Sharp equivalence constants between standard deviation and maximum slope

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(This talk includes work with *Konrad Aguilar*)

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Seminorms

Definition (Norm and seminorm)

Let V be a vector space over $\mathbb{R}$ with zero denoted 0_V. A *seminorm* s on V is a function $s : V \to [0, \infty)$ such that for all $u, v \in V$ and $\alpha \in \mathbb{R}$, we have

1. $s(0_V) = 0$
2. $s(\alpha u) = |\alpha| s(u)$
3. $s(u + v) \leqslant s(u) + s(v)$.

If furthermore $s(u) = 0$ implies that $u = 0_V$, then we call s a *norm*.

Norms and seminorms capture important information of our vectors. For instance, given a vector $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$, the norm

$$\|x\|_2 = \sqrt{\sum_{k=1}^{N} x_k^2}$$

captures the standard Euclidean length of the vector.

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Whereas the norm

$$\|x\|_1 = \sum_{k=1}^{N} |x_k|$$

captures the average value of the entries up to scaling by $\frac{1}{N}$.

Equivalence constants

Although norms calculate different structure, it's important to be able to compare the norms using *equivalence constants*.

Definition (Equivalence constants)

Let s, L be two seminorms on a vector space V over $\mathbb{R}$ such that $s(u) = 0 \iff L(u) = 0$ (this condition is here otherwise s and L would be incomparable). If there exists $\alpha, \beta > 0$ such that

$$\alpha L(u) \leqslant s(u) \leqslant \beta L(u)$$

for all $u \in V$, then we call α, β *equivalence constants*

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We have the following convenient result from Functional Analysis.

Theorem

Let V be a *finite-dimensional* vector space over $\mathbb{R}$. If s, L are two seminorms on V such that $s(u) = 0 \iff L(u) = 0$, then there exist equivalence constants $\alpha, \beta > 0$.

Equivalence constants for $\|x\|_2 = \sqrt{\sum_{k=1}^N x_k^2}$ *and* $\|x\|_1 = \sum_{k=1}^N |x_k|$

These classic norms satisfy

$$\|x\|_2 \leqslant \|x\|_1 \leqslant \sqrt{N}\|x\|_2$$

for all $x \in \mathbb{R}^N$, and so $1, \sqrt{N}$ are equivalence constants. But, one may ask, are these the best? For example, maybe there exists a $0 < \beta < \sqrt{N}$ such that

$$\|x\|_1 \leqslant \beta\|x\|_2$$

which would gives a sharper estimate. The answer is no. This can be verified by the vector

$$\|(1,1,\ldots,1)\|_1 = N = \sqrt{N}\cdot\sqrt{N} = \sqrt{N}\|(1,1,\ldots,1)\|_2$$

and so we can't do better than $\sqrt{N}$ for all $x \in \mathbb{R}^N$. The vector $x = (1,0,\ldots,0)$ gets $\|x\|_2 = \|x\|_1$.

Equivalence constants for $\|x\|_2 = \sqrt{\sum_{k=1}^N x_k^2}$ *and* $\|x\|_1 = \sum_{k=1}^N |x_k|$

These classic norms satisfy

$$\|x\|_2 \leqslant \|x\|_1 \leqslant \sqrt{N}\|x\|_2$$

for all $x \in \mathbb{R}^N$, and so $1, \sqrt{N}$ are the "best" equivalence constants. We have a special name for these.

Definition

Let V be a vector space over $\mathbb{R}$. Let s, L be two seminorms on V such that $s(u) = 0 \iff L(u) = 0$ such that there exist $\alpha, \beta > 0$ such that for all $u \in V$

$$\alpha L(u) \leqslant s(u) \leqslant \beta L(u).$$

We say that α, β are *sharp equivalence constants* if there exist $v, w \in V$ such that

$$\alpha L(v) = s(v) \quad \text{and} \quad s(w) = \beta L(w).$$

Goals

Standard deviation and *maximum slope* from seminorms on the (finite-dimensional) vector space of real-valued functions on a fixed finite set.

We aim to compare standard deviation and maximum slope since they both measure how *non-constant* a function is since these seminorms are zero only on constant functions. We will do the following:

1. Establish equivalence constants α and β on a finite set.
2. Show sharpness on $X = \{0, 1, 2, \ldots, n\}$.
3. Show sharpness on $X = \{a, b, c\}$ where $a, b, c \in \mathbb{R}_{\geqslant 0}$.

