Week 3 Assignment

Task 1

①

$$a_1\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix} + a_2\begin{bmatrix}0 & -1\\ 3 & 1\end{bmatrix} + a_3\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}a_1+a_3 & 2a_1-a_2-a_3\\ 3a_2+2a_3 & a_2-a_3\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{cases} a_1+a_3=0\\ 2a_1-a_2-a_3=0\\ 3a_2+2a_3=0\\ a_2-a_3=0\end{cases} \Rightarrow \begin{cases} a_1=0\\ a_2=0\\ a_3=0\end{cases} \Rightarrow \text{linear independent } \#$$

②

$$a_1\begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix} + a_2\begin{bmatrix}0 & 2\\ 0 & 4\end{bmatrix} + a_3\begin{bmatrix}-2 & 3\\ 2 & 6\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}a_1-2a_3 & 2a_2+3a_3\\ -a_1+2a_3 & 4a_2+6a_3\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{cases} a_1-2a_3=0\\ 2a_2+3a_3=0\\ -a_1+2a_3=0\\ 4a_2+6a_3=0\end{cases} \Rightarrow \begin{cases} a_1=2a_3\\ a_2=-\frac{3}{2}a_3\end{cases}$$

$\Rightarrow$ We can find non-zero solution for a_1, a_2 and a_3

$\Rightarrow$ linear dependent #

③

$$a_1(1,0,-2,1)+a_2(0,-1,1,1)+a_3(-1,2,1,0)+a_4(2,1,-4,4)=(0,0,0,0)$$

$$\Rightarrow (a_1-a_3+2a_4,\ -a_2+2a_3+a_4,\ -2a_1+a_2+a_3-4a_4,\ a_1+a_2+4a_4)=(0,0,0,0)$$

$$\begin{cases} a_1-a_3+2a_4=0 & \cdots ①\\ -a_2+2a_3+a_4=0 & \cdots ②\\ -2a_1+a_2+a_3-4a_4=0 & \cdots ③\\ a_1+a_2+4a_4=0 & \cdots ④\end{cases}$$

$$\Rightarrow \begin{array}{rl} & -2a_1+a_2+a_3-4a_4=0\\ -) & a_1+a_2+4a_4=0\\ \hline & -3a_1+a_3-8a_4=0 \quad \cdots ③-④\\ +) & a_1-a_3+2a_4=0 \quad \cdots ①\\ \hline & -2a_1-6a_4=0 \end{array} \Rightarrow a_1=-3a_4$$

$$\Rightarrow \begin{cases} a_1=-3a_4\\ a_2=-a_4\\ a_3=-a_4\end{cases}$$

$\Rightarrow$ We can find non-zero solution for a_1, a_2, a_3 and a_4

$\Rightarrow$ linear dependent

Task 1

④ $a_1(1,0,-2,1)+a_2(0,-1,1,1)+a_3(-1,2,1,0)+a_4(2,1,2,-2)=0$

$$\Rightarrow \begin{cases} a_1-a_3+2a_4=0 & \cdots ① \\ -a_2+2a_3+a_4=0 & \cdots ② \\ -2a_1+a_2+a_3+2a_4=0 & \cdots ③ \\ a_1+a_2-2a_4=0 & \cdots ④ \end{cases}$$

$① + ③ \Rightarrow -a_1+a_2+4a_4=0$

$① + ③ + ④ \Rightarrow 2a_2-2a_4=0 \Rightarrow a_2=a_4$

$② \Rightarrow -a_4+2a_3+a_4=0 \Rightarrow a_3=0$

$④ \Rightarrow a_1+a_4-2a_4=0 \Rightarrow a_1=a_4$

$① \Rightarrow a_4-a_3+2a_4=0 \Rightarrow a_3=3a_4=0$

$$\Rightarrow \begin{cases} a_1=0 \\ a_2=0 \\ a_3=0 \\ a_4=0 \end{cases}$$

$\Rightarrow$ linear independent #

Task 2

linear dependent $\Leftrightarrow \exists\ \vec{u}_1, \vec{u}_2 \cdots \vec{u}_n \in S$, $a_1 a_2 \cdots a_n \in F$ not all zero such that $a_1\vec{u}_1+a_2\vec{u}_2+\cdots+a_n\vec{u}_n=0$

given $\vec{u}_1=(3,4,5)$ $\vec{u}_2=(9,8,7)$, $a_1=1$, $a_2=2$, $a_3=6$

$\vec{u}_3=(x,y,z)$

$$\Rightarrow \begin{cases} 3+18+6x=0 \\ 4+16+6y=0 \\ 5+14+6z=0 \end{cases} \Rightarrow \begin{cases} x=\frac{-21}{6}=\frac{-7}{2} \\ y=\frac{-20}{6}=\frac{-10}{3} \\ z=\frac{-19}{6} \end{cases}$$

