Quantitative Tools - level I
Fall 2015
Exercises 5 - Interest Rates

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1 Simple interest

Bob borrows $40,000\$$ at the annual interest of 5%. It should be repaid with two equals payments the 4th and 12th month. Bob should pay 0.5% of charge paid the first day.

1. The present value of the loans and the repayments are the same in present value terms. This equality writes :

$$40,000 = \frac{X}{1+\dfrac{4*i}{12}} + \frac{X}{1+i}$$

$$40,000 = X\left(\frac{1}{1+\dfrac{4*i}{12}} + \frac{1}{1+i}\right)$$

$$X = \frac{40,000}{\dfrac{1}{1+\dfrac{4*i}{12}} + \dfrac{1}{1+i}}$$

$$X = 20,508.77$$

2. Bob paid 0.5% of fees, it means that it *actually* borrows $40,000 * (1 - \frac{0.5}{100}) = \$39,800$ to the bank. We write the equality in present value terms with i the effective annual interest rate we are looking for :

$$39,800 = 20,508.77\left(\frac{1}{1+\dfrac{4*i}{12}} + \frac{1}{1+i}\right)$$

$$\frac{39,800}{20,508.77} = \frac{2+\dfrac{16i}{12}}{(1+\dfrac{4i}{12})(1+i)}$$

$$\frac{39,800}{20,508.77} = \frac{2+\dfrac{4i}{3}}{\dfrac{i^2}{3}+\dfrac{4i}{3}+1}$$

$$0 = i^2(\frac{39,800}{3*20,508.77}) + i(\frac{4*39,800}{3*20,508.77} - \frac{4}{3}) + \frac{39,800}{20,508.77} - 2$$

This is a quadratic equation of the form $ax^2 + bx + c$, we compute the discriminant Δ :

$$\begin{aligned}
\Delta &= b^2 - 4 * a * c \\
\Delta &= 1.235076763^2 - 4 * 0.642102524 * (-0.073692428) \\
\Delta &= 1.714686986
\end{aligned}$$

The discriminant is *positive*, thus, the quadratic equation has two solutions, which are :

$$\begin{aligned}
x_1 &= \frac{-b + \sqrt(\Delta)}{2a} \\
x_2 &= \frac{-b - \sqrt(\Delta)}{2a}
\end{aligned}$$

Here, we find :

$$\begin{aligned}
x_1 &= 0.05792 \\
x_2 &= -1.9814
\end{aligned}$$

The negative solution does have any economic sense and we exclude it. The correct answer is an effective interest rate of 5.79%

Alice had a loan of $1,000$\$ to repay in 10 days. She's running out of cash and ask its bank to repay it in 20 days. The annual interest is 5%.

1. The present value of the two repayments should be equal. Let X be the repayment, Alice will make in 20 days. We have :

$$\begin{aligned}
\frac{1000}{1 + \frac{10 * 5}{360 * 100}} &= \frac{X}{1 + \frac{20 * 5}{360 * 100}} \\
1000 \frac{1 + \frac{20 * 5}{360 * 100}}{1 + \frac{10 * 5}{360 * 100}} &= X \\
X &= 1001.39
\end{aligned}$$

A firm decides to repay a loan of $500,000$\$ 10 days before the due date. The commercial discount rate is 10%.

1. We apply the formula from the Lecture for commercial discount

$$\begin{aligned}
CommercialDiscount &= 500,000 \frac{10 * 10}{360 * 100} \\
CommercialDiscount &= 1388.89
\end{aligned}$$

2. We are looking for the date when the two loans are equivalent, that is their *commercial* present values are equal. Let d the number of days between the equivalence date and the date of the repayment of the first loan. The equality of *commercial* present values writes :

$$500,000(1-\frac{d*i}{360}) = 510,000(1-\frac{(d+30)*i}{360})$$
$$\frac{d*i}{360}(510,000-500,000) = 510,000-500,000-\frac{30*i}{360}510,000$$
$$d = \frac{360}{i} - 51*30$$
$$d = 2070$$

The two loans are equivalent 2070 days (5 years and 9 months) before the due date of the first loan.

