (\mu_{i'}(g) \otimes \sigma_{j'}(g))^{-1} = \zeta_m^{i-i'} \zeta_n^{j'-j} A_{\mu;ij,i'j'} \tag{5.39}$$

so invariant configurations can possess non-vanishing components $A_{\mu;ij,i'j'}$ only if

$$\zeta_m^{i-i'} = \zeta(j'-j)_n \tag{5.40}$$
$$\Leftrightarrow \quad (i-i') \mod m = (j'-j) \mod n \tag{5.41}$$
$$\Leftrightarrow \quad j' = j + |i'-i|. \tag{5.42}$$

This means that the submatrices $A_{\mu;ii'}$ has an off-diagonal block form with offset $|i'-i|$. Once again, for a general n, the difficulty is only computational, not conceptual. This can therefore easily be implemented into Mathematica and we get the form of the gauge field for any n.

For the scalar fields, we have

$$X^m_{ij,i'j'} \mapsto R(g)^m{}_n (\mu_i(g) \otimes \sigma_j(g)) X^n_{ij,i'j'} (\mu_{i'}(g) \otimes \sigma_{j'}(g))^{-1} \tag{5.43}$$

$$= \begin{cases} \zeta_m^{i-i'+1}\zeta_n^{j-j'}X^m_{ij,i'j'}, & m=0,1 \\ \zeta_m^{i-i'}\zeta_n^{j-j'+1}X^m_{ij,i'j'}, & m=2,3 \\ \zeta_m^{i-i'-1}\zeta_n^{j-j'-1}X^m_{ij,i'j'}, & m=4,5. \end{cases} \tag{5.44}$$

So a configuration is invariant if and only if the only non-vanishing component satisfy

$$m=0,1: (i-i'+1) \mod m = (j'-j) \mod n \tag{5.45}$$

$$m=2,3: (i-i') \mod m = (j'-j-1) \mod n \tag{5.46}$$

$$m=4,5: (i-i'-1) \mod m = (j'-j+1) \mod n. \tag{5.47}$$

6 | Beyond orbifold singularities

6.1 | Projective representations and discrete torsion

6.2 | Quiver gauge theories deformations and conifold

We can replace the orbifold by a conifold by deforming singular algebraic description of the orbifold with a field into a family smooth surfaces. The total space is then the conifold.

7 | Correspondence between gauge theory and singularity

Above, we presented all the possible orbifold constructions of supersymmetric quiver gauge theories in four dimensions. We started from quotienting the transverse space and we found the corresponding (supersymmetric) gauge theory. In other words, we started from the singularity a found the gauge theory. We can therefore consider that the orbifold singularities are understood. However, not all singularities are orbifold one, such as the conifold for example. We an then ask ourselves how to obtain the gauge theory for more general singularities than the orbifold ones. Is a general approach possible ? On the other hand, we can also study the converse question; is it possible to to obtain the singularity from the gauge theory? And if it is, how so? In general we will see that there is a bijection between the four-dimensional supersymmetric worldvolume gauge theory and the Calabi-Yau singularity. We now detail this bijection.

7.1 | From gauge theory to singularity: forward algorithm

We start with the simplest question: how to recover the singularity from the gauge theory? We already mentioned that the vacuum parameter space of the scalar fields of the gauge theory is the so-called moduli space, denoted $\mathcal{M}$. Because our D3-brane is a point in the Calabi-Yau threefold, the vacuum moduli space $\mathcal{M}$ is the affine coordinates of the Calabi-Yau singularity S.

7.1.1 | $S = \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$

7.1.2 | The conifold

7.2 | From singularity to gauge theory: inverse algorithm

Mathematically, a quiver gauge theory is a representation of a finite quiver with relations. The labels are $\{N_i \in \mathbb{Z}_+\}$, they correspond to the dimension of the vector space $\{V_i\}$. The gauge group is $\prod_i \mathrm{SU}(N_i)$. The gauge fields are self-adjoint arrows $\mathrm{Hom}(V_i, V_i)$ while the matter fields are bi-fundamentals fermions/bosons and are arrows $X_{ij} \in \mathrm{Hom}(V_j, V_i)$. For a quiver with adjacency matrix a_{ij}, the gauge anomaly cancellation condition can be generally expressed as

$$(a_{ij} - a_{ji})N_i = 0. \tag{7.1}$$

At last, there are some relations that arises the superpotential $W(\{X_{ij}\})$. The vacuum is the minima of the superpotential. In other words,

$$\frac{\partial W}{\partial X_{ij}} = 0. \tag{7.2}$$

8 | Determinantal varieties as transverse spaces

8.1 | Basic properties of determinantal varieties

A *determinantal variety* (DV) is a space of matrices with a given upper bound on their ranks. More precisely, given m, n and $r < \min(m, n)$, the DV Y_r of the field K is the set of $m \times n$ matrices over K with rank lower or equal to r:

$$Y_r \equiv \{M \in M_{m\times n}(K) | \operatorname{rank} M \leq r\}. \tag{8.1}$$

Since the rank of a matrix is equal to the dimension of the biggest invertible submatrix, imposing $\operatorname{rank} M \leq r$ is equivalent to the vanishing of its $(r+1) \times (r+1)$ minors, as it also implies tha vanishing of the biggest minors. This naturally qualifies R_r as affine varieties embedded in K^{mn}. Let us denote by $x_{i,j}$ the independent entries of a generic matrix of Y_r, i.e. $x_{i,j}$ are affine coordinates. The minors are therefore homogeneous polynomials of degree $r+1$. The ideal of $K[x_{i,j}]$ generated by these polynomials $(r+1)\times(r+1)$ is called the *determinantal ideal*. Homogeneity the polynomials implies that Y_r can equivalently be seen as a projective variety in $\mathbb{A}^{mn-1}$.