A few definitions

Definition (Standard deviation and maximum slope)

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of real numbers and let f be a real-valued function on X. The *standard deviation* is

$$SD(f) = \max\{|f(x_k) - avg(f)| : k = 1, 2, \ldots, n\}$$

where $avg(f) = \frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n}$. And the *maximum slope* is

$$L_d(f) = \max\left\{\frac{|f(x_j)-f(x_k)|}{|x_j-x_k|} : j, k = 1, 2, \ldots, n, j \neq k\right\}$$

α_n and β_n

Since the vector space of real-valued functions on a fixed finite set is finite dimensional, we have from the intro...

Theorem

Let $n \in \mathbb{N}$ and let $X = \{x_1, x_2, \ldots, x_n\} \subseteq \mathbb{R}$. There exist $\alpha_n, \beta_n > 0$ such that $\alpha_n L_d(f) \leqslant SD(f) \leqslant \beta_n L_d(f)$ for all real-valued functions f defined on X.

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Theorem (Aguilar-Burns, 22)

The following constants satisfy the above theorem.

$$\alpha_n = \frac{1}{2}\min\{|x_j - x_k| : j, k = 1, 2, \ldots, n, j \neq k\}$$

and

$$\beta_n = \frac{1}{n}\max\Gamma_n$$

where $\Gamma_n = \left\{\sum_{j=1, j\neq k}^{n} |x_k - x_j| : k = 1, 2, \ldots, n\right\}$

Proof.

There exist $j, k \in \mathbb{N}$ such that $j \neq k$ and

$$\frac{|f(x_j) - f(x_k)|}{|x_j - x_k|} = L_d(f)$$

$$\frac{|f(x_j) - f(x_k)|}{|x_j - x_k|} = \frac{|f(x_j) - avg(f) + avg(f) - f(x_k)|}{|x_j - x_k|}$$

Triangle inequality:

$$\leqslant \frac{|f(x_j) - avg(f)| + |avg(f) - f(x_k)|}{|x_j - x_k|}$$

Note that $SD(f)$ defined as a max:

$$\leqslant \frac{SD(f) + SD(f)}{|x_j - x_k|}$$

Proof.

$$= \frac{2}{|x_j - x_k|} SD(f)$$

$$L_d(f) \leqslant \max\left\{\frac{2}{|x_l - x_m|} : l, m = 1, \ldots, n \text{ and } l \neq m\right\} SD(f)$$

$$SD(f) \geqslant \frac{1}{\max\left\{\frac{2}{|x_l - x_m|} : l, m = 1, \ldots, n \text{ and } l \neq m\right\}} L_d(f)$$

Thus, we have that:

$$\alpha_n = \frac{1}{2} \min\{|x_l - x_m| : l, m = 1, \ldots, n, l \neq m\}$$

□

Proof.

For β_n, we consider

$$
\begin{aligned}
SD(f) &= \max\{|f(x_l) - avg(f)| : l = 1, 2, \ldots, n\} \\
&= |f(x_k) - avg(f)| \\
&= \left| f(x_k) - \frac{\sum_{j=1}^{n} f(x_j)}{n} \right| \\
&= \frac{1}{n} \left| (n-1) f(x_k) - \sum_{j=1, j\neq k}^{n} f(x_j) \right|
\end{aligned}
$$

Note that there are the same number of $f(x_k)$ as there are in the sum, $n-1$

$$
= \frac{1}{n} \left| \sum_{j=1, j\neq k}^{n} [f(x_k) - f(x_j)] \right|
$$

□

Intro

Finding equivalence constants

Sharpness on natural numbers

Sharpness on X={a,b,c}

Proof.

Triangle inequality:

$$\leqslant \frac{1}{n} \sum_{j=1, j\neq k}^{n} |f(x_k) - f(x_j)|$$

Multiply by 1:

$$= \frac{1}{n} \sum_{j=1, j\neq k}^{n} \left[|x_k - x_j| \frac{|f(x_k) - f(x_j)|}{|x_k - x_j|} \right]$$

By definition of $L_d(f)$ and max:

$$\leqslant \frac{1}{n} \sum_{j=1, j\neq k}^{n} [|x_k - x_j| L_d(f)]$$

Proof.