$\Rightarrow \vec{u}_1=(3,4,5)$, $\vec{u}_2=(9,8,7)$, $\vec{u}_3=\left(-\frac{7}{2}, \frac{-10}{3}, \frac{-19}{6}\right)$ #

Task 3

① False, basis meaning a set of linear independent vectors can spans entire vector space. For R^2, basis 1 $\Rightarrow \{(1,0),(0,1)\}$

basis 2 $\Rightarrow \{(4,0),(2,3)\}$

② True

③ False, For R^3, $S=\{(1,0,0),(2,0,0),(0,1,0)\}$ can not be basis of R^3

S is not linear independent

Task 4

basis $\Leftrightarrow$ (1) linear independent (2) spans entire vector space

① $a_1(1-x^2)+a_2(2+5x+x^2)+a_3(-4x+3x^2)=\vec{0}$

$$\Rightarrow \begin{cases} a_1+2a_2=0 \\ 5a_2-4a_3=0 \\ -a_1+a_2+3a_3=0 \end{cases} \Rightarrow \begin{cases} a_1=-2a_2 \\ a_3=\frac{5}{4}a_2 \end{cases}$$

$-2a_2+a_2+\frac{15}{4}a_2=\frac{11}{4}a_2=0$

$\Rightarrow a_1=a_2=a_3=0 \Rightarrow$ satisfied (1), (2)

$\Rightarrow$ is basis of $P_2(R)$ #

② $a_1(2-4x+x^2)+a_2(3x-x^2)+a_3(6-x^2)=\vec{0}$

$$\Rightarrow \begin{cases} 2a_1+6a_3=0 & \cdots ① \\ -4a_1+3a_2=0 & \cdots ② \\ a_1-a_2-a_3=0 & \cdots ③ \end{cases} \Rightarrow \begin{cases} a_3=-\frac{1}{3}a_1 \\ a_2=\frac{4}{3}a_1 \end{cases}$$

①, ② $\Rightarrow$ ③ $\Rightarrow a_1-\frac{4}{3}a_1+\frac{1}{3}a_1=0$, $a_1 \in R$

$\Rightarrow$ linear dependent

$\Rightarrow$ is not basis of $P_2(R)$ #

Task 4

③ $a_1(1+2x-x^2)+a_2(1+2x^2)+a_3(2+x+x^2)=0$

$$\Rightarrow \begin{cases} a_1+a_2+2a_3=0 & \cdots ① \\ 2a_1+a_3=0 & \cdots ② \\ -a_1+2a_2+a_3=0 & \cdots ③ \end{cases} \Rightarrow$$

② $\Rightarrow a_1 = \frac{1}{2}a_3$

① + ③ $\Rightarrow 3a_2+3a_3=0$

$\Rightarrow a_2=-a_3$

$2\times$① − ② $\Rightarrow 2a_2+3a_3=0$

$\Rightarrow a_2 = \frac{3}{2}a_3$

$a_3=0=a_2=a_1 \Rightarrow$ linear independen

$\Rightarrow$ satisfied (1), (2)

$\Rightarrow$ is basis of $P_2(R)$ #

Task 5

$(a_1,a_2,a_3,a_4,a_5) = (a_3+a_4, a_2, a_3, a_4, a_5) = \vec{0}$

$= a_3(1,0,1,0,0)+a_4(1,0,0,1,0)+a_2(0,1,0,0,0)+a_5(0,0,0,0,1)$

$$\Rightarrow \begin{cases} a_3+a_4=0 \\ a_2=0 \\ a_3=0 \\ a_4=0 \\ a_5=0 \end{cases} \Rightarrow \begin{cases} a_2=0 \\ a_3=0 \\ a_4=0 \\ a_5=0 \end{cases}$$

$\Rightarrow$ linear independent

$\Rightarrow \dim(W) = 4$