Assignment 1

Problem 1:

a) Let us expand the derivative from $x \pm \delta$ and the derivative from $x \pm 2\delta$ of f:

$$\frac{df_2}{dx} = \frac{f(x+\delta) - f(x-\delta)}{2\delta} = \frac{\left(f(x) + f'(x)\cdot\delta + f''(x)\cdot\frac{\delta^2}{2} + \dots\right) - \left(f(x) - f'(x)\delta + f''(x)\cdot\frac{\delta^2}{2}\dots\right)}{2\delta}$$

$$= \frac{1}{\delta}\left(f'(x)\cdot\delta + f'''(x)\frac{\delta^3}{3!}\dots\right)$$

$$= f'(x) + f'''(x)\frac{\delta^2}{3!} + \dots$$

$$\frac{df_4}{dx} = \frac{f(x+2\delta) - f(x-2\delta)}{4\delta} = \frac{\left(f(x) + f'(x)2\delta + f''(x)\cdot\frac{4\delta^2}{2} + \dots\right) - \left(f(x) - f'(x)2\delta + f''(x)\cdot\frac{4\delta^2}{2}\dots\right)}{4\delta}$$

$$= \frac{1}{\delta}\left(\frac{f'(x)}{2}\cdot\delta + f'''(x)\frac{4\delta^3}{3!}\dots\right)$$

$$= f'(x) + f'''(x)\frac{4\delta^2}{3!} + \dots$$

We then want to combine those derivative such that

$$\frac{df_{4,2}}{dx} = \frac{4}{3}\frac{df_2}{dx} - \frac{1}{3}\frac{df_4}{dx} = \frac{4}{3}f'(x) + f'''(x)\frac{4\delta^2}{3\cdot 3!} + \dots - \frac{1}{3}f'(x) - f'''(x)\frac{4\delta^2}{3\cdot 3!} = O(\delta^4)$$

Hence our estimate of the derivative operator is:

$$\frac{df_{4,2}}{dx} = \frac{4}{3}\frac{df_2}{dx} - \frac{1}{3}\frac{df_4}{dx} = O(\delta^4)$$

b) We want δ:

As the derivative operator gives $O(\delta^4)$ then our error is of the form

$$Err \simeq \frac{\epsilon f(x)}{\delta} + f^{(5)}(x)\delta^4 \quad \text{with } \epsilon = 10^{-16} \text{ (As seen in class)}$$

We want to find where Err is the closest to 0 as we differenciate

Hence $Err = 0 \iff -\frac{\epsilon f(x)}{dx^2} + 4f^{(5)}(x)\delta^3 = 0$

$$\iff -\epsilon f(x) + 4f^{(5)}(x)\delta^5 = 0$$

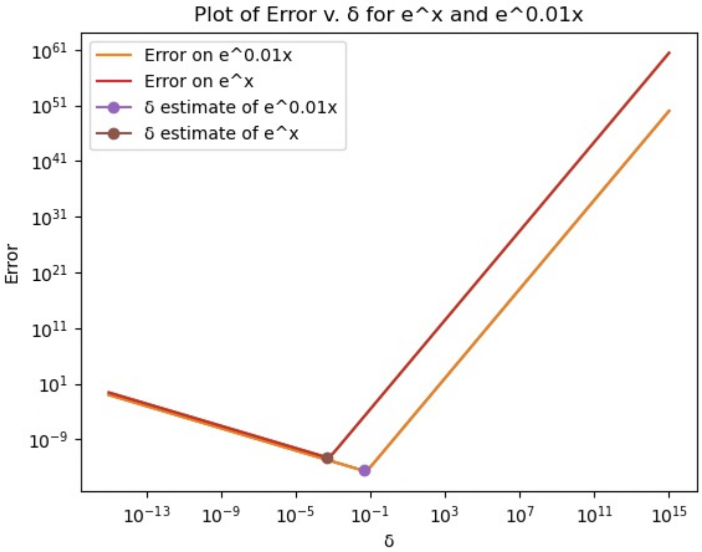
$$\iff \delta = \left(\frac{\epsilon f(x)}{4f^{(5)}(x)}\right)^{1/5}$$

for $f(x) = e^x$ this gives us

$$\delta = \left(\frac{\epsilon e^x}{4 e^x}\right)^{1/5} = \left(\frac{10^{-16}}{4}\right)^{1/5} = 4.8 \times 10^{-4}$$

for $f(x) = e^{0,01x}$ this gives us

$$\delta = \left(\frac{\epsilon e^{0,01 \cdot x}}{4 \cdot 0,01^5 \, e^{0,01x}}\right)^{1/5} = \left(\frac{\epsilon}{4 \cdot (0,01)^5}\right)^{1/5} = 4.8 \times 10^{-2}$$



Problem 2: We have:

$$f' = \frac{f(x+dx) - f(x-dx)}{dx}$$ as our derivative operator.