Denote $\Gamma_n = \left\{\sum_{j=1, j\neq k}^{n} |x_k - x_j| : k = 1, 2, \ldots, n\right\}$

$$SD(f) \leqslant \frac{L_d(f)}{n} \max \Gamma_n$$
$$\beta_n = \frac{1}{n} \max \Gamma_n$$

$\square$

$\alpha L_d(f) = SD(f)$ for $X = \{0, 1, 2, \ldots, n\}$

Recall the equivalence statement $\alpha_n L_d(f) \leqslant SD(f) \leqslant L_d(f)\beta_n$.
$SD(f) = \alpha_{n+1} L_d(f)$

$$\begin{aligned}\alpha_{n+1} &= \frac{1}{2}\min\{|x_j - x_k| : j, k = 1, 2, \ldots, n, n+1, j \neq k\} \\ &= \frac{1}{2}\end{aligned}$$

So, we want that $SD(f) = \frac{1}{2} L_d(f)$. Consider any $r > 0$. For

$$f = \{(0, r), (1, r), (2, r), \ldots, (n-2, r), (n-1, \frac{1}{2} r), (n, \frac{3}{2} r)\}$$

Then $L_d(f) = r$ and $SD(f) = \frac{r}{2} = \frac{1}{2} L_d(f)$.

$SD(f) = \beta L_d(f)$ for $X = \{0, 1, 2, \ldots, n\}$

$$\beta_{n+1} = \frac{1}{n+1} \max \Gamma_{n+1}$$

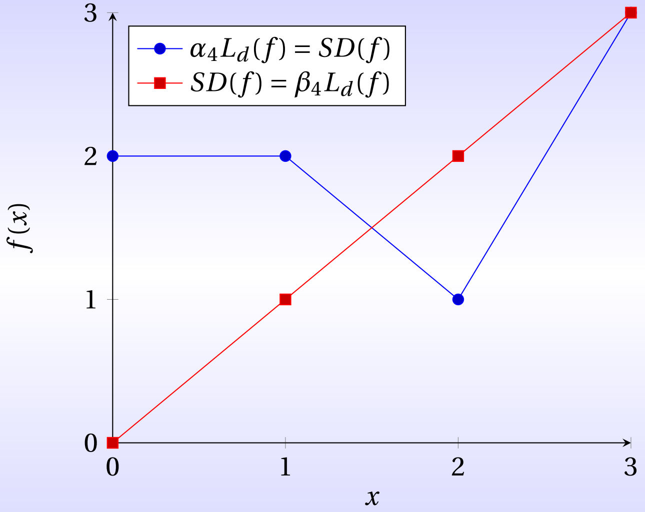
$$\begin{aligned} \max \Gamma_{n+1} &= \max\left\{ \sum_{j=1, j\neq k}^{n+1} |x_k - x_j| : k = 1, 2, \ldots, n, n+1 \right\} \\ &= 1 + 2 + \ldots + n \\ &= \frac{n(n+1)}{2} \end{aligned}$$

$$\beta_{n+1} = \frac{1}{n+1} \frac{n(n+1)}{2} = \frac{n}{2}$$

So, we want that $SD(f) = \frac{n}{2} L_d(f)$. Consider any $r > 0$. For

$$f = \{(0,0), (1, r), (2, 2r), \ldots, (n, nr)\}$$

Then $L_d(f) = r$ and $SD(f) = \frac{rn}{2} = \frac{n}{2} L_d(f)$


$\alpha_4 L_d(f) = SD(f)$
$SD(f) = \beta_4 L_d(f)$
3
2
1
0
0
1
2
3
x
$f(x)$

$\alpha L_d(f) = SD(f)$ for $X = \{a, b, c\}$

Let $b \in (a, \frac{a+c}{2}]$.

$$\alpha_3 = \frac{1}{2}\min\{|a-b|, |a-c|, |b-c|\}$$
$$= \frac{1}{2}\min\{b-a, c-a, c-b\} = \frac{b-a}{2}$$

So we want to satisfy $\frac{b-a}{2}L_d(f) = SD(f)$. For

$$f = \{(a,0), (b,2), (c,1)\}$$

Then,

$$L_d(f) = \max\left\{\frac{|2-0|}{|b-a|}, \frac{|1-0|}{|c-a|}, \frac{|1-2|}{|c-b|}\right\}$$
$$= \max\left\{\frac{2}{b-a}, \frac{1}{c-a}, \frac{1}{c-b}\right\} = \frac{2}{b-a}$$

and $SD(f) = \left(\frac{b-a}{2}\right)\left(\frac{2}{b-a}\right) = 1$. And $avg(f) = \frac{0+1+2}{3} = 1 = SD(f)$.

