

# Notes on Quiver Gauge Theories

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## Abstract

In these notes, we present some basic ideas around the large topic of quiver gauge theories. The goal is to reproduce and regroup the basics of quiver gauge theories. Note that this is a draft, it may contain a lot of typos, errors and imprecisions. It is only meant as a work support.

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## Part I

# Preliminary notions

## 1 | Physical setup

### Brane-world paradigm

We consider our world to be a slice in the ten-dimensional spacetime of type II superstring theory, i.e. the worldvolume of a D3-brane. More precisely, we consider a stack of  $N$  D3-branes in order to have  $U(N)$  Chan-Paton factors resulting in a  $U(N)$  gauge group. To have D3-branes, we need to consider type IIB superstring theory. The spacetime is therefore not necessarily  $\mathbb{R}^{1,9}$  but of the more general form

$$M = \mathbb{R}^{1,3} \times M^{(6)}.$$

This is the so-called *brane-world paradigm*.

### Supersymmetry and Calabi-Yau manifolds

Independently from string theory, we can ask for the worldvolume theory to be supersymmetric. We start from type IIB superstring theory which is 10-dimensional and has  $\mathcal{N} = 2$  supersymmetry so it possesses 32 supercharges. As usual, they transform under the minimal spinor representation (MSR) of the bulk Lorentz group, here  $SO(1,9)$ . In ten dimensions this representation is 8-dimensional (complex) which is why there are  $2(8+8) = 32$  supercharges: 8 transforming in the 8-dimensional MSR and 8 transforming in the 8-dimensional conjugate MSR and whole thing times two since  $\mathcal{N} = 2$ . If we compactify type II string theory on  $\mathbb{R}^{1,3} \times M^{(6)}$  where  $M^{(6)}$  is any 6-dimensional manifold, supersymmetry is broken. The reason for this is that the supercharges now have to transform under the MSR of  $SO(1,3) \times \mathcal{H}(M^{(6)})^{(*)}$ , where  $\mathcal{H}(M^{(6)})$  is the holonomy group of  $M^{(6)}$ . Recall that a generic curved 6-dimensional manifold has  $O(6)$  holonomy and  $SO(6)$  if it is orientable. We only consider the later case. The supercharges must therefore transform under the MSR of  $SO(1,3) \times SO(6)$ . The MSR of  $SO(1,3)$  being **2** and the one of  $SO(6)$  being **4**, we conclude that imposing the spacetime to have the form (1) changes the representation under which the supercharges transform as follows:

$$\mathbf{8} \oplus \bar{\mathbf{8}} \rightarrow (\mathbf{2}_L, \mathbf{4}) \oplus (\mathbf{2}_R, \bar{\mathbf{4}}). \quad (1.1)$$

If we stop here, the residual supercharges might be ill-defined since making a tour around a loop in  $M^{(6)}$  could result in a non-trivial rotation. To solve this problem, we need also to be more restrictive on the holonomy. In fact, it is precisely the holonomy of the transverse space that dictates the number of supersymmetries that are left. Let us now consider a four-dimensional field theory resulting from compactification of the transverse six-dimensional space. The number of supercharges that generate supersymmetries for this theory is the number of Killing spinor (covariantly constant spinor): each Killing spinor contracted with the local supersymmetry current generates a residual supersymmetry<sup>(\*)</sup>. Now the link with holonomy: since  $SO(6) \cong SU(4)$ , minimal spinors can be viewed as having four complex components and as transforming under  $SU(4)$ . Indeed, minimal spinors in six dimensions have four complex components. In order to have one covariantly conserved spinor, we look for the biggest subgroup of  $SU(4)$  that leaves a component of the spinor invariant. This is clearly  $\{e\} \times SU(3) \subset SU(4)$  that acts trivially on the first component. The spinor  $(1,0,0,0)$  is then covariantly conserved. Our transverse space must therefore have  $SU(3)$  holonomy such that the parallel transport of the spinor  $(1,0,0,0)$  under any closed loop is a lower  $SU(3)$  rotation. We conclude that if the transverse Calabi-Yau has  $SU(3)$  holonomy, the worldvolume theory has  $\mathcal{N} = 1$  supersymmetry. The same reasoning can be used to obtain that  $SU(2)$  holonomy implies  $\mathcal{N} = 2$  supersymmetry.

To conclude, preserving any degree of supersymmetry constrains the transverse space  $M^{(6)}$  to be compact, complex, Kähler and to have  $G \subset SU(3)$  holonomy. Namely,  $M^{(6)}$  must be a Calabi-Yau threefold and manifolds are given in appendix 6.

### Non-compact transverse space

If we let the worldvolume of the D3-branes carry the requisite gauge theory while the bulk contains gravity, we can relax the compactness condition and study non-compact threefolds. In other words,  $M^{(6)}$  is taken to be Calabi-Yau variety, instead of a Calabi-Yau manifold. A Calabi-Yau variety is an affine variety that locally models a Calabi-Yau manifold, therefore allowing for singularities.

Using a non-compact transverse space can intuitively be understood as a Kaluza-Klein compactification where we take the size of the compact dimensions to infinity. The four-dimensional gravity coupling constant being inversely proportional to this quantity, there is no gravity in this limit. This makes the analysis much simpler and therefore also serves as an argument to ignore gravity in the worldvolume theory. Consequently, we will mostly ignore gravity and not care about the metric of the spacetime, see appendix B for more details.

### Singular transverse space

The only smooth Calabi-Yau threefold is  $\mathbb{C}^3$  so we are lead to consider singular Calabi-Yau varieties. We usually denote  $S \equiv M^{(6)}$  to remind us of the singular aspect. String theory being a theory of extended objects, it is well-defined on such singularities. In a sense that will be clarified later, this singular structure of the geometry requires to “project” the theory. As a result, the gauge group  $U(N)$  will be broken down into products of smaller gauge groups. The simplest examples of singular Calabi-Yau varieties are Calabi-Yau orbifolds. We will mainly be interested these examples.

From the point of view of the orbifold, the D3-brane is a point. Consequently, the D3-branes parametrize the transverse space. This is the first clue of the crucial relationship between the worldvolume theory and the Calabi-Yau singularity. Eventually, we will see that the classical vacuum of the gauge theory should be, in explicit coordinates, the defining equation of  $S$ . This is precisely the opposite of the projection manipulation we mentioned above: recovering the transverse space from the gauge theory.

Projecting and computing the classical vacua are therefore inverse operations with respect to each other. This suggest a bijection between the singular transverse space and the gauge theory: the former can be computed from the latter and vice-versa. This is called “forward algorithm” and “inverse algorithm”. We will of course discuss this in more details.

### Mathematical formulation

Mathematically, this brane-world paradigm is the realization of branes as supports of vector bundles (sheaf). Gauge theories on branes are intimately related to algebraic constructions of stable bundles, i.e. holomorphic or algebraic vector bundles that are stable in the sense of geometric invariant theory. In particular, D-brane gauge theories manifest as a natural description of symplectic quotients and their resolutions in geometric invariant theory.

To summarize in more mathematical terms, our D-branes, together with the stable vector bundle (sheaf) supported thereupon, resolves the transverse Calabi-Yau orbifold which is the vacuum for the gauge theory on the worldvolume as a GIT quotient (\*).

### Summary

We consider  $N$  D3-branes in type IIB superstring theory carrying a  $U(N)$  gauge group. The transverse space  $S$  is taken to be a non-compact singular Calabi-Yau variety.

## 2 | Supersymmetric Yang-Mills theories

### 2.1 | Vacua space of SYM

Let us consider a supersymmetric gauge theory in  $d = 4$  with  $k$  chiral superfields  $\Phi_i$  ( $i = 1, \dots, k$ ) charged under the gauge group  $G$  in an arbitrary representation  $r_i$ , in which the generators of  $G$  are given by  $T_{r_i}^a$

( $a = 1, \dots, \dim G$ ).  $\mathcal{W}_\alpha$  is the gaugina chiral superfield, containing the field strength. The lagrangian density on superspace is given by

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \Phi^\dagger e^{2V} \Phi + \int d^2\theta \left\{ \frac{\tau}{16\pi i} \text{tr}(\mathcal{W}^\alpha \mathcal{W}_\alpha) + W(\Phi) \right\} + \text{c.c.} \quad (2.1)$$

where we ignored the gauge indices. The superpotential  $W(\Phi)$  is a holomorphic polynomial in the  $\Phi_i$ . The space of vacua is the space of configurations of  $\Phi$  such that the  $D$ -terms and the  $F$ -terms vanish:

$$\boxed{\begin{aligned} D^a &\equiv \sum_i \Phi_i^\dagger T_{r_i}^a \Phi^i = 0, \\ F_i^\dagger &\equiv \frac{\partial W}{\partial \Phi^i} = 0. \end{aligned}} \quad (2.2a) \quad (2.2b)$$

It turns out that the full space of vacua of any supersymmetric gauge theory can be described as an algebraic variety.

## 2.2 | SYM from D-branes

The dynamics of D-branes is described by the Dirac-Born-Infeld action

$$S_{\text{DBI}}[X, F] = -\frac{T_p}{g_s} \int d^{p+1}\sigma \sqrt{-\det_{0 \leq a, b \leq p} (\eta_{ab} + \partial_a X^m \partial_b X_m + 2\pi\alpha' F_{ab})}. \quad (2.3)$$

The latter can be expanded for slowly-varying fields, which is equivalent to passing to the field theory limit  $\alpha' \rightarrow 0$ . The resulting action is the action of a  $U(1)$  gauge theory in  $p+1$  dimensions with  $9-p$  real scalar fields. This action is exactly the same than the one we would obtain by dimensionally-reducing a pure  $U(1)$  Yang-Mills gauge theory in 10 spacetime dimensions with the identification

$$g_{\text{YM}} = g_s T_p^{-1} (2\pi\alpha')^{-2} = \frac{g_s}{\sqrt{\alpha'}} (2\pi\sqrt{\alpha'})^{p-1}. \quad (2.4)$$

This construction can be generalized for multiple D-branes. It now results in a non-abelian theory. The general statement is the following:

The low-energy dynamics of  $N$  parallel, coincident  $Dp$ -branes in flat space is described in static gauge by the dimensional reduction to  $p+1$  dimensions of pure 10d  $\mathcal{N} = 1$  supersymmetric Yang-Mills theory with gauge group  $U(N)$  in ten spacetime dimensions.

Recall that the 10-dimensional action is given by

$$S_{\text{YM}} = \frac{1}{4g_{\text{YM}}^2} \int d^{10}x \left[ \text{tr}(F_{\mu\nu} F^{\mu\nu}) + 2i \text{tr}(\bar{\psi} \Gamma^\mu D_\mu \psi) \right], \quad (2.5)$$

where  $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$  is the non-abelian field strength of the  $U(N)$  gauge field  $A_\mu$ ,  $D_\mu = \partial_\mu - i[A_\mu, \psi]$ ,  $\Gamma^\mu$  are  $16 \times 16$  Dirac matrices (\*), and the  $N \times N$  Hermitian fermion field  $\psi$  is a 16-component Majorana-Weyl spinor of the Lorentz group  $SO(1, 9)$  which transforms under the adjoint representation of the gauge group  $U(N)$ . On-shell, there are eight on-shell bosonic, gauge field degrees of freedom, and eight fermionic degrees of freedom, after imposition of the Dirac equation  $\not{D}\psi = \Gamma^\mu D_\mu \psi = 0$ . One can verify that this action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta_\epsilon A_\mu &= \frac{i}{2} \bar{\epsilon} \Gamma_\mu \psi, \\ \delta_\epsilon \psi &= \frac{1}{2} F_{\mu\nu} [\Gamma^\mu, \Gamma^\nu] \epsilon, \end{aligned}$$

where  $\epsilon$  is an Majorana-Weyl spinor.

Using (2.5), we can construct a supersymmetric Yang-Mills gauge theory in  $p + 1$  dimensions with 16 independent supercharges by dimensional reduction: we take all fields to be independent of the coordinates  $X^{p+1}, \dots, X^9$ , then the ten-dimensional gauge field  $A_\mu$  splits into a  $(p + 1)$ -dimensional  $U(N)$  gauge field  $A_a$  plus  $9 - p$  Hermitian scalar fields  $\Phi^m = X^m/2\pi\alpha'$  in the adjoint representation of  $U(N)$ . The  $Dp$ -brane action is thereby obtained from the dimensionality reduced field theory as

$$S_{Dp} = -\frac{T_p g_s (2\pi\alpha')^2}{4} \int d^{p+1}\sigma \operatorname{tr} \left( F_{ab} F^{ab} + 2D_a \Phi^m D^a \Phi_m + \sum_{m \neq n} [\Phi^m, \Phi^n]^2 + \text{fermions} \right) \quad (2.6)$$

where  $a, b = 0, \dots, p$ ,  $m, n = p + 1, \dots, 9$ . We do not explicitly display the fermionic contributions for the moment. In conclusion, the low-energy brane dynamics is described by a supersymmetric Yang-Mills theory on the  $Dp$ -brane worldvolume which is dynamically coupled to the transverse, adjoint scalar fields  $\Phi^m$ .

The scalar potential is given by

$$V(\Phi) = \sum_{m \neq n} [\Phi^m, \Phi^n]^2. \quad (2.7)$$

It is negative definite because  $[\Phi^m, \Phi^n]^\dagger = [\Phi^n, \Phi^m] = -[\Phi^m, \Phi^n]$ . A classical vacuum of the field theory defined by (2.6) corresponds to a static solution of the equations of motion whereby the potential energy of the system is minimized. It is given by the field configurations which solve simultaneously the equations  $F_{ab} = D_a \Phi^m = \psi^a = 0$  and  $V(\Phi) = 0$ . Since all term in  $V(\Phi)$  have the same sign, the equation  $V(\Phi) = 0$  is equivalent to the equation  $[\Phi^m, \Phi^n] = 0$  for all  $m, n$  and at each point in the  $(p + 1)$ -dimensional worldvolume of the branes. This implies that the  $N \times N$  hermitian matrix fields  $\Phi^m$  are simultaneously diagonalizable by a gauge transformation, so that we may write

$$\Phi^m = U \begin{bmatrix} X_1^m & & & 0 \\ & X_2^m & & \\ & & \ddots & \\ 0 & & & X_N^m \end{bmatrix} U^{-1}, \quad (2.8)$$

the matrix  $U$  is independent of  $m$ . The simultaneous, real eigenvalues  $X_i^m$  give the positions of the  $N$  distinct D-branes in the  $m$ -th transverse direction. It follows that the moduli space of classical vacua for the  $(p + 1)$ -dimensional field theory (2.6) is the quotient space  $(\mathbb{R}^{9-p})^N / S_N$ , where the factors of  $\mathbb{R}$  correspond to the positions of the  $N$   $Dp$ -branes in the  $(9 - p)$ -dimensional transverse space, and  $S_N$  is the symmetric group acting by permutations of the  $N$  coordinates  $X_i$ . The group  $S_N$  corresponds to the residual Weyl symmetry of the  $U(N)$  gauge group acting in (2.8). It represents the permutation symmetry of a system of  $N$  *indistinguishable* D-branes.

From (2.6) one can easily deduce that the masses of the fields corresponding to the off-diagonal matrix elements are given precisely by the distances  $|x_i - x_j|$  between the corresponding branes. This description means that an interpretation of the D-brane configuration in terms of classical geometry is only possible in the classical ground state of the system, whereby the matrices  $\Phi^m$  are simultaneously diagonalizable and the positions of the individual D-branes may be described through their spectrum of eigenvalues. This gives a simple and natural dynamical mechanism for the appearance of “non-commutative geometry” at short distances, where the D-branes cease to have well-defined positions according to classical geometry.

The end of this section has to be rewritten (\*).

## 2.3 | $\mathcal{N} = 4$ super Yang-Mills theory in $D = 4$

### 2.3.1 | Superconformal group $SU(2, 2|4)$ and its representations

Conformal transformations and supersymmetries do not commute so the presence of conformal symmetry in addition to  $\mathcal{N} = 4$  supersymmetry leads to an even larger group of symmetry known as the *superconformal group*. In the  $D = 4, \mathcal{N} = 4$  case, the superconformal group is the super group<sup>1</sup>  $SU(2, 2|4)$ . The different component of the latter are

<sup>1</sup>Supermanifold which is also a group with smooth product and inverse maps.



- **Conformal symmetries:** they form the 15-dimensional subgroup  $\text{SO}(2, 4)$  and are generated by  $P_\mu, M_{\mu\nu}, K_\mu$  and  $D$ .
- **R-symmetry:** they form the 15-dimensional subgroup  $\text{SO}(6)_R$  and are generated by  $T^A$  ( $A = 1, \dots, 15$ ).
- **Poincaré supersymmetries:** they form the 16-dimensional sub group (\*) and are generated by  $Q_\alpha^I$  and  $\bar{Q}_{\dot{\alpha}}^I$ .
- **Conformal supersymmetries:** they form the 16-dimensional subgroup (\*) and are generated by  $S_{\alpha I}$  and  $\bar{S}^{\dot{\alpha} I}$ .

Conformal invariance of this theory can be seen as a consequence of the non-renormalization theorems.

### 2.3.2 | Matter content

For  $D = 4, \mathcal{N} = 4$ , there is only one kind of supermultiplet, the vector multiplet. Therefore, from an  $\mathcal{N} = 4$  perspective, the only  $\mathcal{N} = 4$  is a pure SYM. For extended supersymmetry, it is easier to express it in terms of  $\mathcal{N} = 1$  superfield on  $\mathcal{N} = 1$  superspace instead of looking to construct a superspace for  $\mathcal{N} = 4$ . In this case, we can see that the  $\mathcal{N} = 4$  vector superfield can be expressed in terms of  $\mathcal{N} = 1$  representations as one vector supermultiplet and three chiral scalar supermultiplets:

$$[\mathcal{N} = 4 \text{ vector multiplet}] : V = (\lambda_\alpha, A_\mu, D) \oplus \Phi^A = (\phi^A, \psi_\alpha^A, F^A). \quad (2.9)$$

with  $A = 1, 2, 3$  and

$$\phi^A = \phi_a^A T^a, \quad \psi_\alpha^A = \psi_{\alpha,a}^A T^a, \quad F^A = F_a^A T^a, \quad (2.10)$$

$$\lambda^A = \lambda_a^A T^a, \quad A_\mu^A = A_{\mu,a}^A T^a, \quad D^A = F_a^A T^a, \quad (2.11)$$

$$V = V_a T^a, \quad \Phi^A = \Phi_a^A T^a, \quad (2.12)$$

where  $T^a$  ( $a = 1, \dots, \dim G$ ) are the generators of  $\mathfrak{g}$ . The propagating degrees of freedom are therefore a vector field, three complex scalars and four gauginos. The Lagrangian is very much constrained by  $\mathcal{N} = 4$  supersymmetry. First, the chiral superfields  $\Phi^A$  should transform in the adjoint representation of the gauge group  $G$ , since internal symmetries commute with supersymmetry. This means that all fields transform in the adjoint of  $G$ .

Moreover, there is a large R-symmetry group<sup>2</sup>:  $\text{SU}(4)_R$ . The four Weyl fermions transform in the fundamental of  $\text{SU}(4)_R$ , while the six real scalars in the two times anti-symmetric representation, which is nothing but the fundamental representation of  $\text{SO}(6)$ . The auxiliary fields are singlets under the R-symmetry group. Using  $\mathcal{N} = 1$  superfield formalism the Lagrangian reads

$$\begin{aligned} \mathcal{L}_{\text{SYM}}^{\mathcal{N}=4} = & \frac{1}{32\pi} \text{Im} \left( \tau \int d^4x \text{tr}(W^\alpha W_\alpha) \right) + \int d^2\theta d^2\bar{\theta} \text{tr} \sum_{A=1}^3 \bar{\Phi}^A e^{2gV} \Phi^A \\ & - \int d^2\theta \sqrt{2g} \text{tr} \Phi_1 [\Phi_2, \Phi_3] + \text{h.c.} \end{aligned} \quad (2.13)$$

where as usual  $W_\alpha = -\frac{1}{4} \bar{D}\bar{D}(e^{-V} D_\alpha e^V)$  is the gaugino superfield. This lagrangian is indeed invariant under the superPoincaré algebra and under the gauge transformations

$$e^V \rightarrow e^{i\bar{\Lambda}} e^V e^{-i\Lambda} \quad (\text{which implies that } W_\alpha \rightarrow e^{i\Lambda} W_\alpha e^{-i\Lambda}), \quad (2.14)$$

$$\Phi^A \rightarrow e^{i\Lambda} \Phi^A. \quad (2.15)$$

The large  $\text{SU}(4)_R$  R-symmetry group forbids of having a superpotential. The commutator in the third term of (2.13) appears for the same reason as for the  $\mathcal{N} = 2$  Lagrangian. Notice that the choice of a single  $\mathcal{N} = 1$  supersymmetry generator breaks the full  $\text{SU}(4)_R$  R-symmetry to  $\text{SU}(3) \times \text{U}(1)_R$ . The three

<sup>2</sup>The fact that the scalar fields transform under the fundamental representation of  $\text{SO}(6)$ , which is real, makes the R-symmetry group of the  $\mathcal{N} = 4$  theory being at most  $\text{SU}(4)$  and not  $\text{U}(4)$ , in fact).

chiral superfields transform in the  $\mathbf{3}$  of  $SU(3)$  and have R-charge  $R = 2/3$  under the  $U(1)_R$ . It is an easy but tedious exercise to perform the integration in superspace and get an explicit expression in terms of fields. Finally, one can solve for the auxiliary fields and get an expression where only propagating degrees of freedom are present, and where  $SU(4)_R$  invariance is manifest.

### 2.3.3 | Moduli space and dynamical phases

The scalar potential in (2.13) can be written in a rather compact form in terms of the six real scalars  $X^i$  making up the three complex scalars  $\phi^A$  and reads

$$V(X_1, \dots, X_6) = \frac{1}{2} g^2 \text{tr} \sum_{i,j=1}^6 [X_i, X_j]^2. \quad (2.16)$$

The positive definite behavior of the Cartan-Killing form on the compact gauge algebra  $\mathfrak{g}$  implies that each term in the sum is positive or zero. In other words,  $V = 0$  is equivalent to

$$[X^i, X^j] = 0, \quad i, j = 1, \dots, 6. \quad (2.17)$$

This means that the potential vanishes whenever the scalar fields belong to the Cartan subalgebra of the gauge group  $G$ . At a generic point of the moduli space, the gauge group is broken to  $U(1)^r$  where  $r$  is the rank of  $\mathfrak{g}$ . This equations admit two classes of solutions:

- $\langle X^i \rangle = 0$  for all  $i = 1, \dots, 6$ . This is the *superconformal phase*. Neither the gauge symmetry nor the superconformal symmetry is broken. The physical states and operators are gauge invariant and transform under unitary representations of  $SU(2, 2|4)$ .
- $\langle X^i \rangle \neq 0$  for at least one  $i$ . This is the *spontaneously broken Coulomb phase*. The gauge algebra  $\mathfrak{g}$  is going to be broken to  $U(1)^r$ , where  $r \equiv \text{rank } \mathfrak{g}$ . The low energy behavior is then the one of  $r$  copies of  $\mathcal{N} = 4$   $U(1)$  gauge theories. Superconformal symmetry is spontaneously broken since the non-zero VEV  $\langle X^i \rangle$  sets a scale.

**$\mathcal{N} = 4$  Yang-Mills theory.** There is only one  $D = 4, \mathcal{N} = 4$  Yang-Mills theory and it contains 3  $\mathcal{N} = 1$  chiral scalar supermultiplet and 1  $\mathcal{N} = 1$  vector supermultiplet (up to  $g$  and  $\tau$ ). This theory is conformal and can be recovered from dimensional reduction of  $D = 10, \mathcal{N} = 1$  Yang-Mills on  $\mathbb{T}^6$ .

## 2.4 | Gauge anomaly

The *anomaly degree*  $A(\rho)$  of a representation  $\rho$  is defined as

$$\frac{1}{2} \text{tr}(T_a \{T_b, T_c\}) = A(\rho) d_{abc} \quad (2.18)$$

where  $d_{abc}$  is an invariant symmetric tensor of the Lie algebra of  $G$ , independent of the representation. One can show that  $A(\rho^*) = -A(\rho)$  so self dual representation have  $A(\rho) = 0$  in particular. The only simple Lie groups that allow for a complex non-self-conjugate representation are  $SU(n)$  with  $n \geq 3$ . We can normalize  $d_{abc}$  such that  $A(\rho) = 1$  for the fundamental  $n$ -dimensional representations.

