Web Appendix for "Multiply robust estimation for causal survival analysis with treatment noncompliance"

Web Appendix A Supporting information on the single robust estimators

Web Appendix A.1 Proof of the three nonparametric identification formulas in Result 1

To prove Result 1, we need the following lemma given by Jiang et al. (2022):

Lemma 1. *Under Assumptions 1–3, for any function $h(\boldsymbol{X})$ that has finite moments $\mathbb{E}[h(\boldsymbol{X})|G=g]$ for $g=a,c,n$, the following three sets of balancing conditions hold*
(i) The following expectations are identical and also equal to $\mathbb{E}[h(\boldsymbol{X})|G=c]$.

$$\mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}h(\boldsymbol{X})\right]=\mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{1-S}{1-p_0(\boldsymbol{X})}\frac{1-Z}{1-\pi(\boldsymbol{X})}h(\boldsymbol{X})\right]$$
$$=\mathbb{E}\left[\frac{\frac{SZ}{\pi(\boldsymbol{X})}-\frac{S(1-Z)}{1-\pi(\boldsymbol{X})}}{p_1-p_0}h(\boldsymbol{X})\right]=\mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}h(\boldsymbol{X})\right].$$

(ii) The following expectations are identical and also equal to $\mathbb{E}[h(\boldsymbol{X})|G=n]$.

$$\mathbb{E}\left[\frac{e_n(\boldsymbol{X})}{1-p_1}\frac{1-S}{1-p_0(\boldsymbol{X})}\frac{1-Z}{1-\pi(\boldsymbol{X})}h(\boldsymbol{X})\right]=\mathbb{E}\left[\frac{1-S}{1-p_1}\frac{Z}{\pi(\boldsymbol{X})}h(\boldsymbol{X})\right]$$
$$=\mathbb{E}\left[\frac{1-\frac{SZ}{\pi(\boldsymbol{X})}}{1-p_1}h(\boldsymbol{X})\right]=\mathbb{E}\left[\frac{e_n(\boldsymbol{X})}{1-p_1}h(\boldsymbol{X})\right].$$

(iii) The following expectations are identical and also equal to $\mathbb{E}[h(\boldsymbol{X})|G=a]$.

$$\mathbb{E}\left[\frac{e_a(\boldsymbol{X})}{p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}h(\boldsymbol{X})\right]=\mathbb{E}\left[\frac{S}{p_0}\frac{1-Z}{1-\pi(\boldsymbol{X})}h(\boldsymbol{X})\right]=\mathbb{E}\left[\frac{e_a(\boldsymbol{X})}{p_0}h(\boldsymbol{X})\right].$$

Proof. The is Theorem S1 in Jiang et al. (2022) and we omit the proof. □

Below we only prove the expressions of $\Delta_c(u)$ in Result 1(i)–(iii), where proofs of the expressions of $\Delta_n(u)$ and $\Delta_a(u)$ are similar. Specifically, by the definition of $\Delta_c(u)$, we have that

$$\begin{aligned}\Delta_c(u)=&\mathbb{E}\left[\mathbb{I}(T(1)\geq u)-\mathbb{I}(T(0)\geq u)|G=c\right]\\=&\mathbb{E}[\mathbb{E}[\mathbb{I}(T(1)\geq u)-\mathbb{I}(T(0)\geq u)|G=c,\boldsymbol{X}]|G=c]\\=&\mathbb{E}[\mathbb{E}[\mathbb{I}(T(1)\geq u)|G=c\text{ or }a,\boldsymbol{X}]|G=c]-\mathbb{E}[\mathbb{E}[\mathbb{I}(T(0)\geq u)|G=c\text{ or }n,\boldsymbol{X}]|G=c]\\&\quad\text{(by Assumption 3)}\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}[\mathbb{E}[\mathbb{I}(T \geq u)|Z=1, G=c \text{ or } a, \boldsymbol{X}]|G=c] - \mathbb{E}[\mathbb{E}[\mathbb{I}(T \geq u)|Z=0, G=c \text{ or } n, \boldsymbol{X}]|G=c] \\
&\qquad \text{(by Assumption 1)} \\
&= \mathbb{E}[\mathbb{E}[\mathbb{I}(T \geq u)|Z=1, S=1, \boldsymbol{X}]|G=c] - \mathbb{E}[\mathbb{E}[\mathbb{I}(T \geq u)|Z=0, S=0, \boldsymbol{X}]|G=c] \\
&= \mathbb{E}[\mathcal{S}_{11}(u|\boldsymbol{X})|G=c] - \mathbb{E}[\mathcal{S}_{00}(u|\boldsymbol{X})|G=c] \\
&= \frac{1}{p_1 - p_0}\mathbb{E}\left[\left\{\frac{SZ}{\pi(\boldsymbol{X})} - \frac{S(1-Z)}{1-\pi(\boldsymbol{X})}\right\}\{\mathcal{S}_{11}(u|\boldsymbol{X}) - \mathcal{S}_{00}(u|\boldsymbol{X})\}\right] \quad \text{(by Lemma 1)} \\
&= \mathbb{E}\left[\frac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{p_1 - p_0}\{\mathcal{S}_{11}(u|\boldsymbol{X}) - \mathcal{S}_{00}(u|\boldsymbol{X})\}\right] \quad \text{(by Lemma 1)}.
\end{aligned}
$$

This proves expressions of $\Delta_c(u)$ in Result 1(ii) and Result 1(iii). The expression of $\Delta_c(u)$ in Result 1(i) can be obtained by observing

$$
\begin{aligned}
\Delta_c(u) &= \mathbb{E}\left[\frac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{p_1 - p_0}\mathcal{S}_{11}(u|\boldsymbol{X})\right] - \mathbb{E}\left[\frac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{p_1 - p_0}\mathcal{S}_{00}(u|\boldsymbol{X})\right] \\
&= \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\mathcal{S}_{11}(u|\boldsymbol{X})\mathbb{E}\left[\frac{Z}{\pi(\boldsymbol{X})}|\boldsymbol{X}\right]\right] - \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\mathcal{S}_{00}(u|\boldsymbol{X})\mathbb{E}\left[\frac{1-Z}{1-\pi(\boldsymbol{X})}|\boldsymbol{X}\right]\right] \\
&= \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X})\right] - \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{1-Z}{1-\pi(\boldsymbol{X})}\mathcal{S}_{00}(u|\boldsymbol{X})\right] \\
&= \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X})\mathbb{E}\left[\frac{S}{p_1(\boldsymbol{X})}|Z=1, \boldsymbol{X}\right]\right] \\
&\quad - \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{1-Z}{1-\pi(\boldsymbol{X})}\mathcal{S}_{00}(u|\boldsymbol{X})\mathbb{E}\left[\frac{1-S}{1-p_0(\boldsymbol{X})}|Z=0, \boldsymbol{X}\right]\right] \\
&= \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{Z}{\pi(\boldsymbol{X})}\frac{S}{p_1(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X})\right] - \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{1-S}{1-p_0(\boldsymbol{X})}\mathcal{S}_{00}(u|\boldsymbol{X})\right] \\
&= \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{Z}{\pi(\boldsymbol{X})}\frac{S}{p_1(\boldsymbol{X})}\frac{1}{S^C_{11}(u|\boldsymbol{X})}\mathbb{E}[\mathbb{I}(T \geq u)\mathbb{I}(C \geq u)|Z=1, S=1, \boldsymbol{X}]\right] \\
&\quad - \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{1-S}{1-p_0(\boldsymbol{X})}\frac{1}{S^C_{00}(u|\boldsymbol{X})}\mathbb{E}[\mathbb{I}(T \geq u)\mathbb{I}(C \geq u)|Z=0, S=0, \boldsymbol{X}]\right] \\
&\qquad \text{(by Assumption 4)} \\
&= \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{S^C_{11}(u|\boldsymbol{X})}\right] - \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{1-S}{1-p_0(\boldsymbol{X})}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{S^C_{00}(u|\boldsymbol{X})}\right]
\end{aligned}
$$

This completes the proof of Result 1.

Web Appendix A.2 Explicit expressions of the single robust estimators

Based on Result 1(i), we have that

$$
\begin{aligned}
\widehat{\Delta}^1_c(u) &= \widehat{\mathcal{S}}^1_{1,c}(u) - \widehat{\mathcal{S}}^1_{0,c}(u) \\
&= \mathbb{P}_n\left[\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1 - \widehat{p}_0}\frac{S}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}\right] - \mathbb{P}_n\left[\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1 - \widehat{p}_0}\frac{1-S}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}\right], \\
\widehat{\Delta}^1_n(u) &= \widehat{\mathcal{S}}^1_{1,n}(u) - \widehat{\mathcal{S}}^1_{0,n}(u) \\
&= \mathbb{P}_n\left[\frac{1-S}{1-\widehat{p}_1}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{10})}\right] - \mathbb{P}_n\left[\frac{e_n(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{1-\widehat{p}_1}\frac{1-S}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}\right], \\
\widehat{\Delta}^1_a(u) &= \widehat{\mathcal{S}}^1_{1,a}(u) - \widehat{\mathcal{S}}^1_{0,a}(u)
\end{aligned}
$$

$$=\mathbb{P}_n\left[\frac{e_a(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_0}\frac{S}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}_{11}^C(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}\right]-\mathbb{P}_n\left[\frac{S}{\widehat{p}_0}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}_{01}^C(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{01})}\right].$$

Based on Result 1(ii), we have that

$$\begin{aligned}
\widehat{\Delta}_c^2(u) &= \widehat{\mathcal{S}}_{1,c}^2(u)-\widehat{\mathcal{S}}_{0,c}^2(u)\\
&=\mathbb{P}_n\left[\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1-\widehat{p}_0}\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\right]-\mathbb{P}_n\left[\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1-\widehat{p}_0}\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\right],\\
\widehat{\Delta}_n^2(u) &= \widehat{\mathcal{S}}_{1,n}^2(u)-\widehat{\mathcal{S}}_{0,n}^2(u)\\
&=\mathbb{P}_n\left[\frac{e_n(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{1-\widehat{p}_1}\mathcal{S}_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10})\right]-\mathbb{P}_n\left[\frac{e_n(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{1-\widehat{p}_1}\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\right],\\
\widehat{\Delta}_a^2(u) &= \widehat{\mathcal{S}}_{1,a}^2(u)-\widehat{\mathcal{S}}_{0,a}^2(u)\\
&=\mathbb{P}_n\left[\frac{e_a(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_0}\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\right]-\mathbb{P}_n\left[\frac{e_a(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_0}-\mathcal{S}_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01})\right].
\end{aligned}$$

Based on Result 1(iii), we have that

$$\begin{aligned}
\widehat{\Delta}_c^3(u) &= \widehat{\mathcal{S}}_{1,c}^3(u)-\widehat{\mathcal{S}}_{0,c}^3(u)\\
&=\mathbb{P}_n\left[\left(\frac{SZ}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}-\frac{S(1-Z)}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\right)\frac{\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})}{\widehat{p}_1-\widehat{p}_0}\right]-\mathbb{P}_n\left[\left(\frac{SZ}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}-\frac{S(1-Z)}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\right)\frac{\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})}{\widehat{p}_1-\widehat{p}_0}\right],\\
\widehat{\Delta}_n^3(u) &= \widehat{\mathcal{S}}_{1,n}^3(u)-\widehat{\mathcal{S}}_{0,n}^3(u)\\
&=\mathbb{P}_n\left[\left(1-\frac{SZ}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\right)\frac{\mathcal{S}_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10})}{1-\widehat{p}_1}\right]-\mathbb{P}_n\left[\left(1-\frac{SZ}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\right)\frac{\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})}{1-\widehat{p}_1}\right],\\
\widehat{\Delta}_a^3(u) &= \widehat{\mathcal{S}}_{1,a}^3(u)-\widehat{\mathcal{S}}_{0,a}^3(u)\\
&=\mathbb{P}_n\left[\frac{S(1-Z)}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\frac{\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})}{\widehat{p}_0}\right]-\mathbb{P}_n\left[\frac{S(1-Z)}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\frac{\mathcal{S}_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01})}{\widehat{p}_0}\right].
\end{aligned}$$

Web Appendix B The proposed multiply robust estimator

Web Appendix B.1 Derivation of the multiply robust estimator

In this section, we shall derive the simplified expression of $\psi_{1,c}^{opt}(\mathcal{O})$ shown in equation (8), i.e., the optimal AIPWCC estimating score of $\mathcal{S}_{1,c}(u)$ based on the observed data in the presence of right censoring. As demostrated in Section 3.3 of the paper, the original expression of $\psi_{1,c}^{opt}(\mathcal{O})$ given by Tsiatis (2006, Chapter 10) is

$$\psi_{1,c}^{opt}(\mathcal{O})=\frac{\delta\psi_{1,c}^{mr}(\mathcal{A})}{\mathcal{S}_{ZS}^C(U|\boldsymbol{X})}+\int h^{opt}(r,Z,S,\boldsymbol{X})dM_{ZS}^C(r|\boldsymbol{X}),$$

where $h^{opt}(t,Z,S,\boldsymbol{X}) = \Pi\left\{\frac{\delta\psi_{1,c}^{mr}(\mathcal{A})}{\mathcal{S}_{ZS}^C(U|\boldsymbol{X})}\middle|\mathcal{T}_C\right\}$ and $\mathcal{T}_C = \left\{\int h^*(r,Z,S,\boldsymbol{X})dM_{ZS}^C(r|\boldsymbol{X}):\forall h^*\right\}$ is the nuisance tangent space of the right censoring. According to Theorem 10.4 and Chapter 10.4 of Tsiatis (2006), the optimal choice of $h^{opt}(t,Z,S,\boldsymbol{X})$ can be rewritten as the following conditional expectation form $h^{opt}(t,Z,S,\boldsymbol{X})=\frac{\mathbb{E}[\psi_{1,c}^{mr}(\mathcal{A})|T\geq t,Z,S,\boldsymbol{X}]}{\mathcal{S}_{ZS}^C(t|\boldsymbol{X})}$. This suggests that $\psi_{1,c}^{opt}(\mathcal{O})$ can be rewritten as

$$\psi_{1,c}^{opt}(\mathcal{O}) = \frac{\delta\psi_{1,c}^{mr}(\mathcal{A})}{\mathcal{S}_{ZS}^{C}(U|\boldsymbol{X})} + \int \frac{\mathbb{E}[\psi_{1,c}^{mr}(\mathcal{A})|T \geq r, Z, S, \boldsymbol{X}]}{\mathcal{S}_{ZS}^{C}(r|\boldsymbol{X})} dM_{ZS}^{C}(r|\boldsymbol{X}). \tag{B.1}$$

For simplicity, we suppress the expression of the complete-data estimating score $\psi_{1,c}^{mr}(\mathcal{A})$ introduced in equation (6) as

$$\psi_{1,c}^{mr}(\mathcal{A}) = \frac{e_c(\boldsymbol{X})}{\widetilde{p}_1 - \widetilde{p}_0} \frac{S}{p_1(\boldsymbol{X})} \frac{Z}{\pi(\boldsymbol{X})} \mathbb{I}(T \geq u) + \kappa(u, Z, S, \boldsymbol{X}) \tag{B.2}$$

where we merge all terms in $\psi_{1,c}^{mr}(\mathcal{A})$ in the absence of failure time T into a single term denoted by $\kappa(u, Z, S, \boldsymbol{X})$ such that

$$\begin{aligned}\kappa(u, Z, S, \boldsymbol{X}) = & -\mathcal{S}_{1,c}(u) + \frac{e_c(\boldsymbol{X})}{\widetilde{p}_1 - \widetilde{p}_0}\left(1 - \frac{Z}{\pi(\boldsymbol{X})}\right)\mathcal{S}_{11}(u|\boldsymbol{X}) \\ & + \frac{\mathcal{S}_{11}(u|\mathbf{X})}{\widetilde{p}_1 - \widetilde{p}_0}\left[\frac{1-Z}{1-\pi(\boldsymbol{X})}(S - p_0(\boldsymbol{X})) + \frac{p_0(\boldsymbol{X})}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}(p_1(\boldsymbol{X}) - S)\right]\end{aligned}$$

Substituting equation (B.2) into (B.1) leads to

$$\psi_{1,c}^{opt}(\mathcal{O}) = \frac{e_c(\boldsymbol{X})}{\widetilde{p}_1 - \widetilde{p}_0} \frac{S}{p_1(\boldsymbol{X})} \frac{Z}{\pi(\boldsymbol{X})} \frac{\delta\mathbb{I}(T \geq u)}{\mathcal{S}_{ZS}^{C}(U|\boldsymbol{X})} + \frac{\delta\kappa(u, Z, S, \boldsymbol{X})}{\mathcal{S}_{ZS}^{C}(U|\boldsymbol{X})} \tag{B.3}$$

$$+ \frac{e_c(\boldsymbol{X})}{\widetilde{p}_1 - \widetilde{p}_0} \frac{S}{p_1(\boldsymbol{X})} \frac{Z}{\pi(\boldsymbol{X})} \int_0^\infty \frac{\mathbb{E}[\mathbb{I}(T \geq u)|T \geq r, Z, S, \boldsymbol{X}]}{\mathcal{S}_{ZS}^{C}(r|\boldsymbol{X})} dM_{ZS}^{C}(r|\boldsymbol{X}) \tag{B.4}$$

$$+ \kappa(u, Z, S, \boldsymbol{X}) \int_0^\infty \frac{dM_{ZS}^{C}(r|\boldsymbol{X})}{\mathcal{S}_{ZS}^{C}(r|\boldsymbol{X})} \tag{B.5}$$

Noting $\frac{\delta\mathbb{I}(T\geq u)}{\mathcal{S}_{ZS}^{C}(U|\mathbf{X})} = \frac{\delta\mathbb{I}(U\geq u)}{\mathcal{S}_{ZS}^{C}(U|\mathbf{X})}$, we have that

$$\text{(B.3)} = \frac{e_c(\boldsymbol{X})}{\widetilde{p}_1 - \widetilde{p}_0} \frac{S}{p_1(\boldsymbol{X})} \frac{Z}{\pi(\boldsymbol{X})} \frac{\delta\mathbb{I}(U \geq u)}{\mathcal{S}_{ZS}^{C}(U|\boldsymbol{X})} + \frac{\delta\kappa(u, Z, S, \boldsymbol{X})}{\mathcal{S}_{ZS}^{C}(U|\boldsymbol{X})}. \tag{B.6}$$

By observing that $d\Lambda_{ZS}^{C}(r|\boldsymbol{X}) = -\frac{d\log\mathcal{S}_{ZS}^{C}(u|\boldsymbol{X})}{dr}$, we have $\int_0^t \frac{d\Lambda_{ZS}^{C}(r|\boldsymbol{X})}{\mathcal{S}_{ZS}^{C}(r|\boldsymbol{X})} = \frac{1}{\mathcal{S}_{ZS}^{C}(t|\boldsymbol{X})}$. Therefore, (B.5) can be simplified to

$$\begin{aligned}\text{(B.5)} &= \kappa(u, Z, S, \boldsymbol{X})\left\{\int_0^\infty \frac{dN^C(r)}{\mathcal{S}_{ZS}^{C}(r|\boldsymbol{X})} - \int_0^\infty \frac{\mathbb{I}(U \geq r)d\Lambda_{ZS}^{C}(r|\boldsymbol{X})}{\mathcal{S}_{ZS}^{C}(r|\boldsymbol{X})}\right\} \\ &= \kappa(u, Z, S, \boldsymbol{X})\left\{\frac{1-\delta}{\mathcal{S}_{ZS}^{C}(U|\boldsymbol{X})} - \left(\frac{1}{\mathcal{S}_{ZS}^{C}(U|\boldsymbol{X})} - \frac{1}{\mathcal{S}_{ZS}^{C}(0|\boldsymbol{X})}\right)\right\} \\ &= \kappa(u, Z, S, \boldsymbol{X})\left\{\frac{1-\delta}{\mathcal{S}_{ZS}^{C}(U|\boldsymbol{X})} - \frac{1}{\mathcal{S}_{ZS}^{C}(U|\boldsymbol{X})} + 1\right\} \\ &= \kappa(u, Z, S, \boldsymbol{X})\left\{1 - \frac{\delta}{\mathcal{S}_{ZS}^{C}(U|\boldsymbol{X})}\right\}\end{aligned} \tag{B.7}$$

Next, we can show that

$$\text{(B.4)} = \frac{e_c(\boldsymbol{X})}{\widetilde{p}_1 - \widetilde{p}_0} \frac{S}{p_1(\boldsymbol{X})} \frac{Z}{\pi(\boldsymbol{X})} \int_0^\infty \frac{\mathbb{P}(T \geq u|T \geq r, Z, S, \boldsymbol{X})}{\mathcal{S}_{ZS}^{C}(r|\boldsymbol{X})} dM_{ZS}^{C}(r|\boldsymbol{X})$$

$$
\begin{aligned}
=&\frac{e_c(\boldsymbol{X})}{\widetilde{p}_1-\widetilde{p}_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\int_0^\infty \mathbb{I}(r<u)\frac{\mathcal{S}_{ZS}(u|\boldsymbol{X})}{\mathcal{S}^C_{ZS}(r|\boldsymbol{X})\mathcal{S}_{ZS}(r|\boldsymbol{X})}+\mathbb{I}(r\geq u)\frac{1}{\mathcal{S}^C_{ZS}(r|\boldsymbol{X})}dM^C_{ZS}(r|\boldsymbol{X})\\
=&\frac{e_c(\boldsymbol{X})}{\widetilde{p}_1-\widetilde{p}_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\int_0^u \frac{\mathcal{S}_{ZS}(u|\boldsymbol{X})}{\mathcal{S}^C_{ZS}(r|\boldsymbol{X})\mathcal{S}_{ZS}(r|\boldsymbol{X})}dM^C_{ZS}(r|\boldsymbol{X}) &\text{(B.8)}\\
&+\frac{e_c(\boldsymbol{X})}{\widetilde{p}_1-\widetilde{p}_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\int_u^\infty \frac{dM^C_{ZS}(r|\boldsymbol{X})}{\mathcal{S}^C_{ZS}(r|\boldsymbol{X})}, &\text{(B.9)}
\end{aligned}
$$

where

$$
\begin{aligned}
\text{(B.9)}=&\frac{e_c(\boldsymbol{X})}{\widetilde{p}_1-\widetilde{p}_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\int_u^\infty \frac{dN^C(r)}{\mathcal{S}^C_{ZS}(r|\boldsymbol{X})}+\frac{e_c(\boldsymbol{X})}{\widetilde{p}_1-\widetilde{p}_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\int_u^\infty \frac{\mathbb{I}(U\geq r)d\Lambda^C_{ZS}(r|\boldsymbol{X})}{\mathcal{S}^C_{ZS}(r|\boldsymbol{X})}\\
=&\frac{e_c(\boldsymbol{X})}{\widetilde{p}_1-\widetilde{p}_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathbb{I}(U\geq u)\left\{\frac{1-\delta}{\mathcal{S}^C_{ZS}(U|\boldsymbol{X})}-\int_u^U\frac{d\Lambda^C_{ZS}(r|\boldsymbol{X})}{\mathcal{S}^C_{ZS}(r|\boldsymbol{X})}\right\}\\
=&\frac{e_c(\boldsymbol{X})}{\widetilde{p}_1-\widetilde{p}_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathbb{I}(U\geq u)\left\{\frac{1-\delta}{\mathcal{S}^C_{ZS}(U|\boldsymbol{X})}-\left(\frac{1}{\mathcal{S}^C_{ZS}(U|\boldsymbol{X})}-\frac{1}{\mathcal{S}^C_{ZS}(u|\boldsymbol{X})}\right)\right\}\\
=&\frac{e_c(\boldsymbol{X})}{\widetilde{p}_1-\widetilde{p}_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathbb{I}(U\geq u)\left\{\frac{1}{\mathcal{S}^C_{ZS}(u|\boldsymbol{X})}-\frac{\delta}{\mathcal{S}^C_{ZS}(U|\boldsymbol{X})}\right\}, &\text{(B.10)}
\end{aligned}
$$

Combining the above discussion, we have

$$
\begin{aligned}
\psi^{opt}_{1,c}(\mathcal{O})&=\text{(B.3)}+\text{(B.4)}+\text{(B.5)}\\
&=\text{(B.6)}+\text{(B.8)}+\text{(B.10)}+\text{(B.7)}\\
&=\frac{e_c(\boldsymbol{X})}{\widetilde{p}_1-\widetilde{p}_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\left\{\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{ZS}(u|\boldsymbol{X})}+\mathcal{S}_{ZS}(u|\boldsymbol{X})\int_0^u\frac{dM^C_{ZS}(r|\boldsymbol{X})}{S_{ZS}(r|\boldsymbol{X})S^C_{ZS}(r|\boldsymbol{X})}\right\}+\kappa(u,Z,S,\boldsymbol{X})\\
&=\frac{e_c(\boldsymbol{X})}{\widetilde{p}_1-\widetilde{p}_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\left\{\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X})}+\mathcal{S}_{11}(u|\boldsymbol{X})\int_0^u\frac{dM^C_{11}(r|\boldsymbol{X})}{S_{11}(r|\boldsymbol{X})S^C_{11}(r|\boldsymbol{X})}\right\}+\kappa(u,Z,S,\boldsymbol{X})\\
&=\text{the right-hand side of equation (8) in the paper.} &\text{(B.11)}
\end{aligned}
$$

This completes the derivation.