Let $b \in [\frac{a+c}{2}, c)$.

$$\alpha_3 = \frac{1}{2}\min\{|b-a|, |c-a|, |c-b|\} = \frac{c-b}{2}$$

So we want to satisfy $\frac{c-b}{2} L_d(f) = SD(f)$. For

$$f = \{(a,1),(b,2),(c,0)\}$$

Then

$$\begin{aligned} L_d(f) &= \max\left\{\frac{|2-1|}{|b-a|}, \frac{|0-1|}{|c-a|}, \frac{|0-2|}{|c-b|}\right\} \\ &= \max\left\{\frac{1}{b-a}, \frac{1}{c-a}, \frac{2}{c-b}\right\} = \frac{2}{c-b} \end{aligned}$$

and $SD(f) = \left(\frac{c-b}{2}\right)\left(\frac{2}{c-b}\right) = 1$. And $avg(f) = \frac{1+2+0}{3} = 1 = SD(f)$.

$SD(f) = \beta L_d(f)$ for $X = \{a, b, c\}$

Let $b \in (a, \frac{a+c}{2}]$.

$$\begin{aligned}
\beta_3 &= \frac{1}{3}\max\{|a-b|+|a-c|, |b-a|+|b-c|, |c-a|+|c-b|\} \\
&= \frac{|c-a|+|c-b|}{3} \\
&= \frac{2c-a-b}{3}
\end{aligned}$$

So we want to satisfy $SD(f) = \frac{2c-a-b}{3} L_d(f)$. For

$$f = \{(a,a), (b,b), (c,c)\}$$

Then $L_d(f) = 1$ and so $SD(f) = \frac{2c-a-b}{3}$. Note that $avg(f) = \frac{a+b+c}{3}$.
Thus $SD(f) = \left|c - \frac{a+b+c}{3}\right| = \left|\frac{3c-a-b-c}{3}\right| = \frac{2c-a-b}{3}$.

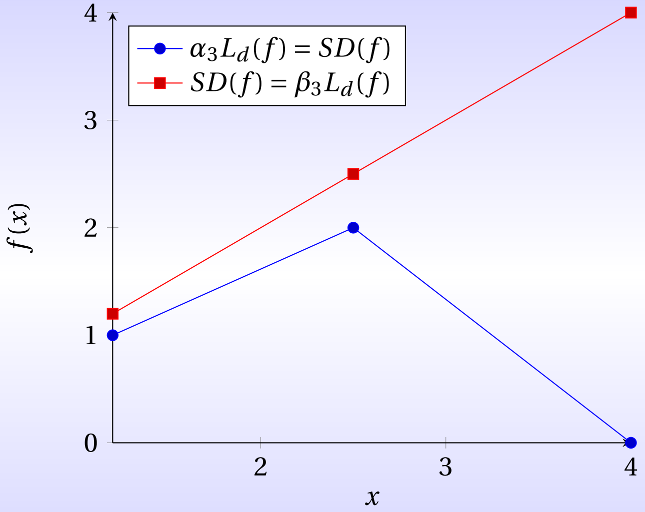
Let $b \in [\frac{a+c}{2}, c)$.

$$\begin{aligned}\beta_3 &= \frac{1}{3}\max\{|a-b|+|a-c|, |b-a|+|b-c|, |c-a|+|c-b|\}\\ &= \frac{|a-b|+|a-c|}{3}\\ &= \frac{c+b-2a}{3}\end{aligned}$$

So we want to satisfy $SD(f) = \frac{c+b-2a}{3}L_d(f)$. For

$$f = \{(a,a),(b,b),(c,c)\}$$

Then $L_d(f) = 1$ and so $SD(f) = \frac{c+b-2a}{3}$. Note that $avg(f) = \frac{a+b+c}{3}$. Thus $SD(f) = \left|a - \frac{a+b+c}{3}\right| = \frac{a+b+c-3a}{3} = \frac{c+b-2a}{3}$.



$\alpha_3 L_d(f) = SD(f)$
$SD(f) = \beta_3 L_d(f)$
$f(x)$
x
0
1
2
3
4
2
3
4

Thank you!

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