TDs - Traitement du Signal

23 Janvier, 2024

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1. TD1

Exercise I:

1. $X(t)$ est périodique à énergie finie.

$$\begin{aligned} R_X(\tau) &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} |X(t)|^2 \mathrm{d}t \\ &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A_0^2 \cos^2(2\pi f_0 t)\mathrm{d}t \\ &= \frac{A_0^2}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (\cos(2\pi f_0 t) + 1)\mathrm{d}t \\ &= \frac{A_0^2}{2T_0}(0 + T_0) \\ &= \frac{A_0^2}{2} \end{aligned}$$

$$\begin{aligned} S_X(f) &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} X(t)X(t-\tau)\mathrm{d}t \\ &= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} A_0^2\cos(2\pi f_0 t)\cos(2\pi f_0(t-\tau))\mathrm{d}t \\ &= \frac{A_0^2}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} (\cos(2\pi f_0(2t-\tau)) + \cos(2\pi f_0(\tau)))\mathrm{d}t \\ &= \frac{A_0^2}{2T_0}(0 + \cos(2\pi f_0(\tau))T_0) \\ &= A_0^2\frac{\cos(2\pi f_0 \tau)}{2} \end{aligned}$$

$$S_x(\tau) = \mathrm{TF}[R_X(\tau)] = \frac{A_0^2}{4}(\delta(f-f_0) + \delta(f+f_0))$$

2. $X(t)$ est aléatoire

$$\begin{aligned} \mathbb{E}_\theta(X(t)) &= \frac{1}{2\pi}\int_0^{2\pi} X(t)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} A_0\cos(2\pi f_0 t + \theta)\mathrm{d}\theta \\ &= \frac{1}{2\pi}A_0 \times 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} R_X(\tau) &= \mathbb{E}_\theta(X(t)X^*(t-\tau)) \\ &= \mathbb{E}_\theta(A_0^2\cos(2\pi f_0 t + \theta)\cos(2\pi f_0(t-\tau) + \theta)) \\ &= \mathbb{E}_\theta\left(\frac{A_0^2}{2}(\cos(2\pi f_0(2t-\tau) + 2\theta) + \cos(2\pi f_0 \tau))\right) \\ &= \frac{A_0^2}{2}(\mathbb{E}_\theta[\cos(2\pi f_0(2t-\tau)+2\theta)] + \mathbb{E}_\theta[\cos(2\pi f_0\tau)]) \\ &= \frac{A_0^2}{2}(0 + \cos(2\pi f_0 \tau)) \\ &= A_0^2\frac{\cos(2\pi f_0\tau)}{2} \end{aligned}$$

$$S_X(f) = \mathrm{TF}(R_X(\tau)) = \frac{A_0^2}{4}(\delta(f-f_0) + \delta(f+f_0))$$

3. $$X(t) = A_0\cos(2\pi f t + \theta)$$

$$\begin{aligned} \mathbb{E}_{f,\theta}[X(t)] &= \mathbb{E}_f[\mathbb{E}_\theta[X(t) \mid f]] \\ &= \mathbb{E}_f[0] \\ &= 0 \end{aligned}$$

$$\begin{aligned} R_X(\tau) &= \mathbb{E}_{f,\theta}[X(t)X(t-\tau)] \\ &= \mathbb{E}_f[\mathbb{E}_\theta[X(t)X(t-\tau) \mid f]] \\ &= \mathbb{E}_f\left[\frac{A_0^2}{2}\cos(2\pi f\tau)\right] \\ &= \frac{A_0^2}{4\Delta f}\left(\frac{1}{2\pi\tau}\sin(2\pi f\tau)\right) \\ &= A_0^{2_{8\pi\Delta f\tau}}(\sin(2\pi(f_0+\Delta f)\tau) - \sin(2\pi(f_0-\Delta f)\tau)) \\ &= \frac{A_0^2}{4\pi\Delta f\tau}\sin(2\pi\Delta f\tau)\cos(2\pi f_0\tau) \\ &= \underbrace{\frac{A_0^2}{2}\operatorname{sinc}(2\pi\Delta f\tau)\cos(2\pi f_0\tau)}_{\text{stationnaire}} \end{aligned}$$

$$S_X(f) = \frac{A_0^2}{2}\left[\frac{1}{2\Delta f}\Pi_{2\Delta f}(f) + \frac{1}{2}\big(\delta(f-f_0) + \delta_f + f_0)\right]$$

Exercise II:

1.

$$X(t) = A(t)\cos(2\pi f_0 t + \theta)$$

$$\begin{aligned} \mathbb{E}(X(t)) &= \mathbb{E}(A(t))\underbrace{\mathbb{E}_\theta[\cos(2\pi f_0 t + \theta)]}_{0} \\ &= 0 \end{aligned}$$

$$\begin{aligned} R_X(\tau) &= \mathbb{E}(X(t)X(t-\tau)) \\ &= \mathbb{E}(A(t)\cos(2\pi f_0 t + \theta))A(t-\tau)\cos(2\pi f_0(t-\tau) + \theta) \\ &= \mathbb{E}(A(t)A(t-\tau))\cos(2\pi f_0 t + \theta)\cos(2\pi f_0(t-\tau) + \theta) \\ &= \mathbb{E}(A(t)A(t-\tau))\mathbb{E}_\theta[\cos(...)\cos(...)] \\ &= \underbrace{R_A(\tau)}_{\text{indép de } t \text{ car A stat.}} \frac{1}{2}\cos(2\pi f_0\tau) \end{aligned}$$

C'est indépendant de t donc $X(t)$ est stationnaire.

$$\begin{aligned} s_X(f) &= \mathrm{TF}(R_X(\tau)) \\ &= \frac{1}{4}s_A(f) \star (\delta(f-f_0) + \delta(f+f_0)) \\ &= \frac{1}{4}(s_A(f-f_0) + s_A(f+f_0)) \end{aligned}$$

1. $$\begin{aligned} R_Y(\tau) &= \mathbb{E}[Y(t)Y^*(t-\tau)] \\ &= \mathbb{E}[X(t)\cos(2\pi f_0 t + \theta)X^*(t-\tau)\cos(2\pi f_0(t-\tau) + \theta)] \\ &= \mathbb{E}\big[A(t)\cos^2(2\pi f_0 t + \theta)A^* t\cos^2(2\pi f_0(t-\tau) + \theta)\big] \\ &= R_A(\tau)\tfrac{1}{4}\mathbb{E}[(1 + \cos(4\pi f_0 t + 2\theta)(1 + \cos(2\pi f_0(t-\tau) + 2\theta)))] \\ &= R_A\tfrac{\tau}{4}\mathbb{E}\big[1 + \cos(2\theta + ...) + \cos(2\theta + ...) + \tfrac{1}{2}\cos(4\pi f_0\tau) + \cos(4\theta + ...)\big] \\ &= R_A\tfrac{\tau}{4}\big(1 + \tfrac{1}{2}\cos(4\pi f_0\tau)\big) \end{aligned}$$