# Notes on Quiver Gauge Theories

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**Abstract**

In these notes, we present some basic ideas around the large topic of quiver gauge theories, more precisely about their brane probes construction. The goal is to reproduce and regroup the basics of these theories for various types of singularities, with increasing level of complexity (orbifold, toric, del Pezzo, etc). Note that this document is only meant as a work support and contains a lot of typos, errors and imprecisions.

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# Contents

# Part I

# Preliminary notions

## 1 | Physical setup

### Brane-world paradigm

We consider our four-dimensional world to be the worldvolume of a D3-brane in the ten-dimensional spacetime of type IIB superstring theory. More precisely, we consider a stack of $N$ D3-branes in order to have $\mathrm{U}(N)$ Chan-Paton factors resulting in a $\mathrm{U}(N)$ gauge group in the worldvolume theory. The spacetime is therefore not necessarily $\mathbb{R}^{1,9}$ but of the more general form

$$M = \mathbb{R}^{1,3} \times M^{(6)}.$$

This is the so-called *brane-world paradigm.*

### Supersymmetry and Calabi-Yau manifolds

Independently from string theory, we can ask for the wolrdvolume theory to be supersymmetric. We start from type IIB superstring theory which is 10-dimensional and has $\mathcal{N} = 2$ supersymmetry so it possesses 32 supercharges. As usual, they transform under the minimal spinor representation (MSR) of the bulk Lorentz group, here $\mathrm{SO}(1, 9)$. In ten dimensions this representation is 8-dimensional (complex) which is why there are $2(8 + 8) = 32$ supercharges: 8 transforming in the 8-dimensional MSR and 8 transforming in the 8-dimensional conjugate MSR and the whole thing times two since $\mathcal{N} = 2$. Compactifying type II string theory on any 6-dimensional manifold $M^{(6)}$ breaks supersymmetry. The reason for this is that the supercharges now have to transform under the MSR of $\mathrm{SO}(1, 3) \times \mathcal{H}(M^{(6)})$, where $\mathcal{H}(M^{(6)})$ is the holonomy group of $M^6$. Actually, the space $\mathbb{R}^{1,3} \times M^{(6)}$ can be viewed as the trivial bundle with base space $\mathbb{R}^{1,3}$ and fibers $M^{(6)}$, sincespinors takes values in the fibers they must transform under the holonomy group of $M^(6)$. A generic curved 6-dimensional manifold has $\mathrm{O}(6)$ holonomy and $\mathrm{SO}(6)$ if it is orientable, as we will always consider. The supercharges must therefore transform under the MSR of $\mathrm{SO}(1, 3)\times\mathrm{SO}(6)$. The MSR of $\mathrm{SO}(1, 3)$ being $\mathbf{2}$ and the one of $\mathrm{SO}(6)$ being $\mathbf{4}$, we conclude that imposing the spacetime to have the form (1) changes the representation under which the supercharges transform in the following way: Why ?

$$\mathbf{8} \oplus \overline{\mathbf{8}} \to (\mathbf{2}_L, \mathbf{4}) \oplus (\mathbf{2}_R, \overline{\mathbf{4}}). \tag{1.1}$$

If we stop here, the residual supercharges might be ill-defined; making a tour around a loop in $M^{(6)}$ could result in a non-trivial rotation. To solve this problem, we need to be more restrictive with the holonomy. In fact, it is precisely the holonomy of the transverse space that dictates the number of residual supersymmetries. To understand this, let us now consider a four-dimensional field theory resulting from compactification of the transverse six-dimensional space. The number of supercharges that generate supersymmetries for this theory is the number of Killing spinors (covariantly constant spinors) because each Killing spinor contracted with the local supersymmetry current generates a residual supersymmetry. Let us now make the link with holonomy: since $\mathrm{SO}(6) \cong \mathrm{SU}(4)$, minimal spinors can be viewed as having four complex components and as transforming under $\mathrm{SU}(4)$. Indeed, minimal spinors in six dimensions have four complex components. In order to have one covariantly conserved spinor, we look for the biggest subgroup of $\mathrm{SU}(4)$ that leaves a component of the spinor invariant. This is clearly $\{e\} \times \mathrm{SU}(3) \subset \mathrm{SU}(4)$ that acts trivially on the first component. The spinor $(1, 0, 0, 0)$ is then covariantly constant. Our transverse space must therefore have $\mathrm{SU}(3)$ holonomy such that the parallel transport of the spinor $(1, 0, 0, 0)$ under any closed loop is a lower $\mathrm{SU}(3)$ rotation. We conclude that if the transverse Calabi-Yau has $\mathrm{SU}(3)$ holonomy, the worldvolume theory has $\mathcal{N} = 1$ supersymmetry. If the holonomy is $\mathrm{SU}(2) \subset \mathrm{SU}(3)$, the spinor $(0, 1, 0, 0)$ is also a Killing spinor which means that we have $\mathcal{N} = 2$ supersymmetry for the worldvolume theory. To summarize, preserving any degree of supersymmetry constrains the transverse space $M^{(6)}$ to be compact, complex, Kähler and to have $G \subset \mathrm{SU}(3)$ holonomy. Namely, $M^{(6)}$ must be a Calabi-Yau threefold, see section 7.

**Non-compact transverse space**

If we let the worldvolume of the D3-branes carry the requisite gauge theory while the bulk contains gravity, we can relax the compactness condition and study non-compact Calabi-Yau threefolds. This makes the analysis much simpler and therefore also serves as an argument to ignore gravity in the worldvolume theory. Consequently, we will mostly ignore gravity and not care about the metric of the spacetime, see appendix C for more details. In this setup we cannot really talk about compactification anymore. Instead we just think of it as a flat space on which lives the gauge theory while gravity only lives in transverse space. To understand intuitively why there is no gravity in this limit, we can think of Kaluza-Klein compactification. The four-dimensional gravity coupling constant is inversely proportional to the size of the compactifying space therefore there is no gravity in the non-compact/infinite-size limit. This more a motivation than a proof.

**Singular transverse space**

The only non-compact smooth Calabi-Yau threefold is $\mathbb{C}^3$, this forces us to consider singular Calabi-Yau varieties is we want more interesting theories. A Calabi-Yau variety is an affine variety that locally models a Calabi-Yau manifold, therefore allowing for singularities. We usually denote $S \equiv M^{(6)}$ to remind us of the singular aspect. String theory being a theory of extended objects, turns out to be it is well-defined on such singularities and even “smoothened” the singularity in some sens. Considering strings on singular geometries requires to “project” the theory obtain from $\mathbb{C}^3$. As a result, the gauge group $\mathrm{U}(N)$ is broken down into products of smaller gauge groups. This “projection” highly depends on the type of singularity (orbifold, toric, del Pezzo, etc) we are considering. While it is relatively straightforward for “simple” singularities (e.g. abelian orbifolds) it quickly gets more complicated or even unknown for others.

From the point of view of the orbifold, the D3-brane is a point, meaning that the D3-branes are really probing the transverse space and, in particular, they parametrize it. This is the first clue of the tight relationship that exists between the worldvolume theory and the transverse singular space. Eventually, we will see that the classical vacuum of the gauge theory should be, in explicit coordinates, the defining equation of $S$. This is precisely the opposite of the projection manipulation we mentioned above: recovering the transverse space from the gauge theory. Projecting and computing the classical vacua are therefore inverse operations with respect to each other. This suggest a bijection between the singular transverse space and the gauge theory: the former can be computed from the latter and vice-versa. This is called “forward algorithm” and “inverse algorithm” respectively.

**Mathematical formulation**

Mathematically, this brane-world paradigm is the realization of branes as supports of vector bundles (sheaf). Gauge theories on branes are intimately related to algebraic constructions of stable bundles, i.e. holomorphic or algebraic vector bundles that are stable in the sense of geometric invariant theory. In particular, D-brane gauge theories manifest as a natural description of symplectic quotients and their resolutions in geometric invariant theory. Together with the stable vector bundle (sheaf) supported thereupon the D-branes resolve the transverse Calabi-Yau orbifold, which is the vacuum for the gauge theory on the worldvolume as a GIT quotient.

**Summary**

We consider $N$ D3-branes in type IIB superstring theory carrying a $\mathrm{U}(N)$ gauge group. The transverse space $S$ is taken to be a non-compact singular Calabi-Yau variety.

# 2 | Supersymmetric Yang-Mills theories

## 2.1 | Vacua space of SYM

Let us consider a supersymmetric gauge theory in $d = 4$ with $k$ chiral sueprfields $\Phi_i$ $(i = 1, \ldots, k)$ charged under the gauge group $G$ in an arbitrary representation $r_i$, in which the generators of $G$ are given by $T^a_{r_i}$

$(a=1,\dots,\dim G)$. $\mathcal{W}_\alpha$ is the gaugina chiral superfield, containing the field strength. The lagrangian density on superspace is given by

$$\mathcal{L}=\int \mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\ \Phi^\dagger e^{2V}\Phi+\int \mathrm{d}^2\theta\left\{\frac{\tau}{16\pi i}\operatorname{tr}(\mathcal{W}^\alpha\mathcal{W}_\alpha)+W(\Phi)\right\}+\text{c.c.} \tag{2.1}$$

where we ignored the gauge indices. The superpotential $W(\Phi)$ is a holomorphic polynomial in the $\Phi_i$. The space of vacua is the space of configurations of $\Phi$ such that the $D$-terms and the $F$-terms vanish:

$$\boxed{\begin{aligned} D^a &\equiv \sum_i \Phi_i^\dagger T^a_{r_i}\Phi^i=0, \\ F_i^\dagger &\equiv \frac{\partial W}{\partial \Phi^i}=0. \end{aligned}} \begin{aligned} &(2.2a)\\ &(2.2b)\end{aligned}$$

In turns out that the full space of vacua of any supersymmetric gauge theory can be described as an algebraic variety.

## 2.2 | $\mathcal{N}=4$ super Yang-Mills theory in $D=4$

### 2.2.1 | Superconformal group SU$(2,2|4)$ and its representations

Conformal transformations and supersymmetries do not commute so the presence of conformal symmetry in addition to $\mathcal{N}=4$ supersymmetry leads to an even larger group of symmetry known as the *superconformal group*. In the $D=4,\mathcal{N}=4$ case, the superconformal group is the super group[1] SU$(2,2|4)$. The different component of the latter are

- **Conformal symmetries**: they form the 15-dimensional subgroup SO$(2,4)$ and are generated by $P_\mu, M_{\mu\nu}, K_\mu$ and $D$.
- **R-symmetry**: they form the 15-dimensional subgroup SO$(6)_R$ and are generated by $T^A$ $(A=1,\dots,15)$.
- **Poincaré supersymmetries**: they form the 16-dimensional sub group and are generated by $Q^I_\alpha$ and $\overline{Q}^I_{\dot\alpha}$. Why ?
- **Conformal supersymmetries**: they form the 16-dimensional subgroup and are generated by $S_{\alpha I}$ and $\overline{S}^{\dot\alpha I}$. Why ?

Conformal invariance of this theory can be seen as a consequence of the non-renormalization theorems.

### 2.2.2 | Matter content

For $D=4,\mathcal{N}=4$, there is only one kind of supermultiplet, the vector multiplet. Therefore, from an $\mathcal{N}=4$ perspective, the only $\mathcal{N}=4$ is a pure SYM. For extended supersymmetry, is is easier to express it in terms of $\mathcal{N}=1$ superfield on $\mathcal{N}=1$ superspace instead of looking to construct a superspace for $\mathcal{N}=4$. In this case, we can see that the $\mathcal{N}=4$ vector superfield can be expressed in terms of $\mathcal{N}=1$ representations as one vector supermultiplet and three chiral scalar supermultiplets:

$$[\mathcal{N}=4 \text{ vector multiplet}]: V=(\lambda_\alpha, A_\mu, D)\oplus\Phi^A=(\phi^A,\psi^A_\alpha,F^A). \tag{2.3}$$

with $A=1,2,3$ and

$$\phi^A=\phi^A_aT^a,\qquad \psi^A_\alpha=\psi^A_{\alpha,a}T^a,\qquad F^A=F^a_aT^a, \tag{2.4}$$
$$\lambda^A=\lambda^A_aT^a,\qquad A^A_\mu=A^A_{\mu,a}T^a,\qquad D^A=F^a_aT^a, \tag{2.5}$$
$$V=V_aT^a,\qquad \Phi^A=\Phi^A_aT^a, \tag{2.6}$$

where $T^a$ $(a=1,\dots,\dim G)$ are the generators of $\mathfrak{g}$. The propagating degrees of freedom are therefore a vector field, three complex scalars and four gauginos. The Lagrangian is very much constrained by

[1] Supermanifold which is also a group with smooth product and inverse maps.

$\mathcal{N}=4$ supersymmetry. First, the chiral superfields $\Phi^A$ should transform in the adjoint representation of the gauge group $G$, since internal symmetries commute with supersymmetry. This means that all fields transform in the adjoint of $G$.

Moreover, there is a large R-symmetry group[2]: $\mathrm{SU}(4)_R$. The four Weyl fermions transform in the fundamental of $\mathrm{SU}(4)_R$, while the six real scalars in the two times anti-symmetric representation, which is nothing but the fundamental representation of SO(6). The auxiliary fields are singlets under the R-symmetry group. Using $\mathcal{N}=1$ superfield formalism the Lagrangian reads

$$\begin{aligned}\mathcal{L}^{\mathcal{N}=4}_{\mathrm{SYM}} = \frac{1}{32\pi}\,\mathrm{Im}\left(\tau\int \mathrm{d}^4x\,\mathrm{tr}(W^\alpha W_\alpha)\right) + \int \mathrm{d}^2\theta\mathrm{d}^2\overline{\theta}\,\mathrm{tr}\sum_{A=1}^{3}\overline{\Phi}^A e^{2gV}\Phi^A \\ - \int \mathrm{d}^2\theta\sqrt{2g}\,\mathrm{tr}\,\Phi_1[\Phi_2,\Phi_3] + \mathrm{h.c.}\end{aligned} \tag{2.7}$$

where as usual $W_\alpha = -\frac{1}{4}\overline{DD}(e^{-V}D_\alpha e^V)$ is the gaugino superfield. This lagrangian is indeed invariant under the superPoincaré algebra and under the gauge transformations

$$e^V \to e^{i\overline{\Lambda}}e^V e^{-i\Lambda} \text{ (which implies that } W_\alpha \to e^{i\Lambda}W_\alpha e^{-i\Lambda}), \tag{2.8}$$

$$\Phi^A \to e^{i\Lambda}\Phi^A. \tag{2.9}$$

The large $\mathrm{SU}(4)_R$ R-symmetry group forbids of having a superpotential. The commutator in the third term of (2.7) appears for the same reason as for the $\mathcal{N}=2$ Lagrangian. Notice that the choice of a single $\mathcal{N}=1$ supersymmetry generator breaks the full $\mathrm{SU}(4)_R$ R-symmetry to $\mathrm{SU}(3)\times\mathrm{U}(1)_R$. The three chiral superfields transform in the **3** of SU(3) and have R-charge $R=2/3$ under the $\mathrm{U}(1)_R$. It is an easy but tedious exercise to perform the integration in superspace and get an explicit expression in terms of fields. Finally, one can solve for the auxiliary fields and get an expression where only propagating degrees of freedom are present, and where $\mathrm{SU}(4)_R$ invariance is manifest.

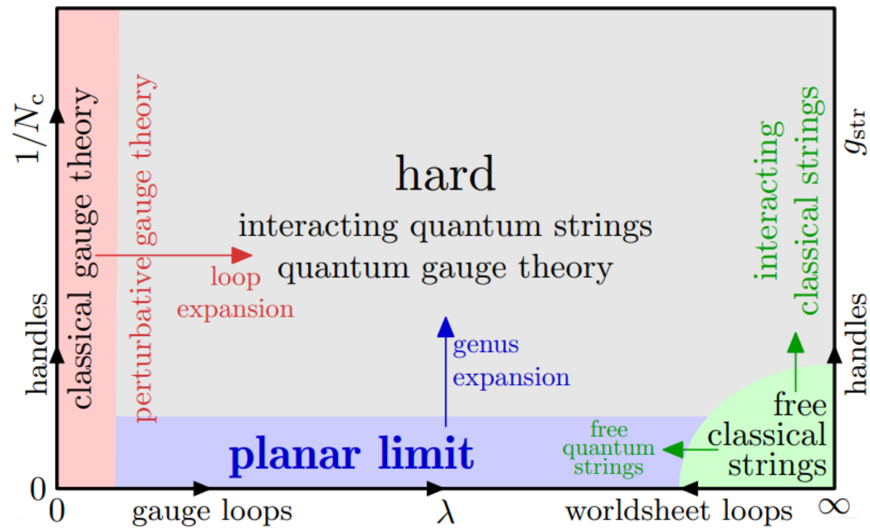



*Figure 1: Map of the parameter space of $\mathcal{N}=4$ SYM or strings on $AdS_5\times S^5$, from [1].*

Explain this diagram.

[2] The fact that the scalar fields transform under the fundamental representation of SO(6), which is real, makes the R-symmetry group of the $\mathcal{N}=4$ theory being at most SU(4) and not U(4), in fact).

### 2.2.3 Moduli space and dynamical phases

The scalar potential in (2.7) can be written in a rather compact form in terms of the six real scalars $X^i$ making up the three complex scalars $\phi^A$ and reads

$$V(X_1,\ldots,X_6)=\frac{1}{2}g^2\operatorname{tr}\sum_{i,j=1}^{6}[X_i,X_j]^2. \tag{2.10}$$

The positive definite behavior of the Cartan-Killing form on the compact gauge algebra $\mathfrak{g}$ implies that each term in the sum is positive or zero. In other words, $V=0$ is equivalent to

$$[X^i,X^j]=0,\qquad i,j=1,\ldots,6. \tag{2.11}$$

This means that the potential vanishes whenever the scalar fields belong to the Cartan subalgebra of the gauge group $G$. At a generic point of the moduli space, the gauge group is broken to $\mathrm{U}(1)^r$ where $r$ is the rank of $\mathfrak{g}$. This equations admit two classes of solutions:

- $\langle X^i\rangle=0$ for all $i=1,\ldots,6$. This is the *superconformal phase*. Neither the gauge symmetry nor the superconformal symmetry is broken. The physical states and operators are gauge invariant and transform under unitary representations of $\mathrm{SU}(2,2|4)$.
- $\langle X^i\rangle\neq 0$ for at least one $i$. This is the *spontaneously broken Coulomb phase*. The gauge algebra $\mathfrak{g}$ is going to be broken to $\mathrm{U}(1)^r$, where $r\equiv\operatorname{rank}\mathfrak{g}$. The low energy behavior is then the one of $r$ copies of $\mathcal{N}=4$ U(1) gauge theories. Superconformal symmetry is spontaneously broken since the non-zero VEV $\langle X^i\rangle$ sets a scale.

> **$\mathcal{N}=4$ Yang-Mills theory.** There is only one $D=4,\mathcal{N}=4$ Yang-Mills theory and it contains 3 $\mathcal{N}=1$ chiral scalar supermutliplet and 1 $\mathcal{N}=1$ vector supermultiplet (up to $g$ and $\tau$). This theory is conformal and can be recovered from dimensional reduction of $D=10,\mathcal{N}=1$ Yang-Mills on $\mathbb{T}^6$.

## 2.3 Gauge anomaly

The *anomaly degree* $A(\rho)$ of a representation $\rho$ is defined as

$$\frac{1}{2}\operatorname{tr}(T_a\{T_b,T_c\})=A(\rho)d_{abc} \tag{2.12}$$

where $d_{abc}$ is an invariant symmetric tensor of the Lie algebra of $G$, independent of the representation. One can show that $A(\rho^*)=-A(\rho)$ so self dual representation have $A(\rho)=0$ in particular. The only simple Lie groups that allow for a complex non-self-conjugate representation are $\mathrm{SU}(n)$ with $n\geq 3$. We can normalize $d_{abc}$ such that $A(\rho)=1$ for the fundamental $n$-dimensional representations.

# 3 Properties of D-branes

## 3.1 SYM from D-branes

The dynamics of D-branes is described by the Dirac-Born-Infeld action

$$S_{\mathrm{DBI}}[X,F]=-\frac{T_p}{g_s}\int \mathrm{d}^{p+1}\sigma\,\sqrt{-\det_{0\leq a,b\leq p}(\eta_{ab}+\partial_a X^m\partial_b X_m+2\pi\alpha' F_{ab})}. \tag{3.1}$$

The latter can be expended for slowly-varying fields, which is equivalent to passing to the field theory limit $\alpha'\to 0$. The resulting action is the action of a U(1) gauge theory in $p+1$ dimensions with $9-p$ real scalar fields. This action is exactly the same than the one we would obtain by dimensionally-reducing a pure U(1) Yang-Mills gauge theory in 10 spacetime dimensions with the identification

$$g_{\mathrm{YM}}=g_sT_p^{-1}(2\pi\alpha')^{-2}=\frac{g_s}{\sqrt{\alpha'}}(2\pi\sqrt{\alpha'})^{p-1}. \tag{3.2}$$

This construction can be generalized for multiple D-branes. It now results in a non-abelian theory. The general statement is the following:

> The low-energy dynamics of $N$ parallel, coicident D$p$-branes in flat space is described in static gauge by the dimensional reduction to $p+1$ dimensions of pure $10d$ $\mathcal{N}=1$ supersymmetric Yang-Mills theory with gauge group $\mathrm{U}(N)$ in ten spacetime dimensions.

Recall that the 10-dimensional action is given by

$$S_{\mathrm{YM}} = \frac{1}{4g_{\mathrm{YM}}^2}\int \mathrm{d}^{10}x\ \left[\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + 2i\,\mathrm{tr}\big(\overline{\psi}\Gamma^\mu D_\mu\psi\big)\right], \tag{3.3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the non-abelian field strength of the $\mathrm{U}(N)$ gauge field $A_\mu$, $D_\mu = \partial_\mu - i[A_\mu,\psi]$, $\Gamma^\mu$ are $16\times16$ Dirac matrices , and the $N\times N$ Hermitian fermion field $\psi$ is a 16-component Majorana-Weyl spinor of the Lorentz group $\mathrm{SO}(1,9)$ which transforms under the adjoint representation of the gauge group $\mathrm{U}(N)$. On-shell, there are eight on-shell bosonic, gauge field degrees of freedom, and eight fermionic degrees of freedom, after imposition of the Dirac equation $\not{D}\psi = \Gamma^\mu D_\mu\psi = 0$. One can verify that this action is invariant under the supersymmetry transformations Why ?

$$\begin{aligned}\delta_\epsilon A_\mu &= \frac{i}{2}\bar{\epsilon}\Gamma_\mu\psi,\\ \delta_\epsilon\psi &= \frac{1}{2}F_{\mu\nu}[\Gamma^\mu,\Gamma^\nu]\epsilon,\end{aligned}$$

where $\epsilon$ is an Majorana-Weyl spinor.

Using (3.3), we can construct a supersymmetric Yanf-Mills gauge theory in $p+1$ dimensions with 16 independent supercharges by dimensional reduction: we take all fields to be independent of the coordinates $X^{p+1},\dots,X^9$, then the ten-dimensional gauge field $A_\mu$ splits into a $(p+1)$-dimensional $\mathrm{U}(N)$ gauge field $A_a$ plus $9-p$ Hermitian scalar fields $\Phi^m = X^m/2\pi\alpha'$ in the adjoint representation of $\mathrm{U}(N)$. The D$p$-brane action is thereby obtained from the dimensionality reduced field theory as

$$S_{\mathrm{D}p} = -\frac{T_p g_s(2\pi\alpha')^2}{4}\int \mathrm{d}^{p+1}\sigma\ \mathrm{tr}\left(F_{ab}F^{ab} + 2D_a\Phi^m D^a\Phi_m + \sum_{m\neq n}[\Phi^m,\Phi^n]^2 + \text{fermions}\right) \tag{3.4}$$

where $a,b = 0,\dots,p$, $m,n = p+1,\dots,9$. We do not explicitly display the fermionic contributions for the moment. In conclusion, the low-energy brane dynamics is described by a supersymmetric Yang-Mills theory on the D$p$-brane worldvolume which is dynamically coupled to the transverse, adjoint scalar fields $\Phi^m$.

The scalar potential is given by

$$V(\Phi) = \sum_{m\neq n}[\Phi^m,\Phi^n]^2. \tag{3.5}$$

It is negative definite because $[\Phi^m,\Phi^n]^\dagger = [\Phi^n,\Phi^m] = -[\Phi^m,\Phi^n]$. A classical vacuum of the field theory defined by (3.4) corresponds to a static solution of the equations of motion whereby the potential energy of the system is minimized. It is given by the field configurations which solve simultaneously the quations $F_{ab} = D_a\Phi^m = \psi^a = 0$ and $V(\Phi) = 0$. Since all term in $V(\Phi)$ have the same sign, the equation $V(\Phi) = 0$ is equivalent to the equation $[\Phi^m,\Phi^n] = 0$ for all $m,n$ and at each point in the $(p+1)$-dimensional worldvolume of the branes. This implies that the $N\times N$ hermitian matrix fields $\Phi^m$ are simultaneously diagonalizable by a gauge transformation, so that we may write

$$\Phi^m = U\begin{bmatrix} X_1^m & & & 0\\ & X_2^m & & \\ & & \ddots & \\ 0 & & & X_N^m\end{bmatrix}U^{-1}, \tag{3.6}$$

the matrix $U$ is independent of $m$. The simultaneous, real eigenvalues $X_i^m$ give the positions of the $N$ distinct D-branes in the $m$-th transverse direction. It follows that the moduli space of classical vacua for the $(p+1)$-dimensional field theory (3.4) is the quotient space $(\mathbb{R}^{9-p})^N/S_N$, where the factors of $\mathbb{R}$ correspond to the positions of the $N$ D$p$-branes in the $(9-p)$-dimensional transverse space, and $S_N$ is the symmetric group acting by permutations of the $N$ coordinates $X_i$. The group $S_N$ corresponds to the residual Weyl symmetry of the $\mathrm{U}(N)$ gauge group acting in (3.6). It represents the permutation symmetry of a system of $N$ *indistinguishable* D-branes.

From (3.4) one can easily deduce that the masses of the fields corresponding to the off-diagonal matrix elements are given precisely by the distances $|x_i - x_j|$ between the corresponding branes. This description means that an interpretation of the D-brane configuration in terms of classical geometry is only possible in the classical ground state of the system, whereby the matrices $\Phi^m$ are simultaneously diagonalizable and the positions of the individual D-branes may be described through their spectrum of eigenvalues. This gives a simple and natural dynamical mechanism for the appearence of "non-commutative geometry" at short distances, where the D-branes cease to have well-defined positions according to classical geometry.

The end of this section has to be rewritten

## 3.2 | D-branes and residual SUSY in type II theories

The minimal irreducible representation in 10 dimensions is a Majorana-Weyl representation of dimension 8. In type II theories, we have $\mathcal{N}=(1,1)$ for IIA and $\mathcal{N}=(2,0)$ for IIB. Because of the string origin of the generators, the two supersymmetry generators $\epsilon_L$ and $\epsilon_R$ (Majorana-Weyl spinors) satisfy

$$\epsilon_L = \Gamma_{11}\epsilon_L, \qquad \epsilon_R = \eta\Gamma_{11}\epsilon_R \tag{3.7}$$

with $\eta = +1$ for IIB and $\eta = -1$ for IIA theory. For a D$p$-brane, the supersymmetry projections is the following:

$$\epsilon_L = \Gamma_0 \dots \Gamma_p \epsilon_R. \tag{3.8}$$

In other words, the supersymmetries with generators of the form

$$Q_\alpha + \Gamma_0 \dots \Gamma_p \overline{Q}_{\dot\alpha} \tag{3.9}$$

are preserved by the D$p$-brane while the one with generators of the form

$$Q_\alpha - \Gamma_0 \dots \Gamma_p \overline{Q}_{\dot\alpha} \tag{3.10}$$

are broken. They violate the boundary conditions. Since there is the same number of generators of the form (3.9) than of the form (3.10), exactly haf of the supersymmetry is broken. The idea that one spacetime direction would break one supercharge could be reasonable if supersymmetries were transforming as vectors which not the case; supercharges transform as spinors. It would also be incompatible with the T-duality because two branes of different dimensions must have the same number of unbroken supercharges if there is a T-duality relating them: the number of unbroken supercharges is the same for all dual descriptions (a necessary condition for the equivalence). And indeed, in the correct theory, that's the case. Every type II D-brane breaks half of the supercharges.

To obtain the previous relations, we start by the ones from M-theory and compactify the 11th direction, getting type IIA theory. $\Gamma_{11}$ then plays the role of the chiral projector in 10 dimensions; the supersymmetry parameters are related by $\epsilon_L = \frac{1}{2}(1+\Gamma_{11})\epsilon$ and $\epsilon_R = \frac{1}{2}(1-\Gamma_{11})\epsilon$. The relations for type IIB theory are then obtained by T-duality. Under a T-duality over the $\widehat{i}$ direction, the supersymmetry parameters transform as

$$\begin{aligned}\epsilon_L &\mapsto \epsilon_L,\\ \epsilon_R &\mapsto \Gamma_i\epsilon_R.\end{aligned}$$

The tension of a D$p$-brane is given by

$$T_p = \frac{1}{(2\pi)^p g_s l_s^{p+1}}. \tag{3.11}$$

This completely fixes the Newton constant: the tension of electric-magnetic duals must satisfy:

$$T_p T_{D-p-4} = \frac{2\pi}{16\pi G_D}. \tag{3.12}$$

In ten dimensions, this gives $G_{10} = 8\pi^6 g_s^2 l_s^8$.

The dualities are defined as follows:

$$\text{S-duality} : g_s \mapsto \frac{1}{g_s}, \qquad l_s^2 \mapsto g_s l_s^2,$$
$$\text{T-duality} : R \mapsto \frac{l_s^2}{R}, \qquad g \mapsto g_s \frac{l_s}{R}.$$

### 3.3 | D-branes wrapping cycles

A D$p$-brane worldvolume $\phi : \Sigma \to X$ in spacetime $X$ *wraps* a cycle $c \in H_{p+1}$ if the pushforward $\phi_*(\Sigma) \in H_\bullet(X)$ of the fundamental class of $\Sigma$ is the class $[c]$ of the given cycle in $X$. If the pushforward is a mutliple of $[c]$, then the branes wraps $c$ multiple times.

## 4 | Algebraic geometry

### 4.1 | Elements

An important idea in algebraic geometry is that is is really the alegbra of function on it that defines a space. For affine varities $X$, this is illustrated by the fact that the structure of $X$ is really contained in its coordinate ring $K[x_1, \dots, x_n]/I(X)$ and by the isomorphism

$$K[x_1, \dots, x_n]|_X = K[x_1, \dots, x_n]/I(X). \tag{4.1}$$

Now an algebraic set $Z(T)$ is irreducible if $I(Z(T))$ is prime. So there is a one-to-one correspondence between prime ideals and affine varieties.

Given an algebraic variety, one can modify the equations continuously by varying some parameters and the variety will be "deformed" accordingly. It is called the *variation of the omplex structure*. The space of all complex deformations of an affine variety $X$ is called the *complex moduli space* of $X$. For a Calabi-Yau manifold, the linearization of the complex moduli space (tangent space) is given by the cohomology group $H^{m-1,1}(X)$, where $m$ is the dimension of $X$. In general, it is much more complicated.

### 4.2 | Divisors and line bundles

A *(Weyl) divisor* $D$ of a complex variety $X$ is a linear combination (formal sum with integer coefficients) of co-dimension one, irreducible subvarieties,

$$D = \sum_i n_i V_i, \qquad n_i \in \mathbb{Z}, V_i \subset X. \tag{4.2}$$

It said to be effective if all $n_i \geq 1$. To any line bundle $L$ with a regular section $s$ (that is, on any open subset $U_\alpha$, $s_\alpha = s|_{U_\alpha}$ is a polyomial in the local coordinates) we can associate a hypersurface $Y \subset X$ defined as

$$Y = \{p \in X | s(p) = 0\}. \tag{4.3}$$

This hypersurface $Y$ can then be decomposed into irreducible parts (affine patches) on which $s_\alpha$ can be factorized in $\mathbb{C}[x_1, \dots, x_n]$ and decomposed in prime ideals $P_i$ of multiplicity $n_i$. Assembling all the $V_i^\alpha$ otgether, we construct co-dimension one subvarities $V_i$ that can be used to form divisors. One can also proceed the other ay around and, given a divisor $D = \sum_i n_i V_i$, define a line bundle $\mathcal{O}_X(D)$ whose sections vanish on each $V_i$ with a zero of order $n_i$. This construction can be generalized to divisors with negative coefficients $n_i < 0$ in which case now have poles of order $n_i$ in $V_i$.

### 4.3 | Singularities and resolutions

A *rational map* from a variety $X$ to another $Y$ is a morphism from a non-empty subset $U \subset X$ to $Y$. Recall that, by definition of the Zariski topology, a non-empty open subset is always dense. Concretely, a rational map can be written in coordinates using ration functions (quotient of polynomials). A *birational map* is an invertible rational map. It induces an isomorphism between two non-empty open subsets. In this case, $X$ and $Y$ are said to be *birationally equivalent*.

