Task 1 $T: V \to W$ is Linear Transform $\Leftrightarrow T(cx+y) = cT(x) + T(y)$ 何昱停

① $T: R^3 \to R^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

Let $x = (a_1, a_2, a_3)$, $y = (b_1, b_2, b_3)$, $c \in F$, $x, y \in R^3$

$$T(cx+y) = T(c(a_1, a_2, a_3) + (b_1, b_2, b_3))$$
$$= T(ca_1+b_1, ca_2+b_2, ca_3+b_3)$$
$$= (c(a_1-a_2)+b_1-b_2), 2ca_3+2b_3)$$

$$cT(x) + T(y) = c(a_1 - a_2, 2a_3) + (b_1 - b_2, 2b_3)$$
$$= (c(a_1-a_2)+(b_1-b_2), 2ca_3+2b_3)$$

$\because T(cx+y) = cT(x) + T(y) \quad \therefore T$ is Linear Transformation

$N(T) = \{x \in V \mid T(x) = 0\}$

$N(T(a_1, a_2, a_3)) = (a_1 - a_2, 2a_3) = (0, 0)$

$\Rightarrow \begin{cases} a_1 = a_2 \\ a_3 = 0 \end{cases} \Rightarrow N(T) = \{(a, a, 0) \mid a \in R\}$

Basis of $N(T) \Rightarrow \{(1,1,0)\} \Rightarrow \text{nullity}(T) = 1$ #

$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(1,0,0), T(0,1,0), T(0,0,1)\})$

$= \text{span}(\{(1,0), (0,-1)\})$

$\Rightarrow \text{rank}(T) = \dim(R(T)) = 2$ #

Thm 2.3: $\text{nullity}(T) + \text{rank}(T) = \dim(V)$ for $T: V \to W$ (V is finite-dim)

$\dim(R^3) = 3 = \text{nullity}(T) + \text{rank}(T) = 1 + 2$ #

one-to-one: $N(T) = \{0\}$ onto: $R(T) = W$ ($T: V \to W$)

$\because N(T) = \{(1,1,0)\}$ $\because \text{rank}(T) = 2 \Rightarrow R(T) = R^2$

$\therefore T$ is not one-to-one # $\therefore T$ is onto #

Task 1

② $T: R^6 \to R^4$ defined by $T(a_1,a_2,a_3,a_4,a_5,a_6) = (2a_1-a_2, a_3+a_2, 0, 0)$

Assume $x = (a_1,a_2,a_3,a_4,a_5,a_6)$ $y = (b_1,b_2,b_3,b_4,b_5,b_6)$

$T(cx+y) = T(ca_1+b_1, ca_2+b_2, ca_3+b_3, ca_4+b_4, ca_5+b_5, ca_6+b_6)$

$= (c(2a_1-a_2)+(2b_1-b_2), c(a_2+a_3)+(b_2+b_3), 0, 0)$

$cT(x)+T(y) = c(2a_1-a_2, a_3+a_2, 0, 0) + (2b_1-b_2, b_3+b_2, 0, 0)$

$= c(2a_1-a_2)+(2b_1-b_2)\ c(a_3+a_2)+b_3+b_2, 0, 0)$

$\because T(cx+y) = cT(x)+T(y)$ $\therefore$ T is Linear Transformation

$N(T) = \{x \in V \mid T(x)=0\}$

$\Rightarrow \begin{cases} 2a_1-a_2=0 \\ a_3+a_2=0 \end{cases} \Rightarrow \begin{cases} a_1=\frac{1}{2}a_2 \\ a_3=-a_2 \end{cases}$

$N(T) = \{(1,2,-2,0,0,0), (0,0,0,1,0,0), (0,0,0,0,1,0), (0,0,0,0,0,1)\}$

nullity$(T) = 4$ #

$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(1,0,0,0,0,0), T(0,1,0,0,0,0), T(0,0,1,0,0,0), T(0,0,0,1,0,0),$
$T(0,0,0,0,1,0), T(0,0,0,0,0,1)\})$

$= \text{span}(\{(2,0,0,0), (0,1,0,0)\})$

rank$(T) = 2$ #

$\because$ dim$(R^6) = 6 =$ nullity(T) + rank$(T) = 4+2=6$ #

one-to-one : $N(T) = \{0\}$

$\because N(T) \neq \{0\}$

$\therefore$ T is not one-to-one #

onto : $R(T) = W$ $(T: V \to W)$

$\because$ rank$(T) = 2 \Rightarrow R(T) = R^2 \neq R^4$

$\therefore$ T is not onto #

Task 1.

