Notes on Quiver Gauge Theories

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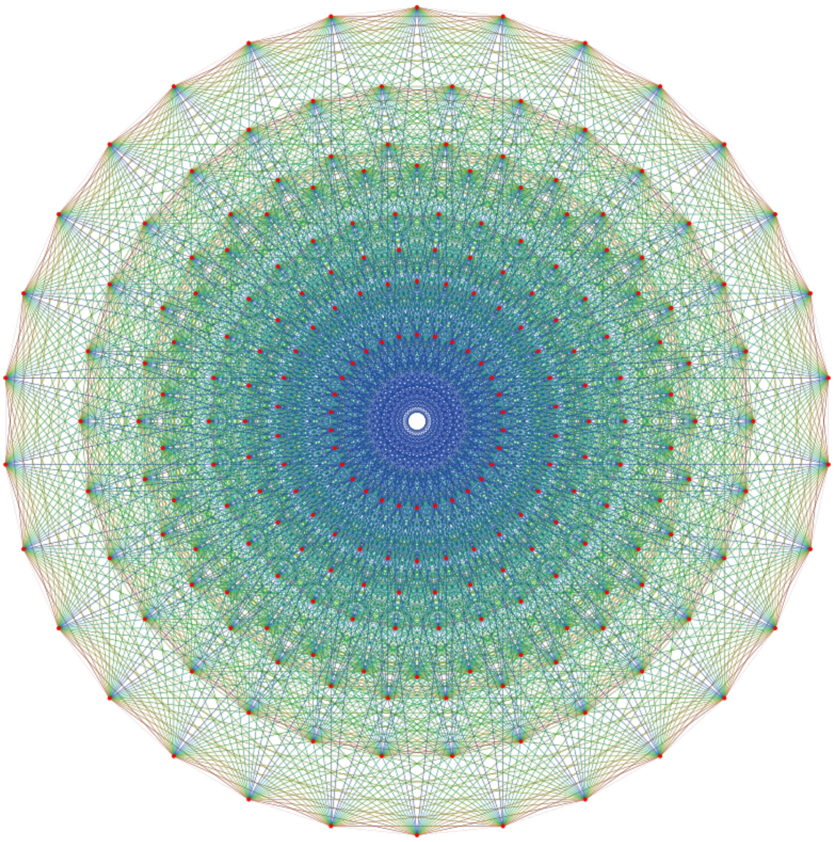

Abstract

In these notes, we present some basic ideas around the large topic of quiver gauge theories, more precisely about their brane probes construction. The goal is to reproduce and regroup the basics of these theories for various types of singularities, with increasing level of complexity (orbifold, toric, del Pezzo, etc). Note that this document is only meant as a work support and contains a lot of typos, errors and imprecisions.

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Contents

Part I

Preliminary notions

1 | Physical setup

Brane-world paradigm

We consider our four-dimensional world to be the worldvolume of a D3-brane in the ten-dimensional spacetime of type IIB superstring theory. More precisely, we consider a stack of N D3-branes in order to have $\mathrm{U}(N)$ Chan-Paton factors resulting in a $\mathrm{U}(N)$ gauge group in the worldvolume theory. The spacetime is therefore not necessarily $\mathbb{R}^{1,9}$ but of the more general form

$$M = \mathbb{R}^{1,3} \times M^{(6)}.$$

This is the so-called *brane-world paradigm.*

Supersymmetry and Calabi-Yau manifolds

Independently from string theory, we can ask for the wolrdvolume theory to be supersymmetric. We start from type IIB superstring theory which is 10-dimensional and has $\mathcal{N} = 2$ supersymmetry so it possesses 32 supercharges. As usual, they transform under the minimal spinor representation (MSR) of the bulk Lorentz group, here $\mathrm{SO}(1,9)$. In ten dimensions this representation is 8-dimensional (complex) which is why there are $2(8+8) = 32$ supercharges: 8 transforming in the 8-dimensional MSR and 8 transforming in the 8-dimensional conjugate MSR and the whole thing times two since $\mathcal{N} = 2$. Compactifying type II string theory on any 6-dimensional manifold $M^{(6)}$ breaks supersymmetry. The reason for this is that the supercharges now have to transform under the MSR of $\mathrm{SO}(1,3) \times \mathcal{H}(M^{(6)})$, where $\mathcal{H}(M^{(6)})$ is the holonomy group of M^6. Actually, the space $\mathbb{R}^{1,3} \times M^{(6)}$ can be viewed as the trivial bundle with base space $\mathbb{R}^{1,3}$ and fibers $M^{(6)}$, sincespinors takes values in the fibers they must transform under the holonomy group of $M^(6)$. A generic curved 6-dimensional manifold has $\mathrm{O}(6)$ holonomy and $\mathrm{SO}(6)$ if it is orientable, as we will always consider. The supercharges must therefore transform under the MSR of $\mathrm{SO}(1,3) \times \mathrm{SO}(6)$. The MSR of $\mathrm{SO}(1,3)$ being $\mathbf{2}$ and the one of $\mathrm{SO}(6)$ being $\mathbf{4}$, we conclude that imposing the spacetime to have the form (1) changes the representation under which the supercharges transform in the following way: Why ?

$$\mathbf{8} \oplus \overline{\mathbf{8}} \to (\mathbf{2}_L, \mathbf{4}) \oplus (\mathbf{2}_R, \overline{\mathbf{4}}). \tag{1.1}$$

If we stop here, the residual supercharges might be ill-defined; making a tour around a loop in $M^{(6)}$ could result in a non-trivial rotation. To solve this problem, we need to be more restrictive with the holonomy. In fact, it is precisely the holonomy of the transverse space that dictates the number of residual supersymmetries. To understand this, let us now consider a four-dimensional field theory resulting from compactification of the transverse six-dimensional space. The number of supercharges that generate supersymmetries for this theory is the number of Killing spinors (covariantly constant spinors) because each Killing spinor contracted with the local supersymmetry current generates a residual supersymmetry. Let us now make the link with holonomy: since $\mathrm{SO}(6) \cong \mathrm{SU}(4)$, minimal spinors can be viewed as having four complex components and as transforming under $\mathrm{SU}(4)$. Indeed, minimal spinors in six dimensions have four complex components. In order to have one covariantly conserved spinor, we look for the biggest subgroup of $\mathrm{SU}(4)$ that leaves a component of the spinor invariant. This is clearly $\{e\} \times \mathrm{SU}(3) \subset \mathrm{SU}(4)$ that acts trivially on the first component. The spinor $(1,0,0,0)$ is then covariantly constant. Our transverse space must therefore have $\mathrm{SU}(3)$ holonomy such that the parallel transport of the spinor $(1,0,0,0)$ under any closed loop is a lower $\mathrm{SU}(3)$ rotation. We conclude that if the transverse Calabi-Yau has $\mathrm{SU}(3)$ holonomy, the worldvolume theory has $\mathcal{N} = 1$ supersymmetry. If the holonomy is $\mathrm{SU}(2) \subset \mathrm{SU}(3)$, the spinor $(0,1,0,0)$ is also a Killing spinor which means that we have $\mathcal{N} = 2$ supersymmetry for the worldvolume theory. To summarize, preserving any degree of supersymmetry constrains the transverse space $M^{(6)}$ to be compact, complex, Kähler and to have $G \subset \mathrm{SU}(3)$ holonomy. Namely, $M^{(6)}$ must be a Calabi-Yau threefold, see section **??**.

Non-compact transverse space

If we let the worldvolume of the D3-branes carry the requisite gauge theory while the bulk contains gravity, we can relax the compactness condition and study non-compact Calabi-Yau threefolds. This makes the analysis much simpler and therefore also serves as an argument to ignore gravity in the worldvolume theory. Consequently, we will mostly ignore gravity and not care about the metric of the spacetime, see appendix C for more details. In this setup we cannot really talk about compactification anymore. Instead we just think of it as a flat space on which lives the gauge theory while gravity only lives in transverse space. To understand intuitively why there is no gravity in this limit, we can think of Kaluza-Klein compactification. The four-dimensional gravity coupling constant is inversely proportional to the size of the compactifying space therefore there is no gravity in the non-compact/infinite-size limit. This more a motivation than a proof.

Singular transverse space

The only non-compact smooth Calabi-Yau threefold is $\mathbb{C}^3$, this forces us to consider singular Calabi-Yau varieties is we want more interesting theories. A Calabi-Yau variety is an affine variety that locally models a Calabi-Yau manifold, therefore allowing for singularities. We usually denote $S \equiv M^{(6)}$ to remind us of the singular aspect. String theory being a theory of extended objects, turns out to be it is well-defined on such singularities and even "smoothened" the singularity in some sens. Considering strings on singular geometries requires to "project" the theory obtain from $\mathbb{C}^3$. As a result, the gauge group $\mathrm{U}(N)$ is broken down into products of smaller gauge groups. This "projection" highly depends on the type of singularity (orbifold, toric, del Pezzo, etc) we are considering. While it is relatively straightforward for "simple" singularities (e.g. abelian orbifolds) it quickly gets more complicated or even unknown for others.

From the point of view of the orbifold, the D3-brane is a point, meaning that the D3-branes are really probing the transverse space and, in particular, they parametrize it. This is the first clue of the tight relationship that exists between the worldvolume theory and the transverse singular space. Eventually, we will see that the classical vacuum of the gauge theory should be, in explicit coordinates, the defining equation of S. This is precisely the opposite of the projection manipulation we mentioned above: recovering the transverse space from the gauge theory. Projecting and computing the classical vacua are therefore inverse operations with respect to each other. This suggest a bijection between the singular transverse space and the gauge theory: the former can be computed from the latter and vice-versa. This is called "forward algorithm" and "inverse algorithm" respectively.

Mathematical formulation

Mathematically, this brane-world paradigm is the realization of branes as supports of vector bundles (sheaf). Gauge theories on branes are intimately related to algebraic constructions of stable bundles, i.e. holomorphic or algebraic vector bundles that are stable in the sense of geometric invariant theory. In particular, D-brane gauge theories manifest as a natural description of symplectic quotients and their resolutions in geometric invariant theory. Together with the stable vector bundle (sheaf) supported thereupon the D-branes resolve the transverse Calabi-Yau orbifold, which is the vacuum for the gauge theory on the worldvolume as a GIT quotient.

Summary

We consider N D3-branes in type IIB superstring theory carrying a $\mathrm{U}(N)$ gauge group. The transverse space S is taken to be a non-compact singular Calabi-Yau variety.