A firm has three loans of $20,000$\$, $30,000$\$ and $15,000$\$ of respective due dates 10, 20 and 30 days. The firm want to repay in one single payment in 45 days. The commercial discount rate is 4%.

1. We write down the equivalence of present values, with X the amount of the single payment.

$$X(1-\frac{45i}{360}) = 20,000(1-\frac{10i}{360}) + 30,000(1-\frac{20i}{360}) + 15,000(1+\frac{30i}{360})$$
$$X = 65,187.05$$

2. We use the same formula, but this time, we know X (=65,000)[1] and we are looking for a number of days d.

$$65,000(1-\frac{di}{360}) = 20,000(1-\frac{10i}{360}) + 30,000(1-\frac{20i}{360}) + 15,000(1-\frac{30i}{360})$$
$$d = \frac{10*20,000+20*30,000+30*15,000}{65,000}$$
$$d = 19.2$$

2 Compound Interests

1.

$$X = 100,000*(1+\frac{5}{100})^3$$
$$X = 115762.5$$

[1]There is a typo in the question, it should be $65,000$ and not $55,000$ as it is the sum of all the face value of the loans

2. Let X be the amount Alice should invest. We have

$$\begin{aligned} X(1+i)^7 &= 1,000 \\ X &= \frac{1000}{(1+i)^7} \\ X &= 710.68133 \end{aligned}$$

Alice should invest 710.69$

3. We have

$$\begin{aligned} \frac{2000}{1000} &= (1+i)^{10} \\ i &= 2^{1/10} - 1 \\ i &= 7.18\% \end{aligned}$$

Alice had a loan of $1,000$$ to repay in 10 days. She's running out of cash and ask its bank to repay it in 20 days. The daily interest is 5%.

1. We write down the PV of the two loans with X the value of the new due repayment. The equality of the PVs writes :

$$\begin{aligned} \frac{1000}{(1+i)^{10}} &= \frac{X}{(1+i)^{20}} \\ X &= 1,000(1+i)^{20-10} \\ X &= 1,000(1+i)^{10} \\ X &= 1628.89 \end{aligned}$$

A firm decides to repay a loan of $500,000$$ 10 days before the due date. The daily discount rate is 10%.

1. We write the equality of PVs with X the value of the new repayment :

$$\begin{aligned} \frac{500,000}{(1+i)^d} &= \frac{X}{(1+i)^{d-10}} \\ X &= 500,000 * (1+i)^{-10} \\ X &= 192,771.64 \end{aligned}$$

2. This question is tricky : if we try to answer by having a look at present values, we won't succeed (see below); thus, the answer is **never**. This is because the difference between

two investments (/loans) is a function of (i) the nominal values of the investments, (ii) the discount rate and (iii) the length between the two payments. Indeed :

$$\frac{500,000}{(1+i)^d} = \frac{510,000}{(1+i)^{d+30}}$$
$$500,000 = \frac{510,000}{(1+i)^{30}}$$
$$500,000 \neq 29,227.36$$

Receiving 500,000 now (or in x days) is always better than receiving 510,000 30 days after (or in $x+30$ days), given an interest rate at 4%. This is also true while looking at simple interest, if we use rational present values instead of commercial present values as we did.

A firm has three loans of $20,000$\$, $30,000$\$ and $15,000$\$ of respective due dates 10, 20 and 30 days. The firm want to repay in one single payment in 45 days. The daily interest rate is 4%.

1. We write the equality of PVs with X the value of the repayment

$$\frac{20,000}{(1+i)^{10}} + \frac{30,000}{(1+i)^{20}} + \frac{15,000}{(1+i)^{30}} = \frac{X}{(1+i)^{45}}$$
$$X = 20,000 * (1+i)^{35} + 30,00 * (1+i)^{25} + 15,000 * (1+i)^{15}$$
$$X = 185,911.02$$

2. We use the equality of PVs :

$$\frac{20,000}{(1+i)^{10}} + \frac{30,000}{(1+i)^{20}} + \frac{15,000}{(1+i)^{30}} = \frac{65,000}{(1+i)^d}$$
$$\frac{65000}{(1+i)^d} = 31,827.67$$
$$d = \frac{ln(\frac{65,000}{31,827.67})}{ln(1+i)}$$
$$d = 18,21 \text{ days}$$