Assignment 1

Problem 1:

a) Let us expand the derivative from $x \pm \delta$ and the derivative from $x \pm 2\delta$ of f:

$$\frac{df_2}{dx} = \frac{f(x+\delta) - f(x-\delta)}{2\delta} = \frac{\left(f(x) + f'(x)\cdot\delta + f''(x)\cdot\frac{\delta^2}{2} + \cdots\right) - \left(f(x) - f'(x)\delta + f''(x)\cdot\frac{\delta^2}{2}\cdots\right)}{2\delta}$$

$$= \frac{1}{\delta}\left(f'(x)\cdot\delta + f'''(x)\frac{\delta^3}{3!}\cdots\right)$$

$$= f'(x) + f'''(x)\frac{\delta^2}{3!} + \cdots$$

$$\frac{df_4}{dx} = \frac{f(x+2\delta) - f(x-2\delta)}{4\delta} = \frac{\left(f(x) + f'(x)\cdot 2\delta + f''(x)\cdot\frac{4\delta^2}{2} + \cdots\right) - \left(f(x) - f'(x)\delta + f''(x)\cdot\frac{4\delta^2}{2}\cdots\right)}{4\delta}$$

$$= \frac{1}{\delta}\left(\frac{f'(x)}{2}\cdot\delta + f'''(x)\frac{4\delta^3}{3!}\cdots\right)$$

$$= f'(x) + f'''(x)\frac{4\delta^2}{3!} + \cdots$$

We then want to combine those derivatives such that

$$\frac{df_{4,2}}{dx} = \frac{4}{3}\frac{df_2}{dx} - \frac{1}{3}\frac{df_4}{dx} = \frac{4}{3}f'(x) + f'''(x)\frac{4\delta^2}{3\cdot 3!} + \cdots - \frac{1}{3}f'(x) - f'''(x)\frac{4\delta^2}{3\cdot 3!}$$

$$= O(\delta^4)$$

Hence our estimate of the first derivative is:

$$\frac{df_{4,2}}{dx} = \frac{4}{3}\frac{df_2}{dx} - \frac{1}{3}\frac{df_4}{dx} = O(\delta^4)$$

b) We want δ:

As the error in the derivative expansion is $O(\delta^4)$ then our error is of the form $Err \simeq \frac{\epsilon f(x)}{\delta} + f^{(5)}(x)\delta^4$

We want to minimize error:

$$\frac{\epsilon f(x)}{\delta} + f^{(5)}(x)\delta^4 = 0 \iff \delta = \sqrt[5]{\frac{\epsilon f(x)}{f^{(5)}(x)}} \iff \delta = \epsilon^{1/5} \iff \delta = 10^{\frac{-16}{5}} \iff \delta = 10^{-3.2}$$

Problem 2: