Subspace Iteration Methods
Application to image compression

1 Reminders and introduction

In the first session of this project, we have seen that this specific algorithm is not efficient in terms of performance and we have presented a more efficient method called **subspace iteration method**, based on an object called *Rayleigh quotient*.

With these latter method, we propose to apply it for image compression. As the images are described as rectangular matrices, this application relies on the Singular Value Decomposition and the best low-rank approximation theorem.

Examples in grayscale of comics images to compress

Theorem: Singular Value Decomposition (SVD)

Let $A \in \mathbb{R}^{q\times p}$ $(q \geq p)$ with full rank, $rank(A) = p$ then we can decompose A as $A = U\Sigma V^T$ with

- $U \in \mathbb{R}^{q\times q}$ is formed of q orthonormal eigenvectors associated to q eigenvalues of AA^T.
- $V \in \mathbb{R}^{p\times p}$ is formed of p orthonormal eigenvectors associated to the p eigenvalues of A^TA.
- $\Sigma \in \mathbb{R}^{q\times p}$ is a rectangular matrix whose non-zero elements on the "diagonal" are the singular values $\sigma_i, i = \{1, .., p\}$ of A and are the square roots of the eigenvalues of bthA^TA and AA^T. A common convention is to arrange the σ_i values in descending order, then the matrix Σ is uniquely determined by A (but U and V are not).

We have the following relations between u_i and v_i:

$$\text{for } i \leq p, v_i = \frac{1}{\sigma_i} A^T u_i \text{ and } u_i = \frac{1}{\sigma_i} A v_i \quad (1)$$

In image processing, the singular values represent the energy of the image. Indeed, the total energy of the image $I \in \mathbb{R}^{q \times p}$ is represented by the following formula:

$$\|I\| = Trace(I^T I) = \sum_{i=1}^{q} \sum_{j=1}^{p} I_{ij}^2 = \sum_{i=1}^{q} \sigma_i^2$$

Theorem: Best low-rank approximation

Let $A \in \mathbb{R}^{q \times p}$ which SVD is $A = \sum_{i=1}^{p} \sigma_i u_i v_i^T$ with $p = rank(A)$.

If $k < p$ and $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ then A_k is the best low-rank approximation of A of rank $k < p$ i.e.:

$$\min_{rank(D)=k} \|A - D\|_F = \|A - A_k\|_F$$

2 Image Compression

Let I of size $q \times p$ be the matrix describing the image.

Let $k < p < q$, we can rebuild the image I using its $k-$low-rank approximation I_k.

To compute the vectors and the singular values needed to achieve this reconstruction, we can either:

1. perform the SVD decomposition of I
2. or, construct $M = II^T$ (or $M = I^T I$ according to the sizes q and p) and compute the k dominant eigenpairs of M then using relations (1), compute the other set of vectors (again according to the size p and q).

Once we have the k singular values, the first k vectors of U and V, compute I_k.

We talk of image compression because instead of having the image represented by an array of size $q \times p$, it is now a triplet (Σ_k, U_k, V_k) with less values when $k < q < p$.

Remark: k could correspond to a number of eigenvalues to achieve a percentage of the trace.

TODO :

Question 1: what are the size of the elements the triplet (Σ_k, U_k, V_k)?

Question 2: Using the different functions that compute the eigenpairs of a matrix (*eig*, *power method*, the 4 versions of the *subspace iteration method*), implement the alternative 2 of the reconstruction while maintaining the displays and difference calculation:

- reconstruct the image, noted I_k based on the k-low rank approximation;
- compute the difference between I and I_k: $\|I - I_k\|$.

You will present the results of the different function with some variation of parameters in a table to answer this question in your report and add some comments.

Bonus question : Dealing with the case of color images