Equivalence constants between deviations and maximum slope

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Introduction

Given a real-valued function, its (maximum or absolute) deviation and maximum slope both calculate how far the function is from not being constant. If we consider a fixed finite subset of the reals and the family of real-valued functions on that set, its deviation and maximum slope must be equivalent as seminorms by finite-dimensionality. Our work focuses on the study of equivalence constants (with respect to both maximum and absolute deviation) for any fixed finite subset of reals.

Norms and Seminorms

Let V be a vector space over $\mathbb{R}$ with zero denoted 0_V. A **seminorm** s on V is a function $s : V \to [0, \infty)$ such that for all $u, v \in V$ and $\alpha \in \mathbb{R}$, we have

1. $s(0_v) = 0$
2. $s(\alpha u) = |\alpha| s(u)$
3. $s(u + v) \leq s(u) + s(v)$

If $s(u) = 0$ implies that $u = 0_V$, then we call s a **norm**.

Equivalence Constants

Let s, L be two seminorms on a vector space V over $\mathbb{R}$ such that $s(u) = 0 \iff L(u) = 0$. If there exists

$$\alpha L(u) \leq s(u) \leq \beta L(u)$$

for all $u \in V$, then we call α, β **equivalence constants**. α, β are **sharp equivalence constants** if there exist $v, w \in V$ such that

$$\alpha L(v) = s(v) \text{ and } s(w) = \beta L(w)$$

Additionally, if V is a finite dimensional vector space, then there exist equivalence constants $\alpha, \beta > 0$.

Maximum Slope

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of $\mathbb{R}$. Consider the vector space of real-functions on X with pointwise addition and scalar multiplication, denoted $F(X, \mathbb{R})$.

For every $f \in F(X, \mathbb{R})$, we denote its **maximum slope** by

$$L(f) = \max\left\{ \frac{|f(x_j) - f(x_k)|}{|x_j - x_k|} : j, k \in \{1, 2, \ldots, n\} \right\}.$$

L is a seminorm on $F(X, \mathbb{R})$ such that

$$L(f) = 0$$

if and only if f is a constant function.

Deviations and Equivalence Constants

Maximum Deviation

Let $F(X, \mathbb{R})$ be a vector space of real-valued functions on a finite subset of $\mathbb{R}$. The **maximum deviation** D_∞ is

$$D_\infty = \|f - avg(f)\|_\infty = \max\{|f(x_k) - avg(f)| : k = 1, 2, \ldots, n\}$$

where

$$avg(f) = \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n}$$

When f is constant, $f = avg(f) \Rightarrow D_\infty(f) = 0$. Since $F(X, \mathbb{R})$ is finite dimensional, we found the following equivalence relation:

$$\alpha_n L(f) \leq D_\infty(f) \leq \beta_n L(f)$$

where

$$\alpha_n = \frac{1}{2} \min\{|x_j - x_k| : j, k = 1, 2, \ldots, n, j \neq k\}$$

$$\beta_n = \frac{1}{n} \max\left\{ \sum_{j=1, j \neq k}^{n} |x_j - x_k| : k = 1, 2, \ldots, n \right\}$$

Absolute Deviation

We define **absolute deviation** D_1 on the finite dimensional vector space $F(X, \mathbb{R})$

$$D_1 = \|f - avg(f)\|_1 = \sum_{k=1}^{N} |f(x_k) - avg(f)|$$

When f is constant, $f = avg(f) \Rightarrow D_1(f) = 0$. We found the following equivalence relation

$$\alpha_n L(f) \leq D_1(f) \leq \beta_n L(f)$$

where

$$\alpha_n = \frac{1}{2} \min\{|x_j - x_k| : j, k = 1, 2, \ldots, n, j \neq k\}$$

$$\beta_n = \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1, j \neq k}^{n} |x_j - x_k|$$

An Example: $f(x) = \sqrt{x}$

Limit of α_n

Let $X = \{1, \frac{1}{2}, \ldots, \frac{1}{2^n}\}$. We have the following relationship from above

$$\frac{1}{2^{n+1}} L(f) \leq D_\infty(f) \leq \left(1 - \frac{1}{2^n}\right) L(f)$$

Let $f(x) = \sqrt{x}$ be defined on X. Thus,

$$L(f) = 2^{\frac{n}{2}}(\sqrt{2} - 1) \Rightarrow \alpha_n L(f) = \frac{\sqrt{2} - 1}{2^{\frac{n+2}{2}}}$$