Web Appendix B.2 Explicit expressions of the multiple robust estimators

$$
\begin{aligned}
\widehat{\mathcal{S}}^{mr}_{0,c}(u)=&\mathbb{P}_n\Bigg\{\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1-\widehat{p}_0}\frac{1-S}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}+\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\int_0^u\frac{d\widehat{M}^C_{00}(r|\boldsymbol{X})}{\mathcal{S}_{00}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\mathcal{S}^C_{00}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}\right]\\
&+\frac{\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})}{\widehat{p}_1-\widehat{p}_0}\left[\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1))+\frac{1-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)-S)\right]\\
&+\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1-\widehat{p}_0}\left(1-\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\Bigg\},\\
\widehat{\mathcal{S}}^{mr}_{1,c}(u)=&\mathbb{P}_n\Bigg\{\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1-\widehat{p}_0}\frac{S}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}+\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\int_0^u\frac{d\widehat{M}^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}\right]\\
&+\frac{\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})}{\widehat{p}_1-\widehat{p}_0}\left[\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)-S)+\frac{p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}))\right]\\
&+\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1-\widehat{p}_0}\left(1-\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\mathcal{S}}^{mr}_{0,n}(u) =& \mathbb{P}_n \Bigg\{ \frac{e_n(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{1-\widehat{p}_1} \frac{1-S}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)} \frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})} \left[\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})} + \mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00}) \int_0^u \frac{d\widehat{M}^C_{00}(r|\boldsymbol{X})}{\mathcal{S}_{00}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\mathcal{S}^C_{00}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})} \right] \\
&+ \frac{\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})}{1-\widehat{p}_1} \left[\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1) - S) + \frac{1-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)} \frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S - p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)) \right] \\
&+ \frac{e_n(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{1-\widehat{p}_1} \left(1 - \frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})} \right) \mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00}) \Bigg\}, \\
\widehat{\mathcal{S}}^{mr}_{1,n}(u) =& \mathbb{P}_n \Bigg\{ \frac{1-S}{1-\widehat{p}_1} \frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})} \left[\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{10})} + \mathcal{S}_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10}) \int_0^u \frac{d\widehat{M}^C_{10}(r|\boldsymbol{X})}{\mathcal{S}_{10}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10})\mathcal{S}^C_{10}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{10})} \right] \\
&+ \frac{1-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}{1-\widehat{p}_1} \left(1 - \frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})} \right) \mathcal{S}_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10}) \Bigg\}, \\
\widehat{\mathcal{S}}^{mr}_{0,a}(u) =& \mathbb{P}_n \Bigg\{ \frac{S}{\widehat{p}_0} \frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})} \left[\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{01})} + \mathcal{S}_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01}) \int_0^u \frac{d\widehat{M}^C_{01}(r|\boldsymbol{X})}{\mathcal{S}_{01}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01})\mathcal{S}^C_{01}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{01})} \right] \\
&+ \frac{p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}{\widehat{p}_0} \left(1 - \frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})} \right) \mathcal{S}_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01}) \Bigg\}, \\
\widehat{\mathcal{S}}^{mr}_{1,a}(u) =& \mathbb{P}_n \Bigg\{ \frac{e_a(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_0} \frac{S}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)} \frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})} \left[\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})} + \mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11}) \int_0^u \frac{d\widehat{M}^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})} \right] \\
&+ \frac{\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})}{\widehat{p}_0} \left[\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S - p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)) + \frac{p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)} \frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}) - S) \right] \\
&+ \frac{e_a(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_0} \left(1 - \frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})} \right) \mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11}) \Bigg\}.
\end{aligned}
$$

Web Appendix B.3 Proof of Result 2

Recall that $\widehat{\boldsymbol{\gamma}}$, $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\theta}}_{zs}$, $\widehat{\boldsymbol{\beta}}_{zs}$ are the point estimates of unknown parameters in the working models $\mathcal{M}_e$, $\mathcal{M}_\pi$, $\mathcal{M}_C$, and $\mathcal{M}_T$ respectively. When $n \to \infty$, assume that they have probability limits $\boldsymbol{\gamma}^*$, $\boldsymbol{\alpha}^*$, $\boldsymbol{\theta}^*_{zs}$, and $\boldsymbol{\beta}^*_{zs}$, respectively. Notice that if $\mathcal{M}_\pi$ is correctly specified, then $\pi(\boldsymbol{X};\boldsymbol{\alpha}^*) = \pi(\boldsymbol{X})$, otherwise $\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)$ generally does not equal to $\pi(\boldsymbol{X})$. Similarly, $p_z(\boldsymbol{X};\boldsymbol{\gamma}^*) = p_z(\boldsymbol{X})$ and $e_g(\boldsymbol{X};\boldsymbol{\gamma}^*) = e_g(\boldsymbol{X})$ if $\mathcal{M}_e$ is correctly specified, $\mathcal{S}^C_{zs}(u|\boldsymbol{X};\boldsymbol{\theta}^*_{zs}) = \mathcal{S}^C_{zs}(u|\boldsymbol{X})$ if $\mathcal{M}_C$ is correctly specified, and $\mathcal{S}_{zs}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{zs}) = \mathcal{S}_{zs}(u|\boldsymbol{X})$ if $\mathcal{M}_T$ is correctly specified. Furthermore, as demonstrated in equation (5) in the paper, $\widehat{p}_0$ and $\widehat{p}_1$ will converge to

$$
\mathbb{E}\left[\frac{(1-Z)(S - p_0(\boldsymbol{X};\boldsymbol{\gamma}^*))}{1-\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)} + p_0(\boldsymbol{X};\boldsymbol{\gamma}^*) \right] \quad \text{and} \quad \mathbb{E}\left[\frac{Z(S - p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}))}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)} + p_1(\boldsymbol{X};\boldsymbol{\gamma}^*) \right],
$$

which equal to p_0 and p_1, respectively, if either $\mathcal{M}_e$ or $\mathcal{M}_\pi$ is correctly specified. In what follows, we prove that proposed multiply robust estimator, $\widehat{\mathcal{S}}^{mr}_{1,c}(u)$, is consistent for $\mathcal{S}_{1,c}$ under $\mathcal{M}_{\pi+e+C} \cup \mathcal{M}_{\pi+T} \cup \mathcal{M}_{e+T}$. Consistency of the multiply robust estimators for other quantities, including $\widehat{\mathcal{S}}^{mr}_{0,c}(u)$, $\widehat{\mathcal{S}}^{mr}_{1,a}(u)$, $\widehat{\mathcal{S}}^{mr}_{0,a}(u)$, $\widehat{\mathcal{S}}^{mr}_{1,n}(u)$, and $\widehat{\mathcal{S}}^{mr}_{0,n}(u)$, can be proved similarly and therefore are omitted here. We introduce the following lemma to facilitate the proof.

Lemma 2. *If assumption 4 holds and $S^C_{zs}(t|\boldsymbol{X};\boldsymbol{\theta}^*_{zs}) > 0$ for all $t \in [0, t_{\max}]$, we have that for any $u \in [0, t_{\max}]$:*

$$
\frac{\mathbb{I}(C \geq T \text{ or } C \geq u)}{\mathcal{S}^C_{zs}(u|\boldsymbol{X};\boldsymbol{\theta}^*_{zs})} = 1 - \int_0^u \frac{dM^C_{zs}(t|\boldsymbol{X};\boldsymbol{\theta}^*_{zs})}{\mathcal{S}^C_{zs}(t|\boldsymbol{X};\boldsymbol{\theta}^*_{zs})}.
$$

even if $\mathcal{M}_C$ is not correctly specified.

Proof. Lemma 2 follows from Lemma 10.4 in Tsiatis (2006) and we omit the proof here. $\square$

In what follows, we shall prove that $\widehat{\mathcal{S}}^{mr}_{1,c}(u)$ converges in probability to $\mathcal{S}_{1,c}(u)$ when either $\mathcal{M}_{\pi+e+C}$, $\mathcal{M}_{e+T}$, or $\mathcal{M}_{\pi+T}$ is correctly specified. This will complete our proof.

(i) When $\mathcal{M}_{\pi+e+C}$ is correctly specified

If $\mathcal{M}_{\pi+e+C}$ is correctly specified, $\widehat{\mathcal{S}}^{mr}_{1,c}(u)$ converges in probability to

$$
\begin{aligned}
&\mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{1}{\mathcal{S}^C_{11}(u|\boldsymbol{X})}\mathbb{I}(U\geq u)\right] && \text{(B.12)}\\
+&\mathbb{E}\left[\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\left(1-\frac{Z}{\pi(\boldsymbol{X})}\right)\right] && \text{(B.13)}\\
+&\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})}{p_1-p_0}\left\{\frac{1-Z}{1-\pi(\boldsymbol{X})}(p_0(\boldsymbol{X})-S)+\frac{p_0(\boldsymbol{X})}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}(S-p_1(\boldsymbol{X}))\right\}\right] && \text{(B.14)}\\
+&\mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\int_0^u\frac{dM^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X})}\right]. && \text{(B.15)}
\end{aligned}
$$

By Result 1, (B.12) $=\mathcal{S}_{1,c}(u)$. Also, applying the law of iterated expectation to (B.13)–(B.15), we have that

$$
\begin{aligned}
\text{(B.13)} =&\mathbb{E}\left\{\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\mathbb{E}\left[1-\frac{Z}{\pi(\boldsymbol{X})}\middle|\boldsymbol{X}\right]\right\}=0,\\
\text{(B.14)} =&\mathbb{E}\left[\mathbb{E}\left\{\frac{1-Z}{1-\pi(\boldsymbol{X})}\mathbb{E}[p_0(\boldsymbol{X})-S|Z=0,\boldsymbol{X}]+\frac{p_0(\boldsymbol{X})}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathbb{E}[S-p_1(\boldsymbol{X})|Z=1,\boldsymbol{X}]\middle|\boldsymbol{X}\right\}\frac{\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})}{p_1-p_0}\right]=0,\\
\text{(B.15)} =&\mathbb{E}\left\{\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_0(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\mathbb{E}\left[\int_0^u\frac{dM^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X})}\middle|Z=1,S=1,\boldsymbol{X}\right]\right\}=0.
\end{aligned}
$$

The last equality in the above set of formulas holds because, conditional on $Z=1, S=1, \boldsymbol{X}$, $\int_0^u\frac{dM^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X})}$ is a martingale and has expectation 0 such that $\mathbb{E}\left[\int_0^u\frac{dM^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X})}\middle|Z=1,S=1,\boldsymbol{X}\right]=0$. Therefore, $\widehat{\mathcal{S}}^{mr}_{1,c}(u)$ converges in probability to (B.12) + (B.13) + (B.14) + (B.15) $=\mathcal{S}_{1,c}(u)$, if $\mathcal{M}_{\pi+e+C}$ is correctly specified.

(ii) When $\mathcal{M}_{e+T}$ is correctly specified

If $\mathcal{M}_{e+T}$ is correctly specified, $\widehat{\mathcal{S}}^{mr}_{1,c}(u)$ converges in probability to

$$
\begin{aligned}
&\mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X};\boldsymbol{\theta}^*_{11})}\right] && \text{(B.16)}\\
+&\mathbb{E}\left[\mathcal{S}_{11}(u|\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\left(1-\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\right)\right] && \text{(B.17)}\\
+&\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{p_1-p_0}\left\{\frac{1-Z}{1-\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}(p_0(\boldsymbol{X})-S)+\frac{p_0(\boldsymbol{X})}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}(S-p_1(\boldsymbol{X}))\right\}\right] && \text{(B.18)}\\
+&\mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathcal{S}_{11}(u|\boldsymbol{X})\int_0^u\frac{dM^C_{11}(r|\boldsymbol{X};\boldsymbol{\theta}^*_{11})}{\mathcal{S}_{11}(r|\boldsymbol{X})\mathcal{S}^C_{11}(r|\boldsymbol{X};\boldsymbol{\theta}^*_{11})}\right]. && \text{(B.19)}
\end{aligned}
$$

By Lemma 2,

$$
\begin{aligned}
\text{(B.16)} =& \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\frac{\mathbb{I}(C\geq u \text{ or } C\geq T)\mathbb{I}(T\geq u)}{\mathcal{S}_{11}^C(u|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right] \\
=& \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathbb{I}(T\geq u)\right] && \text{(B.20)} \\
&- \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathbb{I}(T\geq u)\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}^C(r|\boldsymbol{X};\theta_{11}^*)}\right], && \text{(B.21)} \\
\text{(B.20)} =& \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathbb{E}[\mathbb{I}(T\geq u)|Z=1,S=1,\boldsymbol{X}]\right] \\
=& \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathcal{S}_{11}(u|\boldsymbol{X})\mathbb{E}\left[\frac{S}{p_1(\boldsymbol{X})}\middle| Z=1,\boldsymbol{X}\right]\right] \\
=& \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathcal{S}_{11}(u|\boldsymbol{X})\right].
\end{aligned}
$$

Therefore, by Result 1

$$
\text{(B.17)} + \text{(B.20)} = \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\mathcal{S}_{11}(u|\boldsymbol{X})\right] = \mathcal{S}_{1,c}(u).
$$

Similar to the proof of (B.14) $=0$, we have that (B.18) $=0$. Also, one can show that

$$
\begin{aligned}
&\text{(B.19)} + \text{(B.21)} \\
=& \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\left(\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X})}-\mathbb{I}(T\geq u)\right)\right] \\
=& \mathbb{E}\left\{\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\int_0^u\frac{dN^C(s)}{\mathcal{S}_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X})}-\mathbb{I}(T\geq u)\middle| Z=1,S=1,\boldsymbol{X},C\geq r,T\geq r\right]\right\} \\
&- \mathbb{E}\left\{\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\int_0^u\frac{\mathbb{I}(U\geq u)d\Lambda_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X})}-\mathbb{I}(T\geq u)\middle| Z=1,S=1,\boldsymbol{X},C\geq r,T\geq r\right]\right\}
\end{aligned}
$$

Since

$$
\mathbb{E}[\mathbb{I}(T\geq u)|Z=1,S=1,\boldsymbol{X},C,T\geq r] = \frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X})}, \tag{B.22}
$$

one can conclude (B.19) + (B.21) $=0$. To summarize, if $\mathcal{M}_{e+T}$ is correctly specified, $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$ converges in probability to (B.16) + (B.17) + (B.18) + (B.19) $=\mathcal{S}_{1,c}(u)$.

(iii) When $\mathcal{M}_{\pi+T}$ is correctly specified

Under $\mathcal{M}_{\pi+T}$, $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$ converges in probability to

$$
\begin{aligned}
&\mathbb{E}\left[\frac{e_c(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}_{11}^C(u|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right] && \text{(B.23)} \\
&+\mathbb{E}\left[\mathcal{S}_{11}(u|\boldsymbol{X})\frac{e_c(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1-p_0}\left(1-\frac{1}{\pi(\boldsymbol{X})}\right)\right] && \text{(B.24)} \\
&+\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{p_1-p_0}\left\{\frac{1-Z}{1-\pi(\boldsymbol{X})}(p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)-S)+\frac{p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}(S-p_1(\boldsymbol{X};\boldsymbol{\gamma}^*))\right\}\right] && \text{(B.25)} \\
&+\mathbb{E}\left[\frac{e_c(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X})\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}(u|\boldsymbol{X})\mathcal{S}_{11}^C(u|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right]. && \text{(B.26)}
\end{aligned}
$$

By Lemma 2, and similar to the derivation of (B.20) and (B.21), we have

$$
\begin{aligned}
\text{(B.23)} =&\mathbb{E}\left[\frac{e_c(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(C\geq u \text{ or } C\geq T)\mathbb{I}(T\geq u)}{\mathcal{S}_{11}^C(u|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right]\\
=&\mathbb{E}\left[\frac{e_c(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1-p_0}\frac{p_1(\boldsymbol{X})}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\mathcal{S}_{11}(u|\boldsymbol{X})\right] \quad \text{(B.27)}\\
&-\mathbb{E}\left[\frac{e_c(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}\mathbb{I}(T\geq u)\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right], \quad \text{(B.28)}
\end{aligned}
$$

Also, applying the law of iterative expectation to (B.24), we have

$$
\text{(B.24)} = \mathbb{E}\left\{\mathcal{S}_{11}(u|\boldsymbol{X})\frac{e_c(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1-p_0}\mathbb{E}\left[1-\frac{Z}{\pi(\boldsymbol{X})}\middle|\boldsymbol{X}\right]\right\}=0.
$$

Then, according Result 1, we know that

$$
\text{(B.25)} = \frac{1}{p_1-p_0}\mathbb{E}\left[\left(\frac{ZS}{\pi(\boldsymbol{X})}-\frac{(1-Z)S}{1-\pi(\boldsymbol{X})}\right)\mathcal{S}_{11}(u|\boldsymbol{X})\right]+R=\mathcal{S}_{1,c}(u)+R,
$$

where

$$
\begin{aligned}
R=&\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{p_1-p_0}\left\{\frac{1-Z}{1-\pi(\boldsymbol{X})}p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)-\frac{p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)-\frac{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)-p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{ZS}{\pi(\boldsymbol{X})}\right\}\right]\\
=&\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{p_1-p_0}\left\{p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)-\frac{p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)-\frac{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)-p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}p_1(\boldsymbol{X})\right\}\right]\\
=&-\text{(B.27)}.
\end{aligned}
$$

And similar to the proof of (B.19) + (B.21) = 0, we can show that (B.26) + (B.28) = 0. Therefore, if $\mathcal{M}_{\pi+T}$ is correctly specified, we have that $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$ converges in probability to (B.23) + (B.24) + (B.25) + (B.26) = $\mathcal{S}_{1,c}(u)$.

Web Appendix C A simulation study

We conduct simulations to evaluate the finite-sample performance of the proposed triply robust estimator under ignorable treatment assignment. We consider 500 Monte Carlo replications and for each replication, $n = 1,000$ observations are simulated based on the following data generation process. We first generate five baseline covariates $\boldsymbol{X} = (X_1, X_2, X_3, X_4, X_5)^T$, where $X_1 \sim$ Bernoulli(0.5), $X_2 \sim N(0,1)$, $X_3 \sim N(0,1)$, and $(X_4, X_5)^T = (X_2^2-1, X_3^2-1)^T$. Then, the treatment assignment is generated by $\mathbb{P}(Z=1|\boldsymbol{X}) = 1/(1+e^{-\boldsymbol{\alpha}^T\boldsymbol{X}})$, where $\boldsymbol{\alpha} = [0,0,0,0.5,0.4]^T$. Next, we generate S by the logistic regression $\mathbb{P}(S=1|Z,X) = \frac{\exp(-0.5+Z+\boldsymbol{\gamma}^T\boldsymbol{X})}{1+\exp(-0.5+Z+\boldsymbol{\gamma}^T\boldsymbol{X})}$, where $\boldsymbol{\gamma} = [0,0,0,0.5,0.4]^T$. Within each observed cell $(Z=z, S=s) \in \{0,1\}^{\otimes 2}$, the true failure outcome T is generated by $\mathcal{S}_{zs}(t|\boldsymbol{X}) = \exp(-te^{-1+0.5s+\boldsymbol{\psi}_{zs}^T\boldsymbol{X}})$, where $\boldsymbol{\psi}_{00} = (0,0,0.2,0.4,0.5)^T$, $\boldsymbol{\psi}_{01} = (0,0,0,0.4,0.2)^T$, $\boldsymbol{\psi}_{10} = (0,0,0,0.4,-0.3)^T$, and $\boldsymbol{\psi}_{11} = (0,0,0,-0.3,0.2)^T$. The censoring time is generated by $\mathcal{S}_{zs}^C(t|\boldsymbol{X}) = \exp(-te^{-2+0.3X_4+0.2X_5})$ for each observed cell $(Z=z, S=s) \in \{0,1\}^{\otimes 2}$.

We evaluate performance of the multiply robust estimators under 8 different scenarios depending on correct or incorrect specifications of $\mathcal{M}_\pi$, $\mathcal{M}_e$, $\mathcal{M}_C$, and $\mathcal{M}_T$. For correctly specified $\mathcal{M}_\pi$, $\mathcal{M}_e$, $\mathcal{M}_C$, or $\mathcal{M}_T$, we incorporate all covariates $\boldsymbol{X}$ into the working models $\pi(\boldsymbol{X})$, $e_g(\boldsymbol{X})$, $\mathcal{S}_{zs}^C(u|\boldsymbol{X})$, or $\mathcal{S}_{zs}(u|\boldsymbol{X})$, respectively. Under misspecification of working models, we only adjust for the first 3 covariates, $(X_1, X_2, X_3)^T$. The 8 scenarios are indicated in the first column of Web Table 2. In Scenarios 1–4, the union model $\mathcal{M}_{\pi+e+C} \cup \mathcal{M}_{\pi+T} \cup \mathcal{M}_{e+T}$ is correctly specified so that the multiply robust estimators are consistent. In Scenarios 5–8, subsets of working models are misspecified such that the multiply robust estimators are subject to bias according to Result 2.

Web Table 2 summarizes the simulation results of the multiply robust estimator for the counterfactual survival function among the compliers (evaluated at 5 time points) under assignment to $Z = 0$, $\widehat{\mathcal{S}}_{0,c}^{mr}(u)$. Simulation results for other counterfactural survival functions, including $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$, $\widehat{\mathcal{S}}_{0,n}^{mr}(u)$, $\widehat{\mathcal{S}}_{1,n}^{mr}(u)$, $\widehat{\mathcal{S}}_{0,a}^{mr}(u)$, $\widehat{\mathcal{S}}_{1,a}^{mr}(u)$, are presented in Web Tables 3–7 in the Supplementary Material. In the first four scenarios, the multiply robust estimator provides minimal bias and the associated interval estimator has close-to-nominal coverage rate, even if certain working models are misspecified. This empirically verifies the theoretical prediction in Result 2. In Scenarios 5–8, $\widehat{\mathcal{S}}_{0,c}^{mr}(u)$ exhibits some bias with attenuated coverage rate as an increasing number of working models are misspefied. Simulation results for PSCEs, i.e., the difference between $\widehat{\mathcal{S}}_{1,g}^{mr}(u)$ and $\widehat{\mathcal{S}}_{0,g}^{mr}(u)$ for $g \in \{a, n, c\}$, are similar and therefore omitted for brevity.

