Worksheet on

Orbifold Quiver Gauge Theories

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1 | Physical setup

Brane-world paradigm

We consider our four-dimensional world to be the worldvolume of a D3-brane in the ten-dimensional spacetime of type IIB superstring theory. More precisely, we consider a stack of N D3-branes in order to have U(N) Chan-Paton factors resulting in a U(N) gauge group in the worldvolume theory. The spacetime is therefore not necessarily $\mathbb{R}^{1,9}$ but of the more general form

$$M = \mathbb{R}^{1,3} \times M^{(6)}.$$

This is the so-called *brane-world paradigm.*

Supersymmetry and Calabi-Yau manifolds

Independently from string theory, we can ask for the wolrdvolume theory to be supersymmetric. We start from type IIB superstring theory which is 10-dimensional and has $\mathcal{N} = 2$ supersymmetry so it possesses 32 supercharges. As usual, they transform under the minimal spinor representation (MSR) of the bulk Lorentz group, here SO(1, 9). In ten dimensions this representation is 8-dimensional (complex) which is why there are $2(8 + 8) = 32$ supercharges: 8 transforming in the 8-dimensional MSR and 8 transforming in the 8-dimensional conjugate MSR and the whole thing times two since $\mathcal{N} = 2$. Compactifying type II string theory on any 6-dimensional manifold $M^{(6)}$ breaks supersymmetry. The reason for this is that the supercharges now have to transform under the MSR of $\mathrm{SO}(1, 3) \times \mathcal{H}(M^{(6)})$, where $\mathcal{H}(M^{(6)})$ is the holonomy group of M^6. Actually, the space $\mathbb{R}^{1,3} \times M^{(6)}$ can be viewed as the trivial bundle with base space $\mathbb{R}^{1,3}$ and fibers $M^{(6)}$, sincespinors takes values in the fibers they must transform under the holonomy group of $M^(6)$. A generic curved 6-dimensional manifold has O(6) holonomy and SO(6) if it is orientable, as we will always consider. The supercharges must therefore transform under the MSR of $\mathrm{SO}(1, 3) \times \mathrm{SO}(6)$. The MSR of SO(1, 3) being **2** and the one of SO(6) being **4**, we conclude that imposing the spacetime to have the form (1) changes the representation under which the supercharges transform in the following way: Why ?

$$\mathbf{8} \oplus \overline{\mathbf{8}} \to (\mathbf{2}_L, \mathbf{4}) \oplus (\mathbf{2}_R, \overline{\mathbf{4}}). \tag{1.1}$$

If we stop here, the residual supercharges might be ill-defined; making a tour around a loop in $M^{(6)}$ could result in a non-trivial rotation. To solve this problem, we need to be more restrictive with the holonomy. In fact, it is precisely the holonomy of the transverse space that dictates the number of residual supersymmetries. To understand this, let us now consider a four-dimensional field theory resulting from compactification of the transverse six-dimensional space. The number of supercharges that generate supersymmetries for this theory is the number of Killing spinors (covariantly constant spinors) because each Killing spinor contracted with the local supersymmetry current generates a residual supersymmetry. Let us now make the link with holonomy: since $\mathrm{SO}(6) \cong \mathrm{SU}(4)$, minimal spinors can be viewed as having four complex components and as transforming under $\mathrm{SU}(4)$. Indeed, minimal spinors in six dimensions have four complex components. In order to have one covariantly conserved spinor, we look for the biggest subgroup of $\mathrm{SU}(4)$ that leaves a component of the spinor invariant. This is clearly $\{e\} \times \mathrm{SU}(3) \subset \mathrm{SU}(4)$ that acts trivially on the first component. The spinor $(1,0,0,0)$ is then covariantly constant. Our transverse space must therefore have $\mathrm{SU}(3)$ holonomy such that the parallel transport of the spinor $(1,0,0,0)$ under any closed loop is a lower $\mathrm{SU}(3)$ rotation. We conclude that if the transverse Calabi-Yau has $\mathrm{SU}(3)$ holonomy, the worldvolume theory has $\mathcal{N} = 1$ supersymmetry. If the holonomy is $\mathrm{SU}(2) \subset \mathrm{SU}(3)$, the spinor $(0,1,0,0)$ is also a Killing spinor which means that we have $\mathcal{N} = 2$ supersymmetry for the worldvolume theory. To summarize, preserving any degree of supersymmetry constrains the transverse space $M^{(6)}$ to be compact, complex, Kähler and to have $G \subset \mathrm{SU}(3)$ holonomy. Namely, $M^{(6)}$ must be a Calabi-Yau threefold, see section **??**.

Non-compact transverse space

If we let the worldvolume of the D3-branes carry the requisite gauge theory while the bulk contains gravity, we can relax the compactness condition and study non-compact Calabi-Yau threefolds. This makes the analysis much simpler and therefore also serves as an argument to ignore gravity in the worldvolume theory. Consequently, we will mostly ignore gravity and not care about the metric of the spacetime, see appendix **??** for more details. In this setup we cannot really talk about compactification anymore. Instead we just think of it as a flat space on which lives the gauge theory while gravity only lives in transverse space. To understand intuitively why there is no gravity in this limit, we can think of Kaluza-Klein compactification. The four-dimensional gravity coupling constant is inversely proportional to the size of the compactifying space therefore there is no gravity in the non-compact/infinite-size limit. This more a motivation than a proof.

Singular transverse space

The only non-compact smooth Calabi-Yau threefold is $\mathbb{C}^3$, this forces us to consider singular Calabi-Yau varieties is we want more interesting theories. A Calabi-Yau variety is an affine variety that locally models a Calabi-Yau manifold, therefore allowing for singularities. We usually denote $S \equiv M^{(6)}$ to remind us of the singular aspect. String theory being a theory of extended objects, turns out to be it is well-defined on such singularities and even "smoothened" the singularity in some sens. Considering strings on singular geometries requires to "project" the theory obtain from $\mathbb{C}^3$. As a result, the gauge group $\mathrm{U}(N)$ is broken down into products of smaller gauge groups. This "projection" highly depends on the type of singularity (orbifold, toric, del Pezzo, etc) we are considering. While it is relatively straightforward for "simple" singularities (e.g. abelian orbifolds) it quickly gets more complicated or even unknown for others.

From the point of view of the orbifold, the D3-brane is a point, meaning that the D3-branes are really probing the transverse space and, in particular, they parametrize it. This is the first clue of the tight relationship that exists between the worldvolume theory and the transverse singular space. Eventually, we will see that the classical vacuum of the gauge theory should be, in explicit coordinates, the defining equation of S. This is precisely the opposite of the projection manipulation we mentioned above: recovering the transverse space from the gauge theory. Projecting and computing the classical vacua are therefore inverse operations with respect to each other. This suggest a bijection between the singular transverse space and the gauge theory: the former can be computed from the latter and vice-versa. This is called "forward algorithm" and "inverse algorithm" respectively.

Mathematical formulation

Mathematically, this brane-world paradigm is the realization of branes as supports of vector bundles (sheaf). Gauge theories on branes are intimately related to algebraic constructions of stable bundles, i.e. holomorphic or algebraic vector bundles that are stable in the sense of geometric invariant theory. In particular, D-brane gauge theories manifest as a natural description of symplectic quotients and their resolutions in geometric invariant theory. Together with the stable vector bundle (sheaf) supported thereupon the D-branes resolve the transverse Calabi-Yau orbifold, which is the vacuum for the gauge theory on the worldvolume as a GIT quotient.

Summary

We consider N D3-branes in type IIB superstring theory carrying a $\mathrm{U}(N)$ gauge group. The transverse space S is taken to be a non-compact singular Calabi-Yau variety.

2 | The simplest case: smooth transverse space

2.1 | Generalities

Let us start by considering the simplest configuration were the transverse Calabi-Yau space is non-singular, i.e. it is a proper smooth Calabi-Yau threefold. As mentioned above, the only smooth Calabi-Yau threefold is $S = \mathbb{C}^3$. In this case, the spacetime is simply flat space $\mathbb{R}^{1,9} = \mathbb{R}^{1,3} \times \mathbb{R}^6$ with a choice a complex structure on $\mathbb{R}^6$. From the $\mathrm{U}(N)$ Chan-Paton factors, the worldvolume theory inherits from a $\mathrm{U}(N)$ gauge group. Type IIB superstring theory is a ten-dimensional $\mathcal{N} = 2$ theory so it has 32 supercharges. The presence of the branes breaks the Lorentz symmetry of $\mathbb{R}^{1,9}$ as

$$\mathrm{SO}(1,9) \to \mathrm{SO}(1,3) \times \mathrm{SO}(6), \tag{2.1}$$

whereby breaking half of the supersymmetries, as we explained in the previous section. We are thus left with 16 supercharges. In four dimensions, this corresponds to $\mathcal{N} = 4$. The worldvolume theory for $S = \mathbb{C}^3$ is therefore $D = 4, \mathcal{N} = 4$ $\mathrm{U}(N)$ SCFT gauge theory. This worldvolume theory, obtained in the non-singular case $S = \mathbb{C}^3$, is called the *parent theory*.

Note that the D3-brane will warp the flat space metric to that of $AdS_5 \times S^5$ and the bulk geometry is not strictly $\mathbb{C}^3$. However, as stated above, we are only concerned with the local gauge theory and not with gravitational back-reaction, therefore it suffices to consider S as $\mathbb{C}^3$.

2.2 | Matter content

As discussed in appendix **??**, there is only one $D = 4, \mathcal{N} = 4$ SCFT theory, up to a choice of gauge group G. In our case, $G = \mathrm{U}(N)$. The isometry group of the transverse space $\mathbb{R}^6$ is $\mathrm{SO}(6) \cong \mathrm{SU}(4)$. Since the scalar fields living on the branes are interpreted as its tranverse oscillations, $\mathrm{SO}(6)$ is a global symmetry of the field theory. These global symmetries of worldvolume theory lead to the R-symmetry group $\mathrm{SU}(4)_R$. The only $\mathcal{N} = 4$ supermutliplet can be rewritten in terms of $\mathcal{N} = 1$ supermultiplets as follows:

$$[\mathcal{N} = 4 \text{ vector multiplet}] : V = (\lambda_\alpha, A_\mu, D) \oplus \Phi_A = (\phi^A, \psi^A_\alpha, F^A). \tag{2.2}$$

with $A = 1, 2, 3$. In other words, after removing the auxiliary fields D and F^A, the matter content is

- a $\mathrm{U}(N)$ gauge field A_μ which transforms as a singlets under $\mathrm{SU}(4)_R$:

$$\text{Gauge transformation} : A_\mu \mapsto U A_\mu U^{-1} + U \partial_\mu U^{-1}, \qquad U \in \mathrm{U}(N) \tag{2.3}$$

$$\text{R-symmetry} : A_\mu \mapsto A_\mu. \tag{2.4}$$

  Note that the usual term

- 4 Weyl fermions $\psi^a_\alpha \equiv (\lambda_\alpha, \psi^1_\alpha, \psi^2_\alpha, \psi^3_\alpha,)$ that transform under the adjoint of $\mathrm{U}(N)$ and are mixed together under the representation $\mathbf{4}$ of $\mathrm{SU}(4)_R$. This means that each fermion ψ^a takes values in $\mathfrak{u}(N)$. We denote the components by ψ^a_{IJ} $(I, J = 1, \ldots, N)$. Explicitely:

$$\text{Gauge transformation} : \psi^a \mapsto U \psi^a U^\dagger, \qquad U \in \mathrm{U}(N), \tag{2.5}$$

$$\text{R-symmetry} : \psi^a \mapsto R^a{}_b \psi^b, \qquad R \in \mathrm{SU}(4)_R. \tag{2.6}$$

  Note that this gives us $4N^2$ Weyl fermions in total.

- 3 complex scalar fields ϕ^A transforming under the adjoint representation of $\mathrm{U}(N)$ and under the two-times anti-symmettric representation of $\mathrm{SU}(4)_R$. This means that each ϕ^A takes values in $\mathfrak{u}(N)$ and we denote the components by ϕ^A_{IJ}. Recall that $\mathrm{SU}(4) \cong \mathrm{SO}(6)$ so the action of the R-symmetry can be seen as the $\mathbf{3}$ of $\mathrm{SU}(3) \subset \mathrm{SU}(4)_R$ acting on three complex scalars ϕ^A or equivalently as the $\mathbf{6}$ of $\mathrm{SO}(6)_R$ acting on 6 real scalars X^m, the real and imaginary parts of the ϕ^A. They are interpreted as the oscillations of the branes in the transverse space. Explicitely:

$$\text{Gauge transformation}: X^m \mapsto UX^mU^\dagger, \qquad U \in \mathrm{U}(N), \tag{2.7}$$

$$\text{R-symmetry}: X^m \mapsto R^m{}_n X^n, \qquad R \in \mathrm{SO}(6)_R. \tag{2.8}$$

Note that this gives $6N^2$ real scalars in total. They are the superpartners of the fermions. Detail.

Note that the gauge group $\mathrm{U}(N)$ can also be seen as the group of isometries of the metric space $\mathbb{C}^N$, i.e. $\mathrm{Hom}(\mathbb{C}^N, C^N)$. From this point of view, the transformations (2.3)-(2.8)can be summarized as

$$\boxed{\begin{aligned} A_\mu &\in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N), \\ \psi &\in \mathbf{4} \otimes \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N), \\ X &\in \mathbf{6} \otimes \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N). \end{aligned}} \begin{aligned} (2.9\text{a}) \\ (2.9\text{b}) \\ (2.9\text{c}) \end{aligned}$$

> If the transverse space is non-singular, the only possibility is $S = \mathbb{C}^3$. In this case, the worldvolume theory is therefore $D = 4, \mathcal{N} = 4$ $\mathrm{U}(N)$ SCFT gauge theory. It is called the *parent theory*.

This is famous duality between $\mathcal{N} = 4$ supersymmetric $\mathrm{U}(N)$ Yang-Mills and Type IIB string theory on $\mathrm{AdS}_5 \times S^5$.

3 | Strings on orbifolds

Let us a consider a smooth background space $M^{(6)}$ and Γ a discrete group acting on it. If Γ has no fixed point, the quotient space $M^{(6)}/\Gamma$ is smooth and is a manifold but if the action of Γ has a fixed point then the quotient space $M^{(6)}/\Gamma$ is singular and is an orbifold. Any orbifold can be described as an affine variety therefore it is the description that we will use most often. When the transverse space is singular, the worldvolume theory corresponds to a specific projection of the parent theory that we found in the smooth case $S = \mathbb{C}^3$. We call it the *daughter theory*. This projections depends on the type of singularity that one considers. The simplest case is the the case of orbifold singularity, i.e. when the transverse space is a quotient space with a non-free action.

3.1 | Generalities

We now wish to pick a discrete group Γ and which acts non-trivially on $\mathbb{R}^6$. There are several possibilities:

- $\Gamma \subset \mathrm{SU}(4)$ naturally acts on $\mathbb{R}^6$. This does not require a choice of complex structure. We get an $\mathcal{N} = 0$ theory.
- $\Gamma \subset \mathrm{SU}(3)$ naturally acts on $\mathbb{C}^3$, this also requires a choice of complex structure on $\mathbb{R}^6$. We get an $\mathcal{N} = 1$ theory.
- $\Gamma \subset \mathrm{SU}(2)$ naturally acts on the second factor of $\mathbb{C} \times \mathbb{C}^2$, so this requires a choice of complex structure on $\mathbb{R}^6$. We get an $\mathcal{N} = 2$ theory.