## 3 | Properties of D-branes in type II theories

The minimal irreducible representation in 10 dimensions is a Majorana-Weyl representation of dimension 8. In type II theories, we have  $\mathcal{N} = (1, 1)$  for IIA and  $\mathcal{N} = (2, 0)$  for IIB. Because of the string origin of the generators, the two supersymmetry generators  $\epsilon_L$  and  $\epsilon_R$  (Majorana-Weyl spinors) satisfy

$$\epsilon_L = \Gamma_{11} \epsilon_L, \quad \epsilon_R = \eta \Gamma_{11} \epsilon_R \quad (3.1)$$

with  $\eta = +1$  for IIB and  $\eta = -1$  for IIA theory. For a  $Dp$ -brane, the supersymmetry projections is the following:

$$\epsilon_L = \Gamma_0 \dots \Gamma_p \epsilon_R. \quad (3.2)$$

In other words, the supersymmetries with generators of the form

$$Q_\alpha + \Gamma_0 \dots \Gamma_p \bar{Q}_{\dot{\alpha}} \quad (3.3)$$

are preserved by the D $p$ -brane while the one with generators of the form

$$Q_\alpha - \Gamma_0 \dots \Gamma_p \bar{Q}_{\dot{\alpha}} \quad (3.4)$$

are broken. They violate the boundary conditions. Since there is the same number of generators of the form (3.3) than of the form (3.4), exactly half of the supersymmetry is broken. The idea that one spacetime direction would break one supercharge could be reasonable if supersymmetries were transforming as vectors which not the case; supercharges transform as spinors. It would also be incompatible with the T-duality because two branes of different dimensions must have the same number of unbroken supercharges if there is a T-duality relating them: the number of unbroken supercharges is the same for all dual descriptions (a necessary condition for the equivalence). And indeed, in the correct theory, that's the case. Every type II D-brane breaks half of the supercharges.

To obtain the previous relations, we start by the ones from M-theory and compactify the 11th direction, getting type IIA theory.  $\Gamma_{11}$  then plays the role of the chiral projector in 10 dimensions; the supersymmetry parameters are related by  $\epsilon_L = \frac{1}{2}(1 + \Gamma_{11})\epsilon$  and  $\epsilon_R = \frac{1}{2}(1 - \Gamma_{11})\epsilon$ . The relations for type IIB theory are then obtained by T-duality. Under a T-duality over the  $\hat{i}$  direction, the supersymmetry parameters transform as

$$\begin{aligned} \epsilon_L &\mapsto \epsilon_L, \\ \epsilon_R &\mapsto \Gamma_i \epsilon_R. \end{aligned}$$

The tension of a D $p$ -brane is given by

$$T_p = \frac{1}{(2\pi)^p g_s l_s^{p+1}}. \quad (3.5)$$

This completely fixes the Newton constant: the tension of electric-magnetic duals must satisfy:

$$T_p T_{D-p-4} = \frac{2\pi}{16\pi G_D}. \quad (3.6)$$

In ten dimensions, this gives  $G_{10} = 8\pi^6 g_s^2 l_s^8$ .

The dualities are defined as follows:

$$\begin{aligned} \text{S-duality} : g_s &\mapsto \frac{1}{g_s}, & l_s^2 &\mapsto g_s l_s^2, \\ \text{T-duality} : R &\mapsto \frac{l_s^2}{R}, & g &\mapsto g_s \frac{l_s}{R}. \end{aligned}$$

## 4 | Algebraic geometry

### 4.1 | Elements

An important idea in algebraic geometry is that is is really the algebra of function on it that defines a space. For affine varieties  $X$ , this is illustrated by the fact that the structure of  $X$  is really contained in its coordinate ring  $K[x_1, \dots, x_n]/I(X)$  and by the isomorphism

$$K[x_1, \dots, x_n]|_X = K[x_1, \dots, x_n]/I(X). \quad (4.1)$$

Now an algebraic set  $Z(T)$  is irreducible if  $I(Z(T))$  is prime. So there is a one-to-one correspondence between prime ideals and affine varieties.

Given an algebraic variety, one can modify the equations continuously by varying some parameters and the variety will be “deformed” accordingly. It is called the *variation of the complex structure*. The space of all complex deformations of an affine variety  $X$  is called the *complex moduli space* of  $X$ . For a Calabi-Yau manifold, the linearization of the complex moduli space (tangent space) is given by the cohomology group  $H^{m-1,1}(X)$ , where  $m$  is the dimension of  $X$ . In general, it is much more complicated.

## 4.2 | Divisors and line bundles

A (Weyl) *divisor*  $D$  of a complex variety  $X$  is a linear combination (formal sum with integer coefficients) of co-dimension one, irreducible subvarieties,

$$D = \sum_i n_i V_i, \quad n_i \in \mathbb{Z}, V_i \subset X. \quad (4.2)$$

It said to be effective if all  $n_i \geq 1$ . To any line bundle  $L$  with a regular section  $s$  (that is, on any open subset  $U_\alpha$ ,  $s_\alpha = s|_{U_\alpha}$  is a polynomial in the local coordinates) we can associate a hypersurface  $Y \subset X$  defined as

$$Y = \{p \in X | s(p) = 0\}. \quad (4.3)$$

This hypersurface  $Y$  can then be decomposed into irreducible parts (affine patches) on which  $s_\alpha$  can be factorized in  $\mathbb{C}[x_1, \dots, x_n]$  and decomposed in prime ideals  $P_i$  of multiplicity  $n_i$ . Assembling all the  $V_i^\alpha$  together, we construct co-dimension one subvarieties  $V_i$  that can be used to form divisors. One can also proceed the other way around and, given a divisor  $D = \sum_i n_i V_i$ , define a line bundle  $\mathcal{O}_X(D)$  whose sections vanish on each  $V_i$  with a zero of order  $n_i$ . This construction can be generalized to divisors with negative coefficients  $n_i < 0$  in which case now have poles of order  $n_i$  in  $V_i$ .

## 4.3 | Singularities and resolutions

A *rational map* from a variety  $X$  to another  $Y$  is a morphism from a non-empty subset  $U \subset X$  to  $Y$ . Recall that, by definition of the Zariski topology, a non-empty open subset is always dense. Concretely, a rational map can be written in coordinates using ration functions (quotient of polynomials). A *birational map* is an invertible rational map. It induces an isomorphism between two non-empty open subsets. In this case,  $X$  and  $Y$  are said to be *birationally equivalent*.

The *resolution of a singularity* of an algebraic variety  $V$  is a non-singular variety  $W$  with a proper birational map  $W \rightarrow V$ . For varieties over fields of characteristic 0, it was proven (Hironaka, 1964) that (\*).

## 4.4 | Projective plane curves

In  $\mathbb{CP}^2$ , we consider a hypersurface defined by a single polynomial  $p$  of degree  $d$ . If

$$\frac{\partial p(x)}{\partial x_i} = 0 \quad (4.4)$$

for all  $i$  whenever  $p(x) = 0$ , then the curve is said to be regular, it is a Riemann surface. The latter are classified by their genus and

$$g = \frac{(d-1)(d-2)}{2}. \quad (4.5)$$

For  $d = 3$ , the most general polynomial is

$$\sum_{i+j+k=3} c_{ijk} x_0^i x_1^j x_2^k = 0 \quad (4.6)$$

and defines a torus, also called *elliptic curve*. There 10 independent parameters but 9 of them can be removed by a  $GL(3, \mathbb{C})$  transformation, leaving us with only one complex parameter; the complex structure modulus of the torus.

# 5 | Quivers representations, path algebras, moduli spaces and quiver varieties

## 5.1 | Quivers, path algebras and relations

A quiver  $Q$  is a finite directed graph where loops and multiple arrows between edges are allowed. More precisely, it is a piece of combinatorial data  $(Q_0, Q_1, t, h)$  such that  $Q_0$  is the set of vertices,  $Q_1$  the set of

arrows and  $t, h : Q_1 \rightarrow Q_0$  are the tail and head maps. We can define a notion of product of paths<sup>3</sup> and, provided with the formal sum, the paths form a ring. Given a field  $k$  and an action of  $k$  over this ring, we form an algebra, called the *path algebra* of  $Q$  and denoted by  $kQ$ . It is clear that the path algebra of a quiver is finite-dimensional if and only if  $Q$  has no non-trivial cycles.

To enforce commutativity of some squares inside a quiver, we can use the notion of relation. A *relation* on a quiver  $Q$  is a  $k$ -linear combination of paths from  $Q$ . More precisely, is a subspace of  $kQ$  spanned by linear combinations of paths having a common source and common target, and of length at least 2. A *quiver with relations* is a pair  $(Q, I(R))$  where  $I(R) \subseteq kQ$  is the (two-sided) ideal of the path algebra generated by  $R$ . The path algebra of  $(Q, I)$  is  $kQ/I(R)$ .

**Example.** If  $Q$  is the  $r$ -loop, then a relation is a subspace of  $kQ = k\langle X_1, \dots, X_r \rangle$  spanned by linear combinations of words of length at least 2. For instance, all the commutators  $X_i X_j - X_j X_i$ , then the two elements of  $kQ$  correspond to the same element in  $kQ/I(R)$  if and only their difference is in  $I(R)$ . In this case, this means that they only differ by inverting the product of two variables in their expression. The resulting algebra is then the “commutative version” of  $kQ$ , which is nothing but than the polynomial algebra  $k[X_1, \dots, X_r]$ .

## 5.2 | Quiver representations

A *representation* of a quiver  $Q$  over the field  $k$  is an assignment to every vertex  $i$  as a  $k$ -vector space  $V_i$  and to every arrow  $a$  a linear mapping  $f_a = V_{t(a)} \rightarrow V_{h(a)}$  between the corresponding vector spaces to each arrow. A representation of a quiver with relations  $(Q, R)$  has the same definition but with the additional requirement that the linear maps must preserve the relations(\*). Denoting  $\alpha_i = \dim V_i$ ,  $\alpha = (\alpha_i) \in \mathbb{N}_0^{Q_0}$  is the *dimension vector* of the representation. Quiver representations are an effective combinatorial tool for organizing linear algebraic data. Not only are they naturally related to many algebraic objects such as quantum groups, Kac-Moody algebras, and cluster algebras, but they have also been studied from the geometric point of view, often serving to bridge the gap between representation theory and algebraic geometry.

If  $V = (V_i, f_a)$  and  $W = (W_i, g_a)$  are two finite-dimensional representation of the same quiver with relations  $(Q, R)$ , a morphism  $\psi$  from  $V$  to  $W$  is given by specifying, for every vertex  $i$ , a linear isomorphism  $\psi_i : V_i \rightarrow W_i$  such that for every arrow  $a$   $\psi_{h(a)} \circ f_a = g_a \circ \psi_{t(a)}$ . The *direct sum*  $V \oplus W$  of representations can also be defined as  $(V \oplus W)_i = V_i \oplus W_i$  with the direct sum of the linear mappings. We then say that a representation is *decomposable* if it is isomorphic to the direct sum of non-zero representations and of *finite-type* if it has only finitely many isomorphism classes of indecomposable representations. A very important theorem in indecomposable representations is the following:

**Theorem** (Gabriel). A (connected) quiver is of finite type if and only if its underlying graph (when the directions of the arrows are ignored) is ADE.

To any representation  $V = (V_i, f_a)$  of  $Q$  we can associate a  $k$ -vector space

$$V = \bigoplus_{i \in Q_0} V_i \quad (5.1)$$

equipped with two families of linear self-maps: the projections  $f_i : V \rightarrow V$  ( $i \in Q_0$ ) obtained from the composition  $V \hookrightarrow V_i \rightarrow V$  of the projections with the inclusions, and the maps  $f_a : V \rightarrow V$  ( $a \in Q_1$ ) obtained similarly from the defining maps  $f_a = V_{t(a)} \rightarrow V_{h(a)}$ . One can see that these maps satisfy the relations

$$f_i^2 = f_i, \quad f_i \circ f_j = 0 \ (i \neq j), \quad f_{t(a)} \circ f_a = f_a \circ f_{h(a)} = f_a \quad (5.2)$$

and all other products are zero. In this sense, representations a quiver defines an algebra (the algebra of  $f_i$  and  $f_a$ ). Now comes the whole point to all this is: this algebra is a representation of the path algebra. To see this, we can use the notation  $\rho(e_i) = f_i : V \rightarrow V$  and  $\rho(a) = f_a : V \rightarrow V$  so  $\rho : kQ \rightarrow \text{End}(V)$  and we can show that is is a  $k$ -linear morphism of rings. So  $\rho$  is a representation of the algebra  $kQ$ . So a representation of a quiver  $Q$  gives a representation of the path algebra. The converse can also be shown. We conclude that representations of the path algebra and of the quiver are equivalent. Gabriel's

<sup>3</sup>Note that for every vertex we add a trivial loop.

theorem could then be equivalently phrased as: the path algebra of any quiver has finite representations if and only if it is ADE.

Recall also that giving a  $k$ -linear morphism  $A = (k, R) \rightarrow \text{End}(V)$  is equivalent to giving a structure of  $R$ -module to  $V$ . So from what we discussed above, giving a representation of a quiver is equivalent to giving a module structure to  $V$ . This can be rephrased using a more categorical language by showing that these constructions extend to functors. It is now clear that the finite-dimensional representations of a quiver with relations form a category, which we denote by  $\mathbf{fRep}(kQ, R)$ . If  $(Q, I(R))$  is a quiver with relations and  $A = kQ/I$  its path algebra, then the set of finite-dimensional modules of  $A$  is also a category that we denote by  $\mathbf{fmod}A$ . There is a categorical equivalence

$$\boxed{\mathbf{fRep}(kQ, R) \approx \mathbf{fmod}A.} \quad (5.3)$$

### 5.3 | Geometric invariant theory

### 5.4 | GIT quotient for quiver representation

### 5.5 | Quiver varieties

### 5.6 | Moduli problems

Given a collection of algebra-geometric objects it is natural to try to classify these objects up to equivalence. Naively, given a collection  $\mathcal{A}$  of such objects and an equivalence relation  $\sim$  on  $\mathcal{A}$ , one may ask whether there exists an algebraic variety  $X$  whose points (over the base field  $k$ ) correspond to equivalence classes in  $\mathcal{A}/\sim$ . This approach is flawed, since there may be many such varieties and some are better than others at retaining the relationships between the objects being classified. If, for example, we are interested in classifying all lines through the origin in the complex plane  $\mathbb{C}^2$  up to equality, we can view the corresponding equivalence classes as points of the complex projective line  $\mathbb{P}_{\mathbb{C}}^1$ , but we can also see them as points in the disjoint union  $\mathbb{A}_{\mathbb{C}}^1 \sqcup \{pt\}$  of a line and a point. Ideally, the points of the variety solving a moduli problem should be configured to reflect the relationship between the geometric objects they parametrize.

Thus, in more nuanced approach we try to describe equivalence classes of families of objects of  $\mathcal{A}$ , rather than just of the objects themselves. That is, we look at pairs  $\mathcal{F}, T$ , consisting of a variety  $T$  and a family  $\pi : \mathcal{F} \rightarrow T$  of objects of  $\mathcal{A}$  (the fibers  $\pi^{-1}(t)$  are objects of  $\mathcal{A}$ ) subject to some additional conditions (e.g. that  $\pi$  is flat). Moreover, if the collection of families over a variety  $T$  is denoted by  $\mathcal{A}_T$ , then for any morphism  $f : S \rightarrow T$ , there should be a pullback operation assigning to any family  $\mathcal{F} \in \mathcal{A}_T$  a family  $f^*\mathcal{F} \in \mathcal{A}_S$ . Now, we can extend our moduli problem by introducing an equivalence relation  $\sim_T$  on  $\mathcal{A}_T$  that is compatible with pullback and give us the starting equivalence relation  $\sim$  when  $T$  is  $\text{Spec}k$ .

The solution to such an extended moduli problem, called a *fine moduli space*, consists of a variety  $X$  whose  $k$ -points classify equivalence classes in  $\mathcal{A}/\sim$ , together with a *universal family*  $\mathcal{U} \in \mathcal{A}_X$ , which describes how these equivalence classes relate to each other. More specifically, any family  $\mathcal{F} \in \mathcal{A}_T$  over a variety  $T$  is equivalent to the pullback  $f^*\mathcal{U}$  along a unique morphism  $f : T \rightarrow X$ . In the special case that  $T = \text{Spec}k$ , we obtain that the  $k$ -points of  $X$  are in bijection with equivalence classes in  $\mathcal{A}$ . Furthermore, it turns out that the fine moduli space is unique up to isomorphism.

Considering once again the example of lines through the origin in  $\mathbb{C}^2$ , we see that a family of such lines over a variety  $T$  may be thought of as a line subbundle  $\mathcal{L} \subset \mathbb{C}^2 \times T \rightarrow T$  of the trivial rank 2 vector bundle over  $T$ . The equivalence relation becomes an isomorphism of line subbundles of  $\mathbb{C}^2 \times T$ . The solution to the corresponding moduli problem consists of the complex projective line  $\mathbb{P}_{\mathbb{C}}^1$  together with the tautological line bundle  $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(-1)$ . Indeed, if  $\mathcal{L} \subset \mathbb{C}^2 \times T$  is a subbundle, then its dual  $\mathcal{L}^\vee$  is generated by global sections (specifically, the images of the standard global sections with respect to the surjection  $(\mathbb{C}^2)^\vee \times T \rightarrow \mathcal{L}^\vee$ ). This defines a unique morphism  $f : T \rightarrow \mathbb{P}_{\mathbb{C}}^1$  such that  $\mathcal{L} \simeq f^*\mathcal{O}(-1)$ .

Unfortunately, it is often the case that, for a given class of objects  $\mathcal{A}$ , a fine moduli space either does not exist or requires us to place restrictions on the kinds of objects in  $\mathcal{A}$  we wish to classify. In order to avoid this, we can forget the universal family and look for a nice enough variety with points in bijection

with equivalence classes in  $\mathcal{A}/\sim$ . Alternatively, we can allow for the solution of our moduli problem to no longer be a variety (or even a scheme). The result is a more complicated object called a *stack*.

[This section has to be rewritten \(\\*\)](#).

## 6 | Calabi-Yau manifolds, orbifolds and crepant resolutions

### 6.1 | Kähler, Calabi-Yau structure and moduli spaces

The Kähler form  $\omega$  is a representative of a Dolbeault cohomology class

$$[\omega] \in H^{1,1}(X) \quad (6.1)$$

and  $[\omega]$  is called the *Kähler class* of  $\omega$ .

**Theorem 6.1** (Calabi-Yau). Given  $X$  a compact manifold with trivial canonical bundle, and given a Kähler form  $\tilde{\omega}$  on  $X$ , there exist a unique Ricci-flat metric in the Kähler class of  $\tilde{\omega}$ . That is, a unique Ricci-flat metric defined for some  $\omega \in [\tilde{\omega}]$ .

On the other hand, it is easy to show that Ricci-flatness implies triviality of the canonical bundle. For non-compact manifolds, the theorem does not hold strictly speaking, one must specify boundary conditions at infinity to find a Ricci-flat metric.

Given a Calabi-Yau manifold, we see that there are continuous families of Ricci-flat metrics, one for each cohomology class of  $H^{1,1}(X)$ . One can decompose any cohomology class on a basis  $[\omega]^i$  of the vector space  $H^{1,1}(X)$

$$[\omega] = \sum_{i=1}^{h^{1,1}} \lambda_i [\omega]^i. \quad (6.2)$$

It is called the *Kähler moduli space* of  $X$  and its dimension is denoted by  $h^{1,1}$ .

### 6.2 | Calabi-Yau manifolds

A *Calabi-Yau manifold* of (complex) dimension  $n$  is a compact  $n$ -dimensional Kähler manifold  $M$  satisfying one of the following equivalent conditions:

- the canonical bundle of  $M$  is trivial,
- $M$  has holomorphic  $n$ -form that vanishes nowhere,
- the structure group of the tangent bundle of  $M$  can be reduced from  $U(n)$  to  $SU(n)$ ,
- $M$  has Kähler metric with global holonomy contained in  $SU(n)$ .

It was conjectured by Calabi then proved by Yau that such spaces are necessarily Ricci-flat. In particular, since the first Chern class of CY manifolds is given by

$$c_1 = \frac{1}{2\pi} [\mathcal{R}] \quad (6.3)$$

it implies that  $c_1$  vanishes, the converse is not true.

For a compact  $n$ -dimensional Kähler manifold the following conditions are equivalent to each other:

- the first real Chern class vanishes,
- $M$  has a Kähler metric with vanishing Ricci curvature,
- $M$  has Kähler metric with local holonomy contained in  $SU(n)$ ,
- a positive power of the canonical bundle of  $M$  is trivial,
- $M$  has a finite cover that has trivial canonical bundle,

- $M$  has a finite cover that is a product of a torus and a simply connected manifold with trivial canonical bundle.

They are weaker than the conditions above except when the Kähler manifold is simply connected in which case they are equivalent.

### 6.3 | Calabi-Yau orbifolds

A *Calabi-Yau orbifold* is the quotient of a smooth Calabi-Yau manifold by a discrete group action which generically has fixed points. From an algebraic geometry perspective we can try to resolve the orbifold singularity. A resolution  $(X, \pi)$  of  $\mathbb{C}^n/\Gamma$  is a non-singular complex manifold  $X$  of dimension  $n$  with a proper biholomorphic map

$$\pi : X \rightarrow \mathbb{C}^n/\Gamma \quad (6.4)$$

that induces a biholomorphism between dense open sets. A resolution  $(X, \pi)$  of  $\mathbb{C}^n/\Gamma$  is called a *crepant resolution*<sup>4</sup> if the canonical bundles of  $X$  and  $\mathbb{C}^n/\Gamma$  are isomorphic, i.e.

$$K_X \cong \pi^*(K_{\mathbb{C}^n/\Gamma}).$$

Since Calabi-Yau manifolds have trivial canonical bundle, to obtain a Calabi-Yau structure on  $X$  one must choose a crepant resolution of singularities.

It turns out that the amount of information we know about a crepant resolution of singularities of  $\mathbb{C}^n/\Gamma$  depends dramatically on the dimension  $n$  of the orbifold:

- $n = 2$ : a crepant always exists and is unique. Its topology is entirely described in terms of the finite group  $\Gamma$  (via the McKay correspondence).
- $n = 3$ : a crepant resolution always exists but it is not unique; they are related by flops. However all the crepant resolutions have the same Euler and Betti numbers: the *stringy* Betti and Hodge numbers of the orbifold.
- $n \geq 4$ : very little is known; crepant resolution exists in rather special cases. Many singularity are terminal, which implies that they admit no crepant resolution.

## 7 | Homology

### 7.1 | The idea of Homology

Note that there are in fact many types of homologies, the one we described here is the simplest (which is almost everytime for our needs) and is called *simplicial homology*.

### 7.2 | Simplicial Homology

An  $n$ -*simplex*, denoted by  $\Delta^n = [v_0, \dots, v_n]$  is the smallest convex set in  $\mathbb{R}^m$  containing  $n + 1$  points  $v_0, \dots, v_n$  that do not lie in a hyperplane of dimension less than  $n$ . Or, equivalently, such that  $v_1 - v_0, \dots, v_n - v_0$  are linearly independent. The  $n + 1$  points  $v_0, \dots, v_n$  are the *vertices* of the  $n$ -simplex. A by-product of our ordered notation for the vertices of a simplex is that it determines an orientation for the edges  $[v_i, v_j]$  according to the increasing subscripts. The *faces* of an  $n$ -simplex are all the subsimplices that can be obtained by removing vertices of the original simplex. The order of the vertices of the smaller simplices is taken to be the same as in the original  $n$ -simplex.

Let us now put some simplices together. If  $X$  be a topological space, then a *simplicial complex structure* on  $X$  is a collection of continuous maps  $\sigma_\alpha : \Delta^n \rightarrow X$  called *characteristic maps* such that

- $\sigma_\alpha|_{e^\alpha}$  is injective and that for all  $x \in X$  there exists  $\alpha$  such that  $x \in \text{Im}(\sigma_\alpha|_{e^\alpha})$ ,
- the restriction of any map  $\sigma_\alpha$  to a face of  $\Delta^n$  is another  $\sigma_\beta$ ,

<sup>4</sup>For a resolution of singularities we can define a notion of discrepancy. A crepant resolution is a resolution without discrepancy.



iii) for any subset  $A \subseteq X$ ,  $A$  is open if and only if  $\sigma_\alpha^{-1}(A)$  is open in  $\Delta^n$  for all  $\alpha$ ,

where  $\Delta^n$  is an  $n$ -simplex and  $e^n$  is the interior of  $\Delta^n$ .

Given a simplicial complex structure  $\sigma_\alpha = \Delta^{n(\alpha)} \rightarrow X$  on  $X$ ,  $n$ -chains are finite formal sums  $\sum_\alpha n_\alpha e_\alpha^n$  with coefficient  $n_\alpha \in \mathbb{Z}$ . Equivalently, we can write  $n$ -chains as  $\sum_\alpha n_\alpha \sigma_\alpha$ . We denote by  $\Delta_n(X)$  the free abelian group formed by  $n$ -chains. A basis is given by the open  $n$ -simplices  $e_\alpha^n$  of  $X$ .

For a general simplicial complex structure on  $X$ , we define a *boundary homomorphism*  $\partial_n : \Delta_n(X) \rightarrow \Delta_{n-1}(X)$  by its action on the basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha| [v_0, \dots, \widehat{v_i}, \dots, v_n] \quad (7.1)$$

We can note that the right side of this equations does indeed lies in  $\Delta_{n-1}(X)$  since each restriction  $\sigma_\alpha| [v_0, \dots, \widehat{v_i}, \dots, v_n]$  is the characteristic map of an  $(n-1)$ -simplex of  $X$ .