The *resolution of a singularity* of an algebraic variety $V$ is a non-singular variety $W$ with a proper birational map $W \to V$. For varities over fields of characteristic 0, it was proven (Hironaka, 1964) that . fill in

### 4.4 Projective plane curves

In $\mathbb{CP}^2$, we consider a hypersurface defined by a single polynomial $p$ of degree $d$. If

$$\frac{\partial p(x)}{\partial x_i} = 0 \tag{4.4}$$

for all $i$ whenever $p(x) = 0$, then the curve is said to be regular, it is a Riemann surface. The latter are classified by their genus and

$$g = \frac{(d-1)(d-2)}{2}. \tag{4.5}$$

For $d = 3$, the most geenral polynomial is

$$\sum_{i+j+k=3} c_{ijk} x_0^i x_1^j x_2^k = 0 \tag{4.6}$$

and defines a torus, also called *elliptic curve*. There 10 independant parameters but 9 of them can be removed by a $\mathrm{GL}(3,\mathbb{C})$ transformation, leaving us with only one complex parameter; the complex structrue modulus of the torus.

## 5 Quivers representations, path algebras, moduli spaces and quiver varieties

### 5.1 Quivers, path algebras and relations

A quiver $Q$ is a finite directed graph where loops and multiple arrows between edges are allowed. More precisely, it is a piece of combinatorial data $(Q_0, Q_1, t, h)$ such that $Q_0$ is the set of vertice, $Q_1$ the set of arrows and $t, h : Q_1 \to Q_0$ are the tail and head maps. A *path* of length $m$ is a trivial composition of arrows $a_1 \ldots a_m$ such that $h(a_l) = t(a_{l+1})$ for $l = 1, \ldots, m-1$. It is in particular a *loop* if $h(a_m) = t(a_0)$. We associate to any vertex $i \in Q_0$ a trivial loop $e_i$ going from $i$ to itself. This allows us identify $Q_0$ with the set of paths of lenth 0. Since $Q_1$ is the set of paths with length 1 by definition, we can generalize this notation and denote by $Q_m$ be the set of paths of length $m$.

We define the product of paths as

$$pq \equiv \begin{cases} pq, & h(p) = t(q) \\ 0, & \text{else} \end{cases} \qquad e_i p \equiv \begin{cases} q, & t(p) = i \\ 0, & \text{else} \end{cases} \qquad p e_i \equiv \begin{cases} p, & h(p) = i \\ 0, & \text{else.} \end{cases} \tag{5.1}$$

We are using the convention that $pq$ means $p$ then $q$. Provided with the formal sum, the set of all paths then forms a ring. Given a field $k$ and an action of $k$ aver this ring we form an algebra, called the *path algebra* of $Q$ and denoted by $kQ$. The addition and multiplication of two elements $a$ and $b$

$$a = \sum_{\alpha \in Q_m; a_\alpha \in k} \alpha a_\alpha, \qquad b = \sum_{\beta \in Q_m; a_\beta \in k} \beta b_\beta \tag{5.2}$$

of $kQ$ are therefore defined as

$$a + b = \sum_\alpha \alpha(a_\alpha + b_\alpha), \qquad a \cdot b \sum_{\alpha,\beta} \alpha\beta a_\alpha b_\beta. \tag{5.3}$$

It is clear that the path algebra is finitely generated if and only if $Q_0$ and $Q_1$ are finite and that it is finite-dimensional if and only if $Q$ has no non-trivial cycles. The $Q_m$'s, the set of paths of length $m$, define a gradation of the path algebra $kQ$:

$$kQ = \bigoplus_m kQ_m \qquad \text{with } kQ_m \equiv \bigoplus_{p\in Q_m} pk. \tag{5.4}$$

To enforce commutativity of some squares inside a quiver, we can use the notion of relation. A *relation* on a quiver $Q$ is a $k$-linear combination of paths from $Q$. More precisely, is a subsapce of $kQ$ spanned by linear combinations of paths having a common source and common target, and of length at least 2. A *quiver with relations* is a pair $(Q, I(R))$ where $I(R) \subseteq kQ$ is the (two-sided) ideal of the path algebra generated by $R$. The path algebra of $(Q, I)$ is $kQ/I(R)$.

**Example.** If $Q$ is the $r$-loop, then a relation is a subspace of $kQ = k\langle X_1, \dots, X_r\rangle$ spanned by linear combinations of words of length at least 2. For instance, all the commutators $X_iX_j - X_jX_i$, then the two elements of $kQ$ correspond to the same element in $kQ/I(R)$ if and only their difference is in $I(R)$. In this case, this means that they only differ by inverting the product of two variables in their expression. The resulting algebra is then the "commutative version" of $kQ$, which is nothing but than the polynomial algebra $k[X_1, \dots, X_r]$.

## 5.2 Quiver representations

A *representation* of a quiver $Q$ over the field $k$ is an assignment to every vertex $i$ os a $k$-vector space $V_i$ and to every arrow $a$ a linear mapping $f_a = V_{t(a)} \to V_{h(a)}$ between the corresponding vector spaces to each arrow. A representation of a quiver with relations $(Q, R)$ has the same definition but with the additional requirement that the linear maps must preserve the relations. Denoting $\alpha_i = \dim V_i$, $\alpha = (\alpha_i) \in \mathbb{N}_0^{Q_0}$ Explain. is the *dimension vector* of the representation. Quiver representations are an effective combinatorial tool for organizing linear algebraic data. Not only are they naturally related to many algebraic objects such as quantum groups, Kac-Moody algebras, and cluster algebras, but they have also been studied from the geometric point of view, often serving to bridge the gap between representation theory and algebraic geometry.

If $V = (V_i, f_a)$ and $W = (W_i, g_a)$ are are two finite-dimensional representation of the same quiver with relations $(Q, R)$, a morphism $\psi$ from $V$ to $W$ is given by specifying , for every vertex $i$, a linear isomorphism $\psi_i : V_i \to W_i$ such that for avery arrow $a$ $\psi_{h(a)} \circ f_a = g_a \circ \psi_{t(a)}$. The *direct sum* $V \oplus W$ of representations can also be defined as $(V \oplus W)_i = V_i \oplus W_i$ with the direct sum of the linear mappings. We then say that a representation is *decomposable* is it is isomorphic to the direct sum of non-zero representations and of *finite-type* (or *finite orbit type*) if it has only finitely many isomorphism classes of indecomposable representations. Or, equivalently, if it has only finitely many isomorphism classes of representations for any prescribed dimension vector. A very important theorem in indecomposable representations is the following:

**Theorem** (Gabriel)**.** A (connected) quiver is of fnite type if and only if its underlying undirected graph is a simply-laced Dynkin diagram.

To any representation $V = (V_i, f_a)$ of $Q$ we can associate a $k$-vector space

$$V = \bigoplus_{i\in Q_0} V_i \tag{5.5}$$

equipped with two famillies of linear self-maps: the projections $f_i : V \to V$ $(i \in Q_0)$ obtained from the composition $V \hookrightarrow V_i \to V$ of the projections with the inclisions, and tha maps $f_a : V \to V$ $(a \in Q_1)$ obtained similarly from the defining maps $f_a = V_{t(a) \to V_{h(a)}}$. One can see that these maps satisfy the relations

$$f_i^2 = f_i, \qquad f_i \circ f_j = 0 \ (i \neq j), \qquad f_{t(a)} \circ f_a = f_a \circ f_{h(a)} = f_a \tag{5.6}$$

and all other products are zero. In this sense, representations a quiver defines an algebra (the algebra of $f_i$ and $f_a$). Now comes the whole point to all this is: this algebra is a representation of the path algebra. To see this, we can use the notation $\rho(e_i) = f_i : V \to V$ and $\rho(a) = f_a : V \to V$ so $\rho : kQ \to \mathrm{End}(V)$

and we can show that is is a $k$-linear morphism of rings. So $\rho$ is a representation of the algebra $kQ$. So a representation of a quiver $Q$ gives a representation of the path algebra. Conversely, given the path algebra $kQ$ of a quiver, then $kQ$ is in particular a vector space over $k$ that yields a familly of vector space $\{V_i = e_i V\}_{i \in Q_0}$ and we have a linear map $f_a : V_{t(a)} \to V_{h(a)}$ for any arrow $a \in Q_1$. We conclude that representations of the path algebra and of the quiver are equivalent. Gabriel's theorem could then be equivalently phrased as: the path algebra of any quiver has finite representations if and only if it is ADE.

Recall also that giving a $k$-linear morphism $A = (k, R) \to \mathrm{End}(V)$ is equivalent to giving a structure of $R$-module to $V$. So from what we discussed above, giving a representation of a quiver is equivalent to giving a module structure to $V$. This can be rephrased using a more categorical language by showing that these constructions extend to functors. It is now clear that the finite-dimensional representations of a quiver with relations form a category, which we denote by $\mathtt{fRep}(kQ, R)$. If $(Q, I(R))$ is a quiver with relations and $A = kQ/I$ its path algebra, then the set of finite-dimensional modules of $A$ is also a category that we denote by $\mathtt{fdmod}A$. There is a categorical equivalence

$$\boxed{\mathtt{fRep}(kQ, R) \approx \mathtt{fdmod}A.} \tag{5.7}$$

The representation of any finite-dimensional alegbra can be described by a quiver with relations $(Q, I)$ such that $I$ contains the power of the ideal generated by the arrows.

## 5.3 Geometric invariant theory

## 5.4 GIT quotient for quiver representation

## 5.5 Quiver varities

## 5.6 Moduli problems

Given a collection of algebra-geometric objects it is natural to try to classify these objects up to equivalence. Naively, given a collection $\mathcal{A}$ of such objects and an equivalence relation $\sim$ on $\mathcal{A}$, one may ask whether there exists an algebraic variety $X$ whose points (over the base field $k$) correspond to equivalence classes in $\mathcal{A}/\sim$. This approach is flawed, since there may be many such varieties and some are better than others at retaining the relationships between the objects being classified. If, for example, we are interested in classifying all lines through the origin in the complex plane $\mathbb{C}^2$ up to equality, we can view the corresponding equivalence classes as points of the complex projective line $\mathbb{P}^1_{\mathbb{C}}$, but we can also see them as points in the disjoint union $\mathbb{A}^1_{\mathbb{C}} \sqcup \{pt\}$ of a line and a point. Ideally, the points of the variety solving a moduli problem should be configured to reflect the relationship between the geometric objects they parametrize.

Thus, in more nuanced approach we try to describe equivalence classes of families of objects of $\mathcal{A}$, rather than just of the objects themselves. That is, we look at pairs $\mathcal{F}, T$, consisting of a variety $T$ and a family $\pi : \mathcal{F} \to X$ of objects of $\mathcal{A}$ (the fibers $\pi^{-1}(t)$ are objects of $\mathcal{A}$) subject to some additional conditions (e.g. that $\pi$ is flat). Moreover, if the collection of families over a variety $T$ is denoted by $\mathcal{A}_T$, then for any morphism $f : S \to T$, there should be a pullback operation assigning to any family $\mathcal{F} \in \mathcal{A}_T$ a family $f^*\mathcal{F} \in \mathcal{A}_S$. Now, we can extend our moduli problem by introducing an equivalence relation $\sim_T$ on $A_T$ that is compatible with pullback and give us the starting equivalence relation $\sim$ when $T$ is $\mathrm{Spec}k$.

The solution to such an extended moduli problem, called a *fine moduli space*, consists of a variety $X$ whose $k$-points classify equivalence classes in $\mathcal{A}/\sim$, together with a *universal family* $\mathcal{U} \in \mathcal{A}_X$, which describes how these equivalence classes relate to each other. More specifically, any family $\mathcal{F} \in \mathcal{A}_T$ over a variety $T$ is equivalent to the pullback $f^*\mathcal{U}$ along a unique morphism $f : T \to X$. In the special case that $T = \mathrm{Spec}k$, we obtain that the $k$-points of $X$ are in bijection with equivalence classes in $\mathcal{A}$. Furthermore, it turns out that the fine moduli space is unique up to isomorphism.

Considering once again the example of lines through the origin in $\mathbb{C}^2$, we see that a family of such lines over a variety $T$ may be thought of as a line subbundle $\mathcal{L} \subset \mathbb{C}^2 \times T \to T$ of the trivial rank 2 vector bundle over $T$. The equivalence relation becomes an isomorphism of line subbundles of $\mathbb{C}^2 \times T$. The solution to the corresponding moduli problem consists of the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ together with the tautological line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1)$. Indeed, if $\mathcal{L} \subset \mathbb{C}^2 \times T$ is a subbundle, then its dual $\mathcal{L}^\vee$ is generated by global sections (specifically, the images of the standard global sections with respect to the surjection $(\mathbb{C}^2)^\vee \times T \twoheadrightarrow \mathcal{L}^\vee$). This defines a unique morphism $f : T \to \mathbb{P}^1_{\mathbb{C}}$ such that $\mathcal{L} \simeq f^*\mathcal{O}(-1)$.

Unfortunately, it is often the case that, for a given class of objects $\mathcal{A}$, a fine modduli space either does not exist or requires us to place restrictions on the kinds of objects in $\mathcal{A}$ we wish to classify. In order to avoid this, we can forget the universal family and look for a nice enough variety with points in bijection with equivalence classes in $\mathcal{A}/\sim$. Alternatively, we can allow for the solution of our moduli problem to no longer be a variety (or even a scheme). The result is a more complicated object called a *stack*.

This section has to be rewritten.

# 6 | Quivers in string theory and Yang-mills in graph theory

# 7 | Calabi-Yau manifolds, orbifolds and crepant resolutions

## 7.1 | Kähler, Calabi-Yau structure and moduli spaces

The Kähler form $\omega$ is a representative of a Doleault cohomology class

$$[\omega] \in H^{1,1}(X) \tag{7.1}$$

and $[\omega]$ is called the *Hähler class* of $\omega$.

**Theorem 7.1** (Calabi-Yau)**.** Given $X$ a compact manifold with trivial canonical bundle, and given a Kähler form $\widetilde{\omega}$ on $X$, there exist a unique Ricci-flat metric in the Kähler class of $\widetilde{\omega}$. That is, a unique Ricci-flat metric defined for some $\omega \in [\widetilde{\omega}]$.

On the other hand, it is easy to show that Ricci-flatness implies triviality of the canonical bundle. For non-compact manifolds, the theorem does not hold strictly speaking, one must specify boundary conditions at infinity to find a Ricci-flat metric.

Given a Calabi-Yau anifold, we see that there are continuous famillies of Ricci-flat metrics, one for each cohomology class of $H^{1,1}(X)$. One can decompose any chomology class on a basis $[\omega]^i$ of the vactor space $H^{1,1}(X)$

$$[\omega] = \sum_{i=1}^{h^{1,1}} \lambda_i [\omega]^i. \tag{7.2}$$

It is called the *Kähler moduli space* of $X$ and its dimension is denoed by $h^{1,1}$.

## 7.2 | Calabi-Yau manifolds

A *Calabi-Yau manifold* of (complex) dimension $n$ is a compact $n$-dimensional Kähler manifold $M$ satisfying one of the following equivalent conditions:

- the canonical bundle of $M$ is trivial,
- $M$ has holomorphic $n$-form that vanishes nowhere,
- the structure group of the tangent bundle of $M$ can be reduced from $\mathrm{U}(n)$ to $\mathrm{SU}(n)$,
- $M$ has Kähler metric with global holonomy contained in $\mathrm{SU}(n)$.

It was conjectured by Calabi then prooved by Yau that such spaces are necessarily Ricci-flat. In particular, since the first Chern class of CY manifolds si given by

$$c_1 = \frac{1}{2\pi}[\mathcal{R}] \tag{7.3}$$

it implies that $c_1$ vanishes, the converse is not true.

For a compact $n$-dimensioanl Kähler manifold the following conditions are equivalent to each other:

- the first real Chern class vanishes,
- $M$ has a Kähler metric with vanishing Ricci curvature,
- $M$ has Kähler metric with local holonomy contained in $\mathrm{SU}(n)$.
- a positive power of the canonical bundle of $M$ is trivial,
- $M$ has a finite cover that has trivial canonical bundle,
- $M$ has a finite cover that is a product of a torus and a simply connected manifold with trivial canonical bundle.

They are weaker than the conditions above except when the Kähler manifold is simply connected in which case they are equivalent.

## 7.3 Calabi-Yau orbifolds

A *Calabi-Yau orbifold* is the quotient of a smooth Calabi-Yau manifold by a discrete group action which generically has fixed points. From a algebraic geometry perspective we can try to resolve the orbifold singularity. A resolution $(X, \pi)$ of $\mathbb{C}^n/\Gamma$ is a non-singular complex manifold $X$ of dimension $n$ with a proper biholomorphic map

$$\pi : X \to \mathbb{C}^n/\Gamma \tag{7.4}$$

that induces a biholomorphism between dense open sets. A resolution $(X, \pi)$ of $\mathbb{C}^n/\Gamma$ is called a *crepant resolution*[3] if the canonical bundles of $X$ and $\mathbb{C}^n/\Gamma$ are isomorphic, i.e.

$$K_X \cong \pi^*(K_{\mathbb{C}^n/\Gamma}).$$

Since Calabi-Yau manifolds have trivial canonical bundle, to obtain a Calabi-Yau structure on $X$ one must choose a crepant resolutions of singularities.

It turns out that the amount of information we know about a crepant resolution of singularities of $\mathbb{C}^n/\Gamma$ depends dramatically on the dimension $n$ of the orbifold:

- $n = 2$: a crepant always exists and is unique. Its topology is entirely described in terms of the finite group $\Gamma$ (via the McKay correspondence).
- $n = 3$: a crepant resolution always exists but it is not unique; they are related by flops. However all the crepant resolutions have the same Euler and Betti numbers: the *stringy* Betti and Hodge numbers of the orbifold.
- $n \geq 4$: very little is known; crepant resolution ecists in rather special cases. Many singularity are terminal, which implies that they admit no crepant resolution.

[3]For a resolution of singularities we can define a notion of discrepancy. A crepant resolution is a resolution without discrepancy.

# 8 | McKay correspondence

## 8.1 | Classical correspondence

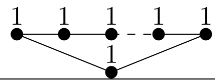
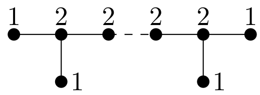
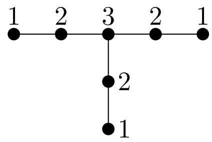
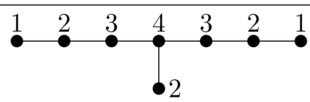
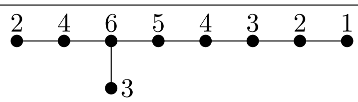
| $\Gamma \subset \mathrm{SU}(2)$ | Platonic solids | McKay graph | Algebraic variety |
|---|---|---|---|
| $\mathbb{Z}_n$ | |   ($n$ nodes) | $z^n + xy = 0$ |
| $2\mathcal{D}_n$ | $n$-polygon |   ($n+3$ nodes) | $x^2 + y^2z + z^{n-1} = 0$ |
| $2\mathcal{T}$ | tetrahedron |   (7 nodes) | $x^2 + y^3 + z^4 = 0$ |
| $2\mathcal{O}$ | cube<br>octahedron |   (8 nodes) | $x^2 + y^3 + yz^3 = 0$ |
| $2\mathcal{I}$ | icosahedron<br>dodecahedron |   (9 nodes) | $x^2 + y^3 + z^5 = 0$ |

*Table 1: Binary polyhedral groups and their McKay graphs.Labels over the vertices are the dimension of the representation. We erase the arrow ends if they go in both directions and erase the label if it is equal to 1.*

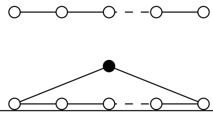
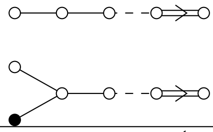
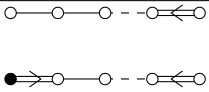
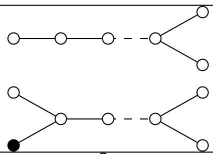
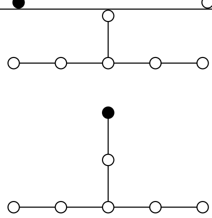
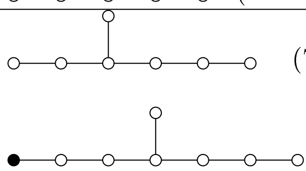
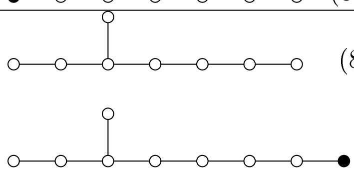
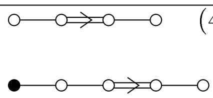
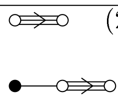
| Simple Lie algebra | Simply laced | Dynkin diagram<br>Extended Dybkin diagram |
|---|---|---|
| $\mathfrak{sl}(n+1,\mathbb{C}), n \geq 1$ | yes | $A_n$ : (n nodes)<br><br>$\widetilde{A}_n$ : ($n+1$ nodes) |
| $\mathfrak{so}(2n+1,\mathbb{R}), n \geq 2$ | no | $B_n$ : ($n$ nodes)<br><br>$\widetilde{B}_n$ : ($n+1$ nodes) |
| $\mathfrak{sp}(2n,\mathbb{C}), n \geq 3$ | no | $C_n$ : ($n$ nodes)<br><br>$\widetilde{C}_n$ : ($n+1$ nodes) |
| $\mathfrak{so}(2n,\mathbb{R}), n \geq 4$ | yes | $D_n$ : ($n$ nodes)<br><br>$\widetilde{D}_n$ : ($n+1$ nodes) |
| $\mathfrak{e}_6$ | yes | $E_6$ : (6 nodes)<br><br>$\widetilde{E}_6$ : (7 nodes) |
| $\mathfrak{e}_7$ | yes | $E_7$ : (7 nodes)<br><br>$\widetilde{E}_7$ : (8 nodes) |
| $\mathfrak{e}_8$ | yes | $E_8$ : (8 nodes)<br><br>$\widetilde{E}_8$ : (9 nodes) |
| $\mathfrak{f}_4$ | no | $F_4$ : (4 nodes)<br><br>$\widetilde{F}_4$ : (5 nodes) |
| $\mathfrak{g}_2$ | no | $G_2$ : (2 nodes)<br><br>$\widetilde{G}_2$ : (3 nodes) |

*Figure 2: Simple Lie algebras and their (extended) Dynkin diagrams. The first four algebras are the classical simple Lie algebras and the last five are the excpetional simple Lie algebras.*

Finally, we can see the following correspondence between the extended Dynkin diagrams and the McKay graphs.

| Simply Lie group | Simply laced Lie algebra | Extended Dybkin diagram | Finite subgroup of SO(3) | Finite subgroup of SU(2) |
|---|---|---|---|---|
| $\mathrm{SU}(n+1)$ | $\mathfrak{sl}(n+1, \mathbb{C})$ | $\widetilde{A}_n$ | $\mathbb{Z}_{n+1}$ | $\mathbb{Z}_{n+1}$ |
| $\mathrm{SO}(2n), \mathrm{Spin}(2n)$ | $\mathfrak{so}(2, \mathbb{R})$ | $\widetilde{D}_n$ | $\mathcal{D}_{2(n-2)}$ | $2\mathcal{D}_{2(n-2)}$ |
| $E6$ | $\mathfrak{e}_6$ | $\widetilde{E_6}$ | $\mathcal{T}$ | $2\mathcal{T}$ |
| $E7$ | $\mathfrak{e}_7$ | $\widetilde{E_7}$ | O | 2O |
| $E8$ | $\mathfrak{e}_8$ | $\widetilde{E_8}$ | $\mathcal{I}$ | $2\mathcal{I}$ |

*Figure 3: Classical McKay correspondence.*

## 8.2 | Geometrical McKay correspondence

The geometrical McKay correspondence is the bijection between the exeptional graph of the blow up of orbifolds $\mathbb{C}^2/\Gamma$ ($\Gamma \subset \mathrm{SU}(2)$) and the McKay graphs of $\Gamma$.

# Part II

# Orbifold singularities

## 9 | The simplest case: smooth transverse space

### 9.1 | Generalities

Let us start by considering the simplest configuration were the transverse Calabi-Yau space is non-singular, i.e. it is a proper smooth Calabi-Yau threefold. As mentioned above, the only smooth Calabi-Yau threefold is $S = \mathbb{C}^3$. In this case, the spacetime is simply flat space $\mathbb{R}^{1,9} = \mathbb{R}^{1,3} \times \mathbb{R}^6$ with a choice a complex structure on $\mathbb{R}^6$. From the $\mathrm{U}(N)$ Chan-Paton factors, the worldvolume theory inherits from a $\mathrm{U}(N)$ gauge group. Type IIB superstring theory is a ten-dimensional $\mathcal{N} = 2$ theory so it has 32 supercharges. The presence of the branes breaks the Lorentz symmetry of $\mathbb{R}^{1,9}$ as

$$\mathrm{SO}(1,9) \to \mathrm{SO}(1,3) \times \mathrm{SO}(6), \tag{9.1}$$

whereby breaking half of the supersymmetries, as we explained in the previous section. We are thus left with 16 supercharges. In four dimensions, this corresponds to $\mathcal{N} = 4$. The worldvolume theory for $S = \mathbb{C}^3$ is therefore $D = 4, \mathcal{N} = 4$ $\mathrm{U}(N)$ SCFT gauge theory. This worldvolume theory, obtained in the non-singular case $S = \mathbb{C}^3$, is called the *parent theory*.

Note that the D3-brane will warp the flat space metric to that of $AdS_5 \times S^5$ and the bulk geometry is not strictly $\mathbb{C}^3$. However, as stated above, we are only concerned with the local gauge theory and not with gravitational back-reaction, therefore it suffices to consider $S$ as $\mathbb{C}^3$.

### 9.2 | Matter content

As discussed in appendix 2.2, there is only one $D = 4, \mathcal{N} = 4$ SCFT theory, up to a choice of gauge group $G$. In our case, $G = \mathrm{U}(N)$. The isometry group of the transverse space $\mathbb{R}^6$ is $\mathrm{SO}(6) \cong \mathrm{SU}(4)$. Since the scalar fields living on the branes are interpreted as its tranverse oscillations, $\mathrm{SO}(6)$ is a global symmetry of the field theory. These global symmetries of worldvolume theory lead to the R-symmetry group $\mathrm{SU}(4)_R$. The only $\mathcal{N} = 4$ supermutliplet can be rewritten in terms of $\mathcal{N} = 1$ supermultiplets as follows:

$$[\mathcal{N} = 4 \text{ vector multiplet}] : V = (\lambda_\alpha, A_\mu, D) \oplus \Phi_A = (\phi^A, \psi^A_\alpha, F^A). \tag{9.2}$$

with $A = 1, 2, 3$. In other words, after removing the auxiliary fields $D$ and $F^A$, the matter content is

- a $\mathrm{U}(N)$ gauge field $A_\mu$ which transforms as a singlets under $\mathrm{SU}(4)_R$:

$$\text{Gauge transformation} : A_\mu \mapsto U A_\mu U^{-1} + U \partial_\mu U^{-1}, \qquad U \in \mathrm{U}(N) \tag{9.3}$$
$$\text{R-symmetry} : A_\mu \mapsto A_\mu. \tag{9.4}$$

  Note that the usual term

- 4 Weyl fermions $\psi^a_\alpha \equiv (\lambda_\alpha, \psi^1_\alpha, \psi^2_\alpha, \psi^3_\alpha,)$ that transform under the adjoint of $\mathrm{U}(N)$ and are mixed together under the representation $\mathbf{4}$ of $\mathrm{SU}(4)_R$. This means that each fermion $\psi^a$ takes values in $\mathfrak{u}(N)$. We denote the components by $\psi^a_{IJ}$ $(I, J = 1, \ldots, N)$. Explicitely:

$$\text{Gauge transformation} : \psi^a \mapsto U \psi^a U^\dagger, \qquad U \in \mathrm{U}(N), \tag{9.5}$$
$$\text{R-symmetry} : \psi^a \mapsto R^a{}_b \psi^b, \qquad R \in \mathrm{SU}(4)_R. \tag{9.6}$$

  Note that this gives us $4N^2$ Weyl fermions in total.

- 3 complex scalar fields $\phi^A$ transforming under the adjoint representation of $\mathrm{U}(N)$ and under the two-times anti-symmetric representation of $\mathrm{SU}(4)_R$. This means that each $\phi^A$ takes values in $\mathfrak{u}(N)$ and we denote the components by $\phi^A_{IJ}$. Recall that $\mathrm{SU}(4) \cong \mathrm{SO}(6)$ so the action of the R-symmetry can be seen as the $\mathbf{3}$ of $\mathrm{SU}(3) \subset \mathrm{SU}(4)_R$ acting on three complex scalars $\phi^A$ or equivalently as the **6** of $\mathrm{SO}(6)_R$ acting on 6 real scalars $X^m$, the real and imaginary parts of the $\phi^A$. They are interpreted as the oscillations of the branes in the transverse space. Explicitely:

$$\text{Gauge transformation} : X^m \mapsto U X^m U^\dagger, \qquad U \in \mathrm{U}(N), \tag{9.7}$$

$$\text{R-symmetry} : X^m \mapsto R^m{}_n X^n, \qquad R \in \mathrm{SO}(6)_R. \tag{9.8}$$

Note that this gives $6N^2$ real scalars in total. They are the superpartners of the fermions. Detail.

Note that the gauge group $\mathrm{U}(N)$ can also be seen as the group of isometries of the metric space $\mathbb{C}^N$, i.e. $\mathrm{Hom}(\mathbb{C}^N, C^N)$. From this point of view, the transformations (9.3)-(9.8)can be summarized as

$$A_\mu \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N), \tag{9.9a}$$

$$\psi \in \mathbf{4} \otimes \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N), \tag{9.9b}$$

$$X \in \mathbf{6} \otimes \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N). \tag{9.9c}$$

If the transverse space is non-singular, the only possibility is $S = \mathbb{C}^3$. In this case, the worldvolume theory is therefore $D = 4, \mathcal{N} = 4$ $\mathrm{U}(N)$ SCFT gauge theory. It is called the *parent theory*.

This is famous duality between $\mathcal{N} = 4$ supersymmetric $\mathrm{U}(N)$ Yang-Mills and Type IIB string theory on $\mathrm{AdS}_5 \times S^5$.

# 10 Strings on orbifolds

Let us a consider a smooth background space $M^{(6)}$ and $\Gamma$ a discrete group acting on it. If $\Gamma$ has no fixed point, the quotient space $M^{(6)}/\Gamma$ is smooth and is a manifold but if the action of $\Gamma$ has a fixed point then the quotient space $M^{(6)}/\Gamma$ is singular and is an orbifold. Any orbifold can be described as an affine variety therefore it is the description that we will use most often. When the transverse space is singular, the worldvolume theory corresponds to a specific projection of the parent theory that we found in the smooth case $S = \mathbb{C}^3$. We call it the *daughter theory*. This projections depends on the type of singularity that one considers. The simplest case is the the case of orbifold singularity, i.e. when the transverse space is a quotient space with a non-free action.

## 10.1 Generalities

We now wish to pick a discrete group $\Gamma$ and which acts non-trivially on $\mathbb{R}^6$. There are several possibilities:

- $\Gamma \subset \mathrm{SU}(4)$ naturally acts on $\mathbb{R}^6$. This does not require a choice of complex structure. We get an $\mathcal{N} = 0$ theory.
- $\Gamma \subset \mathrm{SU}(3)$ naturally acts on $\mathbb{C}^3$, this also requires a choice of complex structure on $\mathbb{R}^6$. We get an $\mathcal{N} = 1$ theory.
- $\Gamma \subset \mathrm{SU}(2)$ naturally acts on the second factor of $\mathbb{C} \times \mathbb{C}^2$, so this requires a choice of complex structure on $\mathbb{R}^6$. We get an $\mathcal{N} = 2$ theory.

We are interested in supersymmetric theories so we take $\Gamma \subset \mathrm{SU}(3)$ with the action

$$\cdot : \begin{pmatrix} \Gamma \times \mathbb{C}^3 & \longrightarrow & \mathbb{C}^3 \\ (\gamma, z) & \longmapsto & \gamma \cdot z \end{pmatrix} \tag{10.1}$$

is the representation of $\Gamma$ coming from the fundamental representation of $\mathrm{SU}(3)$, so $\cdot$ is just the matrix product. We can see that the origin is always a fixed point so this action is never free. Since $\mathbb{C}^3$ is a smooth manifold, this makes $\mathbb{C}^3/\Gamma$ an orbifold. Our choice of $\Gamma$ naturally includes the case $\Gamma \subset \mathrm{SU}(2)$ too (as $\mathrm{SU}(2) \subset \mathrm{SU}(3)$), just not the case $\Gamma \in \mathrm{SO}(6)$. When $\Gamma \subset \mathrm{SU}(2 \subset \mathrm{SU}(3)$, it acts trivially on one component so we write $S = \mathbb{C} \times \mathbb{C}^2/\Gamma$.

If $\Gamma$ is a general finite group the condition that $\mathbb{C}^3/\Gamma$ is an Calabi-Yau orbifold means that there must exist a resolution of this orbifold such that the corresponding smooth space is Calabi-Yau, i.e. a crepant resolution. Existence of such a resolution constrains $\Gamma$ , see section 7. Details ?

