8 Annex

8.1 Thermal model eigenvalue demonstration

In this part we wish to explain the computation of the eigenvalues coming from the thermal graph model devised in the mathematical part.

We define :

$$T = \begin{bmatrix} T_1 & ... & T_i & ... & T_n \end{bmatrix}^\intercal$$

$$C_{-1} = \begin{bmatrix} 1/C_1 & ... & 1/C_i & ... & 1/C_n \end{bmatrix}^\intercal$$

$$1_n = \begin{bmatrix} 1 & ... & 1 \end{bmatrix}^\intercal, \ \ \{T, C_{-1}, 1_n \in \mathbb{R}^n\}$$

The computation of the temporal derivative is given by a function $\frac{dT}{dt} = g(t,T) = f(T)+B(t,T)$. To compute the eigenvalues we focus on the linear part of the problem given by

$$f(T) = C_{-1} \odot ((1_n \times T^\intercal - T \times 1_n{}^\intercal) \odot M_R) \times 1_n),$$

$$f(T) = C_{-1} \odot ((f_1(T) \odot M_R) \times 1_n) = C_{-1} \odot (f_2(T) \times 1_n) = C_{-1} \odot (f_3(T))$$

.
The functions $f_1(T), f_2(T)$ are linear applications of $\mathbb{R}^n \rightarrow \mathbb{R}^{n\times n}$, and there can be represented by a matrices $(A, A_2, A_3) \in \mathbb{R}^{n\times n\times n}$.
This matrix A can be constructed using a basis of $\mathbb{R}$ by the following formula : $A_{i,j,k} = B^k_{i,j}$, with $B^k = f_1(e_k)$, $e = \{e_1, ..., e_n\}$, being an orthonormal basis of the $\mathbb{R}^n$ linear space. Therefore we have

$$(f_1(T))_{i,j} = \sum_{k=1}^{n} A_{i,j,k} \cdot T_k$$

. From the element-wise Hadamard product,

$$(f_2(T))_{i,j} = \sum_{k=1}^{n} A_{i,j,k} \cdot T_k \cdot M_{R_{i,j}}$$

.
The function $f_3(T)$ can then be computed :

$$f_3(T)_i = \sum_{j=1}^{n}\sum_{k=1}^{n} A_{i,j,k} \cdot T_k \cdot M_{R_{i,j}} = \sum_{k=1}^{n}(\sum_{j=1}^{n} A_{i,j,k} \cdot M_{R_{i,j}}) \cdot T_k$$

. As we define the D matrix, such that $D_{i,k} = \sum_{j=1}^{n} A_{i,j,k} \cdot M_{R_{i,j}}$, we get :

$$f_3(T)_i = \sum_{k=1}^{n} D_{i,k} \cdot T_k \ \ therefore \ \ f_3(T) = D \times T$$

$$f(T) = C_{-1} \odot (D \times T) = ((C_{-1} \times 1_n^\intercal) \odot D) \times T = F \times T$$

. These computations are implemented in MATLAB.