#The Integer Quantum Hall Effect

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1 Introduction

The Integer Quantum Hall Effect is an exciting example of classical physics breaking down and quantum mechanics taking over. In this paper we will look at the classical picture, witness its breakdown through experimental results, and then justify these results quantum mechanically.

2 Classical Hall Effect

Consider a moving electron subject to an external magnetic field.

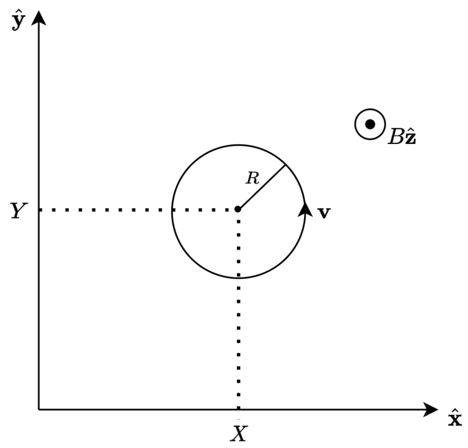

Figure 1: Electron motion in the presence of a uniform magnetic field

The equation of motion reads:

$$-e(\mathbf{v} \times \mathbf{B}) = m\frac{d\mathbf{v}}{dt}. \tag{1}$$

Let's assume $\mathbf{B} = \mathrm{B}\hat{\mathbf{z}}$ $\mathbf{v} = \dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}}$, as in Figure 1. Then Eq. (1) is solved by

$$x(t) = X - Rsin(\omega_B t + \phi) \quad \text{and} \quad y(t) = Y + Rcos(\omega_B t + \phi), \tag{2}$$

where

$$\omega_B = \frac{eB}{m} \tag{3}$$

is known as the *cyclotron frequency.* An electron moving through a magnetic field exhibits circular motion at frequency ω_B. X, Y and ϕ are subject to the initial position of the electron.
Let's now consider a system of many electrons, moving in the xy plane in the presence of a magnetic field along z. If we add an electric field $\mathbf{E}$ and a term for linear friction, the equation of motion is now

$$-e(\mathbf{v}\times\mathbf{B}) - e\mathbf{E} - \frac{m\mathbf{v}}{\tau} = m\frac{d\mathbf{v}}{dt}. \tag{4}$$

This is known as the *Drude Model*, which treats the motion of electrons classically. The scattering time, τ, describes the purity of the substance in which the electron moves. For large values of τ the substance is more pure, leading to negligible friction.
In equilibrium, $d\mathbf{v}/dt = 0$, reducing Eq. (4) to

$$-\frac{e\tau}{m}(\mathbf{v}\times\mathbf{B}) - \mathbf{v} = \frac{e\tau}{m}\mathbf{E}. \tag{5}$$

Since the current density $\mathbf{J} = -en_d\mathbf{v}$, where n_d is the density of electrons, Eq. (5) becomes

$$-\frac{\tau}{n_d m}(\mathbf{J}\times\mathbf{B}) - \frac{1}{en_d}\mathbf{J} = \frac{e\tau}{m}\mathbf{E}. \tag{6}$$

This is a system of two equations (the third equation vanishes due to $J_z = 0$, and $\mathbf{B} = B\hat{\mathbf{z}}$). It can be written in matrix form:

$$\begin{bmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & 1 \end{bmatrix}\begin{bmatrix} J_x \\ J_y \end{bmatrix} = \frac{e^2 n_d \tau}{m}\begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

Let's recast Eq. (6) into the form $\mathbf{J} = \sigma\mathbf{E}$, where

$$\sigma = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{bmatrix}.$$

Importantly, σ is a matrix and not just a number. Its off-diagonal elements are responsible for the connection between current and electric field components that are perpendicular to each other. This encapsulates the *Hall Effect*: an applied electric field E_x induces J_x, which bends into the y direction due to $\mathbf{B}$, thereby inducing E_y.
Upon inverting the matrix on the left hand side, Eq. (6) becomes

$$\begin{bmatrix} J_x \\ J_y \end{bmatrix} = \frac{\sigma_{DC}}{1+(\omega_B\tau)^2}\begin{bmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \end{bmatrix}, \tag{7}$$

where

$$\sigma_{DC} = \frac{e^2 n_d \tau}{m}. \tag{8}$$

As $\mathbf{B} \to 0$, $\omega_B \to 0$ and $\sigma \to \sigma_{DC}\mathbb{1}$, thereby eliminating off-diagonal terms in the conductivity matrix. The resistivity matrix,

$$\rho = \begin{bmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{bmatrix},$$

is found by inverting σ:

$$\rho = \sigma^{-1} = \frac{1}{\sigma_{DC}} \begin{bmatrix} 1 & \omega_B \tau \\ -\omega_B \tau & 1 \end{bmatrix}. \tag{9}$$

We see that

$$\rho_{xx} = \frac{1}{\sigma_{DC}} = \frac{m}{e^2 n_d \tau} \quad \text{and} \quad \rho_{xy} = \frac{\omega_B \tau}{\sigma_{DC}} = \frac{B}{e n_d}. \tag{10}$$

The *Hall Resistance* is defined as

$$R_H = \frac{\rho_{xy}}{B} = \frac{1}{e n_d}, \tag{11}$$

and describes the resistance experienced by current in the x direction due to the y component of the electric field.

