Quantitative Tools - Level 1 - Fall 2015
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Lecture 10

Interest rates

Introduction & Definitions

An interest is the payment of a loan. Historically, it seems that the first loans where loan of cattle or food. It was then *fair* to charge interest since the cattle reproduce themselves and as do seeds. We could think about it as a cost of the loss of the opportunity, indeed the money-lender loose the opportunity to do something else. There are basically two different kind of interest :

- **simple interest** when the interest are paid at the end of the contract. The interest are calculated *pro rata temporis* - in proportion of the loan's duration.
- **compound interest** when the interest are added to the principal so that the interest earns interest from then on.

In this chapter, we will consider that a year has 360 days, divided in twelve months of 30 days each. We use the following notations : Let A_0 be the amount of money invested at the annual interest i, let A_n be the amount of money after n years.

1 Simple Interest

For simple interest, the interest are not added up to the principal.

1.1 Basic formulas - Principle

- After one year we have

$$A_1 = A_0 * (1 + i)$$

- After n years we have

$$A_n = A_0 * (1 + n * i)$$

- Let A_x be the amount of money invested for x months at the annual interest rate i. Then

$$A_x = (1 + \frac{x * i}{12})A_0$$

- Let A_d be the amount of money invested for d days at the annual interest rate i. Then

$$A_d = A_0 * (1 + \frac{d * i}{360})$$

1.2 Use of Basic formulas

Let A_0 be the amount of money invested for d days at the annual interest i. In this subsection, we will derive from the previous formula, formula for the interest rate, the length of the investment, for the present value of the investment, while knowing the others parameters of the problem. Indeed, the problem has four parameters : the amount of money invested, the annual interest rate, the length of the investment and amount of the interests at the end of the investment. Knowing three parameters, we are able to find the fourth.

- If we are looking for the present value of A_d, the money we'll get at the end of the d days, that is expressing A_0 as a function of A_d. We have

$$A_0 = \frac{A_d}{1 + \frac{d * i}{360}}$$

- If we look for the interest rate :

$$i = \frac{A_d - A_o}{A_0} * \frac{360}{d}$$

- If we look for the length d of the investment :

$$d = \frac{A_d - A_0}{A_0} * \frac{360}{i}$$

1.3 Interest checked off - effective rates of investment

When the interest are paid at the beginning of the investment, we used the term *Interest checked off*. Because of this payment, the money investment is not the same. Indeed, we *really* invest only the principal minus the interest checked off. As a consequence, the effective rate of investment has changed.
Let A_0 the capital invest at the beginning, at the annual interest i, during n days, with checked off interest rates.
The amount of money which is indeed invested is :

$$A_0 - \text{the interest paid} = A_0 * (1 - \frac{i * d}{360})$$

The interest rate have also changed, and applying the formula found above, we get a new effective rate of investment i':

$$i' = \frac{A_0 - A_0 * (1 - \frac{i * d}{360})}{A_0 * (1 - \frac{i * d}{360})}$$

$$i' = \frac{i}{1 - \frac{i * d}{360}}$$

1.4 Discount

A discount is a deduction from the usual cost of something, typically given for prompt or advance payment. In finance, it is a percentage deducted from the face value of a bill of exchange or promissory note when the payment is done before the due date. With interest rates, the *timing* matters. Receiveing one hundred euros today allows you to go to the bank, put it on a saving accounts and receive the interest. Receving it in one month is a *loss* of value.

Definition 1.1. Present Value
The present value, PV_A of a amount of money A that will be received (or due) in d days is the amount of money that should be invested now at the interest rate i so that the principal and the interests equals A in d days. We call i the discount rate
Formally,

$$PV_A = \frac{A}{1 + \frac{d * i}{360}}$$

The present value is used by firms for comparing investments. Indeed, there is three things to be compared : the amount of money invested, the returns and the *timing of events*. Thus, in order to make things *comparable* one need to transform all the future flows of money in *present value terms*. The discount rate chosen is usually the rate of a riskless bond (like the US Treasury bonds).