8.1.1 | Computing the dimension

Let us compute the dimension of Y_r seen as an affine variety. We consider the space $\mathbb{A}^{mn} \times \mathbf{Gr}(r, m)$, where $\mathbf{Gr}(r, m)$ is the Grassmannian of r-planes in an m-dimensioanl vector space. Let us define the subsapce

$$Z_r \equiv \{(A, W) | Ax \in W \text{ for all } x \in \mathbb{A}^n\}. \tag{8.2}$$

Y_r and Z_r are birationaly equivalent so $\dim Y_r = \dim Z_r$. We want to compute Z_r. First we notice that Z_r is a vector bundle over $\mathbf{Gr}(r, m)$ and we denote it by $Z_r \xrightarrow{\pi_1} \mathbf{Gr}(r, m)$. Now, over the Grassmannian $\mathbf{Gr}(r, m)$, there is a tautologial vector bundle that we denote by $E_{\mathbf{Gr}} \xrightarrow{\pi_2} \mathbf{Gr}(r, m)$ whose fibers are $\pi_2^{-1}(W) = W \cong \mathbb{R}^r$. Finally, K^m can also be seen as a vector bundle, with fibers $\mathbb{R}^m$. We denote it by $E_{K^n} \xrightarrow{\pi_3} K^n$. From $E_{\mathbf{Gr}}$ and E_{K^n}, we can construct[2] the vector bundle $\mathrm{Hom}(E_{\mathbf{Gr}}, E_{K^n}) \xrightarrow{\pi_4} \mathbf{Gr}(r, m)$. This vector bundle has the same base space and its fibers are $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^r)$ which are exactly the same as the ones of Z_r. So the two vector bundles are isomorphic:

$$Z_r \cong \mathrm{Hom}(E_{\mathbf{Gr}}, E_{K^n}). \tag{8.3}$$

Finally, since the fibers of $\mathrm{Hom}(K^n, E_{\mathbf{Gr}})$ have dimension nr, we find

$$\dim Z_r = \dim \mathrm{Hom}(K^n, E_{\mathbf{Gr}}) = \dim \mathbf{Gr}(r, m) + nr = r(m-r) + nr. \tag{8.4}$$

Finally, we conclude that Y_r is a affine variety of dimension $r(m-r) + nr$.

8.1.2 | Singularity

The varitey Y_r is singlar and the singular locus is contained in the subset of matrices with rank strictly lower than r. Z_r is a resolution (over the open set of matrices with rank exactly r, this map is an isomorphism)(*).

8.1.3 | Action and syzygies

Y_r naturally acts on $G = \mathrm{GL}(m, K) \times \mathrm{GL}(n, K)$

[2] Recall that if E and F are vector bundles over X, then we can construct a new vetor bundle over X, called the Hom-bundle and denoted $\mathrm{Hom}(E, F)$, by defining the fiber over $x \in X$ to be $\mathrm{Hom}(E_x, F_x)$.

A | Supersymmetric Yang-Mills theory from D-branes

The dynamics of D-branes is described by the Dirac-Born-Infeld action

$$S_{\text{DBI}}[X,F] = -\frac{T_p}{g_s}\int \mathrm{d}^{p+1}\sigma\,\sqrt{-\det_{0\leq a,b\leq p}(\eta_{ab}+\partial_a X^m \partial_b X_m + 2\pi\alpha' F_{ab})}. \tag{A.1}$$

The latter can be expended for slowly-varying fields, which is equivalent to passing to the field theory limit $\alpha' \to 0$. The resulting action is the action of a $\mathrm{U}(1)$ gauge theory in $p+1$ dimensions with $9-p$ real scalar fields. This action is exactly the same than the one we would obtain by dimensionally-reducing a pure $\mathrm{U}(1)$ Yang-Mills gauge theory in 10 spacetime dimensions with the identification

$$g_{\text{YM}} = g_s T_p^{-1}(2\pi\alpha')^{-2} = \frac{g_s}{\sqrt{\alpha'}}(2\pi\sqrt{\alpha'})^{p-1}. \tag{A.2}$$

This construction can be generalized for multiple D-branes. It now results in a non-abelian theory. The general statement is the following:

> The low-energy dynamics of N parallel, coicident Dp-branes in flat space is described in static gauge by the dimensional reduction to $p+1$ dimensions of pure $10d$ $\mathcal{N}=1$ supersymmetric Yang-Mills theory with gauge group $\mathrm{U}(N)$ in ten spacetime dimensions.

Recall that the 10-dimensional action is given by

$$S_{\text{YM}} = \frac{1}{4g_{\text{YM}}^2}\int \mathrm{d}^{10}x\,\left[\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + 2i\,\mathrm{tr}\big(\overline{\psi}\Gamma^\mu D_\mu\psi\big)\right], \tag{A.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the non-abelian field strength of the $\mathrm{U}(N)$ gauge field A_μ, $D_\mu = \partial_\mu - i[A_\mu,\psi]$, Γ^μ are 16×16 Dirac matrices (*), and the $N\times N$ Hermitian fermion field ψ is a 16-component Majorana-Weyl spinor of the Lorentz group $\mathrm{SO}(1,9)$ which transforms under the adjoint representation of the gauge group $\mathrm{U}(N)$. On-shell, there are eight on-shell bosonic, gauge field degrees of freedom, and eight fermionic degrees of freedom, after imposition of the Dirac equation $\not{D}\psi = \Gamma^\mu D_\mu \psi = 0$. One can verify that this action is invariant under the supersymmetry transformations

$$\begin{aligned}\delta_\epsilon A_\mu &= \frac{i}{2}\bar{\epsilon}\Gamma_\mu\psi,\\ \delta_\epsilon\psi &= \frac{1}{2}F_{\mu\nu}[\Gamma^\mu,\Gamma^\nu]\epsilon,\end{aligned}$$

where ϵ is an Majorana-Weyl spinor.

Using (A.3), we can construct a supersymmetric Yanf-Mills gauge theory in $p+1$ dimensions with 16 independent supercharges by dimensional reduction: we take all fields to be independent of the coordinates $X^{p+1},\dots,X^9$, then the ten-dimensional gauge field A_μ splits into a $(p+1)$-dimensional $\mathrm{U}(N)$ gauge field A_a plus $9-p$ Hermitian scalar fields $\Phi^m = X^m/2\pi\alpha'$ in the adjoint representation of $\mathrm{U}(N)$. The Dp-brane action is thereby obtained from the dimensionality reduced field theory as

$$S_{\text{D}p} = -\frac{T_p g_s(2\pi\alpha')^2}{4}\int \mathrm{d}^{p+1}\sigma\ \mathrm{tr}\left(F_{ab}F^{ab}) + 2D_a\Phi^m D^a\Phi_m + \sum_{m\neq n}[\Phi^m,\Phi^n]^2 + \text{fermions}\right) \tag{A.4}$$

where $a,b = 0,\dots,p$, $m,n = p+1,\dots,9$. We do not explicitly display the fermionic contributions for the moment. In conclusion, the low-energy brane dynamics is described by a supersymmetric Yang-Mills theory on the Dp-brane worldvolume which is dynamically coupled to the transverse, adjoint scalar fields Φ^m.

