INDENG 290: Assignment, Part I

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Question 1

Test Case 1

Given:

$$\pi_0 = \{1 : 0.5, 3 : 0.5\}$$
$$n = 2$$

We need to determine $\Pr(X_2 = x)$ for any $x \in S$ given the initial state distribution π_0 and $n = 2$.

The transition probabilities are:

$$\Pr(X_{n+1} = x_{n+1} \mid X_n = 0) = \begin{cases} \frac{1}{2} & \text{if } x_{n+1} = 1 \\ \frac{1}{2} & \text{if } x_{n+1} = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Pr(X_{n+1} = x_{n+1} \mid X_n = x_n \neq 0) = \begin{cases} \frac{1}{2} & \text{if } x_{n+1} = x_n + 1 \\ \frac{1}{2x_n} & \text{if } x_{n+1} < x_n \\ 0 & \text{otherwise} \end{cases}$$

Starting with $\pi_0 = \{1 : 0.5, 3 : 0.5\}$:

$$\Pr(X_1 = 0) = \frac{1}{2} \times 0.5 + \frac{1}{6} \times 0.5 = \frac{1}{3}$$

$$\Pr(X_1 = 1) = \frac{1}{6} \times 0.5 = \frac{1}{12}$$

$$\Pr(X_1 = 2) = \frac{1}{2} \times 0.5 + \frac{1}{6} \times 0.5 = \frac{1}{3}$$

$$\Pr(X_1 = 4) = \frac{1}{2} \times 0.5 = \frac{1}{4}$$

For $n = 2$:

$$\Pr(X_2 = 0) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{12} + \frac{1}{4} \times \frac{1}{3} + \frac{1}{8} \times \frac{1}{4} = \frac{31}{96}$$

$$\Pr(X_2 = 1) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{4} \times \frac{1}{3} + \frac{1}{8} \times \frac{1}{4} = \frac{9}{32}$$

$$\Pr(X_2 = 2) = \frac{1}{2} \times \frac{1}{12} + \frac{1}{8} \times \frac{1}{4} = \frac{7}{96}$$
$$\Pr(X_2 = 3) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{8} \times \frac{1}{4} = \frac{19}{96}$$
$$\Pr(X_2 = 5) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

Thus, the expected result π_2 matches the given values.

Final Distribution for Test Case 1:

$$\pi_2 = \{0 : \frac{31}{96}, 1 : \frac{9}{32}, 2 : \frac{7}{96}, 3 : \frac{19}{96}, 5 : \frac{1}{8}\}$$

Test Case 2

Given:

$$\pi_0 = \{0 : 0.5, 1 : 0.5\}$$
$$n = 1$$

Starting with $\pi_0 = \{0 : 0.5, 1 : 0.5\}$:

$$\Pr(X_1 = 0) = \frac{1}{2} \times 0.5 + \frac{1}{2} \times 0.5 = \frac{1}{2}$$
$$\Pr(X_1 = 1) = \frac{1}{2} \times 0.5 = \frac{1}{4}$$
$$\Pr(X_1 = 2) = \frac{1}{2} \times 0.5 = \frac{1}{4}$$

Thus, the expected result π_1 matches the given values.

Final Distribution for Test Case 2:

$$\pi_1 = \{0 : \frac{1}{2}, 1 : \frac{1}{4}, 2 : \frac{1}{4}\}$$

Question 2

In this exercise, we will denote s_k as the number of stocks we still have and a_k as the number of stocks we sell at time t. We will also denote r_k as the reward at time k.

Test Case 1

Given:

$$n = 1$$
$$T = 1$$
$$S_0 = 1$$
$$c = 1$$

Expected Result: 2

Proof:

- Initial stock price $S_0 = 1$
- Number of shares to sell $2n = 2$
- Time horizon $T = 1$
- Cost parameter $c = 1$

At each timestep t, the stock price S_t increases by 1. The number of buyers K_t follows a Binomial distribution $Binomial(1, \frac{1}{2})$.

$$\Pr(K_t = k) = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = \binom{n}{k}\left(\frac{1}{2}\right)^n$$

For $n = 1$, K_t can be either 0 or 1:

$$\Pr(K_t = 0) = \binom{1}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^1 = \frac{1}{2}$$

$$\Pr(K_t = 1) = \binom{1}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^0 = \frac{1}{2}$$

The set of states is :
$S := \{s_0, s_1, s_2\} = \{0, 1, 2\}$
The set of actions is :
$A := \{a_0, a_1, a_2\} = \{0, 1, 2\}$

