Minimal Discriminants of Elliptic Curves with a 4-Isogeny

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Why do we care about Elliptic Curves

Theorem (Pythagorean Theorem 1600 B.C.E.)

Let a, b, and c denote the sides of a right triangle, with c denoting the hypotenuse. Then

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Theorem (Fermat's Last Theorem)

If n is an integer greater than 2, then

$$a^n + b^n = c^n$$

has no nonzero integer solutions.

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- In the early 1990's Andrew Wiles proved Fermat's Last Theorem by using elliptic curves. This proof relied on knowledge of the minimal discriminant of a special kind of elliptic curve.
- Our Goal
  - Our goal for the summer was to make it easier for mathematicians to find the minimal discriminants of elliptic curves with a 4-isogeny. We also worked on finding when such elliptic curves have additive reduction.

Crash Course on Elliptic Curves

We define an **elliptic curve** $E/\mathbb{Q}$ to be given by the following equation

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

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For an elliptic curve E, the **invariants** of the elliptic curve are

defined to be

$$c_4 = a_1^4 + 8a_1^2a_2 - 24a_1a_3 + 16a_2^2 - 48a_4$$
$$c_6 = -\left(a_1^2 + 4a_2\right)^3 + 36\left(a_1^2 + 4a_2\right)\left(2a_4 + a_1a_3\right) - 216\left(a_2^3 + 4a_6\right)$$
$$\Delta = \frac{c_4^3 - c_6^2}{1728} \neq 0.$$

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$$\Delta = \frac{c_4^3 - c_6^2}{1728} \neq 0.$$

The **signature** of an elliptic curve E is $Sig(E) = (c_4, c_6, \Delta)$.

Examples of Elliptic Curves

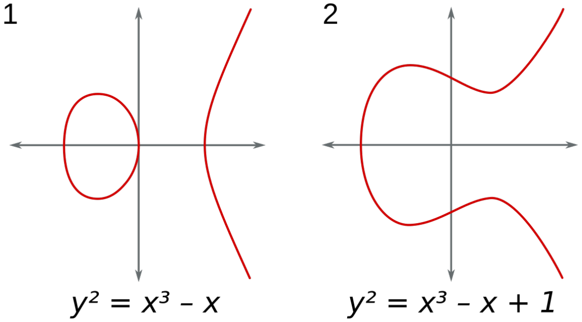
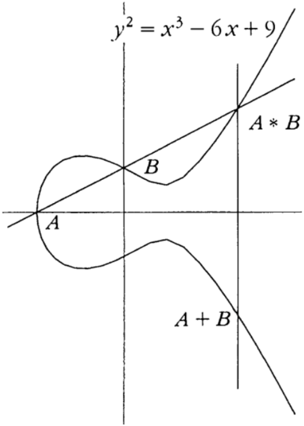
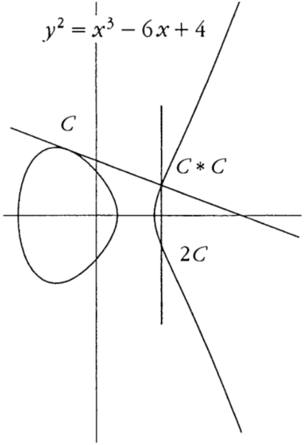


Figure: elliptic curve examples

Group Structures





Elliptic Curves

Given an elliptic curve

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with each a_i a rational number, one can transform and/or scale the graph of E to obtain an isomorphic elliptic curve

$$E' : y^2 + a_1'xy + a_3'y = x^3 + a_2'x^2 + a_4'x + a_6'$$

with the property that each a_i' is an integer and the discriminant Δ' of E' is "minimal" in the sense that $|\Delta'|$ is the smallest discriminant that can be attained from E via translations and/or scalings.

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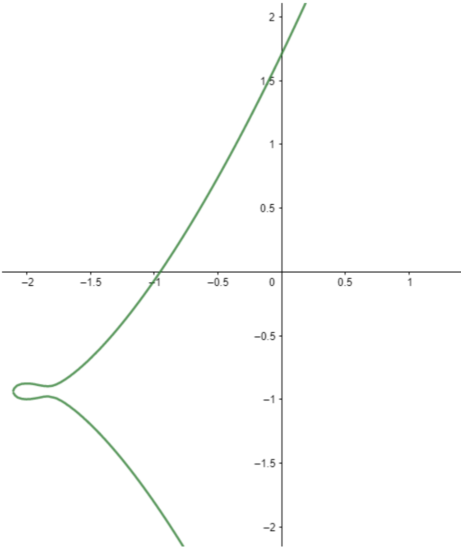
Definition

We say that E' is a **global minimal model** for E, and call Δ' **the minimal discriminant of** E. The signature of E' is **the minimal signature** of E, and is denoted by $\mathrm{sig}_{\min}(E) = (c'_4, c'_6, \Delta')$.