3 Integer Quantum Hall Effect

Eq. (10) implies that as B increases, ρ_{xy} increases linearly and ρ_{xx} remains constant. In 1986, Klaus von Klitzing observed that increasing B to a large enough value while at low temperatures led to strikingly different results.

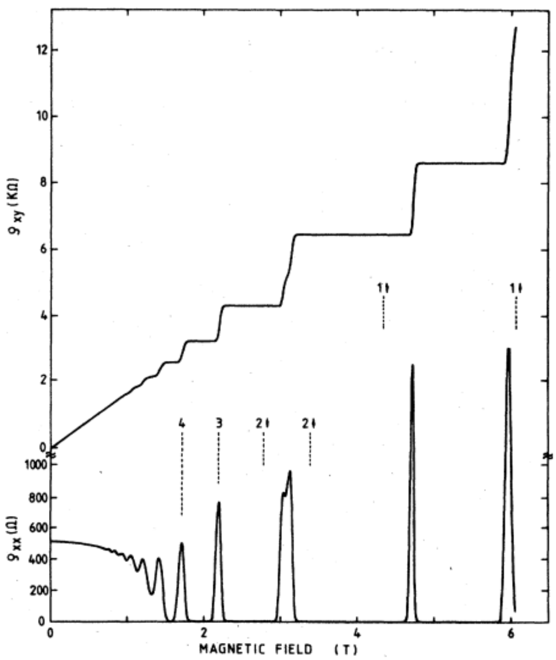

Figure 2: von Klitzing's results of ρ_{xx} and ρ_{xy} as a function of the applied magnetic field[1]

It appears that the transverse resistivity ρ_{xy} takes on quantized values:

$$\rho_{xy} = \frac{2\pi\hbar}{e^2 \nu} \quad \nu \in \mathbf{Z}, \tag{12}$$

[1]K. v Klitzing, G. Dorda, M. Pepper, "New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance", Phys. Rev. Lett. 45 494.

while ρ_{xx} vanishes everywhere except at magnetic field values where ρ_{xy} makes a step. In this section we will explore the origin of this behavior.

3.1 Landau Levels

The Hamiltonian of an electron subject to a magnetic field is given by

$$H = \frac{1}{2m}(\mathbf{p} + e\mathbf{A})^2 \tag{13}$$

where $\mathbf{p} = m\dot{\mathbf{x}} - e\mathbf{A}$. For a uniform magnetic field, and in the Coulomb gauge ($\nabla \cdot \mathbf{A} = 0$), we can say $\mathbf{A} = -\frac{1}{2}(\mathbf{x} \times \mathbf{B})$. Let us define

$$\boldsymbol{\pi} = \mathbf{p} + e\mathbf{A} = m\dot{\mathbf{x}}. \tag{14}$$

We see that

$$\begin{aligned}
[\pi_i, \pi_j] &= [p_i + eA_i, p_j + eA_j] \\
&= e[p_i, A_j] + e[A_i, p_j] \\
&= e[p_i, -\frac{1}{2}\epsilon_{jkl}x_k B_l] + e[-\frac{1}{2}\epsilon_{ist}x_s B_t, p_j] \\
&= -\frac{1}{2}B_l \epsilon_{jkl} e[p_i, x_k] - \frac{1}{2}B_t \epsilon_{ist} e[x_s, p_j] \\
&= -\frac{1}{2}B_l \epsilon_{jkl} e(i\hbar\delta_{ik}) - \frac{1}{2}B_t \epsilon_{ist} e(-i\hbar\delta_{sj}) \\
&= -\frac{1}{2}B_l \epsilon_{jil} e(i\hbar) - \frac{1}{2}B_t \epsilon_{ijt} e(-i\hbar) \\
&= -\frac{1}{2}B_l \epsilon_{jil} e(i\hbar) + \frac{1}{2}B_l \epsilon_{jil} e(-i\hbar) \\
&= -B_l \epsilon_{jil} e(i\hbar) \\
&= (-ieB_k\hbar)\epsilon_{ijk}
\end{aligned}$$