Now, we look at the limits as n approaches infinity. We note that the limit of α_n is zero.

$$\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \frac{1}{2^{n+1}} = 0$$

And the limit of $L(f)$ is infinity

$$\lim_{n \to \infty} L(f) = \lim_{n \to \infty} 2^{\frac{n}{2}}(\sqrt{2} - 1) = \infty$$

Putting the two limits together, the limit is zero.

$$\lim_{n \to \infty} \alpha_n L(f) = \lim_{n \to \infty} \frac{\sqrt{2} - 1}{2^{\frac{n+2}{2}}} = 0$$

Maximum Deviation

Finally, we show that $D_\infty(f)$ must be finite on X

$$\begin{aligned} D_\infty(f) &= \max\{|f(x_k) - avg(f)| : k = 0, 1, 2, \ldots, n\} \\ &= |f(1) - avg(f)| \\ &= \sqrt{1} - \frac{\sqrt{1} + \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{4}} + \ldots + \sqrt{\frac{1}{2^n}}}{n+1} \\ &= 1 - \frac{1}{n+1} \frac{\sqrt{2^{1-n}} - 2}{\sqrt{2} - 2} \end{aligned}$$

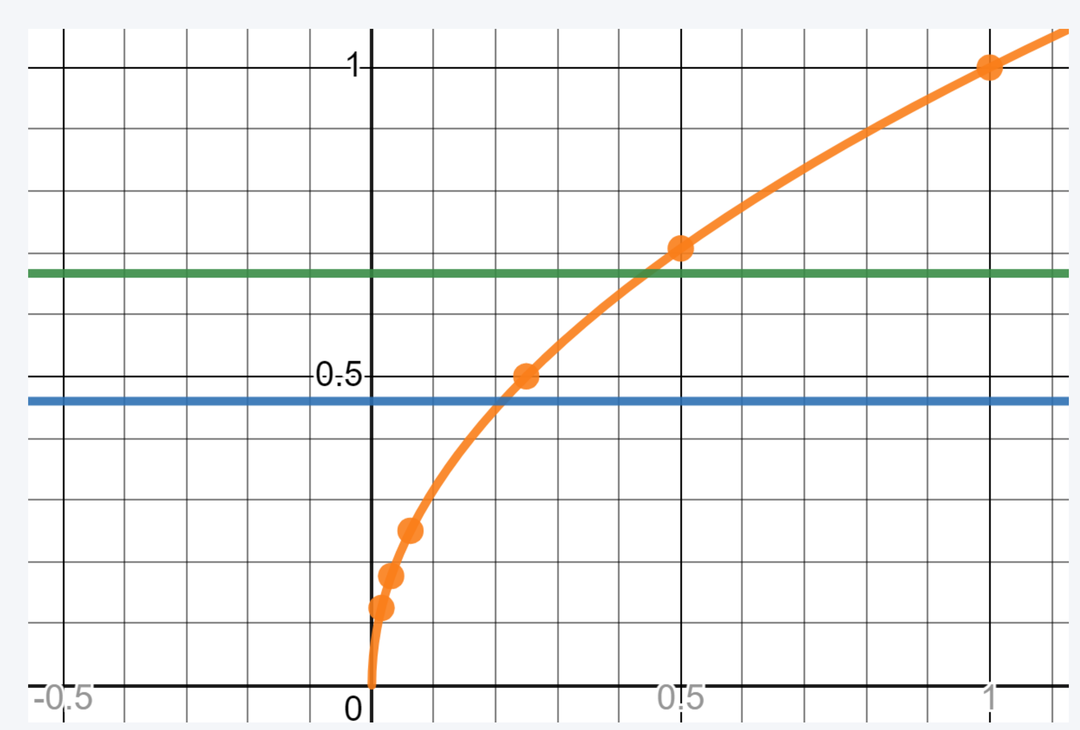

Fig 1. Comparing the average with finite points versus infinite points.

Future Work

We have begun work on refining the β values by testing the too-large values on multiple different sets. We tested the β values on $X = \{3, 9, 27\}, X = \{5, 25, 125\}$, and $X = \{4, 16, 64\}$.

For $X = \{3, 9, 27\}$, we calculated a β_3 value of 32 which is too large by a factor of $\frac{7}{8}$.

