Lista 4 - Álgebra Linear

Exercícios transcritos do livro Gilbert Strang - Álgebra Linear e suas aplicações

Exercício 1. Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. Verify that the trace equals the sum of the eigenvalues, and the determinant equals their product.

Para achar os auto-valores, eu preciso encontrar os λ tal que $det(A - \lambda I) = 0$.

$$\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} = 4 - 5\lambda + \lambda^2 + 2 = \lambda^2 - 5\lambda + 6 = 0$$

$$\lambda \in \{2, 3\}$$

Agora, para os auvto-vetores tenho que achar o espaço nulo da matriz $A - \lambda I$ para cada um dos autovalores $\Rightarrow$

- $\lambda_1 = 2$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = -x_2 \Rightarrow \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \alpha \in \mathbb{R}$$

- $\lambda_2 = 3$

$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2y_1 = y_2 \Rightarrow \alpha \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \alpha \in \mathbb{R}$$

$T(A) = 5 = \lambda_1 + \lambda_2$ e $det(A) = 6 = \lambda_1\lambda_2$

Exercício 5. Find the eigenvalues and the eigenvectors of

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and $\lambda_1\lambda_2\lambda_3$ equals the determinant.

Primeiro para A, nós temos:

$$(A - \lambda I) = \begin{bmatrix} 3-\lambda & 4 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -\lambda \end{bmatrix} \Rightarrow$$

$$\begin{vmatrix} 3-\lambda & 4 & 2 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = 3\lambda(\lambda-1) - \lambda^2(\lambda-1) = (3\lambda - \lambda^2)(\lambda - 1) = 0 \Rightarrow$$

$$\lambda \in \{0, 1, 3\}$$

Agora que temos os auto-valores, preciso encontrar os auto-vetores:

- $\lambda_1 = 0$

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 1 \\ -1 \\ \frac{1}{2} \end{bmatrix}, \alpha \in \mathbb{R}.$$

- $\lambda_2 = 1$

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}.$$

- $\lambda_3 = 3$

$$\begin{bmatrix} 0 & 4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}.$$

Agora para B, nós temos:

$$(B-\lambda I) = \begin{bmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{bmatrix} \Rightarrow$$

$$\begin{vmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = (\lambda+2)(\lambda-2)(2-\lambda) = 0 \Rightarrow$$

$$\lambda \in \{-2, 2\}$$

Agora que temos os auto-valores, preciso encontrar os auto-vetores:

- $\lambda_1 = 2$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 1 \\ \beta \\ 1 \end{bmatrix}, \alpha, \beta \in \mathbb{R}.$$

- $\lambda_2 = -2$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \alpha \in \mathbb{R}.$$

Exercício 8. Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into

$$det(A-\lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\dots(\lambda_n - \lambda) \quad (16),$$

and making a clever choice of λ.

Atribuindo o valor de 0 a λ, temos que:

$$det(A - 0I) = \prod_{i=1}^{n} \lambda_n - 0 \Rightarrow det(A) = \prod_{i=1}^{n} \lambda_n$$

Exercício 9. Show that the trace equals the sum of the eigenvalues, in two steps. First, find the coefficient of $(-\lambda)^{n-1}$ on the right side of equation (16). Next, find all the terms in

$$det(A - \lambda I) = det \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

that involve $(-\lambda)^{n-1}$. They all come from the main diagonal! Find that coefficient of $(-\lambda^{n-1})$ and compare.

De (16), podemos ver que:

$$det(A - \lambda I) = \prod_{i=1}^{n} (\lambda_n - \lambda),$$

ou seja, temos que o coeficiente de $(-\lambda)^{n-1}$ será $\sum_{i=1}^{n} \lambda_n$.

Do que foi dado no enunciado podemos ver que ao calcular o determinante de $A - \lambda I$, o coeficiente do termo $(-\lambda)^{n-1}$ será $\sum_{i=1}^{n} a_{ii}$.

Desse modo, pode-se ver que ao calcularmos o determinante de $A - \lambda I$ de dois modos diferentes, chegamos em dois coeficientes para o termo $(-\lambda)^{n-1}$, logo eles devem ser iguais, pois o determinante é único $\Rightarrow$

$$\Rightarrow T(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

Exercício 11. ***The eigenvalues of** A **equal the eigenvalues of*** A^T. This is because $det(A - \lambda I)$ equals $det(A^T - \lambda I)$. That is true because _____. Show by an example that the eigenvectors of A and A^T are not the same.