2 | Properties of D-branes

2.1 | SYM from D-branes

The dynamics of D-branes is described by the Dirac-Born-Infeld action

$$S_{\mathrm{DBI}}[X,F] = -\frac{T_p}{g_s}\int \mathrm{d}^{p+1}\sigma\ \sqrt{-\det_{0\leq a,b\leq p}(\eta_{ab} + \partial_a X^m \partial_b X_m + 2\pi\alpha' F_{ab})}. \tag{2.1}$$

The latter can be expended for slowly-varying fields, which is equivalent to passing to the field theory limit $\alpha' \to 0$. The resulting action is the action of a U(1) gauge theory in $p+1$ dimensions with $9-p$ real scalar fields. This action is exactly the same than the one we would obtain by dimensionally-reducing a pure U(1) Yang-Mills gauge theory in 10 spacetime dimensions with the identification

$$g_{\mathrm{YM}} = g_s T_p^{-1}(2\pi\alpha')^{-2} = \frac{g_s}{\sqrt{\alpha'}}(2\pi\sqrt{\alpha'})^{p-1}. \tag{2.2}$$

This construction can be generalized for multiple D-branes. It now results in a non-abelian theory. The general statement is the following:

> The low-energy dynamics of N parallel, coicident Dp-branes in flat space is described in static gauge by the dimensional reduction to $p+1$ dimensions of pure $10d$ $\mathcal{N}=1$ supersymmetric Yang-Mills theory with gauge group U(N) in ten spacetime dimensions.

Recall that the 10-dimensional action is given by

$$S_{\mathrm{YM}} = \frac{1}{4g_{\mathrm{YM}}^2}\int \mathrm{d}^{10}x\ \left[\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + 2i\,\mathrm{tr}\big(\overline{\psi}\Gamma^\mu D_\mu\psi\big)\right], \tag{2.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the non-abelian field strength of the U(N) gauge field A_μ, $D_\mu = \partial_\mu - i[A_\mu, \psi]$, Γ^μ are 16×16 Dirac matrices , and the $N\times N$ Hermitian fermion field ψ is a 16-component Why ?
Majorana-Weyl spinor of the Lorentz group SO(1, 9) which transforms under the adjoint representation of the gauge group U(N). On-shell, there are eight on-shell bosonic, gauge field degrees of freedom, and eight fermionic degrees of freedom, after imposition of the Dirac equation $\not{D}\psi = \Gamma^\mu D_\mu \psi = 0$. One can verify that this action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta_\epsilon A_\mu &= \frac{i}{2}\bar{\epsilon}\Gamma_\mu\psi, \\ \delta_\epsilon\psi &= \frac{1}{2}F_{\mu\nu}[\Gamma^\mu, \Gamma^\nu]\epsilon, \end{aligned}$$

where ϵ is an Majorana-Weyl spinor.

Using (2.3), we can construct a supersymmetric Yanf-Mills gauge theory in $p+1$ dimensions with 16 independent supercharges by dimensional reduction: we take all fields to be independent of the coordinates $X^{p+1}, \ldots, X^9$, then the ten-dimensional gauge field A_μ splits into a $(p+1)$-dimensional U(N) gauge field A_a plus $9-p$ Hermitian scalar fields $\Phi^m = X^m/2\pi\alpha'$ in the adjoint representation of U(N). The Dp-brane action is thereby obtained from the dimensionality reduced field theory as

$$S_{\mathrm{D}p} = -\frac{T_p g_s(2\pi\alpha')^2}{4}\int \mathrm{d}^{p+1}\sigma\ \mathrm{tr}\left(F_{ab}F^{ab} + 2D_a\Phi^m D^a\Phi_m + \sum_{m\neq n}[\Phi^m, \Phi^n]^2 + \text{fermions}\right) \tag{2.4}$$

where $a, b = 0, \ldots, p$, $m, n = p+1, \ldots, 9$. We do not explicitly display the fermionic contributions for the moment. In conclusion, the low-energy brane dynamics is described by a supersymmetric Yang-Mills theory on the Dp-brane worldvolume which is dynamically coupled to the transverse, adjoint scalar fields Φ^m.

The scalar potential is given by

$$V(\Phi) = \sum_{m\neq n}[\Phi^m, \Phi^n]^2. \tag{2.5}$$

It is negative definite because $[\Phi^m, \Phi^n]^\dagger = [\Phi^n, \Phi^m] = -[\Phi^m, \Phi^n]$. A classical vacuum of the field theory defined by (2.4) corresponds to a static solution of the equations of motion whereby the potential energy of the system is minimized. It is given by the field configurations which solve simultaneously the quations $F_{ab} = D_a\Phi^m = \psi^a = 0$ and $V(\Phi) = 0$. Since all term in $V(\Phi)$ have the same sign, the equation $V(\Phi) = 0$ is equivalent to the equation $[\Phi^m, \Phi^n] = 0$ for all m, n and at each point in the

$(p+1)$-dimensional worldvolume of the branes. This implies that the $N \times N$ hermitian matrix fields Φ^m are simultaneously diagonalizable by a gauge transformation, so that we may write

$$\Phi^m = U \begin{bmatrix} X_1^m & & & 0 \\ & X_2^m & & \\ & & \ddots & \\ 0 & & & X_N^m \end{bmatrix} U^{-1}, \tag{2.6}$$

the matrix U is independent of m. The simultaneous, real eigenvalues X_i^m give the positions of the N distinct D-branes in the m-th transverse direction. It follows that the moduli space of classical vacua for the $(p+1)$-dimensional field theory (2.4) is the quotient space $(\mathbb{R}^{9-p})^N/S_N$, where the factors of $\mathbb{R}$ correspond to the positions of the N Dp-branes in the $(9-p)$-dimensional transverse space, and S_N is the symmetric group acting by permutations of the N coordinates X_i. The group S_N corresponds to the residual Weyl symmetry of the $\mathrm{U}(N)$ gauge group acting in (2.6). It represents the permutation symmetry of a system of N *indistinguishable* D-branes.

From (2.4) one can easily deduce that the masses of the fields corresponding to the off-diagonal matrix elements are given precisely by the distances $|x_i - x_j|$ between the corresponding branes. This description means that an interpretation of the D-brane configuration in terms of classical geometry is only possible in the classical ground state of the system, whereby the matrices Φ^m are simultaneously diagonalizable and the positions of the individual D-branes may be described through their spectrum of eigenvalues. This gives a simple and natural dynamical mechanism for the appearence of "non-commutative geometry" at short distances, where the D-branes cease to have well-defined positions according to classical geometry.

The end of this section has to be rewritten

2.2 D-branes and residual SUSY in type II theories

The minimal irreducible representation in 10 dimensions is a Majorana-Weyl representation of dimension 8. In type II theories, we have $\mathcal{N} = (1,1)$ for IIA and $\mathcal{N} = (2,0)$ for IIB. Because of the string origin of the generators, the two supersymmetry generators ϵ_L and ϵ_R (Majorana-Weyl spinors) satisfy

$$\epsilon_L = \Gamma_{11}\epsilon_L, \qquad \epsilon_R = \eta\Gamma_{11}\epsilon_R \tag{2.7}$$

with $\eta = +1$ for IIB and $\eta = -1$ for IIA theory. For a Dp-brane, the supersymmetry projections is the following:

$$\epsilon_L = \Gamma_0 \ldots \Gamma_p \epsilon_R. \tag{2.8}$$

In other words, the supersymmetries with generators of the form

$$Q_\alpha + \Gamma_0 \ldots \Gamma_p \overline{Q}_{\dot\alpha} \tag{2.9}$$

are preserved by the Dp-brane while the one with generators of the form

$$Q_\alpha - \Gamma_0 \ldots \Gamma_p \overline{Q}_{\dot\alpha} \tag{2.10}$$

are broken. They violate the boundary conditions. Since there is the same number of generators of the form (2.9) than of the form (2.10), exactly haf of the supersymmetry is broken. The idea that one spacetime direction would break one supercharge could be reasonable if supersymmetries were transforming as vectors which not the case; supercharges transform as spinors. It would also be incompatible with the T-duality because two branes of different dimensions must have the same number of unbroken supercharges if there is a T-duality relating them: the number of unbroken supercharges is the same for all dual descriptions (a necessary condition for the equivalence). And indeed, in the correct theory, that's the case. Every type II D-brane breaks half of the supercharges.

To obtain the previous relations, we start by the ones from M-theory and compactify the 11th direction, getting type IIA theory. Γ_{11} then plays the role of the chiral projector in 10 dimensions; the supersymmetry parameters are related by $\epsilon_L = \frac{1}{2}(1+\Gamma_{11})\epsilon$ and $\epsilon_R = \frac{1}{2}(1-\Gamma_{11})\epsilon$. The relations for type IIB theory are then obtained by T-duality. Under a T-duality over the $\widehat{i}$ direction, the supersymmetry parameters transform as

$$\epsilon_L \mapsto \epsilon_L,$$

$$\epsilon_R \mapsto \Gamma_i \epsilon_R.$$

The tension of a Dp-brane is given by

$$T_p = \frac{1}{(2\pi)^p g_s l_s^{p+1}}. \tag{2.11}$$

This completely fixes the Newton constant: the tension of electric-magnetic duals must satisfy:

$$T_p T_{D-p-4} = \frac{2\pi}{16\pi G_D}. \tag{2.12}$$

In ten dimensions, this gives $G_{10} = 8\pi^6 g_s^2 l_s^8$.

The dualities are defined as follows:

$$\text{S-duality} : g_s \mapsto \frac{1}{g_s}, \qquad l_s^2 \mapsto g_s l_s^2,$$
$$\text{T-duality} : R \mapsto \frac{l_s^2}{R}, \qquad g \mapsto g_s \frac{l_s}{R}.$$

2.3 | D-branes wrapping cycles

A Dp-brane worldvolume $\phi : \Sigma \to X$ in spacetime X *wraps* a cycle $c \in H_{p+1}$ if the pushforward $\phi_*(\Sigma) \in H_\bullet(X)$ of the fundamental class of Σ is the class $[c]$ of the given cycle in X. If the pushforward is a mutliple of $[c]$, then the branes wraps c multiple times.

3 | Algebraic geometry

3.1 | Elements

An important idea in algebraic geometry is that is is really the alegbra of function on it that defines a space. For affine varities X, this is illustrated by the fact that the structure of X is really contained in its coordinate ring $K[x_1, \dots, x_n]/I(X)$ and by the isomorphism

$$K[x_1, \dots, x_n]|_X = K[x_1, \dots, x_n]/I(X). \tag{3.1}$$

Now an algebraic set $Z(T)$ is irreducible if $I(Z(T))$ is prime. So there is a one-to-one correspondence between prime ideals and affine varieties.

Given an algebraic variety, one can modify the equations continuously by varying some parameters and the variety will be "deformed" accordingly. It is called the *variation of the omplex structure*. The space of all complex deformations of an affine variety X is called the *complex moduli space* of X. For a Calabi-Yau manifold, the linearization of the complex moduli space (tangent space) is given by the cohomology group $H^{m-1,1}(X)$, where m is the dimension of X. In general, it is much more complicated.

