Effects on Modal Content in an Optical Cavity due to a General Misalignment

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Introduction

The goal of this report is to understand how the general misalignment of an optical cavity affects its transverse electromagnetic modal content. Starting with an aligned cavity supporting the fundamental Hermite Gaussian mode, misalignment of one mirror is considered through four degrees of freedom. Upon characterizing misaligned cavity geometry, coupling to higher order modes is calculated. Finally, a clipping factor for the captured power is estimated with the goal of accurately scaling the contribution of each mode for a sufficiently large misalignment.

1 Defining Characteristics of Hermite Gaussian Beams in an Optical Cavity

Consider an optical cavity consisting of two mirrors with radius of curvature R facing each other, their centers a distance L apart. The free-space amplitude of the supported Hermite Gaussian beam $A_{mn}(x, y, z)$ is given by the following expression:

$$A_{mn}(x,y,z) = \left[\frac{2}{\pi 2^{m+n} m! n!}\right]^{1/2} \frac{1}{w(z)} H_m\left(\frac{\sqrt{2}x}{w(z)}\right) H_n\left(\frac{\sqrt{2}y}{w(z)}\right) e^{-\frac{x^2+y^2}{w^2(z)}} e^{ik\frac{x^2+y^2}{2R(z)}} e^{-i(m+n+1)\eta(z)}. \tag{1}$$

Let us define all quantities in the above expression.

- $H_n(x)$ are the physicist's Hermite polynomials: $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$.
- $w(z)$ is the beam width: $w(z) = w_0\sqrt{1 + (\frac{z}{z_R})^2}$.
  - ➔ w_0 is the beam width at $z = 0$. For our purposes, $w_0 = \sqrt{\frac{\lambda L}{2\pi}} \left(\frac{R-(L/2)}{(L/2)}\right)^{1/4}$.
  - ➔ z_R is the Rayleigh range: $z_R = \pi\frac{w_0^2}{\lambda}$, where λ is the free space wavelength. For our purposes, $\lambda = 780$ nm.
- k is the wavenumber: $k = \frac{2\pi}{\lambda}$.
- $R(z)$ is the beam's radius of curvature: $R(z) = z\left(1 + (\frac{z_R}{z})^2\right)$
- $\eta(z)$ is the Gouy phase: $\eta(z) = \arctan\left(\frac{z}{z_R}\right)$.

The **cavity mode axis** is defined to be the line which passes through the center of curvature of each mirror. The **cavity mode center** will be defined as the midpoint between the mirrors along the cavity mode axis. Consider a beam described by Eq. 1, with $m = n = 0$. By rotating and/or displacing one of the mirrors, the cavity mode axis rotates and displaces as well. This causes the beam to higher order modes:

$$A_{00} \to \sum_{m,n} c_{mn} A_{mn}, \tag{2}$$

where the values of coefficients c_{mn} depend on the nature of the misalignment. Since A_{mn} form an orthonormal basis and Equation 2 describes a unitary process, we expect $\sum_{m,n} |c_{mn}|^2 = 1$. It is the goal of this report to calculate these coefficients for arbitrary rotational and translational misalignments of one mirror.

2 Finding the Cavity Mode Axis and Center

We will consider an optical cavity in which $R = 2$ mm and $L = 3.95$ mm, so that the center of curvature for each mirror overlaps one another.

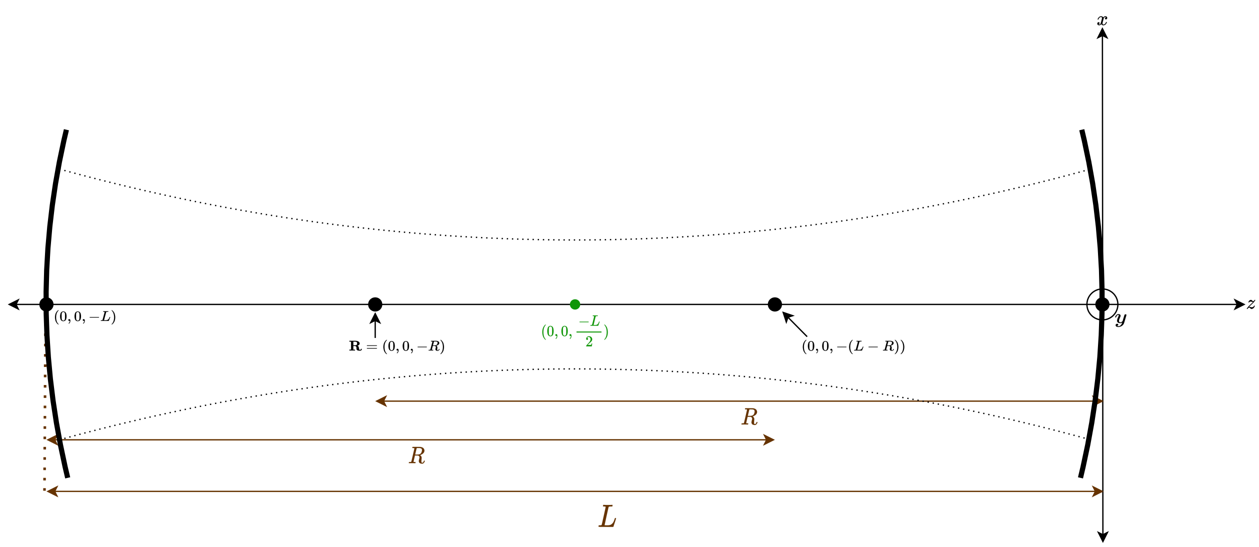

Figure 1: Graphical representation of the optical cavity geometry. Not drawn to scale.

To simply calculations, we will choose a convenient coordinate system. We will place the center of the fixed mirror out at $(0, 0, -L)$ and the center of the movable mirror at the origin, as in Figure 1. From now on we will define the movable mirror to be the right mirror. The curved dotted lines traced between the two mirrors crudely represent the beam width profile along the unperturbed cavity mode axis, which is the z axis by our choice of coordinates. The center of curvature of the right mirror is represented by the vector $\mathbf{R} = (0, 0, -R)$. To calculate the new cavity mode axis and center, we must determine its coordinates after an arbitrary misalignment. We define a 4-dimensional misalignment of the right mirror by the following set of active transformations on $\mathbf{R}$:

1. Rotate $\mathbf{R}$ about the y axis by an angle α in the clockwise direction, then translate along the x axis by an amount dx in the positive direction: $\mathbf{R} \to \mathbf{R'}$.

2. Rotate $\mathbf{R'}$ about the x axis by an angle β in the clockwise direction, then translate along the y axis by an amount dy in the negative direction: $\mathbf{R'} \to \mathbf{R''}$.