We have also studied the performance of the three singly robust estimators for estimating $\mathcal{S}_{0,c}(u)$, and results for $\widehat{\mathcal{S}}_{0,c}^{1}(u)$, $\widehat{\mathcal{S}}_{0,c}^{2}(u)$, and $\widehat{\mathcal{S}}_{0,c}^{3}(u)$ are summarized in Web Tables 8–10 in the Supplementary Material, respectively. Each singly robust estimator has minimal bias only under scenarios when all of working models used for constructing the estimator are correctly specified. For example, $\widehat{\mathcal{S}}_{0,c}^{1}(u)$ exhibits small bias in Scenarios 1 and 2 when $\mathcal{M}_{\pi+e+C}$ is correct, but can be substantially biased in all of the other 6 scenarios. Furthermore, when all working models are correctly specified (Scenario 1), the multiply robust estimator provides smaller Monte Carlo standard errors than $\widehat{\mathcal{S}}_{0,c}^{1}(u)$ but appears to be slightly less efficient than $\widehat{\mathcal{S}}_{0,c}^{2}(u)$ and $\widehat{\mathcal{S}}_{0,c}^{3}(u)$. However, the multiply robust estimator consistently outperforms the singly robust estimator in terms of the bias. Even in Scenarios 5–8 when two or more working models are misspecified, the multiply robust estimator does not appear to provide notably larger bias than the singly robust estimators.

To reflect the study design in ADAPTABLE, we further evaluate the performance of the proposed estimators under a randomized trial. The data generation process follows the observational study setting, expect that Z is now randomly simulated from a Bernoulli distribution with $\mathbb{P}(Z = 1) = 0.5$. The performances of $\widehat{\mathcal{S}}_{0,c}^{mr}(u)$, $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$, $\widehat{\mathcal{S}}_{0,n}^{mr}(u)$, $\widehat{\mathcal{S}}_{1,n}^{mr}(u)$, $\widehat{\mathcal{S}}_{0,a}^{mr}(u)$, $\widehat{\mathcal{S}}_{1,a}^{mr}(u)$, are presented in Web Tables 11–16, respectively. The pattern of simulation results is similar to that in the observational study setting, but now the multiply robust estimator is technically a doubly robust estimator due to randomization of treatment assignments. Specifically, the doubly robust estimator now exhibits minimal bias with nominal coverage rate in Scenarios 1–4 and 7 when the union model $\mathcal{M}_{e+C} \cup \mathcal{M}_{T}$ is correctly specified.

Web Appendix D Sensitivity analysis for principal ignorability

Web Appendix D.1 Nonparametric identification formulas based on the proposed sensitivity functions

The following proposition provides three nonparametric identification strategies for the PSCE with a fixed value of sensitivity functions $(\varepsilon_1(t, \boldsymbol{X}), \varepsilon_0(t, \boldsymbol{X}))$.

Proposition 1. *Suppose that Assumptions 1–2 and 4 hold, $0 < \pi(\boldsymbol{X}) < 1$, $e_g(\boldsymbol{X}) > 0$ for all $g \in \{a, c, n\}$, and $S^C_{zs}(u|\boldsymbol{X}) > 0$ for all $(z, s) \in \{0, 1\}^2$. The following formulas nonparametrically identify the PSCEs for a fixed value of $(\varepsilon_1(t, \boldsymbol{X}), \varepsilon_0(t, \boldsymbol{X}))$:*

(i) Using propensity score, principal score, and survival probability of censoring, we have

$$
\begin{aligned}
\Delta_c(u) = &\mathbb{E}\left[w_{1,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X})}\right] \\
&-\mathbb{E}\left[w_{0,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{1-S}{1-p_0(\boldsymbol{X})}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{00}(u|\boldsymbol{X})}\right], \\
\Delta_n(u) = &\mathbb{E}\left[\frac{1-S}{1-p_1}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{10}(u|\boldsymbol{X})}\right]-\mathbb{E}\left[w_{0,n}(t,\boldsymbol{X})\frac{e_n(\boldsymbol{X})}{1-p_1}\frac{1-S}{1-p_0(\boldsymbol{X})}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{00}(u|\boldsymbol{X})}\right], \\
\Delta_a(u) = &\mathbb{E}\left[w_{1,a}(t,\boldsymbol{X})\frac{e_a(\boldsymbol{X})}{p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X})}\right]-\mathbb{E}\left[\frac{S}{p_0}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{01}(u|\boldsymbol{X})}\right].
\end{aligned}
$$

(ii) Using prinicpal score and survival probability of outcome, we have

$$
\begin{aligned}
\Delta_c(u) &= \mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\{w_{1,c}(u,\boldsymbol{X})\mathcal{S}_{11}(u|\boldsymbol{X})-w_{0,c}(u,\boldsymbol{X})\mathcal{S}_{00}(u|\boldsymbol{X})\}\right], \\
\Delta_n(u) &= \mathbb{E}\left[\frac{e_n(\boldsymbol{X})}{1-p_1}\{\mathcal{S}_{10}(u|\boldsymbol{X})-w_{0,n}(u,\boldsymbol{X})\mathcal{S}_{00}(u|\boldsymbol{X})\}\right], \\
\Delta_a(u) &= \mathbb{E}\left[\frac{e_a(\boldsymbol{X})}{p_0}\{w_{1,a}(u,\boldsymbol{X})\mathcal{S}_{11}(u|\boldsymbol{X})-\mathcal{S}_{01}(u|\boldsymbol{X})\}\right].
\end{aligned}
$$

(iii) Using propensity score and survival probability of outcome, we have

$$
\begin{aligned}
\Delta_c(u) &= \frac{1}{p_1-p_0}\mathbb{E}\left[\left(\frac{SZ}{\pi(\boldsymbol{X})}-\frac{S(1-Z)}{1-\pi(\boldsymbol{X})}\right)\{w_{1,c}(u,\boldsymbol{X})\mathcal{S}_{11}(u|\boldsymbol{X})-w_{0,c}(u,\boldsymbol{X})\mathcal{S}_{00}(u|\boldsymbol{X})\}\right], \\
\Delta_n(u) &= \frac{1}{1-p_1}\mathbb{E}\left[\left(1-\frac{SZ}{\pi(\boldsymbol{X})}\right)\{\mathcal{S}_{10}(u|\boldsymbol{X})-w_{0,n}(u,\boldsymbol{X})\mathcal{S}_{00}(u|\boldsymbol{X})\}\right], \\
\Delta_a(u) &= \frac{1}{p_0}\mathbb{E}\left[\frac{S(1-Z)}{1-\pi(\boldsymbol{X})}\{w_{1,a}(u,\boldsymbol{X})\mathcal{S}_{11}(u|\boldsymbol{X})-\mathcal{S}_{01}(u|\boldsymbol{X})\}\right].
\end{aligned}
$$

Proof. We only prove the expression of $\Delta_c(u)$, where expressions of $\Delta_a(u)$ and $\Delta_n(u)$ can be obtained in a similar way. First, we can show that

$$
\begin{aligned}
&\mathbb{E}[\mathbb{I}(T(1)\geq u)|G=c \text{ or } a,\boldsymbol{X}] \\
=&\frac{e_c(\boldsymbol{X})\mathbb{E}[\mathbb{I}(T(1)\geq u)|G=c,\boldsymbol{X}]+e_a(\boldsymbol{X})\mathbb{E}[\mathbb{I}(T(1)\geq u)|G=a,\boldsymbol{X}]}{e_c(\boldsymbol{X})+e_a(\boldsymbol{X})} \\
=&\frac{e_c(\boldsymbol{X})\mathbb{E}[\mathbb{I}(T(1)\geq u)|G=c,\boldsymbol{X}]+e_a(\boldsymbol{X})\mathbb{E}[\mathbb{I}(T(1)\geq u)|G=c,\boldsymbol{X}]/\varepsilon_1(u,\boldsymbol{X})}{e_c(\boldsymbol{X})+e_a(\boldsymbol{X})} \\
=&\frac{1}{w_{1,c}(u,\boldsymbol{X})}\mathbb{E}[\mathbb{I}(T(1)\geq u)|G=c,\boldsymbol{X}]. \qquad \text{(D.1)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}[\mathbb{I}(T(0)\geq u)|G=c \text{ or } n,\boldsymbol{X}] \\
=&\frac{e_c(\boldsymbol{X})\mathbb{E}[\mathbb{I}(T(0)\geq u)|G=c,\boldsymbol{X}]+e_n(\boldsymbol{X})\mathbb{E}[\mathbb{I}(T(0)\geq u)|G=n,\boldsymbol{X}]}{e_c(\boldsymbol{X})+e_n(\boldsymbol{X})}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{e_c(\boldsymbol{X})\mathbb{E}[\mathbb{I}(T(0) \geq u)|G = c, \boldsymbol{X}] + e_n(\boldsymbol{X})\mathbb{E}[\mathbb{I}(T(0) \geq u)|G = c, \boldsymbol{X}]/\varepsilon_0(u, \boldsymbol{X})}{e_c(\boldsymbol{X}) + e_n(\boldsymbol{X})} \\
&= \frac{1}{w_{0,c}(u, \boldsymbol{X})}\mathbb{E}[\mathbb{I}(T(1) \geq u)|G = c, \boldsymbol{X}]. \qquad \text{(D.2)}
\end{aligned}
$$

Then, by the definition of $\Delta_c(u)$, we have that

$$
\begin{aligned}
\Delta_c(u) =& \mathbb{E}[\mathbb{I}(T(1) \geq u) - \mathbb{I}(T(0) \geq u)|G = c] \qquad \text{(D.3)} \\
=& \mathbb{E}[\mathbb{E}[\mathbb{I}(T(1) \geq u)|G = c, \boldsymbol{X}]|G = c] - \mathbb{E}[\mathbb{E}[\mathbb{I}(T(0) \geq u)|G = c, \boldsymbol{X}]|G = c] \\
=& \mathbb{E}[w_{1,c}(u, \boldsymbol{X})\mathbb{E}[\mathbb{I}(T(1) \geq u)|G = c \text{ or } a, \boldsymbol{X}]|G = c] \\
& - \mathbb{E}[w_{0,c}(u, \boldsymbol{X})\mathbb{E}[\mathbb{I}(T(0) \geq u)|G = c \text{ or } n, \boldsymbol{X}]|G = c] \quad \text{(by (D.1) and (D.2))} \\
=& \frac{1}{p_1 - p_0}\mathbb{E}\left[\left\{\frac{SZ}{\pi(\boldsymbol{X})} - \frac{S(1-Z)}{1-\pi(\boldsymbol{X})}\right\}\{w_{1,c}(u, \boldsymbol{X})\mathcal{S}_{11}(u|\boldsymbol{X}) - w_{0,c}(u, \boldsymbol{X})\mathcal{S}_{00}(u|\boldsymbol{X})\}\right] \qquad \text{(D.4)} \\
& \quad \text{(by Lemma 1)} \\
=& \mathbb{E}\left[\frac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{p_1 - p_0}\{w_{1,c}(u, \boldsymbol{X})\mathcal{S}_{11}(u|\boldsymbol{X}) - w_{0,c}(u, \boldsymbol{X})\mathcal{S}_{00}(u|\boldsymbol{X})\}\right]. \qquad \text{(D.5)} \\
& \quad \text{(by Lemma 1)}
\end{aligned}
$$

Notice that (D.4) and (D.5) are the expressions of $\Delta_c(u)$ in Proposition 1(ii) and (iii), respectively. The expression of $\Delta_c(u)$ in Proposition 1(i) can be obtained by

$$
\begin{aligned}
\Delta_c(u) =& \text{(D.5)} \\
=& \mathbb{E}\left[w_{1,c}(u, \boldsymbol{X})\frac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{p_1 - p_0}\mathcal{S}_{11}(u|\boldsymbol{X})\right] - \mathbb{E}\left[w_{1,c}(u, \boldsymbol{X})\frac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{p_1 - p_0}\mathcal{S}_{00}(u|\boldsymbol{X})\right] \\
=& \mathbb{E}\left[w_{1,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\mathcal{S}_{11}(u|\boldsymbol{X})\mathbb{E}\left[\frac{Z}{\pi(\boldsymbol{X})}|\boldsymbol{X}\right]\right] - \mathbb{E}\left[w_{0,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\mathcal{S}_{00}(u|\boldsymbol{X})\mathbb{E}\left[\frac{1-Z}{1-\pi(\boldsymbol{X})}|\boldsymbol{X}\right]\right] \\
=& \mathbb{E}\left[w_{1,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X})\right] - \mathbb{E}\left[w_{0,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{1-Z}{1-\pi(\boldsymbol{X})}\mathcal{S}_{00}(u|\boldsymbol{X})\right] \\
=& \mathbb{E}\left[w_{1,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X})\mathbb{E}\left[\frac{S}{p_1(\boldsymbol{X})}|Z = 1, \boldsymbol{X}\right]\right] \\
& - \mathbb{E}\left[w_{0,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{1-Z}{1-\pi(\boldsymbol{X})}\mathcal{S}_{00}(u|\boldsymbol{X})\mathbb{E}\left[\frac{1-S}{1-p_0(\boldsymbol{X})}|Z = 0, \boldsymbol{X}\right]\right] \\
=& \mathbb{E}\left[w_{1,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{Z}{\pi(\boldsymbol{X})}\frac{S}{p_1(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X})\right] - \mathbb{E}\left[w_{0,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{1-S}{1-p_0(\boldsymbol{X})}\mathcal{S}_{00}(u|\boldsymbol{X})\right] \\
=& \mathbb{E}\left[w_{1,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{Z}{\pi(\boldsymbol{X})}\frac{S}{p_1(\boldsymbol{X})}\frac{1}{S^C_{11}(u|\boldsymbol{X})}\mathbb{E}[\mathbb{I}(T \geq u)\mathbb{I}(C \geq u)|Z = 1, S = 1, \boldsymbol{X}]\right] \\
& - \mathbb{E}\left[w_{0,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{1-S}{1-p_0(\boldsymbol{X})}\frac{1}{S^C_{00}(u|\boldsymbol{X})}\mathbb{E}[\mathbb{I}(T \geq u)\mathbb{I}(C \geq u)|Z = 0, S = 0, \boldsymbol{X}]\right] \\
& \quad \text{(by Assumption 4)} \\
=& \mathbb{E}\left[w_{1,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{S^C_{11}(u|\boldsymbol{X})}\right] - \mathbb{E}\left[w_{1,c}(u, \boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1 - p_0}\frac{1-S}{1-p_0(\boldsymbol{X})}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{S^C_{00}(u|\boldsymbol{X})}\right].
\end{aligned}
$$

This completes the proof. $\square$

Web Appendix D.2 Explicit forms of the multiply robust estimator without principal ignorability

$$
\begin{aligned}
\widehat{\mathcal{S}}_{0,c}^{mr}(u) =& \mathbb{P}_n\Bigg\{\widehat{w}_{0,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1-\widehat{p}_0}\frac{1-S}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}_{00}^C(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}+\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\int_0^u\frac{d\widehat{M}_{00}^C(r|\boldsymbol{X})}{\mathcal{S}_{00}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\mathcal{S}_{00}^C(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}\right]\\
&+\frac{\widehat{w}_{0,c}^2(u,\boldsymbol{X})}{\varepsilon_0(u,\boldsymbol{X})}\frac{\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})}{\widehat{p}_1-\widehat{p}_0}\left[\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1))+\frac{1-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)-S)\right]\\
&+\widehat{w}_{0,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1-\widehat{p}_0}\left(1-\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\Bigg\}\\
\widehat{\mathcal{S}}_{1,c}^{mr}(u) =& \mathbb{P}_n\Bigg\{\widehat{w}_{1,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1-\widehat{p}_0}\frac{S}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}_{11}^C(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}+\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\int_0^u\frac{d\widehat{M}_{11}^C(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\mathcal{S}_{11}^C(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}\right]\\
&+\frac{\widehat{w}_{1,c}^2(u,\boldsymbol{X})}{\varepsilon_1(u,\boldsymbol{X})}\frac{\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})}{\widehat{p}_1-\widehat{p}_0}\left[\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)-S)+\frac{p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}))\right]\\
&+\widehat{w}_{1,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_1-\widehat{p}_0}\left(1-\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\Bigg\},\\
\widehat{\mathcal{S}}_{0,n}^{mr}(u) =& \mathbb{P}_n\Bigg\{\widehat{w}_{0,n}(u,\boldsymbol{X})\frac{e_n(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{1-\widehat{p}_1}\frac{1-S}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}_{00}^C(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}+\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\int_0^u\frac{d\widehat{M}_{00}^C(r|\boldsymbol{X})}{\mathcal{S}_{00}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\mathcal{S}_{00}^C(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}\right]\\
&+\frac{\widehat{w}_{0,n}^2(u,\boldsymbol{X})}{\varepsilon_0(u,\boldsymbol{X})}\frac{\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})}{1-\widehat{p}_1}\left[\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)-S)+\frac{1-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0))\right]\\
&+\widehat{w}_{0,n}(u,\boldsymbol{X})\frac{e_n(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{1-\widehat{p}_1}\left(1-\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\Bigg\}.\\
\widehat{\mathcal{S}}_{1,a}^{mr}(u) =& \mathbb{P}_n\Bigg\{\widehat{w}_{1,a}(u,\boldsymbol{X})\frac{e_a(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_0}\frac{S}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}_{11}^C(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}+\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\int_0^u\frac{d\widehat{M}_{11}^C(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\mathcal{S}_{11}^C(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}\right]\\
&+\frac{\widehat{w}_{1,a}^2(u,\boldsymbol{X})}{\varepsilon_1(u,\boldsymbol{X})}\frac{\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})}{\widehat{p}_0}\left[\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0))+\frac{p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})-S)\right]\\
&+\widehat{w}_{1,a}(u,\boldsymbol{X})\frac{e_a(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{p}_0}\left(1-\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\Bigg\}.
\end{aligned}
$$

The expressions of $\widehat{\mathcal{S}}_{1,n}^{mr}(u)$ and $\widehat{\mathcal{S}}_{0,a}^{mr}(u)$ are identical to them in Web Appendix B.2.

Web Appendix D.3 Proof of Result 3

We will follow the notation in Web Appendix B.3 to prove that proposed multiply robust estimator in equation (10), $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$, is consistent for $\mathcal{S}_{1,c}$ under $\mathcal{M}_{\pi+e+C}\cup\mathcal{M}_{e+T}$. Consistency of the multiply robust estimators for other quantities in Web Appendix C.2, including $\widehat{\mathcal{S}}_{0,c}^{mr}(u)$, $\widehat{\mathcal{S}}_{1,a}^{mr}(u)$, $\widehat{\mathcal{S}}_{0,a}^{mr}(u)$, $\widehat{\mathcal{S}}_{1,n}^{mr}(u)$, and $\widehat{\mathcal{S}}_{0,n}^{mr}(u)$, can be proved similarly and therefore are omitted here.

In what follows, we shall prove that $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$ provided in equation (10) converges in probability to $\mathcal{S}_{1,c}(u)$ when either $\mathcal{M}_{\pi+e+C}$ or $\mathcal{M}_{e+T}$ is correctly specified.

(i) When $\mathcal{M}_{\pi+e+C}$ is correctly specified

Under $\mathcal{M}_{\pi+e+C}$, $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$ converges in probability to

$$
\mathbb{E}\left[w_{1,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{1}{\mathcal{S}_{11}^C(u|\boldsymbol{X})}\mathbb{I}(U\geq u)\right] \tag{D.6}
$$

$$
+\mathbb{E}\left[w_{1,c}(u,\boldsymbol{X})\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}_{11}^*)\frac{e_c(\boldsymbol{X})}{p_1-p_0}\left(1-\frac{Z}{\pi(\boldsymbol{X})}\right)\right] \tag{D.7}
$$

$$+\frac{1}{p_1-p_0}\mathbb{E}\left\{\frac{w_{1,c}^2(u,\boldsymbol{X})}{\varepsilon_1(u,\boldsymbol{X})}\left[\frac{1-Z}{1-\pi(\boldsymbol{X})}(p_0(\boldsymbol{X})-S)+\frac{p_0(\boldsymbol{X})}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}(S-p_1(\boldsymbol{X}))\right]\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}_{11}^*)\right\} \tag{D.8}$$

$$+\mathbb{E}\left[w_{1,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}_{11}^*)\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}_{11}^*)\mathcal{S}_{11}^C(r|\boldsymbol{X})}\right]. \tag{D.9}$$

Proposition 1(i) suggests that $(D.6)=\mathcal{S}_{1,c}(u)$. Using the law of iterated expectation to the other three terms, we have

$$\begin{aligned}
(\text{D.7}) =&\mathbb{E}\left\{w_{1,c}(u,\boldsymbol{X})\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}_{11}^*)\frac{e_c(\boldsymbol{X})}{p_1-p_0}\mathbb{E}\left[1-\frac{Z}{\pi(\boldsymbol{X})}\middle|\boldsymbol{X}\right]\right\}=0,\\
(\text{D.8}) =&\mathbb{E}\left\{\frac{w_{1,c}^2(u,\boldsymbol{X})}{\varepsilon_1(u,\boldsymbol{X})}\left[\frac{1-Z}{1-\pi(\boldsymbol{X})}\mathbb{E}[p_0(\boldsymbol{X})-S|Z=0,\boldsymbol{X}]+\frac{p_0(\boldsymbol{X})}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathbb{E}[S-p_1(\boldsymbol{X})|Z=1,\boldsymbol{X}]\right]\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}_{11}^*)\right\}=0,\\
(\text{D.9}) =&\mathbb{E}\left\{w_{1,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_0(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}_{11}^*)\mathbb{E}\left[\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}_{11}^*)\mathcal{S}_{11}^C(r|\boldsymbol{X})}\middle|Z=1,S=1,\boldsymbol{X}\right]\right\}=0,
\end{aligned}$$

where the last equality in the above set of formulas holds because, conditional on $Z=1,S=1,\boldsymbol{X}$, $\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}_{11}^*)\mathcal{S}_{11}^C(r|\boldsymbol{X})}$ is a martingale and has expectation 0 such that $\mathbb{E}\left[\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}_{11}^*)\mathcal{S}_{11}^C(r|\boldsymbol{X})}\middle|Z=1,S=1,\boldsymbol{X}\right]=0$. Therefore, we have that $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$ converges to (D.6) + (D.7) + (D.8) + (D.9) = $\mathcal{S}_{1,c}(u)$.

(ii) When $\mathcal{M}_{e+T}$ is correctly specified

Under $\mathcal{M}_{e+T}$, $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$ converges in probability to

$$\mathbb{E}\left[w_{1,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}_{11}^C(u|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right] \tag{D.10}$$

$$+\mathbb{E}\left[w_{1,c}(u,\boldsymbol{X})\mathcal{S}_{11}(u|\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\left(1-\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\right)\right] \tag{D.11}$$

$$+\frac{1}{p_1-p_0}\mathbb{E}\left\{\frac{w_{1,c}^2(u,\boldsymbol{X})}{\varepsilon_1(u,\boldsymbol{X})}\left[\frac{1-Z}{1-\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}(p_0(\boldsymbol{X})-S)+\frac{p_0(\boldsymbol{X})}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}(S-p_1(\boldsymbol{X}))\right]\mathcal{S}_{11}(u|\boldsymbol{X})\right\} \tag{D.12}$$

$$+\mathbb{E}\left[w_{1,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathcal{S}_{11}(u|\boldsymbol{X})\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}(r|\boldsymbol{X})\mathcal{S}_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right]. \tag{D.13}$$

By Lemma 2, and similar to the proof of $(B.20)$ in Web Appendix B.3, we have that

$$\begin{aligned}
(\text{D.10}) =&\mathbb{E}\left[w_{1,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\frac{\mathbb{I}(C\geq u\text{ or }C\geq T)\mathbb{I}(T\geq u)}{\mathcal{S}_{11}^C(u|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right]\\
=&\mathbb{E}\left[w_{1,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathcal{S}_{11}(u|X)\right] &\text{(D.14)}\\
&-\mathbb{E}\left[w_{1,c}(u,\boldsymbol{X})\frac{e_c(\boldsymbol{X})}{p_1-p_0}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathbb{I}(T\geq u)\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right], &\text{(D.15)}
\end{aligned}$$

Also, Proposition 1 suggests that

$$(\text{D.11})+(\text{D.14})=\mathbb{E}\left[\frac{e_c(\boldsymbol{X})}{p_1-p_0}\mathcal{S}_{11}(u|\boldsymbol{X})\right]=\mathcal{S}_{1,c}(u).$$

Similar to the proof of (D.8) = 0, we can show (D.12) = 0. Also, similar to the proof of (B.19) + (B.21) = 0 in Web Appendix B.3, we can obtain (D.13) + (D.15) = 0. Therefore, $\widehat{\mathcal{S}}^{mr}_{1,c}(u)$ converges to (D.10) + (D.11) + (D.12) + (D.13) $= \mathcal{S}_{1,c}(u)$.