3.2 | Divisors and line bundles

A *(Weyl) divisor* D of a complex variety X is a linear combination (formal sum with integer coefficients) of co-dimension one, irreducible subvarieties,

$$D = \sum_i n_i V_i, \qquad n_i \in \mathbb{Z}, V_i \subset X. \tag{3.2}$$

It said to be effective if all $n_i \geq 1$. To any line bundle L with a regular section s (that is, on any open subset U_α, $s_\alpha = s|_{U_\alpha}$ is a polyomial in the local coordinates) we can associate a hypersurface $Y \subset X$ defined as

$$Y = \{p \in X | s(p) = 0\}. \tag{3.3}$$

This hypersurface Y can then be decomposed into irreducible parts (affine patches) on which s_α can be factorized in $\mathbb{C}[x_1, \dots, x_n]$ and decomposed in prime ideals P_i of multiplicity n_i. Assembling all the

V_i^α otgether, we construct co-dimension one subvarities V_i that can be used to form divisors. One can also proceed the other ay around and, given a divisor $D = \sum_i n_i V_i$, define a line bundle $\mathcal{O}_X(D)$ whose sections vanish on each V_i with a zero of order n_i. This construction can be generalized to divisors with negative coefficients $n_i < 0$ in which case now have poles of order n_i in V_i.

3.3 Singularities and resolutions

A *rational map* from a variety X to another Y is a morphism from a non-empty subset $U \subset X$ to Y. Recall that, by definition of the Zariski topology, a non-empty open subset is always dense. Concretely, a rational map can be written in coordinates using ration functions (quotient of polynomials). A *birational map* is an invertible rational map. It induces an isomorphism between two non-empty open subsets. In this case, X and Y are said to be *birationally equivalent*.

The *resolution of a singularity* of an algebraic variety V is a non-singular variety W with a proper birational map $W \to V$. For varities over fields of characteristic 0, it was proven (Hironaka, 1964) that . fill in

3.4 Projective plane curves

In $\mathbb{CP}^2$, we consider a hypersurface defined by a single polynomial p of degree d. If

$$\frac{\partial p(x)}{\partial x_i} = 0 \tag{3.4}$$

for all i whenever $p(x) = 0$, then the curve is said to be regular, it is a Riemann surface. The latter are classified by their genus and

$$g = \frac{(d-1)(d-2)}{2}. \tag{3.5}$$

For $d = 3$, the most geenral polynomial is

$$\sum_{i+j+k=3} c_{ijk} x_0^i x_1^j x_2^k = 0 \tag{3.6}$$

and defines a torus, also called *elliptic curve*. There 10 independant parameters but 9 of them can be removed by a $\mathrm{GL}(3, \mathbb{C})$ transformation, leaving us with only one complex parameter; the complex structrue modulus of the torus.

This section has to be rewritten.

4 Quivers in string theory and Yang-mills in graph theory

Part II

Toric singularities

The next best thing to orbifold singularities is the toric singularities. The a specific class of supersymmetric gauge theories whose space of vacua is toric is called *toric quiver gauge theories*. In this case, the inverse algorithm has been formalized in [1].

5 | Gauged linear sigma model (GLSM)

Witten's gauged linear sigma model provides a physical perspective on toric varieties which provides us with the right approach for the forward algorithm. Let us consider the vectorspace $\mathbb{C}^q$ with complex coordinates $z_1, \ldots, z_q$.

5.1 | Calabi-Yau and non-compactness conditions

6 | Correspondence between gauge theory and singularity

Above, we presented all the possible orbifold constructions of supersymmetric quiver gauge theories in four dimensions. We started from quotienting the transverse space and we found the corresponding (supersymmetric) gauge theory. In other words, we started from the singularity a found the gauge theory. We can therefore consider that the orbifold singularities are understood. However, not all singularities are orbifold one, such as the conifold for example. We can then ask ourselves how to obtain the gauge theory for more general singularities than the orbifold ones. Is a general approach possible ? On the other hand, we can also study the converse question; is it possible to to obtain the singularity from the gauge theory? And if it is, how so? In general we will see that there is a bijection between the four-dimensional supersymmetric worldvolume gauge theory and the Calabi-Yau singularity. We now detail this bijection.

PHYSICS: Gauge Data	Forward Algorithm $\rightleftharpoons$ Inverse Algorithm	MATHEMATICS: Geometry Data
$\Updownarrow$		$\Updownarrow$
QUIVER	$\rightleftharpoons$	Intersection Theory, etc.

Figure 1: Inverse and forward algorithm, from [2].

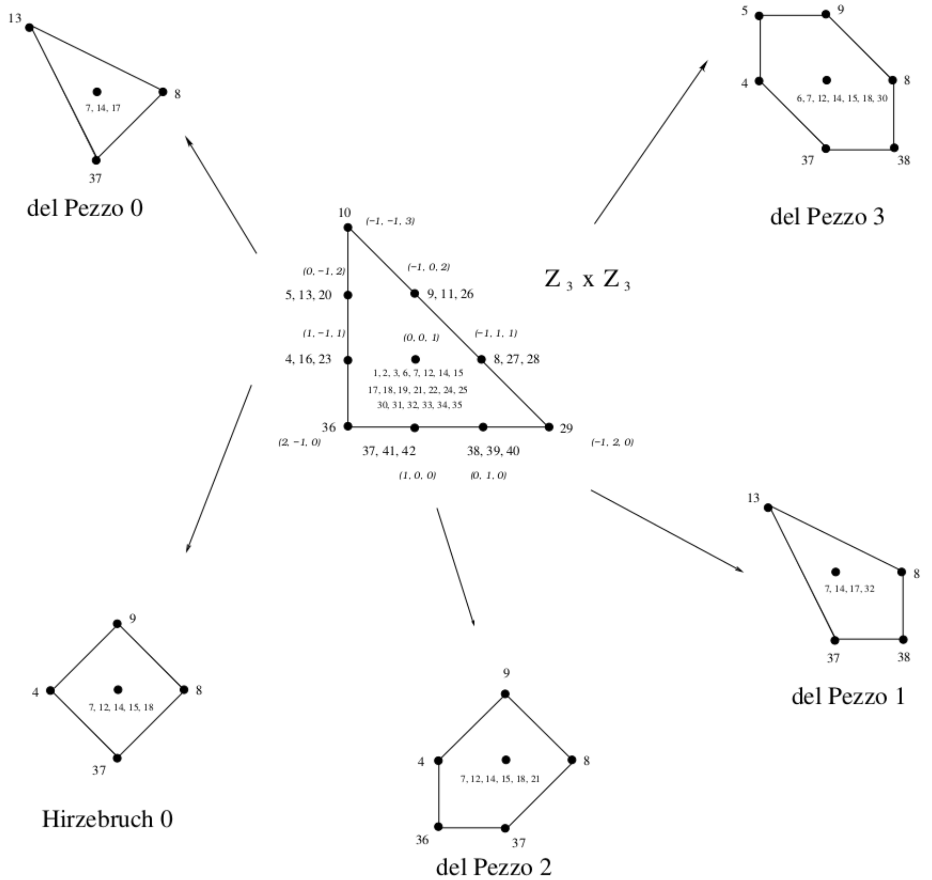
6.1 | From gauge theory to singularity: forward algorithm

We start with the simplest question: how to recover the singularity from the gauge theory? We already mentioned that the vacuum parameter space of the scalar fields of the gauge theory is the so-called moduli space, denoted $\mathcal{M}$. Because our D3-brane is a point in the Calabi-Yau threefold, the vacuum moduli space $\mathcal{M}$ is the affine coordinates of the Calabi-Yau singularity S.

For the ADE $\mathcal{N} = 2$ theories discussed in section **??**, by the Kronheimer-Nakajima construction [**Kronheimer1990**], the moduli space is a hyper-Kähler quotient. In general, the moduli space can be constructed as a *quiver variety*, i.e. a variety constructed from the moduli space of quiver a quiver representation. More rpecisely, given the dimensions of the vector spaces assigned to every vertex, one can form a variety which characterizes all representations of that quiver with those specified dimensions, and consider stability conditions. Let us see some examples of this.

The anomaly free condition is

$$(a_{ij} - a_{ji})n_i = 0. \tag{6.1}$$



Explain more.

6.2 | Forward algorithm for abelina orbifolds

6.3 | From singularity to gauge theory: inverse algorithm

Mathematically, a quiver gauge theory is a representation of a finite quiver with relations. The labels are $\{N_i \in \mathbb{Z}_+\}$, they correspond to the dimension of the vector space $\{V_i\}$. The gauge group is $\prod_i \mathrm{SU}(N_i)$. The gauge fields are self-adjoint arrows $\mathrm{Hom}(V_i, V_i)$ while the matter fields are bi-fundamentals fermions/bosons and are arrows $X_{ij} \in \mathrm{Hom}(V_j, V_i)$. For a quiver with adjacency matrix a_{ij}, the gauge anomaly cancellation condition can be generally expressed as

$$(a_{ij} - a_{ji})N_i = 0. \tag{6.2}$$

At last, there are some relations that arises the superpotential $W(\{X_{ij}\})$. The vacuum is the minima of the superpotential. In other words,

$$\frac{\partial W}{\partial X_{ij}} = 0. \tag{6.3}$$

6.4 | Application to Toric del Pezzo's

7 | Toric duality

Part III

Non-toric singularities

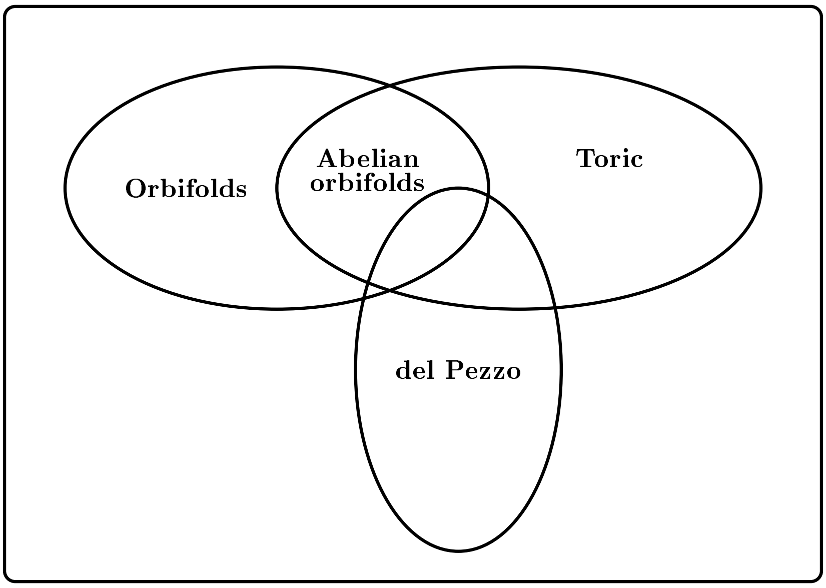


Figure 2: Venn diagram of different types of algebraic singularities.