## 10.2 Twisted sector

What implications does have strings on an orbifold have for the spectrum? In general, we distinguish two kinds of states:

- the *untiwsted states* are the states $\Psi$ that are invariant under the actio of $\Gamma$: $\gamma \cdot \Psi = \Psi$ for all $\gamma \in \Gamma$. They are the generators of the $\Gamma$-invariant part of the Hilbert space and can easily be constrcuted by superposing all the images $\gamma \cdot \Psi_0$.
- closed strings must start and end at the same point, i.e. $X(\tau, \sigma + 2\pi) = X(\tau, \sigma)$. Usually, a strings that starts at connects a point of $M^{(6)}$ to another point of its orbit, i.e. $X(\tau, \sigma+2\pi) = \gamma \cdot [X(\tau, \sigma)]$ would not be allowed. However, in $M^{(6)}/\Gamma$ it is an allowed configuration. Those new states that appear after orbifolding are the *twisted states*. There existsnce is due to the fact that strings are extended objects. One can see that there is one twisted sector per conjugacy classe of $\Gamma$ and that the untwisted states are recovered with $\gamma = e$. details ?

The individual twisted sector quantum states of the strings are localized at the orbifold singularities that the classical configurations (untiwisted sector) enclose.

## 10.3 Projection to daughter theory

Make the link with the ususal setup.

Let us consider a D1-brane in the $x^0, x^1$-plane of the orbifold space $\mathbb{R}^6 \times \mathbb{R}^4/\Gamma$ [2, 3]. We can view the brane as a point in the covering space $\mathbb{R}^4$. If the brane trivially sits at a fixed point, which in our case is usually the origin, there is no problem but if the brane moves away from the origin, this will break the $\Gamma$ symmetry. In order for it to be able to move away from a the origin we therefore also need to add image-branes moving in all the other $\Gamma$-sectors of the covering space (the regions of $\mathbb{C}^n$ that are related by the action of $\Gamma$ and all projected to a unique sector in $\mathbb{C}^n/\Gamma$). Since there are $|\Gamma|$ sectors we need exactly $|\Gamma|$ branes in total, one per sector. From the open string states point of view, this is means adding $|\Gamma|$ Chan-Paton factors, resulting in an $U(|\Gamma|)$ gauge symmetry. $\Gamma$ then acts on the Chan-Paton factors and switch them between themselves. We now need to make sure that the string theory is consistent with that action.

Let us denote by $\rho_{\text{reg}}$ the regular representation of $\Gamma$, which is $|\Gamma|$-dimensional. Concretely, there are three types of open string sector states that one must consider:

- the vector multiplets: $\lambda_V \psi^\mu_{-1/2} |0\rangle$ ($\mu = 0, 1$), where $\lambda_V$ is the Chan-Paton matrix (arbitrary $\mathrm{U}(N)$) matrix, i.e. an arbitrary matrix of $\mathrm{U}(|\Gamma|)$. Invariance under $\Gamma$ means that $\lambda_V$ should satisfy the additional property
$$\rho_{\text{reg}}(\gamma)\lambda_V \rho_{\text{reg}}(\gamma)^{-1} = \lambda_V. \tag{10.2}$$
This constraint will give rise to the gauge group of the theory, transforming in the adjoint of some subgroup of the original gauge group $\mathrm{U}(N)$.
- the hypermultiplet I: $\lambda^{\mathrm{I}}_H \psi^i_{-1/2} |0\rangle$ ($i = 2, \ldots, 5$). They are the rest of the $D = 6, \mathcal{N} = 1$ vectors. From the $\mathcal{N} = 4, D = 2$ point of view, they are the scalar parts of the gauge multiplet and represent the oscillations for the branes in the $X^2, \ldots, X^5$-directions. The matrices $\lambda^{\mathrm{I}}_H$ must satisfy the relations
$$\rho_{\text{reg}}(\gamma)\lambda^{\mathrm{I}}_H \rho_{\text{reg}}(\gamma)^{-1} = \lambda^{\mathrm{I}}_H \tag{10.3}$$
- the hypermultiplet II: $\lambda^{\mathrm{II}}_H \psi^m_{-1/2} |0\rangle$ ($m = 6, \ldots, 9$). The matrices $\lambda^{\mathrm{II}}_H$ must satisfy the relations
$$\rho_{\text{reg}}(\gamma)\lambda^{\mathrm{II}}_H \rho_{\text{reg}}(\gamma)^{-1} = \lambda^{\mathrm{II}}_H \tag{10.4}$$

Note that there is a trivial solution to (10.2): $\lambda_V = \mathbb{1}$. This solution corresponds to an extra $\mathrm{U}(1)$ factor in the gauge group that will always be unbroken. Consequently, the Dynkin digram will always contain an extra $\mathrm{U}(1)$ node, this is what we call the the *extended* (or *affine*)) Dynkin diagram[4] This explains why the quiver that we will obtain will be affine Dynkin diagrams. From the point of view of the worldvolume theory, this means that there is always the possibility of Higgsing away all the vector multiplets except

[4]It can be viewed as the Dynkin diagram for the affine-extended Lie algebra.

the ones corresponding to this trivial U(1). This U(1) is the gauge group of a configuration with a single D3-brane so the hypermultiplet parametrizes the position this brane in the orbifold. From the point of view of the covering space, this brane is a stack of $|\Gamma|$ fractional branes, moving simultaneously in such a way that they are $\Gamma$-images of each other.

Looking only the open string sector is sufficient however to explore covering all the possibilities. One also needs to look at the massless closed strings and in particular at the twisted sector. We will see that these states enter the theory as Fayet-Iliopoulos terms.

Make the link between this discussion and the final projection.

If the transverse space is $S = \mathbb{C}^3/\Gamma$, the field theory must be projected to a theory which also invariant under $\Gamma$, seen as a subgroup of the R-symmetry group. The prescription is straihgt-forward: we can use the elements $\gamma \in \Gamma$ to project out that states that are not $\Gamma$-invariant. That is, if $\rho$ is an embedding of $\Gamma$ in the gauge group $\mathrm{U}(N)$, only the the fields such that

$$\boxed{\begin{aligned} \rho(\gamma)A_\mu\rho(\gamma)^{-1} &= A_\mu, \qquad (10.5a)\\ R(\gamma)\rho(\gamma)\psi_{IJ}\rho(\gamma)^{-1} &= \psi_{IJ}, \qquad (10.5b)\\ R(\gamma)\rho(\gamma)X_{IJ}\rho(\gamma)^{-1} &= X_{IJ} \qquad (10.5c)\end{aligned}}$$

are kept in the spectrum, where $\rho$ is a unitary representation of $\Gamma$ on $\mathbb{C}^N$ and $R = \mathbf{4}, \mathbf{6}$. Let us make two remarks:

- the term $U\partial_\mu U^{-1}$ is absent from 10.5a. Indeed, $\Gamma$ is a finite group and the only smooth functions $x \mapsto \Gamma$ are the constant ones. Consequently, transformations of the gauge field under a finite subgroup of its gauge group cannot depend on $x$.
- the fields that transform non-trivially under R-symmetry also have an extra induced action of $\Gamma$, in agreement with (9.9a)-(9.9c). The R-symmetry untouched by $\Gamma$ will be the resulting R-symmetry of daughter theory.

## 10.4 | Representation theory of the projection

Let $\{(\rho_i, V_i)\}_{i\in I}$ be a complete set of irreducible representations of $\Gamma$. We use the notation $N_i = \dim \rho_i$ the the dimension of the representations, such that $V_i = \mathbb{C}^{N_i}$. The finiteness of $\Gamma$ is crucial in two ways: first, since it is finite it is particular compact and the representations $(\rho_i, V_i)$ can be taken to be unitary. Second, the number of irreducible representations is necessarily finite, i.e. the index $i$ takes a finite number of values.

### 10.4.1 | Embedding of $\Gamma$ in the gauge group

Let us consider a representation of $\Gamma$ on $\mathbb{C}^N$, we denote it $(\rho, \mathbb{C}^N)$ and also take it to be unitary. In that case, $\rho(\gamma) \in \mathrm{U}(N)$. This is what we mean by "the embedding of $\Gamma$ in $\mathrm{U}(N)$". The adjoint representation of $\mathrm{U}(N)$ defined as[5]

$$\mathrm{Ad} : \begin{pmatrix} \mathrm{U}(n) & \longrightarrow & \mathrm{GL}(\mathfrak{u}(N)) \\ U & \longmapsto & \mathrm{Ad}_U \end{pmatrix}, \qquad \mathrm{Ad}_U : \begin{pmatrix} \mathfrak{u}(N) & \longrightarrow & \mathfrak{u}(N) \\ \omega & \longmapsto & \mathrm{Ad}_U(\omega) \equiv U\omega U^{-1} \end{pmatrix}, \tag{10.6}$$

now allows us to act with $\Gamma$ on $\mathfrak{u}(N)$. We use this representation in the expression (10.5a)-(10.5c). More formally, these relations can be rewritten as

$$\mathrm{Ad}_{\rho(\gamma)}A_\mu = A_\mu, \tag{10.7}$$
$$R(\gamma)\mathrm{Ad}_{\rho(\gamma)}\psi_{IJ} = \psi_{IJ}, \tag{10.8}$$
$$R(\gamma)\mathrm{Ad}_{\rho(\gamma)}X_{IJ} = X_{IJ} \tag{10.9}$$

The representation $(\rho, \mathbb{C}^N)$ can be decomposed as follows:

$$(\rho, \mathbb{C}^N) = \bigoplus_{i\in I} (\rho_i, V_i)^{N_i} \tag{10.10}$$

[5]this is well-defined since for all $\omega \in \mathfrak{u}(N)$ and $U \in \mathrm{U}(N)$, $U\omega U^{-1} \in \mathfrak{u}(N)$.

$$= \bigoplus_{i\in I} (\mathbf{1}^{N_i} \otimes \rho_i, \mathbb{C}^{N_i} \otimes V_i) \tag{10.11}$$

were $N_i$ are integer multiplicities ($(\rho_i, V_i)^{N_i} \equiv (\rho_i, V_i)^{\oplus N_i}$) and $\mathbf{1}$ is the trivial representation, so $\Gamma$ acts trivially on the $\mathbb{C}^{N_i}$. We have

$$\sum_i N_i \dim(\rho_i) = N. \tag{10.12}$$

The rewriting (10.11) will be useful later on.

### 10.4.2 | Adjoint representation and bi-fundamental fields

Any object in the fundamental representation $\mathbf{N}$ of $G = \mathrm{U}(N)$ (generated by $\{T_a\}_{a=1,\dots,N^2}$ that are taken to be hermician) has an index $i$ ($i = 1,\dots,N$) and transforms as

$$\delta_a \Phi^i = e^{i(T_a)^i{}_j}\Phi^j. \tag{10.13}$$

An adjoint field has an index $a$ and transforms as

$$\delta_a \Phi^b = -f_{ac}{}^b \Phi^c \tag{10.14}$$

with $[T_a, T_b] = if_{ab}{}^c T_c$. Given any such adjoint field, we can construct an $N\times N$ matrix as $\Phi^i{}_j = \Phi^a (T_a)^i{}_j$. This matrix has $N^2$ independent components, which is exactly the dimension of the adjoint representation. What is the transformation of this matrix? We find

$$\begin{aligned}
\delta_a \Phi^i{}_j &= -f_{ac}{}^b \Phi^c (T_b)^i{}_j &(10.15)\\
&= i(T_a)^i{}_k \Phi^k{}_j - i\Phi^i{}_k (T_a)^k{}_j &(10.16)\\
&= i(T_a)^i{}_k \Phi^k{}_j - i(T_a^*)_j{}^k \Phi^i{}_k, &(10.17)
\end{aligned}$$

in other words, the first index transforms in the fundamental representation $\mathbf{N}$ and the second index transforms in the anti-fundamental transformation $\overline{\mathbf{N}}$. Thus, $\Phi^i{}_j$ transforms under $(\mathbf{N}, \overline{\mathbf{N}}) \equiv \mathbf{N}\otimes\overline{\mathbf{N}}$[6]. This point of view of the adjoint representation is convenient for us. In summary, we found that

$$\boxed{\mathrm{Ad} = \mathbf{N}\otimes\overline{\mathbf{N}}.} \tag{10.18}$$

Taking a step back, we can view write $\mathrm{U}(N)$ as $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)$ and this relation is then nothing more that a particular case of $\mathrm{Hom}(V, W) \cong V \otimes W^*$. Generally speaking, fields transforming under $(\mathbf{N_i}, \overline{\mathbf{N_j}})$ are called *bi-fundamentals fields*. From our discussion, we see that fields transforming in the adjoint representation are in particular bi-fundamental fields.

We end this section by mentioning that the same computations can be done for $\mathrm{SU}(N)$. The only difference is the additional trace cancelling condition. We find that the trace transforms as a scalar, therefore we have $\mathrm{Ad}\oplus 1 = \mathbf{N}\otimes\overline{\mathbf{N}}$ this time.

### 10.4.3 | Invariant configurations: general case

After the projection, the resulting gauge group is given by the $\Gamma$-invariant part of the gauge group, denoted by $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)^\Gamma$. The superscript $\Gamma$ is used to indicate that we only keep the trivial representations in the decomposition, that is, we only keep that subspaces that transform trivially. We want to compute $\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)^\Gamma$. First, we can see that by Schur's lemma Why ?

$$(V_i \otimes V_j^*)^\Gamma = \delta_{ij} \tag{10.19}$$

Using (10.18), we get

$$\mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N)^\Gamma = (\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^\Gamma \tag{10.20}$$

[6] To be more exact, we should say that they transform a "diagonal" version of $\mathbf{N}\otimes\overline{\mathbf{N}}$ since each index in (10.17) transforms with the same group element. We cannot choose to act with $T_a$ on one index and with $T_b$ on the other one for example.

$$= \bigoplus_{i,j\in I} \left((\mathbb{C}^{N_i}\otimes V_i)\otimes(\mathbb{C}^{N_j}\otimes V_j)^*\right)^\Gamma \tag{10.21}$$

$$= \bigoplus_{i,j\in I} \left(\mathbb{C}^{N_i}\otimes(\mathbb{C}^{N_j})^*\otimes V_i\otimes V_j^*\right)^\Gamma \tag{10.22}$$

$$= \bigoplus_{i,j\in I} \left(\mathbb{C}^{N_i}\otimes(\mathbb{C}^{N_j})^*\right)^\Gamma\otimes\left(V_i\otimes V_j^*\right)^\Gamma \tag{10.23}$$

$$= \bigoplus_{i\in I} \mathbb{C}^{N_i}\otimes(\mathbb{C}^{N_i})^* \tag{10.24}$$

so, in complete notations,

$$(\rho,\mathbb{C}^N)^\Gamma = \bigoplus_{i\in I}(\rho_i,\mathbb{C}^{N_i})\otimes(\overline{\rho_i},\mathbb{C}^{N_i}) \tag{10.25}$$

and the daughter gauge group is

$$G_{\text{proj}} = \bigotimes_{i\in I}\mathrm{U}(N_i\dim\rho_i). \tag{10.26}$$

Now it turns out that in the low energy effective field theory the $\mathrm{U}(1)$ factor of every $\mathrm{U}(N_i)$ decouples so the resulting gauge group is in fact Why ?

$$\boxed{G_{\text{proj}} = \bigotimes_{i\in I}\mathrm{SU}(N_i\dim\rho_i).} \tag{10.27}$$

For the matter fields, the reasoning is similar but we now have to take into account the R-symmetry. Let $\mathbf{4}\equiv(\rho_\mathbf{4},V_\mathbf{4})$ be the fundamental representation of $\mathrm{SU}(4)_R$ and $\mathbf{6}\equiv(\rho_\mathbf{6},V_\mathbf{6})$ be the fundamental representation of $\mathrm{SO}(6)_R$. Wish to compute $(V_\mathcal{R}\otimes\mathrm{Hom}(\mathbb{C}^N,\mathbb{C}^N))^\Gamma$ with $\mathcal{R}=\mathbf{4},\mathbf{6}$:

$$(V_\mathcal{R}\otimes\mathrm{Hom}(\mathbb{C}^N,\mathbb{C}^N))^\Gamma = \bigoplus_{i,j\in I}\left(V_\mathcal{R}\otimes(\mathbb{C}^{N_i}\otimes V_i)\otimes(\mathbb{C}^{N_j}\otimes V_j)^*\right)^\Gamma \tag{10.28}$$

$$= \bigoplus_{i,j\in I}\left(V_\mathcal{R}\otimes\mathbb{C}^{N_i}\otimes(\mathbb{C}^{N_j})^*\right)^\Gamma\otimes\left(V_i\otimes V_j^*\right)^\Gamma \tag{10.29}$$

$$= \bigoplus_{i,j\in I} a_{ij}^\mathcal{R}\left(\mathbb{C}^{N_i}\otimes(\mathbb{C}^{N_j})^*\right) \tag{10.30}$$

with

$$\boxed{(\rho_\mathcal{R},V_\mathcal{R})\otimes(\rho_i,V_i) = \bigoplus_{j\in I}a_{ij}^\mathcal{R}(\rho_j,V_j).} \tag{10.31}$$

This expression makes sense because $(\rho_\mathcal{R},V_\mathcal{R})$ is a representation of $\mathrm{SU}(4)$ so it is in particular a representation of $\mathrm{SU}(3)$ and therefore also in particular a representation of $\Gamma$. In more complete notations, we obtained that

$$((\rho_\mathcal{R},V_\mathbb{R})\otimes(\rho,\mathbb{C}^N))^\Gamma = \bigoplus_{i,j\in I}a_{ij}^\mathcal{R}\left((\rho_i,\mathbb{C}^{N_i})\otimes(\overline{\rho}_j,\mathbb{C}^{N_j})\right). \tag{10.32}$$

Using the fact that characters are homomorhisms and the orthogonality of the characters of characters of irreducible non-equivalent representations, we obtain the explicit expression of the coefficient $a_{ij}^\mathcal{R}$:

$$\boxed{a_{ij}^\mathcal{R} = \frac{1}{|\Gamma|}\sum_{\gamma=1}^{r}r_\gamma\chi^\mathcal{R}(\gamma)\chi^i(\gamma)\chi^j(\gamma)^*} \tag{10.33}$$

where $r_\gamma$ is the order of the conjugacy class containing $\gamma$ and $\chi^i$ is the character of $\rho_i$.

In the end, we can see that:

In the daughter theory, the matter fields become a total of $a^{\mathbf{4}}_{ij}$ bi-fundamental fermions $\psi^{ij}_{f_{ij}}$ ($f_{ij} = 1, \dots, a^{\mathbf{4}}_{ij}$) and $a^{\mathbf{6}}_{ij}$ bi-fundamental complex scalars $\phi^{ij}_{f_{ij}}$ ($f_{ij} = 1, \dots, a^{\mathbf{6}}_{ij}$). They all transform in the $(\mathbf{N_i}, \overline{\mathbf{N_j}})$ of $\mathrm{SU}(N_i) \times \mathrm{SU}(N_j)$ under the products of gauge groups.

### 10.4.4 | Bi-index notation

The gauge and the bi-fundamental fields that we obtained in the previous section are easier to manipulate in the *bi-index notation*. Recall that $A_\mu \in \mathfrak{u}(N)$ so it is an $N \times N$ complex matrix and we denote its elements by $A_{\mu;IJ}$ ($I, J = 1, \dots, N$). We saw that this matrix of fields transform under $\Gamma$ as

$$A_\mu \mapsto \rho(\gamma) A_\mu \rho(\gamma)^{-1}, \qquad \gamma \in \Gamma, \tag{10.34}$$

i.e. in the adjoint representation but without the derivative term (as explained above). Now if $\{(\rho_i, V_i)\}_{i \in I}$ is a complete set of irreducible representation of $\gamma$, any representation of $\Gamma$ on $\mathbb{C}^n$ can be decomposed as (10.10). In particular, this provides us with the following partitioning of $N$:

$$N = \sum_{i \in} N_i \dim \rho_i. \tag{10.35}$$

It is very convenient to apply the analogous partitionning for the matrix $A_\mu$. More precisely, we now denote decompose $A_\mu$ into $(\dim \rho_i N_i) \times (\dim \rho_j N_j)$ sub-blocks $A_{\mu;ij}$ where $i, j \in I$ are indices over the irreducible representations. From (**??**), those sub-blocks transform as direct sums of the same irreducible representations:

$$A_{\mu;ij} \mapsto \rho_i(\gamma)^{N_i} A_{\mu;ij} (\rho_j(\gamma)^{-1})^{N_j}, \qquad \gamma \in \Gamma. \tag{10.36}$$

We can go one step further and decompose the sub-blocks $A_{\mu;ij}$ into $\dim \rho_i \times \dim \rho_j$ sub-sub-blocks $A_{\mu;i\alpha_i,j\beta_j}$ where $\alpha_i, \beta_i = 0, \dots N_i - 1$. From (10.36), they transform directly under the irreducible representations:

$$\boxed{A_{\mu;i\alpha_i,j\beta_j} \mapsto \rho_i(\gamma) A_{\mu;i\alpha_i,j\beta_j} \rho_j(\gamma)^{-1}, \qquad \gamma \in \Gamma.} \tag{10.37}$$

In particular, we see that this relations does not depend on the indices $\alpha_i$ and $\beta_j$, they all transform in the same way. This notation is very convenient to compute explicitely the invariant configurations, as we will see in several examples. We use the exact same bi-index notations for $X^m$ and $\psi^a$.

## 10.5 | Field content, quivers and McKay graphs

A convenient way the represent the matter content of a daughter theory is to use *quiver diagrams*. A quiver is a finite oriented graph such that each node $i$ represents a gauge factor $\mathrm{SU}(N_i)$ and each arrow $i \to j$ represents a bi-fundamental field transforming under $(\mathbf{N_i}, \overline{\mathbf{N_j}})$. So, in essence, the arrows represent the vector multiplets (gauge) and the vertices the hypermultiplets (matter). The *adjacency matrix* $A$ of the graph is a $k \times k$ with $k$ being the number of nodes (gauge factors) whose elements $A_{ij}$ are the number of arrows (bi-fundamental fields) from $i$ to $j$. In other words, from (10.32), the adjacency matrix of the fermions has elements $A_{ij} = a^{\mathbf{4}}_{ij}$ and the one of the scalars has elements $A_{ij} = a^{\mathbf{6}}_{ij}$.

On the other hand, given finite group $\Gamma$, a representation $(\rho_W, W)$ and a complete set of irreducible representations $\{(\rho_i, V_i)\}_{i \in I}$ of the latter, one can construct a McKay graph (or quiver) as follows:

1. Draw a vertex for every representation $(\rho_i, V_i)$.
2. For every representation $(\rho_i, V_i)$, compute the decomposition

   $$(\rho_W, V_W) \otimes (\rho_i, V_i) = \bigoplus_j (\rho_j, V_j)^{\oplus n_{ij}}$$

   where $n_{ij}$ is the multiplicity of $(\rho_j, V_j)$ in the decomposition of $(\rho_W, V_W) \otimes (\rho_i, V_i)$.
3. For every $n_{ij} > 0$, draw $n_{ij}$ arrows from the vertex of $(\rho_i, V_i)$ to the one of $(\rho_j, V_j)$.

When $\Gamma \subset \mathrm{SU}(2)$ and that $(\rho_W, W)$ is its defining representation, the McKay graphs are in one-to-one correspondence with the extended Dynkin diagrams of the simply laced Lie algebras. This is the classical McKay correspondence, see appendix 8.

From (10.31) we see that $n_{ij} = a_{ij}^{\mathcal{R}}$, i.e. the matter quivers that we defined previously are exactly the McKay graph associated to the matter representation in question. Put differently, the matter content of the daughter theory is encapsulated in the McKay graphs of $\Gamma$ and with respect to $\mathcal{R}$ with $\mathcal{R} = \mathbf{4}$ for spinors and $\mathcal{R} = \mathbf{6}$ for scalars. This very important point. For example, it will allows us to use known results on McKay graphs such as the McKay correspondence. Not only the field content is a quiver, which is exactly the McKay graph of the finite group, but we also have gauge groups at each vertex and transformations between them for each arrow (these transformations are precisely the fields, see as elements of $\mathrm{Hom}(V_i, V_j)$). So we are actually given a representation of the quiver. Moreover, the vacuum configurations have to solve the F-term and D-term equations. Those algebraic relations constrain the path algebra of the quiver: they are relations.

> A quiver gauge theory is a representations of a quiver with relations.

We mentioned the adjacency matrix as a way to represent a quiver but we will also use sometimes the another matrix called the *incidence matrix* and denoted by $I$. In the later, the columns index the arrows and the rows, the nodes such that the $\alpha$th arrow from node $i$ to $j$ receives a $-1$ in position $I_{i\alpha}$, a $+1$ in position $I_{j\alpha}$ and zero elsewhere.

Another convenient way of representing the same information are the *dimer diagram*, i.e. a graph whose faces represent the gauge groups, the edges represent the bi-fundamental fields and the vertices represent the superpotentials.

## 10.6 Gauge anomaly cancellation

Finally, let us discuss the gauge anomaly cancellation. Our fields transform under the adjoint representation of $\mathrm{SU}(N)$, under the fundamentals of some $\mathrm{SU}(N_i)$ and under the anti-fundamentals of some $\mathrm{SU}(N_i)$. The adjoint representation being real, it is self-conjugate and therefore does not contribute to the anomaly. The other representation however do contribute: the fundamentals of each $\mathrm{SU}(N_i)$ have a $+1$ contribution the anti-fundamentals of each $\mathrm{SU}(N_i)$ have a $-1$ contribution. Anomaly cancellation therefore imposes that the contribution of the fundamental and of the anti-fundamental of $\mathrm{SU}(N_i)$ cancel each other, for each $i \in I$ (see section (2.3)). Since the bi-fundamental representations $(\mathbf{N_i}, \overline{\mathbf{N_j}})$ counts as $N_j$ fundamentals of $\mathrm{SU}(N_i)$ and as $N_i$ anti-fundamentals $\mathrm{SU}(N_j)$, the condition for anomaly cancellation is

$$\sum_{k \in I} a_{jk} N_k - \sum_{i \in I} a_{ij} N_i = 0 \tag{10.38}$$

for every $j \in I$, where $a_{ij}$ are the elements of the adjacency matrix of the quiver. This can be simply rewritten as

$$\boxed{\sum_{i \in I} (a_{ji} - a_{ij}) N_i = 0} \tag{10.39}$$

This is equivalent to say that the vector $(N_0, N_1, N_2, \dots)$ formed by the ranks of the gauge groups must lie in nullspace of the antisymmetrized adjacency matrix. From this nice reformulation we can already see that any "symmetric" quiver, i.e. quiver associated to a symmetric adjacency matrix, is automatically anomaly-free. In other words, a gauge theory is automatically anomaly-free if and only if all arrows come in pairs[7] (with different orientations) in the quiver. If it is not the case, the ranks of the gauge groups must be constrained.

At this stage, we can already predict that all $\mathcal{N} = 2$ orbifold quiver gauge theories will be automatically anomaly-free since their quiver are their McKay graphs in which all arrows come in pairs.

[7] Loops are also allowed since they don't contribute to the anomaly.

## 10.7 A simple example: $S = \mathbb{C}^3/\mathbb{Z}_3$

We illustrate the previous discussion by the simple case where $\Gamma = \mathbb{Z}_3$ acts on $\mathbb{C}^2$ as

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \mapsto \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{10.40}$$

i.e. the transverse space is the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. This simple example is a good first approach in which we will to explain in details each step so that we can go faster afterwards.

### 10.7.1 Projection

Let us consider a representation $(\rho, \mathbb{C}^N)$ of $\mathbb{Z}_3$. A complete set of irreducible representations of $\mathbb{Z}_3$ is given by $\{(\rho_k, V_k)\}_{k=0,1,2}$ with $V_k = \mathbb{C}$ and

$$\rho_k(g) = \zeta_3^k \tag{10.41}$$

where $g$ is the generator of $\mathbb{Z}_3$. The representation $(\rho, V)$ can be decomposed as

$$(\rho, V) = \bigoplus_{i=0}^{2} N_i(\rho_i, V_i). \tag{10.42}$$

In other words, it is equivalent to the representation

$$\rho(g) = \left.\begin{bmatrix} 1 & & & & \cdots & & & & 0 \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & \zeta_3 & & & & & \\ \vdots & & & & \ddots & & & & \vdots \\ & & & & & \zeta_3 & & & \\ & & & & & & \zeta_3^2 & & \\ & & & & & & & \ddots & \\ 0 & & & & \cdots & & & & \zeta_3^2 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} N_0 \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} N_1 \\ \left.\vphantom{\begin{matrix}1\\1\\1\end{matrix}}\right\} N_2 \end{matrix}\right. . \tag{10.43}$$

Since $\dim \rho_i = 1$, we have

$$N_0 + N_1 + N_2 = N. \tag{10.44}$$

The gauge field configurations that are left invariant under the action of $\mathbb{Z}_3$ are therefore the ones that satisfy

$$\rho(g) A_\mu \rho(g)^{-1} = A_\mu. \tag{10.45}$$

We actually want this relation to be true for any element of $\mathbb{Z}_n$ but in this case it is invariant under any element of $\mathbb{Z}_3$ if and only if it is invariant under the generator $g$ of $\mathbb{Z}_3$, so we only need to impose (10.45). The constrained is easily solved by using the bi-index notation $A_{\mu;i\alpha_i,j\beta_j}$ $(i, j = 0, 1, 2, \alpha_i, \beta_i = 1, \ldots, N_i)$ for the component of the gauge fields. From (15.7), we can see that

$$A_{\mu;i\alpha_i,i\beta_j} \mapsto \rho_i(g) A_{\mu;i\alpha_i,j\beta_j} \rho_j(g)^{-1} = \zeta_n^{i-j} A_{\mu;i\alpha_i,j\beta_j}. \tag{10.46}$$

thus only the configurations with $A_{\mu;i\alpha_i,j\beta_j} = 0$ if $i \neq j$ are invariant. The gauge field has therefore a block diagonal form:

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & \\ & A_{\mu;11} & \\ & & A_{\mu;2,2} \end{bmatrix} \tag{10.47}$$

with $A_{\mu;ij} \equiv (A_{\mu;i\alpha_i,j\beta_j})_{\alpha_i=0,\ldots,N_i,\beta_j=0,\ldots,N_j}$. The block $A_{ii}$ transforms under $\mathbb{Z}_3$ under $(\rho_i, V_i)^{N_i}$. Consequently, the gauge group is now broken to

$$G_{\text{proj}} = \mathrm{U}(N_0) \times \mathrm{U}(N_1) \times \mathrm{U}(N_2), \tag{10.48}$$

the biggest subgroup of $\mathrm{U}(N)$ that preserves those form of configurations.

Let us study the scalars. $\mathbb{Z}_3$ acts on the three complex scalars through

$$R(g) = \rho_1^{\oplus 3}(g) = \zeta_3 \mathbb{1}_3 = \begin{bmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{bmatrix} \tag{10.49}$$

or, equivalently, on the real scalars as $R(g) = \zeta_3 \mathbb{1}_6$. According to (9.7)-(9.8), the scalar field configurations that are left invariant satisfy

$$R(g)^m{}_n \rho(g) X^n \rho(g)^{-1} = X^m \tag{10.50}$$

for all $g \in \mathbb{Z}_3$. Using the bi-index notations, this becomes

$$X^m_{i\alpha_i, j\beta_j} \mapsto \zeta_n \delta^m{}_n \rho_i(g) X^n_{i\alpha_i, j\beta_j} \rho_j(g)^{-1} = \zeta_3^{i-j+1} X^m_{i\alpha_i, j\beta_j} \tag{10.51}$$
$$\overline{X}^m_{i\alpha_i, j\beta_j} \mapsto \zeta_3^{-1} \delta^m{}_n \rho_i(g) \overline{X}^n_{i\alpha_i, j\beta_j} \rho_j(g)^{-1} = \zeta_3^{i-j-1} \overline{X}^m_{i\alpha_i, j\beta_j} \tag{10.52}$$

thus only the configurations with $X^n_{i\alpha_i, j\beta_j} = 0$ if $i-j+1 \neq 0$ are left invariant and only the configurations with $\overline{X}^n_{i\alpha_i, j\beta_j} = 0$ if $i-j-1 \neq 0$ are left invariant. The scalar fields $X$ have a block off-diagonal form:

$$X^m = \begin{bmatrix} 0 & X^m_{01} & 0 \\ 0 & 0 & X^m_{12} \\ X^m_{21} & & 0 \end{bmatrix}, \qquad \overline{X}^m = \begin{bmatrix} 0 & 0 & \overline{X}^m_{02} \\ \overline{X}^m_{10} & 0 & 0 \\ 0 & \overline{X}^m_{21} & 0 \end{bmatrix} \tag{10.53}$$

with the block notations

$$X^m_{ij} \equiv (X^m_{i\alpha_i, j\beta_j})_{\alpha_i = 0, \ldots, N_i, \beta_j = 0, \ldots, N_j}$$
$$\overline{X}^m_{ij} \equiv (\overline{X}^m_{i\alpha_i, j\beta_j})_{\alpha_i = 0, \ldots, N_i, \beta_j = 0, \ldots, N_j}.$$

The block $X^m_{ij}$ is an $N_i \times N_j$ block and transforms under the representation $(\mathbf{N_i}, \overline{\mathbf{N}}_{\boldsymbol{j}})$ of $\mathrm{U}(N_i) \times \mathrm{U}(N_j)$:

$$X^m_{i,i+1} \in \mathbf{N_{i+1}} \otimes \overline{\mathbf{N}}_{\boldsymbol{i}} \cong \mathrm{Hom}(V_{i+1}, V_i), \tag{10.54}$$
$$\overline{X}^m_{i+1,i} \in \mathbf{N_i} \otimes \overline{\mathbf{N}}_{\boldsymbol{i+1}} \cong \mathrm{Hom}(V_i, V_{i+1}). \tag{10.55}$$

Let us make an important observation: the form of the scalar fields are the same for every $m = 0, \ldots, 5$. This can be traced back to the fact that $R(g) = \zeta_3 \mathbb{1}_6$ so the R-symmetry action of $\mathbb{Z}_3$ is the same for all $m$.