We are interested in supersymmetric theories so we take $\Gamma \subset \mathrm{SU}(3)$ with the action

$$\cdot : \begin{pmatrix} \Gamma \times \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3 \\ (\gamma, z) & \longmapsto & \gamma \cdot z \end{pmatrix} \tag{3.1}$$

is the representation of Γ coming from the fundamental representation of $\mathrm{SU}(3)$, so $\cdot$ is just the matrix product. We can see that the origin is always a fixed point so this action is never free. Since $\mathbb{C}^3$ is a smooth manifold, this makes $\mathbb{C}^3/\Gamma$ an orbifold. Our choice of Γ naturally includes the case $\Gamma \subset \mathrm{SU}(2)$ too (as $\mathrm{SU}(2) \subset \mathrm{SU}(3)$), just not the case $\Gamma \in \mathrm{SO}(6)$. When $\Gamma \subset \mathrm{SU}(2 \subset \mathrm{SU}(3)$, it acts trivially on one component so we write $S = \mathbb{C} \times \mathbb{C}^2/\Gamma$.

If Γ is a general finite group the condition that $\mathbb{C}^3/\Gamma$ is an Calabi-Yau orbifold means that there must exist a resolution of this orbifold such that the corresponding smooth space is Calabi-Yau, i.e. a crepant resolution. Existence of such a resolution constrains Γ , see section **??**. Details ?

3.2 Twisted sector

What implications does have strings on an orbifold have for the spectrum? In general, we distinguish two kinds of states:

- the *untiwsted states* are the states Ψ that are invariant under the actio of Γ: $\gamma \cdot \Psi = \Psi$ for all $\gamma \in \Gamma$. They are the generators of the Γ-invariant part of the Hilbert space and can easily be constrcuted by superposing all the images $\gamma \cdot \Psi_0$.
- closed strings must start and end at the same point, i.e. $X(\tau, \sigma + 2\pi) = X(\tau, \sigma)$. Usually, a strings that starts at connects a point of $M^{(6)}$ to another point of its orbit, i.e. $X(\tau, \sigma + 2\pi) = \gamma \cdot [X(\tau, \sigma)]$ would not be allowed. However, in $M^{(6)}/\Gamma$ it is an allowed configuration. Those new states that appear after orbifolding are the *twisted states*. There existsnce is due to the fact that strings are extended objects. One can see that there is one twisted sector per conjugacy classe of Γ and that the untwisted states are recovered with $\gamma = e$. details ?

The individual twisted sector quantum states of the strings are localized at the orbifold singularities that the classical configurations (untiwisted sector) enclose.

3.3 Projection to daughter theory

Make the link with the ususal setup.

Let us consider a D1-brane in the x^0, x^1-plane of the orbifold space $\mathbb{R}^6 \times \mathbb{R}^4/\Gamma$ [1, 2]. We can view the brane as a point in the covering space $\mathbb{R}^4$. If the brane trivially sits at a fixed point, which in our case is usually the origin, there is no problem but if the brane moves away from the origin, this will break the Γ symmetry. In order for it to be able to move away from a the origin we therefore also need to add image-branes moving in all the other Γ-sectors of the covering space (the regions of $\mathbb{C}^n$ that are related by the action of Γ and all projected to a unique sector in $\mathbb{C}^n/\Gamma$). Since there are $|\Gamma|$ sectors we need exactly $|\Gamma|$ branes in total, one per sector. From the open string states point of view, this is means adding $|\Gamma|$ Chan-Paton factors, resulting in an $U(|\Gamma|)$ gauge symmetry. Γ then acts on the Chan-Paton factors and switch them between themselves. We now need to make sure that the string theory is consistent with that action.

Let us denote by ρ_{reg} the regular representation of Γ, which is $|\Gamma|$-dimensional. Concretely, there are three types of open string sector states that one must consider:

- the vector multiplets: $\lambda_V \psi^{\mu}_{-1/2} |0\rangle$ ($\mu = 0, 1$), where λ_V is the Chan-Paton matrix (arbitrary $\mathrm{U}(N)$) matrix, i.e. an arbitrary matrix of $\mathrm{U}(|\Gamma|)$. Invariance under Γ means that λ_V should satisfy the additional property
$$\rho_{\mathrm{reg}}(\gamma)\lambda_V \rho_{\mathrm{reg}}(\gamma)^{-1} = \lambda_V. \tag{3.2}$$
This constraint will give rise to the gauge group of the theory, transforming in the adjoint of some subgroup of the original gauge group $\mathrm{U}(N)$.
- the hypermultiplet I: $\lambda^{\mathrm{I}}_H \psi^i_{-1/2} |0\rangle$ ($i = 2, \ldots, 5$). They are the rest of the $D = 6, \mathcal{N} = 1$ vectors. From the $\mathcal{N} = 4, D = 2$ point of view, they are the scalar parts of the gauge multiplet and represent the oscillations for the branes in the $X^2, \ldots, X^5$-directions. The matrices λ^{I}_H must satisfy the relations
$$\rho_{\mathrm{reg}}(\gamma)\lambda^{\mathrm{I}}_H \rho_{\mathrm{reg}}(\gamma)^{-1} = \lambda^{\mathrm{I}}_H \tag{3.3}$$
- the hypermultiplet II: $\lambda^{\mathrm{II}}_H \psi^m_{-1/2} |0\rangle$ ($m = 6, \ldots, 9$). The matrices λ^{II}_H must satisfy the relations
$$\rho_{\mathrm{reg}}(\gamma)\lambda^{\mathrm{II}}_H \rho_{\mathrm{reg}}(\gamma)^{-1} = \lambda^{\mathrm{II}}_H \tag{3.4}$$

Note that there is a trivial solution to (3.2): $\lambda_V = \mathbb{1}$. This solution corresponds to an extra U(1) factor in the gauge group that will always be unbroken. Consequently, the Dynkin digram will always contain an extra U(1) node, this is what we call the the *extended* (or *affine*)) Dynkin diagram[1] This explains why the quiver that we will obtain will be affine Dynkin diagrams. From the point of view of the worldvolume theory, this means that there is always the possibility of Higgsing away all the vector multiplets except the ones corresponding to this trivial U(1). This U(1) is the gauge group of a configuration with a single D3-brane so the hypermultiplet parametrizes the position this brane in the orbifold. From the point of view of the covering space, this brane is a stack of $|\Gamma|$ fractional branes, moving simultaneously in such a way that they are Γ-images of each other.

Looking only the open string sector is sufficient however to explore covering all the possibilities. One also needs to look at the massless closed strings and in particular at the twisted sector. We will see that these states enter the theory as Fayet-Iliopoulos terms.

Make the link between this discussion and the final projection.

If the transverse space is $S = \mathbb{C}^3/\Gamma$, the field theory must be projected to a theory which also invariant under Γ, seen as a subgroup of the R-symmetry group. The prescription is straihgt-forward: we can use the elements $\gamma \in \Gamma$ to project out that states that are not Γ-invariant. That is, if ρ is an embedding of Γ in the gauge group $\mathrm{U}(N)$, only the the fields such that

$$\boxed{\begin{aligned} \rho(\gamma)A_\mu\rho(\gamma)^{-1} &= A_\mu, && (3.5a)\\ R(\gamma)\rho(\gamma)\psi_{IJ}\rho(\gamma)^{-1} &= \psi_{IJ}, && (3.5b)\\ R(\gamma)\rho(\gamma)X_{IJ}\rho(\gamma)^{-1} &= X_{IJ} && (3.5c)\end{aligned}}$$

are kept in the spectrum, where ρ is a unitary representation of Γ on $\mathbb{C}^N$ and $R = \mathbf{4}, \mathbf{6}$. Let us make two remarks:

- the term $U\partial_\mu U^{-1}$ is absent from 3.5a. Indeed, Γ is a finite group and the only smooth functions $x \mapsto \Gamma$ are the constant ones. Consequently, transformations of the gauge field under a finite subgroup of its gauge group cannot depend on x.
- the fields that transform non-trivially under R-symmetry also have an extra induced action of Γ, in agreement with (2.9a)-(2.9c). The R-symmetry untouched by Γ will be the resulting R-symmetry of daughter theory.

3.4 Representation theory of the projection

Let $\{(\rho_i, V_i)\}_{i\in I}$ be a complete set of irreducible representations of Γ. We use the notation $N_i = \dim \rho_i$ the the dimension of the representations, such that $V_i = \mathbb{C}^{N_i}$. The finiteness of Γ is crucial in two ways: first, since it is finite it is particular compact and the representations (ρ_i, V_i) can be taken to be unitary. Second, the number of irreducible representations is necessarily finite, i.e. the index i takes a finite number of values.

3.4.1 Embedding of Γ in the gauge group

Let us consider a representation of Γ on $\mathbb{C}^N$, we denote it $(\rho, \mathbb{C}^N)$ and also take it to be unitary. In that case, $\rho(\gamma) \in \mathrm{U}(N)$. This is what we mean by "the embedding of Γ in $\mathrm{U}(N)$". The adjoint representation of $\mathrm{U}(N)$ defined as[2]

$$\mathrm{Ad} : \begin{pmatrix} \mathrm{U}(n) & \longrightarrow & \mathrm{GL}(\mathfrak{u}(N)) \\ U & \longmapsto & \mathrm{Ad}_U \end{pmatrix}, \qquad \mathrm{Ad}_U : \begin{pmatrix} \mathfrak{u}(N) & \longrightarrow & \mathfrak{u}(N) \\ \omega & \longmapsto & \mathrm{Ad}_U(\omega) \equiv U\omega U^{-1} \end{pmatrix}, \tag{3.6}$$

now allows us to act with Γ on $\mathfrak{u}(N)$. We use this representation in the expression (3.5a)-(3.5c). More formally, these relations can be rewritten as

$$\mathrm{Ad}_{\rho(\gamma)} A_\mu = A_\mu, \tag{3.7}$$

[1] It can be viewed as the Dynkin diagram for the affine-extended Lie algebra.
[2] this is well-defined since for all $\omega \in \mathfrak{u}(N)$ and $U \in \mathrm{U}(N)$, $U\omega U^{-1} \in \mathfrak{u}(N)$.

$$R(\gamma)\mathrm{Ad}_{\rho(\gamma)}\psi_{IJ} = \psi_{IJ}, \tag{3.8}$$
$$R(\gamma)\mathrm{Ad}_{\rho(\gamma)}X_{IJ} = X_{IJ} \tag{3.9}$$

The representation $(\rho, \mathbb{C}^N)$ can be decomposed as follows:

$$(\rho, \mathbb{C}^N) = \bigoplus_{i\in I} (\rho_i, V_i)^{N_i} \tag{3.10}$$
$$= \bigoplus_{i\in I} (\mathbf{1}^{N_i} \otimes \rho_i, \mathbb{C}^{N_i} \otimes V_i) \tag{3.11}$$

were N_i are integer multiplicities $((\rho_i, V_i)^{N_i} \equiv (\rho_i, V_i)^{\oplus N_i})$ and $\mathbf{1}$ is the trivial representation, so Γ acts trivially on the $\mathbb{C}^{N_i}$. We have

$$\sum_i N_i \dim(\rho_i) = N. \tag{3.12}$$

The rewriting (3.11) will be useful later on.

3.4.2 | Adjoint representation and bi-fundamental fields

Any object in the fundamental representation $\mathbf{N}$ of $G = \mathrm{U}(N)$ (generated by $\{T_a\}_{a=1,\dots,N^2}$ that are taken to be hermician) has an index i ($i = 1, \dots, N$) and transforms as

$$\delta_a \Phi^i = e^{i(T_a)^i{}_j}\Phi^j. \tag{3.13}$$

An adjoint field has an index a and transforms as

$$\delta_a \Phi^b = -f_{ac}{}^b\Phi^c \tag{3.14}$$

with $[T_a, T_b] = if_{ab}{}^cT_c$. Given any such adjoint field, we can construct an $N \times N$ matrix as $\Phi^i{}_j = \Phi^a(T_a)^i{}_j$. This matrix has N^2 independent components, which is exactly the dimension of the adjoint representation. What is the transformation of this matrix? We find

$$\delta_a\Phi^i{}_j = -f_{ac}{}^b\Phi^c(T_b)^i{}_j \tag{3.15}$$
$$= i(T_a)^i{}_k\Phi^k{}_j - i\Phi^i{}_k(T_a)^k{}_j \tag{3.16}$$
$$= i(T_a)^i{}_k\Phi^k{}_j - i(T_a^*)_j{}^k\Phi^i{}_k, \tag{3.17}$$

in other words, the first index transforms in the fundamental representation $\mathbf{N}$ and the second index transforms in the anti-fundamental transformation $\overline{\mathbf{N}}$. Thus, $\Phi^i{}_j$ transforms under $(\mathbf{N}, \overline{\mathbf{N}}) \equiv \mathbf{N} \otimes \overline{\mathbf{N}}$[3]. This point of view of the adjoint representation is convenient for us. In summary, we found that

$$\boxed{\mathrm{Ad} = \mathbf{N} \otimes \overline{\mathbf{N}}.} \tag{3.18}$$

Taking a step back, we can view write $\mathrm{U}(N)$ as $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)$ and this relation is then nothing more that a particular case of $\mathrm{Hom}(V, W) \cong V \otimes W^*$. Generally speaking, fields transforming under $(\mathbf{N}_{\boldsymbol{i}}, \overline{\mathbf{N}_{\boldsymbol{j}}})$ are called *bi-fundamentals fields*. From our discussion, we see that fields transforming in the adjoint representation are in particular bi-fundamental fields.

We end this section by mentioning that the same computations can be done for $\mathrm{SU}(N)$. The only difference is the additional trace cancelling condition. We find that the trace transforms as a scalar, therefore we have $\mathrm{Ad} \oplus 1 = \mathbf{N} \otimes \overline{\mathbf{N}}$ this time.

3.4.3 | Invariant configurations: general case

After the projection, the resulting gauge group is given by the Γ-invariant part of the gauge group, denoted by $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)^\Gamma$. The superscript Γ is used to indicate that we only keep the trivial representations in the decomposition, that is, we only keep that subspaces that transform trivially. We want to compute $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)^\Gamma$. First, we can see that by Schur's lemma Why ?