Web Appendix E Sensitivity analysis for monotonicity

Web Appendix E.1 Nonparametric identification formulas based on the proposed sensitivity function

If monotonicity assumption does not hold, defiers will exist and therefore we shall have four principal strata $G = \{c, n, a, d\}$. As we demonstrate in the paper, we can use the following sensitivity function to quantify the extent of deviation from monotonicity:

$$\zeta(\boldsymbol{X}) = \frac{\mathbb{P}(G = d|\boldsymbol{X})}{\mathbb{P}(G = c|\boldsymbol{X})},$$

which is the ratio between the probabilities of defiers and compliers conditional on $\boldsymbol{X}$. To conduct sensitivity analysis, one can consider a parametric forms for $\zeta(\boldsymbol{X})$ based on certain sensitivity parameters $\boldsymbol{\tau}$, and then report the bias-corrected $\Delta_g(u)$-estimator proposed in Appendix E.2 over a range of $\boldsymbol{\tau}$, which summarizes how sensitive the results to departure from monotonicity. In our application study, we consider a constant sensitivity function $\zeta(\boldsymbol{X})$ such that $\zeta(\boldsymbol{X}) = \zeta$ does not depend on the baseline covariates $\boldsymbol{X}$. Otherwise, one can also consider $\zeta(\boldsymbol{X}) = \exp(\boldsymbol{\tau}^T\boldsymbol{X})$ to allow the sensitivity function depending on the baseline covariates. The following proposition presents the propensity scores without the monotonicity assumption:

Proposition 2. *(propensity scores without monotonicity) For a fixed value of $\zeta(\boldsymbol{X})$, we can identify the propensity score for each principal stratum as*

$$e_{\zeta,c}(\boldsymbol{X}) = \frac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{1 - \zeta(\boldsymbol{X})}, \quad e_{\zeta,n}(\boldsymbol{X}) = 1 - p_0(\boldsymbol{X}) - \frac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{1 - \zeta(\boldsymbol{X})},$$
$$e_{\zeta,a}(\boldsymbol{X}) = p_1(\boldsymbol{X}) - \frac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{1 - \zeta(\boldsymbol{X})}, \quad e_{\zeta,d}(\boldsymbol{X}) = \frac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{1 - \zeta(\boldsymbol{X})} - (p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})),$$

where $e_{\zeta,g}(\boldsymbol{X}) = \mathbb{P}(G = g|\boldsymbol{X})$. If we assume that Z has a positive effect on S such that $p_1(\boldsymbol{X}) - p_0(\boldsymbol{X}) > 0$, the sensitivity function $\zeta(\boldsymbol{X})$ is bounded by

$$0 \le \zeta(\boldsymbol{X}) \le 1 - \frac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{min(p_1(\boldsymbol{X}), 1 - p_0(\boldsymbol{X}))} \le 1$$

to ensure that probabilities of each principal stratum are nonnegative (i.e., $e_{\zeta,g}(\boldsymbol{X}) \ge 0$).

Proof. The proof is analogous to the proof of Proposition 5 in Ding and Lu (2017). Specifically, the observed data and the definition of $\zeta(\boldsymbol{X})$ suggests the following set of equations:

$$\begin{cases} e_{\zeta,c}(\boldsymbol{X}) + e_{\zeta,a}(\boldsymbol{X}) = p_1(\boldsymbol{X}), \\ e_{\zeta,d}(\boldsymbol{X}) + e_{\zeta,a}(\boldsymbol{X}) = p_0(\boldsymbol{X}), \\ e_{\zeta,c}(\boldsymbol{X}) + e_{\zeta,a}(\boldsymbol{X}) + e_{\zeta,d}(\boldsymbol{X}) + e_{\zeta,n}(\boldsymbol{X}) = 1, \\ e_{\zeta,d}(\boldsymbol{X}) - \zeta(\boldsymbol{X})e_{\zeta,c}(\boldsymbol{X}) = 0, \end{cases}$$

Solving the above set of equations concludes

$$\begin{cases} e_{\zeta,c}(\boldsymbol{X}) = \dfrac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}, \\ e_{\zeta,n}(\boldsymbol{X}) = 1 - p_0(\boldsymbol{X}) - \dfrac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}, \\ e_{\zeta,a}(\boldsymbol{X}) = p_1(\boldsymbol{X}) - \dfrac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}, \\ e_{\zeta,d}(\boldsymbol{X}) = \dfrac{p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})} - \left(p_1(\boldsymbol{X}) - p_0(\boldsymbol{X})\right). \end{cases}$$

This completes the proof. □

Proposition 2 suggests that there is a upper bound for $\zeta(\boldsymbol{X})$ to ensure the probability of each principal stratum is nonnegative, if we assume that the treatment assignment has a positive effect on the treatment receipt status. This is very plausible in health science studies, and can be empirically verified by the observed data by comparing $\widehat{p}_1(\boldsymbol{X}) - \widehat{p}_0(\boldsymbol{X})$ and 0. The upper bound given by Proposition 2 provides a basis for us to choose the value of $\zeta(\boldsymbol{X})$. For example, if we assume $\zeta(\boldsymbol{X}) = \zeta$ does not depend on $\boldsymbol{X}$, then we can choose a sequence of ζ from

$$\zeta \in \left[0, \min_i \left\{1 - \frac{p_1(\boldsymbol{X}_i) - p_0(\boldsymbol{X}_i)}{\min(p_1(\boldsymbol{X}_i), 1 - p_0(\boldsymbol{X}_i))}\right\}\right] \tag{E.1}$$

where $\min_i\{V_i\}$ is the minimum of V_i for all $i \in \{1, \ldots, n\}$. However, the above upper bound can be very rigid because it requires that the probability of each principal stratum is non-negative conditional on every possible $\boldsymbol{X}$. In practice, this upper bound can be very close to 0 when $\boldsymbol{X}$ contains continuous variables or many categorical variables. Therefore, we suggest using the following marginalized bound instead of (E.1):

$$\zeta \in \left[0, 1 - \frac{p_1 - p_0}{\min(p_1, 1 - p_0)}\right],$$

which ensures that the marginal probability of each principal stratum is non-negative.

If monotonicity does not hold, we need the following generalized principal ignorability assumption instead of the principal ignorability assumption (Assumption 3):

Assumption 5. *(Generalized prinicipal ignorability) For all $u \geq 0$, $z \in \{0, 1\}$, and $g, g' \in \{c, n, a, d\}$, we have that $\mathbb{P}(T(z) \geq u | G = g, \boldsymbol{X}) = \mathbb{P}(T(z) \geq u | G = g', \boldsymbol{X})$.*

Assumption 5 is slightly stronger than the original principal ignorability assumption (Assumption 3). Specifically, Assumption 5 requires that the counterfactual survival probability conditional on $\boldsymbol{X}$ is same across all principal strata, whereas Assumption 3 only requires that the controlled counterfactual survival probability is same between the compliers and always low-dose takers and the treated counterfactural survival is same between the compliers and always high-dose takers, conditional on $\boldsymbol{X}$. The following proposition provides three nonparametric identification strategies for the PSCE with a fixed value of $\zeta(\boldsymbol{X})$.

Proposition 3. *Suppose that Assumptions 1, 4, and 5 hold, $0 < \pi(\boldsymbol{X}) < 1$, $e_{\zeta,g}(\boldsymbol{X}) > 0$ for all $g \in \{a, c, n, d\}$, and $\mathcal{S}^C_{zs}(u|\boldsymbol{X}) > 0$ for all $(z, s) \in \{0, 1\}^2$. Define $e_{\zeta,c} = \mathbb{E}[e_{\zeta,c}(\boldsymbol{X})]$ as the* *marginalized version of $e_{\zeta,c}$. The following formulas nonparametrically identify the PSCEs for a fixed value of $\zeta(\boldsymbol{X})$:*

(i) Using propensity score, principal score, and survival probability of censoring, we have

$$\begin{aligned}
\Delta_c(u) &= \mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X})}\right] - \mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{1-S}{1-\pi_0(\boldsymbol{X})}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{00}(u|\boldsymbol{X})}\right],\\
\Delta_n(u) &= \mathbb{E}\left[\frac{e_{\zeta,n}(\boldsymbol{X})}{e_{\zeta,n}}\frac{1-S}{1-p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{10}(u|\boldsymbol{X})}\right] - \mathbb{E}\left[\frac{e_{\zeta,n}(\boldsymbol{X})}{e_{\zeta,n}}\frac{1-S}{1-p_0(\boldsymbol{X})}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{00}(u|\boldsymbol{X})}\right],\\
\Delta_a(u) &= \mathbb{E}\left[\frac{e_{\zeta,a}(\boldsymbol{X})}{e_{\zeta,a}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X})}\right] - \mathbb{E}\left[\frac{e_{\zeta,a}(\boldsymbol{X})}{e_{\zeta,a}}\frac{S}{p_0(\boldsymbol{X})}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{01}(u|\boldsymbol{X})}\right],\\
\Delta_d(u) &= \mathbb{E}\left[\frac{e_{\zeta,d}(\boldsymbol{X})}{e_{\zeta,d}}\frac{1-S}{1-p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{10}(u|\boldsymbol{X})}\right] - \mathbb{E}\left[\frac{e_{\zeta,d}(\boldsymbol{X})}{e_{\zeta,d}}\frac{S}{p_0(\boldsymbol{X})}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{\mathbb{I}(U \geq u)}{\mathcal{S}^C_{01}(u|\boldsymbol{X})}\right].
\end{aligned}$$

(ii) Using prinicpal score and survival probability of outcome, we have

$$\begin{aligned}
\Delta_c(u) &= \mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\{\mathcal{S}_{11}(u|\boldsymbol{X}) - \mathcal{S}_{00}(u|\boldsymbol{X})\}\right],\\
\Delta_n(u) &= \mathbb{E}\left[\frac{e_{\zeta,n}(\boldsymbol{X})}{e_{\zeta,n}}\{\mathcal{S}_{10}(u|\boldsymbol{X}) - \mathcal{S}_{00}(u|\boldsymbol{X})\}\right],\\
\Delta_a(u) &= \mathbb{E}\left[\frac{e_{\zeta,a}(\boldsymbol{X})}{e_{\zeta,a}}\{\mathcal{S}_{11}(u|\boldsymbol{X}) - \mathcal{S}_{01}(u|\boldsymbol{X})\}\right],\\
\Delta_d(u) &= \mathbb{E}\left[\frac{e_{\zeta,d}(\boldsymbol{X})}{e_{\zeta,d}}\{\mathcal{S}_{10}(u|\boldsymbol{X}) - \mathcal{S}_{01}(u|\boldsymbol{X})\}\right].
\end{aligned}$$

(iii) Using propensity score and survival probability of outcome, we have

$$\begin{aligned}
\Delta_c(u) &= \frac{1}{e_{\zeta,c}}\mathbb{E}\left[\left(\frac{SZ}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})} - \frac{S(1-Z)}{(1-\zeta(\boldsymbol{X}))(1-\pi(\boldsymbol{X}))}\right)\{\mathcal{S}_{11}(u|\boldsymbol{X}) - \mathcal{S}_{00}(u|\boldsymbol{X})\}\right],\\
\Delta_n(u) &= \frac{1}{e_{\zeta,n}}\mathbb{E}\left[\left(1 - \frac{SZ}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})} + \frac{\zeta(\boldsymbol{X})S(1-Z)}{(1-\zeta(\boldsymbol{X}))(1-\pi(\boldsymbol{X}))}\right)\{\mathcal{S}_{10}(u|\boldsymbol{X}) - \mathcal{S}_{00}(u|\boldsymbol{X})\}\right],\\
\Delta_a(u) &= \frac{1}{e_{\zeta,a}}\mathbb{E}\left[\left(\frac{S(1-Z)}{(1-\zeta(\boldsymbol{X}))(1-\pi(\boldsymbol{X}))} - \frac{\zeta(\boldsymbol{X})SZ}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})}\right)\{\mathcal{S}_{11}(u|\boldsymbol{X}) - \mathcal{S}_{01}(u|\boldsymbol{X})\}\right],\\
\Delta_d(u) &= \frac{1}{e_{\zeta,d}}\mathbb{E}\left[\left(\frac{\zeta(\boldsymbol{X})SZ}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})} - \frac{\zeta(\boldsymbol{X})S(1-Z)}{(1-\zeta(\boldsymbol{X}))(1-\pi(\boldsymbol{X}))}\right)\{\mathcal{S}_{10}(u|\boldsymbol{X}) - \mathcal{S}_{01}(u|\boldsymbol{X})\}\right].
\end{aligned}$$

Proof. We only prove the expression of $\Delta_c(u)$, where expressions of $\Delta_a(u)$, $\Delta_n(u)$, and $\Delta_d(u)$ can be obtained similarly. Specifically, by the definition of $\Delta_c(u)$, we have that

$$\begin{aligned}
\Delta_c(u) =& \mathbb{E}\left[\mathbb{I}(T(1) \geq u) - \mathbb{I}(T(0) \geq u)|G = c\right]\\
=& \mathbb{E}[\mathbb{E}[\mathbb{I}(T(1) \geq u) - \mathbb{I}(T(0) \geq u)|G = c, \boldsymbol{X}]|G = c]\\
=& \mathbb{E}[\mathbb{E}[\mathbb{I}(T(1) \geq u)|G = c \text{ or } a, \boldsymbol{X}]|G = c] - \mathbb{E}[\mathbb{E}[\mathbb{I}(T(0) \geq u)|G = c \text{ or } n, \boldsymbol{X}]|G = c]\\
&\quad \text{(by Assumption 5)}\\
=& \mathbb{E}[\mathbb{E}[\mathbb{I}(T \geq u)|Z = 1, G = c \text{ or } a, \boldsymbol{X}]|G = c] - \mathbb{E}[\mathbb{E}[\mathbb{I}(T \geq u)|Z = 0, G = c \text{ or } n, \boldsymbol{X}]|G = c]\\
&\quad \text{(by Assumption 1)}\\
=& \mathbb{E}[\mathbb{E}[\mathbb{I}(T \geq u)|Z = 1, S = 1, \boldsymbol{X}]|G = c] - \mathbb{E}[\mathbb{E}[\mathbb{I}(T \geq u)|Z = 0, S = 0, \boldsymbol{X}]|G = c]
\end{aligned}$$

$$
\begin{aligned}
=&\mathbb{E}[\mathcal{S}_{11}(u|\boldsymbol{X})|G=c]-\mathbb{E}[\mathcal{S}_{00}(u|\boldsymbol{X})|G=c]\\
=&\mathbb{E}[\mathcal{S}_{11}(u|\boldsymbol{X})-\mathcal{S}_{00}(u|\boldsymbol{X})|G=c]\\
=&\mathbb{E}\left[\frac{\mathbb{P}(G=c|\boldsymbol{X})}{\mathbb{P}(G=c)}\left\{\mathcal{S}_{11}(u|\boldsymbol{X})-\mathcal{S}_{00}(u|\boldsymbol{X})\right\}\right]\\
=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\left\{\mathcal{S}_{11}(u|\boldsymbol{X})-\mathcal{S}_{00}(u|\boldsymbol{X})\right\}\right].
\end{aligned}
$$

This proves the expression of $\Delta_c(u)$ in Proposition 3(ii). In addition, one can show that

$$
\begin{aligned}
\mathbb{E}\left[\frac{SZ}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})}\Big|\boldsymbol{X}\right]&=\mathbb{E}\left[\frac{Z}{\pi(\boldsymbol{X})}\mathbb{E}\left[\frac{S}{(1-\zeta(\boldsymbol{X}))}\Big|Z,\boldsymbol{X}\right]\Big|\boldsymbol{X}\right]\\
&=\mathbb{E}\left[\frac{Z}{\pi(\boldsymbol{X})}\mathbb{E}\left[\frac{S}{1-\zeta(\boldsymbol{X})}\Big|Z=1,\boldsymbol{X}\right]\Big|\boldsymbol{X}\right]\\
&=\frac{p_1(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}\mathbb{E}\left[\frac{Z}{\pi(\boldsymbol{X})}\Big|\boldsymbol{X}\right]\\
&=\frac{p_1(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}
\end{aligned}
$$

and analogously

$$
\mathbb{E}\left[\frac{S(1-Z)}{(1-\zeta(\boldsymbol{X}))(1-\pi(\boldsymbol{X}))}\Big|\boldsymbol{X}\right]=\frac{p_0(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})},
$$

which implies that the expression of $\Delta_c(u)$ in Proposition 3(iii) equals to

$$
\begin{aligned}
\Delta_c(u)&=\frac{1}{e_{\zeta,c}}\mathbb{E}\left[\left(\frac{SZ}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})}-\frac{S(1-Z)}{(1-\zeta(\boldsymbol{X}))(1-\pi(\boldsymbol{X}))}\right)\left\{\mathcal{S}_{11}(u|\boldsymbol{X})-\mathcal{S}_{00}(u|\boldsymbol{X})\right\}\right]\\
&=\frac{1}{e_{\zeta,c}}\mathbb{E}\left[\mathbb{E}\left[\frac{SZ}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})}-\frac{S(1-Z)}{(1-\zeta(\boldsymbol{X}))(1-\pi(\boldsymbol{X}))}\Big|\boldsymbol{X}\right]\left\{\mathcal{S}_{11}(u|\boldsymbol{X})-\mathcal{S}_{00}(u|\boldsymbol{X})\right\}\right]\\
&=\frac{1}{e_{\zeta,c}}\mathbb{E}\left[\frac{p_1(\boldsymbol{X})-p_0(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}\left\{\mathcal{S}_{11}(u|\boldsymbol{X})-\mathcal{S}_{00}(u|\boldsymbol{X})\right\}\right]\\
&=\text{expression of } \Delta_c(u) \text{ in Proposition 3(ii).}
\end{aligned}
$$

This verifies the expression of $\Delta_c(u)$ in Proposition 3(iii).

The expression of $\Delta_c(u)$ in Proposition 3(i) can be obtained by observing

$$
\begin{aligned}
\Delta_c(u)=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\mathcal{S}_{11}(u|\boldsymbol{X})\right]-\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\mathcal{S}_{00}(u|\boldsymbol{X})\right]\\
=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\mathcal{S}_{11}(u|\boldsymbol{X})\mathbb{E}\left[\frac{Z}{\pi(\boldsymbol{X})}|\boldsymbol{X}\right]\right]-\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\mathcal{S}_{00}(u|\boldsymbol{X})\mathbb{E}\left[\frac{1-Z}{1-\pi(\boldsymbol{X})}|\boldsymbol{X}\right]\right]\\
=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X})\right]-\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{1-Z}{1-\pi(\boldsymbol{X})}\mathcal{S}_{00}(u|\boldsymbol{X})\right]\\
=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X})\mathbb{E}\left[\frac{S}{p_1(\boldsymbol{X})}|Z=1,\boldsymbol{X}\right]\right]\\
&-\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{1-Z}{1-\pi(\boldsymbol{X})}\mathcal{S}_{00}(u|\boldsymbol{X})\mathbb{E}\left[\frac{1-S}{1-p_0(\boldsymbol{X})}|Z=0,\boldsymbol{X}\right]\right]\\
=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{Z}{\pi(\boldsymbol{X})}\frac{S}{p_1(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X})\right]-\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{1-S}{1-p_0(\boldsymbol{X})}\mathcal{S}_{00}(u|\boldsymbol{X})\right]
\end{aligned}
$$

$$
\begin{aligned}
=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{Z}{\pi(\boldsymbol{X})}\frac{S}{p_1(\boldsymbol{X})}\frac{1}{\mathcal{S}^C_{11}(u|\boldsymbol{X})}\mathbb{E}[\mathbb{I}(T\geq u)\mathbb{I}(C\geq u)|Z=1,S=1,\boldsymbol{X}]\right]\\
&-\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{1-S}{1-p_0(\boldsymbol{X})}\frac{1}{\mathcal{S}^C_{00}(u|\boldsymbol{X})}\mathbb{E}[\mathbb{I}(T\geq u)\mathbb{I}(C\geq u)|Z=0,S=0,\boldsymbol{X}]\right]\\
&\quad\text{(by Assumption 4)}\\
=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X})}\right]-\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{1-S}{1-p_0(\boldsymbol{X})}\frac{1-Z}{1-\pi(\boldsymbol{X})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{00}(u|\boldsymbol{X})}\right]
\end{aligned}
$$

This completes the proof of Proposition 3. □

Web Appendix E.2 The multiple robust estimator without monotonicity

We first restate estimators of the working models before presenting the multiply robust estimator of $\mathcal{S}_{z,g}(u)$. Recall that $p_z(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_z)$, $\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})$, $\mathcal{S}_{zs}(t|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{zs})$, $\mathcal{S}^C_{zs}(t|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{zs})$ are estimators of $p_z(\boldsymbol{X})$, $\pi(\boldsymbol{X})$, $\mathcal{S}_{zs}(t|\boldsymbol{X})$, and $\mathcal{S}^C_{zs}(t|\boldsymbol{X})$, respectively. Also, notice that $d\widehat{M}^C_{zs}(t|\boldsymbol{X})=dN^C(t)-\mathbb{I}(U\geq t)\frac{\widehat{\lambda}^C_{zs}(t|\boldsymbol{X})}{\mathcal{S}^C_{zs}(t|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{zs})}$ is the estimated martingale for the censoring process as defined in Section 3.3, where $\widehat{\lambda}^C_{zs}(t|\boldsymbol{X})=-\frac{d}{dt}\log\mathcal{S}^C_{zs}(t|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{zs})$.

Besides estimators of working models, we also need to estimate the principal score $e_{\zeta,g}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})$ and its marginalized version $e_{\zeta,g}$. Specifically, we consider a plug-in estimator $e_{\zeta,g}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})$ for $e_{\zeta,g}(\boldsymbol{X})$ by replacing the probabilities $\{p_1(\boldsymbol{X}),p_0(\boldsymbol{X})\}$ in their expressions in Proposition 2 with $\{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1),p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)\}$. For $e_{\zeta,g}$, we consider a doubly robust estimator $\widehat{e}_{\zeta,g}$, which are consistent to $e_{\zeta,g}$ under $\mathcal{M}_\pi\cup\mathcal{M}_e$. Construction of $\widehat{e}_{\zeta,g}$ is given in Appendix E.4.

