Secret Hitler Prediction Algorithm

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2022/04/22

1 Introduction

Consider the list of all possible assignments of roles (Liberal, Fascist, or Hitler) to players. Let R_i be the event that the ith role assignment in that list represents the actual role of each player. Furthermore, let G be the event that some game's history (i.e. all legislative sessions, president actions, top-decks, etc.) exactly matches the current game.

We want to find $P(R_i \mid G)$ for all i. By Bayes' theorem,

$$\begin{aligned} P(R_i \mid G) &= \frac{P(G \mid R_i)P(R_i)}{P(G)} \\ &= \frac{P(G \mid R_i)P(R_i)}{\sum_j P(G \cap R_j)} \\ &= \frac{P(G \mid R_i)P(R_i)}{\sum_j P(G \mid R_j)P(R_j)} \end{aligned}$$

Assuming each assignment of roles is equally likely:

$$\begin{aligned} P(R_i \mid G) &= \frac{P(G \mid R_i)P(R_i)}{\sum_j P(G \mid R_j)P(R_i)} \\ &= \frac{P(G \mid R_i)}{\sum_j P(G \mid R_j)} \end{aligned}$$

So we just need to find the probability of the game happening the way it did given each possible assignment of roles.

For simplicity, assume the probability of each legislative session, president action, etc. depends only on the current state of the draw pile (i.e. players' decisions are independent of decisions from previous rounds). Let X_j be a random variable corresponding to the number of Liberal policies remaining in the draw pile at the start of the jth step in the game (either a legislative session under a successfully-elected government, a top-deck, or an executive action). Let n_j be the total number of policies in the draw pile at the start of the jth step. At the beginning of the game, $X_1 = 6$ with a probability of 1 and $n_1 = 17$.

2 Legislative Sessions

2.1 Definitions

Given an assignment of roles, the roles of the President and the Chancellor are known. A legislative session's visible outcome includes four pieces of information: the policy outcome (Liberal or Fascist), the policies that the President claims to have received, the policies that the President claims to have given to the Chancellor, and the policies that the Chancellor claims to have received. There are also two pieces of hidden information: the actual policies received by the President and by the Chancellor. Let:

- R_p be 0 if the President is Fascist, 1 if they are Hitler, and 2 if they are Liberal
- R_c be 0 if the Chancellor is Fascist, 1 if they are Hitler, and 2 if they are Liberal
- Y be 0 if a Fascist policy is passed and 1 if a Liberal policy is passed
- M_1 be the number of Liberal policies that the President claims to have received
- M_2 be the number of Liberal policies that the President claims to have given to the Chancellor
- M_3 be the number of Liberal policies that the Chancellor claims to have received
- A_1 be the actual number of Liberal policies received by the President
- A_2 be the actual number of Liberal policies received by the Chancellor

2.2 Session Outcome

Given roles r_p and r_c, a policy outcome y, and claims m_1, m_2, and m_3, we want to calculate $P(Y = y \cap M_1 = m_1 \cap M_2 = m_2 \cap M_3 = m_3 \mid R_p = r_p \cap R_c = r_c)$. Let Γ be the event that $R_p = r_p \cap R_c = r_c$ and let L be the event that $Y = y \cap M_1 = m_1 \cap M_2 = m_2 \cap M_3 = m_3$. Assume that the state of the draw pile at the start of each legislative session is independent of the players who will be President and Chancellor.

Then

$$
\begin{aligned}
P(L \mid \Gamma) &= \sum_{a_1=0}^{3} P(L \cap A_1 = a_1 \mid \Gamma) \\
&= \sum_{a_1=0}^{3} P(L \mid A_1 = a_1 \cap \Gamma) P(A_1 = a_1 \mid \Gamma) \\
&= \sum_{a_1=0}^{3} P(L \mid A_1 = a_1 \cap \Gamma) P(A_1 = a_1) \\
&= \sum_{a_1=0}^{3} P(L \mid A_1 = a_1 \cap \Gamma) \sum_{x=0}^{6} P(A_1 = a_1 \cap X_j = x) \\
&= \sum_{a_1=0}^{3} P(L \mid A_1 = a_1 \cap \Gamma) \sum_{x=0}^{6} P(X_j = x) P(A_1 = a_1 \mid X_j = x) \\
&= \sum_{a_1=0}^{3} P(L \mid A_1 = a_1 \cap \Gamma) \sum_{x=0}^{6} P(X_j = x) \frac{\binom{x}{a_1}\binom{n_j - x}{3 - a_1}}{\binom{n_j}{3}}
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
P(L \mid A_1 = a_1 \cap \Gamma) &= \sum_{a_2=0}^{2} P(L \cap A_2 = a_2 \mid A_1 = a_1 \cap \Gamma) \\
&= \sum_{a_2=0}^{2} p_{\text{pp}} \cdot p_{\text{pc}} \cdot p_{\text{cp}} \cdot p_{\text{cc}}
\end{aligned}
$$