[3] To be more exact, we should say that they transform a "diagonal" version of $\mathbf{N}\otimes\overline{\mathbf{N}}$ since each index in (3.17) transforms with the same group element. We cannot choose to act with T_a on one index and with T_b on the other one for example.

$$(V_i \otimes V_j^*)^\Gamma = \delta_{ij} \tag{3.19}$$

Using (3.18), we get

$$\begin{aligned}
\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)^\Gamma &= (\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^\Gamma && (3.20)\\
&= \bigoplus_{i,j\in I} \left((\mathbb{C}^{N_i} \otimes V_i) \otimes (\mathbb{C}^{N_j} \otimes V_j)^*\right)^\Gamma && (3.21)\\
&= \bigoplus_{i,j\in I} \left(\mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^* \otimes V_i \otimes V_j^*\right)^\Gamma && (3.22)\\
&= \bigoplus_{i,j\in I} \left(\mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^*\right)^\Gamma \otimes \left(V_i \otimes V_j^*\right)^\Gamma && (3.23)\\
&= \bigoplus_{i\in I} \mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_i})^* && (3.24)
\end{aligned}$$

so, in complete notations,

$$(\rho, \mathbb{C}^N)^\Gamma = \bigoplus_{i\in I} (\rho_i, \mathbb{C}^{N_i}) \otimes (\overline{\rho_i}, \mathbb{C}^{N_i}) \tag{3.25}$$

and the daughter gauge group is

$$G_{\mathrm{proj}} = \bigotimes_{i\in I} \mathrm{U}(N_i \dim \rho_i). \tag{3.26}$$

Now it turns out that in the low energy effective field theory the U(1) factor of every $\mathrm{U}(N_i)$ decouples so the resulting gauge group is in fact Why ?

$$\boxed{G_{\mathrm{proj}} = \bigotimes_{i\in I} \mathrm{SU}(N_i \dim \rho_i).} \tag{3.27}$$

For the matter fields, the reasoning is similar but we now have to take into account the R-symmetry. Let $\mathbf{4} \equiv (\rho_{\mathbf{4}}, V_{\mathbf{4}})$ be the fundamental representation of $\mathrm{SU}(4)_R$ and $\mathbf{6} \equiv (\rho_{\mathbf{6}}, V_{\mathbf{6}})$ be the fundamental representation of $\mathrm{SO}(6)_R$. Wish to compute $(V_{\mathcal{R}} \otimes \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N))^\Gamma$ with $\mathcal{R} = \mathbf{4}, \mathbf{6}$:

$$\begin{aligned}
(V_{\mathcal{R}} \otimes \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N))^\Gamma &= \bigoplus_{i,j\in I} \left(V_{\mathcal{R}} \otimes (\mathbb{C}^{N_i} \otimes V_i) \otimes (\mathbb{C}^{N_j} \otimes V_j)^*\right)^\Gamma && (3.28)\\
&= \bigoplus_{i,j\in I} \left(V_{\mathcal{R}} \otimes \mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^*\right)^\Gamma \otimes \left(V_i \otimes V_j^*\right)^\Gamma && (3.29)\\
&= \bigoplus_{i,j\in I} a_{ij}^{\mathcal{R}} \left(\mathbb{C}^{N_i} \otimes (\mathbb{C}^{N_j})^*\right) && (3.30)
\end{aligned}$$

with

$$\boxed{(\rho_{\mathcal{R}}, V_{\mathcal{R}}) \otimes (\rho_i, V_i) = \bigoplus_{j\in I} a_{ij}^{\mathcal{R}} (\rho_j, V_j).} \tag{3.31}$$

This expression makes sense because $(\rho_{\mathcal{R}}, V_{\mathcal{R}})$ is a representation of SU(4) so it is in particular a representation of SU(3) and therefore also in particular a representation of Γ. In more complete notations, we obtained that

$$((\rho_{\mathcal{R}}, V_{\mathbb{R}}) \otimes (\rho, \mathbb{C}^N))^\Gamma = \bigoplus_{i,j\in I} a_{ij}^{\mathcal{R}} \left((\rho_i, \mathbb{C}^{N_i}) \otimes (\overline{\rho}_j, \mathbb{C}^{N_j})\right). \tag{3.32}$$

Using the fact that characters are homomorhisms and the orthogonality of the characters of characters of irreducible non-equivalent representations, we obtain the explicit expression of the coefficient $a_{ij}^{\mathcal{R}}$:

$$\boxed{a_{ij}^{\mathcal{R}} = \frac{1}{|\Gamma|} \sum_{\gamma=1}^{r} r_\gamma \chi^{\mathcal{R}}(\gamma) \chi^i(\gamma) \chi^j(\gamma)^*} \tag{3.33}$$

where r_γ is the order of the conjugacy class containing γ and χ^i is the character of ρ_i.

In the end, we can see that:

> In the daughter theory, the matter fields become a total of $a^{\mathbf{4}}_{ij}$ bi-fundamental fermions $\psi^{ij}_{f_{ij}}$ ($f_{ij} = 1, \dots, a^{\mathbf{4}}_{ij}$) and $a^{\mathbf{6}}_{ij}$ bi-fundamental complex scalars $\phi^{ij}_{f_{ij}}$ ($f_{ij} = 1, \dots, a^{\mathbf{6}}_{ij}$). They all transform in the $(\mathbf{N_i}, \overline{\mathbf{N_j}})$ of $\mathrm{SU}(N_i) \times \mathrm{SU}(N_j)$ under the products of gauge groups.

3.4.4 | Bi-index notation

The gauge and the bi-fundamental fields that we obtained in the previous section are easier to manipulate in the *bi-index notation*. Recall that $A_\mu \in \mathfrak{u}(N)$ so it is an $N \times N$ complex matrix and we denote its elements by $A_{\mu;IJ}$ ($I, J = 1, \dots, N$). We saw that this matrix of fields transform under Γ as

$$A_\mu \mapsto \rho(\gamma) A_\mu \rho(\gamma)^{-1}, \qquad \gamma \in \Gamma, \tag{3.34}$$

i.e. in the adjoint representation but without the derivative term (as explained above). Now if $\{(\rho_i, V_i)\}_{i\in I}$ is a complete set of irreducible representation of γ, any representation of Γ on $\mathbb{C}^n$ can be decomposed as (3.10). In particular, this provides us with the following partitioning of N:

$$N = \sum_{i\in} N_i \dim \rho_i. \tag{3.35}$$

It is very convenient to apply the analogous partitionning for the matrix A_μ. More precisely, we now denote decompose A_μ into $(\dim \rho_i N_i) \times (\dim \rho_j N_j)$ sub-blocks $A_{\mu;ij}$ where $i, j \in I$ are indices over the irreducible representations. From (**??**), those sub-blocks transform as direct sums of the same irreducible representations:

$$A_{\mu;ij} \mapsto \rho_i(\gamma)^{N_i} A_{\mu;ij} (\rho_j(\gamma)^{-1})^{N_j}, \qquad \gamma \in \Gamma. \tag{3.36}$$

We can go one step further and decompose the sub-blocks $A_{\mu;ij}$ into $\dim \rho_i \times \dim \rho_j$ sub-sub-blocks $A_{\mu;i\alpha_i,j\beta_j}$ where $\alpha_i, \beta_i = 0, \dots N_i - 1$. From (3.36), they transform directly under the irreducible representations:

$$\boxed{A_{\mu;i\alpha_i,j\beta_j} \mapsto \rho_i(\gamma) A_{\mu;i\alpha_i,j\beta_j} \rho_j(\gamma)^{-1}, \qquad \gamma \in \Gamma.} \tag{3.37}$$

In particular, we see that this relations does not depend on the indices α_i and β_j, they all transform in the same way. This notation is very convenient to compute explicitely the invariant configurations, as we will see in several examples. We use the exact same bi-index notations for X^m and ψ^a.

3.5 | Field content, quivers and McKay graphs

A convenient way the represent the matter content of a daughter theory is to use *quiver diagrams*. A quiver is a finite oriented graph such that each node i represents a gauge factor $\mathrm{SU}(N_i)$ and each arrow $i \to j$ represents a bi-fundamental field transforming under $(\mathbf{N_i}, \overline{\mathbf{N_j}})$. So, in essence, the arrows represent the vector multiplets (gauge) and the vertices the hypermultiplets (matter). The *adjacency matrix* A of the graph is a $k \times k$ with k being the number of nodes (gauge factors) whose elements A_{ij} are the number of arrows (bi-fundamental fields) from i to j. In other words, from (3.32), the adjacency matrix of the fermions has elements $A_{ij} = a^{\mathbf{4}}_{ij}$ and the one of the scalars has elements $A_{ij} = a^{\mathbf{6}}_{ij}$.

On the other hand, given finite group Γ, a representation (ρ_W, W) and a complete set of irreducible representations $\{(\rho_i, V_i)\}_{i\in I}$ of the latter, one can construct a McKay graph (or quiver) as follows:

1. Draw a vertex for every representation (ρ_i, V_i).
2. For every representation (ρ_i, V_i), compute the decomposition

$$(\rho_W, V_W) \otimes (\rho_i, V_i) = \bigoplus_j (\rho_j, V_j)^{\oplus n_{ij}}$$

where n_{ij} is the multiplicity of (ρ_j, V_j) in the decomposition of $(\rho_W, V_W) \otimes (\rho_i, V_i)$.

3. For every $n_{ij} > 0$, draw n_{ij} arrows from the vertex of (ρ_i, V_i) to the one of (ρ_j, V_j).

When $\Gamma \subset \mathrm{SU}(2)$ and that (ρ_W, W) is its defining representation, the McKay graphs are in one-to-one correspondence with the extended Dynkin diagrams of the simply laced Lie algebras. This is the classical McKay correspondence, see appendix **??**.

From (3.31) we see that $n_{ij} = a_{ij}^{\mathcal{R}}$, i.e. the matter quivers that we defined previously are exactly the McKay graph associated to the matter representation in question. Put differently, the matter content of the daughter theory is encapsulated in the McKay graphs of Γ and with respect to $\mathcal{R}$ with $\mathcal{R} = \mathbf{4}$ for spinors and $\mathcal{R} = \mathbf{6}$ for scalars. This very important point. For example, it will allows us to use known results on McKay graphs such as the McKay correspondence. Not only the field content is a quiver, which is exactly the McKay graph of the finite group, but we also have gauge groups at each vertex and transformations between them for each arrow (these transformations are precisely the fields, see as elements of $\mathrm{Hom}(V_i, V_j)$). So we are actually given a representation of the quiver. Moreover, the vacuum configurations have to solve the F-term and D-term equations. Those algebraic relations constrain the path algebra of the quiver: they are relations.

> A quiver gauge theory is a representations of a quiver with relations.

We mentioned the adjacency matrix as a way to represent a quiver but we will also use sometimes the another matrix called the *incidence matrix* and denoted by I. In the later, the columns index the arrows and the rows, the nodes such that the αth arrow from node i to j receives a -1 in position $I_{i\alpha}$, a $+1$ in position $I_{j\alpha}$ and zero elsewhere.

Another convenient way of representing the same information are the *dimer diagram*, i.e. a graph whose faces represent the gauge groups, the edges represent the bi-fundamental fields and the vertices represent the superpotentials.

3.6 Gauge anomaly cancellation

Finally, let us discuss the gauge anomaly cancellation. Our fields transform under the adjoint representation of $\mathrm{SU}(N)$, under the fundamentals of some $\mathrm{SU}(N_i)$ and under the anti-fundamentals of some $\mathrm{SU}(N_i)$. The adjoint representation being real, it is self-conjugate and therefore does not contribute to the anomaly. The other representation however do contribute: the fundamentals of each $\mathrm{SU}(N_i)$ have a $+1$ contribution the anti-fundamentals of each $\mathrm{SU}(N_i)$ have a -1 contribution. Anomaly cancellation therefore imposes that the contribution of the fundamental and of the anti-fundamental of $\mathrm{SU}(N_i)$ cancel each other, for each $i \in I$ (see section (**??**)). Since the bi-fundamental representations $(\mathbf{N}_{\boldsymbol{i}}, \overline{\mathbf{N}}_{\boldsymbol{j}})$ counts as N_j fundamentals of $\mathrm{SU}(N_i)$ and as N_i anti-fundamentals $\mathrm{SU}(N_j)$, the condition for anomaly cancellation is

$$\sum_{k \in I} a_{jk} N_k - \sum_{i \in I} a_{ij} N_i = 0 \tag{3.38}$$

for every $j \in I$, where a_{ij} are the elements of the adjacency matrix of the quiver. This can be simply rewritten as

$$\boxed{\sum_{i \in I} (a_{ji} - a_{ij}) N_i = 0} \tag{3.39}$$

This is equivalent to say that the vector $(N_0, N_1, N_2, \dots)$ formed by the ranks of the gauge groups must lie in nullspace of the antisymmetrized adjacency matrix. From this nice reformulation we can already see that any "symmetric" quiver, i.e. quiver associated to a symmetric adjacency matrix, is automatically anomaly-free. In other words, a gauge theory is automatically anomaly-free if and only if all arrows come in pairs[4] (with different orientations) in the quiver. If it is not the case, the ranks of the gauge groups must be constrained.

At this stage, we can already predict that all $\mathcal{N} = 2$ orbifold quiver gauge theories will be automatically anomaly-free since their quiver are their McKay graphs in which all arrows come in pairs.

[4] Loops are also allowed since they don't contribute to the anomaly.

3.7 A simple example: $S = \mathbb{C}^3/\mathbb{Z}_3$

We illustrate the previous discussion by the simple case where $\Gamma = \mathbb{Z}_3$ acts on $\mathbb{C}^2$ as

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{3.40}$$

i.e. the transverse space is the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. This simple example is a good first approach in which we will to explain in details each step so that we can go faster afterwards.

3.7.1 Projection

Let us consider a representation $(\rho, \mathbb{C}^N)$ of $\mathbb{Z}_3$. A complete set of irreducible representations of $\mathbb{Z}_3$ is given by $\{(\rho_k, V_k)\}_{k=0,1,2}$ with $V_k = \mathbb{C}$ and

$$\rho_k(g) = \zeta_3^k \tag{3.41}$$

where g is the generator of $\mathbb{Z}_3$. The representation (ρ, V) can be decomposed as

$$(\rho, V) = \bigoplus_{i=0}^{2} N_i(\rho_i, V_i). \tag{3.42}$$

In other words, it is equivalent to the representation

$$\rho(g) = \begin{bmatrix} 1 & & & & \cdots & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \zeta_3 & & & & & \\ \vdots & & & & \ddots & & & & \vdots \\ & & & & & \zeta_3 & & & \\ & & & & & & \zeta_3^2 & & \\ & & & & & & & \ddots & \\ 0 & & & & \cdots & & & & \zeta_3^2 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} N_0 \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} N_1 \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} N_2 \end{matrix} . \tag{3.43}$$

Since $\dim \rho_i = 1$, we have

$$N_0 + N_1 + N_2 = N. \tag{3.44}$$

The gauge field configurations that are left invariant under the action of $\mathbb{Z}_3$ are therefore the ones that satisfy

$$\rho(g) A_\mu \rho(g)^{-1} = A_\mu. \tag{3.45}$$

We actually want this relation to be true for any element of $\mathbb{Z}_n$ but in this case it is invariant under any element of $\mathbb{Z}_3$ if and only if it is invariant under the generator g of $\mathbb{Z}_3$, so we only need to impose (3.45). The constrained is easily solved by using the bi-index notation $A_{\mu;i\alpha_i,j\beta_j}$ $(i, j = 0, 1, 2, \alpha_i, \beta_i = 1, \ldots, N_i)$ for the component of the gauge fields. From (8.7), we can see that

$$A_{\mu;i\alpha_i,i\beta_j} \mapsto \rho_i(g) A_{\mu;i\alpha_i,j\beta_j} \rho_j(g)^{-1} = \zeta_n^{i-j} A_{\mu;i\alpha_i,j\beta_j}. \tag{3.46}$$

thus only the configurations with $A_{\mu;i\alpha_i,j\beta_j} = 0$ if $i \neq j$ are invariant. The gauge field has therefore a block diagonal form:

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & \\ & A_{\mu;11} & \\ & & A_{\mu;2,2} \end{bmatrix} \tag{3.47}$$

with $A_{\mu;ij} \equiv (A_{\mu;i\alpha_i,j\beta_j})_{\alpha_i=0,\ldots,N_i,\beta_j=0,\ldots,N_j}$. The block A_{ii} transforms under $\mathbb{Z}_3$ under $(\rho_i, V_i)^{N_i}$. Consequently, the gauge group is now broken to

$$G_{\text{proj}} = \mathrm{U}(N_0) \times \mathrm{U}(N_1) \times \mathrm{U}(N_2), \tag{3.48}$$

the biggest subgroup of $\mathrm{U}(N)$ that preserves those form of configurations.