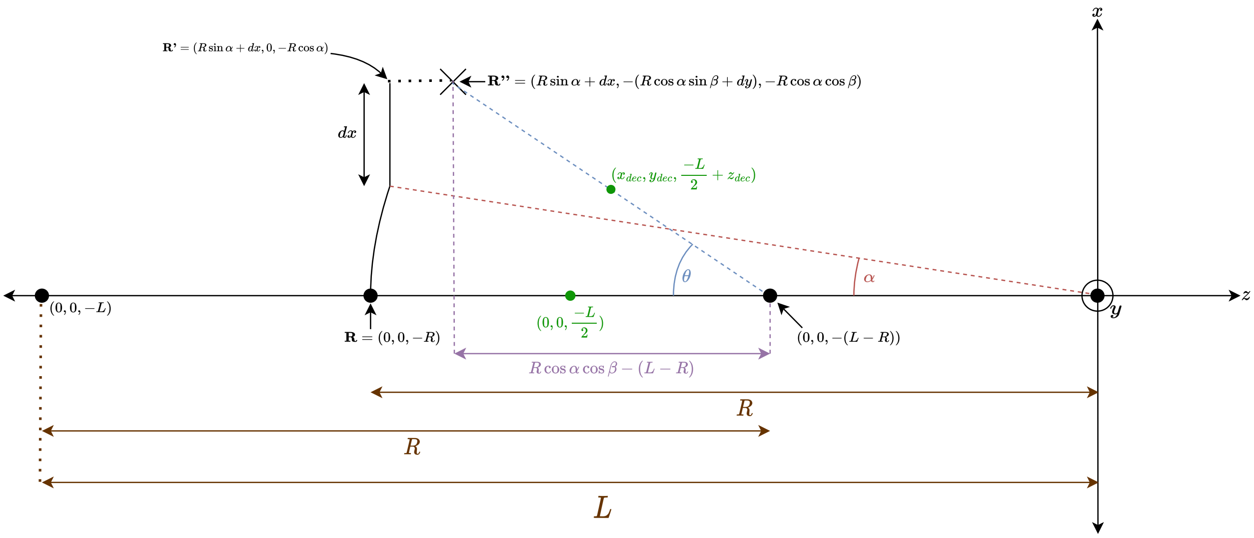

Figure 2: Cavity mode tilt and center after misalignment, as seen in the xz plane.

We are therefore interested in the vector **R"**. Mathematically, we write

$$\mathbf{R"} = R_x(-\beta)\left(R_y(-\alpha)\mathbf{R} + \begin{bmatrix} dx \\ 0 \\ 0 \end{bmatrix}\right) + \begin{bmatrix} 0 \\ -dy \\ 0 \end{bmatrix} = \begin{bmatrix} R\sin\alpha + dx \\ -(R\cos\alpha\sin\beta + dy) \\ -R\cos\alpha\cos\beta \end{bmatrix}, \tag{3}$$

where $R_{x,y}(\gamma)$ are the standard 3D rotation matrices about the x and y axes by an angle γ in the counter-clockwise direction. As seen in Figure 2, the new cavity mode axis makes an angle θ in the xz plane with respect to the unperturbed cavity mode axis. The cavity mode axis makes an additional angle ϕ in the yz plane with respect to the unperturbed cavity mode axis, which can be seen below in Figure 3.

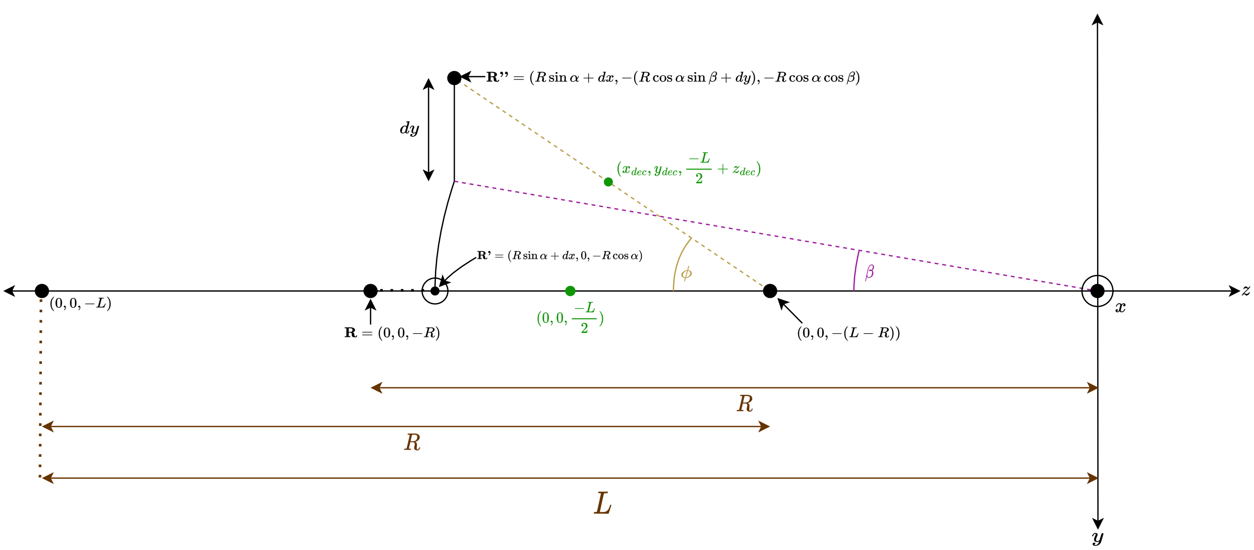

Figure 3: Cavity mode tilt and center after misalignment, as seen in the yz plane.

