Worksheet on

Calabi-Yau Compactification

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1.1 | Kähler, Calabi-Yau structure and moduli spaces

The Kähler form ω is a representative of a Doleault cohomology class

$$[\omega] \in H^{1,1}(X) \tag{1.1}$$

and $[\omega]$ is called the *Hähler class* of ω.

Theorem 1.1 (Calabi-Yau)**.** Given X a compact manifold with trivial canonical bundle, and given a Kähler form $\widetilde{\omega}$ on X, there exist a unique Ricci-flat metric in the Kähler class of $\widetilde{\omega}$. That is, a unique Ricci-flat metric defined for some $\omega \in [\widetilde{\omega}]$.

On the other hand, it is easy to show that Ricci-flatness implies triviality of the canonical bundle. For non-compact manifolds, the theorem does not hold strictly speaking, one must specify boundary conditions at infinity to find a Ricci-flat metric.

Given a Calabi-Yau anifold, we see that there are continuous famillies of Ricci-flat metrics, one for each cohomology class of $H^{1,1}(X)$. One can decompose any chomology class on a basis $[\omega]^i$ of the vactor space $H^{1,1}(X)$

$$[\omega] = \sum_{i=1}^{h^{1,1}} \lambda_i [\omega]^i. \tag{1.2}$$

It is called the *Kähler moduli space* of X and its dimension is denoed by $h^{1,1}$.

1.2 | Calabi-Yau manifolds

A *Calabi-Yau manifold* of (complex) dimension n is a compact n-dimensional Kähler manifold M satisfying one of the following equivalent conditions:

- the canonical bundle of M is trivial,
- M has holomorphic n-form that vanishes nowhere,
- the structure group of the tangent bundle of M can be reduced from $\mathrm{U}(n)$ to $\mathrm{SU}(n)$,
- M has Kähler metric with global holonomy contained in $\mathrm{SU}(n)$.

It was conjectured by Calabi then prooved by Yau that such spaces are necessarily Ricci-flat. In particular, since the first Chern class of CY manifolds si given by

$$c_1 = \frac{1}{2\pi}[\mathcal{R}] \tag{1.3}$$

it implies that c_1 vanishes, the converse is not true.

For a compact n-dimensioanl Kähler manifold the following conditions are equivalent to each other:

- the first real Chern class vanishes,
- M has a Kähler metric with vanishing Ricci curvature,
- M has Kähler metric with local holonomy contained in $\mathrm{SU}(n)$.
- a positive power of the canonical bundle of M is trivial,
- M has a finite cover that has trivial canonical bundle,
- M has a finite cover that is a product of a torus and a simply connected manifold with trivial canonical bundle.

They are weaker than the conditions above except when the Kähler manifold is simply connected in which case they are equivalent.

1.3 Calabi-Yau orbifolds

A *Calabi-Yau orbifold* is the quotient of a smooth Calabi-Yau manifold by a discrete group action which generically has fixed points. From a algebraic geometry perspective we can try to resolve the orbifold singularity. A resolution (X, π) of $\mathbb{C}^n/\Gamma$ is a non-singular complex manifold X of dimension n with a proper biholomorphic map

$$\pi : X \to \mathbb{C}^n/\Gamma \tag{1.4}$$

that induces a biholomorphism between dense open sets. A resolution (X, π) of $\mathbb{C}^n/\Gamma$ is called a *crepant resolution*[1] if the canonical bundles of X and $\mathbb{C}^n/\Gamma$ are isomorphic, i.e.

$$K_X \cong \pi^*(K_{\mathbb{C}^n/\Gamma}).$$

Since Calabi-Yau manifolds have trivial canonical bundle, to obtain a Calabi-Yau structure on X one must choose a crepant resolutions of singularities.

It turns out that the amount of information we know about a crepant resolution of singularities of $\mathbb{C}^n/\Gamma$ depends dramatically on the dimension n of the orbifold:

- $n = 2$: a crepant always exists and is unique. Its topology is entirely described in terms of the finite group Γ (via the McKay correspondence).
- $n = 3$: a crepant resolution always exists but it is not unique; they are related by flops. However all the crepant resolutions have the same Euler and Betti numbers: the *stringy* Betti and Hodge numbers of the orbifold.
- $n \geq 4$: very little is known; crepant resolution ecists in rather special cases. Many singularity are terminal, which implies that they admit no crepant resolution.

[1] For a resolution of singularities we can define a notion of discrepancy. A crepant resolution is a resolution without discrepancy.