Let us study the scalars. $\mathbb{Z}_3$ acts on the three complex scalars through

$$R(g) = \rho_1^{\oplus 3}(g) = \zeta_3 \mathbb{1}_3 = \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} \tag{3.49}$$

or, equivalently, on the real scalars as $R(g) = \zeta_3 \mathbb{1}_6$. According to (2.7)-(2.8), the scalar field configurations that are left invariant satisfy

$$R(g)^m{}_n \rho(g) X^n \rho(g)^{-1} = X^m \tag{3.50}$$

for all $g \in \mathbb{Z}_3$. Using the bi-index notations, this becomes

$$X^m_{i\alpha_i, j\beta_j} \mapsto \zeta_n \delta^m{}_n \rho_i(g) X^n_{i\alpha_i, j\beta_j} \rho_j(g)^{-1} = \zeta_3^{i-j+1} X^m_{i\alpha_i, j\beta_j} \tag{3.51}$$

$$\overline{X}^m_{i\alpha_i, j\beta_j} \mapsto \zeta_3^{-1} \delta^m{}_n \rho_i(g) \overline{X}^n_{i\alpha_i, j\beta_j} \rho_j(g)^{-1} = \zeta_3^{i-j-1} \overline{X}^m_{i\alpha_i, j\beta_j} \tag{3.52}$$

thus only the configurations with $X^n_{i\alpha_i, j\beta_j} = 0$ if $i-j+1 \neq 0$ are left invariant and only the configurations with $\overline{X}^n_{i\alpha_i, j\beta_j} = 0$ if $i-j-1 \neq 0$ are left invariant. The scalar fields X have a block off-diagonal form:

$$X^m = \begin{bmatrix} 0 & X^m_{01} & 0 \\ 0 & 0 & X^m_{12} \\ X^m_{21} & & 0 \end{bmatrix}, \qquad \overline{X}^m = \begin{bmatrix} 0 & 0 & \overline{X}^m_{02} \\ \overline{X}^m_{10} & 0 & 0 \\ 0 & \overline{X}^m_{21} & 0 \end{bmatrix} \tag{3.53}$$

with the block notations

$$X^m_{ij} \equiv (X^m_{i\alpha_i, j\beta_j})_{\alpha_i = 0, \ldots, N_i, \beta_j = 0, \ldots, N_j}$$
$$\overline{X}^m_{ij} \equiv (\overline{X}^m_{i\alpha_i, j\beta_j})_{\alpha_i = 0, \ldots, N_i, \beta_j = 0, \ldots, N_j}.$$

The block X^m_{ij} is an $N_i \times N_j$ block and transforms under the representation $(\mathbf{N_i}, \overline{\mathbf{N}}_{\boldsymbol{j}})$ of $\mathrm{U}(N_i) \times \mathrm{U}(N_j)$:

$$X^m_{i,i+1} \in \mathbf{N_{i+1}} \otimes \overline{\mathbf{N}}_{\boldsymbol{i}} \cong \mathrm{Hom}(V_{i+1}, V_i), \tag{3.54}$$

$$\overline{X}^m_{i+1,i} \in \mathbf{N_i} \otimes \overline{\mathbf{N}}_{\boldsymbol{i+1}} \cong \mathrm{Hom}(V_i, V_{i+1}). \tag{3.55}$$

Let us make an important observation: the form of the scalar fields are the same for every $m = 0, \ldots, 5$. This can be traced back to the fact that $R(g) = \zeta_3 \mathbb{1}_6$ so the R-symmetry action of $\mathbb{Z}_3$ is the same for all m.

Let us now study the four Weyl fermions ψ^a. Continue.

3.7.2 Quiver

We can draw the quiver of this daughter theory. We have three types of bi-fundamental scalar fields:

$$X^m_{01} \in (\mathbf{N_1}, \overline{\mathbf{N}}_{\mathbf{0}}), \quad X^m_{12} \in (\mathbf{N_2}, \overline{\mathbf{N}}_{\mathbf{1}}), \quad X^m_{20} \in (\mathbf{N_0}, \overline{\mathbf{N}}_{\mathbf{2}}). \tag{3.56}$$

In each representation bi-fundamental representation there are six real scalars, i.e. 3 complex scalars. They are each represented by an arrow between the right representations, see fig. 1.

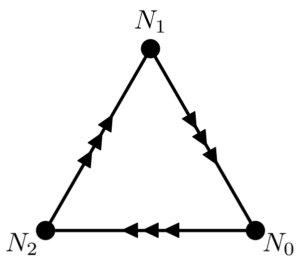


Figure 1: Quiver of the $\mathbb{C}^3/\mathbb{Z}_3$ daughter theory.

The adjacency matrix of this quiver is

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{bmatrix} \tag{3.57}$$

which is coherent with the fact that the coefficients of the McKay decomposition of $\rho_1 \oplus \rho_1 \oplus \rho_1$ are

$$\begin{aligned} n_{00} &= 0, & n_{01} &= 3 & n_{02} &= 0, \\ n_{10} &= 0, & n_{11} &= 0, & n_{12} &= 3, \\ n_{20} &= 3, & n_{21} &= 0, & n_{22} &= 0. \end{aligned} \tag{3.58}$$

3.7.3 | Gauge anomaly cancellation

We get the three following conditions:

$$\mathrm{SU}(N_0) : N_2 - N_1 = 0, \tag{3.59}$$
$$\mathrm{SU}(N_1) : N_0 - N_2 = 0, \tag{3.60}$$
$$\mathrm{SU}(N_2) : N_1 - N_0 = 0. \tag{3.61}$$

which immediately imply

$$N_0 = N_1 = N_2. \tag{3.62}$$

From (3.44), we get that $N_0 = N_1 = N_2 = N/3$, meaning that that the daughter theory has quantum gauge symmetry if and only if the parent theory with has gauge group $\mathrm{SU}(N)$ where N is a multiple of 3. Or, in other words, if the number of D-branes is a multiple of 3.

4 | Conformal invariance

We have discussed the general case and found the resulting field content for strings on any orbifold $\mathbb{C}^3/\Gamma$. The content of those theories was found to be completely determined by the two actions that give ourselves from start:

- the action of Γ on $\mathbb{C}^3$ that defines the orbifold per se and dictates how the "R-symmetry part" of the projection happens. This action is fixed by the choice of orbifold background we want.
- the action of Γ on $\mathbb{C}^N$ (from some arbitrary N) that defines the embedding of Γ in the gauge group $\mathrm{U}(N)$ and and dictates how the "gauge part" of the projection happens. Up until now, this action was completely arbitrary. In other words, the multiplicities N_i are arbitrary, as long as they sum to N, which is arbitrary.

Let is discuss the role of the second representations in more details. If the stack of branes trivially sits at a fixed point, which in our case is usually the origin, there is no problem. On the other hand, if the stack of branes moves away from the origin, this will break the Γ symmetry. The only the configurations that are able to move away from the fixed point are the ones that are still Γ-invariant while doing so, i.e. the configurations where the stack of branes possesses an image-stack in each Γ sector. Each of this stacks must be related by Γ and they move all together in the covering space such as to be set to a unique point in the quotient space. Since there are $|\Gamma|$ sectors we need exactly $|\Gamma|$ branes in total, one per sector. This means that N, the total number of branes in the covering space, cannot be arbitrary anymore. It must be of the form $N = n|\Gamma|$, where n now represents the number of brane in each stack. From the open string states point of view, this is means adding $n|\Gamma|$ Chan-Paton factors, resulting in an $U(n|\Gamma|)$ gauge symmetry. Γ then acts on the Chan-Paton factors and switch them between themselves. We now need to make sure that the string theory is consistent with that action. Let us insist on the fact that we can take N arbitrary but in order to have configurations that can move aways from the fixed point, N must be a multiple of Γ, otherwise the stack of branes just stays at the fixed point.

What is the link between moving axay from the fixed point and conformal invariance.

What does it have to do with the choice of embedding of Γ in the gauge group? If branes move away from the origin .

complete

What are the conditions to get a superconformal daughter theory? The answer lies in the choice of the second action, i.e. on the choice of embedding of Γ in the gauge group. One can show [3] that to have conformal invariance, we have to use the regular representation of Γ. The resulting embedding will be called *regular embedding*. This is only possible of course if $N = n|\Gamma|$. Actually, considering stacks of n branes, Γ acts with ρ_{reg}^n. Recall that the multiplicity of each irredicible representation in the decomposition of the regular representation is the dimension of the irrep, therefore

$$\rho_{\text{reg}}^n = \left(\bigoplus_{i\in I} \rho_i^{n_i} \right)^n = \bigoplus_{i\in I} \rho_i^{nn_i} \tag{4.1}$$

where $n_i \equiv \dim \rho_i$. Comparing to (3.10), this means that $N_i = nn_i$.

The gauge group is now $\mathrm{U}(n|\Gamma|)$ and it is broken to

$$G_{\text{proj}} = \bigotimes_{i\in I} \mathrm{SU}(nn_i) \tag{4.2}$$

at low energy. Let us be clear and mention that any choice of multiplicities N_i is valid and gives a supersymmetric daughter theory, but this theory is conformal only if $N_i = n \dim \rho_i$.

Since the superconformal theories are of particular interest, this case have been studied in some details.

5 Lagrangian

For conformal only ?

Let us describe the lagrangian of the projected theory [3]. We saw that the gauge group is broken to

$$G_{\text{proj}} = \bigotimes_{i\in I} \mathrm{SU}(Nn_i) \tag{5.1}$$

at low energy, with $n_i \equiv \dim \rho_i$. The coupling constant τ_i (including the theta angle as usual) of the ith group is

$$\tau_i = \frac{n_i \tau}{|\Gamma|} \tag{5.2}$$

where τ is the initial $\mathcal{N} = 4$ coupling, or the Type IIB coupling. This implies in particular that

$$\sum_{i\in I} n_i \tau_i = \tau. \tag{5.3}$$

There are Yukawa couplings for each triangle of the quiver consisting of two fermionic arrows and a bosonic arrow and quartic scalar interactions for each square consisting of four bosonic arrows. The coefficient ofthese interactions can be by projecting the $\mathcal{N} = 4$ lagrangian in terms of the field we kept. The Yukawa terms are therefore

$$Y = \sum_{i,j,k\in I} \gamma_{ijk}^{f_{ij},f_{jk},f_{kj}} \operatorname{tr}\left(\psi_{f_{ij}}^{ij} \phi_{f_{jk}}^{jk} \psi_{f_{ki}}^{ki} \right) \tag{5.4}$$

and the quartic scalar interaction terms by

$$V = \sum_{i,j,k,l\in I} \eta_{f_{ij};f_{jk},f_{kl},f_{li}}^{ijkl} \operatorname{tr}\left(\phi_{f_{ij}}^{ij} \phi_{f_{jk}}^{jk} \phi_{f_{kl}}^{kl} \phi_{f_{li}}^{li} \right) \tag{5.5}$$

with

$$\gamma_{ijk}^{f_{ij},f_{jk},f_{kj}} = \Gamma_{\alpha\beta,m} (Y_{f_{ij}})^m{}_{v_i\overline{v}_j} (Y_{f_{jk}})^\beta{}_{v_j\overline{v}_k} (Y_{f_{ki}})^\alpha{}_{v_k\overline{v}_i}, \tag{5.6}$$

$$\eta_{f_{ij};f_{jk},f_{kl},f_{li}}^{ijkl} = (Y_{f_{ij}})^{[m}{}_{v_i\overline{v}_j} (Y_{f_{jk}})^{n]}{}_{v_j\overline{v}_k} (Y^{f_{kl}})^{[m}{}_{v_k\overline{v}_l} (Y_{f_{li}})^{n]}{}_{v_l\overline{v}_i}. \tag{5.7}$$

The coefficients $(Y_{f_{ij}})^\alpha{}_{v_i\overline{v}_j}$ and $(Y_{f_{ij}})^m{}_{v_i\overline{v}_j}$ are to be understood as the f_{ij}th Clebsch-Gordan coefficients of the projection of $\mathbf{4} \otimes \rho_i$ and $\mathbf{6} \otimes \rho_i$ onto ρ_j, and $\Gamma_{\alpha\beta,m}$ is the invariant in $\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{6}$.

When we choose the regular embedding of Γ, the 1-loop beta functions have been computed in [3] and they o indeed vanish [3].

In the same paper, the superpotential inherited from the parent theory is shown to be

$$\mathcal{W} = \sum_{i,j,k\in I} \sum_{f_{ij},f_{jk},f_{ki}} h_{ijk}^{f_{ij},f_{jk},f_{kj}} \operatorname{tr}\left(\phi_{f_{ij}}^{ij}\phi_{f_{jk}}^{jk}\phi_{f_{ki}}^{ki}\right) \tag{5.8}$$

where

$$h_{ijk}^{f_{ij},f_{jk},f_{kj}} = \epsilon_{\alpha\beta\gamma}(Y_{f_{ij}})^{\alpha}{}_{v_i\overline{v}_j}(Y_{f_{jk}})^{\beta}{}_{v_j\overline{v}_k}(Y_{f_{ki}})^{\gamma}{}_{v_k\overline{v}_i} \tag{5.9}$$

and $(Y_{f_{ij}})^{\alpha}{}_{v_i\overline{v}_j}$ is now the f_{ij}th Clebsch-Gordan coefficient of the projection of $\mathbf{3}\otimes\rho_i$ onto ρ_j.

6 | Finitude

For any QFT, the Callan-Symanzik equations dictates the behavior, under the renormalization group flow, of the n-point correlator $G^{(n)}(\{\phi(x_i)\}; M,\lambda)$ forthe quantum fields $\phi(x)$ according the the renormalizationof the coupling λ and momentum scale M:

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right] G^{(n)}(\{\phi(x_i)\}; M,\lambda) = 0, \tag{6.1}$$

where the dimensionless functions β and γ are the β-function and the anomalous dimension respectively. As usuall, the determine how the shifts $\lambda\to\lambda+\delta\lambda$ in the coupling constant and $\phi(1+\delta\eta)\phi$ in the wave function compensate for the shift in the renomalization scale M:

$$\beta(\lambda) \equiv M\frac{\partial\lambda}{\partial M}, \qquad \gamma(\lambda) \equiv -M\frac{\partial\eta}{\partial M}. \tag{6.2}$$

Thee are three possible behaviours for the β-function when λ is small: $\beta(\lambda) > 0$ then the theory has good IR behaviour and admits valid Feynman permutation theory at large distances, $\beta(\lambda) < 0$ then the theory is asymptotically free and has a good permutative behaviour in the UV limit and finally if $\beta(\lambda) = 0$ then the coupling constants do not flow and the renormalized couplings are always equal to the bare ones.

Theories in which no divergences can be associated with the coupling in the ultraviolet are called *finite* theories. To find such theories, supersymmetry is often a very god tool as it indices cancellation of the boon-fermion loop effects. More precisely, $\mathcal{N}=4$ SYM have been shown to be finite to all orders, for $\mathcal{N}=2$ SYM it has been shown that no higher than 1-loop corrections exist for the β-function (Adler-Bardeen theorem) and for $\mathcal{N}=1$ theories the vanishing at 1-loop implies the vanishing at 2-loops. Finite non-supersymmetric theories has however also been porposed (see [4] for references). Of particular interest are theories which, in addition to a vanishing β-function, also have a vanishing anomalous dimensions. These theories are often part of a continuous manifold of scale-invariant theories and are characterized by the existence of exactly marginal operators (and hence dimensionless coupling constants). An important result is that there is a choice of coupling constant such that both the β-function and the anomalous dimensions vanish at first order, then the theory is necessarily finite at all orders.

From (3.31), we must have

$$\dim(\mathcal{R}) r_i = \sum_{j\in I} a_{ij}^{\mathcal{R}} r_j. \tag{6.3}$$

with $r_i \equiv \dim\rho_i$. It was shown (see **??** for references) that this equation necessitates the vanishing of the 1-loop β-function. In addition, we discussed that the remaining SUSY must be in the commutant of Γ in SU(4) R-symmetry of the $\mathcal{N}=4$ parent theory. It was shown in [3, 5] that the 1-loop β-function is proportional to the function $dr_i - a_{ij}^d r_j$, called the *discriminant function*:

$$\beta_{\text{1-loop}} \propto dr_i - a_{ij}^d r_j, \qquad d \equiv 4-\mathcal{N}.$$

Since the vanishing of the β-function signifies finitude, we have in particular the following necessary conditions for finitude:

SUSY	Finitude
$\mathcal{N}=2$	$2r_i = a_{ij}^2 r_j$
$\mathcal{N}=1$	$3r_i = a_{ij}^3 r_j$
$\mathcal{N}=0$	$4r_i = a_{ij}^4 r_j$

Note that in the case $\mathcal{N}=2$ this condition is necessary and sufficient but not for $\mathcal{N}=1$ and $\mathcal{N}=0$ where one also have to chack the superpotential. In the non-supersymmetric case, vanishing of the 1-loop β-function does not necessarily implies the vanishing of the following orders so we consider the notion finiteness in a weaker sense where only the leading order must vanish.