We now have a sequence of homomorphisms of abelian groups

$$\cdots \rightarrow \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \rightarrow \cdots \rightarrow \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \rightarrow 0 \quad (7.2)$$

with  $\partial_{n+1}\partial_n = 0$  for each. Such a sequence is called a *chain complex*. We have extended the chain at the end with  $\partial_0 = 0$ . It follows that  $\text{Im } \partial_{n+1} \subset \text{Ker } \partial_n$ . We then define the  *$n$ th Homology group* of  $X$  to be the quotient group  $H_n(X) = \text{Ker}(\partial_n) / \text{Im } \partial_{n+1}$ . Elements of  $\text{Ker } \partial_n$  are called *cycles* and elements of  $\text{Im } \partial_{n+1}$  are called *boundaries*. Elements of  $H_n$  are cosets of  $\text{Im } \partial_{n+1}$  and are called *homology classes*. This means that two cycles are in the same homology classes if their difference is a boundary.

**Example.** Let us give some usefull examples of Homology groups for various topological spaces:

- the non-trivial Homology groups of the  $n$ -sphere are  $H_0(S^n) = H_n(S^n) = \mathbb{Z}$ .
- the only non-trivial Homology group of the  $n$ -ball is  $H_0(B^n) = \mathbb{Z}$  if  $k = 0$ .
- the non-trivial Homology groups of the 2-torus are  $H_0(T) = H_2(T) = \mathbb{Z}$  and  $H_1(T) = \mathbb{Z}^2$ .
- the non-trivial Homology groups of the complex projective space are  $H_{0 \leq 2k \leq 2n}(\mathbb{CP}^n) = \mathbb{Z}$ .

The  $n$ th simplicial homology group of a topological space  $X$  with a complex simplicial structure  $\Delta$  is always of the form

$$H_n^\Delta(X) = \underbrace{\mathbb{Z} \times \mathbb{Z}}_{b_n} \oplus \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_s}. \quad (7.3)$$

$b_n$  are the Betti numbers. The *Poincaré* polynomial is

$$P_X(x) = \sum_{i \geq 0} b_i x^i \quad (7.4)$$

and the Euler characteristic of  $X$  is given by  $P_X(-1)$ .

## 8 | McKay correspondence

### 8.1 | Classical correspondence

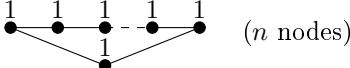
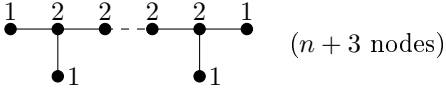
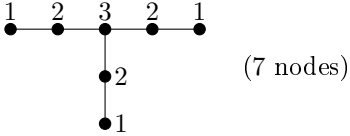
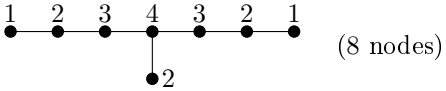
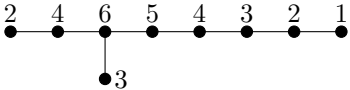
$\Gamma \subset \mathrm{SU}(2)$	Platonic solids	McKay graph	Variety
$\mathbb{Z}_n$		 $(n \text{ nodes})$	$z^n + xy = 0$
$2\mathcal{D}_n$	$n$ -polygon	 $(n + 3 \text{ nodes})$	$x^2 + y^2z + z^{n-1} = 0$
$2\mathcal{T}$	tetrahedron	 $(7 \text{ nodes})$	$x^2 + y^3 + z^4 = 0$
$2\mathcal{O}$	cube octahedron	 $(8 \text{ nodes})$	$x^2 + y^3 + yz^3 = 0$
$2\mathcal{I}$	icosahedron dodecahedron	 $(9 \text{ nodes})$	$x^2 + y^3 + z^5 = 0$

Table 1: Binary polyhedral groups and their McKay graphs. Labels over the vertices are the dimension of the representation. We erase the arrow ends if they go in both directions and erase the label if it is equal to 1.

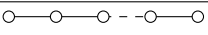
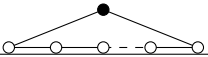
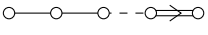
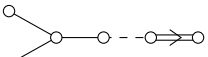
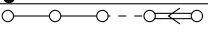
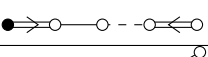
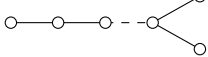
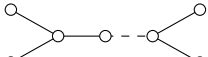
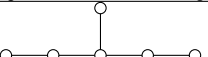
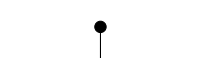
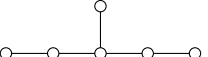
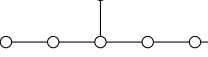

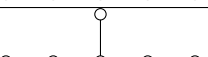

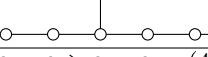

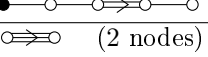
Simple Lie algebra	Simply laced	Dynkin diagram Extended Dynkin diagram
$\mathfrak{sl}(n+1, \mathbb{C}), n \geq 1$	yes	$A_n$ :  ( $n$ nodes) $\tilde{A}_n$ :  ( $n+1$ nodes)
$\mathfrak{so}(2n+1, \mathbb{R}), n \geq 2$	no	$B_n$ :  ( $n$ nodes) $\tilde{B}_n$ :  ( $n+1$ nodes)
$\mathfrak{sp}(2n, \mathbb{C}), n \geq 3$	no	$C_n$ :  ( $n$ nodes) $\tilde{C}_n$ :  ( $n+1$ nodes)
$\mathfrak{so}(2n, \mathbb{R}), n \geq 4$	yes	$D_n$ :  ( $n$ nodes) $\tilde{D}_n$ :  ( $n+1$ nodes)
$\mathfrak{e}_6$	yes	$E_6$ :  (6 nodes) $\tilde{E}_6$ :  (7 nodes)
$\mathfrak{e}_7$	yes	$E_7$ :  (7 nodes) $\tilde{E}_7$ :  (8 nodes)
$\mathfrak{e}_8$	yes	$E_8$ :  (8 nodes) $\tilde{E}_8$ :  (9 nodes)
$\mathfrak{f}_4$	no	$F_4$ :  (4 nodes) $\tilde{F}_4$ :  (5 nodes)
$\mathfrak{g}_2$	no	$G_2$ :  (2 nodes) $\tilde{G}_2$ :  (3 nodes)

Figure 1: Simple Lie algebras and their (extended) Dynkin diagrams. The first four algebras are the classical simple Lie algebras and the last five are the exceptional simple Lie algebras.

Finally, we can see the following correspondence between the extended Dynkin diagrams and the McKay graphs.

Simply Lie group	Simply laced Lie algebra	Extended Dynkin diagram	Finite subgroup of SO(3)	Finite subgroup of SU(2)
$SU(n+1)$	$\mathfrak{sl}(n+1, \mathbb{C})$	$\widetilde{A}_n$	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_{n+1}$
$SO(2n), \text{Spin}(2n)$	$\mathfrak{so}(2n, \mathbb{R})$	$\widetilde{D}_n$	$\mathcal{D}_{2(n-2)}$	$2\mathcal{D}_{2(n-2)}$
$E_6$	$\mathfrak{e}_6$	$\widetilde{E}_6$	$\mathcal{T}$	$2\mathcal{T}$
$E_7$	$\mathfrak{e}_7$	$\widetilde{E}_7$	$\mathcal{O}$	$2\mathcal{O}$
$E_8$	$\mathfrak{e}_8$	$\widetilde{E}_8$	$\mathcal{I}$	$2\mathcal{I}$

Figure 2: Classical McKay correspondence.

## 8.2 | Geometrical McKay correspondence

## 9 | The simplest case: smooth transverse space

### 9.1 | Generalities

Let us start by considering the simplest configuration where the transverse Calabi-Yau space is non-singular, i.e. it is a proper smooth Calabi-Yau threefold. As mentioned above, the only smooth Calabi-Yau threefold is  $S = \mathbb{C}^3$ . In this case, the spacetime is simply flat space  $\mathbb{R}^{1,9} = \mathbb{R}^{1,3} \times \mathbb{R}^6$  with a choice of a complex structure on  $\mathbb{R}^6$ . From the  $U(N)$  Chan-Paton factors, the worldvolume theory inherits from a  $U(N)$  gauge group. Type IIB superstring theory is a ten-dimensional  $\mathcal{N} = 2$  theory so it has 32 supercharges. The presence of the branes breaks the Lorentz symmetry of  $\mathbb{R}^{1,9}$  as

$$SO(1, 9) \rightarrow SO(1, 3) \times SO(6), \quad (9.1)$$

whereby breaking half of the supersymmetries, as we explained in the previous section. We are thus left with 16 supercharges. In four dimensions, this corresponds to  $\mathcal{N} = 4$ . The worldvolume theory for  $S = \mathbb{C}^3$  is therefore  $D = 4, \mathcal{N} = 4$   $U(N)$  SCFT gauge theory. This worldvolume theory, obtained in the non-singular case  $S = \mathbb{C}^3$ , is called the *parent theory*.

Note that the D3-brane will warp the flat space metric to that of  $AdS_5 \times S^5$  and the bulk geometry is not strictly  $\mathbb{C}^3$ . However, as stated above, we are only concerned with the local gauge theory and not with gravitational back-reaction, therefore it suffices to consider  $S$  as  $\mathbb{C}^3$ .

### 9.2 | Matter content

As discussed in appendix 2.3, there is only one  $D = 4, \mathcal{N} = 4$  SCFT theory, up to a choice of gauge group  $G$ . In our case,  $G = U(N)$ . The isometry group of the transverse space  $\mathbb{R}^6$  is  $SO(6) \cong SU(4)$ . Since the scalar fields living on the branes are interpreted as its transverse oscillations,  $SO(6)$  is a global symmetry of the field theory. These global symmetries of worldvolume theory lead to the R-symmetry group  $SU(4)_R$ . The only  $\mathcal{N} = 4$  supermultiplet can be rewritten in terms of  $\mathcal{N} = 1$  supermultiplets as follows:

$$[\mathcal{N} = 4 \text{ vector multiplet}] : V = (\lambda_\alpha, A_\mu, D) \oplus \Phi_A = (\phi^A, \psi_\alpha^A, F^A). \quad (9.2)$$

with  $A = 1, 2, 3$ . In other words, after removing the auxiliary fields  $D$  and  $F^A$ , the matter content is

- a  $U(N)$  gauge field  $A_\mu$  which transforms as a singlet under  $SU(4)_R$ :

$$\text{Gauge transformation : } A_\mu \mapsto UA_\mu U^{-1} + U\partial_\mu U^{-1}, \quad U \in U(N) \quad (9.3)$$

$$\text{R-symmetry : } A_\mu \mapsto A_\mu. \quad (9.4)$$

Note that the usual term

- 4 Weyl fermions  $\psi_\alpha^a \equiv (\lambda_\alpha, \psi_\alpha^1, \psi_\alpha^2, \psi_\alpha^3)$  that transform under the adjoint of  $U(N)$  and are mixed together under the representation **4** of  $SU(4)_R$ . This means that each fermion  $\psi^a$  takes values in  $\mathfrak{u}(N)$ . We denote the components by  $\psi_{IJ}^a$  ( $I, J = 1, \dots, N$ ). Explicitly:

$$\text{Gauge transformation : } \psi^a \mapsto U\psi^a U^\dagger, \quad U \in U(N), \quad (9.5)$$

$$\text{R-symmetry} : \psi^a \mapsto R^a_b \psi^b, \quad R \in \text{SU}(4)_R. \quad (9.6)$$

Note that this gives us  $4N^2$  Weyl fermions in total.

- 3 complex scalar fields  $\phi^A$  transforming under the adjoint representation of  $\text{U}(N)$  and under the two-times anti-symmetric representation of  $\text{SU}(4)_R$ . This means that each  $\phi^A$  takes values in  $\mathfrak{u}(N)$  and we denote the components by  $\phi_{IJ}^A$ . Recall that  $\text{SU}(4) \cong \text{SO}(6)$  so the action of the R-symmetry can be seen as the **3** of  $\text{SU}(3) \subset \text{SU}(4)_R$  acting on three complex scalars  $\phi^A$  or equivalently as the **6** of  $\text{SO}(6)_R$  acting on 6 real scalars  $X^m$ , the real and imaginary parts of the  $\phi^A$ . They are interpreted as the oscillations of the branes in the transverse space. Explicitly:

$$\text{Gauge transformation} : X^m \mapsto U X^m U^\dagger, \quad U \in \text{U}(N), \quad (9.7)$$

$$\text{R-symmetry} : X^m \mapsto R^m_n X^n, \quad R \in \text{SO}(6)_R. \quad (9.8)$$

Note that this gives  $6N^2$  real scalars in total. They are the superpartners of the fermions(\*).

Note that the gauge group  $\text{U}(N)$  can also be seen as the group of isometries of the metric space  $\mathbb{C}^N$ , i.e.  $\text{Hom}(\mathbb{C}^N, \mathbb{C}^N)$ . From this point of view, the transformations (9.3)-(9.8) can be summarized as

$$A_\mu \in \text{Hom}(\mathbb{C}^N, \mathbb{C}^N), \quad (9.9a)$$

$$\psi \in \mathbf{4} \otimes \text{Hom}(\mathbb{C}^N, \mathbb{C}^N), \quad (9.9b)$$

$$X \in \mathbf{6} \otimes \text{Hom}(\mathbb{C}^N, \mathbb{C}^N). \quad (9.9c)$$

If the transverse space is non-singular, the only possibility is  $S = \mathbb{C}^3$ . In this case, the worldvolume theory is therefore  $D = 4, \mathcal{N} = 4$   $\text{U}(N)$  SCFT gauge theory. It is called the *parent theory*.

## Part II

# Orbifold singularities

## 10 | Strings on orbifolds

Let us consider a smooth background space  $M^{(6)}$  and  $\Gamma$  a discrete group acting on it. If  $\Gamma$  has no fixed point, the quotient space  $M^{(6)}/\Gamma$  is smooth and is a manifold but if the action of  $\Gamma$  has a fixed point then the quotient space  $M^{(6)}/\Gamma$  is singular and is an orbifold. Any orbifold can be described as an affine variety therefore it is the description that we will use most often. When the transverse space is singular, the worldvolume theory corresponds to a specific projection of the parent theory that we found in the smooth case  $S = \mathbb{C}^3$ . We call it the *daughter theory*. This projection depends on the type of singularity that one considers. The simplest case is the case of orbifold singularity, i.e. when the transverse space is a quotient space with a non-free action.

### 10.1 | Generalities

We now wish to pick a discrete group  $\Gamma$  and which acts non-trivially on  $\mathbb{R}^6$ . There are several possibilities:

- $\Gamma \subset \text{SU}(4)$  naturally acts on  $\mathbb{R}^6$ . This does not require a choice of complex structure. We get an  $\mathcal{N} = 0$  theory.
- $\Gamma \subset \text{SU}(3)$  naturally acts on  $\mathbb{C}^3$ , this also requires a choice of complex structure on  $\mathbb{R}^6$ . We get an  $\mathcal{N} = 1$  theory.
- $\Gamma \subset \text{SU}(2)$  naturally acts on the second factor of  $\mathbb{C} \times \mathbb{C}^2$ , so this requires a choice of complex structure on  $\mathbb{R}^6$ . We get an  $\mathcal{N} = 2$  theory.

We are interested in supersymmetric theories so we take  $\Gamma \subset \text{SU}(3)$  with the action

$$\cdot : \begin{pmatrix} \Gamma \times \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3 \\ (\gamma, z) & \longmapsto & \gamma \cdot z \end{pmatrix} \quad (10.1)$$

is the representation of  $\Gamma$  coming from the fundamental representation of  $\text{SU}(3)$ , so  $\cdot$  is just the matrix product. We can see that the origin is always a fixed point so this action is never free. Since  $\mathbb{C}^3$  is a smooth manifold, this makes  $\mathbb{C}^3/\Gamma$  an orbifold. Note this case naturally includes the case  $\Gamma \subset \text{SU}(2)$  too (as  $\text{SU}(2) \subset \text{SU}(3)$ ), just not the case  $\Gamma \in \text{SO}(6)$ . When  $\Gamma \subset \text{SU}(2) \subset \text{SU}(3)$ , it acts trivially on one component so we write  $S = \mathbb{C} \times \mathbb{C}^2/\Gamma$ .

If  $\Gamma$  is a general finite group the condition that  $\mathbb{C}^3/\Gamma$  is an Calabi-Yau orbifold means that there must exist a resolution of this orbifold such that the corresponding smooth space is Calabi-Yau, i.e. a crepant resolution. Existence of such a resolution constrains  $\Gamma$  (\*), see appendix 6.

If the transverse space is  $S = \mathbb{C}^3/\Gamma$ , the field theory must be projected to a theory which also invariant under  $\Gamma$ , seen as a subgroup of the R-symmetry group. The prescription is straihg-forward: we can use the elements  $\gamma \in \Gamma$  to project out that states that are not  $\Gamma$ -invariant. That is, if  $\rho$  is an embedding of  $\Gamma$  in the gauge group  $\text{U}(N)$ , only the the fields such that

$$\rho(\gamma)A_\mu\rho(\gamma)^{-1} = A_\mu, \quad (10.2a)$$

$$R(\gamma)\rho(\gamma)\psi_{IJ}\rho(\gamma)^{-1} = \psi_{IJ}, \quad (10.2b)$$

$$R(\gamma)\rho(\gamma)X_{IJ}\rho(\gamma)^{-1} = X_{IJ} \quad (10.2c)$$

are kept in the spectrum, where  $\rho$  is a unitary representation of  $\Gamma$  on  $\mathbb{C}^N$  and  $R = \mathbf{4}, \mathbf{6}$ . Let us make two remarks:

- the term  $U\partial_\mu U^{-1}$  is absent from 10.2a. Indeed,  $\Gamma$  is a finite group and the only smooth functions  $x \mapsto \Gamma$  are the constant ones. Consequently, transformations of the gauge field under a finite subgroup of its gauge group cannot depend on  $x$ .
- the fields that transform non-trivially under R-symmetry also have an extra induced action of  $\Gamma$ , in agreement with (9.9a)-(9.9c). The R-symmetry untouched by  $\Gamma$  will be the resulting R-symmetry of daughter theory.

### 10.1.1 | Further details on the projection

Riemannian symmetric spaces, which are locally isometric to homogeneous spaces  $G/H$  have local holonomy isomorphic to  $H$ .(\*)

## 10.2 | Twisted sector

What implications does have strings on an orbifold have for the spectrum? In general, we distinguish two kinds of states:

- the *untwisted states* are the states  $\Psi$  that are invariant under the actio of  $\Gamma$ :  $\gamma \cdot \Psi = \Psi$  for all  $\gamma \in \Gamma$ . They are the generators of the  $\Gamma$ -invariant part of the Hilbert space and can easily be constrctued by superposing all the images  $\gamma \cdot \Psi_0$ .
- closed strings must start and end at the same point, i.e.  $X(\tau, \sigma + 2\pi) = X(\tau, \sigma)$ . Usually, a strings that starts at connects a point of  $M^{(6)}$  to another point of its orbit, i.e.  $X(\tau, \sigma + 2\pi) = \gamma \cdot [X(\tau, \sigma)]$  would not be allowed. However, in  $M^{(6)}/\Gamma$  it is an allowed configuration. Those new states that appear after orbifolding are the *twisted states*. There existnce is due to the fact that strings are extended objects. One can see that there is one twisted sector per conjugacy classe of  $\Gamma$  (\*) and that the untwisted states are recovered with  $\gamma = e$ .

The individual twisted sector quantum states of the strings are localized at the orbifold singularities that the classical configurations (untwisted sector) enclose.

### 10.3 | Representation theory of the projection

Let  $\{(\rho_i, V_i)\}_{i \in I}$  be a complete set of irreducible representations of  $\Gamma$ . Since  $\Gamma$  is finite, it is particular compact and those representation can be taken to be unitary. We also use the notation  $N_i = \dim \rho_i$  such that  $V_i = \mathbb{C}^{N_i}$ . Moreover,  $i$  takes a finite number of values. Let us consider a representation of  $\Gamma$  on  $\mathbb{C}^N$ , we denote it  $(\rho, \mathbb{C}^N)$  and also take it to be unitary. In that case,  $\rho(\gamma) \in U(N)$ . This is what we mean by “the embedding of  $\Gamma$  in  $U(N)$ ”. The adjoint representation of  $U(N)$  defined as<sup>5</sup>

$$\text{Ad} : \left( \begin{array}{ccc} U(N) & \longrightarrow & \text{GL}(\mathfrak{u}(N)) \\ U & \longmapsto & \text{Ad}_U \end{array} \right), \quad \text{Ad}_U : \left( \begin{array}{ccc} \mathfrak{u}(N) & \longrightarrow & \mathfrak{u}(N) \\ \omega & \longmapsto & \text{Ad}_U(\omega) \equiv U\omega U^{-1} \end{array} \right), \quad (10.3)$$

now allows us to act with  $\Gamma$  on  $\mathfrak{u}(N)$ . We use this representation in the expression (10.2a)-(10.2c). More formally, these relations can be rewritten as

$$\text{Ad}_{\rho(\gamma)} A_\mu = A_\mu, \quad (10.4)$$

$$R(\gamma) \text{Ad}_{\rho(\gamma)} \psi_{IJ} = \psi_{IJ}, \quad (10.5)$$

$$R(\gamma) \text{Ad}_{\rho(\gamma)} X_{IJ} = X_{IJ} \quad (10.6)$$

We can decompose  $(\rho, \mathbb{C}^N)$  as follows:

$$(\rho, \mathbb{C}^N) = \bigoplus_{i \in I} (\rho_i, V_i)^{N_i} \quad (10.7)$$

$$= \bigoplus_{i \in I} (\mathbf{1}^{N_i} \otimes \rho_i, \mathbb{C}^{N_i} \otimes V_i) \quad (10.8)$$

where  $N_i$  are integer multiplicities  $((\rho_i, V_i)^{N_i} \equiv (\rho_i, V_i)^{\oplus N_i})$  and  $\mathbf{1}$  is the trivial representation, so  $\Gamma$  acts trivially on the  $\mathbb{C}^{N_i}$ . We have

$$\sum_i N_i \dim(\rho_i) = N. \quad (10.9)$$

The rewriting (10.8) will be useful later on.

After the projection, the resulting gauge group is given by the  $\Gamma$ -invariant part of the gauge group, that is  $\text{Hom}(\mathbb{C}^N, \mathbb{C}^N)^\Gamma$ . We use the superscript  $\Gamma$  to indicate that we only keep the trivial representations in the decomposition, that is, we only keep that subspaces that transform trivially. First, we can see that by Schur's lemma (\*)

$$(V_i \otimes V_j^*)^\Gamma = \delta_{ij} \quad (10.10)$$

Now since  $\text{Hom}(\mathbb{C}^N, \mathbb{C}^N) \cong \mathbb{C}^N \otimes (\mathbb{C}^N)^*$ , we get

$$\text{Hom}(\mathbb{C}^N, \mathbb{C}^N)^\Gamma = (\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^\Gamma \quad (10.11)$$

$$= \bigoplus_{i,j \in I} ((\mathbb{C}^{N_i} \otimes V_i) \otimes (\mathbb{C}^{N_j} \otimes V_j^*)^\Gamma)^\Gamma \quad (10.12)$$

$$= \bigoplus_{i,j \in I} (\mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^* \otimes V_i \otimes V_j^*)^\Gamma \quad (10.13)$$

$$= \bigoplus_{i,j \in I} (\mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^*)^\Gamma \otimes (V_i \otimes V_j^*)^\Gamma \quad (10.14)$$

$$= \bigoplus_{i \in I} \mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_i})^* \quad (10.15)$$

so, in complete notations,

$$(\rho, \mathbb{C}^N)^\Gamma = \bigoplus_{i \in I} (\rho_i, \mathbb{C}^{N_i}) \otimes (\overline{\rho_i}, \mathbb{C}^{N_i}) \quad (10.16)$$

<sup>5</sup>this is well-defined since for all  $\omega \in \mathfrak{u}(N)$  and  $U \in U(N)$ ,  $U\omega U^{-1} \in \mathfrak{u}(N)$ .

and the daughter gauge group is

$$G_{\text{proj}} = \bigotimes_{i \in I} \text{U}(N_i). \quad (10.17)$$

Now it turns out that in the low energy effective field theory the  $\text{U}(1)$  factor of every  $\text{U}(N_i)$  decouples (\*)so the resulting gauge group is in fact

$$G_{\text{proj}} = \bigotimes_{i \in I} \text{SU}(N_i). \quad (10.18)$$

For the matter fields, the reasoning is similar but we now have to take into account the  $\text{R}$ -symmetry. Let  $\mathbf{4} \equiv (\rho_4, V_4)$  be the fundamental representation of  $\text{SU}(4)_R$  and  $\mathbf{6} \equiv (\rho_6, V_6)$  be the fundamental representation of  $\text{SO}(6)_R$ . Wish to compute  $(V_{\mathcal{R}} \otimes \text{Hom}(\mathbb{C}^N, \mathbb{C}^N))^{\Gamma}$  with  $\mathcal{R} = \mathbf{4}, \mathbf{6}$ :

$$(V_{\mathcal{R}} \otimes \text{Hom}(\mathbb{C}^N, \mathbb{C}^N))^{\Gamma} = \bigoplus_{i,j \in I} (V_{\mathcal{R}} \otimes (\mathbb{C}^{N_i} \otimes V_i) \otimes (\mathbb{C}^{N_j} \otimes V_j)^*)^{\Gamma} \quad (10.19)$$

$$= \bigoplus_{i,j \in I} (V_{\mathcal{R}} \otimes \mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^*)^{\Gamma} \otimes (V_i \otimes V_j^*)^{\Gamma} \quad (10.20)$$

$$= \bigoplus_{i,j \in I} a_{ij}^{\mathcal{R}} (\mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^*) \quad (10.21)$$

with

$$(\rho_{\mathcal{R}}, V_{\mathcal{R}}) \otimes (\rho_i, V_i) = \bigoplus_{j \in I} a_{ij}^{\mathcal{R}} (\rho_j, V_j). \quad (10.22)$$

This expression makes sense because  $(\rho_{\mathcal{R}}, V_{\mathcal{R}})$  is a representation of  $\text{SU}(4)$  so it is in particular a representation of  $\text{SU}(3)$  and therefore also in particular a representation of  $\Gamma$ . In more complete notations, we obtained that

$$((\rho_{\mathcal{R}}, V_{\mathcal{R}}) \otimes (\rho, \mathbb{C}^N))^{\Gamma} = \bigoplus_{i,j \in I} a_{ij}^{\mathcal{R}} ((\rho_i, \mathbb{C}^{N_i}) \otimes (\bar{\rho}_j, \mathbb{C}^{N_j})). \quad (10.23)$$

Using the orthogonality of characters of irreducible non-equivalent representations, we obtain the explicit expression of the coefficient  $a_{ij}^{\mathcal{R}}$ :

$$a_{ij}^{\mathcal{R}} = \frac{1}{|\Gamma|} \sum_{\gamma=1}^r r_{\gamma} \chi^{\mathcal{R}}(\gamma) \chi^i(\gamma) \chi^j(\gamma)^* \quad (10.24)$$

where  $r_{\gamma}$  is the order of the conjugacy class containing  $\gamma$  and  $\chi^i$  is the character of  $\rho_i$ .