where

- $p_{\text{pp}} = P(A_2 = a_2 \mid A_1 = a_1 \cap \Gamma)$
- $p_{\text{pc}} = P(Y = y \mid A_2 = a_2 \cap A_1 = a_1 \cap \Gamma)$
- $p_{\text{cp}} = P(M_1 = m_1 \cap M_2 = m_2 \mid Y = y \cap A_2 = a_2 \cap A_1 = a_1 \cap \Gamma)$
- $p_{\text{cc}} = P(M_3 = m_3 \mid M_1 = m_1 \cap M_2 = m_2 \cap Y = y \cap A_2 = a_2 \cap A_1 = a_1 \cap \Gamma)$

In short, if the actual policies received by the President are known, the probability of a legislative session can be expressed in terms of independent decisions by the President and the Chancellor. p_{pp} represents the policy decision made by the President (i.e. which policy to discard), p_{pc} represents the Chancellor's policy decision, p_{cp} represents the President's claims, and p_{cc} represents the Chancellor's claim. These can all be estimated based on assumptions about players' likely strategies. To find the probability regardless of the actual policies received by the President, sum over all possible agendas.

2.3 Draw Pile

After each legislative session, we want to update the state of the draw pile. That is, we want to determine the PMF of X_{j+1}. If there are fewer than 3 policies remaining in the draw pile, all cards are placed back into the draw pile. Then $n_{j+1} = 17 - \#\text{Liberal policies passed} - \#\text{Fascist policies passed}$ and $X_{j+1} = 6 - \#\text{Liberal policies passed}$ with a probability of 1. Otherwise, $n_{j+1} = n_j - 3$ and the PMF of X_{j+1} can be calculated based on the legislative session outcome and the player roles. As before, assume that the state of the draw pile is independent of the upcoming President and Chancellor. Furthermore, assume that the number of Liberal policies in the draw pile is not relevant to the legislative session except in determining the policies received by the President (i.e. we can ignore X once we know A_1).

$$
\begin{aligned}
& P(X_{j+1} = x \mid L \cap \Gamma) \\
&= \sum_{a_1=0}^{3} P(X_j = x + a_1 \cap A_1 = a_1 \mid L \cap \Gamma) \\
&= \sum_{a_1=0}^{3} \frac{P(X_j = x + a_1 \cap A_1 = a_1 \cap L \mid \Gamma)}{P(L \mid \Gamma)} \\
&= \frac{1}{P(L \mid \Gamma)} \sum_{a_1=0}^{3} P(X_j = x + a_1 \cap A_1 = a_1 \cap L \mid \Gamma) \\
&= \frac{1}{P(L \mid \Gamma)} \sum_{a_1=0}^{3} P(X_j = x + a_1 \mid \Gamma) P(A_1 = a_1 \cap L \mid X_j = x + a_1 \cap \Gamma) \\
&= \frac{1}{P(L \mid \Gamma)} \sum_{a_1=0}^{3} P(X_j = x + a_1) P(A_1 = a_1 \cap L \mid X_j = x + a_1 \cap \Gamma) \\
&= \frac{1}{P(L \mid \Gamma)} \sum_{a_1=0}^{3} P(X_j = x + a_1) P(A_1 = a_1 \mid X_j = x + a_1) P(L \mid A_1 = a_1 \cap X_j = x + a_1 \cap \Gamma) \\
&= \frac{1}{P(L \mid \Gamma)} \sum_{a_1=0}^{3} P(X_j = x + a_1) P(A_1 = a_1 \mid X_j = x + a_1) P(L \mid A_1 = a_1 \cap \Gamma) \\
&= \frac{1}{P(L \mid \Gamma)} \sum_{a_1=0}^{3} P(X_j = x + a_1) \frac{\binom{x+a_1}{a_1}\binom{n_j-x-a_1}{3-a_1}}{\binom{n_j}{3}} P(L \mid A_1 = a_1 \cap \Gamma)
\end{aligned}
$$

3 Top Deck

Let F be the event that the policy enacted was Fascist and let L be the event that it is Liberal. We can calculate the probability of whichever event occurred and update the state of the draw pile accordingly.