(5) $T: M_{1\times2}(R) \to M_{1\times3}(R)$ defined by $T([a_1, a_2]) = [a_1+a_2, 0, 2a_1-a_2]$

Assume $x = [a_1, a_2]$, $y = [b_1, b_2]$

$T(cx+y) = T([ca_1+b_1, ca_2+b_2])$

$= [c(a_1+a_2)+(b_1+b_2), 0, c(2a_1-a_2)+(2b_1-b_2)]$

$cT(x)+T(y) = [ca_1+ca_2, 0, 2ca_1 - ca_2] + [b_1+b_2, 0, 2b_1-b_2]$

$= [c(a_1+a_2)+(b_1+b_2), 0, c(2a_1-a_2)+(2b_1-b_2)]$

$\because T(cx+y) = cT(x)+T(y)$ $\therefore$ T is Linear Transformation

$N(T) = \{x \in V \mid T(x) = 0\}$

$\Rightarrow \begin{cases} a_1+a_2=0 \\ 2a_1-a_2=0 \end{cases} \Rightarrow \begin{cases} a_1=-a_2 \\ a_1=\frac{1}{2}a_2 \end{cases} \Rightarrow a_1=a_2=0$

$\Rightarrow N(T) = \{0\}$ $\Rightarrow$ nullity $= 0$ #

$R(T) = \text{span}(T(\beta)) = \text{span}(\{T([1,0]), T([0,1])\})$

$= \text{span}(\{[1,0,2], [1,0,-1]\})$

$\Rightarrow \text{rank}(T) = 2$ #

$\dim(M_{1\times2}(R)) = 2 = \text{nullity}(T) + \text{rank}(T) = 0+2 = 2$ #

One-to-one: $N(T)=\{0\}$ onto: $R(T) = W$ $(T: V \to W)$

$\because N(T) = \{0\}$ $\because R(T) \neq M_{1\times3}(R)$

$\therefore$ T is one-to-one $\therefore$ T is not onto

Task 2

① $T(a_1,a_2)=(1,a_2)$ Assume $x=(a_1,a_2), y=(b_1,b_2)$

$T(cx+y)=T(ca_1+b_1, ca_2+b_2)$

$=(1, ca_2+b_2)$

$cT(x)+T(y)=c(1,a_2)+(1,b_2)$

$=(c+1, ca_2+b_2)$

$\because T(cx+y)\neq cT(x)+T(y)$ $\therefore$ T is not linear #

② $T(a_1,a_2)=(a_1,a_1^2)$ Assume $x=(a_1,a_2)$, $y=(b_1,b_2)$

$T(cx+y)=T(ca_1+b_1, ca_2+b_2)$

$=(ca_1+b_1, c^2a_1^2+2ca_1b_1+b_1^2)$

$cT(x)+T(y)=c(a_1,a_1^2)+(b_1,b_1^2)$

$=(ca_1+b_1, ca_1^2+b_1^2)$

$\because T(cx+y)\neq cT(x)+T(y)$ $\therefore$ T is not linear #

③ $T(a_1,a_2)=(|a_1|,a_2)$ Assume $x=(a_1,a_2)$ $y=(b_1,b_2)$

$T(cx+y)=T(ca_1+b_1, ca_2+b_2)$

$=(|ca_1+b_1|, ca_2+b_2)$

$cT(x)+T(y)=c(|a_1|,a_2)+(|b_1|,b_2)$

$=(c|a_1|+|b_1|, ca_2+b_2)$

$\because c\in\mathbb{R}$, may be negative

$\Rightarrow T(cx+y)\neq cT(x)+T(y)$

$\therefore$ T is not linear #

Task 3. $T: R^2 \to R^2$ is linear,

$T(1,0) = (1,4)$, $T(1,1) = (2,5)$

① $T(2,3)$?

$(2,3) = a(1,0) + b(1,1)$

$= (a+b, b)$

$\Rightarrow \begin{cases} a+b=2 \\ b=3 \end{cases} \Rightarrow \begin{cases} a=-1 \\ b=3 \end{cases}$

$\Rightarrow T(2,3) = T(-1(1,0) + 3(1,1))$

$= -1T(1,0) + 3T(1,1)$

$= (-1,-4) + (6,15)$

$= (5,11)$ #

② One-to-one $= N(T) = \{0\}$, $N(T) = \{x \in V \mid T(x) = 0\}$

Assume $x = (a_1, a_2) \Rightarrow T(a_1, a_2) = (0,0)$

$T(a_1, a_2) = a_1 T(1,0) + a_2 T(1,1)$

$= (a_1, 4a_1) + (2a_2, 5a_2)$

$= (a_1 + 2a_2, 4a_1 + 5a_2)$

$\Rightarrow \begin{cases} a_1 + 2a_2 = 0 \\ 4a_1 + 5a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = -2a_2 \\ a_1 = -\frac{5}{4}a_2 \end{cases} \Rightarrow a_1 = a_2 = 0$

$\because T(x) = (0,0)$ only if $a_1 = a_2 = 0$

$\therefore T$ is one-to-one #

Task 4 $\quad x = (1,2,1) \quad y = (3,6,3)$

$T(cx) = cT(x)$ if T is Linear Transform

$y = 3x \Rightarrow T(y) = 3T(x)$

$T(x) = (1,1)$

$\Rightarrow 3T(x) = (3,3)$

$\because T(cx) \neq cT(x)$

$\therefore T$ must not a linear transformation

Task 5 $\quad N(T) = \{x \in V \mid T(x) = 0\}$

$N(T) = \{(x,y) \in R^2 \mid x = 0\}$

$T: R^2 \to R^2$

$\because$ by definition the element of $N(T)$ should follows $T(x,y) = (0,0) \Rightarrow \begin{cases} T(x) = (0,0), & x = 0 \\ T(x) \neq (0,0), & \text{otherwise} \end{cases}$

$\therefore$ given example $T: R^2 \to R^2$ defined by $T(a_1, a_2) = (a_1, a_1)$ #