Let us now study the four Weyl fermions $\psi^a$. Continue.

### 10.7.2 | Quiver

We can draw the quiver of this daughter theory. We have three types of bi-fundamental scalar fields:

$$X^m_{01} \in (\mathbf{N_1}, \overline{\mathbf{N}}_{\mathbf{0}}), \quad X^m_{12} \in (\mathbf{N_2}, \overline{\mathbf{N}}_{\mathbf{1}}), \quad X^m_{20} \in (\mathbf{N_0}, \overline{\mathbf{N}}_{\mathbf{2}}). \tag{10.56}$$

In each representation bi-fundamental representation there are six real scalars, i.e. 3 complex scalars. They are each represented by an arrow between the right representations, see fig. 4.

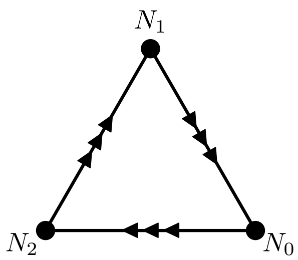



*Figure 4: Quiver of the $\mathbb{C}^3/\mathbb{Z}_3$ daughter theory.*

The adjacency matrix of this quiver is

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{bmatrix} \tag{10.57}$$

which is coherent with the fact that the coefficients of the McKay decomposition of $\rho_1 \oplus \rho_1 \oplus \rho_1$ are

$$\begin{aligned} n_{00} &= 0, & n_{01} &= 3 & n_{02} &= 0, \\ n_{10} &= 0, & n_{11} &= 0, & n_{12} &= 3, \\ n_{20} &= 3, & n_{21} &= 0, & n_{22} &= 0. \end{aligned} \tag{10.58}$$

### 10.7.3 Gauge anomaly cancellation

We get the three following conditions:

$$\mathrm{SU}(N_0) : N_2 - N_1 = 0, \tag{10.59}$$
$$\mathrm{SU}(N_1) : N_0 - N_2 = 0, \tag{10.60}$$
$$\mathrm{SU}(N_2) : N_1 - N_0 = 0. \tag{10.61}$$

which immediately imply

$$N_0 = N_1 = N_2. \tag{10.62}$$

From (10.44), we get that $N_0 = N_1 = N_2 = N/3$, meaning that that the daughter theory has quantum gauge symmetry if and only if the parent theory with has gauge group $\mathrm{SU}(N)$ where $N$ is a multiple of 3. Or, in other words, if the number of D-branes is a multiple of 3.

# 11 Conformal invariance

We have discussed the general case and found the resulting field content for strings on any orbifold $\mathbb{C}^3/\Gamma$. The content of those theories was found to be completely determined by the two actions that give ourselves from start:

- the action of $\Gamma$ on $\mathbb{C}^3$ that defines the orbifold per se and dictates how the "R-symmetry part" of the projection happens. This action is fixed by the choice of orbifold background we want.
- the action of $\Gamma$ on $\mathbb{C}^N$ (from some arbitrary $N$) that defines the embedding of $\Gamma$ in the gauge group $\mathrm{U}(N)$ and and dictates how the "gauge part" of the projection happens. Up until now, this action was completely arbitrary. In other words, the multiplicities $N_i$ are arbitrary, as long as they sum to $N$, which is arbitrary.

Let is discuss the role of the second representations in more details. If the stack of branes trivially sits at a fixed point, which in our case is usually the origin, there is no problem. On the other hand, if the stack of branes moves away from the origin, this will break the $\Gamma$ symmetry. The only the configurations that are able to move away from the fixed point are the ones that are still $\Gamma$-invariant while doing so, i.e. the configurations where the stack of branes possesses a an image-stack in each $\Gamma$ sector. Each of this stacks must be related by $\Gamma$ and they move all together in the covering space such as to be set to a unique point in the quotient space. Since there are $|\Gamma|$ sectors we need exactly $|\Gamma|$ branes in total, one per sector. This means that $N$, the total number of branes in the covering space, cannot be arbitrary anymore. It must be of the form $N = n|\Gamma|$, where $n$ now represents the number of brane in each stack. From the open string states point of view, this is means adding $n|\Gamma|$ Chan-Paton factors, resulting in an $U(n|\Gamma|)$ gauge symmetry. $\Gamma$ then acts on the Chan-Paton factors and switch them between themselves. We now need to make sure that the string theory is consistent with that action. Let us insist on the fact that we can take $N$ arbitrary but in order to have configurations that can move aways from the fixed point, $N$ must be a multiple of $\Gamma$, otherwise the stack of branes juste stays at the fixed point.

What is the link between moving axay from the fixed point and conformal invariance.

What does it have to do with the choice of embedding of $\Gamma$ in the gauge group? If branes move away from the origin .

complete

What are the conditions to get a superconformal daughter theory? The answer lies in the choice of the second action, i.e. on the choice of embedding of $\Gamma$ in the gauge group. One can show [4] that to have conformal invariance, we have to use the regular representation of $\Gamma$. The resulting embedding will be called *regular embedding*. This isonly possible of course if $N = |\Gamma|$. Actually, considering stacks of $N$ branes, we can consider that the rank of the gauge group is $N|\Gamma|$ (we change our $N$) so that the gauge group is $\mathrm{U}(N|\Gamma|)$. Recall that the multiplicity of each irredicible representation in the decomposition of the regular representation is the dimension of the irrep, i.e. $N_i = n_i \equiv \dim \rho_i$ meaning that the gauge group is broken to

$$G_{\text{proj}} = \bigotimes_{i \in I} \mathrm{SU}(N n_i) \tag{11.1}$$

at low energy. Let us be clear and mention that any choice of multiplicities $N_i$ is valid and gives a supersymmetric daughter theory, but this theory is conformal only if $N_i = n \dim \rho_i$.

Since the superconformal theories are of particular interest, this case have been studied in some details.

## 12 Lagrangian

For conformal only ?

Let us describe the lagrangian of the projected theory [4]. We saw that the gauge group is broken to

$$G_{\text{proj}} = \bigotimes_{i \in I} \mathrm{SU}(N n_i) \tag{12.1}$$

at low energy, with $n_i \equiv \dim \rho_i$. The coupling constant $\tau_i$ (including the theta angle as usual) of the $i$th group is

$$\tau_i = \frac{n_i \tau}{|\Gamma|} \tag{12.2}$$

where $\tau$ is the initial $\mathcal{N} = 4$ coupling, or the Type IIB coupling. This implies in particular that

$$\sum_{i \in I} n_i \tau_i = \tau. \tag{12.3}$$

There are Yukawa couplings for each triangle of the quiver consisting of two fermionic arrows and a bosonic arrow and quartic scalar interactions for each square consisting of four bosonic arrows. The coefficient ofthese interactions can be by projecting the $\mathcal{N} = 4$ lagrangian in terms of the field we kept. The Yukawa terms are therefore

$$Y = \sum_{i,j,k \in I} \gamma_{ijk}^{f_{ij}, f_{jk}, f_{kj}} \operatorname{tr}\Big(\psi^{ij}_{f_{ij}} \phi^{jk}_{f_{jk}} \psi^{ki}_{f_{ki}}\Big) \tag{12.4}$$

and the quartic scalar interaction terms by

$$V = \sum_{i,j,k,l \in I} \eta^{ijkl}_{f_{ij}; f_{jk}, f_{kl}, f_{li}} \operatorname{tr}\Big(\phi^{ij}_{f_{ij}} \phi^{jk}_{f_{jk}} \phi^{kl}_{f_{kl}} \phi^{li}_{f_{li}}\Big) \tag{12.5}$$

with

$$\gamma_{ijk}^{f_{ij}, f_{jk}, f_{kj}} = \Gamma_{\alpha\beta,m} (Y_{f_{ij}})^m{}_{v_i \overline{v}_j} (Y_{f_{jk}})^\beta{}_{v_j \overline{v}_k} (Y_{f_{ki}})^\alpha{}_{v_k \overline{v}_i}, \tag{12.6}$$

$$\eta^{ijkl}_{f_{ij}; f_{jk}, f_{kl}, f_{li}} = (Y_{f_{ij}})^{[m}{}_{v_i \overline{v}_j} (Y_{f_{jk}})^{n]}{}_{v_j \overline{v}_k} (Y^{f_{kl}})^{[m}{}_{v_k \overline{v}_l} (Y_{f_{li}})^{n]}{}_{v_l \overline{v}_i}. \tag{12.7}$$

The coefficients $(Y_{f_{ij}})^\alpha{}_{v_i \overline{v}_j}$ and $(Y_{f_{ij}})^m{}_{v_i \overline{v}_j}$ are to be understood as the $f_{ij}$th Clebsch-Gordan coefficients of the projection of $\mathbf{4} \otimes \rho_i$ and $\mathbf{6} \otimes \rho_i$ onto $\rho_j$, and $\Gamma_{\alpha\beta,m}$ is the invariant in $\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{6}$.

When we choose the regular embedding of $\Gamma$, the 1-loop beta functions have been computed in [4] and they o indeed vanish [4].

In the same paper, the superpotential inherited from the parent theory is shown to be

$$\mathcal{W} = \sum_{i,j,k \in I} \sum_{f_{ij}, f_{jk}, f_{ki}} h_{ijk}^{f_{ij}, f_{jk}, f_{kj}} \operatorname{tr}\Big(\phi^{ij}_{f_{ij}} \phi^{jk}_{f_{jk}} \phi^{ki}_{f_{ki}}\Big) \tag{12.8}$$

where

$$h_{ijk}^{f_{ij},f_{jk},f_{kj}} = \epsilon_{\alpha\beta\gamma}(Y_{f_{ij}})^{\alpha}{}_{v_i\bar{v}_j}(Y_{f_{jk}})^{\beta}{}_{v_j\bar{v}_k}(Y_{f_{ki}})^{\gamma}{}_{v_k\bar{v}_i} \tag{12.9}$$

and $(Y_{f_{ij}})^{\alpha}{}_{v_i\bar{v}_j}$ is now the $f_{ij}$th Clebsch-Gordan coefficient of the projection of $\mathbf{3} \otimes \rho_i$ onto $\rho_j$.

# 13 | Finitude

For any QFT, the Callan-Symanzik equations dictates the behavior, under the renormalization group flow, of the $n$-point correlator $G^{(n)}(\{\phi(x_i)\}; M, \lambda)$ forthe quantum fields $\phi(x)$ according the the renormalizationof the coupling $\lambda$ and momentum scale $M$:

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial \lambda} + n\gamma(\lambda)\right] G^{(n)}(\{\phi(x_i)\}; M, \lambda) = 0, \tag{13.1}$$

where the dimensionless functions $\beta$ and $\gamma$ are the $\beta$-function and the anomalous dimension respectively. As usuall, the determine how the shifts $\lambda \to \lambda + \delta\lambda$ in the coupling constant and $\phi(1+\delta\eta)\phi$ in the wave function compensate for the shift in the renomalization scale $M$:

$$\beta(\lambda) \equiv M\frac{\partial \lambda}{\partial M}, \qquad \gamma(\lambda) \equiv -M\frac{\partial \eta}{\partial M}. \tag{13.2}$$

Thee are three possible behaviours for the $\beta$-function when $\lambda$ is small: $\beta(\lambda) > 0$ then the theory has good IR behaviour and admits valid Feynman permutation theory at large distances, $\beta(\lambda) < 0$ then the theory is asymptotically free and has a good permutative behaviour in the UV limit and finally if $\beta(\lambda) = 0$ then the coupling constants do not flow and the renormalized couplings are always equal to the bare ones.

Theories in which no divergences can be associated with the coupling in the ultraviolet are called *finite* theories. To find such theories, supersymmetry is often a very god tool as it indices cancellation of the boon-fermion loop effects. More precisely, $\mathcal{N} = 4$ SYM have been shown to be finite to all orders, for $\mathcal{N} = 2$ SYM it has been shown that no higher than 1-loop corrections exist for the $\beta$-function (Adler-Bardeen theorem) and for $\mathcal{N} = 1$ theories the vanishing at 1-loop implies the vanishing at 2-loops. Finite non-supersymmetric theories has however also been porposed (see [5] for references). Of particular interest are theories which, in addition to a vanishing $\beta$-function, also have a vanishing anomalous dimensions. These theories are often part of a continuous manifold of scale-invariant theories and are characterized by the existence of exactly marginal operators (and hence dimensionless coupling constants). An important result is that there is a choice of coupling constant such that both the $\beta$-function and the anomalous dimensions vanish at first order, then the theory is necessarily finite at all orders.

From (10.31), we must have

$$\dim(\mathcal{R}) r_i = \sum_{j \in I} a_{ij}^{\mathcal{R}} r_j. \tag{13.3}$$

with $r_i \equiv \dim \rho_i$. It was shown (see **??** for references) that this equation necessitates the vanishing of the 1-loop $\beta$-function. In addition, we discussed that the remaining SUSY must be in the commutant of $\Gamma$ in SU(4) R-symmetry of the $\mathcal{N} = 4$ parent theory. It was shown in [4, 6] that the 1-loop $\beta$-function is proportional to the function $dr_i - a_{ij}^d r_j$, called the *discriminant function*:

$$\beta_{\text{1-loop}} \propto dr_i - a_{ij}^d r_j, \qquad d \equiv 4 - \mathcal{N}.$$

Since the vanishing of the $\beta$-function signifies finitude, we have in particular the following necessary conditions for finitude:

| SUSY | Finitude |
|---|---|
| $\mathcal{N} = 2$ | $2r_i = a_{ij}^2 r_j$ |
| $\mathcal{N} = 1$ | $3r_i = a_{ij}^3 r_j$ |
| $\mathcal{N} = 0$ | $4r_i = a_{ij}^4 r_j$ |

Note that in the case $\mathcal{N} = 2$ this condition is necessary and sufficient but not for $\mathcal{N} = 1$ and $\mathcal{N} = 0$ where one also have to chack the superpotential. In the non-supersymmetric case, vanishing of the 1-loop

$\beta$-function does not necessarily implies the vanishing of the following orders so we consider the notion finiteness in a weaker sense where only the leading order must vanish.

Finitude of quiver gauge theories constrcuted in from D-brane probing singularities, the Hanany-Witten setup or geometric engineering is very much linked the the mathematical properties of their quiver. See [7] for more details.

# 14 | Fractional branes

Rework/under this section

We saw that invariant configurations of D-branes naturally give rise to the regular representation of the orbifold group $\Gamma$ on the Chan-Paton factors. Correspondingly, these branes are called *regular branes*. The regular representation is not irreducible and it decomposes as

$$\rho_{\text{reg}} = \bigoplus_{i\in I} N_i \rho_i \tag{14.1}$$

with $N_i = \dim \rho_i$ in terms of the irreducible representations of $\Gamma$. One may then wonder about the existence of a more "elementary" set of D-branes such that the open strings attached to the latter carry Chan-Paton factors indices that transform in an irreducible representation. Those indeed do exist and are called *fractional D-branes*. They are BPS object that carry only a fraction of the charge with respect to the untwisted RR $(p+1)$-form of a regular brane but they are charged with respect to some twisted RR $(p+1)$-form, contrarily to the regular D-branes. They are stuck at the orbifold fixed point (where all twisted fields sit) since sitting elsewhere would require an invariant configuration in the covering space and those correspond to the regular representation, as we mentioned. So they cannot be fractional D-branes.

An important property of fractional D-branes is that they can be interpreted as higher-dimensional branes wrapped on exceptional cycles of the resolved space. In the orbifold limit, these exceptional cycles collapses leaving lower-dimensional fractional branes behind. This suggest a link between the spectrum of the D-branes and the homology of the resolved orbifold space. More precisely, it is linked to its homological K-theory. This is has been studied a lot.

From the point of view of boundary conformal field theory[8], *fractional D-branes* are reflected by boundary states in twisted sectors.

# 15 | $\mathcal{N} = 2$ daughter theories

## 15.1 | Generalities

Quotienting by a finite subgroup $\Gamma$ of SU(2) give rise to the so-called *Klein singularieties*. There also refered as *simple*, *du Val* or *ADE* singularieties, those are all synonims. To find (10.33), the only remaining choice is $\mathcal{R}$. The only thing we now is that these representations must come from the fundamental **4** (for fermions) or the anti-symmetric **6** of SU(4). Now, for a general decomposition

$$\mathcal{R} = \bigoplus_{i\in I} c_i \rho_i \tag{15.1}$$

we must have $\dim \mathcal{R} = \sum_{i\in I} \dim \rho_i$, i.e. the dimension of $\mathcal{R}$ (4 for the fermions and 6 for the bosons) must be partitioned into the dimension of the irreps of $\Gamma$ and, out of those possibilities, we must choose the ones that are define are a representation of the SU(2)SU(4), namely:

$$\begin{array}{ccc} \mathrm{SU}(4) & \to & \mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1) \\ \mathbf{4} & \to & (\mathbf{2},\mathbf{1})_{+1}\oplus(\mathbf{1},\mathbf{2})_{-1} \\ \mathbf{6} & \to & (\mathbf{1},\mathbf{1})_{+2}\oplus(\mathbf{1},\mathbf{1})_{-2}\oplus(\mathbf{2},\mathbf{2})_{0} \end{array} \tag{15.2}$$

[8]CFT on a manifold with boundary. Boundary conditions for open strings are interpreted as coherent states of the corresponding closed string 2d CFT.

where the subscripts correspond to the U(1) factor (i.e. the trace) and in particular the $\pm$ dictates the overall traceless condition. Exploiting these decompositions and paying with the reppresentation properties, one can show that we must have [8]

$$\mathbf{4} = \mathbf{2}_{\text{trivial}} \oplus \mathbf{2} \tag{15.3}$$

$$\mathbf{6} = \mathbf{2}_{\text{trivial}} \oplus \mathbf{2} \oplus \mathbf{2}. \tag{15.4}$$

This limits our attention to only 2-dimensional representations of $\Gamma$. However there are still many possiblities.

$$a_{ij}^{\mathbf{4}} = \tag{15.5}$$

## 15.2 $S = \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$

We consider a representation $(\rho, \mathbb{C}^N)$ of $\mathbb{Z}_n$. We decompose it on the set of irreducible representations of $\mathbb{Z}_n$ as

$$(\rho, \mathbb{C}^n) = \bigoplus_{i=0}^{n-1} N_i(\rho_i, \mathbb{C}). \tag{15.6}$$

In other words, it is equivalent to the representation

$$\rho(g) = \left[\begin{array}{ccccc} 1 & & \cdots & & 0 \\ & \ddots & & & \\ & & 1 & & \\ \vdots & & \ddots & & \vdots \\ & & & \zeta_n^{n-1} & \\ & & & & \ddots \\ 0 & & \cdots & & \zeta_n^{n-1} \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\ \ddots\\ 1\end{array}}\right\} N_0 \\ \vdots \\ \left.\vphantom{\begin{array}{c}\zeta\\ \ddots\\ \zeta\end{array}}\right\} N_{n-1} \end{array} . \tag{15.7}$$

Since $\dim \rho_i = 1$, $\sum_i N_i = N$.

The gauge field configurations that are left invariant under the action of $\mathbb{Z}_n$ are therefore the ones that satisfy

$$\rho(g) A_\mu \rho(g)^{-1} = A_\mu. \tag{15.8}$$

The constrained is easily solved by using the bi-index notation:

$$A_{\mu;i\alpha_i,i\beta_j} \mapsto \rho_i(g) A_{\mu;i\alpha_i,j\beta_j} \rho_j(g)^{-1} = \zeta_n^{i-j} A_{\mu;i\alpha_i,j\beta_j}. \tag{15.9}$$

thus only the configurations with $A_{\mu;i\alpha_i,j\beta_j} = 0$ if $i \neq j$ are invariant. The gauge field has therefore a block diagonal form:

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & & \\ & A_{\mu;11} & & \\ & & \ddots & \\ & & & A_{\mu;n-1,n-1} \end{bmatrix} \tag{15.10}$$

with $A_{\mu;ij} \equiv (A_{\mu;i\alpha_i,j\beta_j})_{\alpha_i=0,\ldots,N_i,\beta_j=0,\ldots,N_j}$. The block $A_{ii}$ transforms under $\mathbb{Z}_n$ as $(\rho_i, V_i)^{N_i}$. For now it is only a simple generalization of the case $\mathbb{C}^3/\mathbb{Z}_3$. This makes sense: projection of the gauge field only depends the discrete group $\Gamma$, not on the way it acts on $\mathbb{C}^3$ because it does not transform under R-symmetry.

The gauge group is now broken to

$$G_{\text{proj}} = \prod_{i=0}^{n-1} \mathrm{U}(N_i). \tag{15.11}$$

Now for the scalar fields. The action of $\mathbb{Z}_n$ that we consider leaves the first component of $\mathbb{C}^3$ untouched so we take the action $\mathbf{1} \oplus \mathbf{2}$ where $\mathbf{2}$ is the usual action of $\mathbb{Z}_n$ on $\mathbb{C}^2$. In other words,

$$R(g) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_n & 0 \\ 0 & 0 & \zeta_n^{-1} \end{bmatrix}. \tag{15.12}$$

Or, equivalently, $R(g)^m{}_n = \delta^m{}_n A_n$ with $A_m = (1, 1, \zeta_n, \zeta_n, \zeta_n^{-1}, \zeta_n^{-1})$. The scalar field configurations that are left invariant satisfy

$$R(g)^m{}_n \rho(g) X^n \rho(g)^{-1} = X^m \tag{15.13}$$

for all $g \in \mathbb{Z}_n$. Using the bi-index notations, this becomes

$$X^m_{i\alpha_i, j\beta_j} \mapsto \delta^m{}_n A_n \rho_i(g) X^m_{i\alpha_i, j\beta_j} \rho_j(g)^{-1} = \delta^m{}_n A_n \zeta_n^{i-j} X^m_{i\alpha_i, j\beta_j} = \begin{cases} \zeta_n^{i-j} X^m_{i\alpha_i, j\beta_j}, & m = 0, 1 \\ \zeta_n^{i-j+1} X^m_{i\alpha_i, j\beta_j}, & m = 2, 3 \\ \zeta_n^{i-j-1} X^m_{i\alpha_i, j\beta_j}, & m = 4, 5 \end{cases} \tag{15.14}$$

$$\overline{X}^m_{i\alpha_i, j\beta_j} \mapsto \delta^m{}_n \overline{A_n} \rho_i(g) \overline{X}^m_{i\alpha_i, j\beta_j} \rho_j(g)^{-1} = \delta^m{}_n \overline{A_n} \zeta_n^{i-j} X^m_{i\alpha_i, j\beta_j} = \begin{cases} \zeta_n^{i-j} X^m_{i\alpha_i, j\beta_j}, & m = 0, 1 \\ \zeta_n^{i-j-1} X^m_{i\alpha_i, j\beta_j}, & m = 2, 3 \\ \zeta_n^{i-j+1} X^m_{i\alpha_i, j\beta_j}, & m = 4, 5 \end{cases} \tag{15.15}$$

thus only the configurations with $X^{0,1}_{i\alpha_i, j\beta_j} = 0$ if $i - j \neq 0$, $X^{2,3}_{i\alpha_i, j\beta_j} = 0$ if $i - j + 1 \neq 0$ and $X^{4,5}_{i\alpha_i, j\beta_j} = 0$ if $i - j - 1 \neq 0$ are left invariant (and similarly for the conjugated fields). The scalar fields $X$ have a the following forms:

$$X^{0,1} = \begin{bmatrix} X^{0,1}_{00} & & 0 \\ & \ddots & \\ 0 & & X^{0,1}_{n-1,n-1} \end{bmatrix}, \tag{15.16}$$

$$X^{2,3} = \begin{bmatrix} 0 & X^{2,3}_{01} & & 0 \\ & \ddots & \ddots & \\ & & 0 & X^{2,3}_{n-2,n-1} \\ X^{2,3}_{n-1,0} & & & 0 \end{bmatrix}, \quad X^{4,5} = \begin{bmatrix} 0 & & & X^{4,5}_{0,n-1} \\ X^{4,5}_{10} & 0 & & \\ & \ddots & \ddots & \\ 0 & & X^{4,5}_{n-1,n-2} & 0 \end{bmatrix}, \tag{15.17}$$

$$\overline{X}^{0,1} = \begin{bmatrix} \overline{X}^{0,1}_{00} & & 0 \\ & \ddots & \\ 0 & & \overline{X}^{0,1}_{n-1,n-1} \end{bmatrix}, \tag{15.18}$$

$$\overline{X}^{2,3} = \begin{bmatrix} 0 & & & \overline{X}^{2,3}_{0,n-1} \\ \overline{X}^{2,3}_{10} & 0 & & \\ & \ddots & \ddots & \\ 0 & & \overline{X}^{2,3}_{n-1,n-2} & 0 \end{bmatrix}, \quad \overline{X}^{4,5} = \begin{bmatrix} 0 & \overline{X}^{4,5}_{01} & & 0 \\ & \ddots & \ddots & \\ & & 0 & \overline{X}^{4,5}_{n-2,n-1} \\ \overline{X}^{4,5}_{n-1,0} & & & 0 \end{bmatrix} \tag{15.19}$$

so $X^m_{ij}$ is an $N_i \times N_j$ block and transforms under the representation $(\mathbf{N_i}, \overline{\mathbf{N}}_{\boldsymbol{j}})$ of $\mathrm{U}(N_i) \times \mathrm{U}(N_j)$:

$$X^{0,1}_{i,i} \in \mathbf{N_i} \otimes \overline{\mathbf{N}}_{\boldsymbol{i}} \cong \mathrm{Hom}(V_i, V_i), \tag{15.20}$$

$$X^{2,3}_{i,i+1} \in \mathbf{N_{i+1}} \otimes \overline{\mathbf{N}}_{\boldsymbol{i}} \cong \mathrm{Hom}(V_{i+1}, V_i), \tag{15.21}$$

$$X^{4,5}_{i+1,i} \in \mathbf{N_i} \otimes \overline{\mathbf{N}}_{\boldsymbol{i+1}} \cong \mathrm{Hom}(V_i, V_{i+1}). \tag{15.22}$$

So the scalar fields are split up in three families depending on the way they transform under R-symmetry. We now see a big difference with the case $\mathbb{C}^3/\mathbb{Z}_3$: since the R-symmetry does not act the same way on each directions in $\mathbb{C}^3$, the invariant scalar field configurations are not the same in each direction either.

Let us draw the quiver for the case $n = 3$ so that we can compare to 4. We have $2 \cdot 9 = 18$ real scalar fields in 9 different representations:

$$X^0_{00}, X^1_{00} \in (\mathbf{N_0}, \overline{\mathbf{N}}_{\mathbf{0}}), \quad X^0_{11}, X^1_{11} \in (\mathbf{N_1}, \overline{\mathbf{N}}_{\mathbf{1}}), \quad X^0_{22}, X^1_{22} \in (\mathbf{N_2}, \overline{\mathbf{N}}_{\mathbf{2}}), \tag{15.23}$$

$$X_{01}^2, X_{01}^3 \in (\mathbf{N_1}, \overline{\mathbf{N_0}}), \quad X_{12}^2, X_{12}^3 \in (\mathbf{N_2}, \overline{\mathbf{N_1}}), \quad X_{20}^2, X_{20}^3 \in (\mathbf{N_0}, \overline{\mathbf{N_2}}), \tag{15.24}$$

$$X_{10}^4, X_{10}^5 \in (\mathbf{N_0}, \overline{\mathbf{N_1}}), \quad X_{21}^4, X_{21}^5 \in (\mathbf{N_1}, \overline{\mathbf{N_2}}), \quad X_{02}^4, X_{02}^5 \in (\mathbf{N_2}, \overline{\mathbf{N_0}}), \tag{15.25}$$

We now only have 1 complex scalar in each representation and the quiver is given by 5.

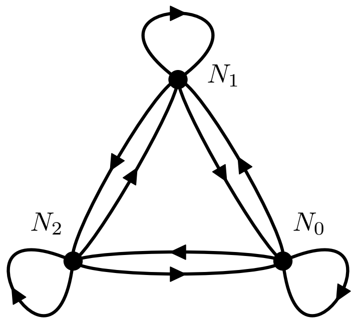



*Figure 5: Quiver of the $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_3$ daughter theory.*

It is easy to see how the construction of the the quiver generalizes for arbitrary $n$. The adjacency matrix is

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix} \tag{15.26}$$

which is, as it should, coincides with the McKay decomposition of $\mathbf{1} \oplus \mathbf{2}$:

$$\begin{array}{ccc} n_{00} = 1, & \ldots, & n_{0,n-1} = 1, \\ \vdots & & \vdots \\ n_{n-1,0} = 1, & \ldots, & n_{n-1,n-1} = 1. \end{array} \tag{15.27}$$

Gauge anomaly cancellation now imposes that

$$N_{i-1} - N_{i-1} + N_{i+1} + N_{i+1} = 0 \tag{15.28}$$

for $i = 0, \ldots, n-1$. Those constrains are always satisfied so the the factors $N_i$ are arbitrary, as long as $\sum_i N_i = N$ of course.

This quiver is easy to scale up for any $n$. We therefore implement that into Mathematica and get the quivers for any $n$.

Looking at (15.20)-(15.20), we see that for each factor in the projected gauge group we have 1 complex scalar transforming in the adjoint (Coulomb branch) and two complex scalars in bi-fundamentals (Higgs branch). The former is the scalar in the $\mathcal{N} = 2$ vector multiplet and the latter are the complex scalars in of two hypermultiplets. The worldvolume theory is therefore $\mathcal{N} = 2$ SYM with two hypermultiplets.

## 15.3 | $S = \mathbb{C} \times \mathbb{C}^2/\mathcal{D}_n$

We quotient $\mathbb{C}^3$ by the binary dihedral group $\mathcal{D}_n$ that acts on the last two components. This group is geneated by two elements that we denote $A$ and $B$. Firther details on its structure and representations are given in appendix B.1. A set of irreducible representations of $\mathcal{D}_n$ is given by

$$\{(\rho_i, V_i)\}_{i=0,\ldots,n+2} \tag{15.29}$$

with $V_i = \mathbb{C}$ for $i = 0, \ldots, 3$ and $V_i = \mathbb{C}^2$ for $i = 4, \ldots, n+2$, so there are 4 one-dimensional representations and $n-1$ two-dimensional representations. It is more convenient to treat the 1-dimensional and 2-dimensional representations idependently. We therefore denote by $\sigma_i$ with $i = 0, \ldots, 3$ the 1-dimensional representations and by $\mu_r$ with $r = 1, \ldots, n-1$ the 2-dimensional ones:

| $a$ | $\sigma_a(A)$ | $\sigma_a(B)$ ($n$ even or odd) |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | $-1$ |
| 2 | $-1$ | 1 or $i$ |
| 3 | $-1$ | $-1$ or $-i$ |

$$\mu_r(A) = \begin{bmatrix} e^{i\frac{\pi}{n}r} & 0 \\ 0 & e^{-i\frac{\pi}{n}r} \end{bmatrix}$$

$$\rho_r(B) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We note that there is a slight difference for the 1-dimensional representations if $n$ is even or odd.

Any representation of $(\rho, \mathbb{C}^N)$ of $\mathcal{D}_n$ can be decomposed as

$$(\rho, \mathbb{C}^n) = \left( \bigoplus_{a=0}^{3} (\sigma_a, \mathbb{C}) \right) \oplus \left( \bigoplus_{r=1}^{n-1} (\mu_r, \mathbb{C}^2) \right). \tag{15.30}$$

The invariant gauge field configurations were found to be of the form

$$A_\mu = \begin{bmatrix}
\times & 0 & 0 & 0 & & & & & & & \cdots & 0 \\
0 & \times & 0 & 0 & & & & & & & & \\
0 & 0 & \times & 0 & & & & & & & & \\
0 & 0 & 0 & \times & & & & & & & & \\
 & & & & x_1 & 0 & & & & & & \\
 & & & & 0 & -x_1 & & & & & & \\
 & & & & & & x_2 & 0 & & & & \\
 & & & & & & 0 & -x_2 & & & & \\
\vdots & & & & & & & & & \ddots & & \vdots \\
 & & & & & & & & & & x_{n-1} & 0 \\
0 & & & & & & & & & & 0 & -x_{n-1}
\end{bmatrix}. \tag{15.31}$$

where each entry $(i, j)$ is an arbitrary block of size $N_i \times N_j$. The explicit derivations are carried out in appendix F.1. The projected gauge group is then

$$G_{\text{proj}} = \mathrm{SU}(N_0) \times \ldots \mathrm{SU}(N_3) \times \mathrm{SU}(2N_4) \times \cdots \times \mathrm{SU}(2N_{n+3}). \tag{15.32}$$

It is interesting to notice the main differences with the case of $\mathbb{Z}_n$:

- since $2\mathcal{D}_n$ is not abelian, it can have irreps of two or more dimension (which it does). This implies that the gauge field bloacks $A_{\mu;ij}$ are not necessarily diagonal. This will have important consequence for the residual gauge group.
- again because some irreps are mode then 1-dimensional, we can act on $\mathbb{C}^3$ with non-diagonal actions (which we do). This means that components of the different complex scalars can be exchanged under R-symmetry.

