TDs - Traitement du Signal

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1. TD1

Exercise 1.1:

1. $X(t)$ est périodique à énergie finie.

$$\begin{aligned}R_X(\tau) &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |X(t)|^2 \mathrm{d}t \\ &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A_0^2 \cos^2(2\pi f_0 t)\mathrm{d}t \\ &= \frac{A_0^2}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (\cos(2\pi f_0 t) + 1)\mathrm{d}t \\ &= \frac{A_0^2}{2T_0}(0 + T_0) \\ &= \frac{A_0^2}{2}\end{aligned}$$

$$\begin{aligned}S_X(f) &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} X(t)X(t-\tau)\mathrm{d}t \\ &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A_0^2 \cos(2\pi f_0 t)\cos(2\pi f_0 (t-\tau))\mathrm{d}t \\ &= \frac{A_0^2}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (\cos(2\pi f_0(2t-\tau)) + \cos(2\pi f_0(\tau)))\mathrm{d}t \\ &= \frac{A_0^2}{2T_0}(0 + \cos(2\pi f_0(\tau))T_0) \\ &= A_0^2 \frac{\cos(2\pi f_0 \tau)}{2}\end{aligned}$$

$$S_x(\tau) = \mathrm{TF}[R_X(\tau)] = \frac{A_0^2}{4}(\delta(f - f_0) + \delta(f + f_0))$$

2. $X(t)$ est aléatoire

$$\begin{aligned}\mathbb{E}_\theta(X(t)) &= \frac{1}{2\pi}\int_0^{2\pi} X(t)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} A_0\cos(2\pi f_0 t + \theta)\mathrm{d}\theta \\ &= \frac{1}{2\pi}A_0 \times 0 \\ &= 0\end{aligned}$$

$$\begin{aligned}R_X(\tau) &= \mathbb{E}_\theta(X(t)X^*(t-\tau)) \\ &= \mathbb{E}_\theta(A_0^2\cos(2\pi f_0 t + \theta)\cos(2\pi f_0(t-\tau) + \theta)) \\ &= \mathbb{E}_\theta\left(\frac{A_0^2}{2}(\cos(2\pi f_0(2t-\tau) + 2\theta) + \cos(2\pi f_0 \tau))\right) \\ &= \frac{A_0^2}{2}(\mathbb{E}_\theta[\cos(2\pi f_0(2t-\tau) + 2\theta)] + \mathbb{E}_\theta[\cos(2\pi f_0\tau)]) \\ &= \frac{A_0^2}{2}(0 + \cos(2\pi f_0 \tau)) \\ &= A_0^2\frac{\cos(2\pi f_0 \tau)}{2}\end{aligned}$$

$$S_X(f) = \mathrm{TF}(R_X(\tau)) = \frac{A_0^2}{4}(\delta(f - f_0) + \delta(f + f_0))$$

3. $$X(t) = A_0\cos(2\pi f t + \theta)$$

$$\begin{aligned}\mathbb{E}_{f,\theta}[X(t)] &= \mathbb{E}_f[\mathbb{E}_\theta[X(t) \mid f]] \\ &= \mathbb{E}_f[0] \\ &= 0\end{aligned}$$

$$\begin{aligned}R_X(\tau) &= \mathbb{E}_{f,\theta}[X(t)X(t-\tau)] \\ &= \mathbb{E}_f[\mathbb{E}_\theta[X(t)X(t-\tau) \mid f]] \\ &= \mathbb{E}_f\left[\frac{A_0^2}{2}\cos(2\pi f \tau)\right] \\ &= \frac{A_0^2}{4\Delta f}\left(\frac{1}{2\pi\tau}\sin(2\pi f\tau)\right) \\ &= A_0^{2_{8\pi\Delta f\tau}}(\sin(2\pi(f_0 + \Delta f)\tau) - \sin(2\pi(f_0 - \Delta f)\tau)) \\ &= \frac{A_0^2}{4\pi\Delta f\tau}\sin(2\pi\Delta f\tau)\cos(2\pi f_0\tau) \\ &= \underbrace{\frac{A_0^2}{2}\operatorname{sinc}(2\pi\Delta f\tau)\cos(2\pi f_0\tau)}_{\text{stationnaire}}\end{aligned}$$

$$S_X(f) = \frac{A_0^2}{2}\left[\frac{1}{2\Delta f}\Pi_{2\Delta f}(f) + \frac{1}{2}(\delta(f - f_0) + \delta_f + f_0)\right]$$