Hence as seen in class: $ERR \simeq \frac{f\epsilon}{dx} + f''' dx^2$

We want f''' hence we want to express f''' in terms of f:

$$f''' = \frac{f''(x+dx) - f''(x-dx)}{2dx}$$

$$f'' = \frac{f'(x+dx) - f'(x-dx)}{2dx}$$

$$f' = \frac{f(x+dx) - f(x-dx)}{2dx}$$

Hence $f''' = \frac{1}{2dx^3}\left[f(x+2dx) - 2f(x+dx) + 2f(x-dx) - f(x-2dx)\right]$

Therefore $ERROR = \frac{f\epsilon}{dx} + \frac{1}{2dx}\left[f(x+2dx) - 2f(x+dx) + 2f(x-dx) - f(x-2dx)\right]$

for $f(x) = x^2$ we get $f'(5) = 9{,}999\,999\,999\,999\,996$

$dx = 5.2 \times 10^{-2}$

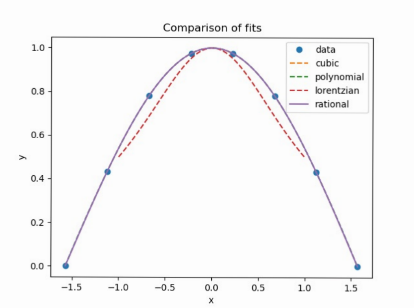
$ERROR = 4.79 \times 10^{-14}$

Problem 4:

The error for the Lorentzian from the rational function fit is supposed to be important as the Lorentzian is not sinusoidal and is not a good estimate.

When we increase the order we get that the polynomial error decreases, the rational error increases and the cubic spline error decreases. It means for polynomial and the cubic spline more points allow better precision.

for $n = 4$ $m = 5$ we get

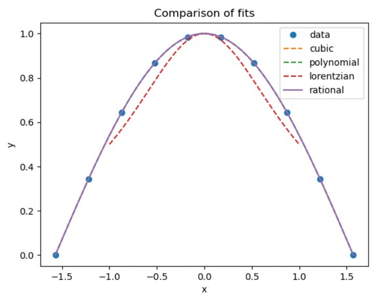


Rational error $= 3,48 \times 10^{-5}$

Polynomial error $= 9,06 \times 10^{-4}$

Cubic spline error $= 2,07 \times 10^{-4}$

for $n = 4$ $m = 7$ we get

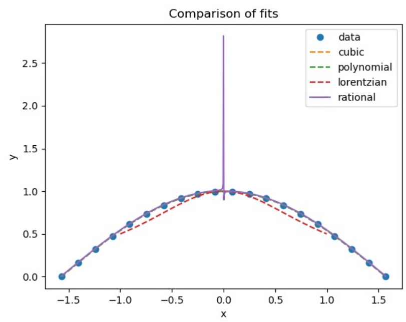


Rational error $= 1,5 \times 10^{-6}$

Polynomial error $= 8,83 \times 10^{-5}$

Cubic spline error $= 5,48 \times 10^{-5}$

for $n = 10$ $m = 11$ we get

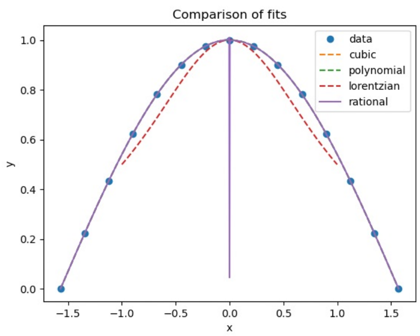


Rational error $= 5,76 \times 10^{-2}$

Polynomial error $= 4.40 \times 10^{-10}$

Cubic spline error $= 1.14 \times 10^{-6}$

for $n = 6$ $m = 10$ we get



Rational error $= 3,01 \times 10^{-2}$

Polynomial error $= 2.5 \times 10^{-14}$

Cubic spline error $= 5.4 \times 10^{-6}$

When we modify with the np.linalg.pinv

for $n = 4$ $m = 5$ we get

Rational error $= 3{,}48 \times 10^{-5}$

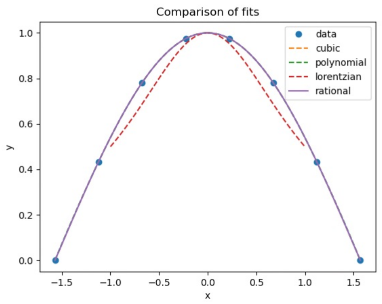
Polynomial error $= 9{,}06 \times 10^{-4}$

Cubic spline error $= 2{,}07 \times 10^{-4}$

without modification

$p = [9{,}9 \times 10^{-1};\ 0;\ -4.05 \times 10^{-1};\ 1.14 \times 10^{-13}]$

$q = [0;\ 9.45 \times 10^{-2};\ 1.42 \times 10^{-14};\ 6.23 \times 10^{-3}]$



for $n = 4$ $m = 5$ we get

Rational error $= 3{,}48 \times 10^{-5}$

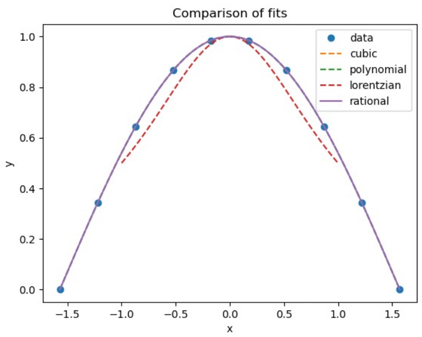
Polynomial error $= 9{,}06 \times 10^{-4}$

Cubic spline error $= 2{,}07 \times 10^{-4}$

with modification

$p = [9.9 \times 10^{-1};\ -2.27 \times 10^{-13};\ -4.05 \times 10^{-1};\ 5.68 \times 10^{-14}]$

$q = [0;\ 9{,}45 \times 10^{-2};\ 1.42 \times 10^{-14};\ 6.23 \times 10^{-3}]$



Changing this parameter will allow to have 0 determinants matrices to have them inverted. Therefore we have less and more precise 0 coefficients.

We understand that by looking at p and q, p has increasing even terms whereas q does not change.