The cavity mode center has also displaced by an amount $(x_{dec}, y_{dec}, z_{dec})$ from its original position, as shown by the two green dots. The cavity mode tilts are given by the following expressions.

$$\boxed{\begin{aligned} \theta &= \arctan\left(\frac{dx + R\sin\alpha}{-(R + R\cos\alpha\cos\beta - L)}\right) \qquad (4a) \\ \phi &= \arctan\left(\frac{R\cos\alpha\sin\beta + dy}{(R + R\cos\alpha\cos\beta - L)}\right) \qquad (4b) \end{aligned}}$$

The cavity mode center displacements $(x_{dec}, y_{dec}, z_{dec})$ are given by the following expressions.

$$\boxed{\begin{aligned} x_{dec} &= \frac{1}{2}(R\sin\alpha + dx) \qquad (5a) \\ y_{dec} &= -\frac{1}{2}(R\cos\alpha\sin\beta + dy) \qquad (5b) \\ z_{dec} &= \frac{R}{2}(1 - \cos\alpha\cos\beta) \qquad (5c) \end{aligned}}$$

From this information, we can fully describe the effective cavity mode axis and center.

3 Calculating the Modal Content of the Misaligned Cavity

How might we use Equations (4a-5c) to predict the components c_{mn} of the cavity output as in Equation 2? Let us exploit the orthonormality of the A_{mn} basis to gain some insight. Firstly, let us change our coordinate system so that the cavity mode center is at the origin and Equation 1 can be used to describe the unperturbed fundamental beam (with $m = n = 0$). Since A_{mn} forms an orthonormal basis,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A^*_{m'n'}(x,y,0)A_{mn}(x,y,0)\,dx\,dy = \delta_{m'm}\delta_{n'n}. \qquad \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \qquad (6)$$

Suppose the right mirror becomes misaligned, causing the cavity mode axis to tilt and cavity mode center to displace according to Equations (4a-5c).