Finitude of quiver gauge theories constrcuted in from D-brane probing singularities, the Hanany-Witten setup or geometric engineering is very much linked the the mathematical properties of their quiver. See [6] for more details.

7 Fractional branes

Rework/under this section

We saw that invariant configurations of D-branes naturally give rise to the regular representation of the orbifold group Γ on the Chan-Paton factors. Correspondingly, these branes are called *regular branes*. The regular representation is not irreducible and it decomposes as

$$\rho_{\text{reg}} = \bigoplus_{i\in I} N_i \rho_i \tag{7.1}$$

with $N_i = \dim \rho_i$ in terms of the irreducible representations of Γ. One may then wonder about the existence of a more "elementary" set of D-branes such that the open strings attached to the latter carry Chan-Paton factors indices that transform in an irreducible representation. Those indeed do exist and are called *fractional D-branes*. They are BPS object that carry only a fraction of the charge with respect to the untwisted RR $(p+1)$-form of a regular brane but they are charged with respect to some twisted RR $(p+1)$-form, contrarily to the regular D-branes. They are stuck at the orbifold fixed point (where all twisted fields sit) since sitting elsewhere would require an invariant configuration in the covering space and those correspond to the regular representation, as we mentioned. So they cannot be fractional D-branes.

An important property of fractional D-branes is that they can be interpreted as higher-dimensional branes wrapped on exceptional cycles of the resolved space. In the orbifold limit, these exceptional cycles collapses leaving lower-dimensional fractional branes behind. This suggest a link between the spectrum of the D-branes and the homology of the resolved orbifold space. More precisely, it is linked to its homological K-theory. This is has been studied a lot.

From the point of view of boundary conformal field theory[5], *fractional D-branes* are reflected by boundary states in twisted sectors.

8 $\mathcal{N}=2$ daughter theories

8.1 Generalities

Quotienting by a finite subgroup Γ of SU(2) give rise to the so-called *Klein singularieties*. There also refered as *simple*, *du Val* or *ADE* singularieties, those are all synonims. To find (3.33), the only remaining choice is $\mathcal{R}$. The only thing we now is that these representations must come from the fundamental **4** (for fermions) or the anti-symmetric **6** of SU(4). Now, for a general decomposition

$$\mathcal{R} = \bigoplus_{i\in I} c_i \rho_i \tag{8.1}$$

[5] CFT on a manifold with boundary. Boundary conditions for open strings are interpreted as coherent states of the corresponding closed string 2d CFT.

we must have $\dim \mathcal{R} = \sum_{i \in I} \dim \rho_i$, i.e. the dimension of $\mathcal{R}$ (4 for the fermions and 6 for the bosons) must be partitioned into the dimension of the irreps of Γ and, out of those possibilities, we must choose the ones that are define are a representation of the SU(2)SU(4), namely:

$$\begin{array}{ccc} \mathrm{SU}(4) & \to & \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1) \\ \mathbf{4} & \to & (\mathbf{2}, \mathbf{1})_{+1} \oplus (\mathbf{1}, \mathbf{2})_{-1} \\ \mathbf{6} & \to & (\mathbf{1}, \mathbf{1})_{+2} \oplus (\mathbf{1}, \mathbf{1})_{-2} \oplus (\mathbf{2}, \mathbf{2})_{0} \end{array} \tag{8.2}$$

where the subscripts correspond to the U(1) factor (i.e. the trace) and in particular the $\pm$ dictates the overall traceless condition. Exploiting these decompositions and paying with the reppresentation properties, one can show that we must have [7]

$$\mathbf{4} = \mathbf{2}_{\text{trivial}} \oplus \mathbf{2} \tag{8.3}$$

$$\mathbf{6} = \mathbf{2}_{\text{trivial}} \oplus \mathbf{2} \oplus \mathbf{2}. \tag{8.4}$$

This limits our attention to only 2-dimensional representations of Γ. However there are still many possiblities.

$$a_{ij}^{\mathbf{4}} = \tag{8.5}$$

8.2 | $S = \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$

We consider a representation $(\rho, \mathbb{C}^N)$ of $\mathbb{Z}_n$. We decompose it on the set of irreducible representations of $\mathbb{Z}_n$ as

$$(\rho, \mathbb{C}^n) = \bigoplus_{i=0}^{n-1} N_i(\rho_i, \mathbb{C}). \tag{8.6}$$

In other words, it is equivalent to the representation

$$\rho(g) = \begin{bmatrix} 1 & & & \cdots & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & \zeta_n^{n-1} & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & \zeta_n^{n-1} \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} N_0 \\ \vdots \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} N_{n-1} \end{array} . \tag{8.7}$$

Since $\dim \rho_i = 1$, $\sum_i N_i = N$.

The gauge field configurations that are left invariant under the action of $\mathbb{Z}_n$ are therefore the ones that satisfy

$$\rho(g) A_\mu \rho(g)^{-1} = A_\mu. \tag{8.8}$$

The constrained is easily solved by using the bi-index notation:

$$A_{\mu; i\alpha_i, i\beta_j} \mapsto \rho_i(g) A_{\mu; i\alpha_i, j\beta_j} \rho_j(g)^{-1} = \zeta_n^{i-j} A_{\mu; i\alpha_i, j\beta_j}. \tag{8.9}$$

thus only the configurations with $A_{\mu; i\alpha_i, j\beta_j} = 0$ if $i \neq j$ are invariant. The gauge field has therefore a block diagonal form:

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & & \\ & A_{\mu;11} & & \\ & & \ddots & \\ & & & A_{\mu;n-1,n-1} \end{bmatrix} \tag{8.10}$$

with $A_{\mu;ij} \equiv (A_{\mu; i\alpha_i, j\beta_j})_{\alpha_i = 0, \ldots, N_i, \beta_j = 0, \ldots, N_j}$. The block A_{ii} transforms under $\mathbb{Z}_n$ as $(\rho_i, V_i)^{N_i}$. For now it is only a simple generalization of the case $\mathbb{C}^3/\mathbb{Z}_3$. This makes sense: projection of the gauge field only depends the discrete group Γ, not on the way it acts on $\mathbb{C}^3$ because it does not transform under R-symmetry.

The gauge group is now broken to

$$G_{\text{proj}} = \prod_{i=0}^{n-1} \mathrm{U}(N_i). \tag{8.11}$$

Now for the scalar fields. The action of $\mathbb{Z}_n$ that we consider leaves the first component of $\mathbb{C}^3$ untouched so we take the action $\mathbf{1} \oplus \mathbf{2}$ where $\mathbf{2}$ is the usual action of $\mathbb{Z}_n$ on $\mathbb{C}^2$. In other words,

$$R(g) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_n & 0 \\ 0 & 0 & \zeta_n^{-1} \end{bmatrix}. \tag{8.12}$$

Or, equivalently, $R(g)^m{}_n = \delta^m{}_n A_n$ with $A_m = (1, 1, \zeta_n, \zeta_n, \zeta_n^{-1}, \zeta_n^{-1})$. The scalar field configurations that are left invariant satisfy

$$R(g)^m{}_n \rho(g) X^n \rho(g)^{-1} = X^m \tag{8.13}$$

for all $g \in \mathbb{Z}_n$. Using the bi-index notations, this becomes

$$X^m_{i\alpha_i, j\beta_j} \mapsto \delta^m{}_n A_n \rho_i(g) X^m_{i\alpha_i, j\beta_j} \rho_j(g)^{-1} = \delta^m{}_n A_n \zeta_n^{i-j} X^m_{i\alpha_i, j\beta_j} = \begin{cases} \zeta_n^{i-j} X^m_{i\alpha_i, j\beta_j}, & m = 0, 1 \\ \zeta_n^{i-j+1} X^m_{i\alpha_i, j\beta_j}, & m = 2, 3 \\ \zeta_n^{i-j-1} X^m_{i\alpha_i, j\beta_j}, & m = 4, 5 \end{cases} \tag{8.14}$$

$$\overline{X}^m_{i\alpha_i, j\beta_j} \mapsto \delta^m{}_n \overline{A_n} \rho_i(g) \overline{X}^m_{i\alpha_i, j\beta_j} \rho_j(g)^{-1} = \delta^m{}_n \overline{A_n} \zeta_n^{i-j} X^m_{i\alpha_i, j\beta_j} = \begin{cases} \zeta_n^{i-j} X^m_{i\alpha_i, j\beta_j}, & m = 0, 1 \\ \zeta_n^{i-j-1} X^m_{i\alpha_i, j\beta_j}, & m = 2, 3 \\ \zeta_n^{i-j+1} X^m_{i\alpha_i, j\beta_j}, & m = 4, 5 \end{cases} \tag{8.15}$$

thus only the configurations with $X^{0,1}_{i\alpha_i, j\beta_j} = 0$ if $i - j \neq 0$, $X^{2,3}_{i\alpha_i, j\beta_j} = 0$ if $i - j + 1 \neq 0$ and $X^{4,5}_{i\alpha_i, j\beta_j} = 0$ if $i - j - 1 \neq 0$ are left invariant (and similarly for the conjugated fields). The scalar fields X have a the following forms:

$$X^{0,1} = \begin{bmatrix} X^{0,1}_{00} & & 0 \\ & \ddots & \\ 0 & & X^{0,1}_{n-1,n-1} \end{bmatrix}, \tag{8.16}$$

$$X^{2,3} = \begin{bmatrix} 0 & X^{2,3}_{01} & & 0 \\ & \ddots & \ddots & \\ & & 0 & X^{2,3}_{n-2,n-1} \\ X^{2,3}_{n-1,0} & & & 0 \end{bmatrix}, \quad X^{4,5} = \begin{bmatrix} 0 & & & X^{4,5}_{0,n-1} \\ X^{4,5}_{10} & 0 & & \\ & \ddots & \ddots & \\ 0 & & X^{4,5}_{n-1,n-2} & 0 \end{bmatrix}, \tag{8.17}$$

$$\overline{X}^{0,1} = \begin{bmatrix} \overline{X}^{0,1}_{00} & & 0 \\ & \ddots & \\ 0 & & \overline{X}^{0,1}_{n-1,n-1} \end{bmatrix}, \tag{8.18}$$

$$\overline{X}^{2,3} = \begin{bmatrix} 0 & & & \overline{X}^{2,3}_{0,n-1} \\ \overline{X}^{2,3}_{10} & 0 & & \\ & \ddots & \ddots & \\ 0 & & \overline{X}^{2,3}_{n-1,n-2} & 0 \end{bmatrix}, \quad \overline{X}^{4,5} = \begin{bmatrix} 0 & \overline{X}^{4,5}_{01} & & 0 \\ & \ddots & \ddots & \\ & & 0 & \overline{X}^{4,5}_{n-2,n-1} \\ \overline{X}^{4,5}_{n-1,0} & & & 0 \end{bmatrix} \tag{8.19}$$

so X^m_{ij} is an $N_i \times N_j$ block and transforms under the representation $(\mathbf{N_i}, \overline{\mathbf{N}}_{\boldsymbol{j}})$ of $\mathrm{U}(N_i) \times \mathrm{U}(N_j)$:

$$X^{0,1}_{i,i} \in \mathbf{N_i} \otimes \overline{\mathbf{N}}_{\boldsymbol{i}} \cong \mathrm{Hom}(V_i, V_i), \tag{8.20}$$

$$X^{2,3}_{i,i+1} \in \mathbf{N_{i+1}} \otimes \overline{\mathbf{N}}_{\boldsymbol{i}} \cong \mathrm{Hom}(V_{i+1}, V_i), \tag{8.21}$$

$$X^{4,5}_{i+1,i} \in \mathbf{N_i} \otimes \overline{\mathbf{N}}_{\boldsymbol{i+1}} \cong \mathrm{Hom}(V_i, V_{i+1}). \tag{8.22}$$

So the scalar fields are split up in three families depending on the way they transform under R-symmetry. We now see a big difference with the case $\mathbb{C}^3/\mathbb{Z}_3$: since the R-symmetry does not act the same way on each directions in $\mathbb{C}^3$, the invariant scalar field configurations are not the same in each direction either.

Let us draw the quiver for the case $n = 3$ so that we can compare to 1. We have $2 \cdot 9 = 18$ real scalar fields in 9 different representations:

$$X^0_{00}, X^1_{00} \in (\mathbf{N_0}, \overline{\mathbf{N}}_\mathbf{0}), \quad X^0_{11}, X^1_{11} \in (\mathbf{N_1}, \overline{\mathbf{N}}_\mathbf{1}), \quad X^0_{22}, X^1_{22} \in (\mathbf{N_2}, \overline{\mathbf{N}}_\mathbf{2}), \tag{8.23}$$

$$X^2_{01}, X^3_{01} \in (\mathbf{N_1}, \overline{\mathbf{N}}_\mathbf{0}), \quad X^2_{12}, X^3_{12} \in (\mathbf{N_2}, \overline{\mathbf{N}}_\mathbf{1}), \quad X^2_{20}, X^3_{20} \in (\mathbf{N_0}, \overline{\mathbf{N}}_\mathbf{2}), \tag{8.24}$$

$$X^4_{10}, X^5_{10} \in (\mathbf{N_0}, \overline{\mathbf{N}}_\mathbf{1}), \quad X^4_{21}, X^5_{21} \in (\mathbf{N_1}, \overline{\mathbf{N}}_\mathbf{2}), \quad X^4_{02}, X^5_{02} \in (\mathbf{N_2}, \overline{\mathbf{N}}_\mathbf{0}), \tag{8.25}$$

We now only have 1 complex scalar in each representation and the quiver is given by 2.

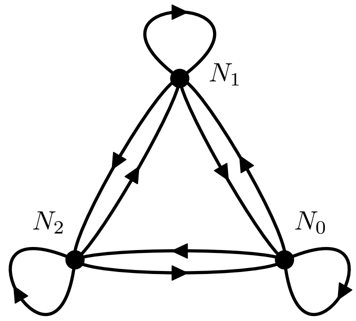


Figure 2: Quiver of the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_3$ daughter theory.

It is easy to see how the construction of the the quiver generalizes for arbitrary n. The adjacency matrix is

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \tag{8.26}$$

which is, as it should, coincides with the McKay decomposition of $\mathbf{1} \oplus \mathbf{2}$:

$$\begin{array}{ccc} n_{00} = 1, & \dots, & n_{0,n-1} = 1, \\ \vdots & & \vdots \\ n_{n-1,0} = 1, & \dots, & n_{n-1,n-1} = 1. \end{array} \tag{8.27}$$

Gauge anomaly cancellation now imposes that

$$N_{i-1} - N_{i-1} + N_{i+1} + N_{i+1} = 0 \tag{8.28}$$

for $i = 0, \dots, n-1$. Those constrains are always satisfied so the the factors N_i are arbitrary, as long as $\sum_i N_i = N$ of course.

This quiver is easy to scale up for any n. We therefore implement that into Mathematica and get the quivers for any n.

Looking at (8.20)-(8.20), we see that for each factor in the projected gauge group we have 1 complex scalar transforming in the adjoint (Coulomb branch) and two complex scalars in bi-fundamentals (Higgs branch). The former is the scalar in the $\mathcal{N} = 2$ vector multiplet and the latter are the complex scalars in of two hypermultiplets. The worldvolume theory is therefore $\mathcal{N} = 2$ SYM with two hypermultiplets.