There are two kinds of discount :

- the rational discount : the difference between an amount of money and its present value.
- the commercial discount : the value of the interests received if it the amount of money is invested at an interest rate equal to discount rate.

Definition 1.2. Rational Discount
Let A_d be the amount of money due in d days. Let i be the discount rate. The rational discount is the difference between A_d and its present value.
That is :

$$\begin{aligned} Rational\ Discount &= A_d - \frac{A_d}{1 + \frac{d*i}{360}} \\ &= A_d \left(1 - \frac{1}{1 + \frac{d*i}{360}} \right) \end{aligned}$$

Definition 1.3. Commercial Discount
Let A_d be the amount of money due in d days. Let i be the discount rate. The commercial discount is :

$$Commercial\ Discount = A_d * \frac{d*i}{360}$$

Definition 1.4. Commercial Present Value
With the *commercial discount* we can define a **commercial present value**, equals to the difference between the money due in d days and the commercial present value.

$$\begin{aligned} Commercial\ Present\ Value &= A_d - A_d * \frac{d*i}{360} \\ &= A_d \left(1 - \frac{d*i}{360} \right) \end{aligned}$$

Definition 1.5. Equivalence of debts contracts
Two debts contracts are equivalent at time t if the commercial present value of the two debts contracts is the same. Date t is **the equivalence date**.
Formally, let A_d and $A_{d'}$ be two debts contracts due in d and d' days respectively. Let i be the discount rate. We have :

$$\begin{gathered} \text{Commercial Present Value of } A_d = \text{Commercial Present value of } A_{d'} \\ A_d(1 - \frac{d*i}{360}) = A_{d'}(1 - \frac{d'*i}{360}) \end{gathered}$$

From that formula, one can deduce $A_{d'}$:

$$A_{d'} = A_d \frac{(1 - \frac{d*i}{360})}{(1 - \frac{d'*i}{360})}$$

One can note that $A_{d'}$ is increasing with d' : the longer the maturity, the higher the interests, which is quite intuitive.

Example 1.1. Extension of maturity
A firm has a repayment due in 30 of 50, 000$ and want to renegotiate the debt contract so that she will pay an amount A_{60} in sixty days. The commercial discount rate is of 5%. We want to find the amount A_{60} due in 60 days.

$$\begin{aligned} 50,000 * (1 - \frac{30*5}{360*100}) &= A_{60} * (1 - \frac{60*5}{360*100}) \\ A_{60} &= 50,000 * \frac{(1 - \frac{30*5}{360*100})}{(1 - \frac{60*5}{360*100})} \\ A_{60} &= 50,210 \end{aligned}$$

Definition 1.6. Average maturity
Let $A_{d_1}, ..., A_{d_n}$, be n due payments in $d_1, ..., d_n$ days respectively. Let A_{d^*} be the sum of all the A_{d_i}. The **average maturity** is the number of days d^* so that a single payment A_{d^*} made in d^* days is equivalent to the n payments made in $d_1, ..., d_n$ days respectively, that is, equals the n payments in commercial present value terms.
Formally :

$$A_{d^*}(1 - \frac{d^* * i}{360 * 100}) = \sum_{j=1}^{n} A_{d_j} * (1 - \frac{i * d_j}{360})$$
$$d^* = \frac{\sum_{j=1}^{n} A_{d_j} * d_j}{A_{d^*}}$$

2 Compound Interest

Definition 2.1. Compount Interest
Compound interests are interests calculated on the initial principal and on the accumulated interest of previous period. Compound interests are *interests on interests.*

2.1 Basic Formulas

Let A_0 be the amount of money invested, A_n the amount of money after n years, i the annual interest rate, and I the interest earned.

$$A_n = A_0 * (1 + i)^n$$

From that formula we can derive :

$$A_0 = \frac{A_n}{(1 + i)^n}$$
$$i = \left(\frac{A_n}{A_0}\right)^{1/n} - 1$$
$$n = \frac{ln(\frac{A_n}{A_0})}{ln(1 + i)}$$
$$I = A_n - A_0$$
$$= A_0((1 + i)^(n) - 1)$$

To Be Continued
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