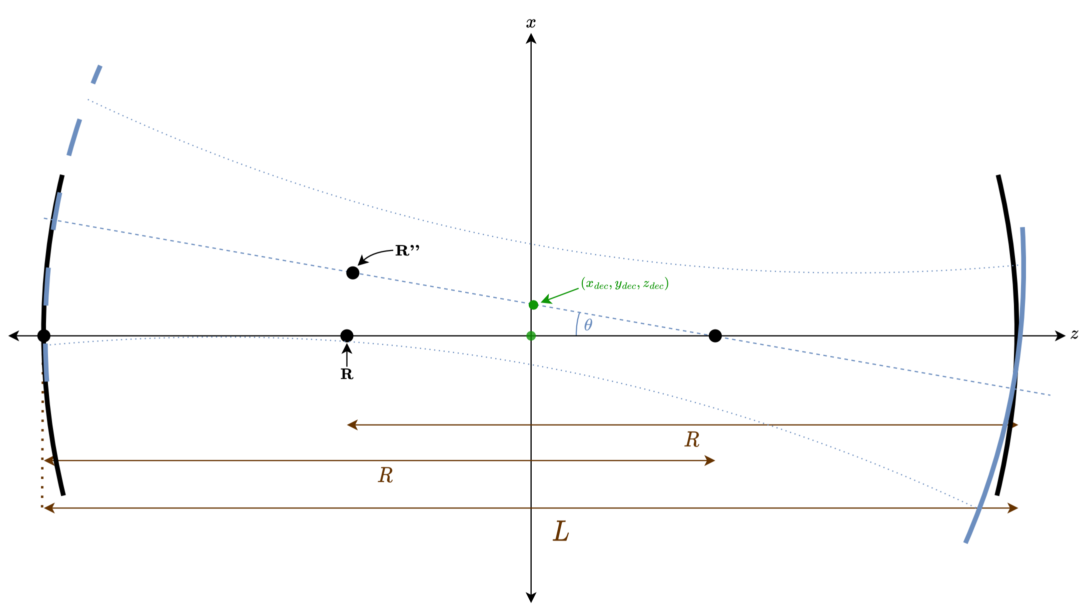

Figure 4: Slightly misaligned cavity, as seen in the xz plane.

If the misalignment is small, we can imagine a new cavity geometry that is skewed according to the new cavity mode axis and center. The skewed cavity would support a set of basis functions similar to those in Equation 1, only they would be centered at the new cavity mode center and rotated by θ and ϕ appropriately. Let us define the skewed basis to be

$$B_{mn}(x, y, z) \equiv A_{mn}(\mathbf{r}_{skew}), \tag{7}$$

where

$$\mathbf{r}_{skew} = R_x(-\phi)R_y(\theta) \begin{bmatrix} x - x_{dec} \\ y - y_{dec} \\ z - z_{dec} \end{bmatrix}. \tag{8}$$

The choice of sign within the argument of each rotation in Equation 8 is needed for consistency with Equations (4a-4b). Since θ (ϕ) is defined to be negative (positive), we must rotate in the same (opposite) direction to achieve a net clockwise rotation in both directions. For convenience, let us define $(\Delta x, \Delta y, \Delta z) \equiv (x - x_{dec}, y - y_{dec}, z - z_{dec})$. Expanding Equation 8,

$$\begin{aligned} \mathbf{r}_{skew} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \\ &= \begin{bmatrix} \Delta x \cos\theta + \Delta z \sin\theta \\ -\Delta x \sin\phi \sin\theta + \Delta y \cos\phi + \Delta z \sin\phi\cos\theta \\ -\Delta x \cos\phi \sin\theta - \Delta y \sin\phi + \Delta z \cos\phi\cos\theta \end{bmatrix}. \end{aligned} \tag{9}$$

We now have a way to express the skewed cavity basis functions in the same coordinate system as the unperturbed basis functions. Here, a hypothesis is made that *the modal content of the misaligned cavity beam output is related to the similarity between bases A_{mn} and B_{mn}*. A natural extension of this idea is that the coefficients c_{mn} are nothing more than an inner product:

$$c_{mn} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A_{00}^*(x, y, 0)B_{mn}(x, y, 0)\ dx\ dy. \tag{10}$$

This model suggests that misalignment in the xz plane causes coupling to modes of higher m, and misalignment in the yz plane causes coupling to modes of higher n in a symmetric manner.

4 Estimating the Clipping of Modes for a Larger Misalignment

Due to the finite size of the mirrors, if the misalignment is sufficiently large, then it is reasonable to assume that the supported intensity reduces, or clips. In this section we look for a way to calculate the clipping of each mode for a general misalignment.

4.1 Finding the Misaligned Cavity Length and Beam Waist

When the right mirror becomes misaligned, the distance between the mirrors' centers, or the **cavity length**, may change. This in turn affects the fundamental waist of the beam, and the waist at the location of the mirrors, as they depend on L. Let us define a vector, **L'**, whose magnitude is the new cavity length and whose direction is along the cavity mode axis. We can find $|\mathbf{L'}| \equiv L'$ from the coordinates of each mirror's center of curvature since we know these points are a distance R from their respective mirrors. Referring to Figure 2 and Equation 3,

$$L' = 2R - \sqrt{(R\sin\alpha + dx)^2 + (R\cos\alpha\sin\beta + dy)^2 + (R\cos\alpha\cos\beta - (L - R))^2}. \tag{11}$$

