APMTH 205 HW 1
Louis Baum
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1 Problem 1

We can set up matrix equations to find the interpolating cubic polynomial in the monomial basis as follows

1.1 monomial basis

$$\begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

For the points (0,0), (1,0), (2,1), (3,2)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

This leads to the solution

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{5}{6} \\ 1 \\ -\frac{1}{6} \end{bmatrix}$$

Corresponding to

$$-\frac{5}{6}x + x^2 - \frac{1}{6}x^3$$

1.2 Lagrange basis

We can simply write down the interpolant from the definition:

$$y_1L_1(x) + y_2L_2(x) + y_3L_3(x) + y_4L_4(x)$$

$$L_3(x) + 2L_4(x)$$

$$\frac{x-0}{2-0}\frac{x-1}{2-1}\frac{x-3}{2-3} + 2\frac{x-0}{3-0}\frac{x-1}{3-1}\frac{x-2}{3-2}$$

if we expand this we get

$$\frac{-3}{6}x^3 + 2x^2 - \frac{9}{6}x + \frac{2}{6}x^3 - x^2 + \frac{4}{6}x$$

which simplifies to

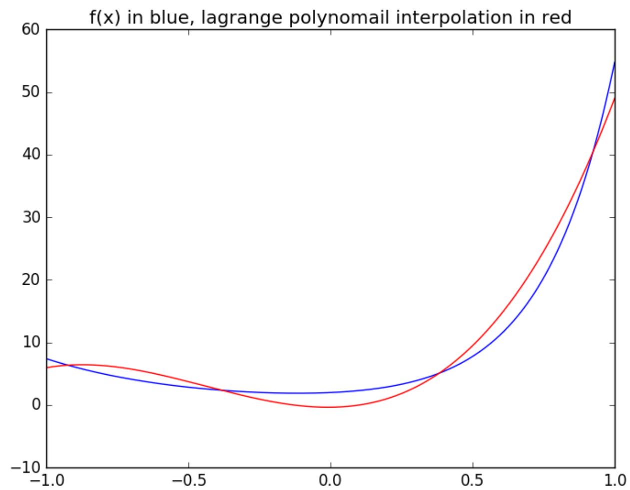
$$-\frac{5}{6}x + x^2 - \frac{1}{6}x^3$$

an equivalent representation

2 Problem 2

a,b

please refer to the python script. It will plot f(x) and the lagrange interpolation and it will print the maximum error.



c

we begin with the definition of error formula:

$$f(x) - p_{n-1}(x) = \frac{f^n(\theta)}{n!} \prod_{i=1}^{n} x - x_i$$

where $f^n(\theta)$ is the maximum of the n^th derivative of f evaluated on the interval $\theta = [-1, 1]$

$|\frac{f^n(\theta)}{n!}|$ is already well defined so we look at $|\prod_{i=1}^{n} x - x_i|$

we use the result from approximation theory from the class notes to claim that the minimum of $|\prod_{i=1}^{n} x - x_i|$ is given by $\frac{1}{2^n}$ achieved by the Chebyshev polynomial $\frac{T_{n+1}(x)}{2^n}$ on the interval $[-1, 1]$

we let $\prod_{i=1}^{n} x - x_i = \frac{T_{n+1}(x)}{2^n}$.

since $T_n(x) = cos(n(cos^{-1}(x)))$ it is clear that $|T_n(x)| \leq 1$

So: $|f(x) - p_{n-1}(x)| = \frac{|f^n(\theta)|}{n!} |\prod_{i=1}^{n} x - x_i|$

$$|f(x) - p_{n-1}(x)| = \frac{|f^n(\theta)|}{n!2^n}$$

d

by using a cubic fit and manipulating the points (x_i) at which the fit was evaluated followed by numerically checking the infinity norm :

$$p_3^{\dagger} = -2.57703 - 2.056x - 31.5292x^2 + 27.8373x^3$$

$$||f - p_3^{\dagger}|| < ||f - p_3||$$

$$5.516 < 5.752$$

3 Problem 3

a

B= $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, C= $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad \kappa(B) = 2 \qquad \kappa(C) = 2 \qquad \kappa(B + C) = 1$

b

B= $\begin{bmatrix} 9 & 0 \\ 0 & -9 \end{bmatrix}$, C= $\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ $\qquad \kappa(B) = 1 \qquad \kappa(C) = 1 \qquad \kappa(B+C) = 190$

c

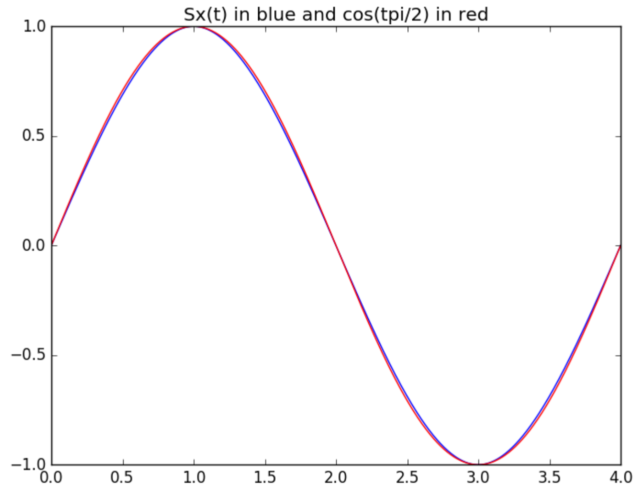
We begin with the definition of condition number:

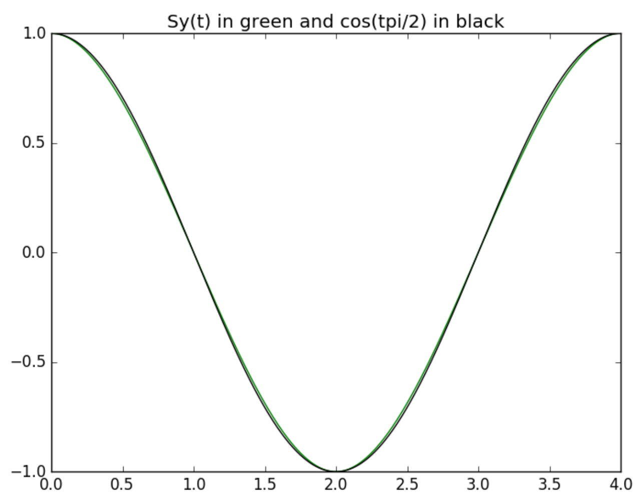
$$k(A) = ||A||||A^{-1}||$$

We know the following about symmetric matrices : $A = R^T DR$ where D is a diagonal matrix with the eigenvalues of A as diagonal entries and R is orthogonal with the eigenvectors of A as its columns.
We know $R^T R = RR^T = I$
$||A|| = \sqrt{(A^T)(A)}$
$(A^T)(A) = ||R^T DR||^2$
We also know that Orthogonal matrices preserve euclidean vector norms.
$||Rv||^2 = ||v||^2$
so that
$||R^T DR||^2 = ||D||^2$
so
$||A|| = \sqrt{||D||^2} = ||D||$
Similarly since
$A^{-1} = R^T D^{-1} R$
$||A^{-1}|| = ||D^{-1}||$
Since we know how to calculate the matrix norm of a diagonal matrix. we can say that the condition number of a symmetric matrix is equal to $\lambda_{max}\lambda_{min}^{-1}$ where λ are the eigenvalues of the matrix A.
We can determine $k(2A) = ||2A||||(2A)^{-1}|| = 2||A||\frac{1}{2}||A^{-1}||$
$k(2A) = k(A)$
We can determine $k(A^2) = ||A^2||||(A^2)^{-1}||$
since we know that k(A) = k(D) where D is the matrix with the eigenvalues of A on the diagonal
$A = R^T DR$ means that $A^2 = R^T DRR^T DR = R^T D^2 R$
again orthogonal matrices preserve euclidian vector norms so
$k(A^2) = k(D^2) = \lambda_{max}^2 \lambda_{min}^{-2} = k(A)^2$

4 Problem 4

Please refer to the python script (APMTH205HW1p4.py).





while I used scipy.interpolate.CublicSpline() - I wrote my own cubic spline script to confirm that it was choosing the boundary conditions properly. (cublic spline.py)

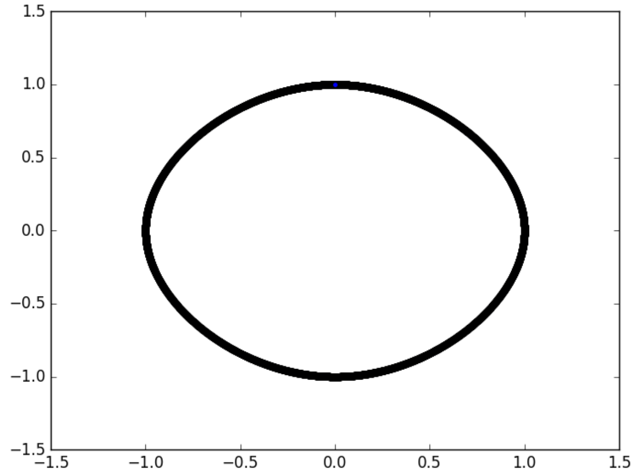


Figure 1: ‘circle’ with the approximation $\pi' \approx 3.04999$

5 Problem 5

Please see the attached code (APMTH205HW1p5.py).

a() will run the regression and plot the regular right image next to the reconstructed right image.

S = 1708

b() will plot the regular left image and the reconstructed left image. it will also print T.

T = 3291

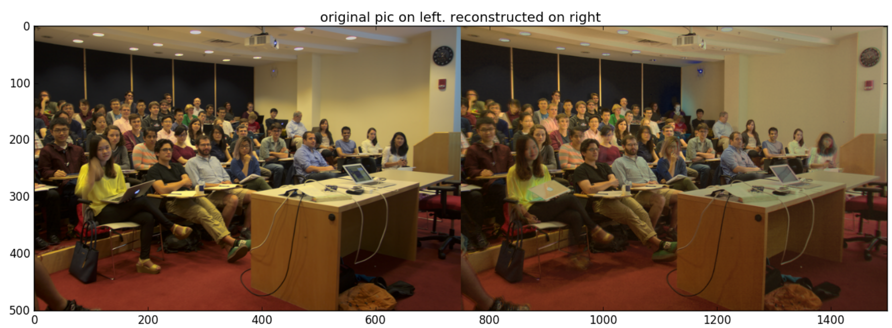


Figure 2: As you can see the reconstructed image tends to be a little less vibrant and a little blurry. Initially I found that there were areas of intense color but after constraining the values of the image to within the 0-255 range of a uint8 these issues largely disappeared. (this clipping marginally improved S)

Figure 3: There are similar issues with the reconstructed left image. but overall it worked well.)