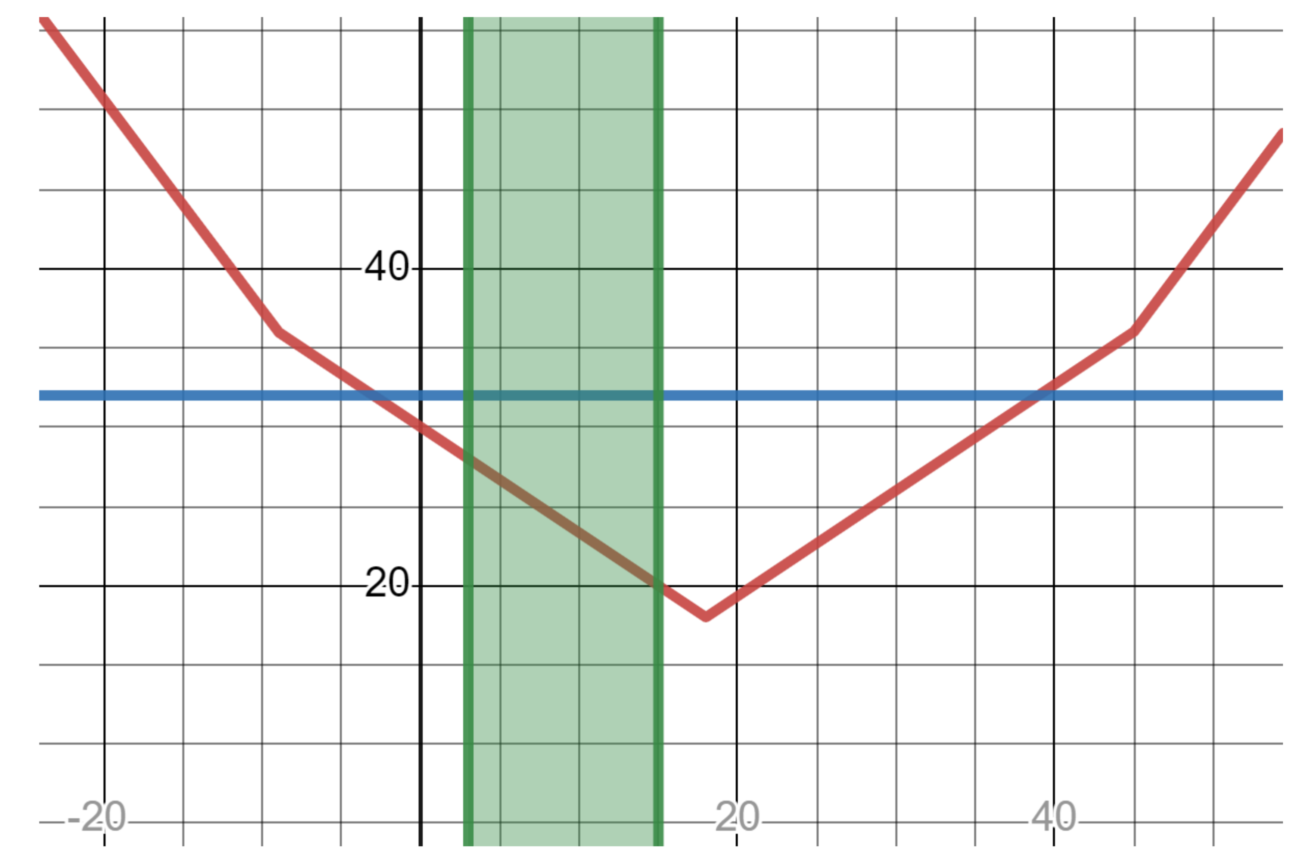

Fig 2. Graphic visualization of the β_3 value for $X = \{3, 9, 27\}$.

For $X = \{4, 16, 64\}$, we calculated a β_3 value of 80 which is too large by a factor of $\frac{9}{10}$.

Finally, For $X = \{5, 25, 125\}$, we calculated a β_3 value of 160 which is too large by a factor of $\frac{11}{12}$.

In general for $X = \{n, n^2, n^3\}$ where $n > 1$ the β_3 value is off by a factor of $\frac{2(n+1)-1}{2(n+1)}$. For this particular sequence of x values, the following equivalence relation will achieve sharpness:

$$D_1(f) \leq \frac{2(n+1) - 1}{2(n+1)} \beta_3 L(f)$$

References

References

[1] J. B. Conway. *A Course in Functional Analysis, Second Edition*, Graduate Texts in Mathematics, Springer Science+Business Media, New York, NY, 2010.

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