Exercise 1.2:

1.

$$X(t) = A(t)\cos(2\pi f_0 t + \theta)$$

$$\begin{aligned}\mathbb{E}(X(t)) &= \mathbb{E}(A(t))\underbrace{\mathbb{E}_\theta[\cos(2\pi f_0 t + \theta)]}_{0} \\ &= 0\end{aligned}$$

$$\begin{aligned}R_X(\tau) &= \mathbb{E}(X(t)X(t-\tau)) \\ &= \mathbb{E}(A(t)\cos(2\pi f_0 t + \theta))A(t-\tau)\cos(2\pi f_0(t-\tau) + \theta) \\ &= \mathbb{E}(A(t)A(t-\tau))\cos(2\pi f_0 t + \theta)\cos(2\pi f_0(t-\tau) + \theta) \\ &= \mathbb{E}(A(t)A(t-\tau))\mathbb{E}_\theta[\cos(...)\cos(...)] \\ &= \underbrace{R_A(\tau)}_{\text{indép de } t \text{ car A stat.}} \frac{1}{2}\cos(2\pi f_0\tau)\end{aligned}$$

C'est indépendant de t donc $X(t)$ est stationnaire.

$$\begin{aligned}s_X(f) &= \mathrm{TF}(R_X(\tau)) \\ &= \frac{1}{4}s_A(f) \star (\delta(f - f_0) + \delta(f + f_0)) \\ &= \frac{1}{4}(s_A(f - f_0) + s_A(f + f_0))\end{aligned}$$

1. $$\begin{aligned}R_Y(\tau) &= \mathbb{E}[Y(t)Y^*(t-\tau)] \\ &= \mathbb{E}[X(t)\cos(2\pi f_0 t + \theta)X^*(t-\tau)\cos(2\pi f_0(t-\tau) + \theta)] \\ &= \mathbb{E}[A(t)\cos^2(2\pi f_0 t + \theta)A^* t\cos^2(2\pi f_0(t-\tau) + \theta)] \\ &= R_A(\tau)\tfrac{1}{4}\mathbb{E}[(1 + \cos(4\pi f_0 t + 2\theta)(1 + \cos(2\pi f_0(t-\tau) + 2\theta)))] \\ &= R_A\tfrac{\tau}{4}\mathbb{E}\left[1 + \cos(2\theta + ...) + \cos(2\theta + ...) + \tfrac{1}{2}\cos(4\pi f_0\tau) + \cos(4\theta + ...)\right] \\ &= R_A\tfrac{\tau}{4}\left(1 + \tfrac{1}{2}\cos(4\pi f_0\tau)\right)\end{aligned}$$

2. TD2

Exercise 2.1:

1. $$\begin{aligned}y(t) &= \frac{1}{T}\int_{t-T}^{t} e^{j\pi 2fu}\mathrm{d}u \\ &= \frac{1}{T}\left[\frac{e^{j\pi 2fu}}{j2\pi f}\right]_{t-T}^{T} \\ &= \frac{1}{T2\pi fj}\left(e^{2j\pi ft} - e^{2j\pi f(t-T)}\right) \\ &= \frac{1}{2\pi fjT}e^{2\pi jft}\left(1 - e^{-2\pi jfT}\right) \\ &= x(t)H(f)\end{aligned}$$

On a $H(f) = \frac{1}{2\pi fjT}\left(1 - e^{-2\pi jfT}\right)$, donc le filtre est linéaire.