The scalar potential is given by

$$V(\Phi) = \sum_{m\neq n}[\Phi^m,\Phi^n]^2. \tag{A.5}$$

It is negative definite because $[\Phi^m, \Phi^n]^\dagger = [\Phi^n, \Phi^m] = -[\Phi^m, \Phi^n]$. A classical vacuum of the field theory defined by (A.4) corresponds to a static solution of the equations of motion whereby the potential energy of the system is minimized. It is given by the field configurations which solve simultaneously the quations $F_{ab} = D_a\Phi^m = \psi^a = 0$ and $V(\Phi) = 0$. Since all term in $V(\Phi)$ have the same sign, the equation $V(\Phi) = 0$ is equivalent to the equation $[\Phi^m, \Phi^n] = 0$ for all m, n and at each point in the $(p+1)$-dimensional worldvolume of the branes. This implies that the $N \times N$ hermitian matrix fields Φ^m are simultaneously diagonalizable by a gauge transformation, so that we may write

$$\Phi^m = U \begin{bmatrix} X_1^m & & & 0 \\ & X_2^m & & \\ & & \ddots & \\ 0 & & & X_N^m \end{bmatrix} U^{-1}, \tag{A.6}$$

the matrix U is independent of m. The simultaneous, real eigenvalues X_i^m give the positions of the N distinct D-branes in the m-th transverse direction. It follows that the moduli space of classical vacua for the $(p+1)$-dimensional field theory (A.4) is the quotient space $(\mathbb{R}^{9-p})^N/S_N$, where the factors of $\mathbb{R}$ correspond to the positions of the N Dp-branes in the $(9-p)$-dimensional transverse space, and S_N is the symmetric group acting by permutations of the N coordinates X_i. The group S_N corresponds to the residual Weyl symmetry of the U(N) gauge group acting in (A.6). It represents the permutation symmetry of a system of N *indistinguishable* D-branes.

From (A.4) one can easily deduce that the masses of the fields corresponding to the off-diagonal matrix elements are given precisely by the distances $|x_i - x_j|$ between the corresponding branes. This description means that an interpretation of the D-brane configuration in terms of classical geometry is only possible in the classical ground state of the system, whereby the matrices Φ^m are simultaneously diagonalizable and the positions of the individual D-branes may be described through their spectrum of eigenvalues. This gives a simple and natural dynamical mechanism for the appearence of "non-commutative geometry" at short distances, where the D-branes cease to have well-defined positions according to classical geometry.

The end of this section has to be rewritten (*).

B | Reminder on $\mathcal{N} = 4$ super Yang-Mills theory in $D = 4$

B.1 | Superconformal group SU$(2, 2|4)$ and its representations

Conformal transformations and supersymmetries do not commute so the presence of conformal symmetry in addition to $\mathcal{N} = 4$ supersymmetry leads to an even larger group of symmetry known as the *superconformal group*. In the $D = 4, \mathcal{N} = 4$ case, the superconformal group is the super group[3] SU$(2, 2|4)$. The different component of the latter are

- **Conformal symmetries**: they form the 15-dimensional subgroup SO$(2, 4)$ and are generated by $P_\mu, M_{\mu\nu}, K_\mu$ and D.
- **R-symmetry**: they form the 15-dimensional subgroup SO$(6)_R$ and are generated by T^A ($A = 1, \ldots, 15$).
- **Poincaré supersymmetries**: they form the 16-dimensional sub group (*)and are generated by Q_α^I and $\overline{Q}_{\dot\alpha}^I$.
- **Conformal supersymmetries**: they form the 16-dimensional subgroup (*)and are generated by $S_{\alpha I}$ and $\overline{S}^{\dot\alpha I}$.

Conformal invariance of this theory can be seen as a consequence of the non-renormalization theorems.

[3] Supermanifold which is also a group with smooth product and inverse maps.

B.2 | Matter content

For $D = 4, \mathcal{N} = 4$, there is one kind of supermultiplet, the vector multiplet. The decomposition of the $\mathcal{N} = 4$ vector superfield in terms of $\mathcal{N} = 1$ representations is as follows:

$$[\mathcal{N} = 4 \text{ vector multiplet}] : V = (\lambda_\alpha, A_\mu, D) \oplus \Phi^A = (\phi^A, \psi^A_\alpha, F^A). \tag{B.1}$$

with $A = 1, 2, 3$, i.e. in terms of one vector supermultiplet and three chiral scalar supermultiplets. The propagating degrees of freedom are therefore a vector field, three complex scalars and four gauginos.

The Lagrangian is very much constrained by $\mathcal{N} = 4$ supersymmetry. First, the chiral superfields Φ^A should transform in the adjoint representation of the gauge group G, since internal symmetries commute with supersymmetry. This means that all fields transform in the adjoint of G.

Moreover, there is a large R-symmetry group, SU$(4)_R$. The four Weyl fermions transform in the fundamental of SU$(4)_R$, while the six real scalars in the two times anti-symmetric representation, which is nothing but the fundamental representation of SO(6). The auxiliary fields are singlets under the R-symmetry group. Using $\mathcal{N} = 1$ superfield formalism the Lagrangian reads

$$\begin{aligned} \mathcal{L}^{\mathcal{N}=4}_{\text{SYM}} = \frac{1}{32\pi} \operatorname{Im}\left(\tau \int \mathrm{d}^4x \operatorname{tr}(W^\alpha W_\alpha)\right) + \int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta} \operatorname{tr} \sum_{A=1}^{3} \overline{\Phi}^A e^{2gV} \Phi^A \\ - \int \mathrm{d}^2\theta \sqrt{2g} \operatorname{tr} \Phi_1 [\Phi_2, \Phi_3] + \text{h.c.} \end{aligned} \tag{B.2}$$

where the commutator in the third term appears for the same reason as for the $\mathcal{N} = 2$ Lagrangian. Notice that the choice of a single $\mathcal{N} = 1$ supersymmetry generator breaks the full SU$(4)_R$ R-symmetry to SU$(3) \times$ U$(1)_R$. The three chiral superfields transform in the **3** of SU(3) and have R-charge $R = 2/3$ under the U$(1)_R$. It is an easy but tedious exercise to perform the integration in superspace and get an explicit expression in terms of fields. Finally, one can solve for the auxiliary fields and get an expression where only propagating degrees of freedom are present, and where SU$(4)_R$ invariance is manifest (the fact that the scalar fields transform under the fundamental representation of SO(6), which is real, makes the R-symmetry group of the $\mathcal{N} = 4$ theory being at most SU(4) and not U(4), in fact).