For a fixed value of $\zeta(\boldsymbol{X})$, the proposed multiply robust estimators of $\mathcal{S}_{z,g}(u)$ are provided as follows

$$
\begin{aligned}
\widehat{\mathcal{S}}^{mr}_{1,c}(u)=&\mathbb{P}_n\Bigg\{\frac{e_{\zeta,c}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}+\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\int_0^u\frac{d\widehat{M}^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}\right]\\
&+\frac{\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})}{\widehat{e}_{\zeta,c}(1-\zeta(\boldsymbol{X}))}\left[\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)-S)+\frac{p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}))\right]\\
&+\frac{e_{\zeta,c}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,c}}\left(1-\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\Bigg\},\\
\widehat{\mathcal{S}}^{mr}_{0,c}(u)=&\mathbb{P}_n\Bigg\{\frac{e_{\zeta,c}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,c}}\frac{1-S}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}+\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\int_0^u\frac{d\widehat{M}^C_{00}(r|\boldsymbol{X})}{\mathcal{S}_{00}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\mathcal{S}^C_{00}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}\right]\\
&+\frac{\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})}{\widehat{e}_{\zeta,c}(1-\zeta(\boldsymbol{X}))}\left[\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1))+\frac{1-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)-S)\right]\\
&+\frac{e_{\zeta,c}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,c}}\left(1-\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\Bigg\},\\
\widehat{\mathcal{S}}^{mr}_{1,n}(u)=&\mathbb{P}_n\Bigg\{\frac{e_{\zeta,n}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,n}}\frac{1-S}{1-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{10})}+\mathcal{S}_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10})\int_0^u\frac{d\widehat{M}^C_{10}(r|\boldsymbol{X})}{\mathcal{S}_{10}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10})\mathcal{S}^C_{10}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{10})}\right]\\
&+\frac{\zeta(\boldsymbol{X})\mathcal{S}_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10})}{\widehat{e}_{\zeta,n}(1-\zeta(\boldsymbol{X}))}\left[\frac{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}{1-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)-S)+\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0))\right]\\
&+\frac{e_{\zeta,n}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,n}}\left(1-\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10})\Bigg\},\\
\widehat{\mathcal{S}}^{mr}_{0,n}(u)=&\mathbb{P}_n\Bigg\{\frac{e_{\zeta,n}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,n}}\frac{1-S}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}+\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\int_0^u\frac{d\widehat{M}^C_{00}(r|\boldsymbol{X})}{\mathcal{S}_{00}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\mathcal{S}^C_{00}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{00})}\right]
\end{aligned}
$$

$$+\frac{\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})}{\widehat{e}_{\zeta,n}(1-\zeta(\boldsymbol{X}))}\left[\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)-S)+\frac{1-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0))\right]$$
$$\left.+\frac{e_{\zeta,n}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,n}}\left(1-\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{00}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{00})\right\},$$

$$\widehat{\mathcal{S}}^{mr}_{1,a}(u)=\mathbb{P}_n\left\{\frac{e_{\zeta,a}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,a}}\frac{S}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}+\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\int_0^u\frac{d\widehat{M}^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{11})}\right]\right.$$
$$+\frac{\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})}{\widehat{e}_{\zeta,a}(1-\zeta(\boldsymbol{X}))}\left[\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0))+\frac{p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})-S)\right]$$
$$\left.+\frac{e_{\zeta,a}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,a}}\left(1-\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{11}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{11})\right\},$$

$$\widehat{\mathcal{S}}^{mr}_{0,a}(u)=\mathbb{P}_n\left\{\frac{e_{\zeta,a}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,a}}\frac{S}{p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{01})}+\mathcal{S}_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01})\int_0^u\frac{d\widehat{M}^C_{01}(r|\boldsymbol{X})}{\mathcal{S}_{01}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01})\mathcal{S}^C_{01}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{01})}\right]\right.$$
$$+\frac{\zeta(\boldsymbol{X})\mathcal{S}_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01})}{\widehat{e}_{\zeta,a}(1-\zeta(\boldsymbol{X}))}\left[\frac{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}{p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0))+\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})-S)\right]$$
$$\left.+\frac{e_{\zeta,a}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,a}}\left(1-\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01})\right\}$$

$$\widehat{\mathcal{S}}^{mr}_{1,d}(u)=\mathbb{P}_n\left\{\frac{e_{\zeta,d}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,d}}\frac{1-S}{1-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{10})}+\mathcal{S}_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10})\int_0^u\frac{d\widehat{M}^C_{10}(r|\boldsymbol{X})}{\mathcal{S}_{10}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10})\mathcal{S}^C_{10}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{10})}\right]\right.$$
$$+\frac{\zeta(\boldsymbol{X})\mathcal{S}_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10})}{\widehat{e}_{\zeta,d}(1-\zeta(\boldsymbol{X}))}\left[\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)-S)+\frac{1-p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}{1-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}))\right]$$
$$\left.+\frac{e_{\zeta,d}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,d}}\left(1-\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{10}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{10})\right\},$$

$$\widehat{\mathcal{S}}^{mr}_{0,d}(u)=\mathbb{P}_n\left\{\frac{e_{\zeta,d}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,d}}\frac{S}{p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}\left[\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{01})}+\mathcal{S}_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01})\int_0^u\frac{d\widehat{M}^C_{01}(r|\boldsymbol{X})}{\mathcal{S}_{01}(r|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01})\mathcal{S}^C_{01}(r|\boldsymbol{X};\widehat{\boldsymbol{\theta}}_{01})}\right]\right.$$
$$+\frac{\zeta(\boldsymbol{X})\mathcal{S}_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01})}{\widehat{e}_{\zeta,d}(1-\zeta(\boldsymbol{X}))}\left[\frac{Z}{\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(S-p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1))+\frac{p_1(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_1)}{p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)}\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})}(p_0(\boldsymbol{X};\widehat{\boldsymbol{\gamma}}_0)-S)\right]$$
$$\left.+\frac{e_{\zeta,d}(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}{\widehat{e}_{\zeta,d}}\left(1-\frac{1-Z}{1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\gamma}})}\right)\mathcal{S}_{01}(u|\boldsymbol{X};\widehat{\boldsymbol{\beta}}_{01})\right\}$$

In Result 4 in the paper, we show that$\widehat{\mathcal{S}}^{mr}_{z,g}(u)$ is consistent to $\mathcal{S}_{z,g}(u)$ under the union model $\mathcal{M}_{\pi+e+C}\cup\mathcal{M}_{\pi+T}\cup\mathcal{M}_{e+T}$.

Web Appendix E.3 Proof of Result 4

Assume that $\widehat{\boldsymbol{\gamma}}$, $\widehat{\boldsymbol{\alpha}}$, $\widehat{\boldsymbol{\theta}}_{zs}$, $\widehat{\boldsymbol{\beta}}_{zs}$ have probability limits $\boldsymbol{\gamma}^*$, $\boldsymbol{\alpha}^*$, $\boldsymbol{\theta}^*_{zs}$, and $\boldsymbol{\beta}^*_{zs}$, respectively. Notice that if $\mathcal{M}_\pi$ is correctly specified, then $\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)=\pi(\boldsymbol{X})$, otherwise $\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)$ generally does not equal to $\pi(\boldsymbol{X})$. Similarly, $p_z(\boldsymbol{X};\boldsymbol{\gamma}^*)=p_z(\boldsymbol{X})$ and $e_{\zeta,g}(\boldsymbol{X};\boldsymbol{\gamma}^*)=e_{\zeta,g}(\boldsymbol{X})$ if $\mathcal{M}_e$ is correctly specified, $\mathcal{S}^C_{zs}(u|\boldsymbol{X};\boldsymbol{\theta}^*_{zs})=\mathcal{S}^C_{zs}(u|\boldsymbol{X})$ if $\mathcal{M}_C$ is correctly specified, and $\mathcal{S}_{zs}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{zs})=\mathcal{S}_{zs}(u|\boldsymbol{X})$ if $\mathcal{M}_T$ is correctly specified. Furthermore, as shown in Appendix E.4, $\widehat{e}_{\zeta,g}$ will converge to $e_{\zeta,g}$ if either $\mathcal{M}_e$ or $\mathcal{M}_\pi$ is correctly specified. In what follows, we prove that proposed multiply robust estimator without monotonicity, $\widehat{\mathcal{S}}^{mr}_{1,c}(u)$, is consistent for $\mathcal{S}_{1,c}$ under $\mathcal{M}_{\pi+e+C}\cup\mathcal{M}_{\pi+T}\cup\mathcal{M}_{e+T}$. Consistency of the multiply robust estimators for other quantities, including $\widehat{\mathcal{S}}^{mr}_{0,c}(u)$, $\widehat{\mathcal{S}}^{mr}_{1,a}(u)$, $\widehat{\mathcal{S}}^{mr}_{0,a}(u)$, $\widehat{\mathcal{S}}^{mr}_{1,n}(u)$, $\widehat{\mathcal{S}}^{mr}_{0,n}(u)$, $\widehat{\mathcal{S}}^{mr}_{1,d}(u)$, and $\widehat{\mathcal{S}}^{mr}_{0,d}(u)$, can be proved similarly and therefore are omitted here.

In what follows, we shall prove that $\widehat{\mathcal{S}}^{mr}_{1,c}(u)$ converges in probability to $\mathcal{S}_{1,c}(u)$ when either

$\mathcal{M}_{\pi+e+C}$, $\mathcal{M}_{e+T}$, or $\mathcal{M}_{\pi+T}$ is correctly specified. This will complete our proof.

(i) When $\mathcal{M}_{\pi+e+C}$ is correctly specified

If $\mathcal{M}_{\pi+e+C}$ is correctly specified, $\widehat{\mathcal{S}}^{mr}_{1,c}(u)$ converges in probability to

$$\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\frac{1}{\mathcal{S}^C_{11}(u|\boldsymbol{X})}\mathbb{I}(U\geq u)\right] \tag{E.2}$$

$$+\mathbb{E}\left[\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\left(1-\frac{Z}{\pi(\boldsymbol{X})}\right)\right] \tag{E.3}$$

$$+\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})}{(1-\zeta(\boldsymbol{X}))e_{\zeta,c}}\left\{\frac{1-Z}{1-\pi(\boldsymbol{X})}(p_0(\boldsymbol{X})-S)+\frac{p_0(\boldsymbol{X})}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}(S-p_1(\boldsymbol{X}))\right\}\right] \tag{E.4}$$

$$+\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\int_0^u\frac{dM^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X})}\right]. \tag{E.5}$$

By Proposition 3(ii), (E.2) $=\mathcal{S}_{1,c}(u)$. Also, applying the law of iterated expectation to (E.3)–(E.5), we have that

$$\text{(E.3)} =\mathbb{E}\left\{\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\mathbb{E}\left[1-\frac{Z}{\pi(\boldsymbol{X})}\middle|\boldsymbol{X}\right]\right\}=0,$$

$$\text{(E.4)} =\mathbb{E}\left[\mathbb{E}\left\{\frac{1-Z}{1-\pi(\boldsymbol{X})}\mathbb{E}[p_0(\boldsymbol{X})-S|Z=0,\boldsymbol{X}]+\frac{p_0(\boldsymbol{X})}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathbb{E}[S-p_1(\boldsymbol{X})|Z=1,\boldsymbol{X}]\middle|\boldsymbol{X}\right\}\frac{\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})}{(1-\zeta(\boldsymbol{X}))e_{\zeta,c}}\right]=0,$$

$$\text{(E.5)} =\mathbb{E}\left\{\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_0(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\mathbb{E}\left[\int_0^u\frac{dM^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X})}\middle|Z=1,S=1,\boldsymbol{X}\right]\right\}=0.$$

The last equality in the above set of formulas holds because, conditional on $Z=1,S=1,\boldsymbol{X}$, $\int_0^u\frac{dM^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X})}$ is a martingale and has expectation 0 such that $\mathbb{E}\Big[\int_0^u\frac{dM^C_{11}(r|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X};\boldsymbol{\beta}^*_{11})\mathcal{S}^C_{11}(r|\boldsymbol{X})}\Big|Z=1,S=1,\boldsymbol{X}\Big]=0$. Therefore, $\widehat{\mathcal{S}}^{mr}_{1,c}(u)$ converges in probability to (E.2) + (E.3) + (E.4) + (E.5) $=\mathcal{S}_{1,c}(u)$, if $\mathcal{M}_{\pi+e+C}$ is correctly specified.

(ii) When $\mathcal{M}_{e+T}$ is correctly specified

If $\mathcal{M}_{e+T}$ is correctly specified, $\widehat{\mathcal{S}}^{mr}_{1,c}(u)$ converges in probability to

$$\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X};\boldsymbol{\theta}^*_{11})}\right] \tag{E.6}$$

$$+\mathbb{E}\left[\mathcal{S}_{11}(u|\boldsymbol{X})\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\left(1-\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\right)\right] \tag{E.7}$$

$$+\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{(1-\zeta(\boldsymbol{X}))e_{\zeta,c}}\left\{\frac{1-Z}{1-\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}(p_0(\boldsymbol{X})-S)+\frac{p_0(\boldsymbol{X})}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}(S-p_1(\boldsymbol{X}))\right\}\right] \tag{E.8}$$

$$+\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathcal{S}_{11}(u|\boldsymbol{X})\int_0^u\frac{dM^C_{11}(r|\boldsymbol{X};\boldsymbol{\theta}^*_{11})}{\mathcal{S}_{11}(r|\boldsymbol{X})\mathcal{S}^C_{11}(r|\boldsymbol{X};\boldsymbol{\theta}^*_{11})}\right]. \tag{E.9}$$

By Lemma 2,

$$\begin{aligned}\text{(E.6)} =&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\frac{\mathbb{I}(C\geq u\text{ or }C\geq T)\mathbb{I}(T\geq u)}{\mathcal{S}^C_{11}(u|\boldsymbol{X};\boldsymbol{\theta}^*_{11})}\right]\\ =&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathbb{I}(T\geq u)\right]\end{aligned} \tag{E.10}$$

$$
-\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathbb{I}(T\geq u)\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}^C(r|\boldsymbol{X};\theta_{11}^*)}\right], \tag{E.11}
$$

$$
\begin{aligned}
\text{(E.10)} =&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathbb{E}[\mathbb{I}(T\geq u)|Z=1,S=1,\boldsymbol{X}]\right]\\
=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathcal{S}_{11}(u|\boldsymbol{X})\mathbb{E}\left[\frac{S}{p_1(\boldsymbol{X})}\middle|Z=1,\boldsymbol{X}\right]\right]\\
=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathcal{S}_{11}(u|\boldsymbol{X})\right].
\end{aligned}
$$

Therefore, by Proposition 3 we have

$$
\text{(E.7)}+\text{(E.10)}=\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\mathcal{S}_{11}(u|\boldsymbol{X})\right]=\mathcal{S}_{1,c}(u).
$$

Similar to the proof of (E.4) $=0$, we have that (E.8) $=0$. Also, one can show that

$$
\begin{aligned}
&\text{(E.9)}+\text{(E.11)}\\
=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\left(\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X})}-\mathbb{I}(T\geq u)\right)\right]\\
=&\mathbb{E}\left\{\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\int_0^u\frac{dN^C(s)}{\mathcal{S}_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X})}-\mathbb{I}(T\geq u)\middle|Z=1,S=1,\boldsymbol{X},C=r,T\geq r\right]\right\}\\
&-\mathbb{E}\left\{\frac{e_{\zeta,c}(\boldsymbol{X})}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X})}\frac{Z}{\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\int_0^u\frac{\mathbb{I}(U\geq u)d\Lambda_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X})}-\mathbb{I}(T\geq u)\middle|Z=1,S=1,\boldsymbol{X},C\geq r,T\geq r\right]\right\}
\end{aligned}
$$

Since

$$
\mathbb{E}[\mathbb{I}(T\geq u)|Z=1,S=1,\boldsymbol{X},C\geq r,T\geq r]=\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{\mathcal{S}_{11}(r|\boldsymbol{X})},
$$

one can conclude (E.9)+(E.11) $=0$. To summarize, if $\mathcal{M}_{e+T}$ is correctly specified, $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$ converges in probability to (E.6) + (E.7) + (E.8) + (E.9) $=\mathcal{S}_{1,c}(u)$.

(iii) When $\mathcal{M}_{\pi+T}$ is correctly specified

Under $\mathcal{M}_{\pi+T}$, $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$ converges in probability to

$$
\begin{aligned}
&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X};\boldsymbol{\gamma}^*)}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(U\geq u)}{\mathcal{S}_{11}^C(u|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right] &\text{(E.12)}\\
&+\mathbb{E}\left[\mathcal{S}_{11}(u|\boldsymbol{X})\frac{e_c(\boldsymbol{X};\boldsymbol{\gamma}^*)}{e_{\zeta,c}}\left(1-\frac{1}{\pi(\boldsymbol{X})}\right)\right] &\text{(E.13)}\\
&+\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{e_{\zeta,c}}\left\{\frac{1-Z}{1-\pi(\boldsymbol{X})}(p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)-S)+\frac{p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}(S-p_1(\boldsymbol{X};\boldsymbol{\gamma}^*))\right\}\right] &\text{(E.14)}\\
&+\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X};\boldsymbol{\gamma}^*)}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}\mathcal{S}_{11}(u|\boldsymbol{X})\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}(u|\boldsymbol{X})\mathcal{S}_{11}^C(u|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right]. &\text{(E.15)}
\end{aligned}
$$

By Lemma 2, and similar to the derivation of (E.10) and (E.11), we have

$$
\begin{aligned}
\text{(E.12)} =&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X};\boldsymbol{\gamma}^*)}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}\frac{\mathbb{I}(C\geq u\text{ or }C\geq T)\mathbb{I}(T\geq u)}{\mathcal{S}_{11}^C(u|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right]\\
=&\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X};\boldsymbol{\gamma}^*)}{e_{\zeta,c}}\frac{p_1(\boldsymbol{X})}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\mathcal{S}_{11}(u|\boldsymbol{X})\right] &\text{(E.16)}
\end{aligned}
$$

$$
-\mathbb{E}\left[\frac{e_{\zeta,c}(\boldsymbol{X};\boldsymbol{\gamma}^*)}{e_{\zeta,c}}\frac{S}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}\mathbb{I}(T\geq u)\int_0^u\frac{dM_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}{\mathcal{S}_{11}^C(r|\boldsymbol{X};\boldsymbol{\theta}_{11}^*)}\right], \tag{E.17}
$$

Also, applying the law of iterative expectation to (E.13), we have

$$
\text{(E.13)}=\mathbb{E}\left\{\mathcal{S}_{11}(u|\boldsymbol{X})\frac{e_{\zeta,c}(\boldsymbol{X};\boldsymbol{\gamma}^*)}{e_{\zeta,c}}\mathbb{E}\left[1-\frac{Z}{\pi(\boldsymbol{X})}\middle|\boldsymbol{X}\right]\right\}=0.
$$

Then, according Proposition 3, we know that

$$
\text{(E.14)}=\frac{1}{e_{\zeta,c}}\mathbb{E}\left[\left(\frac{ZS}{\pi(\boldsymbol{X})}-\frac{(1-Z)S}{1-\pi(\boldsymbol{X})}\right)\mathcal{S}_{11}(u|\boldsymbol{X})\right]+R=\mathcal{S}_{1,c}(u)+R,
$$

where

$$
\begin{aligned}
R=&\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{e_{\zeta,c}}\left\{\frac{1-Z}{1-\pi(\boldsymbol{X})}p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)-\frac{p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{Z}{\pi(\boldsymbol{X})}p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)-\frac{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)-p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}\frac{ZS}{\pi(\boldsymbol{X})}\right\}\right]\\
=&\mathbb{E}\left[\frac{\mathcal{S}_{11}(u|\boldsymbol{X})}{e_{\zeta,c}}\left\{p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)-\frac{p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)-\frac{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)-p_0(\boldsymbol{X};\boldsymbol{\gamma}^*)}{p_1(\boldsymbol{X};\boldsymbol{\gamma}^*)}p_1(\boldsymbol{X})\right\}\right]\\
=&-\text{(E.16)}.
\end{aligned}
$$

And similar to the proof of (E.9) + (E.11) = 0, we can show that (E.15) + (E.17) = 0. Therefore, if $\mathcal{M}_{\pi+T}$ is correctly specified, we have that $\widehat{\mathcal{S}}_{1,c}^{mr}(u)$ converges in probability to (E.12) + (E.13) + (E.14) + (E.15) = $\mathcal{S}_{1,c}(u)$.

Web Appendix E.4 Estimation of the marginalized propensity score without monotonicity

This section proposes a doubly robust estimator of the marginalized propensity score $e_{\zeta,g}=\mathbb{E}[e_{\zeta,g}(\boldsymbol{X})]$. The doubly robust estimator, denoted by $\widehat{e}_{\zeta,g}$, is consistent under $\mathcal{M}_{\pi+e}$, i.e., when the propensity score models (i.e., $\pi(\boldsymbol{X};\boldsymbol{\alpha})$) or the working model for the observed treatment receipt status (i.e., $p_z(\boldsymbol{X};\boldsymbol{\gamma}_z)$) is correctly specified.