As for the direction of **L'**, which we will call $\hat{\mathbf{n}}$, we know that it makes an angle θ in the xz plane and an angle ϕ in the yz plane. Thus,

$$\hat{\mathbf{n}} = (\sin\theta)\hat{\mathbf{x}} + (\sin\phi\cos\theta)\hat{\mathbf{y}} + (\cos\phi\cos\theta)\hat{\mathbf{z}}. \tag{12}$$

We can now calculate the fundamental waist of the misaligned beam, w_0':

$$w_0' = \sqrt{\frac{\lambda L'}{2\pi}} \left(\frac{R - (L'/2)}{(L'/2)}\right)^{1/4}. \tag{13}$$

We can also find the waist of the beam at the location of the mirrors, w_m:

$$w_m = w_0' \sqrt{1 + \left(\frac{(L'/2)}{z_R}\right)^2}. \tag{14}$$

It will be convenient to define quantities L'_{xz} and L'_{yz} as the magnitude of the projection of **L'** into the xz and yz planes, respectively:

$$L'_{xz} = |\mathbf{L'} - (\mathbf{L'} \cdot \hat{\mathbf{y}})|, \tag{15}$$

and

$$L'_{yz} = |\mathbf{L'} - (\mathbf{L'} \cdot \hat{\mathbf{x}})|. \tag{16}$$

4.2 Calculating the Clipping of Modes

The power of a Gaussian beam on the plane of the aligned mirror can be written as

$$P_{total} \propto 2\pi \int_0^\infty e^{-2\rho^2/w_m^2} \rho \, d\rho \tag{17}$$

$$\propto \frac{\pi}{2} w_m^2, \tag{18}$$

where w_m is the waist of the beam at the location of the mirror. For a misaligned cavity, we can estimate the power on the mirror by performing a similar integral over the surface area of the mirror. It is important to note that actual position of the misaligned right mirror will most likely also be misaligned to the effective cavity mode axis which θ and ϕ describe. Therefore, its distance from the *ideal* position on the effective cavity mode axis provides a way to quantify the loss of power in the misaligned beam.

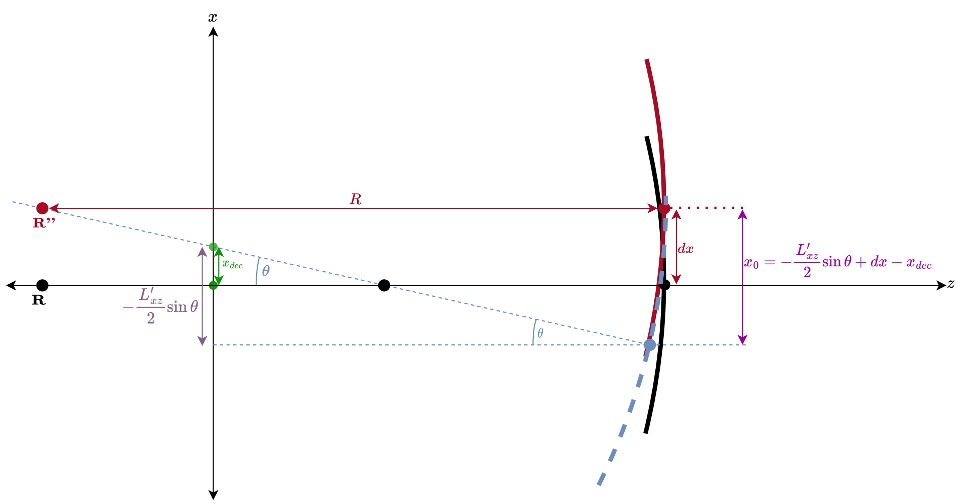

Figure 5: A misaligned mirror (solid red curve) is itself misaligned with the cavity mode axis which θ describes. Its ideal position is depicted by the dashed blue curve, which is a distance x_0 along $\hat{\mathbf{x}}$ from the actual mirror.

