Task 1 Let the nth order basis of $\beta = \beta_n$
$\gamma = \gamma_n$

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① $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$

$\gamma = \{(1,0),(0,1)\}$

$T(1,0,0) = (1,0) = 1\gamma_1$

$T(0,1,0) = (-1,0) = -1\gamma_1$ $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ #

$T(0,0,1) = (0,2) = 2\gamma_2$

② $\beta = \{(1,0,0,0,0,0),(0,1,0,0,0,0),\dots,(0,0,0,0,0,1)\}$

$\gamma = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$

$T(1,0,0,0,0,0) = (2,0,0,0) = 2\gamma_1$

$T(0,1,0,0,0,0) = (-1,1,0,0) = -1\gamma_1 + 1\gamma_2$

$T(0,0,1,0,0,0) = (0,1,0,0) = 1\gamma_2$ $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ #

$T(0,0,0,1,0,0) = (0,0,0,0) = 0$

$T(0,0,0,0,1,0) = (0,0,0,0) = 0$

$T(0,0,0,0,0,1) = (0,0,0,0) = 0$

③ $\beta = \{(1,0),(0,1)\}$

$\gamma = \{(1,0,0),(0,1,0),(0,0,1)\}$

$T(1,0) = (2,3,1) = 2\gamma_1 + 3\gamma_2 + 1\gamma_3$ $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$

$T(0,1) = (-1,4,0) = -1\gamma_1 + 4\gamma_2$

④ $\beta = \{(1,0,0,\cdots,0),(0,1,0,\cdots,0),\cdots,(0,0,\cdots,1)\}$

$\gamma =$ 〃

⑤

$T = (1,0,0,\cdots,0) = (0,0,\cdots,1) = 1\gamma_n$

$T = (0,1,0,\cdots,0) = (0,0,\cdots,0,1,0) = 1\gamma_{n-1}$ $\Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$

$\vdots$

$T = (0,0,\cdots,1) = (1,0,0,\cdots 0) = 1\gamma_1$

(n×n reverse matrix)

Task 2.

$\beta = \{(1,0),(0,1)\}$

$\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$

$T(1,0) = (1,1,2) = a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 \cdots ① \Rightarrow \frac{-1}{3}\gamma_1 + \frac{2}{3}\gamma_3$

$T(0,1) = (-1,0,1) = b_1\gamma_1 + b_2\gamma_2 + b_3\gamma_3 \cdots ② \Rightarrow -\gamma_1 + \gamma_2$

① $\begin{cases} a_1 + 2a_3 = 1 \\ a_1 + a_2 + 2a_3 = 1 \\ a_2 + 3a_3 = 2 \end{cases} \Rightarrow \begin{cases} a_1 = \frac{-1}{3} \\ a_2 = 0 \\ a_3 = \frac{2}{3} \end{cases}$

② $\begin{cases} b_1 + 2b_3 = -1 \\ b_1 + b_2 + 2b_3 = 0 \\ b_2 + 3b_3 = 1 \end{cases} \Rightarrow \begin{cases} b_1 = -1 \\ b_2 = 1 \\ b_3 = 0 \end{cases}$

$[T]_\beta^\gamma = \begin{bmatrix} \frac{-1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{bmatrix}$ #

$\alpha = \{(1,2),(2,3)\}$

$T(1,2) = (-1,1,4) = c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3 \cdots ③ \Rightarrow \frac{-7}{3}\gamma_1 + 2\gamma_2 + \frac{2}{3}\gamma_3$

$T(2,3) = (-1,2,7) = d_1\gamma_1 + d_2\gamma_2 + d_3\gamma_3 \cdots ④ \Rightarrow \frac{-11}{3}\gamma_1 + 3\gamma_2 + \frac{4}{3}\gamma_3$

③ $\begin{cases} c_1 + 2c_3 = -1 \\ c_1 + c_2 + 2c_3 = 1 \\ c_2 + 3c_3 = 4 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{-7}{3} \\ c_2 = 2 \\ c_3 = \frac{2}{3} \end{cases}$

④ $\begin{cases} d_1 + 2d_3 = -1 \\ d_1 + d_2 + 2d_3 = 2 \\ d_2 + 3d_3 = 7 \end{cases} \Rightarrow \begin{cases} d_1 = \frac{-11}{3} \\ d_2 = 3 \\ d_3 = \frac{4}{3} \end{cases}$

$[T]_\alpha^\gamma = \begin{bmatrix} \frac{-7}{3} & \frac{-11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$ #

Task 3 $A_{n\times n} = (a_{ij})$, $B_{n\times n} = (b_{ij})$ (a_{ij} is elements of A , b_{ij} is elements of B)

then $(AB)_{ii} = \sum_{k=1}^{n} a_{ik} b_{ki}$

$$tr(AB) = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk} b_{kj}$$

for BA

$$(BA)_{ii} = \sum_{k=1}^{n} b_{ik} a_{ki}$$

$$tr(BA) = \sum_{j=1}^{n} \sum_{k=1}^{n} b_{jk} a_{kj} = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{kj} b_{jk}$$

$\because$ the max value of j, k is same

$\therefore tr(AB) = tr(BA)$

Task 4

$$\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\beta = \{1, x^2, x^3\}$$

$$\gamma = \{1\}$$

①

$$T(\alpha_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1\alpha_1$$

$$T(\alpha_2) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 1\alpha_3 \Rightarrow [T]_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \#$$

$$T(\alpha_3) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 1\alpha_2$$

$$T(\alpha_4) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 1\alpha_4$$

$$T(A) = \begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix} = 1\alpha_1 + (-1)\alpha_2 + 4\alpha_3 + 2\alpha_4 \Rightarrow [T(A)]_\alpha = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix} \#$$

②

$$T(1) = 1 = \gamma_1$$

$$T(x) = 2 = 2\gamma_1 \Rightarrow [T]_\beta^\gamma = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \#$$

$$T(x^2) = 4 = 4\gamma_1$$

$$T(4x^2-2x+1) = 16-4+1 \Rightarrow [T(f(x))]_\beta^\gamma = [13]$$

$$= 13 = 13\gamma_1$$

where $f(x) = 4x^2-2x+1$ #