As an example, consider the elliptic curve E_{green} : $y^2 + \frac{15}{8}y = x^3 + \frac{23}{4}x^2 + 11x + \frac{49}{8}$. Then $\text{sig}(E_{\text{green}}) = \left(1, \frac{-19}{8}, \frac{-11}{4096}\right)$.



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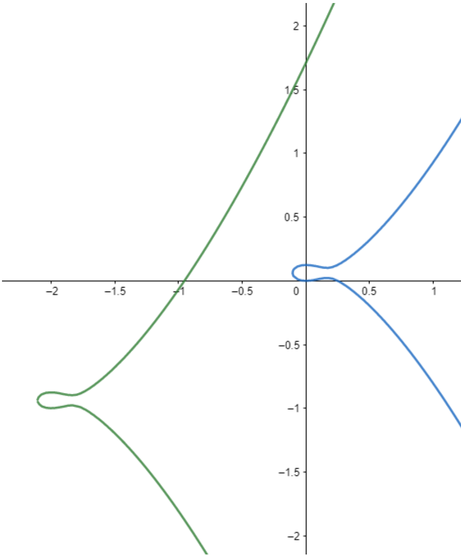
Next, we translate E_{green} to attain $E_{\text{blue}} : y^2 - \frac{1}{8}y = x^3 - \frac{1}{4}x^2$. It turns out that $\text{sig}(E_{\text{green}}) = \text{sig}(E_{\text{blue}})$.



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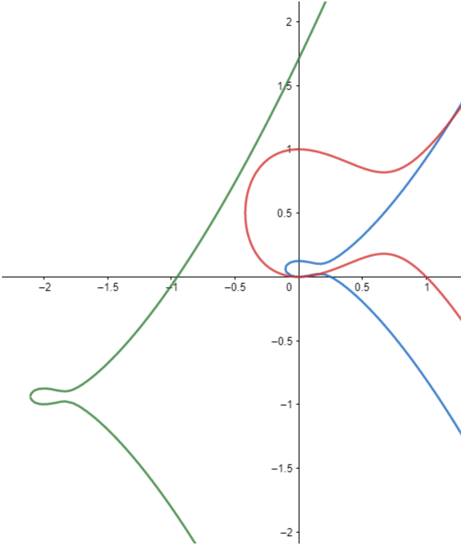
Lastly, we scale E_{blue} to obtain $E_{\text{red}} : y^2 - y = x^3 - x^2$, which is a global minimal model for E_{green}. In particular, $\text{sig}_{\text{min}}(E_{\text{green}}) = \text{sig}(E_{\text{red}}) = (16, -152, -11)$.



Isogenies

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Example

$$E : y^2 = x^3 - 1440x^2 + 108800x$$

The origin $P = (0, 0)$ of E is a point of order 2. Then we get the isogeny $E' = E$ mod P

$$E' : y^2 = x^3 + 2880x^2 + 1638400x$$

Isogenies

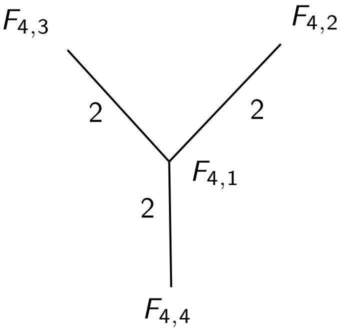
The study of elliptic curves that have isogeny class degree equal to 4 is equivalent to understanding the parameterized elliptic curves $F_{4,i}(a, b, d)$ for $i = 1, 2, 3, 4$ that are given below:

$$F_{4,1}(a, b, d) : y^2 = x^3 + (ad - 16bd)x^2 - 16abd^2x$$
$$F_{4,2}(a, b, d) : y^2 = x^3 + (ad + 8bd)x^2 + 16b^2d^2x$$
$$F_{4,3}(a, b, d) : y^2 = x^3 + (32bd - 2ad)x^2 + (a^2d^2 + 32abd^2 + 256b^2d^2)x$$
$$F_{4,4}(a, b, d) : y^2 = x^3 - (2ad + 64bd)x^2 + a^2d^2x$$

Moreover, $Sig(F_{4,1}(a, b, d)) = (16a^2d^2 + 256abd^2 + 4096b^2d^2,$
$-64a^3d^3 - 1536a^2bd^3 + 24576ab^2d^3 + 262144b^3d^3,$
$4096a^4b^2d^6 + 131072a^3b^3d^6 + 1048576a^2b^4d^6)$

Isogeny Graphs

In the notation $F_{4,i}$, 4 is the **isogeny class degree**, i is an index denoting a vertex on an **isogeny graph**, and each edge corresponds to a 2-isogeny between elliptic curves.