In the end, we can see that:

In the daughter theory, the matter fields become a total of  $a_{ij}^{\mathbf{4}}$  bi-fundamental fermions and  $a_{ij}^{\mathbf{6}}$  bi-fundamental bosons transforming as the  $(\mathbf{N}_i, \bar{\mathbf{N}}_j)$  of  $\text{SU}(N_i) \times \text{SU}(N_j)$  under the products of gauge groups.

## 10.4 | Bi-index notation

The gauge and the bi-fundamental fields that we obtained in the previous section are easier to manipulate in the *bi-index notation*. Recall that  $A_{\mu} \in \mathfrak{u}(N)$  si it is an  $N \times N$  complex matrix and we denote its elements by  $A_{\mu;IJ}$  ( $I, J = 1, \dots, N$ ). We saw that this matrix of fields transform under  $\Gamma$  as

$$A_{\mu} \mapsto \rho(\gamma) A_{\mu} \rho(\gamma)^{-1}, \quad \gamma \in \Gamma, \quad (10.25)$$

i.e. the adjoint representation but without the derivative term. From the decomposition (10.7), it is easier to decompose  $A_{\mu}$  in blocks  $A_{\mu;ij}$  ( $i, j \in I$ ) of size  $N_i \times N_j$  that transform under the representation  $\rho_i^{\oplus N_i} \otimes (\bar{\rho}_j)^{\oplus N_j}$  of  $\text{diag}(\Gamma \times \Gamma) = \Gamma$ :

$$A_{\mu;ij} \mapsto \rho_i(\gamma)^{\oplus N_i} X_{ij}^m (\rho_j(\gamma)^{-1})^{\oplus N_j}, \quad \gamma \in \Gamma. \quad (10.26)$$



Now those blocks that transform under the direct sum of irreducible representations can of course again be divided into blocks that transform directly under the irreducible representations. We denote those  $\dim \rho_i \times \dim \rho_j$  blocks by  $X_{i\alpha_i, j\beta_j}^m$  ( $\alpha_i, \beta_j = 0, \dots, N_i - 1$ ). In the end, we have

$$\boxed{A_{\mu; i\alpha_i, j\beta_j} \mapsto \rho_i(\gamma) A_{\mu; i\alpha_i, j\beta_j} \rho_j(\gamma)^{-1}, \quad \gamma \in \Gamma.} \quad (10.27)$$

This is the equation that is important and that we will use to find the invariant configurations under  $\Gamma$ . We use the exact same bi-index notations for  $X^m$  and  $\psi^a$ .

## 10.5 | Field content, quivers and McKay graphs

A convenient way to represent the matter content of a daughter theory is to use *quiver diagrams*. A quiver is a finite oriented graph such that each node  $i$  represents a gauge factor  $SU(N_i)$  and each arrow  $i \rightarrow j$  represents a bi-fundamental field transforming under  $(\mathbf{N}_i, \overline{\mathbf{N}}_j)$ . The *adjacency matrix*  $A$  of the graph is a  $k \times k$  with  $k$  being the number of nodes (gauge factors) whose elements  $A_{ij}$  are the number of arrows (bi-fundamental fields) from  $i$  to  $j$ . In other words, from (10.23), the adjacency matrix of the fermions has elements  $A_{ij} = a_{ij}^4$  and the one of the scalars has elements  $A_{ij} = a_{ij}^6$ .

On the other hand, given finite group  $\Gamma$ , a representation  $(\rho_W, W)$  and a complete set of irreducible representations  $\{(\rho_i, V_i)\}_{i \in I}$  of the latter, one can construct a McKay graph (or quiver) as follows:

1. Draw a vertex for every representation  $(\rho_i, V_i)$ .
2. For every representation  $(\rho_i, V_i)$ , compute the decomposition

$$(\rho_W, V_W) \otimes (\rho_i, V_i) = \bigoplus_j (\rho_j, V_j)^{\oplus n_{ij}}$$

where  $n_{ij}$  is the multiplicity of  $(\rho_j, V_j)$  in the decomposition of  $(\rho_W, V_W) \otimes (\rho_i, V_i)$ .

3. For every  $n_{ij} > 0$ , draw  $n_{ij}$  arrows from the vertex of  $(\rho_i, V_i)$  to the one of  $(\rho_j, V_j)$ .

When  $\Gamma \subset SU(2)$  and that  $(\rho_W, W)$  is its defining representation, the McKay graphs are in one-to-one correspondence with the extended Dynkin diagrams of the simply laced Lie algebras. This is the classical McKay correspondence, see appendix 8.

From (10.22) we see that  $n_{ij} = a_{ij}^{\mathcal{R}}$ , i.e. the matter quivers that we defined previously are exactly the McKay graph associated to the matter representation in question. Put differently, the matter content of the daughter theory is encapsulated in the McKay graphs of  $\Gamma$  and with respect to  $\mathcal{R}$  with  $\mathcal{R} = 4$  for spinors and  $\mathcal{R} = 6$  for scalars. This very important point allows us to use known results on McKay graphs such as the McKay correspondence for example.

We mentioned the adjacency matrix as a way to represent a quiver but we will also use sometimes the another matrix called the *incidence matrix* and denoted by  $I$ . In the later, the columns index the arrows and the rows, the nodes such that the  $\alpha$ th arrow from node  $i$  to  $j$  receives a  $-1$  in position  $I_{i\alpha}$ , a  $+1$  in position  $I_{j\alpha}$  and zero elsewhere.

## 10.6 | Gauge anomaly cancellation

Finally, let us discuss the gauge anomaly cancellation. Our fields transform under the adjoint representation of  $SU(N)$ , under the fundamentals of  $SU(N_i)$  and under the anti-fundamentals of  $SU(N_i)$ . The adjoint representation being real, it is self-conjugate and therefore does not contribute to the anomaly. Fundamentals of  $SU(N_i)$  have a  $+1$  contribution to the anomaly and anti-fundamentals of  $SU(N_i)$  have a  $-1$  contribution to the anomaly. Anomaly cancellation therefore imposes that the contribution of the fundamental and of the anti-fundamental of  $SU(N_i)$  cancel each other for each  $i = 0, 1, 2$ , see section (2.4). The bi-fundamental representation  $(\mathbf{N}_i, \overline{\mathbf{N}}_j)$  counts as  $N_j$  fundamentals of  $SU(N_i)$  and as  $N_i$  anti-fundamentals  $SU(N_j)$ . The general condition for anomaly cancellation is therefore

$$\boxed{(a_{ij} - a_{ji})n_i = 0} \quad (10.28)$$

where  $a_{ij}$  are the elements of the adjacency matrix of the quiver and  $n_i$  are the node labels. This condition is equivalent to say that the ranks of the gauge groups must lie in nullspace of the antisymmetrized adjacency matrix.

## 10.7 | Quiver gauge theories and quivers

So far we have seen that a quiver gauge theory can be represented by a quiver where the matter content is represented by the arrows and the gauge groups by the nodes. In more mathematical terms, a quiver gauge theory therefore provides us with a representation of a quiver. Moreover, this quiver has relations coming from the interaction terms, in the superpotential. (\*)

## 10.8 | A simple example: $S = \mathbb{C}^3/\mathbb{Z}_3$

We illustrate the previous discussion by the simple case where  $\Gamma = \mathbb{Z}_3$  acts on  $\mathbb{C}^2$  as

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad (10.29)$$

i.e. the transverse space is the orbifold  $\mathbb{C}^3/\mathbb{Z}_3$ . This simple example is a good first approach in which we will to explain in details each step so that we can go faster afterwards.

### 10.8.1 | Projection

Let us consider a representation  $(\rho, \mathbb{C}^N)$  of  $\mathbb{Z}_3$ . A complete set of irreducible representations of  $\mathbb{Z}_3$  is given by  $\{(\rho_k, V_k)\}_{k=0,1,2}$  with  $V_k = \mathbb{C}$  and

$$\rho_k(g) = \zeta_3^k \quad (10.30)$$

where  $g$  is the generator of  $\mathbb{Z}_3$ . The representation  $(\rho, V)$  can be decomposed as

$$(\rho, V) = \bigoplus_{i=0}^2 N_i(\rho_i, V_i). \quad (10.31)$$

In other words, it is equivalent to the representation

$$\rho(g) = \begin{bmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \zeta_3 & & & \\ \vdots & & & & \ddots & & \vdots \\ & & & & & \zeta_3 & \\ & & & & & & \zeta_3^2 \\ & & & & & & \ddots \\ 0 & & & & & & \zeta_3^2 \end{bmatrix} \begin{matrix} \left. \begin{matrix} \\ \\ \\ \\ \\ \end{matrix} \right\} N_0 \\ \left. \begin{matrix} \\ \\ \\ \end{matrix} \right\} N_1 \\ \left. \begin{matrix} \\ \\ \end{matrix} \right\} N_2 \end{matrix} \quad (10.32)$$

Since  $\dim \rho_i = 1$ , we have

$$N_0 + N_1 + N_2 = N. \quad (10.33)$$

The gauge field configurations that are left invariant under the action of  $\mathbb{Z}_3$  are therefore the ones that satisfy

$$\rho(g)A_\mu\rho(g)^{-1} = A_\mu. \quad (10.34)$$

We actually want this relation to be true for any element of  $\mathbb{Z}_n$  but in this case it is invariant under any element of  $\mathbb{Z}_3$  if and only if it is invariant under the generator  $g$  of  $\mathbb{Z}_3$ , so we only need to impose (10.34).

The constrained is easily solved by using the bi-index notation  $A_{\mu;i\alpha_i,j\beta_j}$  ( $i, j = 0, 1, 2, \alpha_i, \beta_i = 1, \dots, N_i$ ) for the component of the gauge fields. From (11.2), we can see that

$$A_{\mu;i\alpha_i,i\beta_j} \mapsto \rho_i(g) A_{\mu;i\alpha_i,j\beta_j} \rho_j(g)^{-1} = \zeta_n^{i-j} A_{\mu;i\alpha_i,j\beta_j}. \quad (10.35)$$

thus only the configurations with  $A_{\mu;i\alpha_i,j\beta_j} = 0$  if  $i \neq j$  are invariant. The gauge field has therefore a block diagonal form:

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & \\ & A_{\mu;11} & \\ & & A_{\mu;2,2} \end{bmatrix} \quad (10.36)$$

with  $A_{\mu;ij} \equiv (A_{\mu;i\alpha_i,j\beta_j})_{\alpha_i=0,\dots,N_i,\beta_j=0,\dots,N_j}$ . The block  $A_{ii}$  transforms under  $\mathbb{Z}_3$  under  $(\rho_i, V_i)^{N_i}$ . Consequently, the gauge group is now broken to

$$G_{\text{proj}} = \text{U}(N_0) \times \text{U}(N_1) \times \text{U}(N_2), \quad (10.37)$$

the biggest subgroup of  $\text{U}(N)$  that preserves those form of configurations.

Let us study the scalars.  $\mathbb{Z}_3$  acts on the three complex scalars through

$$R(g) = \rho_1^{\oplus 3}(g) = \zeta_3 \mathbb{1}_3 = \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} \quad (10.38)$$

or, equivalently, on the real scalars as  $R(g) = \zeta_3 \mathbb{1}_6$ . According to (9.7)-(9.8), the scalar field configurations that are left invariant satisfy

$$R(g)^m {}_n \rho(g) X^n \rho(g)^{-1} = X^m \quad (10.39)$$

for all  $g \in \mathbb{Z}_3$ . Using the bi-index notations, this becomes

$$X_{i\alpha_i,j\beta_j}^m \mapsto \zeta_n \delta_n^m {}_n \rho_i(g) X_{i\alpha_i,j\beta_j}^n \rho_j(g)^{-1} = \zeta_3^{i-j+1} X_{i\alpha_i,j\beta_j}^m \quad (10.40)$$

$$\bar{X}_{i\alpha_i,j\beta_j}^m \mapsto \zeta_3^{-1} \delta_n^m {}_n \rho_i(g) \bar{X}_{i\alpha_i,j\beta_j}^n \rho_j(g)^{-1} = \zeta_3^{i-j-1} \bar{X}_{i\alpha_i,j\beta_j}^m \quad (10.41)$$

thus only the configurations with  $X_{i\alpha_i,j\beta_j}^n = 0$  if  $i-j+1 \neq 0$  are left invariant and only the configurations with  $\bar{X}_{i\alpha_i,j\beta_j}^n = 0$  if  $i-j-1 \neq 0$  are left invariant. The scalar fields  $X$  have a block off-diagonal form:

$$X^m = \begin{bmatrix} 0 & X_{01}^m & 0 \\ 0 & 0 & X_{12}^m \\ X_{21}^m & 0 & 0 \end{bmatrix}, \quad \bar{X}^m = \begin{bmatrix} 0 & 0 & \bar{X}_{02}^m \\ \bar{X}_{10}^m & 0 & 0 \\ 0 & \bar{X}_{21}^m & 0 \end{bmatrix} \quad (10.42)$$

with the block notations

$$X_{ij}^m \equiv (X_{i\alpha_i,j\beta_j}^m)_{\alpha_i=0,\dots,N_i,\beta_j=0,\dots,N_j}$$

$$\bar{X}_{ij}^m \equiv (\bar{X}_{i\alpha_i,j\beta_j}^m)_{\alpha_i=0,\dots,N_i,\beta_j=0,\dots,N_j}.$$

The block  $X_{ij}^m$  is an  $N_i \times N_j$  block and transforms under the representation  $(\mathbf{N}_i, \bar{\mathbf{N}}_j)$  of  $\text{U}(N_i) \times \text{U}(N_j)$ :

$$X_{i,i+1}^m \in \mathbf{N}_{i+1} \otimes \bar{\mathbf{N}}_i \cong \text{Hom}(V_{i+1}, V_i), \quad (10.43)$$

$$\bar{X}_{i+1,i}^m \in \mathbf{N}_i \otimes \bar{\mathbf{N}}_{i+1} \cong \text{Hom}(V_i, V_{i+1}). \quad (10.44)$$

Let us make an important observation: the form of the scalar fields are the same for every  $m = 0, \dots, 5$ . This can be traced back to the fact that  $R(g) = \zeta_3 \mathbb{1}_6$  so the R-symmetry action of  $\mathbb{Z}_3$  is the same for all  $m$ .

Let us now study the four Weyl fermions  $\psi^a. (*)$

### 10.8.2 | Quiver

We can draw the quiver of this daughter theory. We have three types of bi-fundamental scalar fields:

$$X_{01}^m \in (\mathbf{N}_1, \overline{\mathbf{N}}_0), \quad X_{12}^m \in (\mathbf{N}_2, \overline{\mathbf{N}}_1), \quad X_{20}^m \in (\mathbf{N}_0, \overline{\mathbf{N}}_2). \quad (10.45)$$

In each representation bi-fundamental representation there are six real scalars, i.e. 3 complex scalars. They are each represented by an arrow between the right representations, see fig. 3.

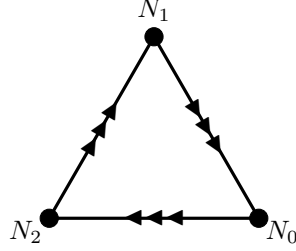


Figure 3: Quiver of the  $\mathbb{C}^3/\mathbb{Z}_3$  daughter theory.

The adjacency matrix of this quiver is

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{bmatrix} \quad (10.46)$$

which is coherent with the fact that the coefficients of the McKay decomposition of  $\rho_1 \oplus \rho_1 \oplus \rho_1$  are

$$\begin{aligned} n_{00} &= 0, & n_{01} &= 3, & n_{02} &= 0, \\ n_{10} &= 0, & n_{11} &= 0, & n_{12} &= 3, \\ n_{20} &= 3, & n_{21} &= 0, & n_{22} &= 0. \end{aligned} \quad (10.47)$$

### 10.8.3 | Gauge anomaly cancellation

We get the three following conditions:

$$\mathrm{SU}(N_0) : N_2 - N_1 = 0, \quad (10.48)$$

$$\mathrm{SU}(N_1) : N_0 - N_2 = 0, \quad (10.49)$$

$$\mathrm{SU}(N_2) : N_1 - N_0 = 0. \quad (10.50)$$

which immediately imply

$$N_0 = N_1 = N_2. \quad (10.51)$$

From (10.33), we get that  $N_0 = N_1 = N_2 = N/3$ , meaning that the daughter theory has quantum gauge symmetry if and only if the parent theory has gauge group  $\mathrm{SU}(N)$  where  $N$  is a multiple of 3. Or, in other words, if the number of D-branes is a multiple of 3.

## 10.9 | $(p+1)$ -dimensional quiver gauge theories

Let us mention that we can generalize our initial brane-world paradigm and consider  $Dp$ -branes in type II string theory (type IIA if  $p$  is even and type IIB if  $p$  is odd) instead of just D3-branes. The spacetime is then of the form

$$M = \mathbb{R}^{1,p} \times \mathbb{R}^{9-p}/\Gamma \quad (10.52)$$

where  $\Gamma$  is a discrete subgroup of  $\mathrm{Spin}(9-p)$ . If  $\Gamma$  is a subgroup of a special holonomy group, we recover a somewhat generalized version of the paradigm that we discussed above. In this case the transverse space is a Calabi-Yau orbifold and some degree of supersymmetry is preserved. Note that the fermionic

and bosonic quivers that coincides. If  $\Gamma$  is not a subgroup of a special holonomy group, then the  $(p+1)$ -dimensional quiver gauge theory that we obtain in the low-energy limit is not supersymmetric. We then have different quivers for the fermions and the bosons, although with the same vertices, by definition.

Recall that the fraction of supercharges that is preserved by compactifying on a Calabi-Yau  $n$ -fold (with  $SU(n)$  holonomy) is  $2^{1-n}$ . Starting from the  $\mathcal{N} = 2$  10-dimensional type IIB string theory with 32 supercharges, this means that

- if we compactify on a 1-fold, we get 32 supercharges in 8 dimensions so  $\mathcal{N} = 2$ ,
- if we compactify on a 2-fold, we get 16 supercharges in 6 dimensions so  $\mathcal{N} = 2$ ,
- if we compactify on a 3-fold, we get 8 supercharges in 4 dimensions so  $\mathcal{N} = 2$ ,
- if we compactify on a 4-fold, we get 4 supercharges in 2 dimensions so  $\mathcal{N} = 4$ .

We will however mostly consider 4-dimensional quiver gauge theories, i.e. living on D3-branes.

## 11 | $\mathcal{N} = 2$ daughter theories

### 11.1 | $S = \mathbb{C} \times \mathbb{C}^2 / \mathbb{Z}_n$

We consider a representation  $(\rho, \mathbb{C}^N)$  of  $\mathbb{Z}_n$ . We decompose it on the set of irreducible representations of  $\mathbb{Z}_n$  as

$$(\rho, \mathbb{C}^N) = \bigoplus_{i=0}^{n-1} N_i(\rho_i, \mathbb{C}). \quad (11.1)$$

In other words, it is equivalent to the representation

$$\rho(g) = \left[ \begin{array}{cccccc} 1 & & \dots & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ \vdots & & & \ddots & & \vdots \\ & & & \zeta_n^{n-1} & & \\ & & & & \ddots & \\ 0 & & \dots & & & \zeta_n^{n-1} \end{array} \right] \left. \begin{array}{l} \left. \begin{array}{c} \vdots \\ \vdots \end{array} \right\} N_0 \\ \left. \begin{array}{c} \vdots \\ \vdots \end{array} \right\} N_{n-1} \end{array} \right\} \quad (11.2)$$

Since  $\dim \rho_i = 1$ ,  $\sum_i N_i = N$ .

The gauge field configurations that are left invariant under the action of  $\mathbb{Z}_n$  are therefore the ones that satisfy

$$\rho(g) A_\mu \rho(g)^{-1} = A_\mu. \quad (11.3)$$

The constrained is easily solved by using the bi-index notation:

$$A_{\mu; i\alpha_i, i\beta_j} \mapsto \rho_i(g) A_{\mu; i\alpha_i, j\beta_j} \rho_j(g)^{-1} = \zeta_n^{i-j} A_{\mu; i\alpha_i, j\beta_j}. \quad (11.4)$$

thus only the configurations with  $A_{\mu; i\alpha_i, j\beta_j} = 0$  if  $i \neq j$  are invariant. The gauge field has therefore a block diagonal form:

$$A_\mu = \begin{bmatrix} A_{\mu; 00} & & & \\ & A_{\mu; 11} & & \\ & & \ddots & \\ & & & A_{\mu; n-1, n-1} \end{bmatrix} \quad (11.5)$$

with  $A_{\mu; ij} \equiv (A_{\mu; i\alpha_i, j\beta_j})_{\alpha_i=0, \dots, N_i, \beta_j=0, \dots, N_j}$ . The block  $A_{ii}$  transforms under  $\mathbb{Z}_n$  as  $(\rho_i, V_i)^{N_i}$ . For now it is only a simple generalization of the case  $\mathbb{C}^3 / \mathbb{Z}_3$ . This makes sense: projection of the gauge field only depends the discrete group  $\Gamma$ , not on the way it acts on  $\mathbb{C}^3$  because it does not transform under R-symmetry.