B.3 | Dynamical phases

The scalar potential in (B.2) can be written in a rather compact form in terms of the six real scalars X^i making up the three complex scalars ϕ^A and reads

$$V = \frac{1}{2} g^2 \operatorname{tr} \sum_{i,j=1}^{6} [X_i, X_j]^2. \tag{B.3}$$

The positive definite behavior of the Cartan-Killing form on the compact gauge algebra $\mathfrak{g}$ implies that each term in the sum is positive or zero. In other words, $V = 0$ is equivalent to

$$[X^i, X^j] = 0, \qquad i, j = 1, \ldots, 6. \tag{B.4}$$

This equations admit two classes of solutions:

- $\langle X^i \rangle = 0$ for all $i = 1, \ldots, 6$. This is the *superconformal phase*. Neither the gauge symmetry nor the superconformal symmetry is broken. The physical states and operators are gauge invariant and transform under unitary representations of SU$(2, 2|4)$.
- $\langle X^i \rangle \neq 0$ for at least one i. This is the *spontaneously broken Coulomb phase*. The gauge algebra $\mathfrak{g}$ is going to be broken to U$(1)^r$, where $r \equiv \operatorname{rank} \mathfrak{g}$. The low energy behavior is then the one of r copies of $\mathcal{N} = 4$ U(1) gauge theories. Superconformal symmetry is spontaneously broken since the non-zero VEV $\langle X^i \rangle$ sets a scale.

$\mathcal{N}=4$ Yang-Mills theory. There is only one $D=4,\mathcal{N}=4$ Yang-Mills theory and it contains 3 $\mathcal{N}=1$ chiral scalar supermutliplet and 1 $\mathcal{N}=1$ vector supermultiplet (up to g and τ). This theory is conformal and can be recovered from dimensional reduction of $D=10,\mathcal{N}=1$ Yang-Mills on $\mathbb{T}^6$.

C | Gauge anomaly

The *anomaly degree* $A(\rho)$ of a representation ρ is defined as

$$\frac{1}{2}\operatorname{tr}(T_a\{T_b,T_c\}) = A(\rho)d_{abc} \tag{C.1}$$

where d_{abc} is an invariant symmetric tensor of the Lie algebra of G, independent of the representation. One can show that $A(\rho^*)=-A(\rho)$ so self dual representation have $A(\rho)=0$ in particular. The only simple Lie groups that allow for a complex non-self-conjugate representation are SU(n) with $n\geq 3$. We can normalize d_{abc} such that $A(\rho)=1$ for the fundamental n-dimensional representations.

D | Properties of D-branes in type II theories

The minimal irreducible representation in 10 dimensions is a Majorana-Weyl representation of dimension 8. In type II theories, we have $\mathcal{N}=(1,1)$ for IIA and $\mathcal{N}=(2,0)$ for IIB. Because of the string origin of the generators, the two supersymmetry generators ϵ_L and ϵ_R (Majorana-Weyl spinors) satisfy

$$\epsilon_L = \Gamma_{11}\epsilon_L, \qquad \epsilon_R = \eta\Gamma_{11}\epsilon_R \tag{D.1}$$

with $\eta=+1$ for IIB and $\eta=-1$ for IIA theory. For a Dp-brane, the supersymmetry projections is the following:

$$\epsilon_L = \Gamma_0\dots\Gamma_p\epsilon_R. \tag{D.2}$$

In other words, the supersymmetries with generators of the form

$$Q_\alpha + \Gamma_0\dots\Gamma_p\overline{Q}_{\dot\alpha} \tag{D.3}$$

are preserved by the Dp-brane while the one with generators of the form

$$Q_\alpha - \Gamma_0\dots\Gamma_p\overline{Q}_{\dot\alpha} \tag{D.4}$$

are broken. They violate the boundary conditions. Since there is the same number of generators of the form (D.3) than of the form (D.4), exactly haf of the supersymmetry is broken. The idea that one spacetime direction would break one supercharge could be reasonable if supersymmetries were transforming as vectors which not the case; supercharges transform as spinors. It would also be incompatible with the T-duality because two branes of different dimensions must have the same number of unbroken supercharges if there is a T-duality relating them: the number of unbroken supercharges is the same for all dual descriptions (a necessary condition for the equivalence). And indeed, in the correct theory, that's the case. Every type II D-brane breaks half of the supercharges.

To obtain the previous relations, we start by the ones from M-theory and compactify the 11th direction, getting type IIA theory. Γ_{11} then plays the role of the chiral projector in 10 dimensions; the supersymmetry parameters are related by $\epsilon_L=\frac{1}{2}(1+\Gamma_{11})\epsilon$ and $\epsilon_R=\frac{1}{2}(1-\Gamma_{11})\epsilon$. The relations for type IIB theory are then obtained by T-duality. Under a T-duality over the $\widehat{i}$ direction, the supersymmetry parameters transform as

$$\begin{aligned}\epsilon_L &\mapsto \epsilon_L,\\ \epsilon_R &\mapsto \Gamma_i\epsilon_R.\end{aligned}$$

The tension of a Dp-brane is given by

$$T_p = \frac{1}{(2\pi)^p g_s l_s^{p+1}}. \tag{D.5}$$

This completely fixes the Newton constant: the tension of electric-magnetic duals must satisfy:

$$T_p T_{D-p-4} = \frac{2\pi}{16\pi G_D}. \tag{D.6}$$

In ten dimensions, this gives $G_{10} = 8\pi^6 g_s^2 l_s^8$.

The dualities are defined as follows:

$$\begin{aligned} \text{S-duality} : g_s \mapsto \frac{1}{g_s}, \qquad l_s^2 \mapsto g_s l_s^2, \\ \text{T-duality} : R \mapsto \frac{l_s^2}{R}, \qquad g \mapsto g_s \frac{l_s}{R}. \end{aligned}$$

E | Some finite subgroups

E.1 | Finite subgroups of SU(2) and SL$(2,\mathbb{C})$

E.1.1 | Finite subgroups

The first thing to recall is that every finite subgroup of SL$(2,\mathbb{C})$ is isomorphic to a subgroup of SU$(2,\mathbb{C})$ and vice-versa, so we equivalently talk about the subgroups of SU(2). The finite subgroups of SU(2), called the *binary polyhedral groups*, are the doubles covers of the finite subgroups of SO(3) that are called *polyhedral groups*. They simply constitutes the symmetries of the Platonic solids. The groups fall into two infinite series, associated to the regular polygons, as well as three exceptional, associated with the 5 regular polyhedra: the tetrahedron (self-dual), the cube (and its dual octahedron), the icosahedron (and its dual dodecahedron).