8 | Non-commutative resolutions

Part IV

Beyond the D-brane probes constructions

There are mainly three methods of constructing finite supersymmetric gauge theories from string theory:

1. Geometrical engineering
2. D-branes probing singularities
3. Hanany-Witten setups.

A | Simplicial Homology

An *n-simplex*, denoted by $\Delta^n = [v_0, \ldots, v_n]$ is the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, \ldots, v_n$ that do not lie in a hyperplane of dimension less than n. Or, equivalently, such that $v_1 - v_0, \ldots, v_n - v_0$ are linearly independant. The $n+1$ point $v_0, \ldots, v_n$ are the *vertices* of the n-simplex. A by-product of our ordered notation for the vertices of a simplex is that it determines an orientation for the edges $[v_i, v_j]$ according to the increasing subscripts. The *faces* of an n-simplex are all the sub-simplices that can obtained by removing vertices of the original simplex. The order of the vertices of the smaller simplices is taken to be same than in the original n-simplex.

Let us now put some simplices together. If X be a topological space, then a *simplicial complex structure* Δ on X is a collection $\Delta = \{\sigma_\alpha\}$ of continuous maps $\sigma_\alpha : \Delta^{n(\alpha)} \to X$ called *characteristic maps* such that

i) $\sigma_\alpha|_{e^{n(\alpha)}}$ is injective and that for all $x \in X$ there exists α such that $x \in \mathrm{Im}(\sigma_\alpha|_{e^{n(\alpha)}})$,

ii) the restriction of any map σ_α to a face of Δ^n is another σ_β,

iii) for any subset $A \subseteq X$, A is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all α,

where Δ^n is an n-simplex and e^n is the interior of Δ^n.

Given a simplicial complex structure Δ on X, n-chains are finite formal sums

$$\sum_\alpha n_\alpha \sigma_\alpha(e^{n(\alpha)}) \tag{A.1}$$

with coefficient $n_\alpha \in \mathbb{Z}$. We denote by $\Delta_n(X)$ the set of all n-chains, it is a free abelian group.

For a general simplicial complex structure on X, we define a *boundary homomorphism* $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by its action on the basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \ldots, \widehat{v_i}, \ldots, v_n]}. \tag{A.2}$$

In particular, homomorphism here means that this maps are linear. We can note that the right side of this equation does indeed lies in $\Delta_{n-1}(X)$ since each restriction $\sigma_\alpha|_{[v_0, \ldots, \widehat{v_i}, \ldots, v_n]}$ is the characteristic map of an $(n-1)$-simplex of X. The most important property of those maps is that $\partial_n \circ \partial_{n+1} = 0$ (symbolically $\partial^2 = 0$), meaning that $\mathrm{Im}\, \partial_{n+1} \subset \mathrm{Ker}\partial_n$. We can then form a sequence of homomorphisms of abelian groups

$$\cdots \to \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \to \cdots \to \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \to 0 \tag{A.3}$$

$\mathrm{Im}\, \partial_{n+1} \subset \mathrm{Ker}\partial_n$ for each n. Chains of homomorphisms satisfying this property are called *chain complex*. We have extended the chain at the end with $\partial_0 = 0$ such that this property is also true at the ends. Elements of $\mathrm{Ker}\partial_n$ are called *n-cycles* and elements of $\mathrm{Im}\, \partial_{n+1}$ are called *n-boundaries*. Note that they are each elements of $\Delta_n(X)$, hence the notation.

Example. If we consider a triangle withe vertices A, B and C, we can put a simplicial complex structure on it consisting of one 2-simplex ($u = [A, B, C]$), three 1-simplices ($a = [A, B], b = [B, C], c = [C, A]$) and three 0-simplices (A, B, C). We can see that

$$\partial_2(u) = [B, C] - [A, C] + [A, B] = a + b + c. \tag{A.4}$$

So we get that $a + b + c$ is a 1-boundary. However, we see that

$$\partial_1(a + b + c) = B - A + C - B + A - C = 0 \tag{A.5}$$

so it is also a 1-cycle.

The fact that image of each map lies the kernel of the next map means that any boundary is also a cycle. As illustrated by the previous example. We can wonder what are the cycles that are not boundaries of any higher-dimensional simplicial complex. This set is precisely the quotient space

$$H_n(X) = \frac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}\,\partial_{n+1}}. \tag{A.6}$$

It naturally inherits a group structure and is called the nth *Homology group* of X. The elements of this space are equivalence classes (cosets of $\mathrm{Im}\,\partial_{n+1}$) of n-cycles where two cycles are considered equivalent if they only differ by a boundary, i.e. if their formal difference is a boundary. These equivalence classes are called *homology classes*.

Example. Let us give some useful examples of Homology groups for various topological spaces:

- the Homology groups of $\mathbb{R}^n$ are all trivial therefore $\chi = 0$.
- the non-trivial Homology groups of the n-sphere are $H_0(S^n) = H_n(S^n) = \mathbb{Z}$ therefore $\chi = (-1)^n - 1$
- the only non-trivial Homology group of the n-ball is $H_0(B^n) = \mathbb{Z}$ therefore $\chi = 1$.
- the non-trivial Homology groups of the 2-torus are $H_0(T) = H_2(T) = \mathbb{Z}$ and $H_1(T) = \mathbb{Z}^2$ therefore $\chi = (1+x)^2$.
- the non-trivial Homology groups of the complex projective space are $H_{0\leq 2k\leq 2n}(\mathbb{CP}^n) = \mathbb{Z}$.
- the only non-trivial Homology groups of any Riemann surface of genus g are $H_0 = \mathbb{Z}, H_1 = \mathbb{Z}^{2g}$ and $H_2 = \mathbb{Z}$, such that $\chi = 2 - 2g$.

Let us make a comment on the zeroth homology group. Since $\mathrm{Ker}\partial_0 = \Delta_0$ by definition, all elements of Δ_0 (i.e. every point of X) are 0-cycles and $H_0(X)$ is the set of 0-cycles that are not the boundary of any chain of 1-simplices. Since any linear combination of 0-simplices (i.e. point) can be seen as the boundary the 1-simples that joins them, we can see that all points that can be joined by a path are equivalent. For a topological space X where all points can be connected by a path this means that the zeroth homology group is necessarily $H_0(X) = \{nA|a \in \mathbb{Z}\} \cong \mathbb{Z}$, where A any point in X. More generally, we can conclude that the number of copies of $\mathbb{Z}_n$ in $H_0(X)$ is the number of path-connected components of X.

For a graph Γ, we can understand from our previous discussions that the only non-trivial homology groups are going to be $H_0(\Gamma) = \mathbb{Z} \times \cdots \times \mathbb{Z}$ where the number of copies is the number of connected components and $H_1(\Gamma) = \mathbb{Z} \times \cdots \times \mathbb{Z}$ where the number of copies is the number of “irreducible” loops.

The nth simplicial homology group of a topological space X with a complex simplicial structure Δ is always of the form

$$H_n^{\Delta}(X) = \underbrace{\mathbb{Z} \times \mathbb{Z}}_{b_n} \oplus \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_s}. \tag{A.7}$$

b_n are the Betti numbers. The *Poincaré polynomial* is

$$P_X(x) = \sum_{i\geq 0} b_i x^i \tag{A.8}$$

and the Euler characteristic of X is given by $P_X(-1)$. Recall that the Euler characteristic is also given by $\chi = 2 - 2g - b - c$, g being the genus, b the number of topological boundaries and c the number of crosscaps.

Note that the homology that we considered is with coefficients in $\mathbb{Z}$, i.e. we started with simplicial chains with coefficients in $\mathbb{Z}$. We can however also consider coefficients in $\mathbb{R}$, in $\mathbb{C}$ or in any ring. In those case however we can easily lose the information about torsion. For example $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ is non trivial but since $2\mathbb{R} = \mathbb{R}$, $\mathbb{R}/2\mathbb{R} = \{0\}$.

B | Some finite subgroups

B.1 | Finite subgroups of $\mathbf{SU}(2)$ and $\mathbf{SL}(2,\mathbb{C})$

B.1.1 | Finite subgroups

The first thing to recall is that every finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ is isomorphic to a subgroup of $\mathrm{SU}(2,\mathbb{C})$ and vice-versa, so we equivalently talk about the subgroups of $\mathrm{SU}(2)$. The finite subgroups of $\mathrm{SU}(2)$, called the *binary polyhedral groups*, are the doubles covers of the finite subgroups of $\mathrm{SO}(3)$ that are called *polyhedral groups*. They simply constitutes the symmetries of the Platonic solids. The groups fall into two infinite series, associated to the regular polygons, as well as three exceptional, associated with the 5 regular polyhedra: the tetrahedron (self-dual), the cube (and its dual octahedron), the icosahedron (and its dual dodecahedron).

More precisely, the finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ are

- $\mathbb{Z}_n$: cyclic group of order n ($n \geq 2$) generated by

$$\begin{bmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{bmatrix} \tag{B.1}$$

- $2\mathcal{D}_n$: *binary dihedral groups* (also known as the *dicyclic group*) of order $4n$ ($n \geq 1$) generated by

$$A \equiv \begin{bmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{bmatrix} \quad \text{and } B \equiv \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{B.2}$$

  One can show that $A^n = B^2$ and that $AB = BA^{-1}$ so that $2\mathcal{D}_n = \{B^b A^a | 0 \leq b \leq 3, 0 \leq a \leq n-1\}$. This rewriting of the most general element of the group will be useful.

- $2\mathcal{T}$: *binary tetrahedral group* of order 24 generated by D_2 and

$$C \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{bmatrix} \tag{B.3}$$

- $2\mathcal{O}$: *binary octahedral group* of order 48 generated by $\mathcal{T}$ and

$$D \equiv \begin{bmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{bmatrix} \tag{B.4}$$

- $2\mathcal{I}$: *binary icosahedral group* of order 120 generated by

$$E \equiv -\frac{1}{\sqrt{5}} \begin{bmatrix} \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{bmatrix} \quad \text{and } F \equiv -\frac{1}{\sqrt{5}} \begin{bmatrix} \zeta_5^2 - \zeta_4^4 & \zeta_5^4 - 1 \\ 1 - \zeta_5 & \zeta_5^3 - \zeta_5 \end{bmatrix} \tag{B.5}$$

with $\zeta_m \equiv e^{i\frac{2\pi}{m}}$ such that $(\zeta_m)^m = 1$. Note that the orders are all divisible by 2. This is because the center of $\mathrm{SU}(2)$ is $\mathbb{Z}_2$.

B.1.2 | Irreducible representations

- $\mathbb{Z}_n$ has n irreducible representations. They are all 1-dimensional (since $\mathbb{Z}_n$ is abelian) and are given by

$$\rho_k(g) = \zeta_n^k \tag{B.6}$$

  with $k = 0, \ldots, n-1$.