The gauge group is now broken to

$$G_{\text{proj}} = \prod_{i=0}^{n-1} \text{U}(N_i). \quad (11.6)$$

Now for the scalar fields. The action of  $\mathbb{Z}_n$  that we consider leaves the first component of  $\mathbb{C}^3$  untouched so we take the action  $\mathbf{1} \oplus \mathbf{2}$  where  $\mathbf{2}$  is the usual action of  $\mathbb{Z}_n$  on  $\mathbb{C}^2$ . In other words,

$$R(g) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_n & 0 \\ 0 & 0 & \zeta_n^{-1} \end{bmatrix}. \quad (11.7)$$

Or, equivalently,  $R(g)^m_n = \delta^m_n A_n$  with  $A_m = (1, 1, \zeta_n, \zeta_n, \zeta_n^{-1}, \zeta_n^{-1})$ . The scalar field configurations that are left invariant satisfy

$$R(g)^m_n \rho(g) X^n \rho(g)^{-1} = X^m \quad (11.8)$$

for all  $g \in \mathbb{Z}_n$ . Using the bi-index notations, this becomes

$$\begin{aligned} X_{i\alpha_i, j\beta_j}^m &\mapsto \delta^m_n A_n \rho_i(g) X_{i\alpha_i, j\beta_j}^m \rho_j(g)^{-1} = \delta^m_n A_n \zeta_n^{i-j} X_{i\alpha_i, j\beta_j}^m = \begin{cases} \zeta_n^{i-j} X_{i\alpha_i, j\beta_j}^m, & m = 0, 1 \\ \zeta_n^{i-j+1} X_{i\alpha_i, j\beta_j}^m, & m = 2, 3 \\ \zeta_n^{i-j-1} X_{i\alpha_i, j\beta_j}^m, & m = 4, 5 \end{cases} \\ \bar{X}_{i\alpha_i, j\beta_j}^m &\mapsto \delta^m_n \bar{A}_n \rho_i(g) \bar{X}_{i\alpha_i, j\beta_j}^m \rho_j(g)^{-1} = \delta^m_n \bar{A}_n \zeta_n^{i-j} \bar{X}_{i\alpha_i, j\beta_j}^m = \begin{cases} \zeta_n^{i-j} \bar{X}_{i\alpha_i, j\beta_j}^m, & m = 0, 1 \\ \zeta_n^{i-j-1} \bar{X}_{i\alpha_i, j\beta_j}^m, & m = 2, 3 \\ \zeta_n^{i-j+1} \bar{X}_{i\alpha_i, j\beta_j}^m, & m = 4, 5 \end{cases} \end{aligned} \quad (11.9)$$

thus only the configurations with  $X_{i\alpha_i, j\beta_j}^{0,1} = 0$  if  $i - j \neq 0$ ,  $X_{i\alpha_i, j\beta_j}^{2,3} = 0$  if  $i - j + 1 \neq 0$  and  $X_{i\alpha_i, j\beta_j}^{4,5} = 0$  if  $i - j - 1 \neq 0$  are left invariant (and similarly for the conjugated fields). The scalar fields  $X$  have a the following forms:

$$X^{0,1} = \begin{bmatrix} X_{00}^{0,1} & & 0 \\ & \ddots & \\ 0 & & X_{n-1, n-1}^{0,1} \end{bmatrix}, \quad (11.11)$$

$$X^{2,3} = \begin{bmatrix} 0 & X_{01}^{2,3} & & 0 \\ & \ddots & \ddots & \\ & & 0 & X_{n-2, n-1}^{2,3} \\ X_{n-1, 0}^{2,3} & & & 0 \end{bmatrix}, \quad X^{4,5} = \begin{bmatrix} 0 & & & X_{0, n-1}^{4,5} \\ X_{10}^{4,5} & 0 & & \\ & \ddots & \ddots & \\ 0 & & X_{n-1, n-2}^{4,5} & 0 \end{bmatrix}, \quad (11.12)$$

$$\bar{X}^{0,1} = \begin{bmatrix} \bar{X}_{00}^{0,1} & & 0 \\ & \ddots & \\ 0 & & \bar{X}_{n-1, n-1}^{0,1} \end{bmatrix}, \quad (11.13)$$

$$\bar{X}^{2,3} = \begin{bmatrix} 0 & & & \bar{X}_{0, n-1}^{2,3} \\ \bar{X}_{10}^{2,3} & 0 & & \\ & \ddots & \ddots & \\ 0 & & \bar{X}_{n-1, n-2}^{2,3} & 0 \end{bmatrix}, \quad \bar{X}^{4,5} = \begin{bmatrix} 0 & \bar{X}_{01}^{4,5} & & 0 \\ & \ddots & \ddots & \\ & & 0 & \bar{X}_{n-2, n-1}^{4,5} \\ \bar{X}_{n-1, 0}^{4,5} & & & 0 \end{bmatrix} \quad (11.14)$$

so  $X_{ij}^m$  is an  $N_i \times N_j$  block and transforms under the representation  $(\mathbf{N}_i, \bar{\mathbf{N}}_j)$  of  $\text{U}(N_i) \times \text{U}(N_j)$ :

$$X_{i,i}^{0,1} \in \mathbf{N}_i \otimes \bar{\mathbf{N}}_i \cong \text{Hom}(V_i, V_i), \quad (11.15)$$

$$X_{i, i+1}^{2,3} \in \mathbf{N}_{i+1} \otimes \bar{\mathbf{N}}_i \cong \text{Hom}(V_{i+1}, V_i), \quad (11.16)$$

$$X_{i+1, i}^{4,5} \in \mathbf{N}_i \otimes \bar{\mathbf{N}}_{i+1} \cong \text{Hom}(V_i, V_{i+1}). \quad (11.17)$$

So the scalar fields are split up in three families depending on the way they transform under R-symmetry. We now see a big difference with the case  $\mathbb{C}^3/\mathbb{Z}_3$ : since the R-symmetry does not act the same way on each directions in  $\mathbb{C}^3$ , the invariant scalar field configurations are not the same in each direction either.

Let us draw the quiver for the case  $n = 3$  so that we can compare to 3. We have  $2 \cdot 9 = 18$  real scalar fields in 9 different representations:

$$X_{00}^0, X_{00}^1 \in (\mathbf{N}_0, \overline{\mathbf{N}}_0), \quad X_{11}^0, X_{11}^1 \in (\mathbf{N}_1, \overline{\mathbf{N}}_1), \quad X_{22}^0, X_{22}^1 \in (\mathbf{N}_2, \overline{\mathbf{N}}_2), \quad (11.18)$$

$$X_{01}^2, X_{01}^3 \in (\mathbf{N}_1, \overline{\mathbf{N}}_0), \quad X_{12}^2, X_{12}^3 \in (\mathbf{N}_2, \overline{\mathbf{N}}_1), \quad X_{20}^2, X_{20}^3 \in (\mathbf{N}_0, \overline{\mathbf{N}}_2), \quad (11.19)$$

$$X_{10}^4, X_{10}^5 \in (\mathbf{N}_0, \overline{\mathbf{N}}_1), \quad X_{21}^4, X_{21}^5 \in (\mathbf{N}_1, \overline{\mathbf{N}}_2), \quad X_{02}^4, X_{02}^5 \in (\mathbf{N}_2, \overline{\mathbf{N}}_0), \quad (11.20)$$

We now only have 1 complex scalar in each representation and the quiver is given by 4.

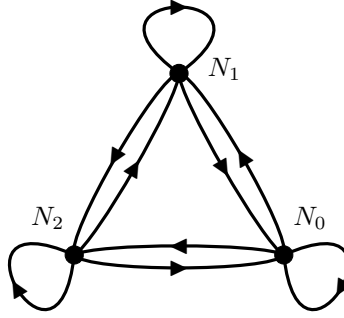


Figure 4: Quiver of the  $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_3$  daughter theory.

It is easy to see how the construction of the quiver generalizes for arbitrary  $n$ . The adjacency matrix is

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \quad (11.21)$$

which is, as it should, coincides with the McKay decomposition of  $\mathbf{1} \oplus \mathbf{2}$ :

$$\begin{aligned} n_{00} = 1, \quad \dots, \quad n_{0,n-1} = 1, \\ \vdots \\ n_{n-1,0} = 1, \quad \dots, \quad n_{n-1,n-1} = 1. \end{aligned} \quad (11.22)$$

Gauge anomaly cancellation now imposes that

$$N_{i-1} - N_{i-1} + N_{i+1} + N_{i+1} = 0 \quad (11.23)$$

for  $i = 0, \dots, n-1$ . Those constraints are always satisfied so the factors  $N_i$  are arbitrary, as long as  $\sum_i N_i = N$  of course.

This quiver is easy to scale up for any  $n$ . We therefore implement that into Mathematica and get the quivers for any  $n$ .

## 11.2 | $S = \mathbb{C} \times \mathbb{C}^2/\mathcal{D}_n$

We quotient  $\mathbb{C}^3$  by the binary dihedral group  $\mathcal{D}_n$  that acts on the last two components. A set of irreducible representations of  $\mathcal{D}_n$  is given by

$$\{(\rho_i, V_i)\}_{i=0, \dots, n+2} \quad (11.24)$$

with  $V_i = \mathbb{C}$  for  $i = 0, \dots, 3$  and  $V_i = \mathbb{C}^2$  for  $i = 4, \dots, n+2$ , so there are 4 one-dimensional representations and  $n-1$  two-dimensional representations. They are explicitly given in section A.1.2. Any representation of  $(\rho, \mathbb{C}^N)$  of  $\mathcal{D}_n$  can be decomposed as

$$()$$
 (11.25)

Using the bi-index notations  $A_{\mu; i\alpha_i, j\beta_j}$  with  $i, j = 0, \dots, n+2$  and  $\alpha_i, \beta_j = 0, \dots, N_i \dim \rho_i - 1$ , the invariant configurations must have  $A_{\mu; ij} = 0$  if  $i \neq j$ , i.e. it must have a diagonal block-form, with block of size  $N_i \times N_i$  for  $i = 0, \dots, 3$  and of size  $2N_i \times 2N_i$  for  $i = 4, \dots, n+1$ . The blocks transforming under the 2-dimensional representations must have the form

$$A_{\mu; ii} = \begin{bmatrix} A_{\mu; i0, i0} & A_{\mu; i0, i1} & \cdots & A_{\mu; i0, i, 2N_i-2} & A_{\mu; i0, i, 2N_i-1} \\ \mp A_{\mu; i0, i1} & A_{\mu; i0, i0} & \cdots & \mp A_{\mu; i0, i, 2N_i-1} & A_{\mu; i0, i, 2N_i-2} \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ A_{\mu; i, 2N_i-2, i, 0} & A_{\mu; i, 2N_i-2, i, 1} & \cdots & A_{\mu; i, 2N_i-2, i, 2N_i-2} & A_{\mu; i, 2N_i-2, i, 2N_i-1} \\ \mp A_{\mu; i, 2N_i-2, i, 1} & A_{\mu; i, 2N_i-2, i, 0} & \cdots & \mp A_{\mu; i, 2N_i-2, i, 2N_i-1} & A_{\mu; i, 2N_i-2, i, 2N_i-2} \end{bmatrix} \quad (11.26)$$

so each block is composed of  $N_i^2$   $2 \times 2$  matrices of the form

$$\begin{bmatrix} A & B \\ \mp B & A \end{bmatrix}. \quad (11.27)$$

We take the “ $-$ ” signs when  $i$  is even and the “ $+$ ” signs when  $i$  is odd. This comes from the fact that the 2-dimensional representations involve alternating signs.

R-symmetry acts on the three complex scalar fields as

$$R(A) = A \quad (11.28)$$

(\*)

### 11.3 | $S = \mathbb{C} \times \mathbb{C}^2/\mathcal{T}, \mathcal{O}, \mathcal{I}$

(\*)

## 12 | $\mathcal{N} = 1$ daughter theories

### 12.1 | $S = \mathbb{C}^3/\mathbb{Z}_n$

Let us now consider the general case of  $\mathbb{Z}_n$  acting on  $\mathbb{C}^3$ , i.e. we want to generalize the case that we treated in section 10.8.

#### 12.1.1 | Actions of $\mathbb{Z}_n$ on $\mathbb{C}^3$

The first thing to do is to specify a representation of  $\mathbb{Z}_n$  on  $\mathbb{C}^3$ . Any representation  $(R, \mathbb{C}^3)$  can be decomposed as

$$R = \bigoplus_{k=0}^{n-1} \rho_k^{\oplus n_k} \quad (12.1)$$

and is therefore equivalent to a block-diagonal representation. This implies that  $\sum_k n_k = 3$  so  $R$  can always be written as

$$R(g) = (\rho_a \oplus \rho_b \oplus \rho_c)(g) = \begin{bmatrix} \zeta_n^a & 0 & 0 \\ 0 & \zeta_n^b & 0 \\ 0 & 0 & \zeta_n^c \end{bmatrix} \quad (12.2)$$

with  $a, b, c$  some arbitrary exponents. We denote the representation (12.2) by  $(a, b, c)$ . Let us make a few remarks on these representations:



- Since we consider  $\mathbb{Z}_n$  as a subgroup of  $SU(3)$ , we must have

$$(a + b + c) \mod n = 0 \quad (12.3)$$

such that  $\det(R(g)) = 1$ .

- Taking  $a$  or  $a + n$  gives the same representation and the same is true for  $b$  and  $c$ , so we can actually restrict ourselves to  $0 < a, b, c < n$ . We can trade the strict inequalities and allow the values 0 or  $n$  if we want to consider trivial representations as well. In this case, we will find that at least one direction of  $\mathbb{C}^3$  is left untouched, i.e. we are in the situation where the orbifold is actually  $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$ , which we will not consider here.
- Permuting  $a, b, c$  gives equivalent representations so we can take  $a, b, c$  to be ordered, so  $0 < a \leq b \leq c < n$ .

The problem has now been reformulated as follows: we are looking for all possible ordered triplets  $(a, b, c)$  such that  $0 < a \leq b \leq c < n$  and (12.3). For  $n = 1$  and  $n = 2$ , it is clear that this is not possible, the possibilities involve at least one trivial representation. For  $n = 3$ , the only possibility is  $(1, 1, 1)$ , the one we used in 10.8. What about bigger values of  $n$ ? First, note that  $0 < a \leq b \leq c < n$  implies that the maximum value of  $a + b + c$  is  $3n - 1$ . From equation (12.3), the only two possibilities are then  $a + b + c = n$  or  $a + b + c = 2n$ . What simplifies the analysis is that the second case can actually be ignored because it is already being taken care of by the first case. Let us explain that in more details: (\*)

Now that we have seen that the only possibility is  $a + b + c = n$ , we find that the maximum value that  $a$  can take is  $\lfloor n/3 \rfloor$ , so  $a = 1, \dots, \lfloor n/3 \rfloor$ .  $c$  can then be expressed in terms of  $a$  and  $b$  as  $c = n - a - b$ . The constraint  $b \leq c$  then implies  $b \leq (n - a)/2$ , i.e.  $b = a, \dots, \lfloor (n - a)/2 \rfloor$ . To summarize, the representations are all the representations of the form

$$(a, b, n - a - b) \quad (12.4)$$

with  $a = 1, \dots, \lfloor n/3 \rfloor$  and  $b = a, \dots, \lfloor (n - a)/2 \rfloor$ . For small values, we get

$$\begin{aligned} \mathbb{Z}_1 &: \text{necessarily involves a trivial representation} \\ \mathbb{Z}_2 &: \text{necessarily involves a trivial representation} \\ \mathbb{Z}_3 &: (1, 1, 1) \\ \mathbb{Z}_4 &: (1, 1, 2) \\ \mathbb{Z}_5 &: (1, 1, 3), (1, 2, 2) (*) \\ \mathbb{Z}_6 &: (1, 1, 4), (1, 2, 3), (2, 2, 2) \\ \mathbb{Z}_7 &: (1, 1, 5), (1, 2, 4), (1, 3, 3), (2, 2, 3) (*) \\ &\vdots \end{aligned}$$

For an arbitrary  $n$ , the total number of different representation is

$$\sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \left\lfloor \frac{n - 3a}{2} \right\rfloor + 1 \right\rfloor = \begin{cases} 3k^2, & \text{if } n = 6k \\ 3k^2 + k, & \text{if } n = 6k + 1, \\ 3k^2 + 2k, & \text{if } n = 6k + 2, \\ 3k^2 + 3k + 1, & \text{if } n = 6k + 3, \\ 3k^2 + 4k + 1, & \text{if } n = 6k + 4, \\ 3k^2 + 5k + 2, & \text{if } n = 6k + 5 \end{cases} \quad (12.5)$$

The details are presented in appendix E.1.

### 12.1.2 | Example: $\mathbb{Z}_5$

For  $\mathbb{Z}_5$ , we saw that there are two nonequivalent ways of acting on  $\mathbb{C}^3$ , see (??). Let us first consider the action  $(1, 1, 3)$

$$R(g) = \begin{bmatrix} \zeta_5 & 0 & 0 \\ 0 & \zeta_5 & 0 \\ 0 & 0 & \zeta_5^3 \end{bmatrix} \quad (12.6)$$

i.e.  $R = \rho_1 \oplus \rho_1 \oplus \rho_3$ .

For the gauge field, the reasoning is exactly the same than for  $\mathbb{Z}_3$  and we get

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & & \\ & \ddots & & \\ & & & A_{\mu;44} \end{bmatrix} \quad (12.7)$$

where each block  $A_{\mu;ii}$  is of size  $N_i \times N_i$ . The projected gauge group is

$$G_{\text{proj}} = \text{U}(N_0) \times \text{U}(N_1) \times \text{U}(N_2) \times \text{U}(N_3) \times \text{U}(N_4). \quad (12.8)$$

The scalar fields transform as

$$X_{i\alpha_i, j\beta_j}^m \rightarrow R(g)^m {}_n \rho_i(g) X_{i\alpha_i, j\beta_j}^n \rho_j(g)^{-1} \quad (12.9)$$

so the invariant configurations must satisfy

$$X_{i\alpha_i, j\beta_j}^m = R(g)^m {}_n \zeta_4^{i-j} \rho_i(g) X_{i\alpha_i, j\beta_j}^n = \begin{cases} \zeta^{i-j+1} X_{i\alpha_i, j\beta_j}^m, & \text{if } m = 0, 1, 2, 3 \\ \zeta^{i-j+3} X_{i\alpha_i, j\beta_j}^m, & \text{if } m = 4, 5. \end{cases} \quad (12.10)$$

and are therefore of the form

$$X^m = \begin{bmatrix} 0 & X_{01}^m & 0 & 0 & 0 \\ 0 & 0 & X_{12}^m & 0 & 0 \\ 0 & 0 & 0 & X_{23}^m & 0 \\ 0 & 0 & 0 & 0 & X_{34}^m \\ X_{40}^m & 0 & 0 & 0 & 0 \end{bmatrix} \quad (12.11)$$

for  $m = 0, 1, 2, 3$  and

$$X^m = \begin{bmatrix} 0 & 0 & 0 & X_{03}^m & 0 \\ 0 & 0 & 0 & 0 & X_{14}^m \\ X_{20}^m & 0 & 0 & 0 & 0 \\ 0 & X_{31}^m & 0 & 0 & 0 \\ 0 & 0 & X_{42}^m & 0 & 0 \end{bmatrix} \quad (12.12)$$

for  $m = 4, 5$ . Gauge anomaly cancellation imposes

$$-N_{i-2} + N_{i+2} - 2N_{i+1} + 2N_{i-1} = 0 \quad (12.13)$$

for  $i = 0, 1, 2, 3, 4$  which implies that

$$N_0 = N_1 = N_2 = N_3 = N_4. \quad (12.14)$$

Now if we choose the representation  $(1, 2, 2)$ , i.e.

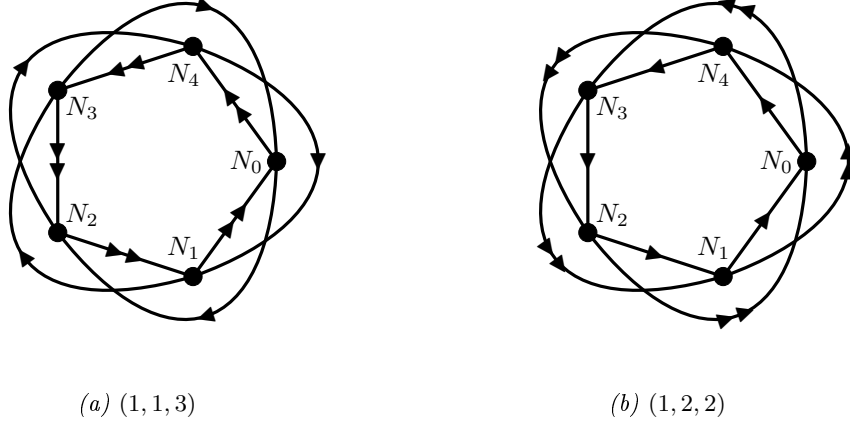
$$R(g) = \begin{bmatrix} \zeta_5 & 0 & 0 \\ 0 & \zeta_5^2 & 0 \\ 0 & 0 & \zeta_5^2 \end{bmatrix} \quad (12.15)$$

we get instead that the invariant field configurations must satisfy

$$X_{i\alpha_i, j\beta_j}^m = R(g)^m {}_n \zeta_4^{i-j} \rho_i(g) X_{i\alpha_i, j\beta_j}^n = \begin{cases} \zeta^{i-j+1} X_{i\alpha_i, j\beta_j}^m, & \text{if } m = 0, 1 \\ \zeta^{i-j+2} X_{i\alpha_i, j\beta_j}^m, & \text{if } m = 2, 3, 4, 5. \end{cases} \quad (12.16)$$

and are therefore of the form

$$X^m = \begin{bmatrix} 0 & X_{01}^m & 0 & 0 & 0 \\ 0 & 0 & X_{12}^m & 0 & 0 \\ 0 & 0 & 0 & X_{23}^m & 0 \\ 0 & 0 & 0 & 0 & X_{34}^m \\ X_{40}^m & 0 & 0 & 0 & 0 \end{bmatrix} \quad (12.17)$$

Figure 5: Quivers of the  $\mathbb{C}^3/\mathbb{Z}_5$  daughter theories.

for  $m = 0, 1$  and

$$X^m = \begin{bmatrix} 0 & 0 & X_{02}^m & 0 & 0 \\ 0 & 0 & 0 & X_{13}^m & 0 \\ 0 & 0 & 0 & 0 & X_{24}^m \\ X_{30}^m & 0 & 0 & 0 & 0 \\ 0 & X_{41}^m & 0 & 0 & 0 \end{bmatrix} \quad (12.18)$$

for  $m = 2, 3, 4, 5$ . Gauge anomaly cancellation imposes

$$-2N_{i+2} + 2N_{i-2} - N_{i+1} + N_{i-1} = 0(*) \quad (12.19)$$

for  $i = 0, 1, 2, 3, 4$  which implies that

$$N_0 = N_1 = N_2 = N_3 = N_4. \quad (12.20)$$

Note that, even we thought that  $(1, 1, 3)$  and  $(1, 2, 2)$  were different representation, that are actually equivalent in the sense that  $(1, 1, 3) + (1, 1, 3) = (2, 2, 1)$ . The two quivers in fig. 7 should therefore be the same. And indeed, upon further inspection, the two are the same. To see this, we can rename the vertices in the second graph as  $N_1 \rightarrow N_3, N_2 \rightarrow N_1, N_3 \rightarrow N_4, N_4 \rightarrow N_2$ . This renaming defines the bijection between the two graphs. A more pragmatic way to see this is to look at the adjacency matrices:

$$a_{(1,1,3)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 2 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \quad a_{(1,2,2)} = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}. \quad (12.21)$$

Changing the names of the vertices is equivalent to swapping line and columns. For example,

### 12.1.3 | General $\mathbb{Z}_n$

Let us now consider the general action

$$R(g) = \begin{bmatrix} \zeta_n^a & 0 & 0 \\ 0 & \zeta_n^b & 0 \\ 0 & 0 & \zeta_n^c \end{bmatrix} \quad (12.22)$$

of  $\mathbb{Z}_n$  on  $\mathbb{C}^3$ , where  $(a, b, c)$  one of the representation that we studied before. In particular, recall that  $a + b + c = n$ . Following the same reasoning than before, we get that the gauge field of the form

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & \\ & \ddots & \\ & & A_{\mu;n-1,n-1} \end{bmatrix}. \quad (12.23)$$

Invariant scalar field configurations transform as

$$X_{i\alpha_i, j\beta_j}^m = \begin{cases} \zeta_n^{i-j+a} X_{i\alpha_i, j\beta_j}^m, & \text{if } m = 0, 1 \\ \zeta_n^{i-j+b} X_{i\alpha_i, j\beta_j}^m, & \text{if } m = 2, 3 \\ \zeta_n^{i-j+c} X_{i\alpha_i, j\beta_j}^m, & \text{if } m = 4, 5 \end{cases} \quad (12.24)$$

so

$$X_{j-a, j}^{0,1} \in (\mathbf{N}_j, \overline{\mathbf{N}}_{j-a}), \quad (12.25)$$

$$X_{j-b, j}^{2,3} \in (\mathbf{N}_j, \overline{\mathbf{N}}_{j-b}), \quad (12.26)$$

$$X_{j-c, j}^{4,5} \in (\mathbf{N}_j, \overline{\mathbf{N}}_{j-c}) \quad (12.27)$$

are the only possible non-vanishing components. This allows us to quickly draw all the possible quivers for a given  $n$ . Once again, the difficulty is only computational, not conceptual. This can therefore easily be implemented into Mathematica and we get the quiver for any  $n$  and any representation  $(a, b, c)$ .