More precisely, the finite subgroups of SL$(2,\mathbb{C})$ are

- $\mathbb{Z}_n$: cyclic group of order n ($n \geq 2$) generated by

$$\begin{bmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{bmatrix} \tag{E.1}$$

- $2\mathcal{D}_n$: *binary dihedral groups* (also known as the *dicyclic group*) of order $4n$ ($n \geq 1$) generated by

$$A \equiv \begin{bmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{bmatrix} \quad \text{and } B \equiv \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{E.2}$$

  One can show that $A^n = B^2$ and that $AB = BA^{-1}$ so that $2\mathcal{D}_n = \{B^b A^a | 0 \leq b \leq 3, 0 \leq a \leq n-1\}$. This rewriting of the most general element of the group will be useful.

- $2\mathcal{T}$: *binary tetrahedral group* of order 24 generated by D_2 and

$$C \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{bmatrix} \tag{E.3}$$

- $2\mathcal{O}$: *binary octahedral group* of order 48 generated by $\mathcal{T}$ and

$$D \equiv \begin{bmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{bmatrix} \tag{E.4}$$

- $2\mathcal{I}$: *binary icosahedral group* of order 120 generated by

$$E \equiv -\frac{1}{\sqrt{5}} \begin{bmatrix} \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{bmatrix} \quad \text{and } F \equiv -\frac{1}{\sqrt{5}} \begin{bmatrix} \zeta_5^2 - \zeta_4^4 & \zeta_5^4 - 1 \\ 1 - \zeta_5 & \zeta_5^3 - \zeta_5 \end{bmatrix} \tag{E.5}$$

with $\zeta_m \equiv e^{i\frac{2\pi}{m}}$ such that $(\zeta_m)^m = 1$. Note that the orders are all divisible by 2. This is because the center of SU(2) is $\mathbb{Z}_2$.

E.1.2 | Irreducible representations

- $\mathbb{Z}_n$ has n irreducible representations. They are all 1-dimensional (since $\mathbb{Z}_n$ is abelian) and are given by
$$\rho_k(g) = \zeta_n^k \tag{E.6}$$
with $k = 0, \dots, n-1$.
- $2\mathcal{D}_n$ has $n+3$ irreducible representations: 4 of dimension 1 and $n-1$ of dimension 2. The 1-dimensional ones are given by

n	$\rho(A)$	$\rho(B)$	$\rho(B^b A^a)$
even	1	1	1
		-1	$(-1)^b$
	-1	1	$(-1)^a$
		-1	$(-1)^{a+b}$
odd	1	1	1
		-1	$(-1)^b$
	-1	i	$(-1)^a i^b$
		$-i$	$(-1)^a(-i)^b$

and the 2-dimensional ones are given binary by
$$\rho_r(A) = \begin{bmatrix} e^{i\frac{\pi}{n}r} & 0 \\ 0 & e^{-i\frac{\pi}{n}r} \end{bmatrix}$$
$$\rho_r(B) = \begin{bmatrix} 0 & (-1)^r \\ 1 & 0 \end{bmatrix}$$
with $r = 1, \dots, n-1$.

E.1.3 | Character tables

conj. class repr.	e	M	M^2	$\dots$	M^{n-1}
conj. class order	1	1	1	$\dots$	1
V_0	1	1	1	$\dots$	1
V_1	1	ζ_n	ζ_n^2	$\dots$	ζ_n^{n-1}
V_2	1	ζ_n^2	ζ_n^4	$\dots$	$\zeta_n^{2(n-1)}$
V_3	1	ζ_n^3	ζ_n^6	$\dots$	$\zeta_n^{3(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
V_{n-1}	1	$\zeta_n^{(n-1)}$	$\zeta_n^{2(n-1)}$	$\dots$	$\zeta_n^{(n-1)^2}$
W	2	$2\cos\left(\frac{2\pi}{n}\right)$	$2\cos\left(\frac{4\pi}{n}\right)$	$\dots$	$2\cos\left(\frac{2\pi(n-1)}{n}\right)$

Table 1: Character table of $\mathbb{Z}_n$.

conj. class repr.	e	B^2	B	BA	A	A^2	$\dots$	A^{n-1}
conj. class order	1	1	n	n	2	2	$\dots$	2
V_0	1	1	1	1	1	1	$\dots$	1
V_1	1	1	-1	-1	1	1	$\dots$	1
V_2	1	1 ou -1	1 ou i	-1 ou $-i$	-1	1	$\dots$	$(-1)^{n-1}$
V_3	1	1 ou -1	-1 ou $-i$	1 ou i	-1	1	$\dots$	$(-1)^{n-1}$
V_4	2	-2	0	0	$2\cos\frac{\pi}{n}$	$2\cos\frac{2\pi}{n}$	$\dots$	$2\cos\frac{(n-1)\pi}{n}$
V_5	2	2	0	0	$2\cos\frac{2\pi}{n}$	$2\cos\frac{4\pi}{n}$	$\dots$	$2\cos\frac{2(n-1)\pi}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
V_{n+2}	2	$2(-1)^{n-1}$	0	0	$2\cos\frac{(n-1)\pi}{n}$	$2\cos\frac{2(n-1)\pi}{n}$	$\dots$	$2\cos\frac{(n-1)^2\pi}{n}$
W	2	-2	0	0	$2\cos\left(\frac{\pi}{n}\right)$	$2\cos\left(2\frac{\pi}{n}\right)$	$\dots$	$2\cos\left(\frac{\pi}{n}(n-1)\right)$

Table 2: Character table of $2\mathcal{D}_n$.

conj. class repr.	e	B^2	B	C	C^2	C^4	C^5
conj. class order	1	1	6	4	4	4	4
V_0	1	1	1	1	1	1	1
V_1	2	-2	0	1	-1	-1	1
V_2	3	3	-1	0	0	0	0
V_3	2	-2	0	$e^{i\frac{2\pi}{3}}$	$-e^{i\frac{2\pi}{3}}$	$-e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$
$V_3^\vee$	2	-2	0	$e^{i\frac{4\pi}{3}}$	$-e^{i\frac{4\pi}{3}}$	$-e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$
V_4	1	1	1	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$
$V_4^\vee$	1	1	1	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$
W	2	-2	0	1	-1	-1	1