As we mentioned previously, there is no constrain on the ranks of the gauge groups coming from the the anomaly condition, it is automatically anomaly-free.

### 15.4 | $S = \mathbb{C} \times \mathbb{C}^2/\mathcal{T}, \mathcal{O}, \mathcal{I}$

Continue.

## 16 | $\mathcal{N} = 1$ daughter theories

### 16.1 | $S = \mathbb{C}^3/\mathbb{Z}_n$

Let us now consider the general case of $\mathbb{Z}_n$ acting on $\mathbb{C}^3$, i.e. we want to generalize the case that we treated in section 10.7.

### 16.1.1 Actions of $\mathbb{Z}_n$ on $\mathbb{C}^3$

The first thing to do is to specify a representation of $\mathbb{Z}_n$ on $\mathbb{C}^3$. Any representation $(R, \mathbb{C}^3)$ can be decomposed as

$$R = \bigoplus_{k=0}^{n-1} \rho_k^{\oplus n_k} \tag{16.1}$$

and is therefore equivalent to a block-diagonal representation. This implies that $\sum_k n_k = 3$ so $R$ can always be written as

$$R(g) = (\rho_a \oplus \rho_b \oplus \rho_c)(g) = \begin{bmatrix} \zeta_n^a & 0 & 0 \\ 0 & \zeta_n^b & 0 \\ 0 & 0 & \zeta_n^c \end{bmatrix} \tag{16.2}$$

with $a, b, c$ some arbitrary exponents. We denote the representation (16.2) by $(a, b, c)$. Let us make a few remarks on these representations:

- Since we consider $\mathbb{Z}_n$ as a subgroup of SU(3), we must have

  $$(a + b + c) \mod n = 0 \tag{16.3}$$

  such that $\det(R(g)) = 1$.
- Taking $a$ or $a + n$ gives the same representation and the same is true for $b$ and $c$, so we we can actually restrict ourselves to $0 < a, b, c < n$. We can trade the strict inequalities and allow the values 0 or $n$ if we want to consider trivial representations as well. In this case, we will find that at least one direction of $\mathbb{C}^3$ is left untouched, i.e. we are in the situation where the orbifold is actually $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_n$, which we will not consider here.
- Permuting $a, b, c$ gives equivalent representations so we can take $a, b, c$ to be ordered, so $0 < a \leq b \leq c < n$.

The problem has now been reformulated as follows: we are looking for all possible ordered triplets $(a, b, c)$ such that $0 < a \leq b \leq c < n$ and (16.3). For $n = 1$ and $n = 2$, it is clear that this is not possible, the possibilities involve at least one trivial representation. For $n = 3$, the only possibility is $(1, 1, 1)$, the one we used in 10.7. What about bigger values of $n$? First, note that $0 < a \leq b \leq c < n$ implies that the maximum value of $a + b + c$ is $3n - 1$. From equation (16.3), the only two possibilities are then $a + b + c = n$ or $a + b + c = 2n$. What simplifies the analysis is that the second case can actually be ignored because it is already being taken care of by the first case. Let us explain that in more details: . details

Now that we have seen that the only possibility is $a + b + c = n$, we find that the maximum value that $a$ can take is $\lfloor n/3 \rfloor$, so $a = 1, \dots, \lfloor n/3 \rfloor$. $c$ can then be expressed in terms of $a$ and $b$ as $c = n - a - b$. The constraint $b \leq c$ then implies $b \leq (n-a)/2$, i.e. $b = a, \dots, \lfloor (n-a)/2 \rfloor$. To summarize, the representations are all the representations of the form

$$(a, b, n - a - b) \tag{16.4}$$

with $a = 1, \dots, \lfloor n/3 \rfloor$ and $b = a, \dots, \lfloor (n - a)/2 \rfloor$. For small values, we get

$$\begin{aligned}
&\mathbb{Z}_1 : \text{necessarily involves a trivial representation} \\
&\mathbb{Z}_2 : \text{necessarily involves a trivial representation} \\
&\mathbb{Z}_3 : (1, 1, 1) \\
&\mathbb{Z}_4 : (1, 1, 2) \\
&\mathbb{Z}_5 : (1, 1, 3), (1, 2, 2) \\
&\mathbb{Z}_6 : (1, 1, 4), (1, 2, 3), (2, 2, 2) \\
&\mathbb{Z}_7 : (1, 1, 5), (1, 2, 4), (1, 3, 3), (2, 2, 3) \\
&\quad \vdots
\end{aligned}$$

For an arbitrary $n$, the total number of different representation is

Correct this.

$$\sum_{a=1}^{\lfloor n/3 \rfloor} \left\lfloor \frac{n-3a}{2} + 1 \right\rfloor = \begin{cases} 3k^2, & \text{if } n = 6k \\ 3k^2 + k, & \text{if } n = 6k+1, \\ 3k^2 + 2k, & \text{if } n = 6k+2, \\ 3k^2 + 3k + 1, & \text{if } n = 6k+3, \\ 3k^2 + 4k + 1, & \text{if } n = 6k+4, \\ 3k^2 + 5k + 2, & \text{if } n = 6k+5 \end{cases} \tag{16.5}$$

The details are presented in appendix F.2.

### 16.1.2 | Example: $\mathbb{Z}_5$

For $\mathbb{Z}_5$, we saw that there are two nonequivalent ways of acting on $\mathbb{C}^3$, see (**??**). Let us first consider the action $(1, 1, 3)$

$$R(g) = \begin{bmatrix} \zeta_5 & 0 & 0 \\ 0 & \zeta_5 & 0 \\ 0 & 0 & \zeta_5^3 \end{bmatrix} \tag{16.6}$$

i.e. $R = \rho_1 \oplus \rho_1 \oplus \rho_3$.

For the gauge field, the reasoning is exactly the same than for $\mathbb{Z}_3$ and we get

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & \\ & \ddots & \\ & & A_{\mu;44} \end{bmatrix} \tag{16.7}$$

where each block $A_{\mu;ii}$ is of size $N_i \times N_i$. The projected gauge group is

$$G_{\text{proj}} = \mathrm{U}(N_0) \times \mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \mathrm{U}(N_3) \times \mathrm{U}(N_4). \tag{16.8}$$

The scalar fields transform as

$$X^m_{i\alpha_i, j\beta_j} \to R(g)^m{}_n \rho_i(g) X^n_{i\alpha_i, j\beta_j} \rho_j(g)^{-1} \tag{16.9}$$

so the invariant configurations must satisfy

$$X^m_{i\alpha_i, j\beta_j} = R(g)^m{}_n \zeta_4^{i-j} \rho_i(g) X^n_{i\alpha_i, j\beta_j} = \begin{cases} \zeta^{i-j+1} X^m_{i\alpha_i, j\beta_j}, & \text{if } m = 0, 1, 2, 3 \\ \zeta^{i-j+3} X^m_{i\alpha_i, j\beta_j}, & \text{if } m = 4, 5. \end{cases} \tag{16.10}$$

and are therefore of the form

$$X^m = \begin{bmatrix} 0 & X^m_{01} & 0 & 0 & 0 \\ 0 & 0 & X^m_{12} & 0 & 0 \\ 0 & 0 & 0 & X^m_{23} & 0 \\ 0 & 0 & 0 & 0 & X^m_{34} \\ X^m_{40} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16.11}$$

for $m = 0, 1, 2, 3$ and

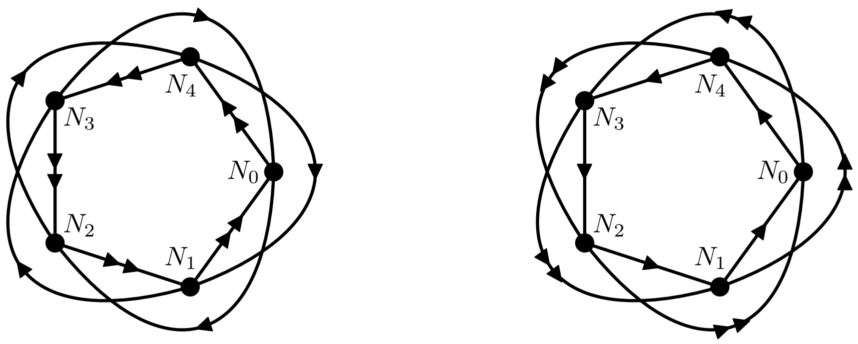
$$X^m = \begin{bmatrix} 0 & 0 & 0 & X^m_{03} & 0 \\ 0 & 0 & 0 & 0 & X^m_{14} \\ X^m_{20} & 0 & 0 & 0 & 0 \\ 0 & X^m_{31} & 0 & 0 & 0 \\ 0 & 0 & X^m_{42} & 0 & 0 \end{bmatrix} \tag{16.12}$$

for $m = 4, 5$. Gauge anomaly cancellation imposes

$$-N_{i-2} + N_{i+2} - 2N_{i+1} + 2N_{i-1} = 0 \tag{16.13}$$

for $i = 0, 1, 2, 3, 4$ which implies that

$$N_0 = N_1 = N_2 = N_3 = N_4. \tag{16.14}$$




*(a)* $(1, 1, 3)$ *(b)* $(1, 2, 2)$

*Figure 6: Quivers of the* $\mathbb{C}^3/\mathbb{Z}_5$ *daughter theories.*

Now if we choose the representation $(1, 2, 2)$, i.e.

$$R(g) = \begin{bmatrix} \zeta_5 & 0 & 0 \\ 0 & \zeta_5^2 & 0 \\ 0 & 0 & \zeta_5^2 \end{bmatrix} \tag{16.15}$$

we get instead that the invariant field configurations must satisfy

$$X^m_{i\alpha_i,j\beta_j} = R(g)^m{}_n \zeta_4^{i-j} \rho_i(g) X^n_{i\alpha_i,j\beta_j} = \begin{cases} \zeta^{i-j+1} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 0, 1 \\ \zeta^{i-j+2} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 2, 3, 4, 5. \end{cases} \tag{16.16}$$

and are therefore of the form

$$X^m = \begin{bmatrix} 0 & X^m_{01} & 0 & 0 & 0 \\ 0 & 0 & X^m_{12} & 0 & 0 \\ 0 & 0 & 0 & X^m_{23} & 0 \\ 0 & 0 & 0 & 0 & X^m_{34} \\ X^m_{40} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{16.17}$$

for $m = 0, 1$ and

$$X^m = \begin{bmatrix} 0 & 0 & X^m_{02} & 0 & 0 \\ 0 & 0 & 0 & X^m_{13} & 0 \\ 0 & 0 & 0 & 0 & X^m_{24} \\ X^m_{30} & 0 & 0 & 0 & 0 \\ 0 & X^m_{41} & 0 & 0 & 0 \end{bmatrix} \tag{16.18}$$

for $m = 2, 3, 4, 5$. Gauge anomaly cancellation imposes

$$-2N_{i+2} + 2N_{i-2} - N_{i+1} + N_{i-1} = 0 \tag{16.19}$$

for $i = 0, 1, 2, 3, 4$ which implies that

$$N_0 = N_1 = N_2 = N_3 = N_4. \tag{16.20}$$

Right conditions?

Note that, even we thought that $(1, 1, 3)$ and $(1, 2, 2)$ were different representation, that are actually equivalent in the sense that $(1, 1, 3) + (1, 1, 3) = (2, 2, 1)$. The two quivers in fig. 9 should therefore be the same. And indeed, upon further inspection, the two are the same. To see this, we can rename the vertices in the second graph as $N_1 \to N_3, N_2 \to N_1, N_3 \to N_4, N_4 \to N_2$. This renaming defines the bijection between the two graphs. A more pragmatic way to see this is to look at the adjacency matrices:

$$a_{(1,1,3)} = \begin{bmatrix} 0 & 0 & 1 & 0 & 2 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \qquad a_{(1,2,2)} = \begin{bmatrix} 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix}. \tag{16.21}$$

Changing the names of the vertices is equivalent to swapping line and columns. For example,

### 16.1.3 General $\mathbb{Z}_n$

Let us now consider the general action

$$R(g) = \begin{bmatrix} \zeta_n^a & 0 & 0 \\ 0 & \zeta_n^b & 0 \\ 0 & 0 & \zeta_n^c \end{bmatrix} \tag{16.22}$$

of $\mathbb{Z}_n$ on $\mathbb{C}^3$, where $(a, b, c)$ one of the representation that we studied before. In particular, recall that $a + b + c = n$. Following the same reasoning than before, we get that the gauge field of of the form

$$A_\mu = \begin{bmatrix} A_{\mu;00} & & \\ & \ddots & \\ & & A_{\mu;n-1,n-1} \end{bmatrix}. \tag{16.23}$$

Invariant scalar field configurations transform as

$$X^m_{i\alpha_i,j\beta_j} = \begin{cases} \zeta_n^{i-j+a} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 0,1 \\ \zeta_n^{i-j+b} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 2,3 \\ \zeta_n^{i-j+c} X^m_{i\alpha_i,j\beta_j}, & \text{if } m = 4,5 \end{cases} \tag{16.24}$$

so

$$X^{0,1}_{j-a,j} \in (\mathbf{N}_{\boldsymbol{j}}, \overline{\mathbf{N}}_{\boldsymbol{j-a}}), \tag{16.25}$$
$$X^{2,3}_{j-b,j} \in (\mathbf{N}_{\boldsymbol{j}}, \overline{\mathbf{N}}_{\boldsymbol{j-b}}), \tag{16.26}$$
$$X^{4,5}_{j-c,j} \in (\mathbf{N}_{\boldsymbol{j}}, \overline{\mathbf{N}}_{\boldsymbol{j-c}}) \tag{16.27}$$

are the only possible non-vanishing components. This allows us to quickly draw all the possible quivers for a given $n$. Once again, the difficulty is only computational, not conceptual. This can therefore easily be implemented into Mathematica and we get the quiver for any $n$ and any representation $(a, b, c)$.

## 16.2 | $S = \mathbb{C}^3/\Delta(3n^2), \Delta(6n^2)$

## 16.3 | $S = \mathbb{C}^3/\Sigma_{36\times3}, \Sigma_{60\times3}, \Sigma_{168\times3}, \Sigma_{216\times3}, \Sigma_{360\times3}$

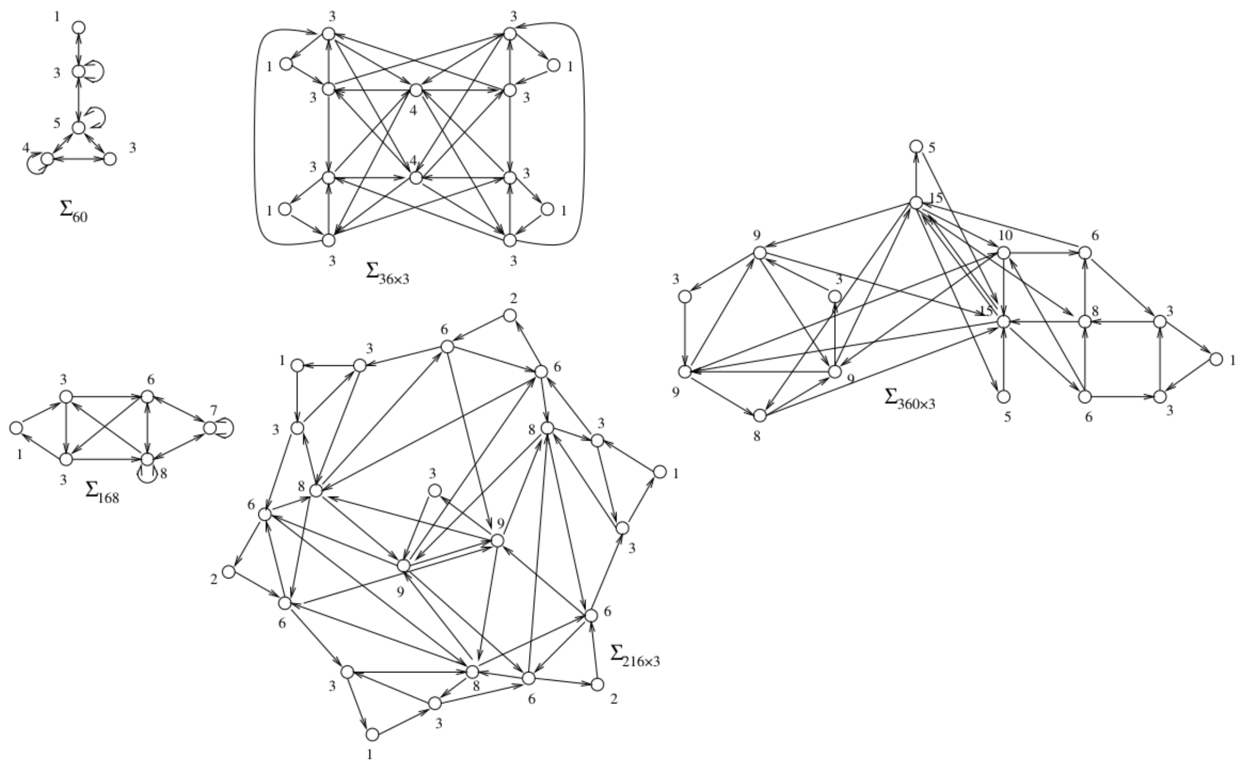



*Figure 7: Quivers of the exceptional finite subgroups of $SU(3)$.*

## 16.4 | $S = \mathbb{C}^3/(\mathbb{Z}_m \times \mathbb{Z}_n)$

### 16.4.1 | Representations of $\mathbb{Z}_m \times \mathbb{Z}_n$

We consider the group $\mathbb{Z}_m \times \mathbb{Z}_n$. Let us denote by $\{\mu_i\}_{i=0,\ldots,m-1}$ and $\{\sigma_j\}_{j=0,\ldots,n-1}$ two complete set of irreducible representation of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively, with

$$\mu(g_1) = \zeta_m^i \tag{16.28}$$

$$\sigma_j(g_2) = \zeta_n^j \tag{16.29}$$

where $g_1$ and $g_2$ are the generating elements of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively. $\mathbb{Z}_m \times \mathbb{Z}_n$ is of order $mn$ and possesses the same number of equivalency classes (abelian). It has therefore $mn$ irreducible representations. Since the group is abelian, they are all of dimension 1. Let us denote by $\{T_k\}_{k=0,\ldots,m+n-1}$ a complete set of irreducible representations. Since we have a product group, for any $k$ there exists indices $i(k)$ and $j(k)$ such that $T_k = \mu_{i(k)} \otimes \sigma_{j(k)}$. We choose the indices $i(k)$ and $j(k)$ such that

$$\begin{aligned}
T_0 &= \mu_0 \otimes \mu_0, \\
T_1 &= \mu_0 \otimes \mu_1, \\
T_2 &= \mu_0 \otimes \mu_2, \\
&\vdots \\
T_{n-1} &= \mu_0 \otimes \mu_{n-1}, \\
T_n &= \mu_1 \otimes \mu_0,
\end{aligned}$$

$$\vdots$$
$$T_{2n-1} = \mu_1 \otimes \mu_{n-1},$$
$$\vdots$$
$$T_{mn-1} = \mu_{m-1} \otimes \mu_{n-1}.$$

That is, we take

$$\begin{cases} i(k) & = \lfloor k/n \rfloor \\ j(k) & = k \mod n \end{cases} \qquad \Leftrightarrow \qquad k = i(k)n + j(k). \tag{16.30}$$

Note that this simply a dictionary between a line notation and a matrix notation and that it is indeed a bijection $k \Leftrightarrow i, j$. In this way, we can proceed to similar manipulations than before, were we used line notations.

Any representation $R$ of $\mathbb{Z}_m \times \mathbb{Z}_n$ can be decomposed as

$$R = \bigoplus_{i,j} N_k T_k = \bigoplus_{i,j} N_{ij} (\mu_i \otimes \sigma_j) \tag{16.31}$$

with $N_k = N_{i(k)j(k)}$. We must have $\sum_k N_k = 3$. In other words, any representation of $\mathbb{Z}_m \times \mathbb{Z}_n$ on $\mathbb{C}^3$ is equivalent to

$$R(g_1, g_2) = [(\mu_a \otimes \sigma_{a'}) \oplus (\mu_{b'} \otimes \sigma_{b'}) \oplus (\mu_c \otimes \sigma_{c'})](g_1, g_2) = \begin{bmatrix} \xi_m^a \xi_n^{a'} & 0 & 0 \\ 0 & \xi_m^b \xi_n^{b'} & 0 \\ 0 & 0 & \xi_m^c \xi_n^{c'} \end{bmatrix}. \tag{16.32}$$

The determinant condition is

$$\xi_m^{a+b+c} = \xi_n^{-a'-b'-c'} \tag{16.33}$$
$$\Leftrightarrow (a+b+c) \mod m = (a'+b'+c') \mod n \tag{16.34}$$

$$R(g_1, g_2) = \begin{bmatrix} \xi_m & 0 & 0 \\ 0 & \xi_n & 0 \\ 0 & 0 & \xi_m^{-1} \xi_n^{-1} \end{bmatrix}. \tag{16.35}$$

### 16.4.2 | Projection

Let us start by the gauge field. We consider a unitary representation $(\rho, \mathbb{C}^N)$ of $\mathbb{Z}_m \times \mathbb{Z}_n$ on $\mathbb{C}^n$ and decompose it as

$$\rho = \bigoplus_{i,j} N_k T_k \tag{16.36}$$

such that

$$\rho(g) = \begin{array}{c} \begin{bmatrix} T_0(g) & & & \cdots & & & 0 \\ & \ddots & & & & & \\ & & T_0(g) & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & T_{mn-1}(g) & & \\ & & & & & \ddots & \\ 0 & & & \cdots & & & T_{mn-1}(g) \end{bmatrix} \end{array} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} N_0 \\ \vdots \\ \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} N_{mn-1} \end{array}. \tag{16.37}$$

We can now use our usual bi-index notation $A_{\mu;k\alpha_k,l\beta_l}$ with $k, l = 0, \ldots mn-1$ and $\alpha_k, \beta_k = 0, \ldots, N_k$ but instead it is more convenient to come back to our matrix notation by writing the block $A_{\mu;k,l}$ as

$A_{\mu;i(k)j(k),i(l)j(l)}$ that we simply denote by $A_{\mu;ij,i'j'}$ with $i, i' \in \{0, \ldots, n-1\}$ and $j, j' \in \{0, \ldots, n-1\}$. So for $m = 2, n = 3$ for example, the link between the two notations is

$$\begin{bmatrix} A_{\mu;00} & A_{\mu;01} & A_{\mu;02} & A_{\mu;03} & A_{\mu;04} & A_{\mu;05} \\ A_{\mu;10} & A_{\mu;11} & A_{\mu;12} & A_{\mu;13} & A_{\mu;14} & A_{\mu;05} \\ A_{\mu;20} & A_{\mu;21} & A_{\mu;22} & A_{\mu;23} & A_{\mu;24} & A_{\mu;05} \\ A_{\mu;30} & A_{\mu;31} & A_{\mu;32} & A_{\mu;33} & A_{\mu;34} & A_{\mu;05} \\ A_{\mu;40} & A_{\mu;41} & A_{\mu;42} & A_{\mu;43} & A_{\mu;44} & A_{\mu;05} \\ A_{\mu;50} & A_{\mu;51} & A_{\mu;52} & A_{\mu;53} & A_{\mu;54} & A_{\mu;05} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{\mu;00,00} & A_{\mu;00,01} & A_{\mu;00,02} \\ A_{\mu;01,00} & A_{\mu;01,01} & A_{\mu;01,02} \\ A_{\mu;02,00} & A_{\mu;02,01} & A_{\mu;02,02} \end{bmatrix} & \begin{bmatrix} A_{\mu;00,10} & A_{\mu;00,11} & A_{\mu;00,12} \\ A_{\mu;01,10} & A_{\mu;01,11} & A_{\mu;01,12} \\ A_{\mu;02,10} & A_{\mu;02,11} & A_{\mu;02,12} \end{bmatrix} \\ \begin{bmatrix} A_{\mu;10,00} & A_{\mu;10,01} & A_{\mu;10,02} \\ A_{\mu;11,00} & A_{\mu;11,01} & A_{\mu;11,02} \\ A_{\mu;12,00} & A_{\mu;12,01} & A_{\mu;12,02} \end{bmatrix} & \begin{bmatrix} A_{\mu;10,10} & A_{\mu;10,11} & A_{\mu;10,12} \\ A_{\mu;11,10} & A_{\mu;11,11} & A_{\mu;11,12} \\ A_{\mu;12,10} & A_{\mu;12,11} & A_{\mu;12,12} \end{bmatrix} \end{bmatrix} \tag{16.38}$$

So instead of considering $A_\mu$ to be a single $mn \times mn$ matrix of element $A_{\mu;kl}$, where $A_{\mu;kl}$ are $N_k \times N_l$ matrices, we consider it as an $m \times m$ where each element $A_{\mu,ii'}$ (line $i$ column $i'$) is itself an $n \times n$ matrices with elements $A_{\mu;ij,i'j'}$ (line $j$ column $j'$), as shown above.

Using these notations, the gauge field transforms as

$$A_{\mu;ij,i'j'} \mapsto (\mu_i(g) \otimes \sigma_j(g)) A_{\mu;ij,i'j'} (\mu_{i'}(g) \otimes \sigma_{j'}(g))^{-1} = \zeta_m^{i-i'} \zeta_n^{j'-j} A_{\mu;ij,i'j'} \tag{16.39}$$

so invariant configurations can possess non-vanishing components $A_{\mu;ij,i'j'}$ only if

$$\zeta_m^{i-i'} = \zeta(j'-j)_n \tag{16.40}$$

$$\Leftrightarrow \quad (i-i') \mod m = (j'-j) \mod n \tag{16.41}$$

$$\Leftrightarrow \quad j' = j + |i'-i|. \tag{16.42}$$

This means that the submatrices $A_{\mu;ii'}$ has an off-diagonal block form with offset $|i'-i|$. Once again, for a general $n$, the difficulty is only computational, not conceptual. This can therefore easily be implemented into Mathematica and we get the form of the gauge field for any $n$.

For the scalar fields, we have

$$X^m_{ij,i'j'} \mapsto R(g)^m{}_n (\mu_i(g) \otimes \sigma_j(g)) X^n_{ij,i'j'} (\mu_{i'}(g) \otimes \sigma_{j'}(g))^{-1} \tag{16.43}$$

$$= \begin{cases} \zeta_m^{i-i'+1} \zeta_n^{j-j'} X^m_{ij,i'j'}, & m = 0, 1 \\ \zeta_m^{i-i'} \zeta_n^{j-j'+1} X^m_{ij,i'j'}, & m = 2, 3 \\ \zeta_m^{i-i'-1} \zeta_n^{j-j'-1} X^m_{ij,i'j'}, & m = 4, 5. \end{cases} \tag{16.44}$$

So a configuration is invariant if and only if the only non-vanishing component satisfy

$$m = 0, 1 : (i - i' + 1) \mod m = (j' - j) \mod n \tag{16.45}$$

$$m = 2, 3 : (i - i') \mod m = (j' - j - 1) \mod n \tag{16.46}$$

$$m = 4, 5 : (i - i' - 1) \mod m = (j' - j + 1) \mod n. \tag{16.47}$$

# 17 | A note about projective representations, discrete torsion and deformations

## 17.1 | Projective representations and discrete torsion

Up until now, and in particular in all of our computations in section **??**, we only considered ordinary representations, i.e. linear representations. We can however use a more general representations such as projective representations for example. That is, representations $\rho$ of $\Gamma$ such that for all $\gamma_1, \gamma_2 \in \Gamma$,

$$\rho(\gamma_1)\rho(\gamma_2) = A(g_1, g_2)\rho(\gamma_1\gamma_2) \tag{17.1}$$

for some factor $A(\gamma_1, \gamma_2)$. The $A(\gamma_1, \gamma_2) = 1$ corresponding to linear representations. For consistency reasons, the factor $A(g_1, g_2)$ cannot be completely arbitrary and must a cocycle condition. As a result, the possibilities for $A(\gamma_1, \gamma_2)$ are classified by the second cohomology group $H^2(\Gamma, \mathbb{C}^*)$, also called *discrete torsion*. There exists a projective representation if and only if the latter does not vanish. This new liberty, whenever admissible, provides new classes of quiver gauge theories that can be remarkably different from the ones we considered up until now, with no discrete torsion.

What happens is that if one turns on an NSNS B-fiels alors the worldvolume, then the moduli space is expected to a non-commutative version of a Calabi-Yau space. This haw the discrete torsion is physically realized. Another, and actually equivalent through T-duality, way of studying gauge theories is to consider D-branes streched between configurations of NS5-branes[9]. This is the*Hanany-Witten setup*.

The discrete tosrion also appears when writing the full open strin gpartition function that inclues the twisted sector where there is ambiguity up to a phase factor. As a consequence from modular invariance, the latter must satisfy certain cocycle conditions. It is precisely the deiscrete torsion. Note that this has been found to be true only for the open string sector.

### 17.2 | Quiver gauge theories deformations and conifold

We can deforming the singular algebraic description of the orbifold with a field into a family smooth surfaces. The resulting total space is the *conifold*.

## 18 | A note on $(p+1)$-dimensional quiver gauge theories

Let us mention that we can generalize our initial brane-world paradigm and consider D$p$-branes in type II string theory (type IIA if $p$ is even and type IIB if $p$ is odd) instead of just D3-branes. The spacetime is then of the form

$$M = \mathbb{R}^{1,p} \times \mathbb{R}^{9-p}/\Gamma \tag{18.1}$$

where $\Gamma$ is a discrete subgroup of $\mathrm{Spin}(9-p)$. If $\Gamma$ is a subgroup of a special holonomy group, we recover a somewhat generalized version of the paradigm that we discussed above. In this case the transverse space is a Calabi-Yau orbifold and some degree of supersymmetry is preserved. Note that the fermionic and bosonic quivers that coincides. If $\Gamma$ is not a subgroup of a special holonomy group, then the $(p+1)$-dimensional quiver gauge theory that we obtain in the low-energy limit is not supersymmetric. We then have different quivers for the fermions and the bosons, although with the same vertices, by definition.

Recall that the fraction of supercharges that is preserved by compactifying on a Calabi-Yau $n$-fold (with $\mathrm{SU}(n)$ holonomy) is $2^{1-n}$. Starting from the $\mathcal{N}=2$ 10-dimensional type IIB string theory with 32 supercharges, this means that

- if we compactify on a 1-fold, we get 32 supercharges in 8 dimensions so $\mathcal{N}=2$,
- if we compactify on a 2-fold, we get 16 supercharges in 6 dimensions so $\mathcal{N}=2$,
- if we compactify on a 3-fold, we get 8 supercharges in 4 dimensions so $\mathcal{N}=2$,
- is we compactify on a 4-fold, we get 4 supercharges in 2 dimensions so $\mathcal{N}=4$.

We will however mostly mostly consider 4-dimensional quiver gauge theories, i.e. living on D3-branes.

[9] D5-branes are charged under the Ramond-Ramond field whose quanta comes from the Ramond-Ramond sector but NS5-branes are charges under the Kalb-Ramonf field whose quanta comes fromthe Neveu-Schwarz. On the worldvolume of an NS5-brane (6-dimensional) propagates propagates a superstring, this called the *little string*.

# 19 | Summary of orbifold worldvolume theories

| SUSY | $\Gamma$ | Gauge group |
|---|---|---|
| $\mathcal{N}=2$ | $\mathbb{Z}_n$ | $(1^n)$ |
| | $\mathbb{Z}_n \times \mathbb{Z}_m$ | $(1^{nm})$ |
| | $\mathcal{D}_n$ | $(1^4, 2^{n-3})$ |
| | $\mathcal{T}$ | $(1^3, 2^3, 3)$ |
| | $\mathcal{O}$ | $(1^2, 2^2, 3^2, 4)$ |
| | $\mathcal{I}$ | $(1, 2^2, 3^2, 4^2, 5, 6)$ |
| $\mathcal{N}=1$ | $T$ | $(1^3, 3)$ |
| | $O$ | $(1, 2^2, 3^2)$ |
| | $I$ | $(1, 3^2, 4, 5)$ |
| | $\Delta_{3n^2}(n=0 \mod 3)$ | $(1^9, 3^{\frac{n^2}{3}-1})$ |
| | $\Delta_{3n^2}(n\neq 0 \mod 3)$ | $(1^3, 3^{\frac{n^2-1}{3}})$ |
| | $\Delta_{6n^2}(n\neq 0 \mod 3)$ | $(1^2, 2, 3^{(2n-1)}, 6^{\frac{n^2-3n+2}{6}})$ |
| | $\Sigma_{168}$ | $(1, 3^2, 6, 7, 8)$ |
| | $\Sigma_{216}$ | $(1^3, 2^3, 3, 8^3)$ |
| | $\Sigma_{36\times 3}$ | $(1^4, 3^8, 4^2)$ |
| | $\Sigma_{216\times 3}$ | $(1^3, 2^3, 3^7, 6^6, 8^3, 9^2)$ |
| | $\Sigma_{360\times 3}$ | $(1, 3^4, 5^2, 6^2, 8^2, 9^3, 10, 15^2)$ |

*Table 2: All supersymmetric orbifold worldvolume theories.*

Add finitude and conformal invariance.