## 12.2 | $S = \mathbb{C}^3/\Delta(3n^2), \Delta(6n^2)$

### 12.3 | $S = \mathbb{C}^3/\Sigma_{36 \times 3}, \Sigma_{60 \times 3}, \Sigma_{168 \times 3}, \Sigma_{216 \times 3}, \Sigma_{360 \times 3}$

### 12.4 | $S = \mathbb{C}^3/(\mathbb{Z}_m \times \mathbb{Z}_n)$

#### 12.4.1 | Representations of $\mathbb{Z}_m \times \mathbb{Z}_n$

We consider the group  $\mathbb{Z}_m \times \mathbb{Z}_n$ . Let us denote by  $\{\mu_i\}_{i=0, \dots, m-1}$  and  $\{\sigma_j\}_{j=0, \dots, n-1}$  two complete set of irreducible representation of  $\mathbb{Z}_m$  and  $\mathbb{Z}_n$  respectively, with

$$\mu(g_1) = \zeta_m^i \quad (12.28)$$

$$\sigma_j(g_2) = \zeta_n^j \quad (12.29)$$

where  $g_1$  and  $g_2$  are the generating elements of  $\mathbb{Z}_m$  and  $\mathbb{Z}_n$  respectively.  $\mathbb{Z}_m \times \mathbb{Z}_n$  is of order  $mn$  and possesses the same number of equivalency classes (abelian). It has therefore  $mn$  irreducible representations. Since the group is abelian, they are all of dimension 1. Let us denote by  $\{T_k\}_{k=0, \dots, m+n-1}$  a complete set of irreducible representations. Since we have a product group, for any  $k$  there exists indices  $i(k)$  and  $j(k)$  such that  $T_k = \mu_{i(k)} \otimes \sigma_{j(k)}$ . We choose the indices  $i(k)$  and  $j(k)$  such that

$$\begin{aligned} T_0 &= \mu_0 \otimes \mu_0, \\ T_1 &= \mu_0 \otimes \mu_1, \\ T_2 &= \mu_0 \otimes \mu_2, \\ &\vdots \\ T_{n-1} &= \mu_0 \otimes \mu_{n-1}, \\ T_n &= \mu_1 \otimes \mu_0, \\ &\vdots \\ T_{2n-1} &= \mu_1 \otimes \mu_{n-1}, \\ &\vdots \\ T_{mn-1} &= \mu_{m-1} \otimes \mu_{n-1}. \end{aligned}$$

That is, we take

$$\begin{cases} i(k) = \lfloor k/n \rfloor \\ j(k) = k \bmod n \end{cases} \Leftrightarrow k = i(k)n + j(k). \quad (12.30)$$

Note that this is simply a dictionary between a line notation and a matrix notation and that it is indeed a bijection  $k \Leftrightarrow i, j$ . In this way, we can proceed to similar manipulations than before, were we used line notations.

Any representation  $R$  of  $\mathbb{Z}_m \times \mathbb{Z}_n$  can be decomposed as

$$R = \bigoplus_{i,j} N_k T_k = \bigoplus_{i,j} N_{ij}(\mu_i \otimes \sigma_j) \quad (12.31)$$

with  $N_k = N_{i(k)j(k)}$ . We must have  $\sum_k N_k = 3$ . In other words, any representation of  $\mathbb{Z}_m \times \mathbb{Z}_n$  on  $\mathbb{C}^3$  is equivalent to

$$R(g_1, g_2) = [(\mu_a \otimes \sigma_{a'}) \oplus (\mu_{b'} \otimes \sigma_{b'}) \oplus (\mu_c \otimes \sigma_{c'})](g_1, g_2) = \begin{bmatrix} \xi_m^a \xi_n^{a'} & 0 & 0 \\ 0 & \xi_m^b \xi_n^{b'} & 0 \\ 0 & 0 & \xi_m^c \xi_n^{c'} \end{bmatrix}. \quad (12.32)$$

The determinant condition is

$$\xi_m^{a+b+c} = \xi_n^{-a'-b'-c'} \quad (12.33)$$

$$\Leftrightarrow (a+b+c) \bmod m = (a'+b'+c') \bmod n \quad (12.34)$$

$$R(g_1, g_2) = \begin{bmatrix} \xi_m & 0 & 0 \\ 0 & \xi_n & 0 \\ 0 & 0 & \xi_m^{-1} \xi_n^{-1} \end{bmatrix}. \quad (12.35)$$

#### 12.4.2 | Projection

Let us start by the gauge field. We consider a unitary representation  $(\rho, \mathbb{C}^N)$  of  $\mathbb{Z}_m \times \mathbb{Z}_n$  on  $\mathbb{C}^n$  and decompose it as

$$\rho = \bigoplus_{i,j} N_k T_k \quad (12.36)$$

such that

$$\rho(g) = \begin{bmatrix} T_0(g) & & \cdots & & & 0 \\ & \ddots & & & & \\ & & T_0(g) & & & \\ \vdots & & & \ddots & & \vdots \\ & & & & T_{mn-1}(g) & \\ & & & & & \ddots \\ 0 & & \cdots & & & T_{mn-1}(g) \end{bmatrix} \left. \begin{array}{l} \vdots \\ \vdots \\ \vdots \end{array} \right\} N_0 \quad (12.37)$$

We can now use our usual bi-index notation  $A_{\mu; k\alpha_k, l\beta_l}$  with  $k, l = 0, \dots, mn-1$  and  $\alpha_k, \beta_k = 0, \dots, N_k$  but instead it is more convenient to come back to our matrix notation by writing the block  $A_{\mu; k, l}$  as  $A_{\mu; i(k)j(k), i(l)j(l)}$  that we simply denote by  $A_{\mu; ij, i'j'}$  with  $i, i' \in \{0, \dots, n-1\}$  and  $j, j' \in \{0, \dots, m-1\}$ . So for  $m=2, n=3$  for example, the link between the two notations is

$$\begin{bmatrix} A_{\mu;00} & A_{\mu;01} & A_{\mu;02} & A_{\mu;03} & A_{\mu;04} & A_{\mu;05} \\ A_{\mu;10} & A_{\mu;11} & A_{\mu;12} & A_{\mu;13} & A_{\mu;14} & A_{\mu;15} \\ A_{\mu;20} & A_{\mu;21} & A_{\mu;22} & A_{\mu;23} & A_{\mu;24} & A_{\mu;25} \\ A_{\mu;30} & A_{\mu;31} & A_{\mu;32} & A_{\mu;33} & A_{\mu;34} & A_{\mu;35} \\ A_{\mu;40} & A_{\mu;41} & A_{\mu;42} & A_{\mu;43} & A_{\mu;44} & A_{\mu;45} \\ A_{\mu;50} & A_{\mu;51} & A_{\mu;52} & A_{\mu;53} & A_{\mu;54} & A_{\mu;55} \end{bmatrix} = \begin{bmatrix} A_{\mu;00,00} & A_{\mu;00,01} & A_{\mu;00,02} \\ A_{\mu;01,00} & A_{\mu;01,01} & A_{\mu;01,02} \\ A_{\mu;02,00} & A_{\mu;02,01} & A_{\mu;02,02} \\ A_{\mu;10,00} & A_{\mu;10,01} & A_{\mu;10,02} \\ A_{\mu;11,00} & A_{\mu;11,01} & A_{\mu;11,02} \\ A_{\mu;12,00} & A_{\mu;12,01} & A_{\mu;12,02} \end{bmatrix} \begin{bmatrix} A_{\mu;00,10} & A_{\mu;00,11} & A_{\mu;00,12} \\ A_{\mu;01,10} & A_{\mu;01,11} & A_{\mu;01,12} \\ A_{\mu;02,10} & A_{\mu;02,11} & A_{\mu;02,12} \\ A_{\mu;10,10} & A_{\mu;10,11} & A_{\mu;10,12} \\ A_{\mu;11,10} & A_{\mu;11,11} & A_{\mu;11,12} \\ A_{\mu;12,10} & A_{\mu;12,11} & A_{\mu;12,12} \end{bmatrix} \quad (12.38)$$

So instead of considering  $A_\mu$  to be a single  $mn \times mn$  matrix of element  $A_{\mu;kl}$ , where  $A_{\mu;kl}$  are  $N_k \times N_l$  matrices, we consider it as an  $m \times m$  where each element  $A_{\mu;ii'}$  (line  $i$  column  $i'$ ) is itself an  $n \times n$  matrices with elements  $A_{\mu;ij, i'j'}$  (line  $j$  column  $j'$ ), as shown above.

Using these notations, the gauge field transforms as

$$A_{\mu; ij, i'j'} \mapsto (\mu_i(g) \otimes \sigma_j(g)) A_{\mu; ij, i'j'} (\mu_{i'}(g) \otimes \sigma_{j'}(g))^{-1} = \zeta_m^{i-i'} \zeta_n^{j'-j} A_{\mu; ij, i'j'} \quad (12.39)$$

so invariant configurations can possess non-vanishing components  $A_{\mu; ij, i'j'}$  only if

$$\zeta_m^{i-i'} = \zeta_n^{j'-j} \quad (12.40)$$

$$\Leftrightarrow (i - i') \bmod m = (j' - j) \bmod n \quad (12.41)$$

$$\Leftrightarrow j' = j + |i' - i|. \quad (12.42)$$

This means that the submatrices  $A_{\mu;ii'}$  has an off-diagonal block form with offset  $|i' - i|$ . Once again, for a general  $n$ , the difficulty is only computational, not conceptual. This can therefore easily be implemented into Mathematica and we get the form of the gauge field for any  $n$ .

For the scalar fields, we have

$$X_{ij,i'j'}^m \mapsto R(g)^m_n (\mu_i(g) \otimes \sigma_j(g)) X_{ij,i'j'}^n (\mu_{i'}(g) \otimes \sigma_{j'}(g))^{-1} \quad (12.43)$$

$$= \begin{cases} \zeta_m^{i-i'+1} \zeta_n^{j-j'} X_{ij,i'j'}^m, & m = 0, 1 \\ \zeta_m^{i-i'} \zeta_n^{j-j'+1} X_{ij,i'j'}^m, & m = 2, 3 \\ \zeta_m^{i-i'-1} \zeta_n^{j-j'-1} X_{ij,i'j'}^m, & m = 4, 5. \end{cases} \quad (12.44)$$

So a configuration is invariant if and only if the only non-vanishing component satisfy

$$m = 0, 1 : (i - i' + 1) \bmod m = (j' - j) \bmod n \quad (12.45)$$

$$m = 2, 3 : (i - i') \bmod m = (j' - j - 1) \bmod n \quad (12.46)$$

$$m = 4, 5 : (i - i' - 1) \bmod m = (j' - j + 1) \bmod n. \quad (12.47)$$

## 13 | A note about projective representations, discrete torsion and deformations

### 13.1 | Projective representations and discrete torsion

Up until now, and in particular in all of our computations in section ??, we only considered ordinary representations, i.e. linear representations. We can however use a more general representations such as projective representations for example. That is, representations  $\rho$  of  $\Gamma$  such that for all  $\gamma_1, \gamma_2 \in \Gamma$ ,

$$\rho(\gamma_1)\rho(\gamma_2) = A(g_1, g_2)\rho(\gamma_1\gamma_2) \quad (13.1)$$

for some factor  $A(\gamma_1, \gamma_2)$ . The  $A(\gamma_1, \gamma_2) = 1$  corresponding to linear representations. For consistency reasons, the factor  $A(g_1, g_2)$  cannot be completely arbitrary and must a cocycle condition. As a result, the possibilities for  $A(\gamma_1, \gamma_2)$  are classified by the second cohomology group  $H^2(\Gamma, \mathbb{C}^*)$ , also called *discrete torsion*. There exists a projective representation if and only if the latter does not vanish. This new liberty, whenever admissible, provides new classes of quiver gauge theories that can be remarkably different from the ones we considered up until now, with no discrete torsion.

What happens is that if one turns on an NSNS B-fiels alors the worldvolume, then the moduli space is expected to a non-commutative version of a Calabi-Yau space. This haw the discrete torsion is physically realized. Another, and actually equivalent through T-duality, way of studying gauge theories is to consider D-branes streched between configurations of NS5-branes<sup>6</sup>. This is the *Hanany-Witten setup*.

The discrete torsion also appears when writing the full open strin gpartition function that includes the twisted sector where there is ambiguity up to a phase factor. As a consequence from modular invariance, the latter must satisfy certain cocycle conditions. It is precisely the deiscrete torsion. Note that this has been found to be true only for the open string sector.

### 13.2 | Quiver gauge theories deformations and conifold

We can deforming the singular algebraic description of the orbifold with a field into a family smooth surfaces. The resulting total space is the *conifold*.

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<sup>6</sup>D5-branes are charged under the Ramond-Ramond field whose quanta comes from the Ramond-Ramond sector but NS5-branes are charges under the Kalb-Ramonf field whose quanta comes from the Neveu-Schwarz. On the worldvolume of an NS5-brane (6-dimensional) propagates a superstring, this called the *little string*.

## Part III

# Toric singularities

The next best thing to orbifold singularities is the toric singularities. The a specific class of supersymmetric gauge theories whose space of vacua is toric is called *toric quiver gauge theories*. In this case, the inverse algorithm has been formalized in [FHH01a].

## 14 | Toric geometry

The interest of toric varieties lies in the fact that all its defining data can be encoded in a simple auxilliary object called a fan. This data is purely combinatoric (discrete quantities) so complicated geometric problems are often reduced to simpler combinatorics problems.

### 14.1 | Cones

Let us consider a lattice  $N = \mathbb{Z}^m$  and  $N_{\mathbb{R}} = N \otimes \mathbb{R}$  the real vector space that we get by allowing real coefficients. Given  $v_1, \dots, v_n$  of the lattice  $N$ , a *cone*<sup>7</sup>  $\sigma \subset N_{\mathbb{R}}$  is a the subset of the vector space  $N_{\mathbb{R}}$  containing all points that are positive linear combination of the vectors  $v_1, \dots, v_n$ , that is

$$\sigma = \left\{ \sum_{i=1}^n a_i v_i \mid a_i \geq 0, a_i \in \mathbb{R} \right\} \quad (14.1)$$

and such that  $\sigma \cap (-\sigma) = \{0\}$ . The last condition is the strong convexity condition and it imposes that our cone has to be “acute”. The dimension of a cone is the dimension of  $\text{Span}(\sigma)$ .

The dual lattice  $N^*$  of  $N$  is the set of linear functionals on  $L$  which take integer values on each point of  $N$ :

$$N^* = \{f \in (\text{Span}(L))^* \mid \forall x \in N, f(x) \in \mathbb{Z}\}. \quad (14.2)$$

We denote by  $M$  the dual lattice of  $N$  and by  $M_{\mathbb{R}}$  the vector space we get from it by allowing real coefficients. For a given cone  $\sigma$  in  $N_{\mathbb{R}}$ , we can define the *dual cone*  $\sigma^\vee$  in  $M_{\mathbb{R}}$  as

$$\sigma^\vee = \{m \in M_{\mathbb{R}} \mid \langle u, m \rangle \geq 0, \forall u \in \sigma\}. \quad (14.3)$$

Let us denote by  $H_m$  the hyperplane gievn by the dual lattice point  $m \in \sigma^\vee$ :  $H_m = \{v \in N_{\mathbb{R}} \mid \langle v, m \rangle = 0\}$  and the closed half-space  $H_m^+ = \{v \in N_{\mathbb{R}} \mid \langle v, m \rangle \geq 0\}$ . We then say that a hyperplane *supports* a cone if the closed half-space of the hyperplane completely contains the cone. A *face* of a cone is the intersection of the cone with a supporting hyperplane. Put differently, a convex subset  $\tau$  is a face of  $\sigma$  if and only whenever  $u, v \in \sigma$  satisfy  $u + v \in \tau$  then  $u \in \tau$  and  $v \in \tau$ . Note that a hyper plane  $H_m$  supports  $\sigma$  if and only if  $m \in \sigma^\vee$ .

### 14.2 | Toric varieties

An *algebraic group* is a group that is also an algebraic variety and such the product and inversion are regular maps on the variety. Any product of  $\mathbb{C}^*$  is an algebraic group that we call *algebraic tori*. An affine variety  $X \subset \mathbb{C}^n$  is a *affine toric variety* if it contains the algebraic torus  $\mathbb{T} = (\mathbb{C}^*)^n$  as a dense open subset such that the action of  $\mathbb{T}$  on itself extends to an action  $\mathbb{T} \times X \rightarrow X$  on  $X$ .

**Example.** Let us enumerate some examples of affine toric varieties.

- $(\mathbb{C}^*)^n$  and  $\mathbb{C}^n$  are naturally provided with an embedding and an action of the torus and are toric varieties

<sup>7</sup>More precisely, a strongly convex rational polyhedral cone.

- $V = Z(x^3 - y^2)$  with torus embedding

$$\left( \begin{array}{ccc} \mathbb{C}^* & \hookrightarrow & V \\ t & \mapsto & (t^2, t^3) \end{array} \right). \quad (14.4)$$

and the action  $t \cdot (u, v) \mapsto (t^2u, t^3v)$ .

### 14.3 | Toric variety of a cone

A *semigroup*  $S$  is a set with an internal associative  $+$  operation and a neutral element 0. It differs from a group in that elements need not have an inverse. A semigroup is *affine* if it can be embedded as a subsemigroup in a lattice  $\mathbb{Z}^m$  (*integral*) and if there exists a finite set  $\mathcal{A}$  such that  $S = \mathbb{N}\mathcal{A}$  (*finitely generated*).

What is interesting is that affine semigroups allows us to construct affine varieties. For this, we need to introduce the notion of *semigroup algebra*  $\mathbb{C}[S]$  associated to any semi group  $S$ . It is the algebra generated by elements  $\chi^u$  indexed by elements  $u \in S$ . The semigroup operation  $+$  induces a multiplication operation for the  $\chi^u$  in  $\mathbb{C}[S]$  as  $\chi^u \cdot \chi^v = \chi^{u+v}$ . For instance, the semigroup algebra of  $S = \mathbb{N}^n$  is simply  $\mathbb{C}[x_1, \dots, x_n]$  and the semigroup algebra of  $S = \{2, 3, \dots\} \subset \mathbb{N}$  is  $\mathbb{C}[x, y]/I(x^3 - y^2)$ . The important result is now that if  $S$  is an affine semigroup then  $\text{Spec}(\mathbb{C}[S])$  is an affine variety.

Given a cone  $\sigma$  and its dual cone  $\sigma^\vee$ ,  $S_\sigma = \sigma^\vee \cap M$  is finitely generated and hence an affine semigroup. In this way, we may obtain an affine variety from a cone as  $U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma])$ , this the *variety if the cone*  $\sigma$ . An important result is that if  $K[V]$  is the coordinate ring of an affine variety  $V$ , then  $V = \text{Spec}(K[V])$ . From this, we see that  $\mathbb{C}[S_\sigma]$  is exactly the coordinate ring of the variety  $U_\sigma$  we are looking for.

We have seen how to get an affine variety  $U_\sigma$  from a cone  $\sigma$ . This variety is in fact toric. If  $n$  denotes the rank of the lattice  $\mathbb{Z}S_\sigma$ , then there is torus action  $\mathbb{T} = (\mathbb{C}^*)^n$  acting on  $U_\sigma$ . The torus that it contained is called the *torus corresponding to a lattice*  $N = \mathbb{Z}^m$  and is  $\mathbb{Z}_N = (\mathbb{C}^*)^n$ , so that the rank of the lattice equals the dimension of the torus:

$$T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* = \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*). \quad (14.5)$$

To torus acts on  $U_\sigma$  as  $t \cdot \gamma : S_\sigma \rightarrow \mathbb{C} : m \mapsto \chi^m(t)\gamma(m)$ , for all  $t \in T_N$  and  $\gamma \in \sigma$  (so  $\gamma : S_\sigma \rightarrow \mathbb{C}$ ). To summarize, the steps to extract the  $m$ -complex dimensional toric variety from a given cone  $\sigma$  in  $N_{\mathbb{R}}$  are the following:

1. find the dual cone  $\sigma^\vee$
2. find the intersection  $S_\sigma \sigma^\vee \cap \mathbb{Z}^m$ , it is a finitely generated semigroup. Find a minimal set of generating vectors  $\{w_1, \dots, w_r\}$
3. find the polynomial ring  $\mathbb{C}[S_\sigma]$  by exponentiating the coordinates of  $S_\sigma$
4. find the relations between the elements of  $\mathbb{C}[S_\sigma]$  corresponding to the generating vectors  $w_i$ . Such a relation looks like

$$\sum_{i \in I} m_i w_i = \sum_{j \in J} m_j w_j, \quad m_i, m_j \in \mathbb{N} \Rightarrow p(x) = \prod_{i \in I} x^{m_i} x_i - \prod_{j \in J} x^{m_j} x_j \quad (14.6)$$

where  $I \cup J = \{1, \dots, r\}$  and  $I \cap J = \emptyset$ . If  $\{p_1, p_2, \dots\}$  is the set of all those relations, then  $I(\{p_1, p_2, \dots\})$  is a prime ideal and  $\mathbb{C}[S_\sigma] = \mathbb{C}[x_1, \dots, x_r]/I(p_1, p_2, \dots)$

5.  $\mathbb{C}[S_\sigma]$  is the coordinate ring of desired the toric variety. We can recover the variety explicitly as  $\text{Spec}(\mathbb{C}[S_\sigma])$ . It might be easier to use the fact that  $\text{Spec}[K[V]] = V$ , so  $\mathbb{C}[S_\sigma]$  is actually the coordinate ring of  $U_\sigma$  and  $U_\sigma = Z(\{p_1, p_2, \dots\})$ .

The toric variety  $U_\sigma$  has the same dimension than  $\sigma$ .

**Example.** If we consider the cone  $\sigma$  generated by  $e_1$  and  $2e_1 - e_2$  in  $\mathbb{Z}^2$  (where  $\{e_1, e_2\}$  is the canonical basis of  $\mathbb{Z}^2$ ), then the dual cone  $\sigma^\vee$  is generated by  $e_1$  and  $e_1 + 2e_2$ . We therefore have  $S_\sigma = \sigma^\vee \cap \mathbb{Z}^2$ . This subset of  $\mathbb{Z}^2$  is spanned by  $e_1, e_1 + e_2$  and  $e_1 + 2e_2$ :

$$S_\sigma = \text{Span}(\{(1, 0), (1, 1), (1, 2)\}). \quad (14.7)$$



Note that, even though it is a 2-dimensional cone, i.e. the intersection of this cone and the lattice actually need three vectors to be completely generated. This is often the case when dealing with lattices. By exponentiating we get the semigroup algebra  $\mathbb{C}[S_\sigma] = \mathbb{C}[x, xy, xy^2]$ . If we denote  $u = x, v = xy$  and  $w = xy^2$ , relation between those three variables is  $uw = v^2$ , so  $\mathbb{C}[S_\sigma] = \mathbb{C}[u, v, w]/I(v^2 - uw)$ . It only remains to compute the spectrum, and we find that

$$\text{Spec}(\mathbb{C}[S_\sigma]) = \mathbb{C}^2/\mathbb{Z}^2. \quad (14.8)$$

In conclusion the toric variety associated to the cone  $\sigma$  is the abelian orbifold  $\mathbb{C}^2/\mathbb{Z}^2$ .

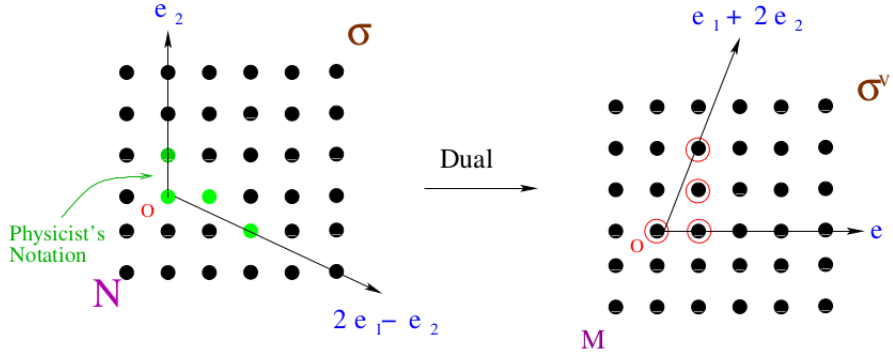


Figure 6: Representations of the cone and dual cone in the lattice  $\mathbb{Z}^2$ .

The previous example illustrate a more genral result:

**Proposition.** All abelian orbifolds are toric varieties.

Note that this result is not that surprising considering the general expression (14.10).

**Example.** Let us start with  $\sigma = \text{Cone}(\{e_1, 2e_1 + e_2\})$ . We then get  $\sigma^\vee = \text{Cone}(\{e_2, e_1 - 2e_2\})$  so  $S_\sigma = \text{Span}_{\mathbb{Z}^+}(\{e_2, e_1 - 2e_2\})$  and  $\mathbb{C}[S_\sigma] = \mathbb{C}[y, xy^{-2}] = \mathbb{C}[u, v]$ .

Now what happens with faces? An inclusion of face  $\tau \subset \sigma$  gives an inclusions of the semigroups  $S_\tau \subset S_\sigma$  and  $\mathbb{C}[S_\tau] \subset \mathbb{C}[S_\sigma]$ . This induces a morphism  $U_\tau \rightarrow U_\sigma$  of affine toric varieties and it happens the this morphism is an open embedding.

**Example.** If  $\sigma = \text{Cone}(e_1, e_2)$  then we find  $U_\sigma = \mathbb{C}^2$ . There are three faces:  $\rho_1 = \text{Cone}(e_1)$ ,  $\rho_2 = \text{Cone}(e_2)$  and the origin 0. These faces can be described by hyperplane  $H_{e_2}, H_{e_1}(*),$  and  $H_{e_1+e_2}$  respectively, so we get

$$\mathbb{C}[S_{\rho_1}] = \mathbb{C}[x, y, y^{-1}], \mathbb{C}[S_{\rho_2}] = \mathbb{C}[x, x^{-1}, y], \mathbb{C}[S_0] = \mathbb{C}[x, y, x^{-1}, y^{-1}]. \quad (14.9)$$

They define the varieties  $U_{\rho_1} = \mathbb{C}^* \times \mathbb{C}$ ,  $U_{\rho_2} = \mathbb{C} \times \mathbb{C}^*$  and  $(\mathbb{C}^*)^2(*),$  which are all open subvarieties of  $\mathbb{C}^2$ .