8.3 | $S = \mathbb{C} \times \mathbb{C}^2/\mathcal{D}_n$

We quotient $\mathbb{C}^3$ by the binary dihedral group $\mathcal{D}_n$ that acts on the last two components. This group is geneated by two elements that we denote A and B. Firther details on its structure and representations are given in appendix A.1. A set of irreducible representations of $\mathcal{D}_n$ is given by

$$\{(\rho_i, V_i)\}_{i=0,\ldots,n+2} \tag{8.29}$$

with $V_i = \mathbb{C}$ for $i = 0, \ldots, 3$ and $V_i = \mathbb{C}^2$ for $i = 4, \ldots, n+2$, so there are 4 one-dimensional representations and $n - 1$ two-dimensional representations. It is more convenient to treat the 1-dimensional and 2-dimensional representations idependently. We therefore denote by σ_i with $i = 0, \ldots, 3$ the 1-dimensional representations and by μ_r with $r = 1, \ldots, n - 1$ the 2-dimensional ones:

a	$\sigma_a(A)$	$\sigma_a(B)$ (n even or odd)
0	1	1
1	1	-1
2	-1	1 or i
3	-1	-1 or $-i$

$$\mu_r(A) = \begin{bmatrix} e^{i\frac{\pi}{n}r} & 0 \\ 0 & e^{-i\frac{\pi}{n}r} \end{bmatrix}$$

$$\rho_r(B) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We note that there is a slight difference for the 1-dimensional representations if n is even or odd.

Any representation of $(\rho, \mathbb{C}^N)$ of $\mathcal{D}_n$ can be decomposed as

$$(\rho, \mathbb{C}^n) = \left(\bigoplus_{a=0}^{3} (\sigma_a, \mathbb{C}) \right) \oplus \left(\bigoplus_{r=1}^{n-1} (\mu_r, \mathbb{C}^2) \right). \tag{8.30}$$

The invariant gauge field configurations were found to be of the form

$$A_\mu = \begin{bmatrix}
\times & 0 & 0 & 0 & & & & & & \cdots & 0 \\
0 & \times & 0 & 0 & & & & & & & \\
0 & 0 & \times & 0 & & & & & & & \\
0 & 0 & 0 & \times & & & & & & & \\
 & & & & x_1 & 0 & & & & & \\
 & & & & 0 & -x_1 & & & & & \\
 & & & & & & x_2 & 0 & & & \\
 & & & & & & 0 & -x_2 & & & \\
\vdots & & & & & & & & \ddots & & \vdots \\
 & & & & & & & & & x_{n-1} & 0 \\
0 & & & & & & & & & 0 & -x_{n-1}
\end{bmatrix}. \tag{8.31}$$

where each entry (i, j) is an arbitrary block of size $N_i \times N_j$. The explicit derivations are carried out in appendix **??**. The projected gauge group is then

$$G_{\text{proj}} = \mathrm{SU}(N_0) \times \ldots \mathrm{SU}(N_3) \times \mathrm{SU}(2N_4) \times \cdots \times \mathrm{SU}(2N_{n+3}). \tag{8.32}$$

It is interesting to notice the main differences with the case of $\mathbb{Z}_n$:

- since $2\mathcal{D}_n$ is not abelian, it can have irreps of two or more dimension (which it does). This implies that the gauge field bloacks $A_{\mu;ij}$ are not necessarily diagonal. This will have important consequence for the residual gauge group.
- again because some irreps are mode then 1-dimensional, we can act on $\mathbb{C}^3$ with non-diagonal actions (which we do). This means that components of the different complex scalars can be exchanged under R-symmetry.

As we mentioned previously, there is no constrain on the ranks of the gauge groups coming from the the anomaly condition, it is automatically anomaly-free.

8.4 | $S = \mathbb{C} \times \mathbb{C}^2/\mathcal{T}, \mathcal{O}, \mathcal{I}$

Continue.

9 | $\mathcal{N} = 1$ daughter theories

9.1 | $S = \mathbb{C}^3/\mathbb{Z}_n$

Let us now consider the general case of $\mathbb{Z}_n$ acting on $\mathbb{C}^3$, i.e. we want to generalize the case that we treated in section 3.7.

9.1.1 | Actions of $\mathbb{Z}_n$ on $\mathbb{C}^3$

The first thing to do is to specify a representation of $\mathbb{Z}_n$ on $\mathbb{C}^3$. Any representation $(R, \mathbb{C}^3)$ can be decomposed as

$$R = \bigoplus_{k=0}^{n-1} \rho_k^{\oplus n_k} \tag{9.1}$$

and is therefore equivalent to a block-diagonal representation. This implies that $\sum_k n_k = 3$ so R can always be written as

$$R(g) = (\rho_a \oplus \rho_b \oplus \rho_c)(g) = \begin{bmatrix} \zeta_n^a & 0 & 0 \\ 0 & \zeta_n^b & 0 \\ 0 & 0 & \zeta_n^c \end{bmatrix} \tag{9.2}$$

with a, b, c some arbitrary exponents. We denote the representation (9.2) by (a, b, c). Let us make a few remarks on these representations:

- Since we consider $\mathbb{Z}_n$ as a subgroup of SU(3), we must have

  $$(a + b + c) \mod n = 0 \tag{9.3}$$

  such that $\det(R(g)) = 1$.
- Taking a or $a + n$ gives the same representation and the same is true for b and c, so we we can actually restrict ourselves to $0 < a, b, c < n$. We can trade the strict inequalities and allow the values 0 or n if we want to consider trivial representations as well. In this case, we will find that at least one direction of $\mathbb{C}^3$ is left untouched, i.e. we are in the situation where the orbifold is actually $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$, which we will not consider here.
- Permuting a, b, c gives equivalent representations so we can take a, b, c to be ordered, so $0 < a \le b \le c < n$.

The problem has now been reformulated as follows: we are looking for all possible ordered triplets (a, b, c) such that $0 < a \le b \le c < n$ and (9.3). For $n = 1$ and $n = 2$, it is clear that this is not possible, the possibilities involve at least one trivial representation. For $n = 3$, the only possibility is $(1, 1, 1)$, the one we used in 3.7. What about bigger values of n? First, note that $0 < a \le b \le c < n$ implies that the maximum value of $a+b+c$ is $3n-1$. From equation (9.3), the only two possibilities are then $a+b+c = n$ or $a + b + c = 2n$. What simplifies the analysis is that the second case can actually be ignored because it is already being taken care of by the first case. Let us explain that in more details: . details

Now that we have seen that the only possibility is $a + b + c = n$, we find that the maximum value that a can take is $\lfloor n/3 \rfloor$, so $a = 1, \dots, \lfloor n/3 \rfloor$. c can then be expressed in terms of a and b as $c = n - a - b$. The constraint $b \le c$ then implies $b \le (n-a)/2$, i.e. $b = a, \dots, \lfloor (n-a)/2 \rfloor$. To summarize, the representations are all the representations of the form

$$(a, b, n - a - b) \tag{9.4}$$

with $a = 1, \dots, \lfloor n/3 \rfloor$ and $b = a, \dots, \lfloor (n - a)/2 \rfloor$. For small values, we get

$$\begin{aligned}
&\mathbb{Z}_1 : \text{necessarily involves a trivial representation} \\
&\mathbb{Z}_2 : \text{necessarily involves a trivial representation} \\
&\mathbb{Z}_3 : (1, 1, 1) \\
&\mathbb{Z}_4 : (1, 1, 2) \\
&\mathbb{Z}_5 : (1, 1, 3), (1, 2, 2) \\
&\mathbb{Z}_6 : (1, 1, 4), (1, 2, 3), (2, 2, 2) \\
&\mathbb{Z}_7 : (1, 1, 5), (1, 2, 4), (1, 3, 3), (2, 2, 3) \\
&\quad \vdots
\end{aligned}$$

For an arbitrary n, the total number of different representation is Correct this.

$$\sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} + 1 \right\rfloor = \begin{cases} 3k^2, & \text{if } n = 6k \\ 3k^2 + k, & \text{if } n = 6k+1, \\ 3k^2 + 2k, & \text{if } n = 6k+2, \\ 3k^2 + 3k + 1, & \text{if } n = 6k+3, \\ 3k^2 + 4k + 1, & \text{if } n = 6k+4, \\ 3k^2 + 5k + 2, & \text{if } n = 6k+5 \end{cases} \tag{9.5}$$

The details are presented in appendix **??**.

9.1.2 Example: $\mathbb{Z}_5$

For $\mathbb{Z}_5$, we saw that there are two nonequivalent ways of acting on $\mathbb{C}^3$, see (**??**). Let us first consider the action $(1,1,3)$

$$R(g) = \begin{bmatrix} \zeta_5 & 0 & 0 \\ 0 & \zeta_5 & 0 \\ 0 & 0 & \zeta_5^3 \end{bmatrix} \tag{9.6}$$

i.e. $R = \rho_1 \oplus \rho_1 \oplus \rho_3$.

For the gauge field, the reasoning is exactly the same than for $\mathbb{Z}_3$ and we get

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & \\ & \ddots & \\ & & A_{\mu;44} \end{bmatrix} \tag{9.7}$$

where each block $A_{\mu;ii}$ is of size $N_i \times N_i$. The projected gauge group is

$$G_{\text{proj}} = \mathrm{U}(N_0) \times \mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \mathrm{U}(N_3) \times \mathrm{U}(N_4). \tag{9.8}$$

The scalar fields transform as

$$X^m_{i\alpha_i, j\beta_j} \to R(g)^m{}_n \rho_i(g) X^n_{i\alpha_i, j\beta_j} \rho_j(g)^{-1} \tag{9.9}$$

so the invariant configurations must satisfy

$$X^m_{i\alpha_i, j\beta_j} = R(g)^m{}_n \zeta_4^{i-j} \rho_i(g) X^n_{i\alpha_i, j\beta_j} = \begin{cases} \zeta^{i-j+1} X^m_{i\alpha_i, j\beta_j}, & \text{if } m = 0,1,2,3 \\ \zeta^{i-j+3} X^m_{i\alpha_i, j\beta_j}, & \text{if } m = 4,5. \end{cases} \tag{9.10}$$

and are therefore of the form

$$X^m = \begin{bmatrix} 0 & X^m_{01} & 0 & 0 & 0 \\ 0 & 0 & X^m_{12} & 0 & 0 \\ 0 & 0 & 0 & X^m_{23} & 0 \\ 0 & 0 & 0 & 0 & X^m_{34} \\ X^m_{40} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9.11}$$

for $m = 0,1,2,3$ and

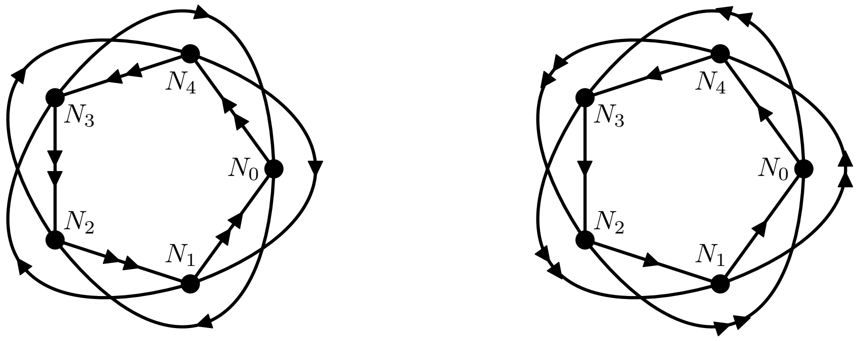
$$X^m = \begin{bmatrix} 0 & 0 & 0 & X^m_{03} & 0 \\ 0 & 0 & 0 & 0 & X^m_{14} \\ X^m_{20} & 0 & 0 & 0 & 0 \\ 0 & X^m_{31} & 0 & 0 & 0 \\ 0 & 0 & X^m_{42} & 0 & 0 \end{bmatrix} \tag{9.12}$$

for $m = 4,5$. Gauge anomaly cancellation imposes

$$-N_{i-2} + N_{i+2} - 2N_{i+1} + 2N_{i-1} = 0 \tag{9.13}$$

for $i = 0,1,2,3,4$ which implies that

$$N_0 = N_1 = N_2 = N_3 = N_4. \tag{9.14}$$



(a) $(1, 1, 3)$ *(b)* $(1, 2, 2)$

Figure 3: Quivers of the $\mathbb{C}^3/\mathbb{Z}_5$ *daughter theories.*

Now if we choose the representation $(1, 2, 2)$, i.e.

$$R(g) = \begin{bmatrix} \zeta_5 & 0 & 0 \\ 0 & \zeta_5^2 & 0 \\ 0 & 0 & \zeta_5^2 \end{bmatrix} \tag{9.15}$$

we get instead that the invariant field configurations must satisfy

$$X^m_{i\alpha_i, j\beta_j} = R(g)^m{}_n \zeta_4^{i-j} \rho_i(g) X^n_{i\alpha_i, j\beta_j} = \begin{cases} \zeta^{i-j+1} X^m_{i\alpha_i, j\beta_j}, & \text{if } m = 0, 1 \\ \zeta^{i-j+2} X^m_{i\alpha_i, j\beta_j}, & \text{if } m = 2, 3, 4, 5. \end{cases} \tag{9.16}$$

and are therefore of the form

$$X^m = \begin{bmatrix} 0 & X^m_{01} & 0 & 0 & 0 \\ 0 & 0 & X^m_{12} & 0 & 0 \\ 0 & 0 & 0 & X^m_{23} & 0 \\ 0 & 0 & 0 & 0 & X^m_{34} \\ X^m_{40} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{9.17}$$

for $m = 0, 1$ and

$$X^m = \begin{bmatrix} 0 & 0 & X^m_{02} & 0 & 0 \\ 0 & 0 & 0 & X^m_{13} & 0 \\ 0 & 0 & 0 & 0 & X^m_{24} \\ X^m_{30} & 0 & 0 & 0 & 0 \\ 0 & X^m_{41} & 0 & 0 & 0 \end{bmatrix} \tag{9.18}$$

for $m = 2, 3, 4, 5$. Gauge anomaly cancellation imposes

$$-2N_{i+2} + 2N_{i-2} - N_{i+1} + N_{i-1} = 0 \tag{9.19}$$

for $i = 0, 1, 2, 3, 4$ which implies that

$$N_0 = N_1 = N_2 = N_3 = N_4. \tag{9.20}$$

Right conditions?

Note that, even we thought that $(1, 1, 3)$ and $(1, 2, 2)$ were different representation, that are actually equivalent in the sense that $(1, 1, 3) + (1, 1, 3) = (2, 2, 1)$. The two quivers in fig. 3 should therefore be the same. And indeed, upon further inspection, the two are the same. To see this, we can rename the vertices in the second graph as $N_1 \to N_3, N_2 \to N_1, N_3 \to N_4, N_4 \to N_2$. This renaming defines the bijection between the two graphs. A more pragmatic way to see this is to look at the adjacency matrices:

$$a_{(1,1,3)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 2 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \qquad a_{(1,2,2)} = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}. \tag{9.21}$$

Changing the names of the vertices is equivalent to swapping line and columns. For example,

9.1.3 | General $\mathbb{Z}_n$

Let us now consider the general action

$$R(g) = \begin{bmatrix} \zeta_n^a & 0 & 0 \\ 0 & \zeta_n^b & 0 \\ 0 & 0 & \zeta_n^c \end{bmatrix} \tag{9.22}$$

of $\mathbb{Z}_n$ on $\mathbb{C}^3$, where (a, b, c) one of the representation that we studied before. In particular, recall that $a + b + c = n$. Following the same reasoning than before, we get that the gauge field of of the form

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & \\ & \ddots & \\ & & A_{\mu;n-1,n-1} \end{bmatrix}. \tag{9.23}$$

Invariant scalar field configurations transform as

$$X^m_{i\alpha_i,j\beta_j} = \begin{cases} \zeta_n^{i-j+a} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 0,1 \\ \zeta_n^{i-j+b} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 2,3 \\ \zeta_n^{i-j+c} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 4,5 \end{cases} \tag{9.24}$$

so

$$X^{0,1}_{j-a,j} \in (\mathbf{N}_{\boldsymbol{j}}, \overline{\mathbf{N}}_{\boldsymbol{j-a}}), \tag{9.25}$$
$$X^{2,3}_{j-b,j} \in (\mathbf{N}_{\boldsymbol{j}}, \overline{\mathbf{N}}_{\boldsymbol{j-b}}), \tag{9.26}$$
$$X^{4,5}_{j-c,j} \in (\mathbf{N}_{\boldsymbol{j}}, \overline{\mathbf{N}}_{\boldsymbol{j-c}}) \tag{9.27}$$

are the only possible non-vanishing components. This allows us to quickly draw all the possible quivers for a given n. Once again, the difficulty is only computational, not conceptual. This can therefore easily be implemented into Mathematica and we get the quiver for any n and any representation (a, b, c).