- $2\mathcal{D}_n$ has $n+3$ irreducible representations: 4 of dimension 1 and $n-1$ of dimension 2. The 1-dimensional ones are given by

n	$\rho(A)$	$\rho(B)$	$\rho(B^b A^a)$
even	1	1	1
		-1	$(-1)^b$
	-1	1	$(-1)^a$
		-1	$(-1)^{a+b}$
odd	1	1	1
		-1	$(-1)^b$
	-1	i	$(-1)^a i^b$
		$-i$	$(-1)^a(-i)^b$

and the 2-dimensional ones are given binary by

$$\rho_r(A) = \begin{bmatrix} e^{i\frac{\pi}{n}r} & 0 \\ 0 & e^{-i\frac{\pi}{n}r} \end{bmatrix}$$
$$\rho_r(B) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with $r = 1, \ldots, n-1$.

B.1.3 Character tables

conj. class repr.	e	M	M^2	$\ldots$	M^{n-1}
conj. class order	1	1	1	$\ldots$	1
V_0	1	1	1	$\ldots$	1
V_1	1	ζ_n	ζ_n^2	$\ldots$	ζ_n^{n-1}
V_2	1	ζ_n^2	ζ_n^4	$\ldots$	$\zeta_n^{2(n-1)}$
V_3	1	ζ_n^3	ζ_n^6	$\ldots$	$\zeta_n^{3(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
V_{n-1}	1	$\zeta_n^{(n-1)}$	$\zeta_n^{2(n-1)}$	$\ldots$	$\zeta_n^{(n-1)^2}$
W	2	$2\cos\left(\frac{2\pi}{n}\right)$	$2\cos\left(\frac{4\pi}{n}\right)$	$\ldots$	$2\cos\left(\frac{2\pi(n-1)}{n}\right)$

Table 1: Character table of $\mathbb{Z}_n$.

conj. class repr.	e	B^2	B	BA	A	A^2	$\ldots$	A^{n-1}
conj. class order	1	1	n	n	2	2	$\ldots$	2
V_0	1	1	1	1	1	1	$\ldots$	1
V_1	1	1	-1	-1	1	1	$\ldots$	1
V_2	1	1 ou -1	1 ou i	-1 ou $-i$	-1	1	$\ldots$	$(-1)^{n-1}$
V_3	1	1 ou -1	-1 ou $-i$	1 ou i	-1	1	$\ldots$	$(-1)^{n-1}$
V_4	2	-2	0	0	$2\cos\frac{\pi}{n}$	$2\cos\frac{2\pi}{n}$	$\ldots$	$2\cos\frac{(n-1)\pi}{n}$
V_5	2	2	0	0	$2\cos\frac{2\pi}{n}$	$2\cos\frac{4\pi}{n}$	$\ldots$	$2\cos\frac{2(n-1)\pi}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
V_{n+2}	2	$2(-1)^{n-1}$	0	0	$2\cos\frac{(n-1)\pi}{n}$	$2\cos\frac{2(n-1)\pi}{n}$	$\ldots$	$2\cos\frac{(n-1)^2\pi}{n}$
W	2	-2	0	0	$2\cos\left(\frac{\pi}{n}\right)$	$2\cos\left(2\frac{\pi}{n}\right)$	$\ldots$	$2\cos\left(\frac{\pi}{n}(n-1)\right)$

Table 2: Character table of $2\mathcal{D}_n$.

conj. class repr.	e	B^2	B	C	C^2	C^4	C^5
conj. class order	1	1	6	4	4	4	4
V_0	1	1	1	1	1	1	1
V_1	2	-2	0	1	-1	-1	1
V_2	3	3	-1	0	0	0	0
V_3	2	-2	0	$e^{i\frac{2\pi}{3}}$	$-e^{i\frac{2\pi}{3}}$	$-e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$
$V_3^\vee$	2	-2	0	$e^{i\frac{4\pi}{3}}$	$-e^{i\frac{4\pi}{3}}$	$-e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$
V_4	1	1	1	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$
$V_4^\vee$	1	1	1	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$
W	2	-2	0	1	-1	-1	1

Table 3: Character table of $2\mathcal{T}$.

conj. class repr.	e	B^2	B	C	C^2	D	BD	D^3
conj. class order	1	1	6	8	8	6	12	6
V_0	1	1	1	1	1	1	1	1
V_1	2	-2	0	1	-1	$-\sqrt{2}$	0	$\sqrt{2}$
V_2	3	3	-1	0	0	1	-1	1
V_3	4	-4	0	-1	1	0	0	0
V_4	3	3	-1	0	0	-1	1	-1
V_5	2	-2	0	1	-1	$\sqrt{2}$	0	$-\sqrt{2}$
V_6	1	1	1	1	1	-1	-1	-1
V_7	2	2	2	-1	-1	0	0	0
W	2	-2	0	1	-1	$-\sqrt{2}$	0	$\sqrt{2}$

Table 4: Character table of $2\mathcal{O}$.

conj. class repr.	e	E^2	E	F	F^2	EF	$(EF)^2$	$(EF)^3$	$(EF)^4$
conj. class order	1	1	30	20	20	12	12	12	12
V_0	1	1	1	1	1	1	1	1	1
V_1	2	-2	0	1	-1	φ^+	$-\varphi^-$	φ^-	$-\varphi^+$
V_2	3	3	-1	0	0	φ^+	φ^-	φ^-	φ^+
V_3	4	-4	0	-1	1	1	-1	1	-1
V_4	5	5	1	-1	-1	0	0	0	0
V_5	6	-6	0	0	0	-1	1	-1	1
V_6	4	4	0	1	1	-1	-1	-1	-1
V_7	2	-2	0	1	-1	φ^-	$-\varphi^+$	φ^+	$-\varphi^-$
V_8	3	3	-1	0	0	φ^-	φ^+	φ^+	φ^-
W	2	-2	0	1	-1	φ^+	$-\varphi^-$	φ^-	$-\varphi^+$

Table 5: Character table of $2\mathcal{I}$, with $\varphi^\pm \equiv (1 \pm \sqrt{5})/2$.

B.2 | Finite subgroups of SU(3)

The finite subgroups of SU(3) are

- the finite subgroups of SU(2)
- $\Delta(3n^2) = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$ and $\Delta(3n^2) = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S^3$
- the exceptional groups

so there are 2 infinite series and 5 exceptional subgroups. Note that they are all divisible by 3 because the center of SU(3) is $\mathbb{Z}_3$.

Theorem. Every abelian finite subgroup of SU(3) is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$.

B.3 | Finite subgroups of SU(4)

See [3].

C | Spacetime geometry: ALE space and orbifolds

Asymptotically locally euclidean (ALE) spaces are a particularly interresting choice of string background to probe with branes for mainly four reasons

(i) they are the resolution (blow-ups) of orbifolds

(ii) there are completely classified: they fall in the ADE classification

(iii) they only break half of the supersymmetry

(iv) they are non-compact therefore we can study them for self-dual type II theory. Why is that ?

Mathematically, an ALE space is complete riemannian n-manifold M such that there exists a compact set $K \subset M$ such that $M \backslash K$ is diffeomorphic to $(\mathbb{R}^n \backslash B_0(R))/G$, where $R \in \mathbb{R}_0^+$ is a radius and $G \subset \mathrm{O}(n)$ a subgroup. Additionally, it is asked that the pulled back metric on $\mathbb{R}^n \backslash B_0(R)$ tends to the euclidean flat metric at infinity.

If one considers string theory an the orbifold $\mathbb{R}^4/\Gamma$ where Γ is a finite sub group of SU(2), massless states appear from the twisted sector. They are precisely the moduli needed the deform the theory to the one with smooth spacetime, i.e. the resolution of the orbifold. In that sense, is said that the strings know about the metric ALE space and that it is said that strings resolve the singularity. The metric of the ALE space can be recovered if the lagrangian of the resulting field theory is explicitely know, such as for the Wess-Zumino-Witten model. However, it is often not the case.

D | Elliptic fibrations

An *elliptic curve* over the complex is a connected riemann surface (connected compact 1-dimensional complex manifold) of genus 1. In other words, it is a complex torus equipped with the structure of a complex manifold, or equivalently with a conformal structure. A complex torus can be defined from a complex number τ, called the *period*, as the quotient $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$. The period characterizes the shape of the torus and, by convention, τ is restricted to the upper-half complex plane $H = \{\tau \in \mathbb{C} | \operatorname{Im}(\tau) > 0\}$. One can see that any complex torus $\mathbb{C}/(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})$ can be put in this form. Even restricted to the space H, the period can still give equivalent tori for different values. In fact, one can show that two tori of periods τ_1 and τ_2 respectively are equivalent if and only their periods are related in the following way:

$$\tau_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \tau 2 \equiv \frac{a\tau + b}{c\tau + d}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{C}). \tag{D.1}$$

These transformations are called *modular transformations*. We see that the matrix of a modular transformation and minus this matrix gives the same transformation of the period therefore the group of modular transformations, i.e. the *modular group*, is not $\mathrm{SL}(2,\mathbb{C})$ but $\mathrm{SL}(2,\mathbb{C})/\{\pm I_2\}$. The groups $\mathrm{SL}(2,\mathbb{C})$ is generated by

$$S \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and } T \equiv \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{D.2}$$

that acts on the period as

$$S \cdot \tau : -\frac{1}{\tau}, \qquad T \cdot \tau = \tau + 1. \tag{D.3}$$

Elliptic curves can be described in a very natural way as a cubic curve in $\mathbb{P}^2$. More precisely, one can construct, using the Weierstrass $\wp$-function[1], an analytic isomorphism between the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and and the following cubic in $\mathbb{P}^2$:

$$E : zy^2 = x^3 + fxz^2 + gz^3. \tag{D.4}$$

For a regular curve E, there is a unique lattice $\mathbb{Z}+\tau\mathbb{Z}$ (up to modular transformations) such that E and $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ are analytic isomorphic as complex Lie groups (through the same map). It is possible to fully classify the elliptic curves by introducing Klein j-invariant, modular curves and modular invariants, which we won't detail here.

An *elliptic fibration* is a bundle of elliptic curves, possibly including singular fibers. These singular fibers have fit in the ADE classification. Over the base space $\mathbb{P}^1$ parametrized by τ, there is a countably infinite number of singularities at $\operatorname{Im}\tau = +\infty$ and each such singularity is of type A or D. For finite values of τ however there is a finite number of singularities. Namely, there are seven singularities: $A_0, A_1, A_2, D_4, E_6, E_7, E_8$. When z approaches one of the singularities, the torus fibre degenerates in a specific way that depends on the type of the singularity. For example, for a type A_n singularity, the torus is pinched in $n+1$ places so as to become a necklace of $n+1$ 2-spheres that intersects at those pinched points. The resulting singular surface is then described by

$$y^2 = x^3 + f(z)x + g(z) \tag{D.5}$$

for some specific polynomials f and g.

