Worksheet on

The McKay Correspondence

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1.1 | Classical correspondence

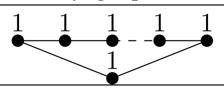
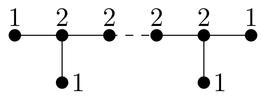
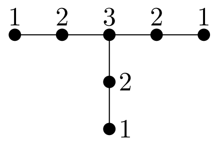
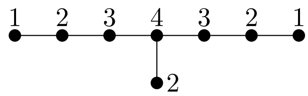
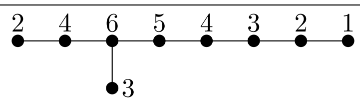
$\Gamma \subset \mathrm{SU}(2)$	Platonic solids	McKay graph	Algebraic variety
$\mathbb{Z}_n$		 (n nodes)	$z^n + xy = 0$
$2\mathcal{D}_n$	n-polygon	 ($n+3$ nodes)	$x^2 + y^2 z + z^{n-1} = 0$
$2\mathcal{T}$	tetrahedron	 (7 nodes)	$x^2 + y^3 + z^4 = 0$
$2\mathcal{O}$	cube octahedron	 (8 nodes)	$x^2 + y^3 + yz^3 = 0$
$2\mathcal{I}$	icosahedron dodecahedron	 (9 nodes)	$x^2 + y^3 + z^5 = 0$

Table 1: Binary polyhedral groups and their McKay graphs.Labels over the vertices are the dimension of the representation. We erase the arrow ends if they go in both directions and erase the label if it is equal to 1*.*

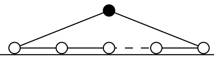
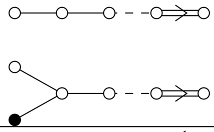
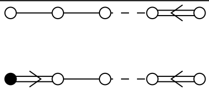
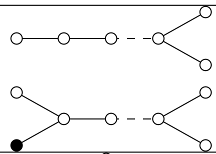
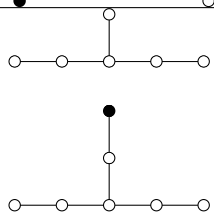
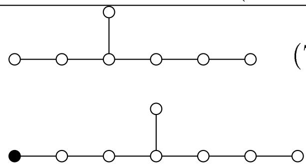
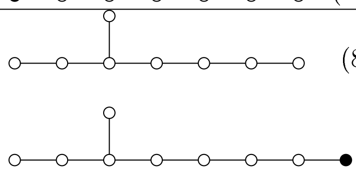
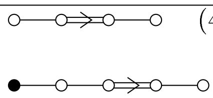
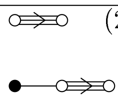
Simple Lie algebra	Simply laced	Dynkin diagram Extended Dybkin diagram
$\mathfrak{sl}(n+1,\mathbb{C}), n \geq 1$	yes	A_n : (n nodes)  $\widetilde{A}_n$: ($n+1$ nodes)
$\mathfrak{so}(2n+1,\mathbb{R}), n \geq 2$	no	B_n : (n nodes)  $\widetilde{B}_n$: ($n+1$ nodes)
$\mathfrak{sp}(2n,\mathbb{C}), n \geq 3$	no	C_n : (n nodes)  $\widetilde{C}_n$: ($n+1$ nodes)
$\mathfrak{so}(2n,\mathbb{R}), n \geq 4$	yes	D_n : (n nodes)  $\widetilde{D}_n$: ($n+1$ nodes)
$\mathfrak{e}_6$	yes	E_6 : (6 nodes)  $\widetilde{E}_6$: (7 nodes)
$\mathfrak{e}_7$	yes	E_7 : (7 nodes)  $\widetilde{E}_7$: (8 nodes)
$\mathfrak{e}_8$	yes	E_8 : (8 nodes)  $\widetilde{E}_8$: (9 nodes)
$\mathfrak{f}_4$	no	F_4 : (4 nodes)  $\widetilde{F}_4$: (5 nodes)
$\mathfrak{g}_2$	no	G_2 : (2 nodes)  $\widetilde{G}_2$: (3 nodes)

Figure 1: Simple Lie algebras and their (extended) Dynkin diagrams. The first four algebras are the classical simple Lie algebras and the last five are the excpetional simple Lie algebras.

Finally, we can see the following correspondence between the extended Dynkin diagrams and the McKay graphs.

Simply Lie group	Simply laced Lie algebra	Extended Dybkin diagram	Finite subgroup of SO(3)	Finite subgroup of SU(2)
$\mathrm{SU}(n+1)$	$\mathfrak{sl}(n+1,\mathbb{C})$	$\widetilde{A}_n$	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_{n+1}$
$\mathrm{SO}(2n), \mathrm{Spin}(2n)$	$\mathfrak{so}(2,\mathbb{R})$	$\widetilde{D}_n$	$\mathcal{D}_{2(n-2)}$	$2\mathcal{D}_{2(n-2)}$
$E6$	$\mathfrak{e}_6$	$\widetilde{E_6}$	$\mathcal{T}$	$2\mathcal{T}$
$E7$	$\mathfrak{e}_7$	$\widetilde{E_7}$	O	2O
$E8$	$\mathfrak{e}_8$	$\widetilde{E_8}$	$\mathcal{I}$	$2\mathcal{I}$

Figure 2: Classical McKay correspondence.

1.2 Geometrical McKay correspondence

The geometrical McKay correspondence is the bijection between the exeptional graph of the blow up of orbifolds $\mathbb{C}^2/\Gamma$ ($\Gamma \subset \mathrm{SU}(2)$) and the McKay graphs of Γ.