3.1 Fascist Policy

The probability that a Fascist policy is enacted is

$$\begin{aligned} P(F) &= \sum_{x=0}^{6} P(F \cap X_j = x) \\ &= \sum_{x=0}^{6} P(X_j = x) P(F \mid X_j = x) \\ &= \sum_{x=0}^{6} P(X_j = x) \frac{n_j - x}{n_j} \end{aligned}$$

The PMF of X_{j+1} is

$$\begin{aligned} P(X_{j+1} = x \mid F) &= P(X_j = x \mid F) \\ &= P(F \mid X_j = x) \frac{P(X_j = x)}{P(F)} \\ &= \frac{n_j - x}{n_j} \frac{P(X_j = x)}{P(F)} \end{aligned}$$

3.2 Liberal Policy

The probability that a Liberal policy is selected is

$$\begin{aligned} P(L) &= \sum_{x=0}^{6} P(L \cap X_j = x) \\ &= \sum_{x=0}^{6} P(X_j = x) P(L \mid X_j = x) \\ &= \sum_{x=0}^{6} P(X_j = x) \frac{x}{n_j} \end{aligned}$$

In this case, the PMF of X_{j+1} is

$$\begin{aligned} P(X_{j+1} = x \mid L) &= P(X_j = x + 1 \mid L) \\ &= P(L \mid X_j = x + 1) \frac{P(X_j = x + 1)}{P(L)} \\ &= \frac{x + 1}{n_j} \frac{P(X_j = x + 1)}{P(L)} \end{aligned}$$

4 Executive Actions

4.1 Investigation

Let R_p be the role of the President and let R_t be the role of the target of the investigation. Let Y be 1 if the President claims that the target is Fascist (i.e. accuses them) and 0 otherwise. Given a President role r_p, a target role r_t, and an outcome y, we want to find $P(R_t = r_t \cap Y = y \mid R_p = r_r)$. This can be expressed as the product $P(R_t = r_t \mid R_p = r_p)P(Y = y \mid R_t = r_t \cap R_p = r_p)$. In other words, we need to know how likely a President is to choose a target of the given role and then how likely they are to accuse their target after seeing their party affiliation. Each probability can be estimated ahead of time. For the President's choice of target, assume that:

- Fascists are less likely to investigate each other and extremely unlikely to investigate Hitler
- Liberals choose at random
- Hitler chooses at random (since investigations only happen when there are at least seven players, at which point Hitler does not know the identities of the Fascists)

For the decision to accuse or not accuse, assume that:

- Fascists will sometimes accuse Liberals, but are unlikely to accuse another Fascist and very unlikely to accuse Hitler
- Hitler is unlikely to make accusations in general and is more likely to accuse a Fascist than a Liberal
- Liberals will always tell the truth

4.2 Policy Peek

Let R_p be the role of the President, let A be the actual number of Liberal policies observed by the President, and let M be the number of Liberal policies reported by the President. Given a President identity r_p and a reported number of Liberal policies m, we want to find the probability of that outcome and update the state of the draw pile. Let Γ be the event that $R_p = r_p$.

4.2.1 Policy Peek Outcome

$$P(M = m \mid \Gamma) = \sum_{x=0}^{6} P(M = m \cap X_j = x \mid \Gamma)$$

where

$$\begin{aligned} P(M = m \cap X_j = x \mid \Gamma) &= P(X_j = x \mid \Gamma) P(M = m \mid X_j = x \cap \Gamma) \\ &= P(X_j = x) P(M = m \mid X_j = x \cap \Gamma) \end{aligned}$$

and

$$\begin{aligned} & P(M = m \mid X_j = x \cap \Gamma) \\ &= \sum_{a=0}^{3} P(M = m \cap A = a \mid X_j = x \cap \Gamma) \\ &= \sum_{a=0}^{3} P(A = a \mid X_j = x \cap \Gamma) P(M = m \mid A = a \cap X_j = x \cap \Gamma) \\ &= \sum_{a=0}^{3} P(A = a \mid X_j = x) P(M = m \mid A = a \cap \Gamma) \\ &= \sum_{a=0}^{3} \frac{\binom{x}{a}\binom{n_j - x}{3-a}}{\binom{n_j}{3}} P(M = m \mid A = a \cap \Gamma) \end{aligned}$$

4.2.2 Draw Pile

$$\begin{aligned} P(X_{j+1} = x \mid M = m \cap \Gamma) &= P(X_j = x \mid M = m \cap \Gamma) \\ &= \frac{P(M = m \cap X_j = x \mid \Gamma)}{P(M = m \mid \Gamma)} \end{aligned}$$