D.1 | Connection to Physics: Seiberg-Witten theory

Due to the fact that the period τ of the torus can be interpreted as the coupling constant of a supersymmetric gauge theory, it is possible to make very fruitful links with physics. The starting idea is to associate to any singularity discussed above an $\mathcal{N}=2$ SYM theory with the global symmetry group of corresponding type. That is, if the singularity is of type A, then the global symmetry group is $\mathrm{SU}(n)$ because the Dynkin diagram of its Lie algebra corresponds to type singularities through the geometric McKay correspondence. The most interesting theories are the strongly coupled ones, i.e. the one corresponding to one of the seven singularities at finite distance in τ-space. Indeed, $\tau = \frac{i}{g^2} + \theta$ so the limit $\operatorname{Im}\tau \to +\infty$ is the weak coupling limit $g \to 0$.

As an example, we can consider the original Saiberg-Witten theory, i.e. $\mathcal{N}=2$ SYM with $\mathrm{SU}(2)$ gauge group, $\mathrm{SO}(8,\mathbb{C})$ global symmetry and four hypermultiplets. $\mathrm{SO}(8)$ corresponds to a singularity of type $\mathcal{D}_4$. The name *Seiberg-Witten theory* can be generalized to include all aforementioned strongly coupled ADE theories.

E | Determinantal varieties as transverse spaces

E.1 | Basic properties of determinantal varieties

A *determinantal variety* (DV) is a space of matrices with a given upper bound on their ranks. More precisely, given m, n and $r < \min(m,n)$, the DV Y_r of the field K is the set of $m \times n$ matrices over K with rank lower or equal to r:

$$Y_r \equiv \{M \in M_{m\times n}(K) | \operatorname{rank} M \leq r\}. \tag{E.1}$$

Recall that a k-minor is the determinant of a $k \times k$ sub-matrix and that the rank of a matrix is equal to the biggest integer such that there is a non-vanishing minor of that size. Imposing $\operatorname{rank} M \leq r$ is therefore equivalent to the vanishing of its $(r+1)\times(r+1)$ minors, as it also implies tha vanishing of the biggest minors. This naturally qualifies R_r as affine varieties embedded in K^{mn}.

[1] The Weierstrass $\wp$-function $\wp(z,\tau) = \frac{1}{z^3} + \sum_{w\in(\mathbb{Z}+\tau\mathbb{Z})\setminus\{0\}} \left(\frac{1}{(w-z)^2} - \frac{1}{w^2}\right)$ is meromorphic, has double poles at the lattice points $w \in \mathbb{Z}+\tau\mathbb{Z}$, is doubly-periodic: $\wp(z+1,\tau) = \wp(z,\tau)$, $\wp(z+\tau,\tau) = \wp(z,\tau)$ and its derivative $\wp'$ (with respect to z) has a pole of order 3 at the origin. They satisfy the Weierstrass equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ where $g_2(\tau) = 60G_4(\tau)$ and $g_3(\tau) = 140G_6(\tau)$ are defined in terms of the Einstein series $G_{2k}(\tau) = \sum_{w\in(\mathbb{Z}+\tau\mathbb{Z})\setminus\{0\}} w^{-2\tau}$.

Let us denote by $X = (x_{ij})$ an arbitrary $m \times n$ matrix. The independent entries x_{ij} are affine coordinates. The $(r+1) \times (r+1)$ minors are therefore homogeneous polynomials of degree $r+1$. The *determinantal ideal* $I_{r+1}(X)$ is the ideal of $k[X]$ generated by these polynomials. The cooridnate ring is

$$R = k[X]/I_{r+1}(X) \tag{E.2}$$

Homogeneity the polynomials implies that Y_r can equivalently be seen as a projective variety in $\mathbb{A}^{mn-1}$.

E.1.1 | Computing the dimension

Let us compute the dimension of Y_r seen as an affine variety. We consider the space $\mathbb{A}^{mn} \times \mathbf{Gr}(r, m)$, where $\mathbf{Gr}(r, m)$ is the Grassmannian of r-planes in an m-dimensioanl vector space. Let us define the subsapce

$$Z_r \equiv \{(A, W) | Ax \in W \text{ for all } x \in \mathbb{A}^n\}. \tag{E.3}$$

Y_r and Z_r are birationaly equivalent so $\dim Y_r = \dim Z_r$. We want to compute Z_r. First we notice that Z_r is a vector bundle over $\mathbf{Gr}(r, m)$ and we denote it by $Z_r \xrightarrow{\pi_1} \mathbf{Gr}(r, m)$. Now, over the Grassmannian $\mathbf{Gr}(r, m)$, there is a tautologial vector bundle that we denote by $E_{\mathbf{Gr}} \xrightarrow{\pi_2} \mathbf{Gr}(r, m)$ whose fibers are $\pi_2^{-1}(W) = W \cong \mathbb{R}^r$. Finally, K^m can also be seen as a vector bundle, with fibers $\mathbb{R}^m$. We denote it by $E_{K^n} \xrightarrow{\pi_3} K^n$. From $E_{\mathbf{Gr}}$ and E_{K^n}, we can construct[2] the vector bundle $\mathrm{Hom}(E_{\mathbf{Gr}}, E_{K^n}) \xrightarrow{\pi_4} \mathbf{Gr}(r, m)$. This vector bundle has the same base space and its fibers are $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^r)$ which are exactly the same as the ones of Z_r. So the two vector bundles are isomorphic:

$$Z_r \cong \mathrm{Hom}(E_{\mathbf{Gr}}, E_{K^n}). \tag{E.4}$$

Finally, since the fibers of $\mathrm{Hom}(K^n, E_{\mathbf{Gr}})$ have dimension nr, we find

$$\dim Z_r = \dim \mathrm{Hom}(K^n, E_{\mathbf{Gr}}) = \dim \mathbf{Gr}(r, m) + nr = r(m-r) + nr. \tag{E.5}$$

Finally, we conclude that Y_r is a affine variety of dimension $r(m-r) + nr$.

m	n	r	$\dim_{\mathbb{C}} Y_r$
2	2	1	3
3	2	1	4
3	3	1	5
3	3	2	8
4	2	1	5
4	3	1	6
4	3	2	10
4	4	1	7
4	4	2	12
4	4	3	15

E.1.2 | Singularity

Determinantal varieties are singular and possess non-commutative resolutions. Y_r is singlar and the singular locus is contained in the subset of matrices with rank strictly lower than r. Z_r is a resolution (over the open set of matrices with rank exactly r, this map is an isomorphism), it is called the *Springer desingularization* of $\mathrm{Spec} R$.

E.1.3 | Action and syzygies

Y_r naturally acts on $G = \mathrm{GL}(m, K) \times \mathrm{GL}(n, K)$

[2] Recall that if E and F are vector bundles over X, then we can construct a new vetor bundle over X, called the Hom-bundle and denoted $\mathrm{Hom}(E, F)$, by defining the fiber over $x \in X$ to be $\mathrm{Hom}(E_x, F_x)$.

E.2 | Young's lattice

Young's lattice is a lattice Y formed by all integer partitions ordered by inclusion of their Young tableau. It is generally used to to describe the irreducible representation sof the symmetric group[3] S_n together with their branching properties. Conventionnally, Young's lattice is depicted in a Hasse diagram, i.e. with element of the same rank shown at the same height and with links such that the descendance of two elements is the union and the parent is the intersection.

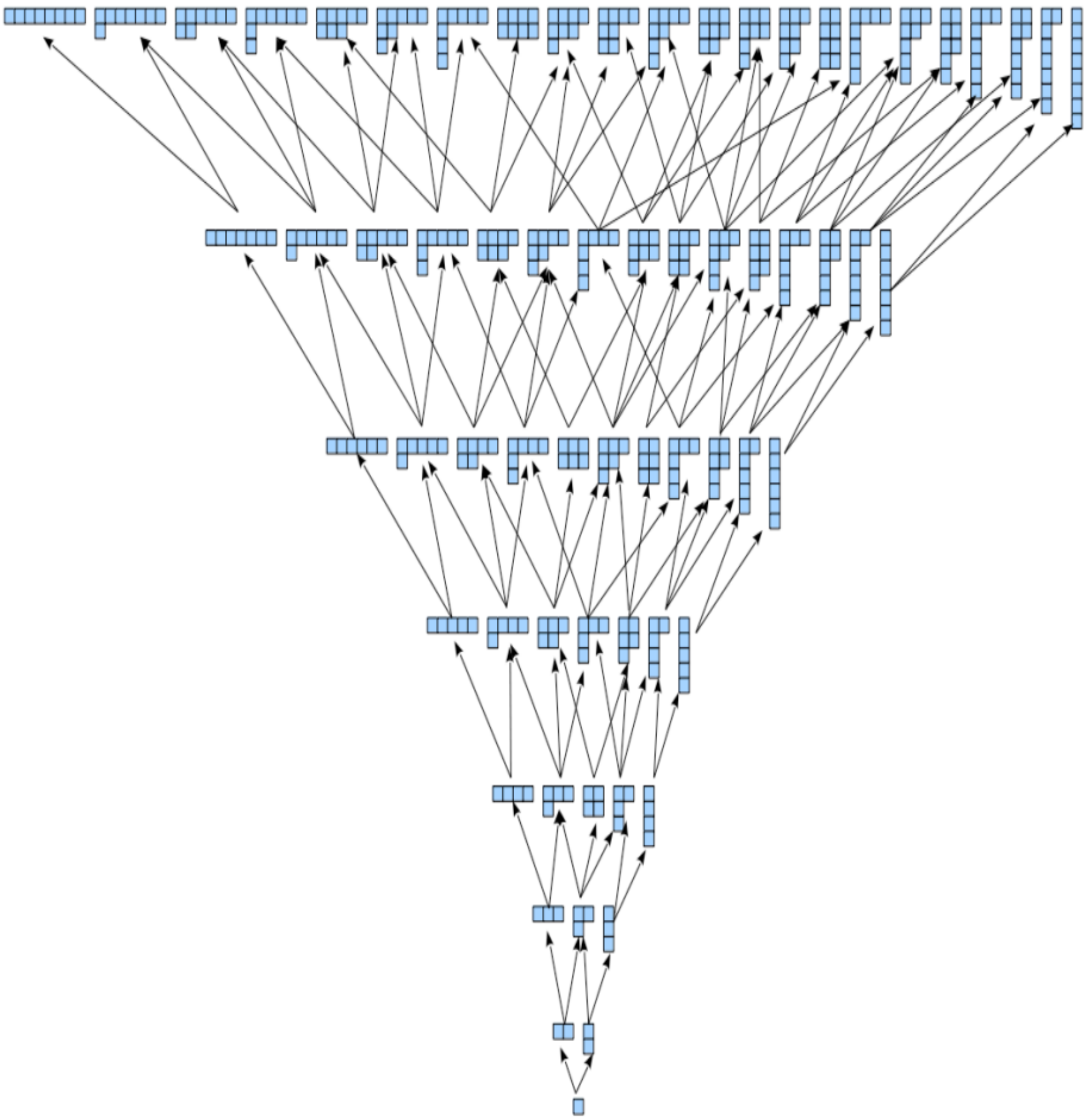


Figure 3: Young's lattice.