According to Proposition 2, $e_{\zeta,g}$ has the following explicit forms

$$
\begin{aligned}
e_{\zeta,c}=\mathbb{E}\left[\frac{p_1(\boldsymbol{X})-p_0(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}\right],\quad & e_{\zeta,n}=1-\mathbb{E}[p_0(\boldsymbol{X})]-\mathbb{E}\left[\frac{p_1(\boldsymbol{X})-p_0(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}\right],\\
e_{\zeta,a}=\mathbb{E}[p_1(\boldsymbol{X})]-\mathbb{E}\left[\frac{p_1(\boldsymbol{X})-p_0(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}\right],\quad & e_{\zeta,d}=\mathbb{E}\left[\frac{p_1(\boldsymbol{X})-p_0(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}\right]-\mathbb{E}[p_1(\boldsymbol{X})]+\mathbb{E}[p_0(\boldsymbol{X})],
\end{aligned}
$$

which can be re-expressed as

$$
\begin{aligned}
e_{\zeta,c}=p_{\zeta,1}-p_{\zeta,0},\quad & e_{\zeta,n}=1-p_0-p_{\zeta,1}+p_{\zeta,0},\\
e_{\zeta,a}=p_1-p_{\zeta,1}+p_{\zeta,0},\quad & e_{\zeta,d}=p_{\zeta,1}-p_{\zeta,0}-p_1+p_0,
\end{aligned}
$$

where $p_{\zeta,z}=\mathbb{E}\left[\frac{p_z(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}\right]$ for $z\in\{0,1\}$. In Section 3.1, we provided a doubly robust estimator $\widehat{p}_z$ of p_z under $\mathcal{M}_{\pi+e}$. In terms of $p_{\zeta,z}$, we propose the following estimator following the same strategy used in estimation of p_z:

$$\widehat{p}_{\zeta,0} = \mathbb{P}_n\left[\frac{(1-Z)(S-p_0(\boldsymbol{X};\widehat{\gamma}_0))}{(1-\zeta(\boldsymbol{X}))(1-\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}}))} + \frac{p_0(\boldsymbol{X};\widehat{\gamma}_0)}{1-\zeta(\boldsymbol{X})}\right],$$

$$\widehat{p}_{\zeta,1} = \mathbb{P}_n\left[\frac{Z(S-p_1(\boldsymbol{X};\widehat{\gamma}_1))}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X};\widehat{\boldsymbol{\alpha}})} + \frac{p_1(\boldsymbol{X};\widehat{\gamma}_1)}{1-\zeta(\boldsymbol{X})}\right].$$

Note that in a special scenario when the sensitivity function is constant such that $\zeta(\boldsymbol{X}) = \zeta$, $\widehat{p}_{\zeta,0}$ and $\widehat{p}_{\zeta,1}$ can be simplified to

$$\widehat{p}_{\zeta,0} = \frac{\widehat{p}_0}{1-\zeta} \text{ and } \widehat{p}_{\zeta,1} = \frac{\widehat{p}_1}{1-\zeta}.$$

After obtaining $\widehat{p}_z$ and $\widehat{p}_{\zeta,z}$, the estimators of $e_{\zeta,g}$ can be obtained by

$$\widehat{e}_{\zeta,c} = \widehat{p}_{\zeta,1} - \widehat{p}_{\zeta,0}, \quad \widehat{e}_{\zeta,n} = 1 - \widehat{p}_0 - \widehat{p}_{\zeta,1} + \widehat{p}_{\zeta,0},$$
$$\widehat{e}_{\zeta,a} = \widehat{p}_1 - \widehat{p}_{\zeta,1} + \widehat{p}_{\zeta,0}, \quad \widehat{e}_{\zeta,d} = \widehat{p}_{\zeta,1} - \widehat{p}_{\zeta,0} - \widehat{p}_1 + \widehat{p}_0,$$

Below we prove that $\widehat{p}_{\zeta,1}$ is a doubly robust estimator of $p_{\zeta,1}$ under $\mathcal{M}_{\pi+e}$, where doubly robustness of $\widehat{p}_{\zeta,0}$ can be proved in a similar way. Specifically, one can verify that $\widehat{p}_{\zeta,1}$ converges in probability to

$$\widetilde{p}_{\zeta,1} = \mathbb{E}\left[\frac{Z(S-p_1(\boldsymbol{X};\boldsymbol{\gamma}_1^*))}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)} + \frac{p_1(\boldsymbol{X};\boldsymbol{\gamma}_1^*)}{1-\zeta(\boldsymbol{X})}\right],$$

where $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}$ if $\mathcal{M}_\pi$ is correct and $\boldsymbol{\gamma}_1^* = \boldsymbol{\gamma}_1$ if $\mathcal{M}_e$ is correct. If $\mathcal{M}_e$ is correct, one can show that

$$\begin{aligned}
\widetilde{p}_{\zeta,1} &= \mathbb{E}\left[\frac{Z(S-p_1(\boldsymbol{X}))}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)} + \frac{p_1(\boldsymbol{X})}{1-\zeta(\boldsymbol{X})}\right] \\
&= p_{\zeta,1} + \mathbb{E}\left[\frac{Z}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathbb{E}\left[S-p_1(\boldsymbol{X})\Big|Z,\boldsymbol{X}\right]\right] \\
&= p_{\zeta,1} + \mathbb{E}\left[\frac{Z}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)}\mathbb{E}\left[S-p_1(\boldsymbol{X})\Big|Z=1,\boldsymbol{X}\right]\right] \\
&= p_{\zeta,1} + \mathbb{E}\left[\frac{Z}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X};\boldsymbol{\alpha}^*)} \times 0\right] \\
&= p_{\zeta,1},
\end{aligned}$$

regardless of whether $\mathcal{M}_\pi$ is correctly specified or not. On the other hand, if $\mathcal{M}_\pi$ is correct, one can show that

$$\begin{aligned}
\widetilde{p}_{\zeta,1} &= \mathbb{E}\left[\frac{Z(S-p_1(\boldsymbol{X};\boldsymbol{\gamma}_1^*))}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})} + \frac{p_1(\boldsymbol{X};\boldsymbol{\gamma}_1^*)}{1-\zeta(\boldsymbol{X})}\right] \\
&= \mathbb{E}\left[\frac{ZS}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})}\right] + \mathbb{E}\left[\left(\frac{\pi(\boldsymbol{X})-Z}{\pi(\boldsymbol{X})}\right)\frac{p_1(\boldsymbol{X};\boldsymbol{\gamma}_1^*)}{1-\zeta(\boldsymbol{X})}\right] \\
&= \mathbb{E}\left[\frac{Z\mathbb{E}[S|Z,\boldsymbol{X}]}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})}\right] + \mathbb{E}\left[0 \times \frac{p_1(\boldsymbol{X};\boldsymbol{\gamma}_1^*)}{1-\zeta(\boldsymbol{X})}\right] \\
&= \mathbb{E}\left[\frac{Z\mathbb{E}[S|Z=1,\boldsymbol{X}]}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})}\right] = \mathbb{E}\left[\frac{Zp_1(\boldsymbol{X})}{(1-\zeta(\boldsymbol{X}))\pi(\boldsymbol{X})}\right] \\
&= \mathbb{E}\left[E\left[\frac{Z}{\pi(\boldsymbol{X})}\Big|\boldsymbol{X}\right]\frac{p_1(\boldsymbol{X})}{(1-\zeta(\boldsymbol{X}))}\right] = \mathbb{E}\left[\frac{p_1(\boldsymbol{X})}{(1-\zeta(\boldsymbol{X}))}\right] \\
&= p_{\zeta,1}
\end{aligned}$$

regardless of whether $\mathcal{M}_e$ is correctly specified or not. This concludes double robustness of $\widehat{p}_{\zeta,1}$.

Web Appendix F Web Tables and Figures

Web Table 1: A literature review for identification and inference of causal effects with noncompliance and survival outcomes.

Literature	Assumptions	Covariates	Methods	Sensitivity Analysis
Baker (1998)	Mono+ER	No	Parametric	No
Frangakis and Rubin (1999)	Mono+ER	No	Nonparametric	No
Loeys and Goetghebeur (2003)	Mono+ER	No	Semiparametric	No
Nie et al. (2011)	Mono+ER	No	Nonparametric	No
Yu et al. (2015)	Mono+ER	Yes	Semiparametric	No
Cuzick et al. (2007)	Mixture model	Yes	Semiparametric	No
Wei et al. (2021)	Mono+ER	Yes	Nonparametric	No
The present work	Mono+PI	Yes	Semiparametric	Yes

* Abbreviations: ER, exclusion restriction; Mono: monotonicity; PI: principal ignorability

† The second column displays key causal assumptions used in each publication, where all publications require randomization of treatment assignment and (conditional) independent censoring assumption and both assumptions are omitted in this table. It is shown that the majority of existing literature uses a combination of monotonicity and exclusion restriction assumptions to identify related estimands. One exception is Cuzick et al. (2007) who uses a mixture model with proportional hazard assumptions for the failure time outcome.

‡ The third column displays whether the literature allows for incorporating baseline covariates. The fourth assumption displays whether the proposed methodology is parametric, semiparametric, or fully nonparametric. The last column shows whether appropriate sensitivity analysis strategies are developed to address potential violation of the causal assumptions in the second column.

Web Table 2: Simulation results of $\widehat{\mathcal{S}}^{mr}_{0,c}(u)$, where $\mathcal{M}_e, \mathcal{M}_\pi, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, propensity score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or misspecified (denoted by a 'F' label).

		u	$\mathcal{S}_{0,c}(u)$	$\widehat{\mathcal{S}}^{mr}_{0,c}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.695	0.695	0.000	0.050	0.946
$\mathcal{M}_e$	T	2	0.517	0.518	0.000	0.048	0.960
$\mathcal{M}_\pi$	T	3	0.397	0.395	-0.002	0.046	0.960
$\mathcal{M}_T$	T	4	0.309	0.306	-0.003	0.039	0.968
$\mathcal{M}_C$	T	5	0.245	0.242	-0.003	0.036	0.964
Scenario 2		1	0.695	0.693	-0.002	0.048	0.960
$\mathcal{M}_e$	T	2	0.517	0.517	0.000	0.048	0.972
$\mathcal{M}_\pi$	T	3	0.397	0.394	-0.002	0.046	0.970
$\mathcal{M}_T$	F	4	0.309	0.307	-0.003	0.041	0.960
$\mathcal{M}_C$	T	5	0.245	0.243	-0.002	0.037	0.970
Scenario 3		1	0.695	0.693	-0.002	0.172	0.936
$\mathcal{M}_e$	F	2	0.517	0.517	-0.001	0.133	0.946
$\mathcal{M}_\pi$	T	3	0.397	0.394	-0.003	0.108	0.956
$\mathcal{M}_T$	T	4	0.309	0.306	-0.004	0.084	0.966
$\mathcal{M}_C$	F	5	0.245	0.241	-0.003	0.068	0.960
Scenario 4		1	0.695	0.692	-0.003	0.042	0.962
$\mathcal{M}_e$	T	2	0.517	0.517	0.000	0.044	0.954
$\mathcal{M}_\pi$	F	3	0.397	0.395	-0.002	0.043	0.962
$\mathcal{M}_T$	T	4	0.309	0.308	-0.002	0.038	0.968
$\mathcal{M}_C$	F	5	0.245	0.244	-0.001	0.035	0.976
Scenario 5		1	0.695	0.711	0.016	0.040	0.918
$\mathcal{M}_e$	F	2	0.517	0.537	0.019	0.045	0.948
$\mathcal{M}_\pi$	T	3	0.397	0.413	0.016	0.046	0.950
$\mathcal{M}_T$	F	4	0.309	0.322	0.013	0.046	0.964
$\mathcal{M}_C$	T	5	0.245	0.256	0.011	0.045	0.966
Scenario 6		1	0.695	0.742	0.047	0.030	0.692
$\mathcal{M}_e$	T	2	0.517	0.575	0.058	0.035	0.696
$\mathcal{M}_\pi$	F	3	0.397	0.452	0.055	0.036	0.696
$\mathcal{M}_T$	F	4	0.309	0.361	0.051	0.034	0.736
$\mathcal{M}_C$	F	5	0.245	0.292	0.048	0.032	0.758
Scenario 7		1	0.695	0.595	-0.100	0.036	0.224
$\mathcal{M}_e$	F	2	0.517	0.438	-0.079	0.032	0.340
$\mathcal{M}_\pi$	F	3	0.397	0.335	-0.062	0.030	0.488
$\mathcal{M}_T$	T	4	0.309	0.262	-0.047	0.026	0.612
$\mathcal{M}_C$	F	5	0.245	0.209	-0.036	0.025	0.714
Scenario 8		1	0.695	0.761	0.066	0.024	0.246
$\mathcal{M}_e$	F	2	0.517	0.600	0.083	0.029	0.228
$\mathcal{M}_\pi$	F	3	0.397	0.479	0.082	0.029	0.234
$\mathcal{M}_T$	F	4	0.309	0.387	0.078	0.029	0.264
$\mathcal{M}_C$	F	5	0.245	0.318	0.073	0.029	0.328

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 3: Simulation results of the multiply robust estimation of $\mathcal{S}_{1,c}(u)$, where $\mathcal{M}_e, \mathcal{M}_\pi, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, propensity score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{1,c}(u)$	$\widehat{\mathcal{S}}_{1,c}^{mr}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.528	0.528	0.000	0.037	0.936
$\mathcal{M}_e$	T	2	0.294	0.292	-0.002	0.035	0.942
$\mathcal{M}_\pi$	T	3	0.171	0.168	-0.003	0.030	0.940
$\mathcal{M}_T$	T	4	0.103	0.100	-0.003	0.025	0.944
$\mathcal{M}_C$	T	5	0.064	0.061	-0.003	0.021	0.918
Scenario 2		1	0.528	0.528	0.000	0.037	0.940
$\mathcal{M}_e$	T	2	0.294	0.293	-0.001	0.035	0.942
$\mathcal{M}_\pi$	T	3	0.171	0.168	-0.003	0.030	0.946
$\mathcal{M}_T$	F	4	0.103	0.100	-0.003	0.024	0.944
$\mathcal{M}_C$	T	5	0.064	0.061	-0.003	0.021	0.906
Scenario 3		1	0.528	0.527	0.000	0.038	0.934
$\mathcal{M}_e$	F	2	0.294	0.291	-0.003	0.036	0.944
$\mathcal{M}_\pi$	T	3	0.171	0.166	-0.005	0.031	0.928
$\mathcal{M}_T$	T	4	0.103	0.098	-0.005	0.027	0.928
$\mathcal{M}_C$	F	5	0.064	0.057	-0.007	0.027	0.918
Scenario 4		1	0.528	0.528	0.000	0.037	0.938
$\mathcal{M}_e$	T	2	0.294	0.292	-0.002	0.035	0.944
$\mathcal{M}_\pi$	F	3	0.171	0.168	-0.003	0.031	0.952
$\mathcal{M}_T$	T	4	0.103	0.100	-0.002	0.025	0.950
$\mathcal{M}_C$	F	5	0.064	0.062	-0.002	0.022	0.916
Scenario 5		1	0.528	0.537	0.009	0.031	0.928
$\mathcal{M}_e$	F	2	0.294	0.312	0.018	0.030	0.910
$\mathcal{M}_\pi$	T	3	0.171	0.189	0.019	0.029	0.942
$\mathcal{M}_T$	F	4	0.103	0.120	0.017	0.026	0.954
$\mathcal{M}_C$	T	5	0.064	0.077	0.013	0.027	0.932
Scenario 6		1	0.528	0.526	-0.001	0.037	0.938
$\mathcal{M}_e$	T	2	0.294	0.288	-0.006	0.035	0.938
$\mathcal{M}_\pi$	F	3	0.171	0.161	-0.010	0.030	0.932
$\mathcal{M}_T$	F	4	0.103	0.092	-0.011	0.024	0.916
$\mathcal{M}_C$	F	5	0.064	0.053	-0.011	0.019	0.856
Scenario 7		1	0.528	0.528	0.000	0.038	0.932
$\mathcal{M}_e$	F	2	0.294	0.292	-0.002	0.038	0.944
$\mathcal{M}_\pi$	F	3	0.171	0.168	-0.003	0.033	0.926
$\mathcal{M}_T$	T	4	0.103	0.100	-0.002	0.029	0.940
$\mathcal{M}_C$	F	5	0.064	0.061	-0.003	0.024	0.934
Scenario 8		1	0.528	0.534	0.006	0.031	0.938
$\mathcal{M}_e$	F	2	0.294	0.303	0.009	0.031	0.932
$\mathcal{M}_\pi$	F	3	0.171	0.177	0.006	0.028	0.946
$\mathcal{M}_T$	F	4	0.103	0.106	0.004	0.023	0.944
$\mathcal{M}_C$	F	5	0.064	0.065	0.001	0.020	0.916

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 4: Simulation results of the multiply robust estimation of $\mathcal{S}_{0,n}(u)$, where $\mathcal{M}_e, \mathcal{M}_\pi, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, propensity score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{0,n}(u)$	$\widehat{\mathcal{S}}_{0,n}^{mr}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.747	0.746	-0.001	0.025	0.942
$\mathcal{M}_e$	T	2	0.578	0.577	-0.001	0.030	0.920
$\mathcal{M}_\pi$	T	3	0.456	0.455	-0.001	0.029	0.948
$\mathcal{M}_T$	T	4	0.364	0.364	0.000	0.029	0.950
$\mathcal{M}_C$	T	5	0.293	0.293	0.000	0.028	0.956
Scenario 2		1	0.747	0.747	0.000	0.026	0.934
$\mathcal{M}_e$	T	2	0.578	0.578	0.000	0.030	0.928
$\mathcal{M}_\pi$	T	3	0.456	0.456	0.000	0.030	0.954
$\mathcal{M}_T$	F	4	0.364	0.365	0.000	0.029	0.944
$\mathcal{M}_C$	T	5	0.293	0.294	0.000	0.028	0.942
Scenario 3		1	0.747	0.746	-0.001	0.024	0.950
$\mathcal{M}_e$	F	2	0.578	0.578	-0.001	0.030	0.924
$\mathcal{M}_\pi$	T	3	0.456	0.456	-0.001	0.029	0.954
$\mathcal{M}_T$	T	4	0.364	0.365	0.000	0.029	0.948
$\mathcal{M}_C$	F	5	0.293	0.294	0.000	0.028	0.954
Scenario 4		1	0.747	0.746	-0.001	0.025	0.946
$\mathcal{M}_e$	T	2	0.578	0.577	-0.001	0.029	0.930
$\mathcal{M}_\pi$	F	3	0.456	0.455	-0.001	0.029	0.946
$\mathcal{M}_T$	T	4	0.364	0.364	0.000	0.028	0.940
$\mathcal{M}_C$	F	5	0.293	0.293	0.000	0.028	0.936
Scenario 5		1	0.747	0.737	-0.010	0.027	0.932
$\mathcal{M}_e$	F	2	0.578	0.567	-0.012	0.030	0.902
$\mathcal{M}_\pi$	T	3	0.456	0.446	-0.011	0.030	0.936
$\mathcal{M}_T$	F	4	0.364	0.355	-0.009	0.029	0.920
$\mathcal{M}_C$	T	5	0.293	0.286	-0.007	0.028	0.934
Scenario 6		1	0.747	0.777	0.030	0.023	0.720
$\mathcal{M}_e$	T	2	0.578	0.616	0.038	0.029	0.718
$\mathcal{M}_\pi$	F	3	0.456	0.495	0.039	0.029	0.736
$\mathcal{M}_T$	F	4	0.364	0.402	0.038	0.030	0.768
$\mathcal{M}_C$	F	5	0.293	0.329	0.035	0.030	0.774
Scenario 7		1	0.747	0.783	0.036	0.027	0.720
$\mathcal{M}_e$	F	2	0.578	0.608	0.030	0.032	0.822
$\mathcal{M}_\pi$	F	3	0.456	0.480	0.023	0.031	0.864
$\mathcal{M}_T$	T	4	0.364	0.383	0.019	0.031	0.902
$\mathcal{M}_C$	F	5	0.293	0.308	0.014	0.030	0.908
Scenario 8		1	0.747	0.766	0.020	0.023	0.860
$\mathcal{M}_e$	F	2	0.578	0.603	0.024	0.029	0.836
$\mathcal{M}_\pi$	F	3	0.456	0.481	0.025	0.029	0.850
$\mathcal{M}_T$	F	4	0.364	0.389	0.024	0.029	0.876
$\mathcal{M}_C$	F	5	0.293	0.316	0.023	0.029	0.872

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 5: Simulation results of the multiply robust estimation of $\mathcal{S}_{1,n}(u)$, where $\mathcal{M}_e, \mathcal{M}_\pi, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, propensity score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{1,n}(u)$	$\widehat{\mathcal{S}}_{1,n}^{mr}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.692	0.693	0.001	0.038	0.928
$\mathcal{M}_e$	T	2	0.492	0.490	-0.002	0.041	0.948
$\mathcal{M}_\pi$	T	3	0.355	0.354	-0.001	0.042	0.926
$\mathcal{M}_T$	T	4	0.261	0.260	-0.002	0.039	0.940
$\mathcal{M}_C$	T	5	0.193	0.192	-0.001	0.037	0.912
Scenario 2		1	0.692	0.693	0.001	0.038	0.920
$\mathcal{M}_e$	T	2	0.492	0.490	-0.002	0.041	0.938
$\mathcal{M}_\pi$	T	3	0.355	0.354	-0.001	0.042	0.924
$\mathcal{M}_T$	F	4	0.261	0.260	-0.001	0.040	0.948
$\mathcal{M}_C$	T	5	0.193	0.192	0.000	0.037	0.916
Scenario 3		1	0.692	0.693	0.001	0.038	0.924
$\mathcal{M}_e$	F	2	0.492	0.490	-0.002	0.041	0.944
$\mathcal{M}_\pi$	T	3	0.355	0.354	-0.001	0.042	0.924
$\mathcal{M}_T$	T	4	0.261	0.260	-0.002	0.039	0.940
$\mathcal{M}_C$	F	5	0.193	0.192	-0.001	0.037	0.914
Scenario 4		1	0.692	0.694	0.002	0.037	0.920
$\mathcal{M}_e$	T	2	0.492	0.491	-0.001	0.041	0.948
$\mathcal{M}_\pi$	F	3	0.355	0.355	0.000	0.042	0.918
$\mathcal{M}_T$	T	4	0.261	0.260	-0.001	0.039	0.944
$\mathcal{M}_C$	F	5	0.193	0.193	0.000	0.037	0.918
Scenario 5		1	0.692	0.693	0.001	0.038	0.920
$\mathcal{M}_e$	F	2	0.492	0.490	-0.002	0.041	0.936
$\mathcal{M}_\pi$	T	3	0.355	0.354	-0.001	0.042	0.922
$\mathcal{M}_T$	F	4	0.261	0.260	-0.001	0.040	0.950
$\mathcal{M}_C$	T	5	0.193	0.192	0.000	0.037	0.914
Scenario 6		1	0.692	0.684	-0.007	0.038	0.922
$\mathcal{M}_e$	T	2	0.492	0.484	-0.008	0.041	0.950
$\mathcal{M}_\pi$	F	3	0.355	0.351	-0.004	0.042	0.920
$\mathcal{M}_T$	F	4	0.261	0.259	-0.002	0.039	0.930
$\mathcal{M}_C$	F	5	0.193	0.193	0.000	0.037	0.910
Scenario 7		1	0.692	0.707	0.015	0.040	0.894
$\mathcal{M}_e$	F	2	0.492	0.499	0.006	0.044	0.938
$\mathcal{M}_\pi$	F	3	0.355	0.358	0.003	0.045	0.914
$\mathcal{M}_T$	T	4	0.261	0.259	-0.002	0.042	0.944
$\mathcal{M}_C$	F	5	0.193	0.190	-0.003	0.039	0.918
Scenario 8		1	0.692	0.685	-0.007	0.038	0.924
$\mathcal{M}_e$	F	2	0.492	0.484	-0.008	0.041	0.944
$\mathcal{M}_\pi$	F	3	0.355	0.351	-0.005	0.042	0.922
$\mathcal{M}_T$	F	4	0.261	0.257	-0.004	0.039	0.926
$\mathcal{M}_C$	F	5	0.193	0.191	-0.002	0.037	0.908

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 6: Simulation results of the multiply robust estimation of $\mathcal{S}_{0,a}(u)$ under an observational study setting (Part I), where $\mathcal{M}_e, \mathcal{M}_\pi, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, propensity score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{0,a}(u)$	$\widehat{\mathcal{S}}_{0,a}^{mr}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.458	0.459	0.001	0.043	0.944
$\mathcal{M}_e$	T	2	0.256	0.256	0.000	0.035	0.926
$\mathcal{M}_\pi$	T	3	0.152	0.151	-0.001	0.028	0.946
$\mathcal{M}_T$	T	4	0.094	0.092	-0.002	0.024	0.932
$\mathcal{M}_C$	T	5	0.059	0.057	-0.002	0.019	0.924
Scenario 2		1	0.458	0.458	0.000	0.094	0.940
$\mathcal{M}_e$	T	2	0.256	0.256	0.000	0.067	0.942
$\mathcal{M}_\pi$	T	3	0.152	0.152	0.000	0.049	0.954
$\mathcal{M}_T$	F	4	0.094	0.093	-0.001	0.038	0.948
$\mathcal{M}_C$	T	5	0.059	0.058	-0.001	0.030	0.966
Scenario 3		1	0.458	0.462	0.003	0.049	0.948
$\mathcal{M}_e$	F	2	0.256	0.257	0.001	0.038	0.934
$\mathcal{M}_\pi$	T	3	0.152	0.152	0.000	0.029	0.942
$\mathcal{M}_T$	T	4	0.094	0.093	-0.001	0.025	0.928
$\mathcal{M}_C$	F	5	0.059	0.058	-0.002	0.020	0.922
Scenario 4		1	0.458	0.461	0.003	0.039	0.950
$\mathcal{M}_e$	T	2	0.256	0.258	0.002	0.034	0.928
$\mathcal{M}_\pi$	F	3	0.152	0.153	0.000	0.029	0.930
$\mathcal{M}_T$	T	4	0.094	0.093	-0.001	0.024	0.930
$\mathcal{M}_C$	F	5	0.059	0.057	-0.002	0.020	0.950
Scenario 5		1	0.458	0.461	0.002	0.064	0.934
$\mathcal{M}_e$	F	2	0.256	0.257	0.001	0.046	0.944
$\mathcal{M}_\pi$	T	3	0.152	0.152	0.000	0.035	0.948
$\mathcal{M}_T$	F	4	0.094	0.093	-0.001	0.029	0.926
$\mathcal{M}_C$	T	5	0.059	0.058	-0.001	0.023	0.946
Scenario 6		1	0.458	0.559	0.100	0.038	0.308
$\mathcal{M}_e$	T	2	0.256	0.337	0.081	0.039	0.428
$\mathcal{M}_\pi$	F	3	0.152	0.211	0.059	0.037	0.574
$\mathcal{M}_T$	F	4	0.094	0.135	0.041	0.033	0.694
$\mathcal{M}_C$	F	5	0.059	0.088	0.029	0.027	0.822
Scenario 7		1	0.458	0.496	0.038	0.038	0.822
$\mathcal{M}_e$	F	2	0.256	0.281	0.025	0.036	0.906
$\mathcal{M}_\pi$	F	3	0.152	0.166	0.014	0.032	0.914
$\mathcal{M}_T$	T	4	0.094	0.101	0.007	0.027	0.926
$\mathcal{M}_C$	F	5	0.059	0.062	0.003	0.022	0.948
Scenario 8		1	0.458	0.558	0.099	0.038	0.322
$\mathcal{M}_e$	F	2	0.256	0.336	0.080	0.039	0.458
$\mathcal{M}_\pi$	F	3	0.152	0.209	0.057	0.037	0.620
$\mathcal{M}_T$	F	4	0.094	0.133	0.039	0.033	0.742
$\mathcal{M}_C$	F	5	0.059	0.086	0.026	0.027	0.844