Initialisation:

$$V_1^*(s_0) = 0$$
$$V_1^*(s_1) = \frac{1}{2}$$
$$V_1^*(s_2) = 1$$

We are looking for $V_0^*(s_2) = max_{a \in A} Q_0^*(s_2, a)$

For $a = a_0$:

$$\begin{aligned}
Q_0^*(s_2, a_0) &= r_o(s_2, a_0) + \sum_{s' \in S} \Pr(S_1 = s' \mid s_2, A_0 = a_0) \times V_1^*(s') \\
&= r_0(s_2, a_0) + V_1^*(s_2) \\
&= 1
\end{aligned}$$

For $a = a_1$:

$$\begin{aligned}
Q_0^*(s_2, a_1) &= r_o(s_2, a_1) + \sum_{s' \in S} \Pr(S_1 = s' \mid s_2, A_0 = a_1) \times V_1^*(s') \\
&= r_0(s_2, a_1) + \frac{1}{2} \times V_1^*(s_1) + \frac{1}{2} \times V_1^*(s_2) \\
&= \frac{1}{2} \times S_0 + \frac{1}{2} \times (-c) + \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1 \\
&= \frac{3}{4}
\end{aligned}$$

For $a = a_2$:

$$\begin{aligned}
Q_0^*(s_2, a_2) &= r_o(s_2, a_2) + \sum_{s' \in S} \Pr(S_1 = s' \mid s_2, A_0 = a_2) \times V_1^*(s') \\
&= \frac{1}{2} \times (S_0 - c) + \frac{1}{2} \times (-2c) + \frac{1}{2} \times V_1^*(s_1) + \frac{1}{2} \times V_1^*(s_2) \\
&= -\frac{1}{4}
\end{aligned}$$

Finally, $V_0^*(s_2) = max_{a \in A} Q_0^*(s_2, a) = 1$

Test Case 2

Given:

$$n = 1$$

$$T = 1$$

$$S_0 = 3$$

$$c = 2$$

Expected Result:

Proof:

- Initial stock price $S_0 = 3$

- Number of shares to sell $2n = 2$
- Time horizon $T = 1$
- Cost parameter $c = 3$

At each timestep t, the stock price S_t increases by 1. The number of buyers K_t follows a Binomial distribution $Binomial(1, \frac{1}{2})$.

$$\Pr(K_t = k) = \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = \binom{n}{k} \left(\frac{1}{2}\right)^n$$

For $n = 1$, K_t can be either 0 or 1:

$$\Pr(K_t = 0) = \binom{1}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^1 = \frac{1}{2}$$

$$\Pr(K_t = 1) = \binom{1}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^0 = \frac{1}{2}$$

The set of states is :
$S := \{s_0, s_1, s_2\} = \{0, 1, 2\}$
The set of actions is :
$A := \{a_0, a_1, a_2\} = \{0, 1, 2\}$

Initialisation:

$$V_1^*(s_0) = 0$$
$$V_1^*(s_1) = 2$$
$$V_1^*(s_2) = 4$$

We are looking for $V_0^*(s_2) = max_{a \in A} Q_0^*(s_2, a)$

For $a = a_0$:

$$\begin{aligned} Q_0^*(s_2, a_0) &= r_o(s_2, a_0) + \sum_{s' \in S} \Pr(S_1 = s' \mid s_2, A_0 = a_0) \times V_1^*(s') \\ &= r_0(s_2, a_0) + V_1^*(s_2) \\ &= 4 \end{aligned}$$

For $a = a_1$:

$$
\begin{aligned}
Q_0^*(s_2, a_1) &= r_o(s_2, a_1) + \sum_{s' \in S} \Pr(S_1 = s' \mid s_2, A_0 = a_1) \times V_1^*(s') \\
&= r_0(s_2, a_1) + \frac{1}{2} \times V_1^*(s_1) + \frac{1}{2} \times V_1^*(s_2) \\
&= \frac{1}{2} \times S_0 + \frac{1}{2} \times (-c) + \frac{1}{2} \times 2 + \frac{1}{2} \times 4 \\
&= \frac{7}{2}
\end{aligned}
$$

For $a = a_2$:

$$
\begin{aligned}
Q_0^*(s_2, a_2) &= r_o(s_2, a_2) + \sum_{s' \in S} \Pr(S_1 = s' \mid s_2, A_0 = a_2) \times V_1^*(s') \\
&= \frac{1}{2} \times (S_0 - c) + \frac{1}{2} \times (-2c) + \frac{1}{2} \times V_1^*(s_1) + \frac{1}{2} \times V_1^*(s_2) \\
&= \frac{3}{2}
\end{aligned}
$$

Finally, $V_0^*(s_2) = max_{a \in A} Q_0^*(s_2, a) = 4$