Young's lattice possess the folling symmetry: the partition $n + n - 1 + \ldots 2 + 1$ of the nth triangular number has a Young diagram that looks like a staircase. If we now only keep the elements whose hull is contained in this staircase, we get a subset of Young's lattice. When rank-embedded, this subset clearly has the expected bilateral symmetry of Young's lattice but also a rotational symmetry, which appear more clearly if we move away from this rank-embedding. The rotation group of order $n + 1$ acts on this poset[4]. Since it has both a bilateral and a rotational symmetry it must also have a dihedral symmetry

[3]Two permutations of S_n are equivalent if and only they have they have the same number of cycles of the same sizes. Therefore, the quivalence classes of the symmetric group S_n are parametrized by the partitions of n, i.e. by Young diagrams.

[4]Partially ordered set.

and, indeed, the dihedral group $\mathcal{D}_{n+1}$ acts faithfully on this set.

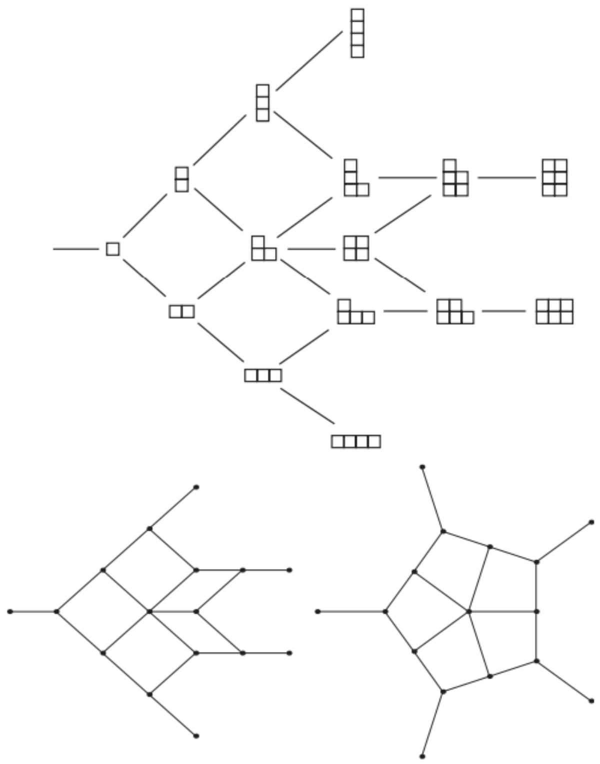


Figure 4: Example of dihedral symmetry for $n = 4$.

F | Some derivations

F.1 | Invariant configurations for $\mathbb{C} \times \mathbb{C}^2/2\mathcal{D}_n$

F.1.1 | Gauge field

To find the invariant configurations of the gauge group, we use use the bi-index notation and split the sub-blocks of A_μ in four categories depending on the dimensionality of the representations that they transform in. Note that it is only necessary to check the invariance under the two generators of $2\mathcal{D}_n$ to ensure invariance under the whole group.

- components $A_{\mu;a\alpha_a,b\beta_b}$ are 1×1 blocks that transform as $A_{\mu;a\alpha_a,b\beta_b} \mapsto \sigma_a(\gamma)A_{\mu;a\alpha_a,b\beta_b}\sigma_b(\gamma)^{-1}$. It follows that only the component with $a, b = 0, 1$ or $a, b = 2, 3$ can be non-zero to have invariance under A. For invariance under B, we find that only the component with $a, b = 0, 2$ or $a, b = 1, 3$ can be non-zero if n is even and only the component with $a = b$ if n is odd. In conclusion, the invarint configuration under A and B are of the form

$$(A_{\mu;ab}) = \begin{bmatrix} \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \end{bmatrix} \tag{F.1}$$

regardless of the parity of n.

- components $A_{\mu;a\alpha_a,r\beta_r}$ are 1×2 blocks that transform as $A_{\mu;a\alpha_a,r\beta_r} \mapsto \sigma_a(\gamma)A_{\mu;a\alpha_a,r\beta_r}\mu_r(\gamma)^{-1}$. More explicitely, each block is of the form $\begin{bmatrix} x_1 & x_2 \end{bmatrix}$ and transforms as

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \mapsto \sigma_a(A) \begin{bmatrix} x_1\zeta_{2n}^{-r} & x_2\zeta_{2n}^{r} \end{bmatrix}$$

under the generator A. This never invariant unless $a = b = 0$. There is no need to check the invariance under B since all these component are already all zero.

- components $A_{\mu;r\alpha_r,a\beta_a}$ are 2×1. The situation is exactly the samme as in the previous point: they must all vanish.
- components $A_{\mu;r\alpha_r,s\beta_s}$ are 2×2 blocks that transform as $A_{\mu;r\alpha_r,s\beta_s} \mapsto \mu_r(\gamma) A_{\mu;r\alpha_r,s\beta_s} \mu_s(\gamma)^{-1}$. Generically speaking, an invariant block under A must satisfy

$$\begin{bmatrix} \zeta_{2n}^r & 0 \\ 0 & \zeta_{2n}^{-r} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} \zeta_{2n}^{-s} & 0 \\ 0 & \zeta_{2n}^{s} \end{bmatrix} = \begin{bmatrix} x_1 \zeta_{2n}^{r-s} & x_2 \zeta_{2n}^{r+s} \\ x_3 \zeta_{2n}^{-r-s} & x_4 \zeta_{2n}^{-r+s} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}. \tag{F.2}$$

  There are two possibilities to have non-vanishing component: $\zeta_{2n}^{r-s} = 1$ and $x_2 = x_3 = 0$ or $\zeta_{2n}^{r+s} = 1$ and $x_1 = x_4 = 0$ but the latter is actually not possible since $r, s = 1, \ldots, n-1$. To we find that the blocks must be of the form

$$A_{\mu;r\alpha_r,s\beta_s} = \begin{bmatrix} \times & 0 \\ 0 & \times \end{bmatrix} \tag{F.3}$$

  if $r = s$ and vanishing otherwise. For invariance under B, a blocks must satisfy

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -x_4 & x_3 \\ x_2 & -x_1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}. \tag{F.4}$$

  which is only possible if $x_1 - x_4$ and $x_2 = x_3$. Finally, we find that invariance under A and B imposes the block to be of the form

$$A_{\mu;r\alpha_r,s\beta_s} = \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \tag{F.5}$$

  if $r = s$ and vanishing otherwise.

The invariant gauge field configurations were found to be of the form

$$A_\mu = \begin{bmatrix} \times & 0 & 0 & 0 & & & & & & \cdots & 0 \\ 0 & \times & 0 & 0 & & & & & & & \\ 0 & 0 & \times & 0 & & & & & & & \\ 0 & 0 & 0 & \times & & & & & & & \\ & & & & x_1 & 0 & & & & & \\ & & & & 0 & -x_1 & & & & & \\ & & & & & & x_2 & 0 & & & \\ & & & & & & 0 & -x_2 & & & \\ \vdots & & & & & & & & \ddots & & \vdots \\ & & & & & & & & & x_{n-1} & 0 \\ 0 & & & & & & & & & 0 & -x_{n-1} \end{bmatrix} \tag{F.6}$$

where each entry (i, j) is an arbitrary block of size $N_i \times N_j$.

Scalar fields

For the real scalar fields X^m, we need the action of $2\mathcal{D}_n$ on $\mathbb{C}^3$:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \xmapsto{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_{2n} & 0 \\ 0 & 0 & \zeta_{2n}^{-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \qquad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \xmapsto{B} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}. \tag{F.7}$$

The partitionning of X^m is similar to A_μ. The additional difficulty is come from R-symmetry. Since it acts differently on the different components, we have the study themalmost one by one.

- the fields X^0 and X^1 are left untouched by R-symmetry, meaning that the invariant configurations have the same form than the gauge field, i.e. (F.6).
- $X^{2,3}_{a\alpha_a,b\beta_b}$ transforms under A as $X^{2,3}_{a\alpha_a,b\beta_b} \mapsto \xi_{2n} \sigma_a(A) X^{2,3}_{a\alpha_a,b\beta_b} \sigma_b(A)^{-1}$. The only configurations that are left invariant are therefore the ones such that $\xi_{2n} \sigma_a(A) \sigma_b(A)^{-1} = 1$, which is never the case. So $X^{2,3}_{a\alpha_a,b\beta_b} = 0$ for all $a, b = 0, \ldots, 3$.

- $X^{2,3}_{a\alpha_a,k\beta_k}$ transforms under A as $X^{2,3}_{a\alpha_a,k\beta_k}\mapsto \xi_{2n}\sigma_a(A)X^{2,3}_{a\alpha_a,k\beta_k}\mu_k(A)^{-1}$. More explicitely, if we denote a block$X^{2,3}_{a\alpha_a,k\beta_k}$ by $\begin{bmatrix}x_1 & x_2\end{bmatrix}$, we get

$$\begin{bmatrix}\xi_{2n}^{k+1}\sigma_a(A)x_1 & \xi_{2n}^{-k+1}\sigma_a(A)x_2\end{bmatrix}=\begin{bmatrix}x_1 & x_2\end{bmatrix} \tag{F.8}$$

therefore we can have $x_1\neq 0$ iff $\sigma_a(A)=-1$(i.e. $a=2,3$) and $k=n-1$, and we can have $x_1\neq 0$ iff $\sigma_a(A)=1$(i.e. $a=0,1$) and $k=1$.

- $X^{2,3}_{k\alpha_k,b\beta_b}$ transforms under A as $X^{2,3}_{k\alpha_k,b\beta_b}\mapsto \xi_{2n}\mu_k(A)X^{2,3}_{k\alpha_k,b\beta_b}\sigma_a(A)^{-1}$. Similarly to the previous case, we can write the blocks $X^{2,3}_{k\alpha_k,a\beta_a}$ as $\begin{bmatrix}x_1\\ x_2\end{bmatrix}$, we get

$$\begin{bmatrix}\xi_{2n}^{k+1}\sigma_a(A)x_1\\ \xi_{2n}^{-k+1}\sigma_a(A)x_2\end{bmatrix}=\begin{bmatrix}x_1\\ x_2\end{bmatrix} \tag{F.9}$$

therefore the conditions are exactly the same: we can have $x_1\neq 0$ iff $\sigma_a(A)=-1$(i.e. $a=2,3$) and $k=n-1$, and we can have $x_1\neq 0$ iff $\sigma_a(A)=1$(i.e. $a=0,1$) and $k=1$.

