

1 Simple interest

Bob borrows 40,000\$ at the annual interest of 5%. It should be repaid with two equals payments the 4th and 12th month. Bob should pay 0.5% of charge paid the first day.

1. The present value of the loans and the repayments are the same in present value terms. This equality writes :

$$40,000 = \frac{X}{1 + \frac{4 * i}{12}} + \frac{X}{1 + i}$$

$$40,000 = X \left(\frac{1}{1 + \frac{4 * i}{12}} + \frac{1}{1 + i} \right)$$

$$X = \frac{40,000}{\frac{1}{1 + \frac{4 * i}{12}} + \frac{1}{1 + i}}$$

$$X = 20,508.77$$

2. Bob paid 0.5% of fees, it means that it *actually* borrows $40,000 * (1 - \frac{0.5}{100}) = \$39,800$ to the bank. We write the equality in present value terms with i the effective annual interest rate we are looking for :

$$39,800 = 20,508.77 \left(\frac{1}{1 + \frac{4 * i}{12}} + \frac{1}{1 + i} \right)$$

$$\frac{39,800}{20,508.77} = \frac{2 + \frac{16i}{12}}{(1 + \frac{4i}{12})(1 + i)}$$

$$\frac{39,800}{20,508.77} = \frac{2 + \frac{4i}{3}}{i^2 + \frac{4i}{3} + 1}$$

$$0 = i^2 \left(\frac{39,800}{3 * 20,508.77} \right) + i \left(\frac{4 * 39,800}{3 * 20,508.77} - \frac{4}{3} \right) + \frac{39,800}{20,508.77} - 2$$

This is a quadratic equation of the form $ax^2 + bx + c$, we compute the discriminant Δ :

$$\Delta = b^2 - 4 * a * c$$

$$\Delta = 1.235076763^2 - 4 * 0.642102524 * (-0.073692428)$$

$$\Delta = 1.714686986$$

The discriminant is *positive*, thus, the quadratic equation has two solutions, which are :

$$x_1 = \frac{-b + \sqrt{(\Delta)}}{2a}$$

$$x_2 = \frac{-b - \sqrt{(\Delta)}}{2a}$$

Here, we find :

$$x_1 = 0.05792$$

$$x_2 = -1.9814$$

The negative solution does have any economic sense and we exclude it. The correct answer is an effective interest rate of 5.79%

Alice had a loan of 1,000\$ to repay in 10 days. She's running out of cash and ask its bank to repay it in 20 days. The annual interest is 5%.

1. The present value of the two repayments should be equal. Let X be the repayment, Alice will make in 20 days. We have :

$$\begin{aligned} \frac{1000}{1 + \frac{10 * 5}{360 * 100}} &= \frac{X}{1 + \frac{20 * 5}{360 * 100}} \\ 1000 \frac{1 + \frac{20 * 5}{360 * 100}}{1 + \frac{10 * 5}{360 * 100}} &= X \\ X &= 1001.39 \end{aligned}$$

A firm decides to repay a loan of 500,000\$ 10 days before the due date. The commercial discount rate is 10%.

1. We apply the formula from the Lecture for commercial discount

$$\begin{aligned} CommercialDiscount &= 500,000 \frac{10 * 10}{360 * 100} \\ CommercialDiscount &= 1388.89 \end{aligned}$$

2. We are looking for the date when the two loans are equivalent, that is their *commercial* present values are equal. Let d the number of days between the equivalence date and the date of the repayment of the first loan. The equality of *commercial* present values writes :

$$\begin{aligned}
 500,000\left(1 - \frac{d * i}{360}\right) &= 510,000\left(1 - \frac{(d + 30) * i}{360}\right) \\
 \frac{d * i}{360}(510,000 - 500,000) &= 510,000 - 500,000 - \frac{30 * i}{360}510,000 \\
 d &= \frac{360}{i} - 51 * 30 \\
 d &= 2070
 \end{aligned}$$

The two loans are equivalent 2070 days (5 years and 9 months) before the due date of the first loan.

A firm has three loans of 20,000\$, 30,000\$ and 15,000\$ of respective due dates 10, 20 and 30 days. The firm want to repay in one single payment in 45 days. The commercial discount rate is 4%.