Kraus's Theorem

Theorem (Kraus's Theorem, 1989)

Let $\alpha, \beta, \gamma \in \mathbb{Z}$ with $\alpha^3 - \beta^2 = 1728\gamma$ with $\alpha \neq 0$. Then there is an elliptic curve, E, with integer coefficients and with $Sig(E) = (\alpha, \beta, \gamma)$ if and only if

1. $v_3(\beta) \neq 2$
2. *Either* $\beta \equiv -1 \mod 4$ *or both* $v_2(\alpha) \geq 4$ *and* $\beta \equiv 0, 8 \mod 32$.

More About Isomorphic Elliptic Curves

Definition (Isomorphic)

Let E and E' be elliptic curves over $\mathbb{Q}$. We say that E and E' are **isomorphic**, denoted $E \cong E'$, if and only if there exist $u, r, s, w \in \mathbb{Q}, u \neq 0$ such that we have a map

$$E \longrightarrow E' \text{ where } (x, y) \longmapsto (u^2x + r, u^3y + u^2sx + w).$$

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Denote $Sig(E) = (c_4, c_6, \Delta)$ and $Sig(E') = (c_4', c_6', \Delta')$. If $E \cong E'$, then we have the following relationship

$$c_4' = u^{-4}c_4,\ c_6' = u^{-6}c_6,\ \Delta' = u^{-12}\Delta$$

Results!

Theorem (A.,B.,N., 2023)

Let $a, b, d \in \mathbb{Z}$ with $\gcd(a, b) = 1$ and d squarefree. If $F_{4,i}(a, b, d)$ is an elliptic curve with discriminant $\Delta_{4,i}$, then the minimal discriminant of $F_{4,i}(a, b, d)$ is $u_i^{-12}\Delta_{4,i}$ where u_i is given below.

$v_2(a)$	Additional conditions		(u_1, u_2, u_3, u_4)
≥ 8	$bd \equiv 3 \mod 4$		$(8, 4, 8, 16)$
	$bd \not\equiv 3 \mod 4$		$(4, 2, 4, 8)$
$6, 7$			$(4, 2, 4, 8)$
5	d is even		$(4, 2, 4, 8)$
	d is odd		$(4, 2, 4, 4)$
4	$v_2(a + 16b) \geq 8$	$bd \equiv 1 \mod 4$	$(8, 4, 16, 8)$
		$bd \not\equiv 1 \mod 4$	$(8, 4, 8, 4)$
	$v_2(a + 16b) < 8$	d is even	$(8, 4, 8, 4)$
		d odd, $v_2((a + 16b)^2 - 256ab) \geq 12$	$(8, 4, 8, 4)$
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3	d is even		$(4, 2, 4, 4)$
	d is odd		$(2, 2, 2, 2)$
2			$(2, 2, 2, 2)$
1	d is even		$(2, 2, 2, 2)$
	d is odd		$(1, 1, 1, 1)$
0	$a \equiv 1 \mod 4$		$(2, 2, 2, 2)$
	$a \not\equiv 1 \mod 4$		$(1, 1, 1, 1)$

Our Theorem!!

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Example

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$$v_2(a + 16b) = v_2(16 + 16(-17)) = v_2(16(1 - 17)) = 8$$

and

$$bd \equiv 17 \cdot 4 \mod 4 \equiv 1 \mod 4$$

By the table, we have $(u_1, u_2, u_3, u_4) = (8, 4, 16, 8)$. As a consequence, we have that

$$\Delta_1^{min} = 8^{-12}(2^{36} \cdot 5^6 \cdot 17^2) = 5^6 \cdot 17^2$$
$$\Delta_2^{min} = 4^{-12}(-1 \cdot 2^{24} \cdot 5^6 \cdot 17^4) = -1 \cdot 5^6 \cdot 17^4$$
$$\Delta_3^{min} = 16^{-12}(2^{48} \cdot 5^6 \cdot 17) = 5^6 \cdot 17$$
$$\Delta_4^{min} = 8^{-12}(2^{36} \cdot 5^6 \cdot 17) = 5^6 \cdot 17$$

Whats Next?

We just looked at the family of elliptic curves with isogeny class degree 4, but there are so many other families of elliptic curves. Namely, those with isogeny class degree n where

$$n \in \{2, 3, \cdots, 10, 12, 13, 16, 18, 25\}$$

Acknowledgements

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