# 20 | Classical McKay correspondence from strings

We mentionned before the equivalence between fractional branes stuck at orbifold singularities and wrapped branes on the blow-up resolution. From the point of view of the worldvolume theory, this equivalence is exhibited by going from the Higgs branch to the Coulomb branch. The first step to understand this was done by Kronheimer. He showed that the resolution of the orbifold $\mathbb{C}^2/\Gamma$ with $\Gamma$ a finite group of SU(2) is precisely the generic form of the gauge orbit of the direct product of $\mathrm{U}(N_i)$ factors. From the QGT point of view, the product of those circle group is the gauge group of the theory and each factor is associated to a vertex of the quiver. The fields of the theory are organized as a linear representation into a direct sum of $\mathrm{Hom}(V_i, V_j)$ for each edge. If we pick one field and follow it around as the gauge group transforms it, the space swept out is the gauge orbit of that field. Kronheimer then showed that, if the quiver is a Dynkin diagram, this orbit is $\mathbb{C}^2/\Gamma$.

Now in general, gauge theories with with simple Lie groups (such as $\mathrm{SU}(N)$, $E_8$,etc) are more interesting than the ones with gauge groups that are direct products so how could we relate the two? The mechanism the relates the two classes of theories is SSB, or *Higgsing*. Indeed, one may embed the fields configurations in a higher-dimensional field configuration space (see them as submatrices of bigger matrices) on which acts the bigger (simple) Lie group. To from this group to a product of small ones. Fixing this submatrix-structure then singles out every gauge group factor one by one as the stabilizer subgroups of every submatrix. The final step is $\mathcal{N}=2$ super Yang-Mills theories (Seiberg-Witten theory) which have a potential such that its vacua (more precisely the Higgs branch part) break a simple Lie group down to a Dynkin diagram QGT. The Coulomb branch is supposed to behave in a similar way. To summarize, the relation between simple Lie groups and the finite subgroups of SU(2) is the following:

1. start with $\mathcal{N}=2$ SYM with gauge group that is a simple Lie group,
2. let it spontaneously find its vacuum,
3. consider the orbit space of the remaining spontaneously broken symmetry group,
4. the latter space is the resolution of the orbifold of $\mathbb{C}^2/\Gamma$.

All the details are presented in [9].

# Part III

# Toric singularities

The next best thing to orbifold singularities is the toric singularities. The a specific class of supersymmetric gauge theories whose space of vacua is toric is called *toric quiver gauge theories*. In this case, the inverse algorithm has been formalized in [10].

## 21 | Toric geometry

The interest of toric varieties lies in the fact that all its defining data can be encoded in a simple auxilliary object called a fan. This data is purely combinatoric (discrete quantities) so complicated geometric problems are often reduced to simpler combinatorics problems.

### 21.1 | Cones

Let us consider a lattice $N = \mathbb{Z}^m$ and $N_{\mathbb{R}} = N \otimes \mathbb{R}$ the real vector space that we get by allowing real coefficients. Given $v_1, \ldots, v_n$ of the lattice $N$, a *cone*[10] $\sigma \subset N_{\mathbb{R}}$ is a the subset of the vector space $N_{\mathbb{R}}$ containing all points that are positive linear combination of the vectors $v_1, \ldots, v_n$, that is

$$\sigma = \left\{ \sum_{i=1}^{n} a_i v_i | a_i \geq 0, a_i \in \mathbb{R} \right\} \tag{21.1}$$

and such that $\sigma \cap (-\sigma) = \{0\}$. The last condition is the strong convexity condition and it imposes that our cone has to be "acute". The dimension of a cone is the dimension of $\mathrm{Span}(\sigma)$.

The dual lattice $N^*$ of $N$ is the set of linear functionals on $L$ which take integer values on each point of $N$:

$$N^* = \{f \in (\mathrm{Span}(L))^* | \forall x \in N, f(x) \in \mathbb{Z}\}. \tag{21.2}$$

We denote by $M$ the dual lattice of $N$ and by $M_{\mathbb{R}}$ the vector space we get from it by allowing real coefficients. For a given cone $\sigma$ in $N_{\mathbb{R}}$, we can define the *dual cone* $\sigma^\vee$ in $M_{\mathbb{R}}$ as

$$\sigma^\vee = \{m \in M_{\mathbb{R}} | \langle u, m \rangle \geq 0,\ \forall u \in \sigma\}. \tag{21.3}$$

Let us denote by $H_m$ the hyperplane gievn by the dual lattice point $m \in \sigma^\vee$: $H_m = \{v \in N_{\mathbb{R}} | \langle v, m \rangle = 0\}$ and the closed halp-space $H_m^+ = \{v \in N_{\mathbb{R}} | \langle v, m \rangle \geq 0\}$. We then say that a hyperplane *suports* a cone if the closed half-space of the hyperplane completely contains the cone. A *face* of a cone is the intersection of the cone with a supporting hyperplane. Put differently, a convex subset $\tau$ is a face of $\sigma$ if and only whenever $u, v \in \sigma$ satisfy $u + v \in \tau$ then $i \in \tau$ and $v \in \tau$. Note that a hyper plane $H_m$ supports $\sigma$ if and only if $m \in \sigma^\vee$.

### 21.2 | Toric varieties

An *algebraic group* is a group that is also an algebraic variety and such the product and inversion are regular maps on the variety. Any product of $\mathbb{C}^*$ is an algebraic group that we call *algebraic tori*. An affine variety $X \subset \mathbb{C}^n$ is a *affine toric variety* if it contains the algebraic torus $\mathbb{T} = (\mathbb{C}^*)^n$ as a dense open subset such that the action of $\mathbb{T}$ on itself extends to an action $\mathbb{T} \times X \to X$ on $X$.

**Example.** Let us enumerate some examples of affine toric varieties.

- $(\mathbb{C}^*)^n$ and $\mathbb{C}^n$ are naturally provided with an embeddingand an action of the torus and are toric varieties

[10] More precisely, a strongly convex rational polyhedral cone.

- $V = Z(x^3 - y^2)$ with torus embedding

$$\left( \begin{array}{ccc} \mathbb{C}^* & \hookrightarrow & V \\ t & \longmapsto & (t^2, t^3)) \end{array} \right). \tag{21.4}$$

and the action $t \cdot (u, v) \mapsto (t^2 u, t^3 v)$.

## 21.3 Constructing toric affine varieties

### 21.3.1 Toric variety of a cones

A *semigroup* $S$ is a set with an internal associative $+$ operation and a neutral element 0. It differs from a group in that elements need not have an inverse. A semigroup is *affine* if it can be embedded as a subsemigroup in a lattice $\mathbb{Z}^m$ (*integral*) and if there exists a finite set $\mathscr{A}$ such that $S = \mathbb{N}\mathscr{A}$ (*finitely generated*).

What is interesting is that affine semigroups allows us to construct affine varities. For this, we need to introduce the notion of *semigroup algebra* $\mathbb{C}[S]$ associated to any semi group $S$. It is the algebra generated by elements $\chi^u$ indexed by elements $u \in S$. The semigroup operation $+$ the induces a multiplication operation for the $\chi^u$ in $\mathbb{C}[S]$ as $\chi^u \cdot \chi^v = \chi^{u+v}$. For instance, the semigroup algebra of $S = \mathbb{N}^n$ is simply $\mathbb{C}[x_1, \dots, x_n]$ and the semigroup algebra of $S = \{2, 3, \dots\} \subset \mathbb{N}$ is $\mathbb{C}[x, y]/I(x^3 - y^2)$. The important result is now that if $S$ is an affine semigroup then $\mathrm{Spec}(\mathbb{C}[S])$ is an affine variety.

Given a cone $\sigma$ and its dual cone $\sigma^\vee$, $S_\sigma = \sigma^\vee \cap M$ is finitely generated and hence an affine semigroup. In this way, we may obtain an affine variety from a cone as $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$, this the *variety if the cone* $\sigma$. An important result is that if $K[V]$ is the coordinate ring of an affine variety $V$, then $V = \mathrm{Spec}(K[V])$. From this, we see that $\mathbb{C}[S_\sigma]$ is exactly the coordinate ring of the variety $U_\sigma$ we are looking for.

We have seen how to get an affine variety $U_\sigma$ from a cone $\sigma$. This variety is in fact toric. If $n$ denotes the rank of the lattice $\mathbb{Z}S_\sigma$, then there is torus action $\mathbb{T} = (\mathbb{C}^*)^n$ acting on $U_\sigma$. The torus that it contained is called the *torus corresponding to a lattice* $N = \mathbb{Z}^m$ and is $\mathbb{Z}_N = (\mathbb{C}^*)^n$, so that the rank of the lattice equals the dimension of the torus:

$$T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*). \tag{21.5}$$

To torus acts on $U_\sigma$ as $t \cdot \gamma : S_\sigma \to \mathbb{C} : m \mapsto \chi^m(t)\gamma(m)$, for all $t \in T_N$ and $\gamma \in \sigma$ (so$\gamma : S_\sigma \to \mathbb{C}$). To summarize, the steps to extract the $m$-complex dimensional toric variety from a given cone $\sigma$ in $N_\mathbb{R}$ are the following:

1. find the dual cone $\sigma^\vee$
2. find the intersection $S_\sigma = \sigma^\vee \cap \mathbb{Z}^m$, it is a finitely generated semigroup. Find a minimal set of generating vectors $\{w_1, \dots, w_r\}$
3. find the polynomial ring $\mathbb{C}[S_\sigma]$ by exponentiating the coordinates of $S_\sigma$
4. find the relations between the elements of $\mathbb{C}[S_\sigma]$ corresponding to the generating vectors $w_i$. Such a relation looks like

$$\sum_{i \in I} m_i w_i = \sum_{j \in J}^{r} m_j w_j, \quad m_i, m_j \in \mathbb{N} \Rightarrow p(x) = \prod_{i \in I} x^{m_i} x_i - \prod_{j \in J} x^{m_j} x_j \tag{21.6}$$

   where $I \cup J = \{1, \dots, r\}$ and $I \cap J = 0$. If $\{p_1, p_2, \dots\}$ is the set of all those relations, then $I(\{p_1, , p_2, \dots\})$ is a prime ideal and $\mathbb{C}[S_\sigma] = \mathbb{C}[x_1, \dots, x_r]/I(p_1, p_2, \dots)$
5. $\mathbb{C}[S_\sigma]$ is the coordinate ring of desired the toric variety. We can recover the variety explicitely as $\mathrm{Spec}(\mathbb{C}[S_\sigma])$. It might be easier to use the fact that $\mathrm{Spec}[K[V]] = V$, so $\mathbb{C}[S_\sigma]$ is actually the coordinate ring of $U_\sigma$ and $U_\sigma = Z(\{p_1, p_2, \dots\})$.

The vtoric variety $U_\sigma$ has the same dimension than $\sigma$.

**Example.** If we consider the cone $\sigma$ generated by $e_2$ and $2e_1 - e_2$ in $\mathbb{Z}^2$ (where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{Z}^2$), then the dual cone $\sigma^\vee$ is generated by $e_1$ and $e_1 + 2e_2$. We therefore have $S_\sigma = \sigma^\vee \cap \mathbb{Z}^2$. Ths subset of $\mathbb{Z}^2$ is spanned by $e_1, e_1 + e_2$ and $e_1 + 2e_1$:

$$S_\sigma = \mathrm{Span}(\{(1,0),(1,1),(1,2)\}). \tag{21.7}$$

Note that, even though it is a 2-dimensional cone, i.e. the intersection of this cone and the lattice actually need three vectors to be completely generated. This is often the case when dealing with lattices. By exponentiating we get the semigroup algebra $\mathbb{C}[S_\sigma] = \mathbb{C}[x, xy, xy^2]$. If we denote $u = x, v = xy$ and $w = xy^2$, relation between those three variables is $uw = v^2$, so $\mathbb{C}[S_\sigma] = \mathbb{C}[u,v,w]/I(v^2 - uw)$. It only remains to compute the spectrum, and we find that

$$\mathrm{Spec}(\mathbb{C}[S_\sigma]) = \mathbb{C}^2/\mathbb{Z}^2. \tag{21.8}$$

In conclusion the toric varitey associated to the cone $\sigma$ is the abelian orbifold $\mathbb{C}^2/\mathbb{Z}^2$.

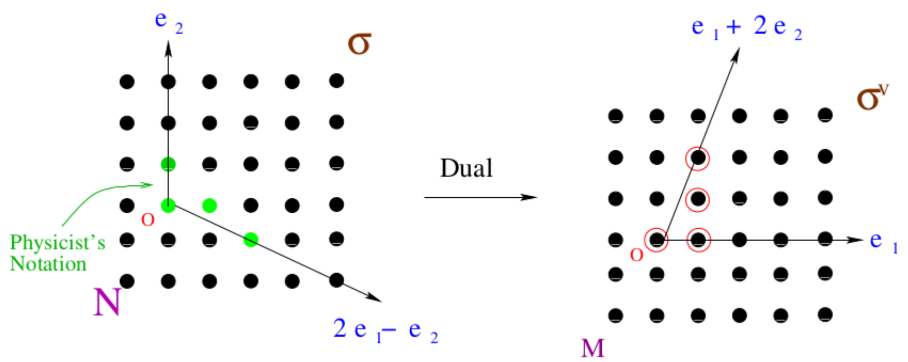



*Figure 8: Representations of the cone and dual cone in the lattice $\mathbb{Z}^2$.*

The previous example illistrate a more genral result:

**Proposition.** All abelian orbifolds are toric varities.

Note that this result is not that surprising considering the general expression (21.10).

**Example.** Let us start with $\sigma = \mathrm{Cone}(\{e_1, 2e_1 + e_2\})$. We then get $\sigma^\vee = \mathrm{Cone}(\{e_2, e_1 - 2e_2\})$ so $S_\sigma = \mathrm{Span}_{\mathbb{Z}^+}(\{e_2, e_1 - 2e_2\})$ and $\mathbb{C}[S_\sigma] = \mathbb{C}[y, xy^{-2}] = \mathbb{C}[u, v]$.

Now what happens with faces? An inclusion of face $\tau \subset \sigma$ gives an inclusions of the semigroups $S_\sigma \subset S_\tau$ and $\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_\tau]$. This induces a morphism $U_\tau \to U_\sigma$ of affine toric varieties and it happens the this morphism is an open embedding.

**Example.** If $\sigma = \mathrm{Cone}(e_1, e_2)$ then we find $U_\sigma = \mathbb{C}^2$. There are three faces: $\rho_1 = \mathrm{Cone}(e_1)$, $\rho_2 = \mathrm{Cone}(e_2)$ and the origin 0. These faces can be described by hyperplane $H_{e_2}, H_{e_1}$ and $H_{e_1+e_2}$ respectively, so we get Why ?

$$\mathbb{C}[S_{\rho_1}] = \mathbb{C}[x, y, y^{-1}], \mathbb{C}[S_{\rho_2}] = \mathbb{C}[x, x^{-1}, y], \mathbb{C}[S_0] = \mathbb{C}[x, y, x^{-1}, y^{-1}]. \tag{21.9}$$

They define the varieties $U_{\rho_1} = \mathbb{C}^* \times \mathbb{C}$, $U_{\rho_1} = \mathbb{C} \times \mathbb{C}^*$ and $(\mathbb{C}^*)^2$, which are all open subvarieties of $\mathbb{C}^2$. Why ?

## 21.4 Toric variety of a fan

A *fan* is a collection $\Delta$ of cones in $N_\mathbb{R}$ such that each face of a cone is also a cone and the intersection of two cones is a face of each. Those two requirement imply in particular that the intersection of two cones in a fan is again a cone in the fan. Note that to any cone corresponds a fan containing the cone itself and all its faces.

For a cone $\sigma$ in $N_\mathbb{R}$ the variety $U_\sigma$ contains the torus $T_N$ so fan in $N_\mathbb{R}$ produces a collection of varieties that all contain th same torus $T_N$. Gluing together the affine varities we obtain $X_\Delta$, it is clear that $T_N$ is an open subset of $X_\Delta$ and that $T_N$ acts on $X_\Delta$. This is called the *toric variety of a fan*. For the toric varitey of a fan, each cone in the fan turns out to be determines an orbit $O(\sigma)$ of the torus.

Generally speaking, an $m$-dimensional algebraic variety of a fan can be obtained as as particular holomorphic quotient of $\mathbb{C}^n$. If $\mathbb{Z}_\Delta \subset \mathbb{C}^n$ a set of points and $G \cong (\mathbb{C}^*)^{n-m} \times \Gamma$ is the group formed by the algebraic torus and an abelian discrete group $\Gamma$, then

$$\boxed{X_\Delta = \frac{\{\mathbb{C}^n \backslash \mathbb{Z}_\Delta\}}{G}.} \tag{21.10}$$

Let us explain this construction.

If $\Delta$ is an $m$-dimensional fan, we denote by $\Delta(j)$ $(j \leq n)$ the collection of $j$-dimensional cones in $\Delta$. The set $\Delta(1) \subset \Delta$ is then the collection of all one-dimensional cones in $\Delta$, i.e. the set of edges. To each cone $\sigma \in \Delta(1)$ is associated a unique vector $v_\sigma \in N$ that generates the sublattice $\sigma \cap N$. This vactor is called the *primitive generator* of $\sigma$. Let $v_1, \ldots, v_n$ be the primitive generators of of each $\sigma \in \Delta(1)$. Note that we always have $n \geq m$ since $\Delta$ is $m$-dimensional. Each vector $v_i$ has component $v_i^k$ $(i = 1, \ldots, n, k = 1, \ldots, m)$ so we can construct an $m \times n$ matrix by putting the vectors in columns:

$$V = \begin{bmatrix} v_1^1 & \cdots & v_n^1 \\ \vdots & & \vdots \\ v_1^m & \cdots & v_n^m \end{bmatrix} \tag{21.11}$$

This defines a linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$ whose kernel is $\mathbb{C}^{n-m}$. Let $Q^a$ $(a = 1, \ldots, n-m)$ be vectors that generate this kernel, then we must have

$$VQ^a = 0 \Leftrightarrow \sum_{i=1}^n v_i^k Q_i^a = 0, \qquad k = 1, \ldots, m \tag{21.12}$$

for all $a = 1, \ldots, n-m$. We now put that on the side for a moment. To each edge $\sigma \in \Delta(1)$ with associate a complex coordinate $x_\sigma$ and consider the new map

$$\phi : \left( \begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^m \\ (z_1, \ldots, z_n) & \longmapsto & (\prod_{i=1}^n z_i^{v_i^1}, \ldots, \prod_{i=1}^n z_i^{v_i^m}) \end{array} \right). \tag{21.13}$$

It is clear that the kernel of $\phi$ is $\widetilde{G} \equiv \mathrm{Ker}(\phi) = (\mathbb{C}^*)^{n-m}$. One can see that this kernel naturally acts on $\mathbb{C}^n$ as Why ?

$$\left( \begin{array}{ccc} (\mathbb{C}^*)_a \times \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ (\lambda, z_1, \ldots, z_n) & \longmapsto & (\lambda^{Q_1^a} z_1, \ldots, \lambda^{Q_n^a} z_n) \end{array} \right) \tag{21.14}$$

where $Q_i^a$ are the component ofthe vectors $Q^a$ defined above. We defined the action of $\widetilde{G} = (\mathbb{C}^*)^{n-m}$ on $\mathbb{C}^n$for each factor separately for simplicity.

The $n$ vecotrs $v_i$ generate over $\mathbb{Z}$ a sublattice of $N$ that we denote $N'$. The discrete group obtained by taking the quotient of $N$ by this sublattice

$$\Gamma = N/N' \tag{21.15}$$

is a subgroup of $G$ and taking the quotient by $\Gamma$ gives rise to the orbifold singularities.

The last piece of data in the construction is the zero set $Z_\Delta$. Let us denote by $\mathcal{S}$ any subset of $\Delta(1)$ that does not generate a cone in $\Delta$ and $V(\mathcal{S}) \subset \mathbb{C}^n$ the linear subspace defined by setting $x_\sigma = 0$ for all $\sigma \in \mathcal{S}$, i.e. the hyperplane generated by the edges that does not generate a cone in $\Delta$. We the denote $Z_\Delta$ the union of all those hyerplanes, i.e. of all the $V(\mathcal{S})$. Our toric variety is defined as a qutient of $\mathbb{C}^n \backslash Z_\Delta$.

To summarize, we can construct affine toric variety $X_\Delta$ from the $m$-fan $\Delta$ by following these steps:

- find the primitive generators $v_1, \dots, v_n$ of the 1-cones of $\Delta$
- find the $n-m$ vectors $Q^a$ such that $\sum_i v_i^k Q_i^a = 0$
- compute $\Gamma = N/N'$
- find the subsets of $\{v_1, \dots, v_n\}$ that don't generate in $\Delta$ and compute $Z_\Delta$
- the toric affine variety is $X_\Delta = (\mathbb{C}^n \backslash Z_\Delta)/((\mathbb{C}^*)^{n-m} \times \Gamma)$.

**Example.** If we consider the 2-dimensional fan $\Delta = \mathrm{Fan}(e_1, e_2, -e_1 - e_2)$ in $N = \mathbb{Z}^3$ (so $m = 2$), we have three one-dimensional cones generated by $v_1 = e_1, v_2 = e_2$ and $v_3 = -e_2 - e_2$ (so $n = 3$) and the component matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}. \tag{21.16}$$

We only have one vector $Q^a$ to find and since it must satisfy the relation (21.12), i.e. in this case $Q_1 v_1 + Q_2 v_2 + Q_3 v_3 = 0$, it implies that $Q = (1, 1, 1)$. The group action of $\widetilde{G} = \mathbb{C}^*$ on the $\mathbb{C}^3$ is then

$$(z_1, z_2, z_3) \mapsto (\lambda z_1, \lambda z_2 \lambda z_3). \tag{21.17}$$

Since $v_1, v_2$ and $v_3$ generate all of $\mathbb{Z}^2$, $\Gamma = \{0\}$ is the trivial group and $G = \widetilde{G} \times \Gamma \cong\cong \widetilde{G} = \mathbb{C}^*$. There is no subset of $\{v_1, v_2, v_3\}$ that generates a cone which is not in $\Delta$ (because we defined $\Delta$ as the fan generated by those vectors in the first place) therefore $Z_\Delta = \{(0, 0, 0)\}$. Finally, the affine toric variety of $\Delta$ is Why?

$$X_\Delta = \frac{\mathbb{C}^3 \backslash Z_\Delta}{G} = \frac{(\mathbb{C}^*)^3}{\mathbb{C}^*} = \mathbb{CP}^2. \tag{21.18}$$

**Proposition.** A toric variety $X_\Delta$ is compact if and only its fan $\Delta$ span the $N_\mathbb{R}$.

### 21.4.1 | Fan from a toric variety

## 21.5 | More general constructions

In general, an affine variety canbe defined by an ideal in $\mathbb{C}[x_1, \dots, x_n]$. In the same way, a toric affine variety can be defined by a *toric ideal* (or *binomial ideal*), i.e. an ideal in $\mathbb{C}[x_1, \dots, x_n]$ generated by binomials (polynoamils with precisely two non-zero coefficients).

**Theorem.** Let $V$ be an affine variety. The following are equivalent:

- $V$ is toric,
- $V = \mathrm{Spec}(\mathbb{C}[S])$ for an affine semigroup $S$,
- $V = V(I)$ for a toric ideal.

This is a very powerfull theorem since is states that an affine avariety is toric if and only if the generating polynomials of its ideal are binomials.

## 21.6 | Relation between fan varieties, cone varities and local coordinates

Given an $m$-dimensional cone spanned by $n$ vectors $v_1, \dots, v_n$, we want to find the local coordinates associated to it. Since the the toric variety can be expressend as the quotient by $G$, the local coordinates should $G$-invariant polynomials, that is polynomials $x = z_1^{n_1} \dots z_n^{n_n}$ such that

$$x \mapsto \lambda^{\sum_i Q_i^a n_i} x = x \tag{21.19}$$

under $G$. But remember that (21.12) so we can take $n_i = \langle w, v_i \rangle$ for any $w \in M$. We foud that the local coordinates are in one-to-one correspondence with the dual lattice $M$.

Given a fan and its fan toric variety, we can a toric affine variety to each top-dimensional cone. These cone toric varities will be affine patches of the fan variety. The transition functions between these patches are also encoded in the initial fan. Let us explain those statement in greater details. If $\sigma_i \in \Delta$ are the top-dimensional cones of a given fan $\Delta$, we can construct their toric variety $U_{\sigma_i}$ by following the procedure that we presented above. The toric varieties $U_{\sigma_i}$ act as affine pacthes and can be patched together to form $X_\Delta$. Now suppose that $\tau$ is a face of both $\sigma_i$ and $\sigma_j$, one can show that

$$\sigma_{i,j}^\vee \subset \tau^\vee \quad \Rightarrow \quad \mathbb{C}[\sigma_{i,j}^\vee \cap M] \subset \mathbb{C}[\tau^\vee \cap M] \quad \Rightarrow \quad U_\tau \subset U_{\sigma_i} \cap U_{\sigma_j} \tag{21.20}$$

so the affine set associated to the fae $\tau$ is in the intser section of the affine sets of the cones. The relation between local coordinates $x^(i)$ of $U_{\sigma_i}$ and $x^(j)$ of $U_{\sigma_j}$ can then be read from the relations between the generators of $\sigma_{i,j}^\vee \cap M$ and $\sigma_{i,j}^\vee \cap M$:

$$\sum_{l=1}^{r_i} q_l w_l^{(i)} = \sum_{k=1}^{r_j} q_k w_k^{(j)}, \quad q_l, q_k \in \mathbb{Z} \quad \Rightarrow \quad \prod_{l=1}^{r_i} (x_l^{(i)})^{q_l} = \prod_{k=1}^{r_j} (x_k^{(j)})^{q_k}. \tag{21.21}$$

We see that those transition functions are always rational functions. This is crucial property of toric varieties.

**Example.** Going back to the fan of $\mathbb{CP}^2$, we see that there are three top-dimensional cones $\sigma_1, \sigma_2$ and $\sigma_3$. For each we have $U_{\sigma_i} = \mathbb{C}^2$. Applying (21.21) we find that the transition functions between the cooridnates $(x_1, x_2)$ of $U_{\sigma_1}$ and $(y_1, y_2)$ of $U_{\sigma_2}$ are

$$x_1 = \frac{y_1}{y_2}, \quad x_2 = \frac{1}{y_2}. \tag{21.22}$$

## 21.7 Calabi-Yau toric varities

As motivated in the beginning, we are mainly interested in Calabi-You affine varities, so Calabi-Yau affine toric varieties in this case. A very convenient property of the toric varieties is that the CY condition is translated into a simple condition on the combinatoric data of the variety.

Recall that a divisor of an affine variety is a linear combination of codimension-one irreducible subvarieties. A *toric divisor* is a divisor invariant under the action of $G$. Using the homogeneous coordinates $(z_i)$, we can easily construct $G$-invariant subvarieties. The simple algebraic subsets

$$\{(z_1, \dots, z_n) | z_i = 0 \forall i \in I \subset \{1, \dots, n\}\} \tag{21.23}$$

are $G$-invariant so the subvarieties

$$D_i = \{z_i = 0\} \cap X_\Delta \tag{21.24}$$

are toric divisors. Even stronger, one can actually show that they generate the full group of divisors of $X_\Delta$ and that if $X_\Delta$ is smooth with canonical bundle $K_X$, we have

$$K_X = \mathcal{O}\left(-\sum_{i=1}^n D_i\right). \tag{21.25}$$

This important result allows us us to state the Calabi-Yau condition (triviality of the canonical bundle) in a very simple way. The only thing left is to see that any $G$-invairant function is asection of the trivial bundle and that $K_X = \mathcal{O}\left(-\sum_{i=1}^n D_i\right)$ is trivial if and only if

$$G : z_1 \dots z_n \mapsto \lambda^{\sum_{i=1}^n Q_i^a} z_1 \dots z_n \quad \Leftrightarrow \sum_{i=1}^n Q_i^a = 0. \tag{21.26}$$

This last condition is equivalent to the existence of a vector $w \in M$ such that $\langle w, v_i \rangle = 1$ for all $v_i$ in the fan. Finally, we get the follwing criteria:

**Proposition.** The toric variety $X_\Delta$ is CY if and only if all the vectors $v_i$ in $\Delta$ end on the same hyperplane in $N$, which happens if and only if $\sum_{i=1}^n Q_i^a = 0$ for $a = 1, \dots, n-m$.

Following from the proposition on compactness of toric varieties that we gave before, this implies tha toric CY varieties are necessarily non-compact.

| $X_\Delta$ | $\dim_{\mathbb{C}}$ | generating polynomial | compact | CY | $\Delta$ |
|---|---|---|---|---|---|
| $\mathbb{C}^2$ | 2 | | no | yes | $\mathrm{Fan}(e_1, e_2)$ |
| $\mathbb{C}^2/\mathbb{Z}_k$ | 2 | $uv - w^2$ | no | yes | $\mathrm{Fan}(e_2, ke_1 - e_2)$ |
| $\mathbb{C}^3/\mathbb{Z}_k$ | 3 | | no | yes | $\mathrm{Fan}(e_2, ke_1 - e_2, e_3)$ |
| $\mathcal{C}_0$ | 3 | $x_1x_2 - x_3x_4$ | no | yes | $\mathrm{Fan}(e_3, e_1 + e_3, e_1 + e_2 + e_3, e_2 + e_3)$ |
| SSP | | | | | |

*Table 3: Useful list of toric varieties.*

## 21.8 Toric diagrams and p-q webs

For CY toric varieties, the combinatoric information encoded in the fan can be expressed in term of a reduced lattice of one less dimension. This is espacially convenient for us since we are mainly concerned about toric CY threefolds, which can therefore be fully encoded in the 2-dimensional lattice. Indeand, instead of drawing a 3-dimensional fan, we can simply project iton the special plane defined by $\langle w, v_i\rangle = 1$, and we get the *toric diagram*.

We can also write the dual of the toric diagram by replacing each line with an orthogonal line. This is called the *peq-web*. They have a nice physical interpreatation as webs intersecting 5-banes.

## 21.9 Singularities

A cone is said to be *smooth* if it is generated by part of a lattice basis.

**Theorem.** A cone $\sigma$ is smooth of and only if $U_\sigma$ is smooth.

**Example.** The affine toric variety $Z(x^2 - y^3)$ discuss above is not smooth.

**Proposition.** The varitey $X_\Delta$ is is non-singular if and only if $\Delta$ is a smooth fan.

A toric variety is nonsingular if its cones of maximal dimension are generated by a basis of the lattice. This implies that every toric variety has a resolution of singularities given by another toric variety, which can be constructed by subdividing the maximal cones into cones of non-singular toric varieties.

# 22 Important examples

## 22.1 The conifold

The singular conifold $\mathcal{C}_0$ can be viewed as the toric variety of the fan $\Delta = \mathrm{Fan}(e_3, e_1{+}e_3, e_1{+}e_2{+}e_3, e_2{+}e_3)$ in $\mathbb{Z}^3$. This fan contains 10 cones on total. Denoting the four generating vectors as $v_1, v_2, v_3$ and $v_4$, we find that the only charge vector $Q$ must satisfy $Q_1v_1 + Q_2v_2 + Q_3v_3 + Q_4v_4 = 0$ which gives $Q = (1, -1, 1-1)$ so $\widetilde{G} = \mathbb{C}^*$ acts as

$$(z_1, z_2, z_3, z_4) \mapsto (\lambda z_1, \lambda^{-1}z_2\lambda z_3, \lambda^{-1}z_4). \tag{22.1}$$

We also find that $\Gamma = \{0\}$ is the trivial group and that $Z_\Delta = Z(z_1, z_3) \cup Z(z_2, z_4)$. Finally,

$$\mathcal{C}_0 = \frac{\mathbb{C}^3\backslash Z_\Delta}{\mathbb{C}^*}. \tag{22.2}$$

Note that, by definition we also have $\mathcal{C}_0 = Z(x_1x_4 - x_2x_3)$ so it is generated by a binomial andistherefore expected to be toric.

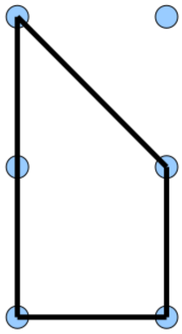


*Figure 10: Toric diagram of the SPP.*

*Figure 9: Toric diagram and pq-web of the conifold.*

### 22.2 The suspended pinch point

### 22.3 del Pezzo surfaces

## 23 Gauged linear sigma model (GLSM)

Witten's gauged linear sigma model provides a physical perspective on toric varieties which provides us with the right approach for the forward algorithm. Let us consider the vectorspace $\mathbb{C}^q$ with complex coordinates $z_1, \ldots, z_q$.