1. We write down the equivalence of present values, with X the amount of the single payment.

$$\begin{aligned}
 X\left(1 - \frac{45i}{360}\right) &= 20,000\left(1 - \frac{10i}{360}\right) + 30,000\left(1 - \frac{20i}{360}\right) + 15,000\left(1 - \frac{30i}{360}\right) \\
 X &= 65,187.05
 \end{aligned}$$

2. We use the same formula, but this time, we know X ($=65,000$)¹ and we are looking for a number of days d .

$$\begin{aligned}
 65,000\left(1 - \frac{di}{360}\right) &= 20,000\left(1 - \frac{10i}{360}\right) + 30,000\left(1 - \frac{20i}{360}\right) + 15,000\left(1 - \frac{30i}{360}\right) \\
 d &= \frac{10 * 20,000 + 20 * 30,000 + 30 * 15,000}{65,000} \\
 d &= 19.2
 \end{aligned}$$

2 Compound Interests

- 1.

$$\begin{aligned}
 X &= 100,000 * \left(1 + \frac{5}{100}\right)^3 \\
 X &= 115762.5
 \end{aligned}$$

¹There is a typo in the question, it should be 65,000 and not 55,000 as it is the sum of all the face value of the loans

2. Let X be the amount Alice should invest. We have

$$\begin{aligned} X(1+i)^7 &= 1,000 \\ X &= \frac{1000}{(1+i)^7} \\ X &= 710.68133 \end{aligned}$$

Alice should invest 710.69\$

3. We have

$$\begin{aligned} \frac{2000}{1000} &= (1+i)^{10} \\ i &= 2^{1/10} - 1 \\ i &= 7.18\% \end{aligned}$$

Alice had a loan of 1,000\$ to repay in 10 days. She's running out of cash and ask its bank to repay it in 20 days. The daily interest is 5%.

1. We write down the PV of the two loans with X the value of the new due repayment. The equality of the PVs writes :

$$\begin{aligned} \frac{1000}{(1+i)^{10}} &= \frac{X}{(1+i)^{20}} \\ X &= 1,000(1+i)^{20-10} \\ X &= 1,000(1+i)^{10} \\ X &= 1628.89 \end{aligned}$$

A firm decides to repay a loan of 500,000\$ 10 days before the due date. The daily discount rate is 10%.

1. We write the equality of PVs with X the value of the new repayment :

$$\begin{aligned} \frac{500,000}{(1+i)^d} &= \frac{X}{(1+i)^{d-10}} \\ X &= 500,000 * (1+i)^{-10} \\ X &= 192,771.64 \end{aligned}$$

2. This question is tricky : if we try to answer by having a look at present values, we won't succeed (see below); thus, the answer is **never**. This is because the difference between

two investments (/loans) is a function of (i) the nominal values of the investments, (ii) the discount rate and (iii) the length between the two payments. Indeed :

$$\begin{aligned}\frac{500,000}{(1+i)^d} &= \frac{510,000}{(1+i)^{d+30}} \\ 500,000 &= \frac{510,000}{(1+i)^{30}} \\ 500,000 &\neq 29,227.36\end{aligned}$$

Receiving 500,000 now (or in x days) is always better than receiving 510,000 30 days after (or in $x + 30$ days). This is also true while looking at simple interest, if we use rational present values instead of commercial present values as we did.

A firm has three loans of 20,000\$, 30,000\$ and 15,000\$ of respective due dates 10, 20 and 30 days. The firm want to repay in one single payment in 45 days. The daily interest rate is 4%.

1. We write the equality of PVs with X the value of the repayment

$$\begin{aligned}\frac{20,000}{(1+i)^{10}} + \frac{30,000}{(1+i)^{20}} + \frac{15,000}{(1+i)^{30}} &= \frac{X}{(1+i)^{45}} \\ X &= 20,000 * (1+i)^{35} + 30,00 * (1+i)^{25} + 15,000 * (1+i)^{15} \\ X &= 185,911.02\end{aligned}$$

2. We use the equality of PVs :

$$\begin{aligned}\frac{20,000}{(1+i)^{10}} + \frac{30,000}{(1+i)^{20}} + \frac{15,000}{(1+i)^{30}} &= \frac{65,000}{(1+i)^d} \\ \frac{65000}{(1+i)^d} &= 31,827.67 \\ d &= \frac{\ln(\frac{65,000}{31,827.67})}{\ln(1+i)} \\ d &= 18,21 \text{ days}\end{aligned}$$