General remarks

Objectives

1. learn how to solve problems numerically
2. Use numerical methods as a tool to investigate and learn about interesting problems
3. Get familiar with tools that could be used for research project

Work in teams during afternoon sessions
- complete during session, but maybe more work needed
- ok to work w/ other teams; don't leave anyone out

Work evaluated individually
- Jupyter notebooks w/ places to add code / plots / discussion
- Discuss w/ others freely, but what you turn in should be your own work
- turn in completed notebooks for all projects at the end of term

Use python + dedalus

Projects

1. Intro to numerical methods + differential eq.
2. Ocean mixing + river pollution
3. Tsunami
4. storms

Here, focus on numerical methods to solve partial differential equations (PDEs)

Need to discretize space and time derivatives

PDE solvers are classified based on spatial discretization

Methods:

1. Finite difference (FD)

Approximate function and derivatives on grid w/ spacing Δx

recall
$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{y(x+\delta x) - y(x)}{\delta x}$$

inspired by this, approximate
$$\frac{dy}{dx} \simeq \frac{y(x+\Delta x) - y(x-\Delta x)}{2\Delta x}$$

"centered difference"

2. Spectral Method

Apply a spectral (e.g. Fourier) transform to the PDE and solve for the coefficients of the spectral basis.

Fourier transform

$$\hat{y}(k) \equiv \int_{-\infty}^{\infty} y(x) e^{-ikx} dx$$

e.g. consider $\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = 0$

F.T. $\Rightarrow$ $\int_{-\infty}^{\infty}\left(\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x}\right)e^{-ikx} dx = 0$

$$\frac{\partial \hat{y}}{\partial t} + c\underbrace{\int_{-\infty}^{\infty} \frac{\partial y}{\partial x} e^{-ikx} dx}_{} = 0$$

$$= \underbrace{\left[ye^{-ikx}\right]_{-\infty}^{\infty}}_{=0 \text{ if } \lim_{x\to\pm\infty} y = 0} - \int_{-\infty}^{\infty} y(x)(-ik)e^{-ikx} dx$$

hence $\frac{\partial \hat{y}}{\partial t} + ik\hat{y} = 0$

diffusion equation:

$$\frac{\partial y}{\partial t} = K\frac{\partial^2 y}{\partial x^2} = K\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)$$

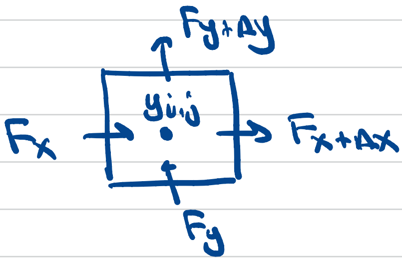
$$\Longrightarrow \frac{\partial \hat{y}}{\partial t} = -Kk^2\hat{y}$$

PRO: Allows <u>accurate</u> and <u>fast</u> solution of PDEs.

CON: Boundary conditions more difficult to impose

3 Finite volume

Calculate the evolution of a quantity in a cell by evaluating the <u>fluxes</u> through the cell edges, typically using FD method.



$$\frac{dy}{dt} = F_x - F_{x+\Delta x} + F_y - F_{y+\Delta y}$$

PRO: Conservation laws satisfied on discrete grid
CON: Irregular grids w/ unequal # faces difficult

4. Finite Element

Represent function within cells w/ polynomial

PRO: Allows irregular grids, good for complex geometries
CON: conservation laws difficult to maintain

Timestepping

Consider $\frac{dy}{dt} = f(\underline{y}, t)$

discretize time : $t = t_0, t_1, \ldots, t_n, t_{n+1}, \ldots t_N$

denote $y_n = y(t_n)$, etc.

Suppose we know $\underline{y}$ at $t_0, t_1, \ldots t_n$ and want to find $\underline{y}(t_{n+1})$

Explicit methods evaluate f based on y at previous times $y(t_0), y(t_1), \ldots y(t_n)$

Implicit methods evaluate f using $\underline{y}(t_{n+1})$

Examples

Explicit Euler :
$$\frac{y_{n+1} - y_n}{\Delta t} = f(y_n, t_n)$$
$$(\Delta t = t_{n+1} - t_n)$$

Crank-Nicolson (semi-implicit)
$$\frac{y_{n+1} - y_n}{\Delta t} = \frac{1}{2}\left(f(y_n, t_n) + f(y_{n+1}, t_{n+1}) \right)$$