Table 3: Character table of $2\mathcal{T}$.

conj. class repr.	e	B^2	B	C	C^2	D	BD	D^3
conj. class order	1	1	6	8	8	6	12	6
V_0	1	1	1	1	1	1	1	1
V_1	2	-2	0	1	-1	$-\sqrt{2}$	0	$\sqrt{2}$
V_2	3	3	-1	0	0	1	-1	1
V_3	4	-4	0	-1	1	0	0	0
V_4	3	3	-1	0	0	-1	1	-1
V_5	2	-2	0	1	-1	$\sqrt{2}$	0	$-\sqrt{2}$
V_6	1	1	1	1	1	-1	-1	-1
V_7	2	2	2	-1	-1	0	0	0
W	2	-2	0	1	-1	$-\sqrt{2}$	0	$\sqrt{2}$

Table 4: Character table of $2\mathcal{O}$.

conj. class repr.	e	E^2	E	F	F^2	EF	$(EF)^2$	$(EF)^3$	$(EF)^4$
conj. class order	1	1	30	20	20	12	12	12	12
V_0	1	1	1	1	1	1	1	1	1
V_1	2	-2	0	1	-1	φ^+	$-\varphi^-$	φ^-	$-\varphi^+$
V_2	3	3	-1	0	0	φ^+	φ^-	φ^-	φ^+
V_3	4	-4	0	-1	1	1	-1	1	-1
V_4	5	5	1	-1	-1	0	0	0	0
V_5	6	-6	0	0	0	-1	1	-1	1
V_6	4	4	0	1	1	-1	-1	-1	-1
V_7	2	-2	0	1	-1	φ^-	$-\varphi^+$	φ^+	$-\varphi^-$
V_8	3	3	-1	0	0	φ^-	φ^+	φ^+	φ^-
W	2	-2	0	1	-1	φ^+	$-\varphi^-$	φ^-	$-\varphi^+$

Table 5: Character table of $2\mathcal{I}$, with $\varphi^\pm \equiv (1 \pm \sqrt{5})/2$.

E.2 | Finite subgroups of SU(3)

The finite subgroups of SU(3) are

- the finite subgroups of SU(2)

so there are 2 infinite series and 5 exceptional subgroups. Note that they are all divisible by 3 because the center of SU(3) is $\mathbb{Z}_3$.

F | The McKay correspondence

F.1 | Classical correspondence

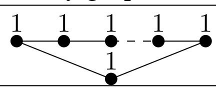
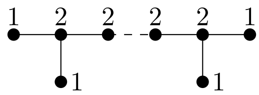
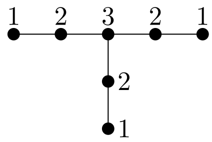
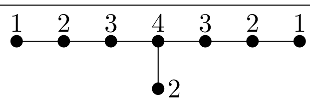
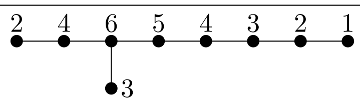
$\Gamma \subset$ SU(2)	Platonic solids	McKay graph	Variety
$\mathbb{Z}_n$		 (n nodes)	$z^n + xy = 0$
$2\mathcal{D}_n$	n-polygon	 ($n+3$ nodes)	$x^2 + y^2 z + z^{n-1} = 0$
$2\mathcal{T}$	tetrahedron	 (7 nodes)	$x^2 + y^3 + z^4 = 0$
$2\mathcal{O}$	cube octahedron	 (8 nodes)	$x^2 + y^3 + yz^3 = 0$
$2\mathcal{I}$	icosahedron dodecahedron	 (9 nodes)	$x^2 + y^3 + z^5 = 0$

Table 6: Binary polyhedral groups and their McKay graphs.Labels over the vertices are the dimension of the representation. We erase the arrow ends if they go in both directions and erase the label if it is equal to 1.

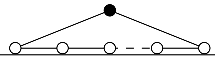
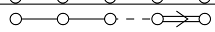
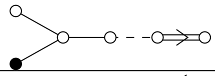
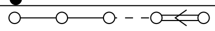
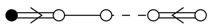
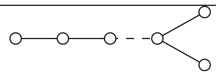
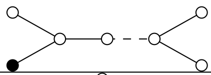
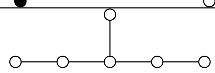
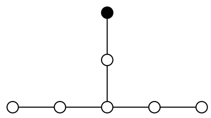
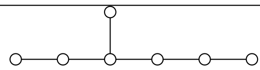
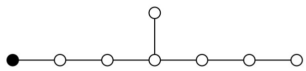
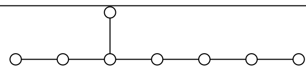
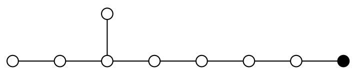
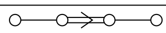
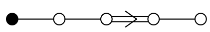
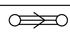
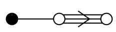
Simple Lie algebra	Simply laced	Dynkin diagram Extended Dybkin diagram
$\mathfrak{sl}(n+1,\mathbb{C}), n \geq 1$	yes	A_n : (n nodes) $\widetilde{A}_n$:  ($n+1$ nodes)
$\mathfrak{so}(2n+1,\mathbb{R}), n \geq 2$	no	B_n :  (n nodes) $\widetilde{B}_n$:  ($n+1$ nodes)
$\mathfrak{sp}(2n,\mathbb{C}), n \geq 3$	no	C_n :  (n nodes) $\widetilde{C}_n$:  ($n+1$ nodes)
$\mathfrak{so}(2n,\mathbb{R}), n \geq 4$	yes	D_n :  (n nodes) $\widetilde{D}_n$:  ($n+1$ nodes)
$\mathfrak{e}_6$	yes	E_6 :  (6 nodes) $\widetilde{E}_6$:  (7 nodes)
$\mathfrak{e}_7$	yes	E_7 :  (7 nodes) $\widetilde{E}_7$:  (8 nodes)
$\mathfrak{e}_8$	yes	E_8 :  (8 nodes) $\widetilde{E}_8$:  (9 nodes)
$\mathfrak{f}_4$	no	F_4 :  (4 nodes) $\widetilde{F}_4$:  (5 nodes)
$\mathfrak{g}_2$	no	G_2 :  (2 nodes) $\widetilde{G}_2$:  (3 nodes)

Figure 4: Simple Lie algebras and their (extended) Dynkin diagrams. The first four algebras are the classical simple Lie algebras and the last five are the excpetional simple Lie algebras.