9.2 | $S = \mathbb{C}^3/\Delta(3n^2), \Delta(6n^2)$

9.3 | $S = \mathbb{C}^3/\Sigma_{36\times3}, \Sigma_{60\times3}, \Sigma_{168\times3}, \Sigma_{216\times3}, \Sigma_{360\times3}$

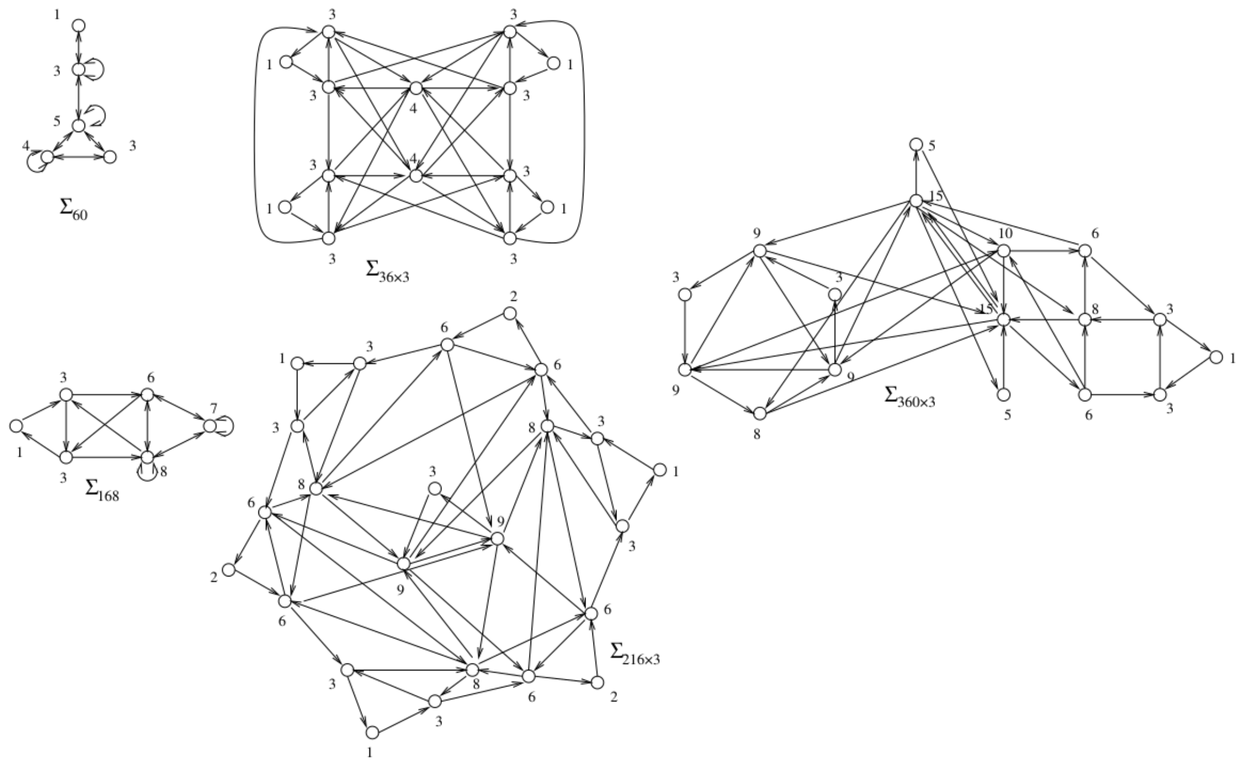


Figure 4: Quivers of the exceptional finite subgroups of $SU(3)$.

9.4 | $S = \mathbb{C}^3/(\mathbb{Z}_m \times \mathbb{Z}_n)$

9.4.1 | Representations of $\mathbb{Z}_m \times \mathbb{Z}_n$

We consider the group $\mathbb{Z}_m \times \mathbb{Z}_n$. Let us denote by $\{\mu_i\}_{i=0,\dots,m-1}$ and $\{\sigma_j\}_{j=0,\dots,n-1}$ two complete set of irreducible representation of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively, with

$$\mu(g_1) = \zeta_m^i \tag{9.28}$$

$$\sigma_j(g_2) = \zeta_n^j \tag{9.29}$$

where g_1 and g_2 are the generating elements of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively. $\mathbb{Z}_m \times \mathbb{Z}_n$ is of order mn and possesses the same number of equivalency classes (abelian). It has therefore mn irreducible representations. Since the group is abelian, they are all of dimension 1. Let us denote by $\{T_k\}_{k=0,\dots,m+n-1}$ a complete set of irreducible representations. Since we have a product group, for any k there exists indices $i(k)$ and $j(k)$ such that $T_k = \mu_{i(k)} \otimes \sigma_{j(k)}$. We choose the indices $i(k)$ and $j(k)$ such that

$$\begin{aligned}
T_0 &= \mu_0 \otimes \mu_0, \\
T_1 &= \mu_0 \otimes \mu_1, \\
T_2 &= \mu_0 \otimes \mu_2, \\
&\vdots \\
T_{n-1} &= \mu_0 \otimes \mu_{n-1}, \\
T_n &= \mu_1 \otimes \mu_0,
\end{aligned}$$

$$\vdots$$
$$T_{2n-1} = \mu_1 \otimes \mu_{n-1},$$
$$\vdots$$
$$T_{mn-1} = \mu_{m-1} \otimes \mu_{n-1}.$$

That is, we take

$$\begin{cases} i(k) &= \lfloor k/n \rfloor \\ j(k) &= k \mod n \end{cases} \qquad \Leftrightarrow \qquad k = i(k)n + j(k). \tag{9.30}$$

Note that this simply a dictionary between a line notation and a matrix notation and that it is indeed a bijection $k \Leftrightarrow i, j$. In this way, we can proceed to similar manipulations than before, were we used line notations.

Any representation R of $\mathbb{Z}_m \times \mathbb{Z}_n$ can be decomposed as

$$R = \bigoplus_{i,j} N_k T_k = \bigoplus_{i,j} N_{ij}(\mu_i \otimes \sigma_j) \tag{9.31}$$

with $N_k = N_{i(k)j(k)}$. We must have $\sum_k N_k = 3$. In other words, any representation of $\mathbb{Z}_m \times \mathbb{Z}_n$ on $\mathbb{C}^3$ is equivalent to

$$R(g_1, g_2) = [(\mu_a \otimes \sigma_{a'}) \oplus (\mu_{b'} \otimes \sigma_{b'}) \oplus (\mu_c \otimes \sigma_{c'})](g_1, g_2) = \begin{bmatrix} \xi_m^a \xi_n^{a'} & 0 & 0 \\ 0 & \xi_m^b \xi_n^{b'} & 0 \\ 0 & 0 & \xi_m^c \xi_n^{c'} \end{bmatrix}. \tag{9.32}$$

The determinant condition is

$$\xi_m^{a+b+c} = \xi_n^{-a'-b'-c'} \tag{9.33}$$
$$\Leftrightarrow (a+b+c) \mod m = (a'+b'+c') \mod n \tag{9.34}$$

$$R(g_1, g_2) = \begin{bmatrix} \xi_m & 0 & 0 \\ 0 & \xi_n & 0 \\ 0 & 0 & \xi_m^{-1}\xi_n^{-1} \end{bmatrix}. \tag{9.35}$$

9.4.2 | Projection

Let us start by the gauge field. We consider a unitary representation $(\rho, \mathbb{C}^N)$ of $\mathbb{Z}_m \times \mathbb{Z}_n$ on $\mathbb{C}^n$ and decompose it as

$$\rho = \bigoplus_{i,j} N_k T_k \tag{9.36}$$

such that

$$\rho(g) = \left.\begin{bmatrix} T_0(g) & & \cdots & & & 0 \\ & \ddots & & & & \\ & & T_0(g) & & & \\ \vdots & & & \ddots & & \vdots \\ & & & T_{mn-1}(g) & & \\ & & & & \ddots & \\ 0 & & \cdots & & & T_{mn-1}(g) \end{bmatrix}\begin{matrix} \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} N_0 \\ \vdots \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} N_{mn-1} \end{matrix}\right. . \tag{9.37}$$

We can now use our usual bi-index notation $A_{\mu;k\alpha_k,l\beta_l}$ with $k, l = 0, \ldots mn-1$ and $\alpha_k, \beta_k = 0, \ldots, N_k$ but instead it is more convenient to come back to our matrix notation by writing the block $A_{\mu;k,l}$ as

$A_{\mu;i(k)j(k),i(l)j(l)}$ that we simply denote by $A_{\mu;ij,i'j'}$ with $i,i' \in \{0,\dots,n-1\}$ and $j,j' \in \{0,\dots,n-1\}$. So for $m=2, n=3$ for example, the link between the two notations is

$$\begin{bmatrix} A_{\mu;00} & A_{\mu;01} & A_{\mu;02} & A_{\mu;03} & A_{\mu;04} & A_{\mu;05} \\ A_{\mu;10} & A_{\mu;11} & A_{\mu;12} & A_{\mu;13} & A_{\mu;14} & A_{\mu;05} \\ A_{\mu;20} & A_{\mu;21} & A_{\mu;22} & A_{\mu;23} & A_{\mu;24} & A_{\mu;05} \\ A_{\mu;30} & A_{\mu;31} & A_{\mu;32} & A_{\mu;33} & A_{\mu;34} & A_{\mu;05} \\ A_{\mu;40} & A_{\mu;41} & A_{\mu;42} & A_{\mu;43} & A_{\mu;44} & A_{\mu;05} \\ A_{\mu;50} & A_{\mu;51} & A_{\mu;52} & A_{\mu;53} & A_{\mu;54} & A_{\mu;05} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{\mu;00,00} & A_{\mu;00,01} & A_{\mu;00,02} \\ A_{\mu;01,00} & A_{\mu;01,01} & A_{\mu;01,02} \\ A_{\mu;02,00} & A_{\mu;02,01} & A_{\mu;02,02} \end{bmatrix} & \begin{bmatrix} A_{\mu;00,10} & A_{\mu;00,11} & A_{\mu;00,12} \\ A_{\mu;01,10} & A_{\mu;01,11} & A_{\mu;01,12} \\ A_{\mu;02,10} & A_{\mu;02,11} & A_{\mu;02,12} \end{bmatrix} \\ \begin{bmatrix} A_{\mu;10,00} & A_{\mu;10,01} & A_{\mu;10,02} \\ A_{\mu;11,00} & A_{\mu;11,01} & A_{\mu;11,02} \\ A_{\mu;12,00} & A_{\mu;12,01} & A_{\mu;12,02} \end{bmatrix} & \begin{bmatrix} A_{\mu;10,10} & A_{\mu;10,11} & A_{\mu;10,12} \\ A_{\mu;11,10} & A_{\mu;11,11} & A_{\mu;11,12} \\ A_{\mu;12,10} & A_{\mu;12,11} & A_{\mu;12,12} \end{bmatrix} \end{bmatrix} \tag{9.38}$$

So instead of considering A_μ to be a single $mn \times mn$ matrix of element $A_{\mu;kl}$, where $A_{\mu;kl}$ are $N_k \times N_l$ matrices, we consider it as an $m \times m$ where each element $A_{\mu,ii'}$ (line i column i') is itself an $n \times n$ matrices with elements $A_{\mu;ij,i'j'}$ (line j column j'), as shown above.

Using these notations, the gauge field transforms as

$$A_{\mu;ij,i'j'} \mapsto (\mu_i(g) \otimes \sigma_j(g)) A_{\mu;ij,i'j'} (\mu_{i'}(g) \otimes \sigma_{j'}(g))^{-1} = \zeta_m^{i-i'} \zeta_n^{j'-j} A_{\mu;ij,i'j'} \tag{9.39}$$

so invariant configurations can possess non-vanishing components $A_{\mu;ij,i'j'}$ only if

$$\zeta_m^{i-i'} = \zeta(j'-j)_n \tag{9.40}$$

$$\Leftrightarrow \quad (i-i') \mod m = (j'-j) \mod n \tag{9.41}$$

$$\Leftrightarrow \quad j' = j + |i'-i|. \tag{9.42}$$

This means that the submatrices $A_{\mu;ii'}$ has an off-diagonal block form with offset $|i'-i|$. Once again, for a general n, the difficulty is only computational, not conceptual. This can therefore easily be implemented into Mathematica and we get the form of the gauge field for any n.

For the scalar fields, we have

$$X^m_{ij,i'j'} \mapsto R(g)^m{}_n (\mu_i(g) \otimes \sigma_j(g)) X^n_{ij,i'j'} (\mu_{i'}(g) \otimes \sigma_{j'}(g))^{-1} \tag{9.43}$$

$$= \begin{cases} \zeta_m^{i-i'+1} \zeta_n^{j-j'} X^m_{ij,i'j'}, & m = 0,1 \\ \zeta_m^{i-i'} \zeta_n^{j-j'+1} X^m_{ij,i'j'}, & m = 2,3 \\ \zeta_m^{i-i'-1} \zeta_n^{j-j'-1} X^m_{ij,i'j'}, & m = 4,5. \end{cases} \tag{9.44}$$

So a configuration is invariant if and only if the only non-vanishing component satisfy

$$m = 0,1 : (i-i'+1) \mod m = (j'-j) \mod n \tag{9.45}$$

$$m = 2,3 : (i-i') \mod m = (j'-j-1) \mod n \tag{9.46}$$

$$m = 4,5 : (i-i'-1) \mod m = (j'-j+1) \mod n. \tag{9.47}$$

10 | A note about projective representations, discrete torsion and deformations

10.1 | Projective representations and discrete torsion

Up until now, and in particular in all of our computations in section **??**, we only considered ordinary representations, i.e. linear representations. We can however use a more general representations such as projective representations for example. That is, representations ρ of Γ such that for all $\gamma_1, \gamma_2 \in \Gamma$,

$$\rho(\gamma_1)\rho(\gamma_2) = A(g_1, g_2)\rho(\gamma_1\gamma_2) \tag{10.1}$$

for some factor $A(\gamma_1, \gamma_2)$. The $A(\gamma_1, \gamma_2) = 1$ corresponding to linear representations. For consistency reasons, the factor $A(g_1, g_2)$ cannot be completely arbitrary and must a cocycle condition. As a result, the possibilities for $A(\gamma_1, \gamma_2)$ are classified by the second cohomology group $H^2(\Gamma, \mathbb{C}^*)$, also called *discrete torsion*. There exists a projective representation if and only if the latter does not vanish. This new liberty, whenever admissible, provides new classes of quiver gauge theories that can be remarkably different from the ones we considered up until now, with no discrete torsion.

What happens is that if one turns on an NSNS B-fiels alors the worldvolume, then the moduli space is expected to a non-commutative version of a Calabi-Yau space. This haw the discrete torsion is physically realized. Another, and actually equivalent through T-duality, way of studying gauge theories is to consider D-branes streched between configurations of NS5-branes[6]. This is the*Hanany-Witten setup*.