Sabemos que $det(M) = det(M^T)$ pra qualquer matriz M. Podemos então dizer que $det(A - \lambda I) = det\left[(A - \lambda I)^T\right]$. Porém, $(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$. Logo, temos que $det(A - \lambda I) = det\left[(A - \lambda I)^T\right] = det(A^T - \lambda I)$.

$$\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} \Rightarrow \lambda^2 - 6\lambda - 7 = 0 \Rightarrow \lambda \in \{-1, 7\}$$

- $\lambda = -1$

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

- $\lambda = 7$

$$\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 3 & 5 \end{bmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} \Rightarrow \lambda^2 - 6\lambda - 7 = 0 \Rightarrow \lambda \in \{-1, 7\}$$

- $\lambda = -1$

$$\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

- $\lambda = 7$

$$\begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Logo os eigenvectors são diferentes.

Exercício 12. Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Primeiro acharei os eigenvalues:

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = \lambda^2 - 25 = 0 \Rightarrow \lambda = \pm 5 \begin{vmatrix} a-\lambda & b \\ b & a-\lambda \end{vmatrix} = \lambda^2 - 2a\lambda + a^2 - b^2 \Rightarrow \lambda = a \pm b$$

Agora precisamos achar os eigenvectors:

Primeira Matriz:

- 5:

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

- -5:

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \alpha \in \mathbb{R}$$

Segunda Matriz:

- $a + b$

$$\begin{bmatrix} -b & b \\ b & -b \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

- $a - b$

$$\begin{bmatrix} b & b \\ b & b \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \alpha \in \mathbb{R}$$

Exercício 21. Compute the eigenvalues and eigenvectors of A and A^{-1}:

$$A = \begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

A^{-1} has the ___ eigenvectors as A. When A has eigenvalues λ_1 and λ_2, its inverse has eigenvalues ___.

Primeiro vamos calcular os eigenvalues:

A:

$$\begin{vmatrix} -\lambda & 2 \\ 2 & 3-\lambda \end{vmatrix} \Rightarrow \lambda \in \{-1, 4\}$$

B:

$$\begin{vmatrix} -\frac{3}{4}-\lambda & \frac{1}{2} \\ \frac{1}{2} & -\lambda \end{vmatrix} \Rightarrow \lambda \in \{-1, \tfrac{1}{4}\}$$

Agora vamos achar os eigenvectors:

Primeira Matriz:

- -1:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

- -4:

$$\begin{bmatrix} 4 & 2 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \alpha \in \mathbb{R}$$

Segunda Matriz:

- -1:

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

- -4:

$$\begin{bmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \alpha \in \mathbb{R}$$

<u>**Conclusão**</u>: Os eigenvectors de A e A^{-1} são iguais e os eigenvalues de A^{-1} são os inversos dos eigenvalues de A.

Exercício 39. Challenge problem: *Is there a real 2 by 2 matrix* (other than I) *with* $A^3 = I$? Its eigenvalues must satisfy $\lambda^3 = I$. They can be $e^{2\pi i/3}$ and $e^{-2\pi i/3}$. What trace and determinant would this give? Contruct A.

A matriz A seria a matriz de rotação de 120°. Como seus eigenvalues são $e^{2\pi i/3}$ e $e^{-2\pi i/3}$, o traço será a soma desses 2, ou seja $e^{2\pi i/3} + e^{-2\pi i/3}$ e o determinante seria o produto, ou seja $e^{2\pi i/3}e^{-2\pi i/3} = e^0 = 1$

Exercício 40. There are six 3 by 3 permutation matrices P. What numbers can be the *determinants* of P? What numbers can be *pivots*? What numbers can be the *trace* of P? What *four numbers* can be eigenvalues of P?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

O determinante pode ser 0 ou -1.

O único número que pode ser o pivô é o 1.

O traço pode ser 3, 1 e 0.

Fazendo as contas que fizemos até agora na lista toda (ficou no meu rascunho), os eigenvalues podem ser 1, -1, i, $-i$.