Week 3

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Task 1 Invertible $\Leftrightarrow$ ① one-to-one $(N(T)=\{0\})$

② onto $(R(T)=W)$

① $T: R^2 \to R^3$ $T(a_1,a_2)=(2a_1-a_2, 3a_1+4a_2, a_1)$

$N(T)=\{x\in V \mid T(x)=0\}$

$\Rightarrow \begin{cases} 2a_1-a_2=0 \\ 3a_1+4a_2=0 \\ a_1=0 \end{cases} \Rightarrow a_1=a_2=a_3=0$

$\Rightarrow N(T)=\{0\} \to$ one-to-one

$R(T)=\text{span}(T(\beta))$

$\beta=\{(1,0),(0,1)\}$

$\Rightarrow R(T)=\text{span}(\{(2,3,1),(-1,4,0)\})$

$\text{rank}(T)=2 \Rightarrow R(T)\neq R^3$

$\Rightarrow$ T is not onto

$\Rightarrow$ T is not invertible #

② $T: R^3\to R^3$ $T(a_1,a_2,a_3)=(3a_1-2a_3, a_2, 3a_1+4a_2)$

$N(T)=\{x\in V \mid T(x)=0\}$

$\Rightarrow \begin{cases} 3a_1-2a_3=0 \\ a_2=0 \\ 3a_1+4a_2=0 \end{cases} \Rightarrow a_1=a_2=a_3=0$

$\Rightarrow N(T)=\{0\} \Rightarrow$ T is one-to-one

$R(T)=\text{Span}(T(\beta))$

$\beta=\{(1,0,0),(0,1,0),(0,0,1)\}$

$\Rightarrow R(T)=\text{Span}(\{(3,0,3),(0,1,4),(-2,0,0)\})$

$\text{rank}(T)=3 \Rightarrow R(T)=R^3$

$\Rightarrow$ T is onto

$\Rightarrow$ T is one-to-one & onto $\to$ T is invertible #

③ $T: M_{2\times2}(R)\to P_2(R)$ $T\begin{pmatrix} a & b \\ c & d\end{pmatrix}=a+2bx+(c+d)x^2$

$N(T)=\{x\in V \mid T(x)=0\}$

$\Rightarrow \begin{cases} a=0 \\ 2b=0 \\ c+d=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=-d \end{cases}$

$\Rightarrow N(T)=\{(0,0,c,-c)\}$

$\Rightarrow$ nullity(T) = 1

$\Rightarrow$ T is not one-to-one $\Rightarrow$ T is not invertible #

Task 2. isomorphic $\Leftrightarrow$ $\dim(V) = \dim(W) < \infty$
$(V \sim W)$

① $\mathbb{R}^3$, $P_3(\mathbb{R})$

$\dim(\mathbb{R}^3) = 3 \neq \dim(P_3(\mathbb{R})) = 4 \Rightarrow$ not isomorphic #

② $\mathbb{R}^4$, $P_3(\mathbb{R})$

$\dim(\mathbb{R}^4) = 4 = \dim(P_3(\mathbb{R})) = 4 \Rightarrow$ isomorphic #

③ $V = \{A \in M_{2\times 2}(\mathbb{R})$ where $tr(A) = 0\}$ and $\mathbb{R}^3$

$tr(A) = 0 \Rightarrow A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \Rightarrow \beta_V = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \Rightarrow \dim(V) = 3$

$\dim(V) = 3 = \dim(\mathbb{R}^3) = 3 \Rightarrow$ isomorphic #

Task 3 $T: V \to W$ is invertible $\Leftrightarrow$ $\exists U: W \to V \ni UT = I_V$ and $TU = I_W$

① $A^2 = 0$

Assume A is invertible $\Rightarrow A^{-1}A = I$

$\Rightarrow A^{-1}A^2 = A^{-1}AA = IA = 0$

$\Rightarrow A = 0$ $\because$ A^{-1} is not exists where $A = 0$ $\therefore$ A is not invertible #

② $AB = 0$ where B is nonzero $n \times n$ matrix

Assume A is invertible $\Rightarrow A^{-1}A = I$

$A^{-1}AB = IB = 0$

$\Rightarrow B = 0 \Rightarrow$ not following the condition. B is nonzero $n\times n$ matrix

$\Rightarrow A$ is not invertible #

Task 4. $T: V \to W$ where $\dim(V) = \dim(W) = n$ and T is linear

Wanted: T is isomorphic $\Leftrightarrow$ $T(\beta)$ is a basis for W

① T is isomorphic $\Rightarrow$ $T(\beta)$ is a basis for W

T is isomorphic $\Leftrightarrow$ T is invertible $\Leftrightarrow$ T is one-to-one and onto

$\because$ T is one-to-one $\Rightarrow$ $N(T) = \{0\}$ $\Rightarrow$ $T(\beta)$ is Linear Independent

$\because$ T is onto $\Rightarrow$ $R(T) = \text{span}(T(\beta)) = W$ $\Rightarrow$ $T(\beta)$ can generate W

$\therefore$ $T(\beta)$ is a basis for W #

② $T(\beta)$ is a basis for W $\Rightarrow$ T is isomorphic

$T(\beta)$ is a basis for W $\Leftrightarrow$ 1. $T(\beta)$ is Linear Independent $\Leftrightarrow$ T is one-to-one

2. $T(\beta)$ can generate W $\Leftrightarrow$ T is onto

$\therefore$ T is invertible $\Rightarrow$ T is isomorphic #