- $X^{2,3}_{k\alpha_k,l\beta_l}$ transforms under A as $X^{2,3}_{k\alpha_k,l\beta_l}\mapsto \xi_{2n}\mu_k(A)X^{2,3}_{k\alpha_k,l\beta_l}\mu_l(A)^{-1}$. Again, we can write the blocks $X^{2,3}_{k\alpha_k,l\beta_l}$ as $\begin{bmatrix}x_1 & x_2\\ x_3 & x_4\end{bmatrix}$ and we get

$$\begin{bmatrix}\zeta_{2n}^{k-l+1}x_1 & \zeta_{2n}^{k+l+1}x_2\\ \zeta_{2n}^{-k-l+1}x_3 & \zeta_{2n}^{-k+l+1}x_4\end{bmatrix}=\begin{bmatrix}x_1 & x_2\\ x_3 & x_4\end{bmatrix} \tag{F.10}$$

therefore, we can have

  - $x_1\neq 0$ iff $l=k+1$,
  - $x_2\neq 0$ iff $l=-k-1$ (not possible),
  - $x_3\neq 0$ iff $l=-k+1$ (not possible),
  - $x_4\neq 0$ iff $l=k-1$.

For $X^{4,5}_{i\alpha_i,j\beta_j}$, the reasonning is exactly the same but with the R-symmetry acting as ζ_{2n}^{-1} instead of ζ_{2n}. After similar computations, we get that the components $X^{4,5}_{a\alpha_a,b\beta_b}$ must be all vanishing too and the components $X^{4,5}_{a\alpha_a,k\beta_k}=\begin{bmatrix}x_1 & x_2\end{bmatrix}$ can have $x_1\neq 0$ iff $a=0,1$ and $k=1$ and $x_2\neq 0$ iff $a=2,3$ and $k=n-1$. The same goes for the components $X^{4,5}_{k\alpha_k,b\beta_b}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ and, at last, for the components $X^{4,5}_{k\alpha_k,l\beta_l}=\begin{bmatrix}x_1 & x_2\\ x_3 & x_4\end{bmatrix}$, we find

  - $x_1\neq 0$ iff $l=k-1$,
  - $x_2\neq 0$ iff $l=-k+1$ (not possible),
  - $x_3\neq 0$ iff $l=-k-1$ (not possible),
  - $x_4\neq 0$ iff $l=k+1$.

We have established what configurations are invariant under the generator A, equivalently under the subgroup of $2\mathcal{D}_n$ generated by A. What about B? The action of R-symmetry for B is more tiresome because it is not diagonal, see (F.7). This implies that components get exchanged. More precisely, recall our notations $z_1=X^0+iX^1$, etc, if we rewrite (F.7) in terms of real components, we get that

$$\begin{bmatrix}X^0\\ X^1\\ X^2\\ X^3\\ X^4\\ X^5\end{bmatrix}\overset{B}{\longmapsto}\begin{bmatrix}X^0\\ X^1\\ -X^5\\ X^4\\ -X^3\\ X^2\end{bmatrix}. \tag{F.11}$$

For the components $X^2_{a\alpha_a,b\beta_b}$, this implies that $X^2_{a\alpha_a,b\beta_b} = -\sigma_a(B)\sigma_b(B)^{-1}X^5_{a\alpha_a,b\beta_b}$. This completely fixes $X^5_{a\alpha_a,b\beta_b}$ in terms of $X^2_{a\alpha_a,b\beta_b}$. The can be done the other components of X^2, they we find that they all determine the ones of X^5. Without fully splitting each fields into components, we see that we must have

$$X^2_{ij} = -\rho_i(B)X^5_{ij}\rho_j(B)^{-1}, \tag{F.12}$$
$$X^3_{ij} = \rho_i(B)X^4_{ij}\rho_j(B)^{-1}, \tag{F.13}$$
$$X^4_{ij} = -\rho_i(B)X^3_{ij}\rho_j(B)^{-1}, \tag{F.14}$$
$$X^5_{ij} = \rho_i(B)X^2_{ij}\rho_j(B)^{-1}, \tag{F.15}$$
$$\tag{F.16}$$

bto have invariance under B. This equations imply in particular that $X^2_{kl} = -\rho_k(B^2)X^2_{kl}\rho_l(B^2)^{-1}$. Since, $\rho_k(B^2) = -\mathbb{1}_{2\times 2}$ for every k, we get that all components X^2_{kl} must be vanishing. In turn, this implies the components $X^{3,4,5}_{kl}$ must also all vanish. This cannot be true.

F.2

We want to compute the sum

$$\sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} + 1 \right\rfloor = \left\lfloor \frac{n}{3} \right\rfloor + \sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} \right\rfloor. \tag{F.17}$$

Let us write $n \in \mathbb{N}$ as $n = 3m + r$ with $r = 0, 1$ or 2 and $m \in \mathbb{N}$. Regardless of r, we have $\lfloor n/3 \rfloor = m$ and

$$\sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} \right\rfloor = \sum_{a=1}^{m} \left\lfloor \frac{3}{2}(m-a) + \frac{r}{2} \right\rfloor = \sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + \frac{r}{2} \right\rfloor. \tag{F.18}$$

- if $r = 0$, then (F.18) becomes

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a \right\rfloor = \sum_{a=0}^{m-1} a + \sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = \frac{(m-1)m}{2} + \sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor. \tag{F.19}$$

  Now if m is even, we have

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = 2 \sum_{a=0}^{\lfloor \frac{m-1}{2} \rfloor} a = 2 \sum_{a=0}^{\frac{m}{2}-1} a = \left(\frac{m}{2} - 1\right) \frac{m}{2} \tag{F.20}$$

  and if m is odd,

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = 2 \sum_{a=0}^{\lfloor \frac{m-2}{2} \rfloor} a + \left\lfloor \frac{m-1}{2} \right\rfloor = 2 \sum_{a=0}^{\frac{m-3}{2}} a + \frac{m-1}{2} = \frac{(m-1)^2}{4} \tag{F.21}$$

  so

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = \begin{cases} \left(\frac{m}{2} - 1\right) \frac{m}{2}, & \text{if } m \text{ is even} \\ \frac{(m-1)^2}{4}, & \text{if } m \text{ is odd} \end{cases}. \tag{F.22}$$

  and

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3a}{2} \right\rfloor = \begin{cases} \frac{m(3m-4)}{4}, & \text{if } m \text{ is even} \\ \frac{(m-1)(3m-1)}{4}, & \text{if } m \text{ is odd} \end{cases}. \tag{F.23}$$

- if $r = 1$, then (F.18) becomes

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + \frac{1}{2} \right\rfloor = \sum_{a=0}^{m-1} a + \sum_{a=0}^{m-1} \left\lfloor \frac{a+1}{2} \right\rfloor = \frac{(m-1)m}{2} + \sum_{a=0}^{m-1} \left\lfloor \frac{a+1}{2} \right\rfloor \tag{F.24}$$

and
$$\sum_{a=0}^{m-1}\left\lfloor\frac{a+1}{2}\right\rfloor=\sum_{a=1}^{m}\left\lfloor\frac{a}{2}\right\rfloor=\sum_{a=0}^{m}\left\lfloor\frac{a}{2}\right\rfloor=\begin{cases}\frac{m^2}{4}, & \text{if } m \text{ is even}\\ \frac{m^2-1}{4}, & \text{if } m \text{ is odd}\end{cases} \tag{F.25}$$
by (F.22) so
$$\sum_{a=0}^{m-1}\left\lfloor\frac{3}{2}a+\frac{1}{2}\right\rfloor=\begin{cases}\frac{m(3m-2)}{4}, & \text{if } m \text{ is even}\\ \frac{3m^2-2m-1}{4}, & \text{if } m \text{ is odd}\end{cases} \tag{F.26}$$

- if $r=2$, then (F.18) becomes
$$\sum_{a=0}^{m-1}\left\lfloor\frac{3}{2}a+1\right\rfloor=m+\sum_{a=0}^{m-1}\left\lfloor\frac{3}{2}a\right\rfloor. \tag{F.27}$$
so
$$\sum_{a=0}^{m-1}\left\lfloor\frac{3}{2}a+1\right\rfloor=\begin{cases}\frac{3m^2}{4}, & \text{if } m \text{ is even}\\ \frac{3m^2+1}{4}, & \text{if } m \text{ is odd}\end{cases} \tag{F.28}$$
from (F.23).

Finally, we can write $m=2k$ if m if even and $m=2k+1$ if m is odd in order to distinguish the six different cases. We get

$$a(n)\equiv\sum_{a=1}^{\lfloor n/3\rfloor}\left\lfloor\frac{n-3a}{2}+1\right\rfloor=\begin{cases}2k+\frac{2k(6k-4)}{4}, & \text{if } n=6k\\ 2k+\frac{2k(6k-2)}{4}, & \text{if } n=6k+1,\\ 2k+\frac{12k^2}{4}, & \text{if } n=6k+2,\\ (2k+1)+\frac{2k(6k+2)}{4}, & \text{if } n=6k+3,\\ (2k+1)+\frac{3(2k+1)^2-2(2k+1)-1}{4}, & \text{if } n=6k+4,\\ (2k+1)+\frac{3(2k+1)^2+1}{4}, & \text{if } n=6k+5\end{cases} \tag{F.29}$$
$$=\begin{cases}3k^2, & \text{if } n=6k\\ 3k^2+k, & \text{if } n=6k+1,\\ 3k^2+2k, & \text{if } n=6k+2,\\ 3k^2+3k+1, & \text{if } n=6k+3,\\ 3k^2+4k+1, & \text{if } n=6k+4,\\ 3k^2+5k+2, & \text{if } n=6k+5\end{cases}. \tag{F.30}$$

Starting from $n=1$, the first value of this sequence is : $0,0,1,1,2,3,4,5,7,8,10,12,\ldots$. Uppon further analysis, this correspond to the sequence A001399, that have several interpretations:

- the number of partitions of n into at most 3 parts. This makes sense with our initial problem: finding all the a,b,c's such that $a+b+c=n$,
- the number of connected graphs with 3 nodes and n edges (where multiple edges between the same nodes are allowed),
- the number of non-negative solutions to $b+2c+3d=n$,

as well as many others. Finally, we note that we can simply write
$$a(n)=\text{round}\left(\frac{n^2}{12}\right). \tag{F.31}$$

G | References guide

- General strings and D-branes: [4],[5],[6]
- He: review: [2], thesis: [7]

- Orbifold construction for $\Gamma \subset \mathrm{SU}(2)$: type A : [8], type D and E: [**PhysRevD.55.6382**]
- Orbifold construction for $\Gamma \subset \mathrm{SU}(3)$:[9]
- Orbifold construction for $\Gamma \subset \mathrm{SU}(4)$:[3]
- Formalization of projection to daughter theories: [10],[11]
- Quivers representations and varieties: [12],[13]
- On toric varieties: [14],[15]([16],[17])
- Forward algorithm for toric singularities devlopments: [18],[19],[20],[21],[22], formalization: [23],[24],[25]
- Toric diagrams, dimer diagrams and Higgsing: [26]
- Fractional branes: [27]
- Formalization of inverse algorithm for toric singularities: [1]
- geometry and K3 surfaces: [28]
- general review of SYM and their brane description: [29]
- Hanany-Witten setup: [30]
- geometric engineering: [31],[32],[33]
- link between graph theory and Yang-Mills (constructed from string theory with the three different methods) to study the finiteness of the theories: [3]

Todo list

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