### 23.1 Calabi-Yau and non-compactness conditions

## 24 Correspondence between gauge theory and singularity

Above, we presented all the possible orbifold constructions of supersymmetric quiver gauge theories in four dimensions. We started from quotienting the transverse space and we found the corresponding (supersymmetric) gauge theory. In other words, we started from the singularity a found the gauge theory. We can therefore consider that the orbifold singularities are understood. However, not all singularities are orbifold one, such as the conifold for example. We can then ask ourselves how to obtain the gauge theory for more general singularities than the orbifold ones. Is a general approach possible ? On the other hand, we can also study the converse question; is it possible to to obtain the singularity from the gauge theory? And if it is, how so? In general we will see that there is a bijection between the four-dimensional supersymmetric worldvolume gauge theory and the Calabi-Yau singularity. We now detail this bijection.

| | | |
|---|---|---|
| | Forward Algorithm | |
| PHYSICS: Gauge Data | $\rightleftharpoons$ | MATHEMATICS: Geometry Data |
| | Inverse Algorithm | |
| $\Updownarrow$ | | $\Updownarrow$ |
| QUIVER | $\rightleftharpoons$ | Intersection Theory, etc. |

*Figure 11: Inverse and forward algorithm, from [11].*

## 24.1 From gauge theory to singularity: forward algorithm

We start with the simplest question: how to recover the singularity from the gauge theory? We already mentioned that the vacuum parameter space of the scalar fields of the gauge theory is the so-called moduli space, denoted $\mathcal{M}$. Because our D3-brane is a point in the Calabi-Yau threefold, the vacuum moduli space $\mathcal{M}$ is the affine coordinates of the Calabi-Yau singularity $S$.

For the ADE $\mathcal{N} = 2$ theories discussed in section 15, by the Kronheimer-Nakajima construction [**Kronheimer1990**], the moduli space is a hyper-Kähler quotient. In general, the moduli space can be constructed as a *quiver variety*, i.e. a variety constructed from the moduli space of quiver a quiver representation. More rpecisely, given the dimensions of the vector spaces assigned to every vertex, one can form a variety which characterizes all representations of that quiver with those specified dimensions, and consider stability conditions. Let us see some examples of this.

The anomaly free condition is

$$(a_{ij} - a_{ji})n_i = 0. \tag{24.1}$$

Explain more.

## 24.2 Forward algorithm for abelina orbifolds

## 24.3 From singularity to gauge theory: inverse algorithm

Mathematically, a quiver gauge theory is a representation of a finite quiver with relations. The labels are $\{N_i \in \mathbb{Z}_+\}$, they correspond to the dimension of the vector space $\{V_i\}$. The gauge group is $\prod_i \mathrm{SU}(N_i)$. The gauge fields are self-adjoint arrows $\mathrm{Hom}(V_i, V_i)$ while the matter fields are bi-fundamentals fermions/bosons and are arrows $X_{ij} \in \mathrm{Hom}(V_j, V_i)$. For a quiver with adjacency matrix $a_{ij}$, the gauge anomaly cancellation condition can be generally expressed as

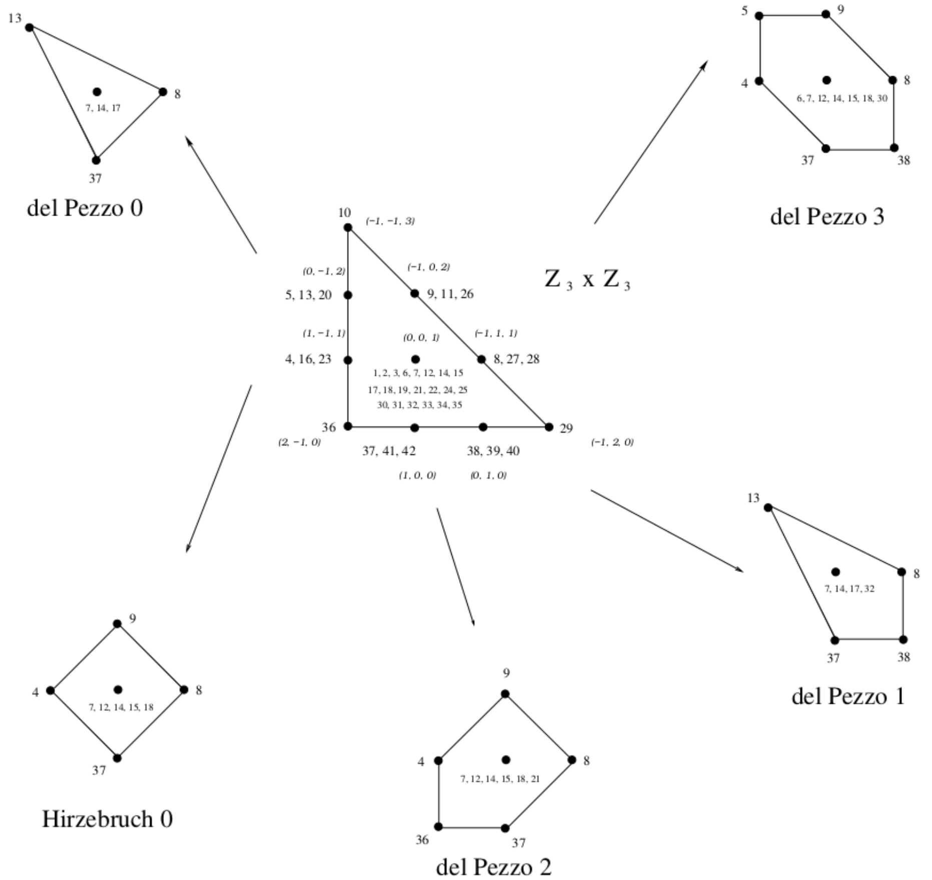
$$(a_{ij} - a_{ji})N_i = 0. \tag{24.2}$$

At last, there are some relations that arises the superpotential $W(\{X_{ij}\})$. The vacuum is the minima of the superpotential. In other words,

$$\frac{\partial W}{\partial X_{ij}} = 0. \tag{24.3}$$

## 24.4 Application to Toric del Pezzo's

# 25 Toric duality



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7, 12, 14, 15, 18
37
Hirzebruch 0
9
4
8
7, 12, 14, 15, 18, 21
36
37
del Pezzo 2

# Part IV
# Non-toric singularities

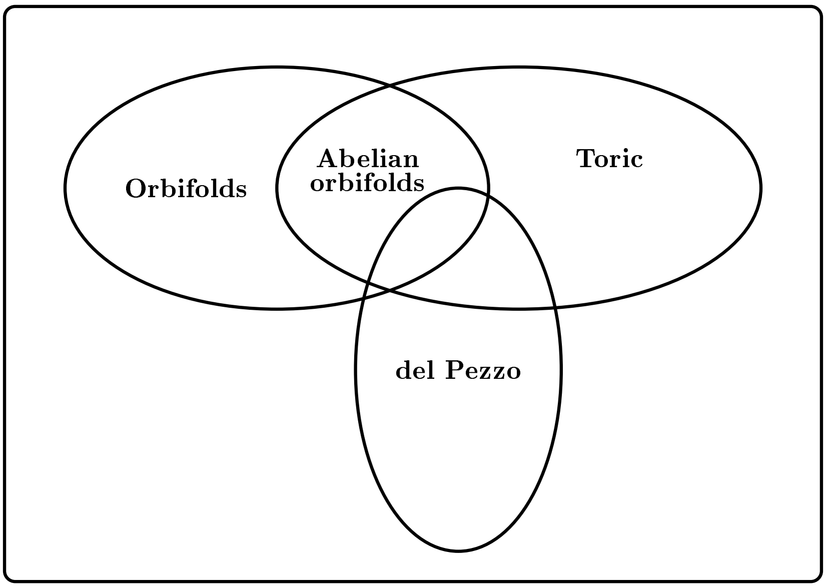


*Figure 12: Venn diagram of different types of algebraic singularities.*

## 26 | Non-commutative resolutions

# Part V

# Beyond the D-brane probes constructions

There are mainly three methods of constructing finite supersymmetric gauge theories from string theory:

1. Geometrical engineering
2. D-branes probing singularities
3. Hanany-Witten setups.

# A | Simplicial Homology

An *n-simplex*, denoted by $\Delta^n = [v_0, \ldots, v_n]$ is the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, \ldots, v_n$ that do not lie in a hyperplane of dimension less than $n$. Or, equivalently, such that $v_1 - v_0, \ldots, v_n - v_0$ are linearly independant. The $n+1$ point $v_0, \ldots, v_n$ are the *vertices* of the $n$-simplex. A by-product of our ordered notation for the vertices of a simplex is that it determines an orientation for the edges $[v_i, v_j]$ according to the increasing subscripts. The *faces* of an $n$-simplex are all the sub-simplices that can obtained by removing vertices of the original simplex. The order of the vertices of the smaller simplices is taken to be same than in the original $n$-simplex.

Let us now put some simplices together. If $X$ be a topological space, then a *simplicial complex structure* $\Delta$ on $X$ is a collection $\Delta = \{\sigma_\alpha\}$ of continuous maps $\sigma_\alpha : \Delta^{n(\alpha)} \to X$ called *characteristic maps* such that

i) $\sigma_\alpha|_{e^{n(\alpha)}}$ is injective and that for all $x \in X$ there exists $\alpha$ such that $x \in \mathrm{Im}(\sigma_\alpha|_{e^{n(\alpha)}})$,

ii) the restriction of any map $\sigma_\alpha$ to a face of $\Delta^n$ is another $\sigma_\beta$,

iii) for any subset $A \subseteq X$, $A$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in $\Delta^n$ for all $\alpha$,

where $\Delta^n$ is an $n$-simplex and $e^n$ is the interior of $\Delta^n$.

Given a simplicial complex structure $\Delta$ on $X$, $n$-chains are finite formal sums

$$\sum_\alpha n_\alpha \sigma_\alpha(e^{n(\alpha)}) \tag{A.1}$$

with coefficient $n_\alpha \in \mathbb{Z}$. We denote by $\Delta_n(X)$ the set of all $n$-chains, it is a free abelian group.

For a general simplicial complex structure on $X$, we define a *boundary homomorphism* $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$ by its action on the basis elements:

$$\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[v_0, \ldots, \widehat{v_i}, \ldots, v_n]}. \tag{A.2}$$

In particular, homomorphism here means that this maps are linear. We can note that the right side of this equation does indeed lies in $\Delta_{n-1}(X)$ since each restriction $\sigma_\alpha|_{[v_0, \ldots, \widehat{v_i}, \ldots, v_n]}$ is the characteristic map of an $(n-1)$-simplex of $X$. The most important property of those maps is that $\partial_n \circ \partial_{n+1} = 0$ (symbolically $\partial^2 = 0$), meaning that $\mathrm{Im}\, \partial_{n+1} \subset \mathrm{Ker}\partial_n$. We can then form a sequence of homomorphisms of abelian groups

$$\cdots \to \Delta_{n+1}(X) \xrightarrow{\partial_{n+1}} \Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \to \cdots \to \Delta_1(X) \xrightarrow{\partial_1} \Delta_0(X) \to 0 \tag{A.3}$$

$\mathrm{Im}\, \partial_{n+1} \subset \mathrm{Ker}\partial_n$ for each $n$. Chains of homomorphisms satisfying this property are called *chain complex*. We have extended the chain at the end with $\partial_0 = 0$ such that this property is also true at the ends. Elements of $\mathrm{Ker}\partial_n$ are called *n-cycles* and elements of $\mathrm{Im}\, \partial_{n+1}$ are called *n-boundaries*. Note that they are each elements of $\Delta_n(X)$, hence the notation.

**Example.** If we consider a triangle withe vertices $A, B$ and $C$, we can put a simplicial complex structure on it consisting of one 2-simplex ($u = [A, B, C]$), three 1-simplices ($a = [A, B], b = [B, C], c = [C, A]$) and three 0-simplices ($A, B, C$). We can see that

$$\partial_2(u) = [B, C] - [A, C] + [A, B] = a + b + c. \tag{A.4}$$

So we get that $a + b + c$ is a 1-boundary. However, we see that

$$\partial_1(a + b + c) = B - A + C - B + A - C = 0 \tag{A.5}$$

so it is also a 1-cycle.

The fact that image of each map lies the kernel of the next map means that any boundary is also a cycle. As illustrated by the previous example. We can wonder what are the cycles that are not boundaries of any higher-dimensional simplicial complex. This set is precisely the quotient space

$$H_n(X) = \frac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}\,\partial_{n+1}}. \tag{A.6}$$

It naturally inherits a group structure and is called the $n$th *Homology group* of $X$. The elements of this space are equivalence classes (cosets of $\mathrm{Im}\,\partial_{n+1}$) of $n$-cycles where two cycles are considered equivalent if they only differ by a boundary, i.e. if their formal difference is a boundary. These equivalence classes are called *homology classes.*

**Example.** Let us give some useful examples of Homology groups for various topological spaces:

- the Homology groups of $\mathbb{R}^n$ are all trivial therefore $\chi = 0$.
- the non-trivial Homology groups of the $n$-sphere are $H_0(S^n) = H_n(S^n) = \mathbb{Z}$ therefore $\chi = (-1)^n - 1$
- the only non-trivial Homology group of the $n$-ball is $H_0(B^n) = \mathbb{Z}$ therefore $\chi = 1$.
- the non-trivial Homology groups of the 2-torus are $H_0(T) = H_2(T) = \mathbb{Z}$ and $H_1(T) = \mathbb{Z}^2$ therefore $\chi = (1+x)^2$.
- the non-trivial Homology groups of the complex projective space are $H_{0\leq 2k\leq 2n}(\mathbb{CP}^n) = \mathbb{Z}$.
- the only non-trivial Homology groups of any Riemann surface of genus $g$ are $H_0 = \mathbb{Z}, H_1 = \mathbb{Z}^{2g}$ and $H_2 = \mathbb{Z}$, such that $\chi = 2 - 2g$.

Let us make a comment on the zeroth homology group. Since $\mathrm{Ker}\partial_0 = \Delta_0$ by definition, all elements of $\Delta_0$ (i.e. every point of $X$) are 0-cycles and $H_0(X)$ is the set of 0-cycles that are not the boundary of any chain of 1-simplices. Since any linear combination of 0-simplices (i.e. point) can be seen as the boundary the 1-simples that joins them, we can see that all points that can be joined by a path are equivalent. For a topological space $X$ where all points can be connected by a path this means that the zeroth homology group is necessarily $H_0(X) = \{nA | a \in \mathbb{Z}\} \cong \mathbb{Z}$, where $A$ any point in $X$. More generally, we can conclude that the number of copies of $\mathbb{Z}_n$ in $H_0(X)$ is the number of path-connected components of $X$.

For a graph $\Gamma$, we can understand from our previous discussions that the only non-trivial homology groups are going to be $H_0(\Gamma) = \mathbb{Z} \times \cdots \times \mathbb{Z}$ where the number of copies is the number of connected components and $H_1(\Gamma) = \mathbb{Z} \times \cdots \times \mathbb{Z}$ where the number of copies is the number of "irreducible" loops.

The $n$th simplicial homology group of a topological space $X$ with a complex simplicial structure $\Delta$ is always of the form

$$H_n^{\Delta}(X) = \underbrace{\mathbb{Z} \times \mathbb{Z}}_{b_n} \oplus \mathbb{Z}_{q_1} \times \cdots \times \mathbb{Z}_{q_s}. \tag{A.7}$$

$b_n$ are the Betti numbers. The *Poincaré polynomial* is

$$P_X(x) = \sum_{i\geq 0} b_i x^i \tag{A.8}$$

and the Euler characteristic of $X$ is given by $P_X(-1)$. Recall that the Euler characteristic is also given by $\chi = 2 - 2g - b - c$, $g$ being the genus, $b$ the number of topological boundaries and $c$ the number of crosscaps.

Note that the homology that we considered is with coefficients in $\mathbb{Z}$, i.e. we started with simplicial chains with coefficients in $\mathbb{Z}$. We can however also consider coefficients in $\mathbb{R}$, in $\mathbb{C}$ or in any ring. In those case however we can easily lose the information about torsion. For example $\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ is non trivial but since $2\mathbb{R} = \mathbb{R}$, $\mathbb{R}/2\mathbb{R} = \{0\}$.

# B | Some finite subgroups

## B.1 | Finite subgroups of SU(2) and SL(2, $\mathbb{C}$)

### B.1.1 | Finite subgroups

The first thing to recall is that every finite subgroup of SL(2, $\mathbb{C}$) is isomorphic to a subgroup of SU(2, $\mathbb{C}$) and vice-versa, so we equivalently talk about the subgroups of SU(2). The finite subgroups of SU(2), called the *binary polyhedral groups*, are the doubles covers of the finite subgroups of SO(3) that are called *polyhedral groups*. They simply constitutes the symmetries of the Platonic solids. The groups fall into two infinite series, associated to the regular polygons, as well as three exceptional, associated with the 5 regular polyhedra: the tetrahedron (self-dual), the cube (and its dual octahedron), the icosahedron (and its dual dodecahedron).

More precisely, the finite subgroups of SL(2, $\mathbb{C}$) are

- $\mathbb{Z}_n$ : cyclic group of order $n$ ($n \geq 2$) generated by

$$\begin{bmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{bmatrix} \tag{B.1}$$

- $2\mathcal{D}_n$ : *binary dihedral groups* (also known as the *dicyclic group*) of order $4n$ ($n \geq 1$) generated by

$$A \equiv \begin{bmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{bmatrix} \quad \text{and } B \equiv \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \tag{B.2}$$

  One can show that $A^n = B^2$ and that $AB = BA^{-1}$ so that $2\mathcal{D}_n = \{B^b A^a | 0 \leq b \leq 3, 0 \leq a \leq n-1\}$. This rewriting of the most general element of the group will be useful.

- $2\mathcal{T}$ : *binary tetrahedral group* of order 24 generated by $D_2$ and

$$C \equiv \frac{1}{\sqrt{2}} \begin{bmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{bmatrix} \tag{B.3}$$

- $2\mathcal{O}$ : *binary octahedral group* of order 48 generated by $\mathcal{T}$ and

$$D \equiv \begin{bmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{bmatrix} \tag{B.4}$$

- $2\mathcal{I}$ : *binary icosahedral group* of order 120 generated by

$$E \equiv -\frac{1}{\sqrt{5}} \begin{bmatrix} \zeta_5^4 - \zeta_5 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{bmatrix} \quad \text{and } F \equiv -\frac{1}{\sqrt{5}} \begin{bmatrix} \zeta_5^2 - \zeta_4^4 & \zeta_5^4 - 1 \\ 1 - \zeta_5 & \zeta_5^3 - \zeta_5 \end{bmatrix} \tag{B.5}$$

with $\zeta_m \equiv e^{i\frac{2\pi}{m}}$ such that $(\zeta_m)^m = 1$. Note that the orders are all divisible by 2. This is because the center of SU(2) is $\mathbb{Z}_2$.

### B.1.2 | Irreducible representations

- $\mathbb{Z}_n$ has $n$ irreducible representations. They are all 1-dimensional (since $\mathbb{Z}_n$ is abelian) and are given by

$$\rho_k(g) = \zeta_n^k \tag{B.6}$$

  with $k = 0, \ldots, n-1$.

- $2\mathcal{D}_n$ has $n+3$ irreducible representations: 4 of dimension 1 and $n-1$ of dimension 2. The 1-dimensional ones are given by

| $n$ | $\rho(A)$ | $\rho(B)$ | $\rho(B^bA^a)$ |
|---|---|---|---|
| even | 1 | 1 | 1 |
| | | $-1$ | $(-1)^b$ |
| | -1 | 1 | $(-1)^a$ |
| | | $-1$ | $(-1)^{a+b}$ |
| odd | 1 | 1 | 1 |
| | | $-1$ | $(-1)^b$ |
| | -1 | $i$ | $(-1)^a i^b$ |
| | | $-i$ | $(-1)^a(-i)^b$ |

and the 2-dimensional ones are given binary by

$$\rho_r(A) = \begin{bmatrix} e^{i\frac{\pi}{n}r} & 0 \\ 0 & e^{-i\frac{\pi}{n}r} \end{bmatrix}$$
$$\rho_r(B) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with $r = 1, \dots, n-1$.

### B.1.3 Character tables

| conj. class repr. | $e$ | $M$ | $M^2$ | $\dots$ | $M^{n-1}$ |
|---|---|---|---|---|---|
| conj. class order | 1 | 1 | 1 | $\dots$ | 1 |
| $V_0$ | 1 | 1 | 1 | $\dots$ | 1 |
| $V_1$ | 1 | $\zeta_n$ | $\zeta_n^2$ | $\dots$ | $\zeta_n^{n-1}$ |
| $V_2$ | 1 | $\zeta_n^2$ | $\zeta_n^4$ | $\dots$ | $\zeta_n^{2(n-1)}$ |
| $V_3$ | 1 | $\zeta_n^3$ | $\zeta_n^6$ | $\dots$ | $\zeta_n^{3(n-1)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | |
| $V_{n-1}$ | 1 | $\zeta_n^{(n-1)}$ | $\zeta_n^{2(n-1)}$ | $\dots$ | $\zeta_n^{(n-1)^2}$ |
| $W$ | 2 | $2\cos\left(\frac{2\pi}{n}\right)$ | $2\cos\left(\frac{4\pi}{n}\right)$ | $\dots$ | $2\cos\left(\frac{2\pi(n-1)}{n}\right)$ |

*Table 4: Character table of $\mathbb{Z}_n$.*

| conj. class repr. | $e$ | $B^2$ | $B$ | $BA$ | $A$ | $A^2$ | $\dots$ | $A^{n-1}$ |
|---|---|---|---|---|---|---|---|---|
| conj. class order | 1 | 1 | $n$ | $n$ | 2 | 2 | $\dots$ | 2 |
| $V_0$ | 1 | 1 | 1 | 1 | 1 | 1 | $\dots$ | 1 |
| $V_1$ | 1 | 1 | $-1$ | $-1$ | 1 | 1 | $\dots$ | 1 |
| $V_2$ | 1 | 1 ou $-1$ | 1 ou $i$ | $-1$ ou $-i$ | $-1$ | 1 | $\dots$ | $(-1)^{n-1}$ |
| $V_3$ | 1 | 1 ou $-1$ | $-1$ ou $-i$ | 1 ou $i$ | $-1$ | 1 | $\dots$ | $(-1)^{n-1}$ |
| $V_4$ | 2 | $-2$ | 0 | 0 | $2\cos\frac{\pi}{n}$ | $2\cos\frac{2\pi}{n}$ | $\dots$ | $2\cos\frac{(n-1)\pi}{n}$ |
| $V_5$ | 2 | 2 | 0 | 0 | $2\cos\frac{2\pi}{n}$ | $2\cos\frac{4\pi}{n}$ | $\dots$ | $2\cos\frac{2(n-1)\pi}{n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $V_{n+2}$ | 2 | $2(-1)^{n-1}$ | 0 | 0 | $2\cos\frac{(n-1)\pi}{n}$ | $2\cos\frac{2(n-1)\pi}{n}$ | $\dots$ | $2\cos\frac{(n-1)^2\pi}{n}$ |
| $W$ | 2 | $-2$ | 0 | 0 | $2\cos\left(\frac{\pi}{n}\right)$ | $2\cos\left(2\frac{\pi}{n}\right)$ | $\dots$ | $2\cos\left(\frac{\pi}{n}(n-1)\right)$ |

*Table 5: Character table of $2\mathcal{D}_n$.*

| conj. class repr. | $e$ | $B^2$ | $B$ | $C$ | $C^2$ | $C^4$ | $C^5$ |
|---|---|---|---|---|---|---|---|
| conj. class order | 1 | 1 | 6 | 4 | 4 | 4 | 4 |
| $V_0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $V_1$ | 2 | $-2$ | 0 | 1 | $-1$ | $-1$ | 1 |
| $V_2$ | 3 | 3 | $-1$ | 0 | 0 | 0 | 0 |
| $V_3$ | 2 | $-2$ | 0 | $e^{i\frac{2\pi}{3}}$ | $-e^{i\frac{2\pi}{3}}$ | $-e^{i\frac{4\pi}{3}}$ | $e^{i\frac{4\pi}{3}}$ |
| $V_3^\vee$ | 2 | $-2$ | 0 | $e^{i\frac{4\pi}{3}}$ | $-e^{i\frac{4\pi}{3}}$ | $-e^{i\frac{2\pi}{3}}$ | $e^{i\frac{2\pi}{3}}$ |
| $V_4$ | 1 | 1 | 1 | $e^{i\frac{2\pi}{3}}$ | $e^{i\frac{2\pi}{3}}$ | $e^{i\frac{4\pi}{3}}$ | $e^{i\frac{4\pi}{3}}$ |
| $V_4^\vee$ | 1 | 1 | 1 | $e^{i\frac{4\pi}{3}}$ | $e^{i\frac{4\pi}{3}}$ | $e^{i\frac{2\pi}{3}}$ | $e^{i\frac{2\pi}{3}}$ |
| $W$ | 2 | $-2$ | 0 | 1 | $-1$ | $-1$ | 1 |

*Table 6: Character table of $2\mathcal{T}$.*

| conj. class repr. | $e$ | $B^2$ | $B$ | $C$ | $C^2$ | $D$ | $BD$ | $D^3$ |
|---|---|---|---|---|---|---|---|---|
| conj. class order | 1 | 1 | 6 | 8 | 8 | 6 | 12 | 6 |
| $V_0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $V_1$ | 2 | $-2$ | 0 | 1 | $-1$ | $-\sqrt{2}$ | 0 | $\sqrt{2}$ |
| $V_2$ | 3 | 3 | $-1$ | 0 | 0 | 1 | $-1$ | 1 |
| $V_3$ | 4 | $-4$ | 0 | $-1$ | 1 | 0 | 0 | 0 |
| $V_4$ | 3 | 3 | $-1$ | 0 | 0 | $-1$ | 1 | $-1$ |
| $V_5$ | 2 | $-2$ | 0 | 1 | $-1$ | $\sqrt{2}$ | 0 | $-\sqrt{2}$ |
| $V_6$ | 1 | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ |
| $V_7$ | 2 | 2 | 2 | $-1$ | $-1$ | 0 | 0 | 0 |
| $W$ | 2 | $-2$ | 0 | 1 | $-1$ | $-\sqrt{2}$ | 0 | $\sqrt{2}$ |

*Table 7: Character table of $2\mathcal{O}$.*

| conj. class repr. | $e$ | $E^2$ | $E$ | $F$ | $F^2$ | $EF$ | $(EF)^2$ | $(EF)^3$ | $(EF)^4$ |
|---|---|---|---|---|---|---|---|---|---|
| conj. class order | 1 | 1 | 30 | 20 | 20 | 12 | 12 | 12 | 12 |
| $V_0$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $V_1$ | 2 | $-2$ | 0 | 1 | $-1$ | $\varphi^+$ | $-\varphi^-$ | $\varphi^-$ | $-\varphi^+$ |
| $V_2$ | 3 | 3 | $-1$ | 0 | 0 | $\varphi^+$ | $\varphi^-$ | $\varphi^-$ | $\varphi^+$ |
| $V_3$ | 4 | $-4$ | 0 | $-1$ | 1 | 1 | $-1$ | 1 | $-1$ |
| $V_4$ | 5 | 5 | 1 | $-1$ | $-1$ | 0 | 0 | 0 | 0 |
| $V_5$ | 6 | $-6$ | 0 | 0 | 0 | $-1$ | 1 | $-1$ | 1 |
| $V_6$ | 4 | 4 | 0 | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ |
| $V_7$ | 2 | $-2$ | 0 | 1 | $-1$ | $\varphi^-$ | $-\varphi^+$ | $\varphi^+$ | $-\varphi^-$ |
| $V_8$ | 3 | 3 | $-1$ | 0 | 0 | $\varphi^-$ | $\varphi^+$ | $\varphi^+$ | $\varphi^-$ |
| $W$ | 2 | $-2$ | 0 | 1 | $-1$ | $\varphi^+$ | $-\varphi^-$ | $\varphi^-$ | $-\varphi^+$ |

*Table 8: Character table of $2\mathcal{I}$, with $\varphi^\pm \equiv (1 \pm \sqrt{5})/2$.*

## B.2 Finite subgroups of SU(3)

The finite subgroups of SU(3) are

- the finite subgroups of SU(2)
- $\Delta(3n^2) = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3$ and $\Delta(3n^2) = (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S^3$
- the exceptional groups

so there are 2 infinite series and 5 exceptional subgroups. Note that they are all divisible by 3 because the center of SU(3) is $\mathbb{Z}_3$.

**Theorem.** Every abelian finite subgroup of SU(3) is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_n$.

## B.3 | Finite subgroups of SU(4)

See [7].

# C | Spacetime geometry: ALE space and orbifolds

Asymptotically locally euclidean (ALE) spaces are a particularly interresting choice of string background to probe with branes for mainly four reasons

(i) they are the resolution (blow-ups) of orbifolds

(ii) there are completely classified: they fall in the ADE classification

(iii) they only break half of the supersymmetry

(iv) they are non-compact therefore we can study them for self-dual type II theory. Why is that ?

Mathematically, an ALE space is complete riemannian $n$-manifold $M$ such that there exists a compact set $K \subset M$ such that $M \backslash K$ is diffeomorphic to $(\mathbb{R}^n \backslash B_0(R))/G$, where $R \in \mathbb{R}_0^+$ is a radius and $G \subset \mathrm{O}(n)$ a subgroup. Additionally, it is asked that the pulled back metric on $\mathbb{R}^n \backslash B_0(R)$ tends to the euclidean flat metric at infinity.

If one considers string theory an the orbifold $\mathbb{R}^4/\Gamma$ where $\Gamma$ is a finite sub group of SU(2), massless states appear from the twisted sector. They are precisely the moduli needed the deform the theory to the one with smooth spacetime, i.e. the resolution of the orbifold. In that sense, is said that the strings know about the metric ALE space and that it is said that strings resolve the singularity. The metric of the ALE space can be recovered if the lagrangian of the resulting field theory is explicitely know, such as for the Wess-Zumino-Witten model. However, it is often not the case.

# D | Elliptic fibrations

An *elliptic curve* over the complex is a connected riemann surface (connected compact 1-dimensional complex manifold) of genus 1. In other words, it is a complex torus equipped with the structure of a complex manifold, or equivalently with a conformal structure. A complex torus can be defined from a complex number $\tau$, called the *period*, as the quotient $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$. The period characterizes the shape of the torus and, by convention, $\tau$ is restricted to the upper-half complex plane $H = \{\tau \in \mathbb{C} | \operatorname{Im}(\tau) > 0\}$. One can see that any complex torus $\mathbb{C}/(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$ can be put in this form. Even restricted to the space $H$, the period can still give equivalent tori for different values. In fact, one can show that two tori of periods $\tau_1$ and $\tau_2$ respectively are equivalent if and only their periods are related in the following way:

$$\tau_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \tau 2 \equiv \frac{a\tau + b}{c\tau + d}, \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C}). \tag{D.1}$$

These transformations are called *modular transformations*. We see that the matrix of a modular transformation and minus this matrix gives the same transformation of the period therefore the group of modular transformations, i.e. the *modular group*, is not $\mathrm{SL}(2,\mathbb{C})$ but $\mathrm{SL}(2,\mathbb{C})/\{\pm I_2\}$. The groups $\mathrm{SL}(2,\mathbb{C})$ is generated by

$$S \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and } T \equiv \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{D.2}$$

that acts on the period as

$$S \cdot \tau : -\frac{1}{\tau}, \qquad T \cdot \tau = \tau + 1. \tag{D.3}$$

Elliptic curves can be described in a very natural way as a cubic curve in $\mathbb{P}^2$. More precisely, one can construct, using the Weierstrass $\wp$-function[11], an analytic isomorphism between the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and and the following cubic in $\mathbb{P}^2$:

$$E : zy^2 = x^3 + fxz^2 + gz^3. \tag{D.4}$$

For a regular curve $E$, there is a unique lattice $\mathbb{Z}+\tau\mathbb{Z}$ (up to modular transformations) such that $E$ and $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ are analytic isomorphic as complex Lie groups (through the same map). It is possible to fully classify the elliptic curves by introducing Klein $j$-invariant, modular curves and modular invariants, which we won't detail here.

An *elliptic fibration* is a bundle of elliptic curves, possibly including singular fibers. These singular fibers have fit in the ADE classification. Over the base space $\mathbb{P}^1$ parametrized by $\tau$, there is a countably infinite number of singularities at $\operatorname{Im}\tau = +\infty$ and each such singularity is of type $A$ or $D$. For finite values of $\tau$ however there is a finite number of singularities. Namely, there are seven singularities: $A_0, A_1, A_2, D_4, E_6, E_7, E_8$. When $z$ approaches one of the singularities, the torus fibre degenerates in a specific way that depends on the type of the singularity. For example, for a type $A_n$ singularity, the torus is pinched in $n+1$ places so as to become a necklace of $n+1$ 2-spheres that intersects at those pinched points. The resulting singular surface is then described by

$$y^2 = x^3 + f(z)x + g(z) \tag{D.5}$$

for some specific polynomials $f$ and $g$.