In Figure 5, it can be seen that the actual position of the right mirror is a distance x_0 along $\hat{\mathbf{x}}$ from its ideal position on the cavity mode axis:

$$x_0 = -\frac{L'_{xz}}{2}\sin\theta + dx - x_{dec}. \tag{19}$$

Similarly, the misaligned mirror will be a distance y_0 along $\hat{\mathbf{y}}$ from its ideal position on the cavity mode axis:

$$y_0 = -\left(\frac{L'_{yz}}{2}\sin\phi + dy + y_{dec}\right). \tag{20}$$

The choice of global and relative signs in Equations (19-20) is once again for consistency with the choice of signs in Equations (4a-5c). Since we are interested in the power on the mirror, we should replace the upper integral limit in Equation 17 with r, the radius of the finite-sized mirror ($r = 0.5$ mm). In Cartesian coordinates,

$$P_{total} \propto \int_{-r}^{r} dx \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} e^{-2x^2/w_m^2} e^{-2y^2/w_m^2}\, dy. \tag{21}$$

The remaining power after misalignment, $P_{clipped}$, can be found by displacing the integrand by x_0 along $\hat{\mathbf{x}}$ and y_0 along $\hat{\mathbf{y}}$:

$$P_{clipped} \propto \int_{-r}^{r} dx \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} e^{-2(x-x_0)^2/(w_m\cos\delta)^2} e^{-2(y-y_0)^2/(w_m\cos\xi)^2}\, dy, \tag{22}$$

where

$$\delta \equiv |\theta| - |\alpha| \quad \text{and} \quad \xi \equiv |\phi| - |\beta|. \tag{23}$$

To extend this calculation to higher order modes, we will add dependence on indices m and n to the integrand of Equation 22. Let us define w_x and w_y to be the waist of the beam along x and y respectively, at the position of the mirror. Then for a higher order mode (m,n),

$$w_x \to w_x\sqrt{2m+1} \quad \text{and} \quad w_y \to w_y\sqrt{2n+1}.$$

Identifying the waist in the Gaussian over x (y) as w_x (w_y), we can finally write

$$\boxed{P_{total}(m,n) \propto \int_{-r}^{r} dx \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} e^{-2x^2/\left(w_m\sqrt{2m+1}\right)^2} e^{-2y^2/\left(w_m\sqrt{2n+1}\right)^2}\, dy} \tag{24}$$

$$\boxed{P_{clipped}(m,n) \propto \int_{-r}^{r} dx \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} e^{-2(x-x_0)^2/\left(w_m\sqrt{2m+1}\cos\delta\right)^2} e^{-2(y-y_0)^2/\left(w_m\sqrt{2n+1}\cos\xi\right)^2}\, dy.} \tag{25}$$

The mode-dependent clipping factor, κ_{mn}, is the ratio

$$\kappa_{mn} = \sqrt{\frac{P_{clipped}(m,n)}{P_{total}(m,n)}}. \tag{26}$$

It then follows to multiply the coefficients c_{mn} by κ_{mn} to accurately reduce the contribution of each mode due to sufficiently large misalignment:

$$A_{00} \to \sum_{m,n} \kappa_{mn} c_{mn} A_{mn}. \tag{27}$$

Conclusion

For a general misalignment described by parameters (α, dx, β, dy), an effective cavity mode axis and center are created and described by Equations (4a-5c). The effective cavity supports a new set of basis functions, B_{mn}, which are transformed versions of the A_{mn} basis functions according to Equations (7-9). The components of the cavity beam output, c_{mn}, are then estimated to be an inner product between the two bases, as in Equation 10. For sufficiently large misalignment, power is lost from each component, thereby reducing c_{mn} by a mode-dependent factor, κ_{mn}. These calculations imply that misalignment in the x direction will cause coupling to higher order modes in m, and misalignment in the y direction causes coupling to higher order modes in n. This should be expected due to the symmetric nature of the cavity geometry and the Hermite Gaussian modes it supports.