De plus,

$$\begin{aligned}H(f) &\underset{\text{angle moitié}}{=} \frac{e^{-j\pi fT}}{2\pi jfT}\left(e^{j\pi fT} - e^{-j\pi fT}\right) \\ &= e^{-j\pi fT}\frac{\sin(\pi fT)}{T\pi f} \\ &= e^{-j\pi fT}\operatorname{sinc}(\pi fT)\end{aligned}$$

Ainsi,

$$\begin{aligned}h(t) &= \mathrm{TF}^{-1}(H(f)) \\ &= \delta\left(t - \frac{T}{2}\right) \star \Pi_T(f)\frac{1}{T} \\ &= \frac{1}{T}\Pi_T\left(t - \frac{T}{2}\right)\end{aligned}$$

2. La réponse impulsionnelle est :
   - réelle
   - causale ($\Leftrightarrow h(t) = 0, \forall t \geq 0$)
   - stable ($\Leftrightarrow \int h(f)\mathrm{d}f < \infty$)

Exercise 2.2:

1. On a $s_y(f) = |H(f)|^2 s_{x(f)}$

$$\begin{aligned}P_{Y_s} &= \int_{\mathbb{R}} s_{y_s}(f)\mathrm{d}f \\ &= \int_{\mathbb{R}} S(f)|H(f)|^2\mathrm{d}f\end{aligned}$$

$$S(f) = \frac{A^2}{4}(\delta(f - f_0) - \delta(f + f_0))$$

$$\begin{aligned}P_{Y_s} &= \int_{\mathbb{R}} \frac{A^2}{4}\frac{\delta(f - f_0) - \delta(f + f_0)}{|\theta + j2\pi f|^2} \\ &= \int_{\mathbb{R}} \frac{A^2}{4}\frac{\delta(f - f_0) - \delta(f + f_0)}{\theta^2 + (2\pi f)^2}\mathrm{d}f \\ &= \frac{A^2}{4}\left(\frac{1}{\theta^2} + 4\pi^2 f^2\right)\end{aligned}$$

$$\begin{aligned}P_{Y_B} &= \int_{\mathbb{R}} \frac{N_0}{2}\frac{1}{\theta^2 + 4(\pi f)^2}\mathrm{d}f \\ &= \frac{N_0}{2}\int_{\mathbb{R}} \frac{1}{\theta^2 + 4\pi^2 f^2}\mathrm{d}f \\ &= \frac{N_0}{2\theta}\int_{\mathbb{R}} \frac{1}{1 + \left(2\pi\frac{f}{\theta}\right)^2}\mathrm{d}f \\ &\underset{\begin{cases} u = 2\pi\frac{f}{\theta} \\ \mathrm{d}u = 2\pi\frac{\mathrm{d}f}{\theta}\end{cases}}{=} \frac{N_0}{2\theta}\int_{\mathbb{R}}\left(\frac{1}{1 + u^2}\right)\mathrm{d}u \\ &= \frac{N_0}{4\pi\theta}[\arctan(u)]_{\mathbb{R}} \\ &= \frac{N_0}{4\pi\theta}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \\ &= \frac{N_0}{4\theta}\end{aligned}$$

Ainsi

$$\begin{aligned}\mathrm{RSB} = \frac{P_{Y_s}}{P_{y_B}} &= \frac{A^2}{2}\frac{\frac{1}{\theta^2 + 4\pi^2 f_0^2}}{\frac{N_0}{4\theta}} \\ &= 2A^2\frac{\theta}{N_0}\frac{1}{\theta^2 + 4\pi^2 f_0^2}\end{aligned}$$

2. $$\begin{aligned}\mathrm{RSB}'(\theta) &= 0 \\ &\Leftrightarrow \theta = 2\pi f_0\end{aligned}$$

Exercise 2.3:

1. $$\begin{aligned}Y(t) &= e^{X(t)} \\ \Leftrightarrow \mathbb{E}[Y(t)] &= \mathbb{E}\left[e^{X(t)}\right]\end{aligned}$$

On a $\mathbb{E}\left[e^{uZ}\right] = e^{mu + \sigma^2\frac{u^2}{2}}$

Ici $Z = X(t)$ et $u = 1$

Puis

$$\begin{aligned}Y(t) &= e^{X(t)} \\ \Leftrightarrow \mathbb{E}[Y(t)] &= e^{\frac{\sigma^2}{2}}\end{aligned}$$

2. $$\begin{aligned}V &= \mathbb{E}[Y(t) - \mathbb{E}(Y(t))] \\ &= \mathbb{E}\left[e^{X(t)}\right] - \mathbb{E}\left[e^{\frac{\sigma^2}{2}}\right]^2 \\ &= \mathbb{E}\left[e^{2X(t)}\right] - e^{\sigma^2}\end{aligned}$$