The discrete tosrion also appears when writing the full open strin gpartition function that inclues the twisted sector where there is ambiguity up to a phase factor. As a consequence from modular invariance, the latter must satisfy certain cocycle conditions. It is precisely the deiscrete torsion. Note that this has been found to be true only for the open string sector.

10.2 Quiver gauge theories deformations and conifold

We can deforming the singular algebraic description of the orbifold with a field into a family smooth surfaces. The resulting total space is the *conifold*.

11 A note on $(p+1)$-dimensional quiver gauge theories

Let us mention that we can generalize our initial brane-world paradigm and consider Dp-branes in type II string theory (type IIA if p is even and type IIB if p is odd) instead of just D3-branes. The spacetime is then of the form

$$M = \mathbb{R}^{1,p} \times \mathbb{R}^{9-p}/\Gamma \tag{11.1}$$

where Γ is a discrete subgroup of $\mathrm{Spin}(9-p)$. If Γ is a subgroup of a special holonomy group, we recover a somewhat generalized version of the paradigm that we discussed above. In this case the transverse space is a Calabi-Yau orbifold and some degree of supersymmetry is preserved. Note that the fermionic and bosonic quivers that coincides. If Γ is not a subgroup of a special holonomy group, then the $(p+1)$-dimensional quiver gauge theory that we obtain in the low-energy limit is not supersymmetric. We then have different quivers for the fermions and the bosons, although with the same vertices, by definition.

Recall that the fraction of supercharges that is preserved by compactifying on a Calabi-Yau n-fold (with $\mathrm{SU}(n)$ holonomy) is 2^{1-n}. Starting from the $\mathcal{N}=2$ 10-dimensional type IIB string theory with 32 supercharges, this means that

- if we compactify on a 1-fold, we get 32 supercharges in 8 dimensions so $\mathcal{N}=2$,
- if we compactify on a 2-fold, we get 16 supercharges in 6 dimensions so $\mathcal{N}=2$,
- if we compactify on a 3-fold, we get 8 supercharges in 4 dimensions so $\mathcal{N}=2$,
- is we compactify on a 4-fold, we get 4 supercharges in 2 dimensions so $\mathcal{N}=4$.

We will however mostly mostly consider 4-dimensional quiver gauge theories, i.e. living on D3-branes.

[6]D5-branes are charged under the Ramond-Ramond field whose quanta comes from the Ramond-Ramond sector but NS5-branes are charges under the Kalb-Ramonf field whose quanta comes fromthe Neveu-Schwarz. On the worldvolume of an NS5-brane (6-dimensional) propagates propagates a superstring, this called the *little string*.

12 | Summary of orbifold worldvolume theories

SUSY	Γ	Gauge group
$\mathcal{N}=2$	$\mathbb{Z}_n$	(1^n)
	$\mathbb{Z}_n \times \mathbb{Z}_m$	(1^{nm})
	$\mathcal{D}_n$	$(1^4, 2^{n-3})$
	$\mathcal{T}$	$(1^3, 2^3, 3)$
	$\mathcal{O}$	$(1^2, 2^2, 3^2, 4)$
	$\mathcal{I}$	$(1, 2^2, 3^2, 4^2, 5, 6)$
$\mathcal{N}=1$	T	$(1^3, 3)$
	O	$(1, 2^2, 3^2)$
	I	$(1, 3^2, 4, 5)$
	$\Delta_{3n^2}(n = 0 \mod 3)$	$(1^9, 3^{\frac{n^2}{3}-1})$
	$\Delta_{3n^2}(n \neq 0 \mod 3)$	$(1^3, 3^{\frac{n^2-1}{3}})$
	$\Delta_{6n^2}(n \neq 0 \mod 3)$	$(1^2, 2, 3^{(2n-1)}, 6^{\frac{n^2-3n+2}{6}})$
	Σ_{168}	$(1, 3^2, 6, 7, 8)$
	Σ_{216}	$(1^3, 2^3, 3, 8^3)$
	$\Sigma_{36\times3}$	$(1^4, 3^8, 4^2)$
	$\Sigma_{216\times3}$	$(1^3, 2^3, 3^7, 6^6, 8^3, 9^2)$
	$\Sigma_{360\times3}$	$(1, 3^4, 5^2, 6^2, 8^2, 9^3, 10, 15^2)$

Table 1: All supersymmetric orbifold worldvolume theories.

Add finitude and conformal invariance.

13 | Classical McKay correspondence from strings

We mentionned before the equivalence between fractional branes stuck at orbifold singularities and wrapped branes on the blow-up resolution. From the point of view of the worldvolume theory, this equivalence is exhibited by going from the Higgs branch to the Coulomb branch. The first step to understand this was done by Kronheimer. He showed that the resolution of the orbifold $\mathbb{C}^2/\Gamma$ with Γ a finite group of SU(2) is precisely the generic form of the gauge orbit of the direct product of $\mathrm{U}(N_i)$ factors. From the QGT point of view, the product of those circle group is the gauge group of the theory and each factor is associated to a vertex of the quiver. The fields of the theory are organized as a linear representation into a direct sum of $\mathrm{Hom}(V_i, V_j)$ for each edge. If we pick one field and follow it around as the gauge group transforms it, the space swept out is the gauge orbit of that field. Kronheimer then showed that, if the quiver is a Dynkin diagram, this orbit is $\mathbb{C}^2/\Gamma$.

Now in general, gauge theories with with simple Lie groups (such as $\mathrm{SU}(N)$, E_8,etc) are more interesting than the ones with gauge groups that are direct products so how could we relate the two? The mechanism the relates the two classes of theories is SSB, or *Higgsing*. Indeed, one may embed the fields configurations in a higher-dimensional field configuration space (see them as submatrices of bigger matrices) on which acts the bigger (simple) Lie group. To from this group to a product of small ones. Fixing this submatrix-structure then singles out every gauge group factor one by one as the stabilizer subgroups of every submatrix. The final step is $\mathcal{N}=2$ super Yang-Mills theories (Seiberg-Witten theory) which have a potential such that its vacua (more precisely the Higgs branch part) break a simple Lie group down to a Dynkin diagram QGT. The Coulomb branch is supposed to behave in a similar way. To summarize, the relation between simple Lie groups and the finite subgroups of SU(2) is the following:

1. start with $\mathcal{N}=2$ SYM with gauge group that is a simple Lie group,
2. let it spontaneously find its vacuum,
3. consider the orbit space of the remaining spontaneously broken symmetry group,
4. the latter space is the resolution of the orbifold of $\mathbb{C}^2/\Gamma$.

All the details are presented in [8].

A | Some finite subgroups

A.1 | Finite subgroups of SU(2) and SL$(2,\mathbb{C})$

A.1.1 | Finite subgroups

The first thing to recall is that every finite subgroup of SL$(2,\mathbb{C})$ is isomorphic to a subgroup of SU$(2,\mathbb{C})$ and vice-versa, so we equivalently talk about the subgroups of SU(2). The finite subgroups of SU(2), called the *binary polyhedral groups*, are the doubles covers of the finite subgroups of SO(3) that are called *polyhedral groups*. They simply constitutes the symmetries of the Platonic solids. The groups fall into two infinite series, associated to the regular polygons, as well as three exceptional, associated with the 5 regular polyhedra: the tetrahedron (self-dual), the cube (and its dual octahedron), the icosahedron (and its dual dodecahedron).

More precisely, the finite subgroups of SL$(2,\mathbb{C})$ are

- $\mathbb{Z}_n$: cyclic group of order n ($n \geq 2$) generated by
$$\begin{bmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{bmatrix} \tag{A.1}$$
- $2\mathcal{D}_n$: *binary dihedral groups* (also known as the *dicyclic group*) of order $4n$ ($n \geq 1$) generated by
$$A \equiv \begin{bmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{bmatrix} \quad \text{and } B \equiv \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{A.2}$$
One can show that $A^n = B^2$ and that $AB = BA^{-1}$ so that $2\mathcal{D}_n = \{B^b A^a | 0 \leq b \leq 3, 0 \leq a \leq n-1\}$. This rewriting of the most general element of the group will be useful.
- $2\mathcal{T}$: *binary tetrahedral group* of order 24 generated by D_2 and
$$C \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{bmatrix} \tag{A.3}$$
- $2\mathcal{O}$: *binary octahedral group* of order 48 generated by $\mathcal{T}$ and
$$D \equiv \begin{bmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{bmatrix} \tag{A.4}$$
- $2\mathcal{I}$: *binary icosahedral group* of order 120 generated by
$$E \equiv -\frac{1}{\sqrt{5}} \begin{bmatrix} \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{bmatrix} \quad \text{and } F \equiv -\frac{1}{\sqrt{5}} \begin{bmatrix} \zeta_5^2 - \zeta_4^4 & \zeta_5^4 - 1 \\ 1 - \zeta_5 & \zeta_5^3 - \zeta_5 \end{bmatrix} \tag{A.5}$$

with $\zeta_m \equiv e^{i\frac{2\pi}{m}}$ such that $(\zeta_m)^m = 1$. Note that the orders are all divisible by 2. This is because the center of SU(2) is $\mathbb{Z}_2$.

A.1.2 | Irreducible representations

- $\mathbb{Z}_n$ has n irreducible representations. They are all 1-dimensional (since $\mathbb{Z}_n$ is abelian) and are given by
$$\rho_k(g) = \zeta_n^k \tag{A.6}$$
with $k = 0, \ldots, n-1$.

- $2\mathcal{D}_n$ has $n+3$ irreducible representations: 4 of dimension 1 and $n-1$ of dimension 2. The 1-dimensional ones are given by

n	$\rho(A)$	$\rho(B)$	$\rho(B^b A^a)$
even	1	1	1
		-1	$(-1)^b$
	-1	1	$(-1)^a$
		-1	$(-1)^{a+b}$
odd	1	1	1
		-1	$(-1)^b$
	-1	i	$(-1)^a i^b$
		$-i$	$(-1)^a(-i)^b$

and the 2-dimensional ones are given binary by

$$\rho_r(A) = \begin{bmatrix} e^{i\frac{\pi}{n}r} & 0 \\ 0 & e^{-i\frac{\pi}{n}r} \end{bmatrix}$$
$$\rho_r(B) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with $r = 1, \ldots, n-1$.

A.1.3 | Character tables

conj. class repr.	e	M	M^2	$\ldots$	M^{n-1}
conj. class order	1	1	1	$\ldots$	1
V_0	1	1	1	$\ldots$	1
V_1	1	ζ_n	ζ_n^2	$\ldots$	ζ_n^{n-1}
V_2	1	ζ_n^2	ζ_n^4	$\ldots$	$\zeta_n^{2(n-1)}$
V_3	1	ζ_n^3	ζ_n^6	$\ldots$	$\zeta_n^{3(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
V_{n-1}	1	$\zeta_n^{(n-1)}$	$\zeta_n^{2(n-1)}$	$\ldots$	$\zeta_n^{(n-1)^2}$
W	2	$2\cos\left(\frac{2\pi}{n}\right)$	$2\cos\left(\frac{4\pi}{n}\right)$	$\ldots$	$2\cos\left(\frac{2\pi(n-1)}{n}\right)$

Table 2: Character table of $\mathbb{Z}_n$.

conj. class repr.	e	B^2	B	BA	A	A^2	$\ldots$	A^{n-1}
conj. class order	1	1	n	n	2	2	$\ldots$	2
V_0	1	1	1	1	1	1	$\ldots$	1
V_1	1	1	-1	-1	1	1	$\ldots$	1
V_2	1	1 ou -1	1 ou i	-1 ou $-i$	-1	1	$\ldots$	$(-1)^{n-1}$
V_3	1	1 ou -1	-1 ou $-i$	1 ou i	-1	1	$\ldots$	$(-1)^{n-1}$
V_4	2	-2	0	0	$2\cos\frac{\pi}{n}$	$2\cos\frac{2\pi}{n}$	$\ldots$	$2\cos\frac{(n-1)\pi}{n}$
V_5	2	2	0	0	$2\cos\frac{2\pi}{n}$	$2\cos\frac{4\pi}{n}$	$\ldots$	$2\cos\frac{2(n-1)\pi}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
V_{n+2}	2	$2(-1)^{n-1}$	0	0	$2\cos\frac{(n-1)\pi}{n}$	$2\cos\frac{2(n-1)\pi}{n}$	$\ldots$	$2\cos\frac{(n-1)^2\pi}{n}$
W	2	-2	0	0	$2\cos\left(\frac{\pi}{n}\right)$	$2\cos\left(2\frac{\pi}{n}\right)$	$\ldots$	$2\cos\left(\frac{\pi}{n}(n-1)\right)$

Table 3: Character table of $2\mathcal{D}_n$.

conj. class repr.	e	B^2	B	C	C^2	C^4	C^5
conj. class order	1	1	6	4	4	4	4
V_0	1	1	1	1	1	1	1
V_1	2	-2	0	1	-1	-1	1
V_2	3	3	-1	0	0	0	0
V_3	2	-2	0	$e^{i\frac{2\pi}{3}}$	$-e^{i\frac{2\pi}{3}}$	$-e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$
$V_3^\vee$	2	-2	0	$e^{i\frac{4\pi}{3}}$	$-e^{i\frac{4\pi}{3}}$	$-e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$
V_4	1	1	1	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$
$V_4^\vee$	1	1	1	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{4\pi}{3}}$	$e^{i\frac{2\pi}{3}}$	$e^{i\frac{2\pi}{3}}$
W	2	-2	0	1	-1	-1	1

Table 4: Character table of $2\mathcal{T}$.

conj. class repr.	e	B^2	B	C	C^2	D	BD	D^3
conj. class order	1	1	6	8	8	6	12	6
V_0	1	1	1	1	1	1	1	1
V_1	2	-2	0	1	-1	$-\sqrt{2}$	0	$\sqrt{2}$
V_2	3	3	-1	0	0	1	-1	1
V_3	4	-4	0	-1	1	0	0	0
V_4	3	3	-1	0	0	-1	1	-1
V_5	2	-2	0	1	-1	$\sqrt{2}$	0	$-\sqrt{2}$
V_6	1	1	1	1	1	-1	-1	-1
V_7	2	2	2	-1	-1	0	0	0
W	2	-2	0	1	-1	$-\sqrt{2}$	0	$\sqrt{2}$

Table 5: Character table of $2\mathcal{O}$.

conj. class repr.	e	E^2	E	F	F^2	EF	$(EF)^2$	$(EF)^3$	$(EF)^4$
conj. class order	1	1	30	20	20	12	12	12	12
V_0	1	1	1	1	1	1	1	1	1
V_1	2	-2	0	1	-1	φ^+	$-\varphi^-$	φ^-	$-\varphi^+$
V_2	3	3	-1	0	0	φ^+	φ^-	φ^-	φ^+
V_3	4	-4	0	-1	1	1	-1	1	-1
V_4	5	5	1	-1	-1	0	0	0	0
V_5	6	-6	0	0	0	-1	1	-1	1
V_6	4	4	0	1	1	-1	-1	-1	-1
V_7	2	-2	0	1	-1	φ^-	$-\varphi^+$	φ^+	$-\varphi^-$
V_8	3	3	-1	0	0	φ^-	φ^+	φ^+	φ^-
W	2	-2	0	1	-1	φ^+	$-\varphi^-$	φ^-	$-\varphi^+$

Table 6: Character table of $2\mathcal{I}$, with $\varphi^\pm \equiv (1 \pm \sqrt{5})/2$.

A.2 Finite subgroups of SU(3)

The finite subgroups of SU(3) are

- the finite subgroups of SU(2)
- $\Delta(3n^2) = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$ and $\Delta(3n^2) = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S^3$
- the exceptional groups

so there are 2 infinite series and 5 exceptional subgroups. Note that they are all divisible by 3 because the center of SU(3) is $\mathbb{Z}_3$.

Theorem. Every abelian finite subgroup of SU(3) is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$.

A.3 | Finite subgroups of SU(4)

See [6].

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