## 14.4 | More general constructions

In general, an affine variety canbe defined by an ideal in  $\mathbb{C}[x_1, \dots, x_n]$ . In the same way, a toric affine variety can be defined by a *toric ideal* (or *binomial ideal*), i.e. an ideal in  $\mathbb{C}[x_1, \dots, x_n]$  generated by binomials (polynoamils with precisely two non-zero coefficients).

**Theorem.** Let  $V$  be an affine variety. The following are equivalent:

- $V$  is toric,
- $V = \text{Spec}(\mathbb{C}[S])$  for an affine semigroup  $S$ ,
- $V = V(I)$  for a toric ideal.

This is a very powerfull theorem since is states that an affine avariety is toric if and only if the generating polynomials of its ideal are binomials.

### 14.5 | Toric variety of a fan

A *fan* is a collection  $\Delta$  of cones in  $N_{\mathbb{R}}$  such that each face of a cone is also a cone and the intersection of two cones is a face of each. Those two requirements imply in particular that the intersection of two cones in a fan is again a cone in the fan. Note that to any cone corresponds a fan containing the cone itself and all its faces.

For a cone  $\sigma$  in  $N_{\mathbb{R}}$  the variety  $U_{\sigma}$  contains the torus  $T_N$  so fan in  $N_{\mathbb{R}}$  produces a collection of varieties that all contain the same torus  $T_N$ . Gluing together the affine varieties we obtain  $X_{\Delta}$ , it is clear that  $T_N$  is an open subset of  $X_{\Delta}$  and that  $T_N$  acts on  $X_{\Delta}$ . This is called the *toric variety of a fan*. For the toric variety of a fan, each cone in the fan turns out to be determines an orbit  $O(\sigma)$  of the torus.

Generally speaking, an  $m$ -dimensional algebraic variety of a fan can be obtained as a particular holomorphic quotient of  $\mathbb{C}^n$ . If  $\mathbb{Z}_{\Delta} \subset \mathbb{C}^n$  a set of points and  $G \cong (\mathbb{C}^*)^{n-m} \times \Gamma$  is the group formed by the algebraic torus and an abelian discrete group  $\Gamma$ , then

$$X_{\Delta} = \frac{\{\mathbb{C}^n \setminus \mathbb{Z}_{\Delta}\}}{G}. \quad (14.10)$$

Let us explain this construction.

If  $\Delta$  is an  $m$ -dimensional fan, we denote by  $\Delta(j)$  ( $j \leq n$ ) the collection of  $j$ -dimensional cones in  $\Delta$ . The set  $\Delta(1) \subset \Delta$  is then the collection of all one-dimensional cones in  $\Delta$ . Each cone in  $\Delta(1)$  is generated by a vector of  $N = \mathbb{Z}^n$  and we denote those generating vectors by  $v_1, \dots, v_n$ . Note that we always have  $n \geq m$ . Each vector  $v_i$  has component  $v_i^k$  ( $i = 1, \dots, n, k = 1, \dots, m$ ) so we can construct an  $m \times n$  matrix by putting the vectors in columns:

$$\begin{bmatrix} v_1^1 & \cdots & v_n^1 \\ \vdots & & \vdots \\ v_1^m & \cdots & v_n^m \end{bmatrix} \quad (14.11)$$

The kernel of this map from  $\mathbb{C}^n$  to  $\mathbb{C}^m$  is  $\mathbb{C}^{n-m}$ . Let  $Q^a$  ( $a = 1, \dots, n-m$ ) be vectors that generate this kernel, i.e.

$$\sum_{i=1}^n v_i^k Q_i^a = 0 \quad (14.12)$$

for all  $a = 1, \dots, n-m$  and  $k = 1, \dots, m$ . We can now consider the map

$$\phi : \begin{pmatrix} \mathbb{C}^n & \longrightarrow & \mathbb{C}^m \\ (z_1, \dots, z_n) & \longmapsto & (\prod_{i=1}^n z_i^{v_i^1}, \dots, \prod_{i=1}^n z_i^{v_i^m}) \end{pmatrix}. \quad (14.13)$$

and it is clear that its kernel is  $\tilde{G} \equiv \text{Ker}(\phi) = (\mathbb{C}^*)^{n-m} (*)$ . One can see that this kernel naturally acts on  $\mathbb{C}^n$  as

$$\begin{pmatrix} (\mathbb{C}^*)^a \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ (\lambda, z_1, \dots, z_n) & \longmapsto & (\lambda^{Q_1^a} z_1, \dots, \lambda^{Q_n^a} z_n) \end{pmatrix} \quad (14.14)$$

where  $Q_i^a$  are the component of the vectors  $Q^a$  defined above. We defined the action of  $\tilde{G} = (\mathbb{C}^*)^{n-m}$  on  $\mathbb{C}^n$  for each factor separately for simplicity.

The  $n$  vectors  $v_i$  generate over  $\mathbb{Z}$  a sublattice of  $N$  that we denote  $N'$ . The discrete group obtained by taking the quotient of  $N$  by this sublattice

$$\Gamma = N/N' \quad (14.15)$$

is a subgroup of  $G$  and taking the quotient by  $\Gamma$  gives rise to the orbifold singularities.

The last piece of data in the construction is the zero set  $Z_{\Delta}$ . For any subset of  $\Delta(1)$  (corresponding to the vectors  $v_{i_1}, \dots, v_{i_l}$ ) which does not generate a cone in  $\Delta$ , we define the algebraic set  $V_{i_1, \dots, i_l} = Z(z_{i_1}, \dots, z_{i_l})$ , i.e. the hyperplane of  $\mathbb{C}^n$  defined by  $z_{i_1} = \dots = z_{i_l} = 0$ .  $Z_{\Delta}$  is then defined as the union of all such  $V_{i_1, \dots, i_l}$ .

**Example.** If we consider the 2-dimensional fan  $\Delta = \text{Fan}(e_1, e_2, -e_1 - e_2)$  in  $N = \mathbb{Z}^3$  (so  $m = 2$ ), we have three one-dimensional cones generated by  $v_1 = e_1, v_2 = e_2$  and  $v_3 = -e_1 - e_2$  (so  $n = 3$ ) and the component matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}. \quad (14.16)$$

We only have one vector  $Q^a$  to find and since it must satisfy the relation (14.12), i.e. in this case  $Q_1 v_1 + Q_2 v_2 + Q_3 v_3 = 0$ , it implies that  $Q = (1, 1, 1)$ . The group action of  $\tilde{G} = \mathbb{C}^*$  on the  $\mathbb{C}^3$  is then

$$(z_1, z_2, z_3) \mapsto (\lambda z_1, \lambda z_2, \lambda z_3). \quad (14.17)$$

Since  $v_1, v_2$  and  $v_3$  generate all of  $\mathbb{Z}^2$ ,  $\Gamma = \{0\}$  is the trivial group and  $G = \tilde{G} \times \Gamma \cong \tilde{G} = \mathbb{C}^*$ . There is no subset of  $\{v_1, v_2, v_3\}$  that generates a cone which is not in  $\Delta$  (because we defined  $\Delta$  as the fan generated by those vectors in the first place) therefore  $Z_\Delta = \{(0, 0, 0)\}^*$ . Finally, the affine toric variety of  $\Delta$  is

$$X_\Delta = \frac{\mathbb{C}^3 \setminus Z_\Delta}{G} = \frac{(\mathbb{C}^*)^3}{\mathbb{C}^*} = \mathbb{CP}^2. \quad (14.18)$$

**Proposition.** A toric variety  $\Delta$  is compact if and only its fan  $\Delta$  span the  $N_{\mathbb{R}}$ .

## 14.6 | Relation between fan varieties, cone varieties and local coordinates

Given an  $m$ -dimensional cone spanned by  $n$  vectors  $v_1, \dots, v_n$ , we want to find the local coordinates associated to it. Since the toric variety can be expressed as the quotient by  $G$ , the local coordinates should be  $G$ -invariant polynomials, that is polynomials  $x = z_1^{n_1} \dots z_n^{n_n}$  such that

$$x \mapsto \lambda^{\sum_i Q_i^a n_i} x = x \quad (14.19)$$

under  $G$ . But remember that (14.12) so we can take  $n_i = \langle w, v_i \rangle$  for any  $w \in M$ . We found that the local coordinates are in one-to-one correspondence with the dual lattice  $M$ .

Given a fan and its fan toric variety, we can associate a toric affine variety to each top-dimensional cone. These cone toric varieties will be affine patches of the fan variety. The transition functions between these patches are also encoded in the initial fan. Let us explain those statements in greater details. If  $\sigma_i \in \Delta$  are the top-dimensional cones of a given fan  $\Delta$ , we can construct their toric variety  $U_{\sigma_i}$  by following the procedure that we presented above. The toric varieties  $U_{\sigma_i}$  act as affine patches and can be patched together to form  $X_\Delta$ . Now suppose that  $\tau$  is a face of both  $\sigma_i$  and  $\sigma_j$ , one can show that

$$\sigma_{i,j}^\vee \subset \tau^\vee \Rightarrow \mathbb{C}[\sigma_{i,j}^\vee \cap M] \subset \mathbb{C}[\tau^\vee \cap M] \Rightarrow U_\tau \subset U_{\sigma_i} \cap U_{\sigma_j} \quad (14.20)$$

so the affine set associated to the face  $\tau$  is in the intersection of the affine sets of the cones. The relation between local coordinates  $x^{(i)}$  of  $U_{\sigma_i}$  and  $x^{(j)}$  of  $U_{\sigma_j}$  can then be read from the relations between the generators of  $\sigma_{i,j}^\vee \cap M$  and  $\sigma_{i,j}^\vee \cap M$ :

$$\sum_{l=1}^{r_i} q_l w_l^{(i)} = \sum_{k=1}^{r_j} q_k w_k^{(j)}, \quad q_l, q_k \in \mathbb{Z} \Rightarrow \prod_{l=1}^{r_i} (x_l^{(i)})^{q_l} = \prod_{k=1}^{r_j} (x_k^{(j)})^{q_k}. \quad (14.21)$$

We see that those transition functions are always rational functions. This is a crucial property of toric varieties.

**Example.** Going back to the fan of  $\mathbb{CP}^2$ , we see that there are three top-dimensional cones  $\sigma_1, \sigma_2$  and  $\sigma_3$ . For each we have  $U_{\sigma_i} = \mathbb{C}^2$ . Applying (14.21) we find that the transition functions between the coordinates  $(x_1, x_2)$  of  $U_{\sigma_1}$  and  $(y_1, y_2)$  of  $U_{\sigma_2}$  are

$$x_1 = \frac{y_1}{y_2}, \quad x_2 = \frac{1}{y_2}. \quad (14.22)$$

## 14.7 | Calabi-Yau toric varieties

As motivated in the beginning, we are mainly interested in Calabi-Yau affine varieties, so Calabi-Yau affine toric varieties in this case. A very convenient property of the toric varieties is that the CY condition is translated into a simple condition on the combinatoric data of the variety.

Recall that a divisor of an affine variety is a linear combination of codimension-one irreducible subvarieties. A *toric divisor* is a divisor invariant under the action of  $G$ . Using the homogeneous coordinates  $(z_i)$ , we can easily construct  $G$ -invariant subvarieties.

**Proposition.** The toric variety  $X_\Delta$  is CY if and only if all the vectors  $v_i$  in  $\Delta$  end on the same hyperplane in  $N$ , which happens if and only if  $\sum_{i=1}^n Q_i^a = 0$  for  $a = 1, \dots, n - m$ .

Following from the proposition on compactness of toric varieties that we gave before, this implies that toric CY varieties are necessarily non-compact.

## 14.8 | Toric diagrams and p-q webs

For CY toric varieties, the combinatoric information encoded in the fan can be expressed in terms of a reduced lattice of one less dimension. This is especially convenient for us since we are mainly concerned about toric CY threefolds, which can therefore be fully encoded in the 2-dimensional lattice. Indeed, instead of drawing a 3-dimensional fan, we can simply project it on the special plane defined by  $\langle w, v_i \rangle = 1$ , and we get the *toric diagram*.

We can also write the dual of the toric diagram by replacing each line with an orthogonal line. This is called the *peq-web*. They have a nice physical interpretation as webs intersecting 5-banes.

## 14.9 | Singularities

A cone is said to be *smooth* if it is generated by part of a lattice basis.

**Theorem.** A cone  $\sigma$  is smooth if and only if  $U_\sigma$  is smooth.

**Example.** The affine toric variety  $Z(x^2 - y^3)$  discussed above is not smooth.

**Proposition.** The variety  $X_\Delta$  is non-singular if and only if  $\Delta$  is a smooth fan.

A toric variety is nonsingular if its cones of maximal dimension are generated by a basis of the lattice. This implies that every toric variety has a resolution of singularities given by another toric variety, which can be constructed by subdividing the maximal cones into cones of non-singular toric varieties.

## 14.10 | Example 1: the conifold

The singular conifold  $\mathcal{C}_0$  can be viewed as the toric variety of the fan  $\Delta = \text{Fan}(e_3, e_1 + e_3, e_1 + e_2 + e_3, e_2 + e_3)$  in  $\mathbb{Z}^3$ . This fan contains 10 cones in total. Denoting the four generating vectors as  $v_1, v_2, v_3$  and  $v_4$ , we find that the only charge vector  $Q$  must satisfy  $Q_1 v_1 + Q_2 v_2 + Q_3 v_3 + Q_4 v_4 = 0$  which gives  $Q = (1, -1, 1, -1)$  so  $\tilde{G} = \mathbb{C}^*$  acts as

$$(z_1, z_2, z_3, z_4) \mapsto (\lambda z_1, \lambda^{-1} z_2 \lambda z_3, \lambda^{-1} z_4). \quad (14.23)$$

We also find that  $\Gamma = \{0\}$  is the trivial group and that  $Z_\Delta = Z(z_1, z_3) \cup Z(z_2, z_4)$ . Finally,

$$\mathcal{C}_0 = \frac{\mathbb{C}^3 \setminus Z_\Delta}{\mathbb{C}^*}. \quad (14.24)$$

Note that, by definition we also have  $\mathcal{C}_0 = Z(x_1 x_4 - x_2 x_3)$  so it is generated by a binomial and is therefore expected to be toric.

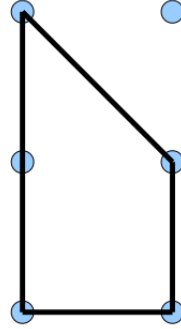


Figure 8: Toric diagram of the SPP.



Figure 7: Toric diagram and pq-web of the conifold.

## 14.11 | Examples 1: the suspended pinch point

## 15 | Gauged linear sigma model (GLSM)

Witten's gauged linear sigma model provides a physical perspective on toric varieties which provides us with the right approach for the forward algorithm. Let us consider the vectorspace  $\mathbb{C}^q$  with complex coordinates  $z_1, \dots, z_q$ .

### 15.1 | Calabi-Yau and non-compactness conditions

## 16 | Correspondence between gauge theory and singularity

Above, we presented all the possible orbifold constructions of supersymmetric quiver gauge theories in four dimensions. We started from quotienting the transverse space and we found the corresponding (supersymmetric) gauge theory. In other words, we started from the singularity and found the gauge theory. We can therefore consider that the orbifold singularities are understood. However, not all singularities are orbifold one, such as the conifold for example. We can then ask ourselves how to obtain the gauge theory for more general singularities than the orbifold ones. Is a general approach possible? On the other hand, we can also study the converse question; is it possible to obtain the singularity from the gauge theory? And if it is, how so? In general we will see that there is a bijection between the four-dimensional supersymmetric worldvolume gauge theory and the Calabi-Yau singularity. We now detail this bijection.

Forward Algorithm		
PHYSICS: Gauge Data	$\rightleftharpoons$	MATHEMATICS: Geometry Data
Inverse Algorithm		
$\Updownarrow$ QUIVER	$\rightleftharpoons$	$\Updownarrow$ Intersection Theory, etc.

Figure 9: Inverse and forward algorithm, from [He04].

### 16.1 | From gauge theory to singularity: forward algorithm

We start with the simplest question: how to recover the singularity from the gauge theory? We already mentioned that the vacuum parameter space of the scalar fields of the gauge theory is the so-called moduli space, denoted  $\mathcal{M}$ . Because our D3-brane is a point in the Calabi-Yau threefold, the vacuum moduli space  $\mathcal{M}$  is the affine coordinates of the Calabi-Yau singularity  $S$ .

For the ADE  $\mathcal{N} = 2$  theories discussed in section 11, by the Kronheimer-Nakajima construction [Kronheimer1990], the moduli space is a hyper-Kähler quotient. In general, the moduli space can be constructed as a *quiver variety*, i.e. a variety constructed from the moduli space of quiver a quiver representation. More rpecisely, given the dimensions of the vector spaces assigned to every vertex, one can form a variety which characterizes all representations of that quiver with those specified dimensions, and consider stability conditions. Let us see some examples of this.

The anomaly free condition is

$$(a_{ij} - a_{ji})n_i = 0(*). \quad (16.1)$$

### 16.2 | Forward algorithm for abelina orbifolds

### 16.3 | From singularity to gauge theory: inverse algorithm

Mathematically, a quiver gauge theory is a representation of a finite quiver with relations. The labels are  $\{N_i \in \mathbb{Z}_+\}$ , they correspond to the dimension of the vector space  $\{V_i\}$ . The gauge group is  $\prod_i \text{SU}(N_i)$ . The gauge fields are self-adjoint arrows  $\text{Hom}(V_i, V_i)$  while the matter fields are bi-fundamentals fermions/bosons and are arrows  $X_{ij} \in \text{Hom}(V_j, V_i)$ . For a quiver with adjacency matrix  $a_{ij}$ , the gauge anomaly cancellation condition can be generally expressed as

$$(a_{ij} - a_{ji})N_i = 0. \quad (16.2)$$

At last, there are some relations that arises the superpotential  $W(\{X_{ij}\})$ . The vacuum is the minima of the superpotential. In other words,

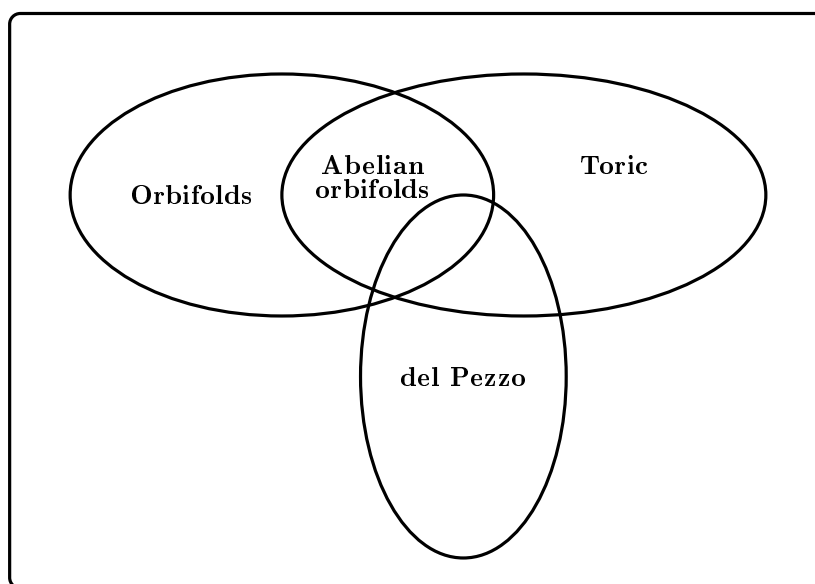
$$\frac{\partial W}{\partial X_{ij}} = 0. \quad (16.3)$$

### 16.3.1 | Application to Toric del Pezzo's

## 17 | Toric duality

### Part IV

## Beyond toric singularities



*Figure 10: Venn diagram of different types of algebraic singularities.*

## 18 | Non-commutative resolutions

## A | Some finite subgroups

### A.1 | Finite subgroups of $\mathrm{SU}(2)$ and $\mathrm{SL}(2, \mathbb{C})$

#### A.1.1 | Finite subgroups

The first thing to recall is that every finite subgroup of  $\mathrm{SL}(2, \mathbb{C})$  is isomorphic to a subgroup of  $\mathrm{SU}(2, \mathbb{C})$  and vice-versa, so we equivalently talk about the subgroups of  $\mathrm{SU}(2)$ . The finite subgroups of  $\mathrm{SU}(2)$ , called the *binary polyhedral groups*, are the doubles covers of the finite subgroups of  $\mathrm{SO}(3)$  that are called *polyhedral groups*. They simply constitutes the symmetries of the Platonic solids. The groups fall into two infinite series, associated to the regular polygons, as well as three exceptional, associated with the 5 regular polyhedra: the tetrahedron (self-dual), the cube (and its dual octahedron), the icosahedron (and its dual dodecahedron).

More precisely, the finite subgroups of  $\mathrm{SL}(2, \mathbb{C})$  are

- $\mathbb{Z}_n$  : cyclic group of order  $n$  ( $n \geq 2$ ) generated by

$$\begin{bmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{bmatrix} \quad (\text{A.1})$$

- $2\mathcal{D}_n$  : *binary dihedral groups* (also known as the *dicyclic group*) of order  $4n$  ( $n \geq 1$ ) generated by

$$A \equiv \begin{bmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{bmatrix} \quad \text{and} \quad B \equiv \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad (\text{A.2})$$

One can show that  $A^n = B^2$  and that  $AB = BA^{-1}$  so that  $2\mathcal{D}_n = \{B^b A^a | 0 \leq b \leq 3, 0 \leq a \leq n-1\}$ . This rewriting of the most general element of the group will be useful.

- $2\mathcal{T}$  : *binary tetrahedral group* of order 24 generated by  $D_2$  and

$$C \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{bmatrix} \quad (\text{A.3})$$

- $2\mathcal{O}$  : *binary octahedral group* of order 48 generated by  $\mathcal{T}$  and

$$D \equiv \begin{bmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{bmatrix} \quad (\text{A.4})$$

- $2\mathcal{I}$  : *binary icosahedral group* of order 120 generated by

$$E \equiv -\frac{1}{\sqrt{5}} \begin{bmatrix} \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{bmatrix} \quad \text{and} \quad F \equiv -\frac{1}{\sqrt{5}} \begin{bmatrix} \zeta_5^2 - \zeta_5^4 & \zeta_5^4 - 1 \\ 1 - \zeta_5 & \zeta_5^3 - \zeta_5 \end{bmatrix} \quad (\text{A.5})$$

with  $\zeta_m \equiv e^{i\frac{2\pi}{m}}$  such that  $(\zeta_m)^m = 1$ . Note that the orders are all divisible by 2. This is because the center of  $\mathrm{SU}(2)$  is  $\mathbb{Z}_2$ .

#### A.1.2 | Irreducible representations

- $\mathbb{Z}_n$  has  $n$  irreducible representations. They are all 1-dimensional (since  $\mathbb{Z}_n$  is abelian) and are given by

$$\rho_k(g) = \zeta_n^k \quad (\text{A.6})$$

with  $k = 0, \dots, n-1$ .



- $2\mathcal{D}_n$  has  $n + 3$  irreducible representations: 4 of dimension 1 and  $n - 1$  of dimension 2. The 1-dimensional ones are given by

$n$	$\rho(A)$	$\rho(B)$	$\rho(B^b A^a)$
even	1	1	1
		-1	$(-1)^b$
	-1	1	$(-1)^a$
		-1	$(-1)^{a+b}$
odd	1	1	1
		-1	$(-1)^b$
	-1	$i$	$(-1)^a i^b$
		$-i$	$(-1)^a (-i)^b$

and the 2-dimensional ones are given binary by

$$\rho_r(A) = \begin{bmatrix} e^{i\frac{\pi}{n}r} & 0 \\ 0 & e^{-i\frac{\pi}{n}r} \end{bmatrix}$$

$$\rho_r(B) = \begin{bmatrix} 0 & (-1)^r \\ 1 & 0 \end{bmatrix}$$

with  $r = 1, \dots, n - 1$ .