## D.1 | Connection to Physics: Seiberg-Witten theory

Due to the fact that the period $\tau$ of the torus can be interpreted as the coupling constant of a supersymmetric gauge theory, it is possible to make very fruitful links with physics. The starting idea is to associate to any singularity discussed above an $\mathcal{N}=2$ SYM theory with the global symmetry group of corresponding type. That is, if the singularity is of type $A$, then the global symmetry group is $\mathrm{SU}(n)$ because the Dynkin diagram of its Lie algebra corresponds to type singularities through the geometric McKay correspondence. The most interesting theories are the strongly coupled ones, i.e. the one corresponding to one of the seven singularities at finite distance in $\tau$-space. Indeed, $\tau = \frac{i}{g^2} + \theta$ so the limit $\operatorname{Im}\tau \to +\infty$ is the weak coupling limit $g \to 0$.

As an example, we can consider the original Saiberg-Witten theory, i.e. $\mathcal{N}=2$ SYM with $\mathrm{SU}(2)$ gauge group, $\mathrm{SO}(8,\mathbb{C})$ global symmetry and four hypermultiplets. $\mathrm{SO}(8)$ corresponds to a singularity of type $\mathcal{D}_4$. The name *Seiberg-Witten theory* can be generalized to include all aforementioned strongly coupled ADE theories.

# E | Determinantal varieties as transverse spaces

## E.1 | Basic properties of determinantal varieties

A *determinantal variety* (DV) is a space of matrices with a given upper bound on their ranks. More precisely, given $m, n$ and $r < \min(m,n)$, the DV $Y_r$ of the field $K$ is the set of $m \times n$ matrices over $K$ with rank lower or equal to $r$:

$$Y_r \equiv \{M \in M_{m\times n}(K) | \operatorname{rank} M \leq r\}. \tag{E.1}$$

Recall that a $k$-minor is the determinant of a $k \times k$ sub-matrix and that the rank of a matrix is equal to the biggest integer such that there is a non-vanishing minor of that size. Imposing $\operatorname{rank} M \leq r$ is therefore equivalent to the vanishing of its $(r+1)\times(r+1)$ minors, as it also implies tha vanishing of the biggest minors. This naturally qualifies $R_r$ as affine varieties embedded in $K^{mn}$.

[11] The Weierstrass $\wp$-function $\wp(z,\tau) = \frac{1}{z^3} + \sum_{w\in(\mathbb{Z}+\tau\mathbb{Z})\setminus\{0\}} \left(\frac{1}{(w-z)^2} - \frac{1}{w^2}\right)$ is meromorphic, has double poles at the lattice points $w \in \mathbb{Z}+\tau\mathbb{Z}$, is doubly-periodic: $\wp(z+1,\tau) = \wp(z,\tau)$, $\wp(z+\tau,\tau) = \wp(z,\tau)$ and its derivative $\wp'$ (with respect to $z$) has a pole of order 3 at the origin. They satisfy the Weierstrass equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ where $g_2(\tau) = 60G_4(\tau)$ and $g_3(\tau) = 140G_6(\tau)$ are defined in terms of the Einstein series $G_{2k}(\tau) = \sum_{w\in(\mathbb{Z}+\tau\mathbb{Z})\setminus\{0\}} w^{-2\tau}$.

Let us denote by $X = (x_{ij})$ an arbitrary $m \times n$ matrix. The independent entries $x_{ij}$ are affine coordinates. The $(r+1) \times (r+1)$ minors are therefore homogeneous polynomials of degree $r+1$. The *determinantal ideal* $I_{r+1}(X)$ is the ideal of $k[X]$ generated by these polynomials. The cooridnate ring is

$$R = k[X]/I_{r+1}(X) \tag{E.2}$$

Homogeneity the polynomials implies that $Y_r$ can equivalently be seen as a projective variety in $\mathbb{A}^{mn-1}$.

### E.1.1 | Computing the dimension

Let us compute the dimension of $Y_r$ seen as an affine variety. We consider the space $\mathbb{A}^{mn} \times \mathbf{Gr}(r,m)$, where $\mathbf{Gr}(r,m)$ is the Grassmannian of $r$-planes in an $m$-dimensioanl vector space. Let us define the subsapce

$$Z_r \equiv \{(A, W) | Ax \in W \text{ for all } x \in \mathbb{A}^n\}. \tag{E.3}$$

$Y_r$ and $Z_r$ are birationaly equivalent so $\dim Y_r = \dim Z_r$. We want to compute $Z_r$. First we notice that $Z_r$ is a vector bundle over $\mathbf{Gr}(r,m)$ and we denote it by $Z_r \xrightarrow{\pi_1} \mathbf{Gr}(r,m)$. Now, over the Grassmannian $\mathbf{Gr}(r,m)$, there is a tautologial vector bundle that we denote by $E_{\mathbf{Gr}} \xrightarrow{\pi_2} \mathbf{Gr}(r,m)$ whose fibers are $\pi_2^{-1}(W) = W \cong \mathbb{R}^r$. Finally, $K^m$ can also be seen as a vector bundle, with fibers $\mathbb{R}^m$. We denote it by $E_{K^n} \xrightarrow{\pi_3} K^n$. From $E_{\mathbf{Gr}}$ and $E_{K^n}$, we can construct[12] the vector bundle $\mathrm{Hom}(E_{\mathbf{Gr}}, E_{K^n}) \xrightarrow{\pi_4} \mathbf{Gr}(r,m)$. This vector bundle has the same base space and its fibers are $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^r)$ which are exactly the same as the ones of $Z_r$. So the two vector bundles are isomorphic:

$$Z_r \cong \mathrm{Hom}(E_{\mathbf{Gr}}, E_{K^n}). \tag{E.4}$$

Finally, since the fibers of $\mathrm{Hom}(K^n, E_{\mathbf{Gr}})$ have dimension $nr$, we find

$$\dim Z_r = \dim \mathrm{Hom}(K^n, E_{\mathbf{Gr}}) = \dim \mathbf{Gr}(r,m) + nr = r(m-r) + nr. \tag{E.5}$$

Finally, we conclude that $Y_r$ is a affine variety of dimension $r(m-r)+nr$.

| $m$ | $n$ | $r$ | $\dim_{\mathbb{C}} Y_r$ |
|---|---|---|---|
| 2 | 2 | 1 | 3 |
| 3 | 2 | 1 | 4 |
| 3 | 3 | 1 | 5 |
| 3 | 3 | 2 | 8 |
| 4 | 2 | 1 | 5 |
| 4 | 3 | 1 | 6 |
| 4 | 3 | 2 | 10 |
| 4 | 4 | 1 | 7 |
| 4 | 4 | 2 | 12 |
| 4 | 4 | 3 | 15 |

### E.1.2 | Singularity

Determinantal varieties are singular and possess non-commutative resolutions. $Y_r$ is singlar and the singular locus is contained in the subset of matrices with rank strictly lower than $r$. $Z_r$ is a resolution (over the open set of matrices with rank exactly $r$, this map is an isomorphism), it is called the *Springer desingularization* of Spec$R$.

### E.1.3 | Action and syzygies

$Y_r$ naturally acts on $G = \mathrm{GL}(m,K) \times \mathrm{GL}(n,K)$

[12]Recall that if $E$ and $F$ are vector bundles over $X$, then we can construct a new vetor bundle over $X$, called the Hom-bundle and denoted $\mathrm{Hom}(E,F)$, by defining the fiber over $x \in X$ to be $\mathrm{Hom}(E_x, F_x)$.

## E.2 | Young's lattice

*Young's lattice* is a lattice $Y$ formed by all integer partitions ordered by inclusion of their Young tableau. It is generally used to to describe the irreducible representation sof the symmetric group[13] $S_n$ together with their branching properties. Conventionnally, Young's lattice is depicted in a Hasse diagram, i.e. with element of the same rank shown at the same height and with links such that the descendance of two elements is the union and the parent is the intersection.

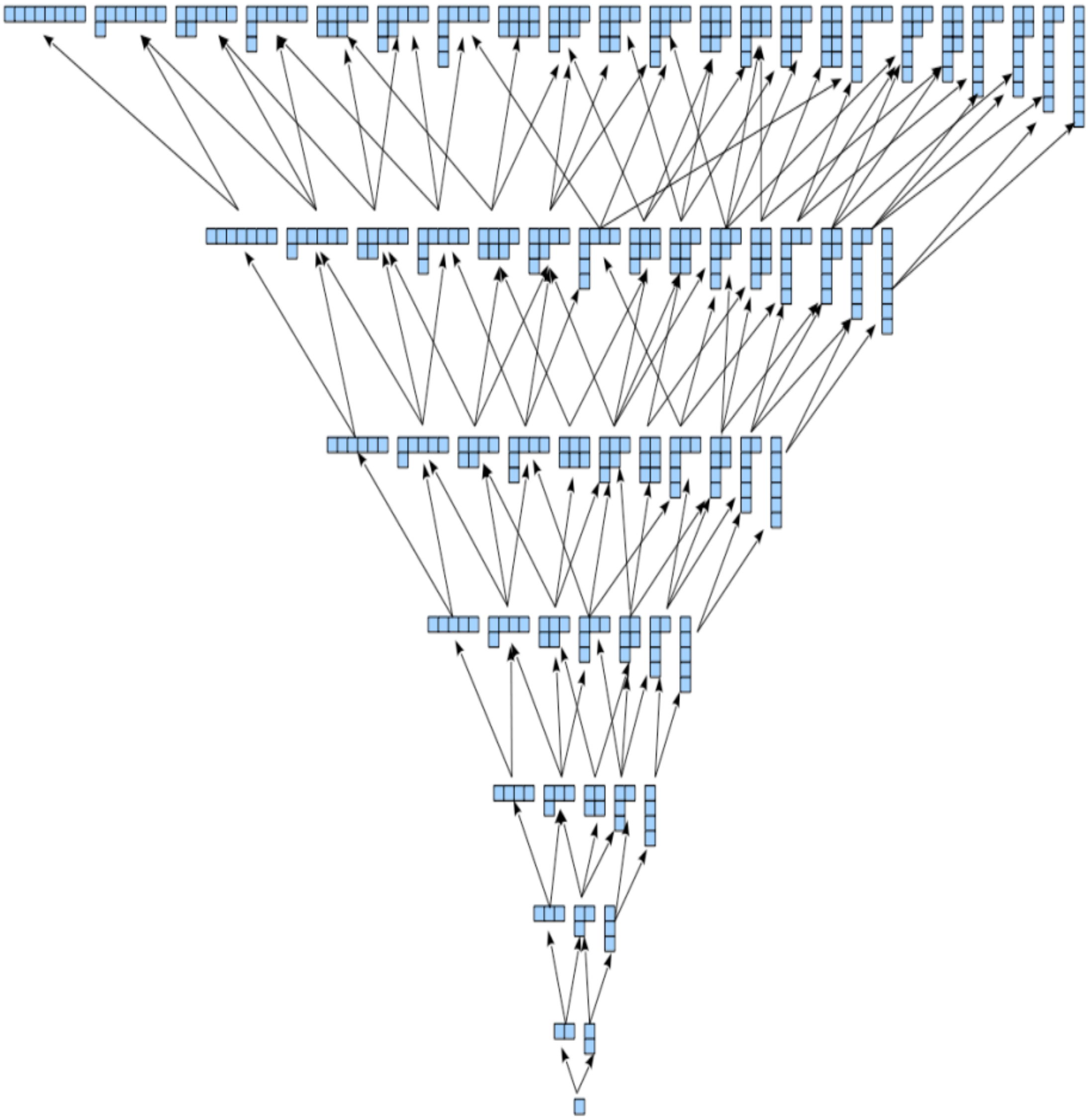

*Figure 13: Young's lattice.*

Young's lattice possess the folling symmetry: the partition $n + n - 1 + \ldots 2 + 1$ of the $n$th triangular number has a Young diagram that looks like a staircase. If we now only keep the elements whose hull is contained in this staircase, we get a subset of Young's lattice. When rank-embedded, this subset clearly has the expected bilateral symmetry of Young's lattice but also a rotational symmetry, which appear more clearly if we move away from this rank-embedding. The rotation group of order $n + 1$ acts on this poset[14]. Since it has both a bilateral and a rotational symmetry it must also have a dihedral symmetry

[13]Two permutations of $S_n$ are equivalent if and only they have they have the same number of cycles of the same sizes. Therefore, the quivalence classes of the symmetric group $S_n$ are parametrized by the partitions of $n$, i.e. by Young diagrams.

[14]Partially ordered set.

and, indeed, the dihedral group $\mathcal{D}_{n+1}$ acts faithfully on this set.

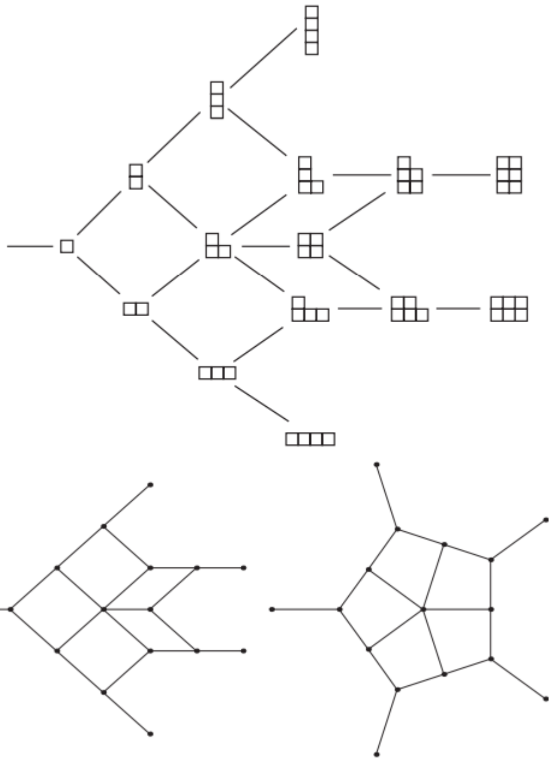

*Figure 14: Example of dihedral symmetry for $n = 4$.*

# F | Some derivations

## F.1 | Invariant configurations for $\mathbb{C} \times \mathbb{C}^2/2\mathcal{D}_n$

### F.1.1 | Gauge field

To find the invariant configurations of the gauge group, we use use the bi-index notation and split the sub-blocks of $A_\mu$ in four categories depending on the dimensionality of the representations that they transform in. Note that it is only necessary to check the invariance under the two generators of $2\mathcal{D}_n$ to ensure invariance under the whole group.

- components $A_{\mu;a\alpha_a,b\beta_b}$ are $1 \times 1$ blocks that transform as $A_{\mu;a\alpha_a,b\beta_b} \mapsto \sigma_a(\gamma) A_{\mu;a\alpha_a,b\beta_b} \sigma_b(\gamma)^{-1}$. It follows that only the component with $a, b = 0, 1$ or $a, b = 2, 3$ can be non-zero to have invariance under $A$. For invariance under $B$, we find that only the component with $a, b = 0, 2$ or $a, b = 1, 3$ can be non-zero if $n$ is even and only the component with $a = b$ if $n$ is odd. In conclusion, the invarint configuration under $A$ and $B$ are of the form

$$(A_{\mu;ab}) = \begin{bmatrix} \times & 0 & 0 & 0 \\ 0 & \times & 0 & 0 \\ 0 & 0 & \times & 0 \\ 0 & 0 & 0 & \times \end{bmatrix} \tag{F.1}$$

  regardless of the parity of $n$.

- components $A_{\mu;a\alpha_a,r\beta_r}$ are $1 \times 2$ blocks that transform as $A_{\mu;a\alpha_a,r\beta_r} \mapsto \sigma_a(\gamma) A_{\mu;a\alpha_a,r\beta_r} \mu_r(\gamma)^{-1}$. More explicitely, each block is of the form $\begin{bmatrix} x_1 & x_2 \end{bmatrix}$ and transforms as

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \mapsto \sigma_a(A) \begin{bmatrix} x_1 \zeta_{2n}^{-r} & x_2 \zeta_{2n}^{r} \end{bmatrix}$$

under the generator $A$. This never invariant unless $a = b = 0$. There is no need to check the invariance under $B$ since all these component are already all zero.

- components $A_{\mu;r\alpha_r,a\beta_a}$ are $2 \times 1$. The situation is exactly the samme as in the previous point: they must all vanish.
- components $A_{\mu;r\alpha_r,s\beta_s}$ are $2 \times 2$ blocks that transform as $A_{\mu;r\alpha_r,s\beta_s} \mapsto \mu_r(\gamma) A_{\mu;r\alpha_r,s\beta_s} \mu_s(\gamma)^{-1}$. Generically speaking, an invariant block under $A$ must satisfy

$$\begin{bmatrix} \zeta_{2n}^{r} & 0 \\ 0 & \zeta_{2n}^{-r} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} \zeta_{2n}^{-s} & 0 \\ 0 & \zeta_{2n}^{s} \end{bmatrix} = \begin{bmatrix} x_1 \zeta_{2n}^{r-s} & x_2 \zeta_{2n}^{r+s} \\ x_3 \zeta_{2n}^{-r-s} & x_4 \zeta_{2n}^{-r+s} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}. \tag{F.2}$$

  There are two possibilities to have non-vanishing component: $\zeta_{2n}^{r-s} = 1$ and $x_2 = x_3 = 0$ or $\zeta_{2n}^{r+s} = 1$ and $x_1 = x_4 = 0$ but the latter is actually not possible since $r, s = 1, \ldots, n-1$. To we find that the blocks must be of the form

$$A_{\mu;r\alpha_r,s\beta_s} = \begin{bmatrix} \times & 0 \\ 0 & \times \end{bmatrix} \tag{F.3}$$

  if $r = s$ and vanishing otherwise. For invariance under $B$, a blocks must satisfy

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -x_4 & x_3 \\ x_2 & -x_1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}. \tag{F.4}$$

  which is only possible if $x_1 - x_4$ and $x_2 = x_3$. Finally, we find that invariance under $A$ and $B$ imposes the block to be of the form

$$A_{\mu;r\alpha_r,s\beta_s} = \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \tag{F.5}$$

  if $r = s$ and vanishing otherwise.

The invariant gauge field configurations were found to be of the form

$$A_\mu = \begin{bmatrix} \times & 0 & 0 & 0 & & & & & & \cdots & 0 \\ 0 & \times & 0 & 0 & & & & & & & \\ 0 & 0 & \times & 0 & & & & & & & \\ 0 & 0 & 0 & \times & & & & & & & \\ & & & & x_1 & 0 & & & & & \\ & & & & 0 & -x_1 & & & & & \\ & & & & & & x_2 & 0 & & & \\ & & & & & & 0 & -x_2 & & & \\ \vdots & & & & & & & & \ddots & & \vdots \\ & & & & & & & & & x_{n-1} & 0 \\ 0 & & & & & & & & & 0 & -x_{n-1} \end{bmatrix} \tag{F.6}$$

where each entry $(i, j)$ is an arbitrary block of size $N_i \times N_j$.

### Scalar fields

For the real scalar fields $X^m$, we need the action of $2\mathcal{D}_n$ on $\mathbb{C}^3$:

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \overset{A}{\longmapsto} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta_{2n} & 0 \\ 0 & 0 & \zeta_{2n}^{-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \qquad \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \overset{B}{\longmapsto} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}. \tag{F.7}$$

The partitionning of $X^m$ is similar to $A_\mu$. The additional difficulty is come from R-symmetry. Since it acts differently on the different components, we have the study themalmost one by one.

- the fields $X^0$ and $X^1$ are left untouched by R-symmetry, meaning that the invariant configurations have the same form than the gauge field, i.e. (F.6).
- $X^{2,3}_{a\alpha_a,b\beta_b}$ transforms under $A$ as $X^{2,3}_{a\alpha_a,b\beta_b} \mapsto \xi_{2n} \sigma_a(A) X^{2,3}_{a\alpha_a,b\beta_b} \sigma_b(A)^{-1}$. The only configurations that are left invariant are therefore the ones such that $\xi_{2n} \sigma_a(A) \sigma_b(A)^{-1} = 1$, which is never the case. So $X^{2,3}_{a\alpha_a,b\beta_b} = 0$ for all $a, b = 0, \ldots, 3$.

- $X^{2,3}_{a\alpha_a,k\beta_k}$ transforms under $A$ as $X^{2,3}_{a\alpha_a,k\beta_k} \mapsto \xi_{2n}\sigma_a(A)X^{2,3}_{a\alpha_a,k\beta_k}\mu_k(A)^{-1}$. More explicitely, if we denote a block$X^{2,3}_{a\alpha_a,k\beta_k}$ by $\begin{bmatrix} x_1 & x_2 \end{bmatrix}$, we get

$$\begin{bmatrix} \xi_{2n}^{k+1}\sigma_a(A)x_1 & \xi_{2n}^{-k+1}\sigma_a(A)x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \tag{F.8}$$

therefore we can have $x_1 \neq 0$ iff $\sigma_a(A) = -1$(i.e. $a = 2,3$) and $k = n-1$, and we can have $x_1 \neq 0$ iff $\sigma_a(A) = 1$(i.e. $a = 0,1$) and $k = 1$.

- $X^{2,3}_{k\alpha_k,b\beta_b}$ transforms under $A$ as $X^{2,3}_{k\alpha_k,b\beta_b} \mapsto \xi_{2n}\mu_k(A)X^{2,3}_{k\alpha_k,b\beta_b}\sigma_a(A)^{-1}$. Similarly to the previous case, we can write the blocks $X^{2,3}_{k\alpha_k,a\beta_a}$ as $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we get

$$\begin{bmatrix} \xi_{2n}^{k+1}\sigma_a(A)x_1 \\ \xi_{2n}^{-k+1}\sigma_a(A)x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{F.9}$$

therefore the conditions are exactly the same: we can have $x_1 \neq 0$ iff $\sigma_a(A) = -1$(i.e. $a = 2,3$) and $k = n-1$, and we can have $x_1 \neq 0$ iff $\sigma_a(A) = 1$(i.e. $a = 0,1$) and $k = 1$.

- $X^{2,3}_{k\alpha_k,l\beta_l}$ transforms under $A$ as $X^{2,3}_{k\alpha_k,l\beta_l} \mapsto \xi_{2n}\mu_k(A)X^{2,3}_{k\alpha_k,l\beta_l}\mu_l(A)^{-1}$. Again, we can write the blocks $X^{2,3}_{k\alpha_k,l\beta_l}$ as $\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ and we get

$$\begin{bmatrix} \zeta_{2n}^{k-l+1}x_1 & \zeta_{2n}^{k+l+1}x_2 \\ \zeta_{2n}^{-k-l+1}x_3 & \zeta_{2n}^{-k+l+1}x_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \tag{F.10}$$

therefore, we can have

  - $x_1 \neq 0$ iff $l = k+1$,
  - $x_2 \neq 0$ iff $l = -k-1$ (not possible),
  - $x_3 \neq 0$ iff $l = -k+1$ (not possible),
  - $x_4 \neq 0$ iff $l = k-1$.

For $X^{4,5}_{i\alpha_i,j\beta_j}$, the reasonning is exactly the same but with the $R$-symmetry acting as $\zeta_{2n}^{-1}$ instead of $\zeta_{2n}$. After similar computations, we get that the components $X^{4,5}_{a\alpha_a,b\beta_b}$ must be all vanishing too and the components $X^{4,5}_{a\alpha_a,k\beta_k} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ can have $x_1 \neq 0$ iff $a = 0,1$ and $k = 1$ and $x_2 \neq 0$ iff $a = 2,3$ and $k = n-1$. The same goes for the components $X^{4,5}_{k\alpha_k,b\beta_b} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and, at last, for the components $X^{4,5}_{k\alpha_k,l\beta_l} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, we find

  - $x_1 \neq 0$ iff $l = k-1$,
  - $x_2 \neq 0$ iff $l = -k+1$ (not possible),
  - $x_3 \neq 0$ iff $l = -k-1$ (not possible),
  - $x_4 \neq 0$ iff $l = k+1$.

We have established what configurations are invariant under the generator $A$, equivalently under the subgroup of $2\mathcal{D}_n$ generated by $A$. What about $B$? The action of $R$-symmetry for $B$ is more tiresome because it is not diagonal, see (F.7). This implies that components get exchanged. More precisely, recall our notations $z_1 = X^0 + iX^1$, etc, if we rewrite (F.7) in terms of real components, we get that

$$\begin{bmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \\ X^4 \\ X^5 \end{bmatrix} \overset{B}{\longmapsto} \begin{bmatrix} X^0 \\ X^1 \\ -X^5 \\ X^4 \\ -X^3 \\ X^2 \end{bmatrix}. \tag{F.11}$$

For the components $X^2_{a\alpha_a,b\beta_b}$, this implies that $X^2_{a\alpha_a,b\beta_b} = -\sigma_a(B)\sigma_b(B)^{-1}X^5_{a\alpha_a,b\beta_b}$. This completely fixes $X^5_{a\alpha_a,b\beta_b}$ in terms of $X^2_{a\alpha_a,b\beta_b}$. The can be done the other components of $X^2$, they we find that they all determine the ones of $X^5$. Without fully splitting each fields into components, we see that we must have

$$X^2_{ij} = -\rho_i(B)X^5_{ij}\rho_j(B)^{-1}, \tag{F.12}$$
$$X^3_{ij} = \rho_i(B)X^4_{ij}\rho_j(B)^{-1}, \tag{F.13}$$
$$X^4_{ij} = -\rho_i(B)X^3_{ij}\rho_j(B)^{-1}, \tag{F.14}$$
$$X^5_{ij} = \rho_i(B)X^2_{ij}\rho_j(B)^{-1}, \tag{F.15}$$
$$\tag{F.16}$$

bto have invariance under $B$. This equations imply in particular that $X^2_{kl} = -\rho_k(B^2)X^2_{kl}\rho_l(B^2)^{-1}$. Since, $\rho_k(B^2) = -\mathbb{1}_{2\times 2}$ for every $k$, we get that all components $X^2_{kl}$ must be vanishing. In turn, this implies the components $X^{3,4,5}_{kl}$ must also all vanish. This cannot be true.

## F.2

We want to compute the sum

$$\sum_{a=1}^{\lfloor n/3\rfloor} \left\lfloor \frac{n-3a}{2} + 1 \right\rfloor = \left\lfloor \frac{n}{3} \right\rfloor + \sum_{a=1}^{\lfloor n/3\rfloor} \left\lfloor \frac{n-3a}{2} \right\rfloor. \tag{F.17}$$

Let us write $n \in \mathbb{N}$ as $n = 3m + r$ with $r = 0, 1$ or $2$ and $m \in \mathbb{N}$. Regardless of $r$, we have $\lfloor n/3 \rfloor = m$ and

$$\sum_{a=1}^{\lfloor n/3\rfloor} \left\lfloor \frac{n-3a}{2} \right\rfloor = \sum_{a=1}^{m} \left\lfloor \frac{3}{2}(m-a) + \frac{r}{2} \right\rfloor = \sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + \frac{r}{2} \right\rfloor. \tag{F.18}$$

- if $r = 0$, then (F.18) becomes

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a \right\rfloor = \sum_{a=0}^{m-1} a + \sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = \frac{(m-1)m}{2} + \sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor. \tag{F.19}$$

  Now if $m$ is even, we have

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = 2 \sum_{a=0}^{\lfloor \frac{m-1}{2} \rfloor} a = 2 \sum_{a=0}^{\frac{m}{2}-1} a = \left(\frac{m}{2} - 1\right)\frac{m}{2} \tag{F.20}$$

  and if $m$ is odd,

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = 2 \sum_{a=0}^{\lfloor \frac{m-2}{2} \rfloor} a + \left\lfloor \frac{m-1}{2} \right\rfloor = 2 \sum_{a=0}^{\frac{m-3}{2}} a + \frac{m-1}{2} = \frac{(m-1)^2}{4} \tag{F.21}$$

  so

$$\sum_{a=0}^{m-1} \left\lfloor \frac{a}{2} \right\rfloor = \begin{cases} \left(\frac{m}{2} - 1\right)\frac{m}{2}, & \text{if } m \text{ is even} \\ \frac{(m-1)^2}{4}, & \text{if } m \text{ is odd} \end{cases}. \tag{F.22}$$

  and

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3a}{2} \right\rfloor = \begin{cases} \frac{m(3m-4)}{4}, & \text{if } m \text{ is even} \\ \frac{(m-1)(3m-1)}{4}, & \text{if } m \text{ is odd} \end{cases}. \tag{F.23}$$

- if $r = 1$, then (F.18) becomes

$$\sum_{a=0}^{m-1} \left\lfloor \frac{3}{2}a + \frac{1}{2} \right\rfloor = \sum_{a=0}^{m-1} a + \sum_{a=0}^{m-1} \left\lfloor \frac{a+1}{2} \right\rfloor = \frac{(m-1)m}{2} + \sum_{a=0}^{m-1} \left\lfloor \frac{a+1}{2} \right\rfloor \tag{F.24}$$

and

$$\sum_{a=0}^{m-1}\left\lfloor\frac{a+1}{2}\right\rfloor=\sum_{a=1}^{m}\left\lfloor\frac{a}{2}\right\rfloor=\sum_{a=0}^{m}\left\lfloor\frac{a}{2}\right\rfloor=\begin{cases}\frac{m^2}{4}, & \text{if } m \text{ is even}\\ \frac{m^2-1}{4}, & \text{if } m \text{ is odd}\end{cases} \tag{F.25}$$

by (F.22) so

$$\sum_{a=0}^{m-1}\left\lfloor\frac{3}{2}a+\frac{1}{2}\right\rfloor=\begin{cases}\frac{m(3m-2)}{4}, & \text{if } m \text{ is even}\\ \frac{3m^2-2m-1}{4}, & \text{if } m \text{ is odd}\end{cases} \tag{F.26}$$

- if $r=2$, then (F.18) becomes

$$\sum_{a=0}^{m-1}\left\lfloor\frac{3}{2}a+1\right\rfloor=m+\sum_{a=0}^{m-1}\left\lfloor\frac{3}{2}a\right\rfloor. \tag{F.27}$$

so

$$\sum_{a=0}^{m-1}\left\lfloor\frac{3}{2}a+1\right\rfloor=\begin{cases}\frac{3m^2}{4}, & \text{if } m \text{ is even}\\ \frac{3m^2+1}{4}, & \text{if } m \text{ is odd}\end{cases} \tag{F.28}$$

from (F.23).

Finally, we can write $m=2k$ if $m$ if even and $m=2k+1$ if $m$ is odd in order to distinguish the six different cases. We get

$$a(n)\equiv\sum_{a=1}^{\lfloor n/3\rfloor}\left\lfloor\frac{n-3a}{2}+1\right\rfloor=\begin{cases}2k+\frac{2k(6k-4)}{4}, & \text{if } n=6k\\ 2k+\frac{2k(6k-2)}{4}, & \text{if } n=6k+1,\\ 2k+\frac{12k^2}{4}, & \text{if } n=6k+2,\\ (2k+1)+\frac{2k(6k+2)}{4}, & \text{if } n=6k+3,\\ (2k+1)+\frac{3(2k+1)^2-2(2k+1)-1}{4}, & \text{if } n=6k+4,\\ (2k+1)+\frac{3(2k+1)^2+1}{4}, & \text{if } n=6k+5\end{cases} \tag{F.29}$$

$$=\begin{cases}3k^2, & \text{if } n=6k\\ 3k^2+k, & \text{if } n=6k+1,\\ 3k^2+2k, & \text{if } n=6k+2,\\ 3k^2+3k+1, & \text{if } n=6k+3,\\ 3k^2+4k+1, & \text{if } n=6k+4,\\ 3k^2+5k+2, & \text{if } n=6k+5\end{cases}. \tag{F.30}$$

Starting from $n=1$, the first value of this sequence is : $0,0,1,1,2,3,4,5,7,8,10,12,\ldots$. Uppon further analysis, this correspond to the sequence A001399, that have several interpretations:

- the number of partitions of $n$ into at most 3 parts. This makes sense with our initial problem: finding all the $a,b,c$'s such that $a+b+c=n$,
- the number of connected graphs with 3 nodes and $n$ edges (where multiple edges between the same nodes are allowed),
- the number of non-negative solutions to $b+2c+3d=n$,

as well as many others. Finally, we note that we can simply write

$$a(n)=\text{round}\left(\frac{n^2}{12}\right). \tag{F.31}$$

# G References guide

- General strings and D-branes: [12],[13],[14]
- He: review: [11], thesis: [15]

- Orbifold construction for $\Gamma \subset \mathrm{SU}(2)$: type $A$ : [16], type $D$ and $E$: [**PhysRevD.55.6382**]
- Orbifold construction for $\Gamma \subset \mathrm{SU}(3)$:[8]
- Orbifold construction for $\Gamma \subset \mathrm{SU}(4)$:[7]
- Formalization of projection to daughter theories: [4],[17]
- Quivers representations and varieties: [18],[19]
- On toric varieties: [20],[21]([22],[23])
- Forward algorithm for toric singularities devlopments: [24],[25],[26],[27],[28], formalization: [29],[30],[31]
- Toric diagrams, dimer diagrams and Higgsing: [32]
- Fractional branes: [33]
- Formalization of inverse algorithm for toric singularities: [10]
- geometry and K3 surfaces: [34]
- general review of SYM and their brane description: [35]
- Hanany-Witten setup: [36]
- geometric engineering: [6],[37],[38]
- link between graph theory and Yang-Mills (constructed from string theory with the three different methods) to study the finiteness of the theories: [7]

# Todo list

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