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 7: Simulation results of the multiply robust estimation of $\mathcal{S}_{1,a}(u)$, where $\mathcal{M}_e, \mathcal{M}_\pi, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, propensity score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{1,a}(u)$	$\widehat{\mathcal{S}}_{1,a}^{mr}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.543	0.542	-0.001	0.032	0.940
$\mathcal{M}_e$	T	2	0.328	0.326	-0.002	0.030	0.946
$\mathcal{M}_\pi$	T	3	0.212	0.210	-0.002	0.028	0.938
$\mathcal{M}_T$	T	4	0.146	0.140	-0.005	0.029	0.904
$\mathcal{M}_C$	T	5	0.105	0.097	-0.008	0.028	0.876
Scenario 2		1	0.543	0.542	-0.001	0.031	0.924
$\mathcal{M}_e$	T	2	0.328	0.324	-0.004	0.030	0.946
$\mathcal{M}_\pi$	T	3	0.212	0.207	-0.006	0.029	0.930
$\mathcal{M}_T$	F	4	0.146	0.136	-0.010	0.029	0.876
$\mathcal{M}_C$	T	5	0.105	0.090	-0.015	0.028	0.818
Scenario 3		1	0.543	0.542	-0.001	0.038	0.932
$\mathcal{M}_e$	F	2	0.328	0.325	-0.002	0.039	0.926
$\mathcal{M}_\pi$	T	3	0.212	0.210	-0.002	0.039	0.920
$\mathcal{M}_T$	T	4	0.146	0.141	-0.005	0.038	0.886
$\mathcal{M}_C$	F	5	0.105	0.099	-0.006	0.038	0.878
Scenario 4		1	0.543	0.543	0.000	0.031	0.930
$\mathcal{M}_e$	T	2	0.328	0.327	-0.001	0.030	0.954
$\mathcal{M}_\pi$	F	3	0.212	0.212	0.000	0.028	0.928
$\mathcal{M}_T$	T	4	0.146	0.144	-0.002	0.026	0.918
$\mathcal{M}_C$	F	5	0.105	0.101	-0.004	0.024	0.924
Scenario 5		1	0.543	0.538	-0.006	0.031	0.940
$\mathcal{M}_e$	F	2	0.328	0.315	-0.013	0.028	0.918
$\mathcal{M}_\pi$	T	3	0.212	0.195	-0.017	0.027	0.870
$\mathcal{M}_T$	F	4	0.146	0.125	-0.021	0.024	0.788
$\mathcal{M}_C$	T	5	0.105	0.081	-0.024	0.023	0.722
Scenario 6		1	0.543	0.545	0.002	0.033	0.938
$\mathcal{M}_e$	T	2	0.328	0.329	0.001	0.032	0.956
$\mathcal{M}_\pi$	F	3	0.212	0.209	-0.004	0.030	0.920
$\mathcal{M}_T$	F	4	0.146	0.135	-0.011	0.028	0.878
$\mathcal{M}_C$	F	5	0.105	0.088	-0.017	0.026	0.838
Scenario 7		1	0.543	0.536	-0.007	0.031	0.926
$\mathcal{M}_e$	F	2	0.328	0.313	-0.015	0.029	0.902
$\mathcal{M}_\pi$	F	3	0.212	0.195	-0.017	0.027	0.864
$\mathcal{M}_T$	T	4	0.146	0.127	-0.019	0.024	0.830
$\mathcal{M}_C$	F	5	0.105	0.085	-0.020	0.022	0.768
Scenario 8		1	0.543	0.540	-0.003	0.030	0.946
$\mathcal{M}_e$	F	2	0.328	0.318	-0.010	0.029	0.928
$\mathcal{M}_\pi$	F	3	0.212	0.196	-0.016	0.027	0.886
$\mathcal{M}_T$	F	4	0.146	0.124	-0.022	0.025	0.812
$\mathcal{M}_C$	F	5	0.105	0.078	-0.027	0.023	0.712

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 8: Simulation results of $\widehat{\mathcal{S}}^1_{0,c}(u)$, where $\mathcal{M}_e, \mathcal{M}_\pi, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, propensity score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{0,c}(u)$	$\widehat{\mathcal{S}}^1_{0,c}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.697	0.695	-0.002	0.052	0.934
$\mathcal{M}_e$	T	2	0.519	0.518	-0.001	0.048	0.948
$\mathcal{M}_\pi$	T	3	0.397	0.397	0.000	0.046	0.932
$\mathcal{M}_T$	T	4	0.310	0.310	0.000	0.040	0.942
$\mathcal{M}_C$	T	5	0.245	0.245	0.000	0.036	0.936
Scenario 2		1	0.697	0.695	-0.002	0.052	0.934
$\mathcal{M}_e$	T	2	0.519	0.518	-0.001	0.048	0.948
$\mathcal{M}_\pi$	T	3	0.397	0.397	0.000	0.046	0.932
$\mathcal{M}_T$	F	4	0.310	0.310	0.000	0.040	0.942
$\mathcal{M}_C$	T	5	0.245	0.245	0.000	0.036	0.936
Scenario 3		1	0.697	1.040	0.344	0.258	0.022
$\mathcal{M}_e$	F	2	0.519	0.797	0.278	0.202	0.024
$\mathcal{M}_\pi$	T	3	0.397	0.624	0.226	0.165	0.038
$\mathcal{M}_T$	T	4	0.310	0.495	0.185	0.134	0.046
$\mathcal{M}_C$	F	5	0.245	0.396	0.151	0.108	0.104
Scenario 4		1	0.697	0.769	0.072	0.062	0.762
$\mathcal{M}_e$	T	2	0.519	0.592	0.073	0.059	0.758
$\mathcal{M}_\pi$	F	3	0.397	0.465	0.068	0.055	0.758
$\mathcal{M}_T$	T	4	0.310	0.371	0.062	0.048	0.760
$\mathcal{M}_C$	F	5	0.245	0.298	0.054	0.044	0.774
Scenario 5		1	0.697	1.040	0.344	0.258	0.022
$\mathcal{M}_e$	F	2	0.519	0.797	0.278	0.202	0.024
$\mathcal{M}_\pi$	T	3	0.397	0.624	0.226	0.165	0.038
$\mathcal{M}_T$	F	4	0.310	0.495	0.185	0.134	0.046
$\mathcal{M}_C$	T	5	0.245	0.396	0.151	0.108	0.104
Scenario 6		1	0.697	0.769	0.072	0.062	0.762
$\mathcal{M}_e$	T	2	0.519	0.592	0.073	0.059	0.758
$\mathcal{M}_\pi$	F	3	0.397	0.465	0.068	0.055	0.758
$\mathcal{M}_T$	F	4	0.310	0.371	0.062	0.048	0.760
$\mathcal{M}_C$	F	5	0.245	0.298	0.054	0.044	0.774
Scenario 7		1	0.697	0.766	0.069	0.023	0.188
$\mathcal{M}_e$	F	2	0.519	0.603	0.084	0.029	0.188
$\mathcal{M}_\pi$	F	3	0.397	0.481	0.084	0.029	0.202
$\mathcal{M}_T$	T	4	0.310	0.389	0.080	0.029	0.238
$\mathcal{M}_C$	F	5	0.245	0.317	0.072	0.029	0.274
Scenario 8		1	0.697	0.766	0.069	0.023	0.188
$\mathcal{M}_e$	F	2	0.519	0.603	0.084	0.029	0.188
$\mathcal{M}_\pi$	F	3	0.397	0.481	0.084	0.029	0.202
$\mathcal{M}_T$	F	4	0.310	0.389	0.080	0.029	0.238
$\mathcal{M}_C$	F	5	0.245	0.317	0.072	0.029	0.274

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 9: Simulation results of $\widehat{\mathcal{S}}^2_{0,c}(u)$, where $\mathcal{M}_e, \mathcal{M}_\pi, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, propensity score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{0,c}(u)$	$\widehat{\mathcal{S}}^2_{0,c}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.697	0.695	-0.002	0.048	0.948
$\mathcal{M}_e$	T	2	0.519	0.520	0.001	0.047	0.936
$\mathcal{M}_\pi$	T	3	0.397	0.401	0.003	0.045	0.950
$\mathcal{M}_T$	T	4	0.310	0.315	0.005	0.040	0.948
$\mathcal{M}_C$	T	5	0.245	0.250	0.005	0.036	0.948
Scenario 2		1	0.697	0.762	0.065	0.171	0.662
$\mathcal{M}_e$	T	2	0.519	0.597	0.078	0.146	0.604
$\mathcal{M}_\pi$	T	3	0.397	0.476	0.079	0.125	0.578
$\mathcal{M}_T$	F	4	0.310	0.386	0.076	0.110	0.572
$\mathcal{M}_C$	T	5	0.245	0.314	0.070	0.097	0.582
Scenario 3		1	0.697	0.697	0.000	0.166	0.916
$\mathcal{M}_e$	F	2	0.519	0.522	0.003	0.126	0.932
$\mathcal{M}_\pi$	T	3	0.397	0.402	0.005	0.100	0.948
$\mathcal{M}_T$	T	4	0.310	0.315	0.006	0.078	0.954
$\mathcal{M}_C$	F	5	0.245	0.250	0.006	0.063	0.958
Scenario 4		1	0.697	0.582	-0.115	0.090	0.772
$\mathcal{M}_e$	T	2	0.519	0.329	-0.190	0.071	0.372
$\mathcal{M}_\pi$	F	3	0.397	0.206	-0.192	0.060	0.216
$\mathcal{M}_T$	T	4	0.310	0.136	-0.174	0.049	0.154
$\mathcal{M}_C$	F	5	0.245	0.093	-0.152	0.041	0.142
Scenario 5		1	0.697	0.758	0.061	0.100	0.668
$\mathcal{M}_e$	F	2	0.519	0.594	0.075	0.091	0.568
$\mathcal{M}_\pi$	T	3	0.397	0.474	0.077	0.083	0.592
$\mathcal{M}_T$	F	4	0.310	0.384	0.074	0.076	0.542
$\mathcal{M}_C$	T	5	0.245	0.313	0.068	0.069	0.588
Scenario 6		1	0.697	1.188	0.491	0.109	0.000
$\mathcal{M}_e$	T	2	0.519	0.932	0.413	0.094	0.000
$\mathcal{M}_\pi$	F	3	0.397	0.748	0.350	0.081	0.000
$\mathcal{M}_T$	F	4	0.310	0.610	0.300	0.072	0.000
$\mathcal{M}_C$	F	5	0.245	0.503	0.258	0.065	0.000
Scenario 7		1	0.697	0.374	-0.323	0.063	0.000
$\mathcal{M}_e$	F	2	0.519	0.212	-0.307	0.051	0.000
$\mathcal{M}_\pi$	F	3	0.397	0.133	-0.264	0.042	0.002
$\mathcal{M}_T$	T	4	0.310	0.088	-0.221	0.034	0.006
$\mathcal{M}_C$	F	5	0.245	0.061	-0.184	0.028	0.006
Scenario 8		1	0.697	0.760	0.063	0.024	0.336
$\mathcal{M}_e$	F	2	0.519	0.596	0.077	0.029	0.266
$\mathcal{M}_\pi$	F	3	0.397	0.478	0.081	0.029	0.192
$\mathcal{M}_T$	F	4	0.310	0.390	0.080	0.029	0.190
$\mathcal{M}_C$	F	5	0.245	0.322	0.077	0.028	0.172

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 10: Simulation results of the estimation of $\widehat{\mathcal{S}}^3_{0,c}(u)$, where $\mathcal{M}_e, \mathcal{M}_\pi, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, propensity score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{0,c}(u)$	$\widehat{\mathcal{S}}^3_{0,c}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.697	0.695	-0.002	0.047	0.930
$\mathcal{M}_e$	T	2	0.519	0.519	0.000	0.046	0.932
$\mathcal{M}_\pi$	T	3	0.397	0.399	0.002	0.044	0.928
$\mathcal{M}_T$	T	4	0.310	0.313	0.004	0.038	0.948
$\mathcal{M}_C$	T	5	0.245	0.249	0.004	0.035	0.936
Scenario 2		1	0.697	0.765	0.068	0.035	0.456
$\mathcal{M}_e$	T	2	0.519	0.601	0.082	0.035	0.364
$\mathcal{M}_\pi$	T	3	0.397	0.480	0.083	0.034	0.360
$\mathcal{M}_T$	F	4	0.310	0.389	0.080	0.033	0.360
$\mathcal{M}_C$	T	5	0.245	0.318	0.073	0.032	0.376
Scenario 3		1	0.697	1.050	0.350	0.263	0.024
$\mathcal{M}_e$	F	2	0.519	0.779	0.260	0.197	0.032
$\mathcal{M}_\pi$	T	3	0.397	0.601	0.203	0.157	0.052
$\mathcal{M}_T$	T	4	0.310	0.472	0.163	0.124	0.076
$\mathcal{M}_C$	F	5	0.245	0.376	0.131	0.100	0.148
Scenario 4		1	0.697	0.694	-0.003	0.041	0.948
$\mathcal{M}_e$	T	2	0.519	0.518	-0.001	0.043	0.938
$\mathcal{M}_\pi$	F	3	0.397	0.398	0.001	0.042	0.926
$\mathcal{M}_T$	T	4	0.310	0.313	0.003	0.037	0.952
$\mathcal{M}_C$	F	5	0.245	0.248	0.003	0.034	0.928
Scenario 5		1	0.697	1.200	0.504	0.307	0.020
$\mathcal{M}_e$	F	2	0.519	0.944	0.425	0.244	0.018
$\mathcal{M}_\pi$	T	3	0.397	0.755	0.358	0.203	0.018
$\mathcal{M}_T$	F	4	0.310	0.613	0.303	0.165	0.018
$\mathcal{M}_C$	T	5	0.245	0.501	0.256	0.136	0.018
Scenario 6		1	0.697	0.764	0.067	0.024	0.298
$\mathcal{M}_e$	T	2	0.519	0.600	0.081	0.030	0.294
$\mathcal{M}_\pi$	F	3	0.397	0.479	0.082	0.031	0.292
$\mathcal{M}_T$	F	4	0.310	0.388	0.079	0.031	0.322
$\mathcal{M}_C$	F	5	0.245	0.317	0.073	0.031	0.342
Scenario 7		1	0.697	0.666	-0.031	0.028	0.804
$\mathcal{M}_e$	F	2	0.519	0.496	-0.023	0.028	0.840
$\mathcal{M}_\pi$	F	3	0.397	0.382	-0.016	0.026	0.910
$\mathcal{M}_T$	T	4	0.310	0.300	-0.009	0.024	0.922
$\mathcal{M}_C$	F	5	0.245	0.239	-0.006	0.023	0.940
Scenario 8		1	0.697	0.763	0.066	0.023	0.250
$\mathcal{M}_e$	F	2	0.519	0.600	0.081	0.029	0.226
$\mathcal{M}_\pi$	F	3	0.397	0.480	0.082	0.030	0.228
$\mathcal{M}_T$	F	4	0.310	0.389	0.080	0.030	0.244
$\mathcal{M}_C$	F	5	0.245	0.318	0.073	0.029	0.270

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 11: Simulation results of the multiply robust estimation of $\mathcal{S}_{0,c}(u)$ in a randomized controlled trial setting, where $\mathcal{M}_e, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{0,c}(u)$	$\widehat{\mathcal{S}}_{0,c}^{mr}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.458	0.460	0.001	0.036	0.960
$\mathcal{M}_e$	T	2	0.255	0.256	0.001	0.033	0.930
$\mathcal{M}_T$	T	3	0.152	0.150	-0.001	0.030	0.926
$\mathcal{M}_C$	T	4	0.094	0.092	-0.002	0.025	0.918
		5	0.059	0.057	-0.002	0.021	0.906
Scenario 2		1	0.458	0.460	0.002	0.036	0.954
$\mathcal{M}_e$	T	2	0.255	0.257	0.002	0.034	0.930
$\mathcal{M}_T$	F	3	0.152	0.151	0.000	0.030	0.920
$\mathcal{M}_C$	T	4	0.094	0.093	-0.001	0.025	0.916
		5	0.059	0.057	-0.002	0.021	0.912
Scenario 3		1	0.458	0.460	0.002	0.036	0.958
$\mathcal{M}_e$	F	2	0.255	0.256	0.001	0.033	0.928
$\mathcal{M}_T$	T	3	0.152	0.151	-0.001	0.030	0.932
$\mathcal{M}_C$	F	4	0.094	0.092	-0.002	0.025	0.922
		5	0.059	0.057	-0.002	0.021	0.908
Scenario 4		1	0.458	0.460	0.002	0.036	0.962
$\mathcal{M}_e$	T	2	0.255	0.256	0.001	0.034	0.930
$\mathcal{M}_T$	T	3	0.152	0.151	-0.001	0.030	0.930
$\mathcal{M}_C$	F	4	0.094	0.092	-0.002	0.026	0.918
		5	0.059	0.057	-0.002	0.021	0.900
Scenario 5		1	0.458	0.460	0.001	0.036	0.958
$\mathcal{M}_e$	F	2	0.255	0.256	0.001	0.034	0.936
$\mathcal{M}_T$	F	3	0.152	0.151	-0.001	0.030	0.916
$\mathcal{M}_C$	T	4	0.094	0.092	-0.002	0.025	0.914
		5	0.059	0.057	-0.002	0.021	0.906
Scenario 6		1	0.458	0.469	0.011	0.037	0.922
$\mathcal{M}_e$	T	2	0.255	0.266	0.011	0.035	0.932
$\mathcal{M}_T$	F	3	0.152	0.160	0.008	0.032	0.936
$\mathcal{M}_C$	F	4	0.094	0.099	0.005	0.027	0.920
		5	0.059	0.062	0.003	0.023	0.922
Scenario 7		1	0.458	0.460	0.002	0.036	0.952
$\mathcal{M}_e$	F	2	0.255	0.256	0.001	0.034	0.934
$\mathcal{M}_T$	T	3	0.152	0.151	-0.001	0.030	0.928
$\mathcal{M}_C$	F	4	0.094	0.092	-0.002	0.026	0.920
		5	0.059	0.057	-0.002	0.021	0.902
Scenario 8		1	0.458	0.468	0.010	0.037	0.930
$\mathcal{M}_e$	F	2	0.255	0.266	0.011	0.035	0.934
$\mathcal{M}_T$	F	3	0.152	0.159	0.008	0.032	0.934
$\mathcal{M}_C$	F	4	0.094	0.099	0.005	0.027	0.914
		5	0.059	0.062	0.003	0.023	0.912

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 12: Simulation results of the multiply robust estimation of $\mathcal{S}_{1,c}(u)$ in a randomized controlled trial setting, where $\mathcal{M}_e, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{1,c}(u)$	$\widehat{\mathcal{S}}_{1,c}^{mr}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.528	0.528	0.000	0.037	0.936
$\mathcal{M}_e$	T	2	0.294	0.292	-0.002	0.035	0.942
$\mathcal{M}_T$	T	3	0.171	0.168	-0.003	0.030	0.940
$\mathcal{M}_C$	T	4	0.103	0.100	-0.003	0.025	0.944
		5	0.064	0.061	-0.003	0.021	0.918
Scenario 2		1	0.528	0.528	0.000	0.037	0.940
$\mathcal{M}_e$	T	2	0.294	0.293	-0.001	0.035	0.942
$\mathcal{M}_T$	F	3	0.171	0.168	-0.003	0.030	0.946
$\mathcal{M}_C$	T	4	0.103	0.100	-0.003	0.024	0.944
		5	0.064	0.061	-0.003	0.021	0.906
Scenario 3		1	0.528	0.527	0.000	0.038	0.934
$\mathcal{M}_e$	F	2	0.294	0.291	-0.003	0.036	0.944
$\mathcal{M}_T$	T	3	0.171	0.166	-0.005	0.031	0.928
$\mathcal{M}_C$	F	4	0.103	0.098	-0.005	0.027	0.928
		5	0.064	0.057	-0.007	0.027	0.918
Scenario 4		1	0.528	0.528	0.000	0.037	0.938
$\mathcal{M}_e$	T	2	0.294	0.292	-0.002	0.035	0.944
$\mathcal{M}_T$	T	3	0.171	0.168	-0.003	0.031	0.952
$\mathcal{M}_C$	F	4	0.103	0.100	-0.002	0.025	0.950
		5	0.064	0.062	-0.002	0.022	0.916
Scenario 5		1	0.528	0.537	0.009	0.031	0.928
$\mathcal{M}_e$	F	2	0.294	0.312	0.018	0.030	0.910
$\mathcal{M}_T$	F	3	0.171	0.189	0.019	0.029	0.942
$\mathcal{M}_C$	T	4	0.103	0.120	0.017	0.026	0.954
		5	0.064	0.077	0.013	0.027	0.932
Scenario 6		1	0.528	0.526	-0.001	0.037	0.938
$\mathcal{M}_e$	T	2	0.294	0.288	-0.006	0.035	0.938
$\mathcal{M}_T$	F	3	0.171	0.161	-0.010	0.030	0.932
$\mathcal{M}_C$	F	4	0.103	0.092	-0.011	0.024	0.916
		5	0.064	0.053	-0.011	0.019	0.856
Scenario 7		1	0.528	0.528	0.000	0.038	0.932
$\mathcal{M}_e$	F	2	0.294	0.292	-0.002	0.038	0.944
$\mathcal{M}_T$	T	3	0.171	0.168	-0.003	0.033	0.926
$\mathcal{M}_C$	F	4	0.103	0.100	-0.002	0.029	0.940
		5	0.064	0.061	-0.003	0.024	0.934
Scenario 8		1	0.528	0.534	0.006	0.031	0.938
$\mathcal{M}_e$	F	2	0.294	0.303	0.009	0.031	0.932
$\mathcal{M}_T$	F	3	0.171	0.177	0.006	0.028	0.946
$\mathcal{M}_C$	F	4	0.103	0.106	0.004	0.023	0.944
		5	0.064	0.065	0.001	0.020	0.916