### A.1.3 | Character tables

conj. class repr.	$e$	$M$	$M^2$	...	$M^{n-1}$
conj. class order	1	1	1	...	1
$V_0$	1	1	1	...	1
$V_1$	1	$\zeta_n$	$\zeta_n^2$	...	$\zeta_n^{n-1}$
$V_2$	1	$\zeta_n^2$	$\zeta_n^4$	...	$\zeta_n^{2(n-1)}$
$V_3$	1	$\zeta_n^3$	$\zeta_n^6$	...	$\zeta_n^{3(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$V_{n-1}$	1	$\zeta_n^{(n-1)}$	$\zeta_n^{2(n-1)}$	...	$\zeta_n^{(n-1)^2}$
$W$	2	$2 \cos\left(\frac{2\pi}{n}\right)$	$2 \cos\left(\frac{4\pi}{n}\right)$	...	$2 \cos\left(\frac{2\pi(n-1)}{n}\right)$

Table 2: Character table of  $\mathbb{Z}_n$ .

conj. class repr.	$e$	$B^2$	$B$	$BA$	$A$	$A^2$	...	$A^{n-1}$
conj. class order	1	1	$n$	$n$	2	2	...	2
$V_0$	1	1	1	1	1	1	...	1
$V_1$	1	1	-1	-1	1	1	...	1
$V_2$	1	1 ou -1	1 ou $i$	-1 ou - $i$	-1	1	...	$(-1)^{n-1}$
$V_3$	1	1 ou -1	-1 ou - $i$	1 ou $i$	-1	1	...	$(-1)^{n-1}$
$V_4$	2	-2	0	0	$2 \cos \frac{\pi}{n}$	$2 \cos \frac{2\pi}{n}$	...	$2 \cos \frac{(n-1)\pi}{n}$
$V_5$	2	2	0	0	$2 \cos \frac{2\pi}{n}$	$2 \cos \frac{4\pi}{n}$	...	$2 \cos \frac{2(n-1)\pi}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$V_{n+2}$	2	$2(-1)^{n-1}$	0	0	$2 \cos \frac{(n-1)\pi}{n}$	$2 \cos \frac{2(n-1)\pi}{n}$	...	$2 \cos \frac{(n-1)^2\pi}{n}$
$W$	2	-2	0	0	$2 \cos\left(\frac{\pi}{n}\right)$	$2 \cos\left(\frac{2\pi}{n}\right)$	...	$2 \cos\left(\frac{\pi}{n}(n-1)\right)$

Table 3: Character table of  $2\mathcal{D}_n$ .

conj. class repr.	$e$	$B^2$	$B$	$C$	$C^2$	$C^4$	$C^5$
conj. class order	1	1	6	4	4	4	4
$V_0$	1	1	1	1	1	1	1
$V_1$	2	-2	0	1	-1	-1	1
$V_2$	3	3	-1	0	0	0	0
$V_3$	2	-2	0	$e^{i\frac{2\pi}{3}}$	$-e^{i\frac{2\pi}{3}}$	$-e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$
$V_3^\vee$	2	-2	0	$e^{i\frac{4\pi}{3}}$	$-e^{i\frac{4\pi}{3}}$	$-e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$
$V_4$	1	1	1	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$
$V_4^\vee$	1	1	1	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$
$W$	2	-2	0	1	-1	-1	1

Table 4: Character table of  $2\mathcal{T}$ .

conj. class repr.	$e$	$B^2$	$B$	$C$	$C^2$	$D$	$BD$	$D^3$
conj. class order	1	1	6	8	8	6	12	6
$V_0$	1	1	1	1	1	1	1	1
$V_1$	2	-2	0	1	-1	$-\sqrt{2}$	0	$\sqrt{2}$
$V_2$	3	3	-1	0	0	1	-1	1
$V_3$	4	-4	0	-1	1	0	0	0
$V_4$	3	3	-1	0	0	-1	1	-1
$V_5$	2	-2	0	1	-1	$\sqrt{2}$	0	$-\sqrt{2}$
$V_6$	1	1	1	1	1	-1	-1	-1
$V_7$	2	2	2	-1	-1	0	0	0
$W$	2	-2	0	1	-1	$-\sqrt{2}$	0	$\sqrt{2}$

Table 5: Character table of  $2\mathcal{O}$ .

conj. class repr.	$e$	$E^2$	$E$	$F$	$F^2$	$EF$	$(EF)^2$	$(EF)^3$	$(EF)^4$
conj. class order	1	1	30	20	20	12	12	12	12
$V_0$	1	1	1	1	1	1	1	1	1
$V_1$	2	-2	0	1	-1	$\varphi^+$	$-\varphi^-$	$\varphi^-$	$-\varphi^+$
$V_2$	3	3	-1	0	0	$\varphi^+$	$\varphi^-$	$\varphi^-$	$\varphi^+$
$V_3$	4	-4	0	-1	1	1	-1	1	-1
$V_4$	5	5	1	-1	-1	0	0	0	0
$V_5$	6	-6	0	0	0	-1	1	-1	1
$V_6$	4	4	0	1	1	-1	-1	-1	-1
$V_7$	2	-2	0	1	-1	$\varphi^-$	$-\varphi^+$	$\varphi^+$	$-\varphi^-$
$V_8$	3	3	-1	0	0	$\varphi^-$	$\varphi^+$	$\varphi^+$	$\varphi^-$
$W$	2	-2	0	1	-1	$\varphi^+$	$-\varphi^-$	$\varphi^-$	$-\varphi^+$

Table 6: Character table of  $2\mathcal{I}$ , with  $\varphi^\pm \equiv (1 \pm \sqrt{5})/2$ .

## A.2 | Finite subgroups of $SU(3)$

The finite subgroups of  $SU(3)$  are

- the finite subgroups of  $SU(2)$

so there are 2 infinite series and 5 exceptional subgroups. Note that they are all divisible by 3 because the center of  $SU(3)$  is  $\mathbb{Z}_3$ .

## B | Spacetime geometry: ALE space and orbifolds

Asymptotically locally euclidean (ALE) spaces are a particularly interesting choice of string background to probe with branes for mainly four reasons

- (i) they are the resolution (blow-ups) of orbifolds
- (ii) there are completely classified: they fall in the ADE classification
- (iii) they only break half of the supersymmetry
- (iv) they are non-compact therefore we can study them for self-dual type II theory<sup>(\*)</sup>.

Mathematically, an ALE space is complete riemannian  $n$ -manifold  $M$  such that there exists a compact set  $K \subset M$  such that  $M \setminus K$  is diffeomorphic to  $(\mathbb{R}^n \setminus B_0(R))/G$ , where  $R \in \mathbb{R}_0^+$  is a radius and  $G \subset O(n)$  a subgroup. Additionally, it is asked that the pulled back metric on  $\mathbb{R}^n \setminus B_0(R)$  tends to the euclidean flat metric at infinity.

If one considers string theory on the orbifold  $\mathbb{R}^4/\Gamma$  where  $\Gamma$  is a finite sub group of  $SU(2)$ , massless states appear from the twisted sector. They are precisely the moduli needed to deform the theory to the one with smooth spacetime, i.e. the resolution of the orbifold. In that sense, it is said that the strings know about the metric ALE space and that it is said that strings resolve the singularity. The metric of the ALE space can be recovered if the lagrangian of the resulting field theory is explicitly known, such as for the Wess-Zumino-Witten model. However, it is often not the case.

## C | Graphs

The *dimer diagram* of a quiver gauge theory is a graph whose faces represent the gauge groups, the edges represent the bi-fundamental fields and the vertices represent the superpotentials.

## D | Determinantal varieties as transverse spaces

### D.1 | Basic properties of determinantal varieties

A *determinantal variety* (DV) is a space of matrices with a given upper bound on their ranks. More precisely, given  $m, n$  and  $r < \min(m, n)$ , the DV  $Y_r$  of the field  $K$  is the set of  $m \times n$  matrices over  $K$  with rank lower or equal to  $r$ :

$$Y_r \equiv \{M \in M_{m \times n}(K) | \text{rank } M \leq r\}. \quad (\text{D.1})$$

Recall that a  $k$ -minor is the determinant of a  $k \times k$  sub-matrix and that the rank of a matrix is equal to the biggest integer such that there is a non-vanishing minor of that size. Imposing  $\text{rank } M \leq r$  is therefore equivalent to the vanishing of its  $(r+1) \times (r+1)$  minors, as it also implies the vanishing of the biggest minors. This naturally qualifies  $Y_r$  as affine varieties embedded in  $K^{mn}$ .

Let us denote by  $X = (x_{ij})$  an arbitrary  $m \times n$  matrix. The independent entries  $x_{ij}$  are affine coordinates. The  $(r+1) \times (r+1)$  minors are therefore homogeneous polynomials of degree  $r+1$ . The *determinantal ideal*  $I_{r+1}(X)$  is the ideal of  $k[X]$  generated by these polynomials. The coordinate ring is

$$R = k[X]/I_{r+1}(X) \quad (\text{D.2})$$

Homogeneity of the polynomials implies that  $Y_r$  can equivalently be seen as a projective variety in  $\mathbb{A}^{mn-1}$ .

#### D.1.1 | Computing the dimension

Let us compute the dimension of  $Y_r$  seen as an affine variety. We consider the space  $\mathbb{A}^{mn} \times \mathbf{Gr}(r, m)$ , where  $\mathbf{Gr}(r, m)$  is the Grassmannian of  $r$ -planes in an  $m$ -dimensional vector space. Let us define the subspace

$$Z_r \equiv \{(A, W) | Ax \in W \text{ for all } x \in \mathbb{A}^n\}. \quad (\text{D.3})$$

$Y_r$  and  $Z_r$  are birationally equivalent so  $\dim Y_r = \dim Z_r$ . We want to compute  $Z_r$ . First we notice that  $Z_r$  is a vector bundle over  $\mathbf{Gr}(r, m)$  and we denote it by  $Z_r \xrightarrow{\pi_1} \mathbf{Gr}(r, m)$ . Now, over the Grassmannian  $\mathbf{Gr}(r, m)$ , there is a tautological vector bundle that we denote by  $E_{\mathbf{Gr}} \xrightarrow{\pi_2} \mathbf{Gr}(r, m)$  whose fibers are  $\pi_2^{-1}(W) = W \cong \mathbb{R}^r$ . Finally,  $K^m$  can also be seen as a vector bundle, with fibers  $\mathbb{R}^m$ . We denote it by  $E_{K^m} \xrightarrow{\pi_3} K^m$ . From  $E_{\mathbf{Gr}}$  and  $E_{K^m}$ , we can construct<sup>8</sup> the vector bundle  $\text{Hom}(E_{\mathbf{Gr}}, E_{K^m}) \xrightarrow{\pi_4} \mathbf{Gr}(r, m)$ . This vector bundle has the same base space and its fibers are  $\text{Hom}(\mathbb{R}^m, \mathbb{R}^r)$  which are exactly the same as the ones of  $Z_r$ . So the two vector bundles are isomorphic:

$$Z_r \cong \text{Hom}(E_{\mathbf{Gr}}, E_{K^m}). \quad (\text{D.4})$$

Finally, since the fibers of  $\text{Hom}(K^n, E_{\mathbf{Gr}})$  have dimension  $nr$ , we find

$$\dim Z_r = \dim \text{Hom}(K^n, E_{\mathbf{Gr}}) = \dim \mathbf{Gr}(r, m) + nr = r(m - r) + nr. \quad (\text{D.5})$$

Finally, we conclude that  $Y_r$  is a affine variety of dimension  $r(m - r) + nr$ .

$m$	$n$	$r$	$\dim_{\mathbb{C}} Y_r$
2	2	1	3
3	2	1	4
3	3	1	5
3	3	2	8
4	2	1	5
4	3	1	6
4	3	2	10
4	4	1	7
4	4	2	12
4	4	3	15

### D.1.2 | Singularity

Determinantal varieties are singular and possess non-commutative resolutions.  $Y_r$  is singular and the singular locus is contained in the subset of matrices with rank strictly lower than  $r$ .  $Z_r$  is a resolution (over the open set of matrices with rank exactly  $r$ , this map is an isomorphism), it is called the *Springer desingularization* of  $\text{Spec} R$ .

### D.1.3 | Action and syzygies

$Y_r$  naturally acts on  $G = \text{GL}(m, K) \times \text{GL}(n, K)$

## D.2 | Young's lattice

*Young's lattice* is a lattice  $Y$  formed by all integer partitions ordered by inclusion of their Young tableau. It is generally used to describe the irreducible representation of the symmetric group<sup>9</sup>  $S_n$  together with their branching properties. Conventionnally, Young's lattice is depicted in a Hasse diagram, i.e. with element of the same rank shown at the same height and with links such that the descendance of two elements is the union and the parent is the intersection.

<sup>8</sup>Recall that if  $E$  and  $F$  are vector bundles over  $X$ , then we can construct a new vector bundle over  $X$ , called the Hom-bundle and denoted  $\text{Hom}(E, F)$ , by defining the fiber over  $x \in X$  to be  $\text{Hom}(E_x, F_x)$ .

<sup>9</sup>Two permutations of  $S_n$  are equivalent if and only if they have the same number of cycles of the same sizes. Therefore, the equivalence classes of the symmetric group  $S_n$  are parametrized by the partitions of  $n$ , i.e. by Young diagrams.

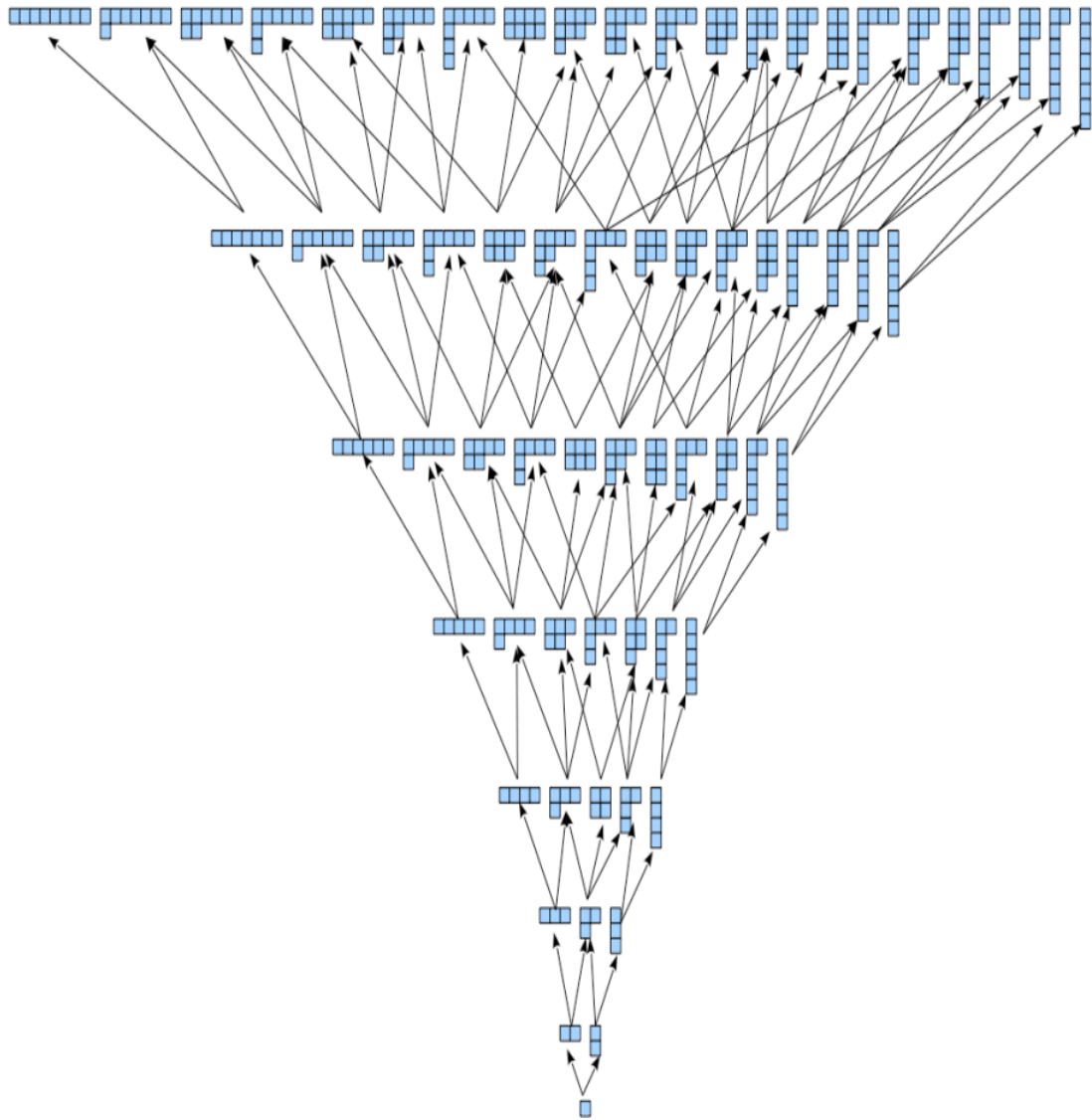


Figure 11: Young's lattice.

Young's lattice possess the following symmetry: the partition  $n + n - 1 + \dots 2 + 1$  of the  $n$ th triangular number has a Young diagram that looks like a staircase. If we now only keep the elements whose hull is contained in this staircase, we get a subset of Young's lattice. When rank-embedded, this subset clearly has the expected bilateral symmetry of Young's lattice but also a rotational symmetry, which appears more clearly if we move away from this rank-embedding. The rotation group of order  $n + 1$  acts on this poset<sup>10</sup>. Since it has both a bilateral and a rotational symmetry it must also have a dihedral symmetry and, indeed, the dihedral group  $\mathcal{D}_{n+1}$  acts faithfully on this set.

<sup>10</sup>Partially ordered set.

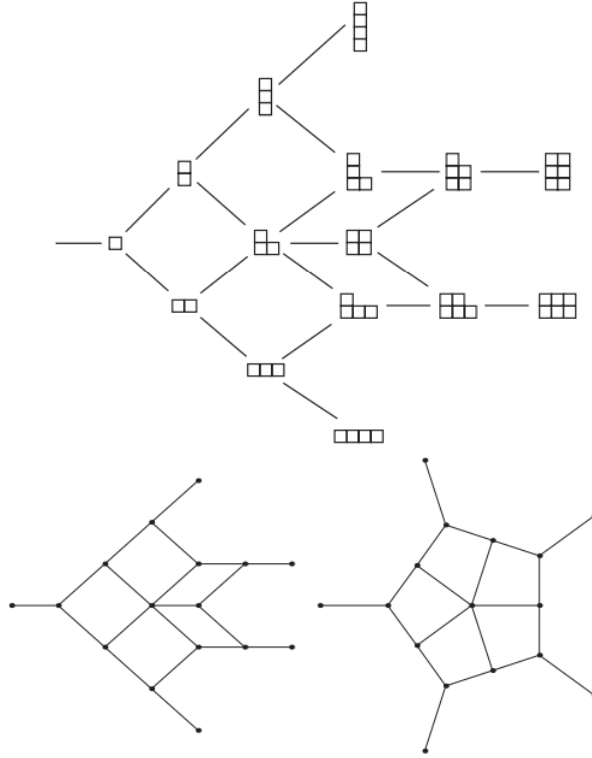


Figure 12: Example of dihedral symmetry for  $n = 4$ .

## E | Some derivations

### E.1

We want to compute the sum

$$\sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} + 1 \right\rfloor = \left\lfloor \frac{n}{3} \right\rfloor + \sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} \right\rfloor. \quad (\text{E.1})$$

Let us write  $n \in \mathbb{N}$  as  $n = 3m + r$  with  $r = 0, 1$  or  $2$  and  $m \in \mathbb{N}$ . Regardless of  $r$ , we have  $\lfloor n/3 \rfloor = m$  and

$$\sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} \right\rfloor = \sum_{a=1}^m \left\lfloor \frac{3}{2}(m-a) + \frac{r}{2} \right\rfloor = \sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + \frac{r}{2} \right\rfloor. \quad (\text{E.2})$$

- if  $r = 0$ , then (E.2) becomes

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a \right\rfloor = \sum_{a=0}^{m-1} a + \sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = \frac{(m-1)m}{2} + \sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor. \quad (\text{E.3})$$

Now if  $m$  is even, we have

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = 2 \sum_{a=0}^{\lfloor \frac{m-1}{2} \rfloor} a = 2 \sum_{a=0}^{\frac{m}{2}-1} a = \left( \frac{m}{2} - 1 \right) \frac{m}{2} \quad (\text{E.4})$$

and if  $m$  is odd,

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = 2 \sum_{a=0}^{\lfloor \frac{m-2}{2} \rfloor} a + \left\lfloor \frac{m-1}{2} \right\rfloor = 2 \sum_{a=0}^{\frac{m-3}{2}} a + \frac{m-1}{2} = \frac{(m-1)^2}{4} \quad (\text{E.5})$$

so

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = \begin{cases} \left( \frac{m}{2} - 1 \right) \frac{m}{2}, & \text{if } m \text{ is even} \\ \frac{(m-1)^2}{4}, & \text{if } m \text{ is odd} \end{cases}. \quad (\text{E.6})$$

and

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3a}{2} \right\rfloor = \begin{cases} \frac{m(3m-4)}{4}, & \text{if } m \text{ is even} \\ \frac{(m-1)(3m-1)}{4}, & \text{if } m \text{ is odd} \end{cases}. \quad (\text{E.7})$$

- if  $r = 1$ , then (E.2) becomes

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + \frac{1}{2} \right\rfloor = \sum_{a=0}^{m-1} a + \sum_{a=0}^{m-1} \left\lfloor \frac{a+1}{2} \right\rfloor = \frac{(m-1)m}{2} + \sum_{a=0}^{m-1} \left\lfloor \frac{a+1}{2} \right\rfloor \quad (\text{E.8})$$

and

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a+1}{2} \right\rfloor = \sum_{a=1}^m \left\lfloor \frac{a}{2} \right\rfloor = \sum_{a=0}^m \left\lfloor \frac{a}{2} \right\rfloor = \begin{cases} \frac{m^2}{4}, & \text{if } m \text{ is even} \\ \frac{m^2-1}{4}, & \text{if } m \text{ is odd} \end{cases} \quad (\text{E.9})$$

by (E.6) so

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + \frac{1}{2} \right\rfloor = \begin{cases} \frac{m(3m-2)}{4}, & \text{if } m \text{ is even} \\ \frac{3m^2-2m-1}{4}, & \text{if } m \text{ is odd} \end{cases} \quad (\text{E.10})$$

- if  $r = 2$ , then (E.2) becomes

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + 1 \right\rfloor = m + \sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a \right\rfloor. \quad (\text{E.11})$$

so

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + 1 \right\rfloor = \begin{cases} \frac{3m^2}{4}, & \text{if } m \text{ is even} \\ \frac{3m^2+1}{4}, & \text{if } m \text{ is odd} \end{cases} \quad (\text{E.12})$$

from (E.7).

Finally, we can write  $m = 2k$  if  $m$  is even and  $m = 2k + 1$  if  $m$  is odd in order to distinguish the six different cases. We get

$$a(n) \equiv \sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} + 1 \right\rfloor = \begin{cases} 2k + \frac{2k(6k-4)}{4}, & \text{if } n = 6k \\ 2k + \frac{2k(6k-2)}{4}, & \text{if } n = 6k + 1, \\ 2k + \frac{12k^2}{4}, & \text{if } n = 6k + 2, \\ (2k+1) + \frac{2k(6k+2)}{4}, & \text{if } n = 6k + 3, \\ (2k+1) + \frac{3(2k+1)^2-2(2k+1)-1}{4}, & \text{if } n = 6k + 4, \\ (2k+1) + \frac{3(2k+1)^2+1}{4}, & \text{if } n = 6k + 5 \end{cases} \quad (\text{E.13})$$

$$= \begin{cases} 3k^2, & \text{if } n = 6k \\ 3k^2 + k, & \text{if } n = 6k + 1, \\ 3k^2 + 2k, & \text{if } n = 6k + 2, \\ 3k^2 + 3k + 1, & \text{if } n = 6k + 3, \\ 3k^2 + 4k + 1, & \text{if } n = 6k + 4, \\ 3k^2 + 5k + 2, & \text{if } n = 6k + 5 \end{cases}. \quad (\text{E.14})$$

Starting from  $n = 1$ , the first value of this sequence is : 0, 0, 1, 1, 2, 3, 4, 5, 7, 8, 10, 12, ... Upon further analysis, this correspond to the sequence [A001399](#), that have several interpretations:

- the number of partitions of  $n$  into at most 3 parts. This makes sense with our initial problem: finding all the  $a, b, c$ 's such that  $a + b + c = n$ ,

- the number of connected graphs with 3 nodes and  $n$  edges (where multiple edges between the same nodes are allowed),
- the number of non-negative solutions to  $b + 2c + 3d = n$ ,

as well as many others. Finally, we note that we can simply write

$$a(n) = \text{round} \left( \frac{n^2}{12} \right). \quad (\text{E.15})$$



## F | References guide

- He: review: [He04], thesis: [He02]
- Orbifold construction for  $\Gamma \subset \mathrm{SU}(2)$ : type  $A$ : [DM96], type  $D$  and  $E$ : [JM97]
- Orbifold construction for  $\Gamma \subset \mathrm{SU}(3)$ : [HH99]
- Formalization of projection to daughter theories: [LNV98],[KS98]
- On toric varieties: [CLS11],[Clo09]([Ful93],[Oda88])
- Forward algorithm for toric singularities: developments: [Asp94],[DGM97],[MP98],[Bea+00],[DD00], formalization: [FHH01b],[FHH01c],[Fen+02]
- Toric diagrams, dimer diagrams and Higgsing: [AFP08]
- Fractional branes: [Arg+08]
- Quiver representations and quiver varieties: [Kir16]
- Formalization of inverse algorithm for toric singularities: [FHH01a]

The bests:

- [Ber21]
- [DL99a; DL99b]
- [He04]
- [Bri]
- [Clo09]

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