Finally, we can see the following correspondence between the extended Dynkin diagrams and the McKay graphs.

Simply Lie group	Simply laced Lie algebra	Extended Dybkin diagram	Finite subgroup of SO(3)	Finite subgroup of SU(2)
$\mathrm{SU}(n+1)$	$\mathfrak{sl}(n+1,\mathbb{C})$	$\widetilde{A}_n$	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_{n+1}$
$\mathrm{SO}(2n), \mathrm{Spin}(2n)$	$\mathfrak{so}(2,\mathbb{R})$	$\widetilde{D}_n$	$\mathcal{D}_{2(n-2)}$	$2\mathcal{D}_{2(n-2)}$
$E6$	$\mathfrak{e}_6$	$\widetilde{E}_6$	$\mathcal{T}$	$2\mathcal{T}$
$E7$	$\mathfrak{e}_7$	$\widetilde{E}_7$	O	2O
$E8$	$\mathfrak{e}_8$	$\widetilde{E}_8$	$\mathcal{I}$	$2\mathcal{I}$

Figure 5: Classical McKay correspondence.

F.2 | Geometrical McKay correspondence

G | Algebraic geometry

G.1 | Singularities and resolutions

A *rational map* from a variety X to another Y is a morphism from a non-empty subset $U \subset X$ to Y. Recall that, by definition of the Zariski topology, a non-empty open subset is always dense. Concretely, a rational map can be written in coordinates using ration functions (quotient of polynomials). A *birational map* is an invertible rational map. It induces an isomorphism between two non-empty open subsets. In this case, X and Y are said to be *birationally equivalent*.

The *resolution of a singularity* of an algebraic variety V is a non-singular variety W with a proper birational map $W \to V$. For varities over fields of characteristic 0, it was proven (Hironaka, 1964) that (*).

G.2 | Path algebras

H | Calabi-Yau manifolds, orbifolds and crepant resolutions

Simply put, a *Calabi-Yau manifold* is a Kähler manifold with trivial canonical bundle or, equivalently, with a Kähler metric whose global holonomy is contained in $\mathrm{SU}(n)$. This is equivalent to having a trivial canonical bundle.

A *Calabi-Yau orbifold* is the quotient of a smooth Calabi-Yau manifold by a discrete group action which generically has fixed points. From a algebraic geometry perspective we can try to resolve the orbifold singularity. A resolution (X, π) of $\mathbb{C}^n/\Gamma$ is a non-singular complex manifold X of dimension n with a proper biholomorphic map

$$\pi : X \to \mathbb{C}^n/\Gamma \tag{H.1}$$

that induces a biholomorphism between dense open sets. A resolution (X, π) of $\mathbb{C}^n/\Gamma$ is called a *crepant resolution*[4] if the canonical bundles of X and $\mathbb{C}^n/\Gamma$ are isomorphic, i.e.

$$K_X \cong \pi^*(K_{\mathbb{C}^n/\Gamma}).$$

Since Calabi-Yau manifolds have trivial canonical bundle, to obtain a Calabi-Yau structure on X one must choose a crepant resolutions of singularities.

It turns out that the amount of information we know about a crepant resolution of singularities of $\mathbb{C}^n/\Gamma$ depends dramatically on the dimension n of the orbifold:

- $n = 2$: a crepant always exists and is unique. Its topology is entirely described in terms of the finite group Γ (via the McKay correspondence).
- $n = 3$: a crepant resolution always exists but it is not unique; they are related by flops. However all the crepant resolutions have the same Euler and Betti numbers: the *stringy* Betti and Hodge numbers of the orbifold.

[4]For a resolution of singularities we can define a notion of discrepancy. A crepant resolution is a resolution without discrepancy.

- $n \geq 4$: very little is known; crepant resolution ecists in rather special cases. Many singularity are terminal, which implies that they admit no crepant resolution.

I | Elements of graph theory

I.1 | Quiver diagrams

A *quiver* $Q = (E, V, t, h)$ is a piece of combinatorial data where E is a set of of edges, V is a set of vertices, and $h, t : V \to E$ are two maps that respectively assign a head and a tail to every arrow. A *morphism* $m = (m_V, m_E)$ between two quivers (V, E, t, h) and (V', E', t', h') consists in two maps

$$m_V : V \to V' \tag{I.1}$$
$$m_E : E \to E' \tag{I.2}$$

such that $m_V \circ t = t' \circ m_E$ and $m_V \circ h = h' \circ m_E$.

The *line graph* $L(G)$ of a graph G is the graph such that

- each vertex of $L(G)$ represent a line of G
- two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common endpoint in G.

In other words, it is the intersection graph of the edges of G, where each edge by the set of its two endpoints. Whitney graph isomorphism theorem states that two graphs are isomorphic if and only if their line graphs are isomorphic, with only one exception: K_3 and $K_{1,3}$.

I.2 | Toric diagrams

I.3 | Dimer diagrams

A *bipartite graph* is a graph such that all vertices can be couloured black or white such that any vertex of one color is only adjacent to vertices of the other color. A *matching* is a subset of edges without common vertices. A mathcing is *perfect* of every vertex is covered. The statistical mechanics of random perfect matchings is called a *dimer model*. Ghe difference between any two closed perfect matching is a closed curve.

The *dimer diagram* of a quiver gauge theory is a graph whose faces represent the gauge groups, the edges represent the bi-fundamental fields and the vertices represent the superpotentials.

J | Toric and non-toric singularities

All abelian orbifolds are toric varities.

K | Spacetime geometry: ALE space and orbifolds

Asymptotically locally euclidean (ALE) spaces are a particularly interresting choice of string background to probe with branes for mainly four reasons

(i) they are the resolution (blow-ups) of orbifolds

(ii) there are completelyclassified: they fall in the ADE classification

(iii) they only break half of the supersymmetry

(iv) they are non-compact therefore we can study them for self-dual type II theory(*).

Mathematically, an ALE space is complete riemannian n-manifold M such that there exists a compact set $K \subset M$ such that $M \backslash K$ is diffeomorphic to $(\mathbb{R}^n \backslash B_0(R))/G$, where $R \in \mathbb{R}_0^+$ is a radius and $G \subset \mathrm{O}(n)$ a subgroup. Additionally, it is asked that the pulled back metric on $\mathbb{R}^n \backslash B_0(R)$ tends to the euclidean flat metric at infinity.

