Notes on Quiver Gauge Theories

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Abstract

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1 | The brane-world paradigm

We consider our world to be a slice in the ten-dimensional spacetime of type II superstring theory, i.e. the worldvolume of a D3-brane. More precisely we consider a stack of n $D3$-branes carrying a $\mathrm{U}(n)$ gauge group. The spacetime is therefore not necessarily $\mathbb{R}^{1,9}$ but of the form

$$M = \mathbb{R}^{1,3} \times M^{(6)}.$$

This is the so called *brane-world paradigm.*

Independently from string theory, we can require to have $\mathcal{N} = 1$ supersymmetry in four dimensions. This constrains six-dimensional space of compactification $M^{(6)}$ to be compact, complex, Kähler and to have SU(3) Holonomy. In other words, $M^{(6)}$ must be a Calabi-Yau threefold. If we let the worldvolume of the D3-branes carry the requisite gauge theory while the bulk contains gravity, we can relax the compactness condition an study non-compact threefolds[1]. In other words, $M^{(6)}$ is an affine variety that localy models a Calabi-Yau threefold. This makes the analysis much simpler and therefore also serves as an argument to ignore gravity on the brane. Thus we have four dimensional- D3-branes on which there is a $\mathrm{U}(n)$ gauge group and transverse to which gravity propagates.

The only smooth Calabi-Yau threefold being $\mathbb{C}^3$, we are lead to consider singular Calabi-Yau manifolds or, more precisely, Calabi-Yau orbifolds that we usually denote $S \equiv M^{(6)}$ String theory being a theory of extended objects, it is well-defined in such singularities. We will see that this singular structure of the geometry will break $\mathrm{U}(n)$ into products of gauge groups.

From the point of view of the orbifold, the D3-brane is a point. Consequently, there is a crucial relationship between the D3-brane worldvolume theory and the Calabi-Yau singularity: the former parametrizes the latter. In other words, the classical vacuum of the gauge theory should be, in explicit coordinates, the defining equation of S.

Mathematically, this brane-world paradigm is the realization of branes as supports of vector bundles (sheaf). Gauge theories on branes are intimately related to algebraic constructions of stable bundles. In particular, D-brane gauge theories manifest as a natural description of symplectic quotients and their resolutions in geometric invariant theory.

To summarize in more mathematical terms, our D-brane, together with the stable vector bundle (sheaf) supported thereupon, resolves the transverse Calabi-Yau orbifold which is the vacuum for the gauge theory on the worldvolume as a GIT quotient.

2 | The simplest case : $S = \mathbb{C}^3$

2.1 | Generalities

Let us consider the simplest non-compact Calabi-Yau threefold: $S = \mathbb{C}^3$. In this case, the spacetime is simply flat space $\mathbb{R}^{1,9} = \mathbb{R}^{1,3} \times \mathbb{R}^6$ with a choice a complex structure on $\mathbb{R}^6$. As mentioned above, the worldvolume theory has a $\mathrm{U}(n)$ gauge group. Type IIB superstring theory is a ten-dimensional $\mathcal{N} = 2$ theory so it has 32 supercharges. The presence of the breaks the Lorentz symmetry of $\mathbb{R}^{1,9}$ as

$$\mathrm{SO}(1,9) \to \mathrm{SO}(1,3) \times \mathrm{SO}(6), \tag{2.1}$$

[1]Intuitively, this can be understood as a Kaluza-Klein compactification where we take the size of the compact dimensions to infinity. The four-dimensional gravity coupling constant being inversely proportional to this quantity, there is no gravity in this limit.

whereby breaking half of the supersymmetries and we are left with 16 supercharges for the worldvolume theory. In four dimensions, this corresponds to $\mathcal{N} = 4$. We have therefore $\mathcal{N} = 4$ U(n) SCFT gauge theory on the worldvolume.

Note that the D3-brane will warp the flat space metric to that of $AdS_5 \times S^5$ and the bulk geometry is not strictly $\mathbb{C}^3$. However, as stated above, we are only concerned with the local gauge theory and not with gravitational back-reaction, therefore it sufices to consider S as $\mathbb{C}^3$.

2.2 | Matter content

A | Calabi-Yau orbifolds and crepant resolutions

Simply put, as *Calabi-Yau manifold* is a Kähler manifold with trivial canonical bundle or, equivalently, with a Kähler metric whose global holonomy is contained in $SU(n)$. A *Calabi-Yau orbifold* is the quotient of a smooth Calabi-Yau manifold by a discrete group action which generically has fixed points. From a geometrical perspective we can try to resolve the orbifold singularity. A resolution (X, π) of $\mathbb{C}^n/\Gamma$ is a non-singular complex manifold X of dimension n with a proper biholomorphic map

$$\pi : X \to \mathbb{C}^n/\Gamma \tag{A.1}$$

that induces a biholomorphism between dense open sets.

Definition A.1. A resolution (X, π) of $\mathbb{C}^n/\Gamma$ is called a *crepant resolution*[2] if the canonical bundles of X and $\mathbb{C}^n/\Gamma$ are isomorphic, i.e.

$$K_X \cong \pi^*(K_{\mathbb{C}^n/\Gamma}).$$

Since Calabi-Yau manifolds have trivial canonical bundle, to obtain a Calabi-Yau structure on X one must choose a crepant resolutions of singularities.

It turns out that the amount of information we know about a crepant resolution of singularities of $\mathbb{C}^n/\Gamma$ depends dramatically on the dimension n of the orbifold. For $n = 3$, a crepant resolution always exists but it is not unique; they are related by flops. However all the crepant resolutions have the same Euler and Betti numbers: the *stringy* Betti and Hodge numbers of the orbifold.

[2]For a resolution of singularities we can define a notion of discrepancy. A crepant resolution is a resolution without discrepancy.

List of Markers

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