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 13: Simulation results of the multiply robust estimation of $\mathcal{S}_{0,n}(u)$ in a randomized controlled trial setting, where $\mathcal{M}_e, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{0,n}(u)$	$\widehat{\mathcal{S}}_{0,n}^{mr}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.746	0.746	0.000	0.027	0.942
$\mathcal{M}_e$	T	2	0.580	0.579	-0.001	0.031	0.936
$\mathcal{M}_T$	T	3	0.457	0.457	0.000	0.032	0.942
$\mathcal{M}_C$	T	4	0.364	0.365	0.001	0.033	0.916
		5	0.293	0.294	0.001	0.032	0.926
Scenario 2		1	0.746	0.746	0.000	0.027	0.952
$\mathcal{M}_e$	T	2	0.580	0.578	-0.002	0.032	0.926
$\mathcal{M}_T$	F	3	0.457	0.456	0.000	0.032	0.940
$\mathcal{M}_C$	T	4	0.364	0.365	0.001	0.034	0.910
		5	0.293	0.294	0.000	0.032	0.924
Scenario 3		1	0.746	0.746	0.000	0.026	0.940
$\mathcal{M}_e$	F	2	0.580	0.578	-0.002	0.031	0.940
$\mathcal{M}_T$	T	3	0.457	0.456	0.000	0.032	0.944
$\mathcal{M}_C$	F	4	0.364	0.365	0.001	0.033	0.920
		5	0.293	0.294	0.000	0.031	0.926
Scenario 4		1	0.746	0.746	0.000	0.026	0.948
$\mathcal{M}_e$	T	2	0.580	0.579	-0.001	0.031	0.944
$\mathcal{M}_T$	T	3	0.457	0.457	0.000	0.032	0.932
$\mathcal{M}_C$	F	4	0.364	0.366	0.002	0.033	0.914
		5	0.293	0.294	0.001	0.032	0.924
Scenario 5		1	0.746	0.728	-0.018	0.026	0.880
$\mathcal{M}_e$	F	2	0.580	0.557	-0.022	0.031	0.850
$\mathcal{M}_T$	F	3	0.457	0.436	-0.021	0.032	0.868
$\mathcal{M}_C$	T	4	0.364	0.346	-0.018	0.032	0.874
		5	0.293	0.277	-0.017	0.030	0.880
Scenario 6		1	0.746	0.748	0.002	0.027	0.950
$\mathcal{M}_e$	T	2	0.580	0.583	0.004	0.032	0.934
$\mathcal{M}_T$	F	3	0.457	0.463	0.007	0.032	0.926
$\mathcal{M}_C$	F	4	0.364	0.373	0.009	0.034	0.914
		5	0.293	0.302	0.009	0.033	0.918
Scenario 7		1	0.746	0.746	0.000	0.027	0.944
$\mathcal{M}_e$	F	2	0.580	0.578	-0.002	0.031	0.932
$\mathcal{M}_T$	T	3	0.457	0.457	0.000	0.032	0.942
$\mathcal{M}_C$	F	4	0.364	0.365	0.001	0.033	0.922
		5	0.293	0.294	0.001	0.032	0.930
Scenario 8		1	0.746	0.731	-0.015	0.027	0.896
$\mathcal{M}_e$	F	2	0.580	0.564	-0.016	0.031	0.890
$\mathcal{M}_T$	F	3	0.457	0.444	-0.012	0.032	0.912
$\mathcal{M}_C$	F	4	0.364	0.355	-0.009	0.033	0.908
		5	0.293	0.286	-0.007	0.031	0.904

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 14: Simulation results of the multiply robust estimation of $\mathcal{S}_{1,n}(u)$ in a randomized controlled trial setting, where $\mathcal{M}_e, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{1,n}(u)$	$\widehat{\mathcal{S}}_{1,n}^{mr}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.694	0.692	-0.002	0.032	0.926
$\mathcal{M}_e$	T	2	0.493	0.489	-0.004	0.037	0.942
$\mathcal{M}_T$	T	3	0.356	0.353	-0.003	0.038	0.916
$\mathcal{M}_C$	T	4	0.261	0.259	-0.002	0.036	0.924
		5	0.193	0.192	-0.001	0.034	0.920
Scenario 2		1	0.694	0.692	-0.002	0.033	0.928
$\mathcal{M}_e$	T	2	0.493	0.489	-0.004	0.037	0.942
$\mathcal{M}_T$	F	3	0.356	0.353	-0.003	0.038	0.932
$\mathcal{M}_C$	T	4	0.261	0.259	-0.002	0.036	0.924
		5	0.193	0.192	-0.001	0.034	0.930
Scenario 3		1	0.694	0.692	-0.002	0.032	0.926
$\mathcal{M}_e$	F	2	0.493	0.489	-0.004	0.037	0.946
$\mathcal{M}_T$	T	3	0.356	0.353	-0.003	0.038	0.920
$\mathcal{M}_C$	F	4	0.261	0.259	-0.002	0.036	0.922
		5	0.193	0.192	-0.001	0.034	0.924
Scenario 4		1	0.694	0.692	-0.002	0.032	0.926
$\mathcal{M}_e$	T	2	0.493	0.490	-0.004	0.037	0.940
$\mathcal{M}_T$	T	3	0.356	0.354	-0.003	0.038	0.916
$\mathcal{M}_C$	F	4	0.261	0.259	-0.002	0.036	0.920
		5	0.193	0.193	-0.001	0.034	0.922
Scenario 5		1	0.694	0.692	-0.002	0.033	0.928
$\mathcal{M}_e$	F	2	0.493	0.489	-0.004	0.037	0.944
$\mathcal{M}_T$	F	3	0.356	0.353	-0.003	0.038	0.922
$\mathcal{M}_C$	T	4	0.261	0.259	-0.002	0.036	0.928
		5	0.193	0.192	-0.001	0.034	0.930
Scenario 6		1	0.694	0.693	-0.001	0.033	0.932
$\mathcal{M}_e$	T	2	0.493	0.491	-0.003	0.037	0.942
$\mathcal{M}_T$	F	3	0.356	0.354	-0.003	0.039	0.916
$\mathcal{M}_C$	F	4	0.261	0.258	-0.002	0.036	0.920
		5	0.193	0.191	-0.002	0.034	0.920
Scenario 7		1	0.694	0.692	-0.002	0.033	0.928
$\mathcal{M}_e$	F	2	0.493	0.490	-0.004	0.037	0.940
$\mathcal{M}_T$	T	3	0.356	0.353	-0.003	0.038	0.920
$\mathcal{M}_C$	F	4	0.261	0.259	-0.002	0.036	0.926
		5	0.193	0.193	-0.001	0.034	0.930
Scenario 8		1	0.694	0.693	-0.001	0.033	0.928
$\mathcal{M}_e$	F	2	0.493	0.490	-0.003	0.037	0.940
$\mathcal{M}_T$	F	3	0.356	0.354	-0.003	0.039	0.912
$\mathcal{M}_C$	F	4	0.261	0.258	-0.002	0.036	0.922
		5	0.193	0.191	-0.003	0.034	0.924

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

Web Table 15: Simulation results of the multiply robust estimation of $\mathcal{S}_{0,a}(u)$ in a randomized controlled trial setting, where $\mathcal{M}_e, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

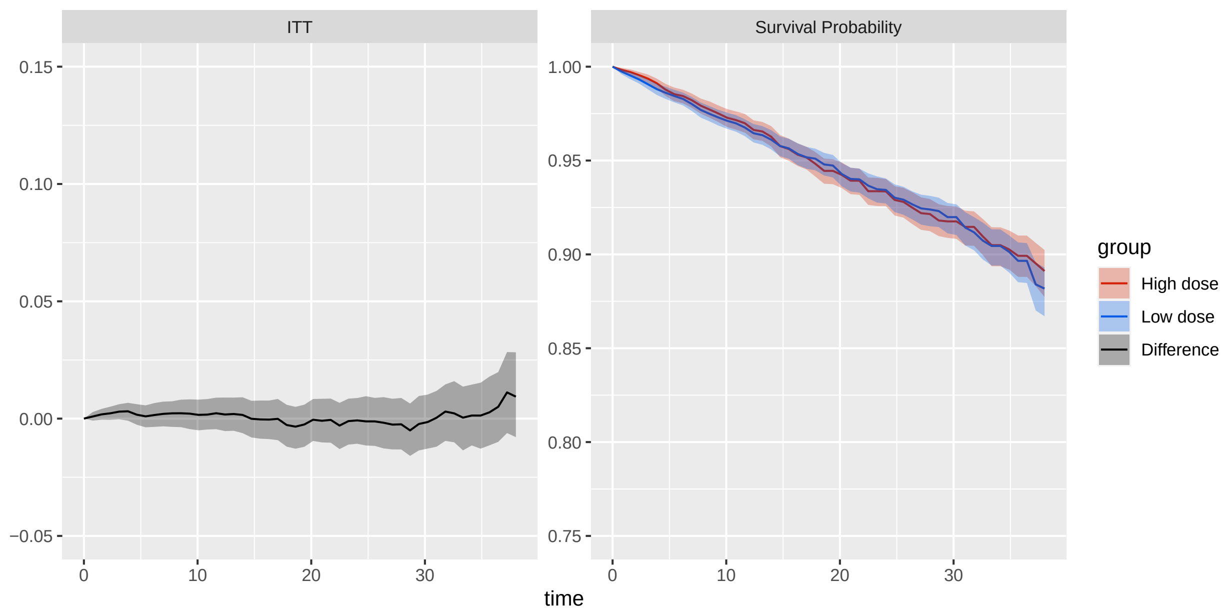
		u	$\mathcal{S}_{0,a}^{mr}(u)$	$\widehat{\mathcal{S}}_{0,a}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.458	0.460	0.002	0.036	0.960
$\mathcal{M}_e$	T	2	0.255	0.256	0.001	0.033	0.928
$\mathcal{M}_T$	T	3	0.152	0.150	-0.001	0.030	0.928
$\mathcal{M}_C$	T	4	0.093	0.092	-0.001	0.025	0.918
		5	0.059	0.057	-0.002	0.021	0.906
Scenario 2		1	0.458	0.460	0.002	0.036	0.954
$\mathcal{M}_e$	T	2	0.255	0.257	0.001	0.034	0.932
$\mathcal{M}_T$	F	3	0.152	0.151	0.000	0.030	0.920
$\mathcal{M}_C$	T	4	0.093	0.093	-0.001	0.025	0.916
		5	0.059	0.057	-0.002	0.021	0.912
Scenario 3		1	0.458	0.460	0.002	0.036	0.960
$\mathcal{M}_e$	F	2	0.255	0.256	0.001	0.033	0.928
$\mathcal{M}_T$	T	3	0.152	0.151	-0.001	0.030	0.932
$\mathcal{M}_C$	F	4	0.093	0.092	-0.001	0.025	0.924
		5	0.059	0.057	-0.002	0.021	0.908
Scenario 4		1	0.458	0.460	0.002	0.036	0.962
$\mathcal{M}_e$	T	2	0.255	0.256	0.001	0.034	0.932
$\mathcal{M}_T$	T	3	0.152	0.151	-0.001	0.030	0.928
$\mathcal{M}_C$	F	4	0.093	0.092	-0.002	0.026	0.920
		5	0.059	0.057	-0.002	0.021	0.900
Scenario 5		1	0.458	0.460	0.002	0.036	0.958
$\mathcal{M}_e$	F	2	0.255	0.256	0.001	0.034	0.936
$\mathcal{M}_T$	F	3	0.152	0.151	-0.001	0.030	0.916
$\mathcal{M}_C$	T	4	0.093	0.092	-0.001	0.025	0.916
		5	0.059	0.057	-0.002	0.021	0.906
Scenario 6		1	0.458	0.469	0.011	0.037	0.922
$\mathcal{M}_e$	T	2	0.255	0.266	0.011	0.035	0.934
$\mathcal{M}_T$	F	3	0.152	0.160	0.008	0.032	0.936
$\mathcal{M}_C$	F	4	0.093	0.099	0.006	0.027	0.926
		5	0.059	0.062	0.003	0.023	0.922
Scenario 7		1	0.458	0.460	0.002	0.036	0.952
$\mathcal{M}_e$	F	2	0.255	0.256	0.001	0.034	0.934
$\mathcal{M}_T$	T	3	0.152	0.151	-0.001	0.030	0.928
$\mathcal{M}_C$	F	4	0.093	0.092	-0.001	0.026	0.922
		5	0.059	0.057	-0.002	0.021	0.904
Scenario 8		1	0.458	0.468	0.011	0.037	0.928
$\mathcal{M}_e$	F	2	0.255	0.266	0.011	0.035	0.934
$\mathcal{M}_T$	F	3	0.152	0.159	0.008	0.032	0.934
$\mathcal{M}_C$	F	4	0.093	0.099	0.005	0.027	0.918
		5	0.059	0.062	0.003	0.023	0.912

Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.

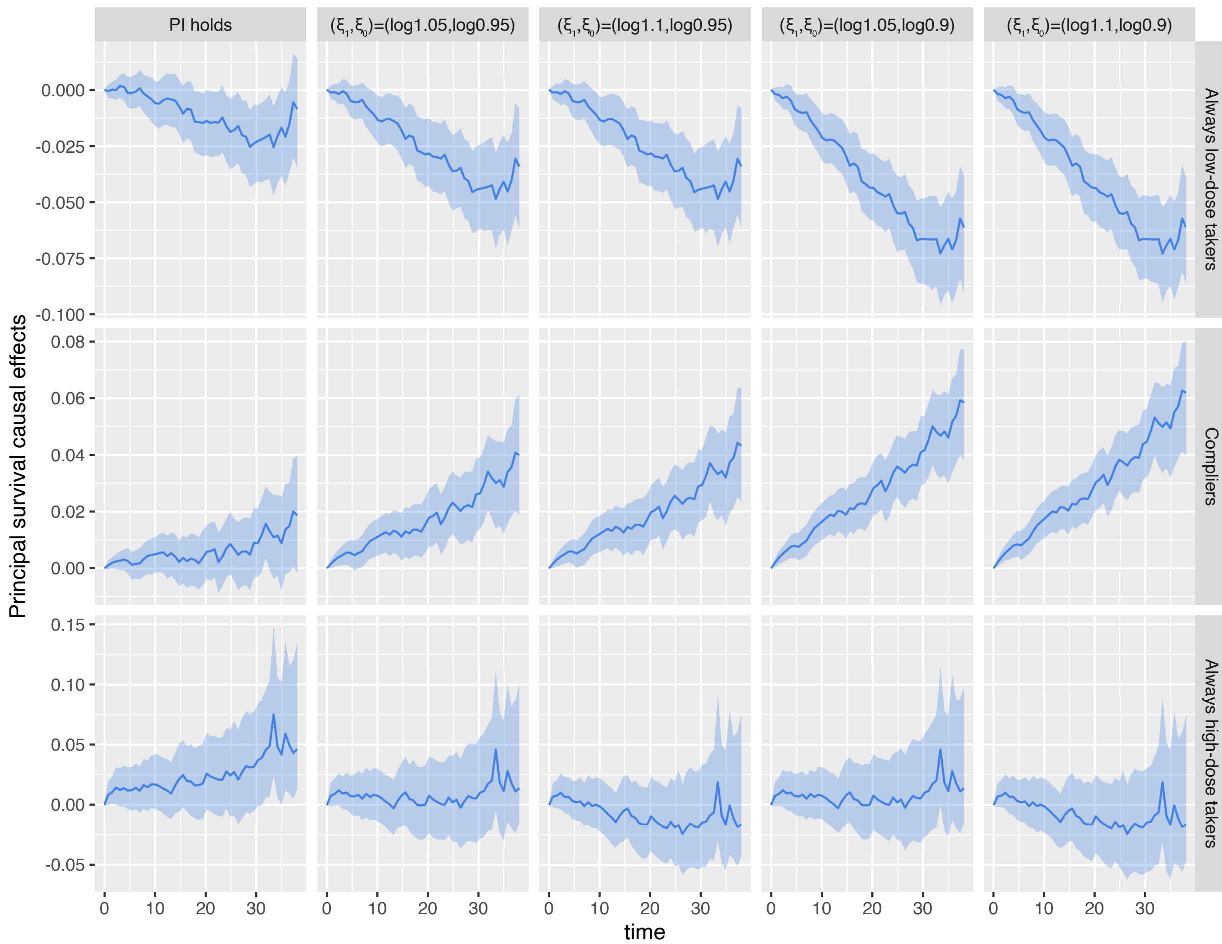
Web Table 16: Simulation results of the multiply robust estimation of $\mathcal{S}_{1,a}(u)$ in a randomized controlled trial setting, where $\mathcal{M}_e, \mathcal{M}_T$ and $\mathcal{M}_C$ indicate whether the working model for principal score, time-to-event and time-to-censoring is correctly specified (denoted by a 'T' label) or not (denoted by a 'F' label).

		u	$\mathcal{S}_{1,a}(u)$	$\widehat{\mathcal{S}}^{mr}_{1,a}(u)$	Bias	Monte Carlo SE	Coverage Rate
Scenario 1		1	0.543	0.542	-0.001	0.031	0.942
$\mathcal{M}_e$	T	2	0.328	0.325	-0.003	0.031	0.936
$\mathcal{M}_T$	T	3	0.212	0.207	-0.006	0.030	0.918
$\mathcal{M}_C$	T	4	0.145	0.139	-0.006	0.030	0.902
		5	0.105	0.096	-0.009	0.031	0.878
Scenario 2		1	0.543	0.542	-0.001	0.031	0.940
$\mathcal{M}_e$	T	2	0.328	0.323	-0.005	0.032	0.940
$\mathcal{M}_T$	F	3	0.212	0.203	-0.010	0.033	0.896
$\mathcal{M}_C$	T	4	0.145	0.132	-0.013	0.031	0.856
		5	0.105	0.087	-0.018	0.031	0.806
Scenario 3		1	0.543	0.543	-0.001	0.030	0.934
$\mathcal{M}_e$	F	2	0.328	0.325	-0.002	0.030	0.932
$\mathcal{M}_T$	T	3	0.212	0.208	-0.005	0.030	0.914
$\mathcal{M}_C$	F	4	0.145	0.140	-0.005	0.029	0.898
		5	0.105	0.098	-0.007	0.028	0.882
Scenario 4		1	0.543	0.544	0.000	0.031	0.936
$\mathcal{M}_e$	T	2	0.328	0.327	0.000	0.030	0.942
$\mathcal{M}_T$	T	3	0.212	0.211	-0.002	0.029	0.928
$\mathcal{M}_C$	F	4	0.145	0.143	-0.002	0.027	0.926
		5	0.105	0.102	-0.003	0.026	0.910
Scenario 5		1	0.543	0.537	-0.006	0.030	0.932
$\mathcal{M}_e$	F	2	0.328	0.312	-0.016	0.029	0.884
$\mathcal{M}_T$	F	3	0.212	0.190	-0.022	0.029	0.838
$\mathcal{M}_C$	T	4	0.145	0.120	-0.025	0.026	0.748
		5	0.105	0.078	-0.027	0.025	0.692
Scenario 6		1	0.543	0.538	-0.005	0.031	0.944
$\mathcal{M}_e$	T	2	0.328	0.311	-0.017	0.032	0.900
$\mathcal{M}_T$	F	3	0.212	0.187	-0.026	0.029	0.808
$\mathcal{M}_C$	F	4	0.145	0.115	-0.030	0.025	0.712
		5	0.105	0.072	-0.033	0.022	0.636
Scenario 7		1	0.543	0.544	0.000	0.031	0.926
$\mathcal{M}_e$	F	2	0.328	0.327	-0.001	0.030	0.932
$\mathcal{M}_T$	T	3	0.212	0.210	-0.002	0.029	0.924
$\mathcal{M}_C$	F	4	0.145	0.143	-0.002	0.027	0.906
		5	0.105	0.102	-0.003	0.026	0.910
Scenario 8		1	0.543	0.534	-0.010	0.031	0.924
$\mathcal{M}_e$	F	2	0.328	0.303	-0.025	0.030	0.834
$\mathcal{M}_T$	F	3	0.212	0.177	-0.035	0.027	0.726
$\mathcal{M}_C$	F	4	0.145	0.107	-0.038	0.023	0.604
		5	0.105	0.065	-0.039	0.020	0.496

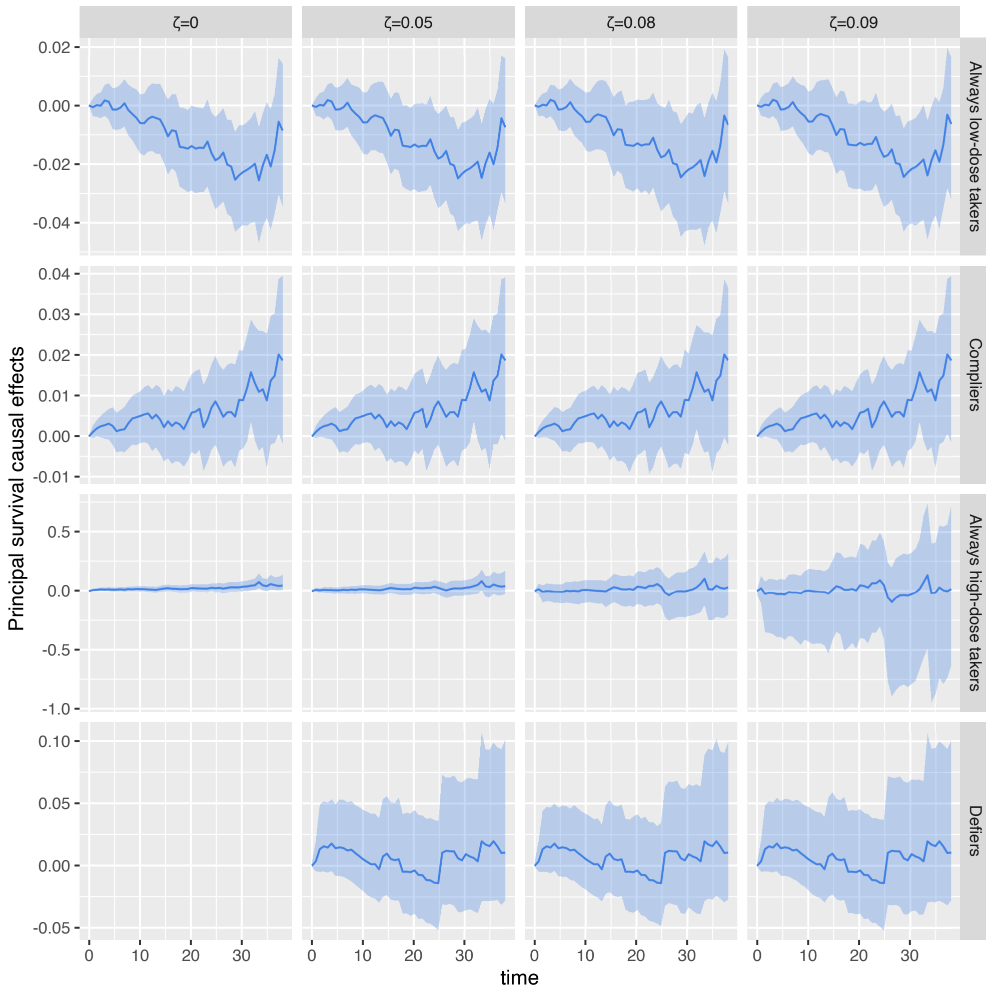
Monte Carlo SE: empirical standard error; Coverage Rate: 95% bootstrap confidence interval coverage rate.



Web Figure 1: The ITT effects and corresponding survival probability curve, ADAPTABLE trial, 2016–2020.



Web Figure 2: Sensitivity of the PSCEs to the violation of the principal ignorability assumption, in the scenario that the always low-dose takers are healthiest group and the always high-dose takers are the unhealthiest group. The PSCEs under principal ignorability assumption are added in the first column as benchmark.



Web Figure 3: Sensitivity of the principal survival causal effects to the violation of the monotonicity assumption. Point estimates and 95% confidence intervals for the principal survival causal effects (PSCEs) among $\zeta \in \{0, 0.05, 0.08, 0.09\}$ are provided, where $\zeta = 0$ indicates that monotonicity assumption holds. Notice that the defiers does not exists when $\zeta = 0$ and therefore the PSCEs among defiers are not defined under $\zeta = 0$.

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