If one considers string theory an the orbifold $\mathbb{R}^4/\Gamma$ where Γ is a finite sub group of SU(2), massless states appear from the twisted sector. They are precisely the moduli needed the deform the theory to the one with smooth spacetime, i.e. the resolution of the orbifold. In that sense, is said that the strings know about the metric ALE space and that it is said that strings resolve the singularity. The metric of the ALE space can be recovered if the lagrangian of the resulting field theory is explicitely know, such as for the Wess-Zumino-Witten model. However, it is often not the case.

L | Some derivations

L.1 |

We want to compute the sum

$$\sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} + 1 \right\rfloor = \left\lfloor \frac{n}{3} \right\rfloor + \sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} \right\rfloor . \tag{L.1}$$

Let us write $n \in \mathbb{N}$ as $n = 3m + r$ with $r = 0, 1$ or 2 and $m \in \mathbb{N}$. Regardless of r, we have $\lfloor n/3 \rfloor = m$ and

$$\sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} \right\rfloor = \sum_{a=1}^{m} \left\lfloor \frac{3}{2}(m-a) + \frac{r}{2} \right\rfloor = \sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + \frac{r}{2} \right\rfloor . \tag{L.2}$$

- if $r = 0$, then (L.2) becomes

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a \right\rfloor = \sum_{a=0}^{m-1} a + \sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = \frac{(m-1)m}{2} + \sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor . \tag{L.3}$$

  Now if m is even, we have

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = 2 \sum_{a=0}^{\left\lfloor \frac{m-1}{2} \right\rfloor} a = 2 \sum_{a=0}^{\frac{m}{2}-1} a = \left(\frac{m}{2} - 1\right) \frac{m}{2} \tag{L.4}$$

  and if m is odd,

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = 2 \sum_{a=0}^{\left\lfloor \frac{m-2}{2} \right\rfloor} a + \left\lfloor \frac{m-1}{2} \right\rfloor = 2 \sum_{a=0}^{\frac{m-3}{2}} a + \frac{m-1}{2} = \frac{(m-1)^2}{4} \tag{L.5}$$

  so

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = \begin{cases} \left(\frac{m}{2} - 1\right) \frac{m}{2}, & \text{if } m \text{ is even} \\ \frac{(m-1)^2}{4}, & \text{if } m \text{ is odd} \end{cases} . \tag{L.6}$$

  and

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3a}{2} \right\rfloor = \begin{cases} \frac{m(3m-4)}{4}, & \text{if } m \text{ is even} \\ \frac{(m-1)(3m-1)}{4}, & \text{if } m \text{ is odd} \end{cases} . \tag{L.7}$$

- if $r = 1$, then (L.2) becomes

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + \frac{1}{2} \right\rfloor = \sum_{a=0}^{m-1} a + \sum_{a=0}^{m-1} \left\lfloor \frac{a+1}{2} \right\rfloor = \frac{(m-1)m}{2} + \sum_{a=0}^{m-1} \left\lfloor \frac{a+1}{2} \right\rfloor \tag{L.8}$$

  and

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a+1}{2} \right\rfloor = \sum_{a=1}^{m} \left\lfloor \frac{a}{2} \right\rfloor = \sum_{a=0}^{m} \left\lfloor \frac{a}{2} \right\rfloor = \begin{cases} \frac{m^2}{4}, & \text{if } m \text{ is even} \\ \frac{m^2-1}{4}, & \text{if } m \text{ is odd} \end{cases} \tag{L.9}$$

by (L.6) so

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + \frac{1}{2} \right\rfloor = \begin{cases} \frac{m(3m-2)}{4}, & \text{if } m \text{ is even} \\ \frac{3m^2-2m-1}{4}, & \text{if } m \text{ is odd} \end{cases} \tag{L.10}$$

- if $r = 2$, then (L.2) becomes

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + 1 \right\rfloor = m + \sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a \right\rfloor . \tag{L.11}$$

so

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + 1 \right\rfloor = \begin{cases} \frac{3m^2}{4}, & \text{if } m \text{ is even} \\ \frac{3m^2+1}{4}, & \text{if } m \text{ is odd} \end{cases} \tag{L.12}$$

from (L.7).

Finally, we can write $m = 2k$ if m if even and $m = 2k+1$ if m is odd in order to distinguish the six different cases. We get

$$a(n) \equiv \sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} + 1 \right\rfloor = \begin{cases} 2k + \frac{2k(6k-4)}{4}, & \text{if } n = 6k \\ 2k + \frac{2k(6k-2)}{4}, & \text{if } n = 6k+1, \\ 2k + \frac{12k^2}{4}, & \text{if } n = 6k+2, \\ (2k+1) + \frac{2k(6k+2)}{4}, & \text{if } n = 6k+3, \\ (2k+1) + \frac{3(2k+1)^2-2(2k+1)-1}{4}, & \text{if } n = 6k+4, \\ (2k+1) + \frac{3(2k+1)^2+1}{4}, & \text{if } n = 6k+5 \end{cases} \tag{L.13}$$

$$= \begin{cases} 3k^2, & \text{if } n = 6k \\ 3k^2 + k, & \text{if } n = 6k+1, \\ 3k^2 + 2k, & \text{if } n = 6k+2, \\ 3k^2 + 3k + 1, & \text{if } n = 6k+3, \\ 3k^2 + 4k + 1, & \text{if } n = 6k+4, \\ 3k^2 + 5k + 2, & \text{if } n = 6k+5 \end{cases} . \tag{L.14}$$

Starting from $n = 1$, the first value of this sequence is : $0, 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, \ldots$. Uppon further analysis, this correspond to the sequence A001399, that have several interpretations:

- the number of partitions of n into at most 3 parts. This makes sense with our initial problem: finding all the a, b, c's such that $a + b + c = n$,
- the number of connected graphs with 3 nodes and n edges (where multiple edges between the same nodes are allowed),
- the number of non-negative solutions to $b + 2c + 3d = n$,

as well as many others. Finally, we note that we can simply write

$$a(n) = \text{round}\left(\frac{n^2}{12}\right) . \tag{L.15}$$

M | References guide

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- Orbifold construction for $\Gamma \subset \mathrm{SU}(2)$: type A : [DM96], type D and E: [JM97]
- Orbifold construction for $\Gamma \subset \mathrm{SU}(3)$:[HH99]
- Formalization of projection to daughter theories: [LNV98],[KS98]
- On toric singularities: [Ful93],[Oda88]
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- Fractional branes: [Arg+08]
- Quiver representations and quiver varities: [Kir16]

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