Thus,

$$[\pi_x, \pi_y] = -ieB\hbar \tag{15}$$

for $\mathbf{B} = B\hat{\mathbf{z}}$.
Let's introduce operators a and $a^\dagger$ as follows:

$$a = \frac{1}{\sqrt{2eB\hbar}}(\pi_x - i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2eB\hbar}}(\pi_x + i\pi_y), \tag{16}$$

which satisfy

$$[a, a^\dagger] = 1. \tag{17}$$

This is reminiscent of the quantum mechanical simple harmonic oscillator. We can rewrite Eq. (13) in terms of a and $a^\dagger$:

$$\begin{aligned}
H &= \frac{1}{2m}(\mathbf{p}+e\mathbf{A})^2 \\
&= \frac{1}{2m}\boldsymbol{\pi}\cdot\boldsymbol{\pi} \\
&= \frac{1}{2m}\left((a+a^\dagger)\sqrt{\frac{eB\hbar}{2}}\right)^2 + \frac{1}{2m}\left((a-a^\dagger)i\sqrt{\frac{eB\hbar}{2}}\right)^2 \\
&= \frac{\hbar eB}{m}\left(a^\dagger a+\frac{1}{2}\right) \\
H &= \hbar\omega_B\left(a^\dagger a+\frac{1}{2}\right). \qquad (18)
\end{aligned}$$

The Hamiltonian is in exactly the same form as for a simple harmonic oscillator with $\omega = \omega_B$. We can then write

$$a^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \quad \text{and} \quad a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \tag{19}$$

where $|n\rangle$ solves Eq. (18) with energy eigenvalue:

$$E_n = \hbar\omega_B(n+\frac{1}{2}). \tag{20}$$

Let's now choose $\mathbf{A} = xB\hat{\mathbf{y}}$, such that $\mathbf{B} = \nabla\times\mathbf{A} = B\hat{\mathbf{z}}$. Eq. (13) becomes

$$H = \frac{1}{2m}\left(p_x^2 + (p_y+eBx)^2\right) \tag{21}$$

With our choice of $\mathbf{A}$, the eigenstates of H are eigenstates of p_y. We can then look for separable solutions of the form

$$\psi_k(x,y) = e^{iky}u_k(x). \tag{22}$$

The time-independent Schrödinger Equation reads:

$$\begin{aligned}
H\psi_k(x,y) &= \left[\frac{1}{2m}\left(p_x^2+(p_y+eBx)^2\right)\right]\psi_k(x,y) \\
&= \left[\frac{1}{2m}\left(p_x^2+(\hbar k+eBx)^2\right)\right]\psi_k(x,y) \\
&= \left[\frac{p_x^2}{2m}+\frac{1}{2m}(\hbar k+eBx)^2\right]\psi_k(x,y) \\
&= \left[\frac{p_x^2}{2m}+\frac{1}{2}m\omega_B^2(x+l_B^2k)^2\right]\psi_k(x,y) \qquad \left(l_B = \sqrt{\frac{\hbar}{eB}}\right) \\
&= H_k\psi_k(x,y).
\end{aligned}$$

Here,

$$H_k = \frac{p_x^2}{2m}+\frac{1}{2}m\omega_B^2(x+l_B^2k)^2 \tag{23}$$

is the Hamiltonian of a simple harmonic oscillator, centered at $x = -l_B^2 k$. The eigenstates of Eq. (21) will then look like

$$\psi_{n,k}(x, y) \propto e^{iky} H_n(x + l_B^2 k) e^{-(x+l_B^2 k)^2/2l_B^2}, \tag{24}$$

where $H_n(x)$ are the Hermite polynomials. The corresponding energy eigenvalues are

$$E_n = \hbar\omega_B(n + \frac{1}{2})$$

as in Eq. (20). These energies are the *Landau Levels.*
Note that there is a degeneracy here: our wavefunctions are labeled by n and k, while the energy only depends on n. To get an idea of how large the degeneracy is, let's restrict the system to a rectangle in the xy plane:

$$0 \leq x \leq L_x \quad \textit{and} \quad 0 \leq y \leq L_y. \tag{25}$$

Since the wavefunction follows a plane wave in the y direction, k takes on quantized units of $2\pi/L_y$. The restriction of $0 \leq x \leq L_x$ implies

$$-\frac{L_x}{l_B^2} \leq k \leq 0 \tag{26}$$

since our wavefunction is exponentially centered at $x = -l_B^2 k$ by Eqs. (23) and (24). We can find the number of allowed of states N in this range:

$$N = \frac{(L_x/l_B^2)}{(2\pi/L_y)} = L_x L_y \frac{eB}{2\pi\hbar}. \tag{27}$$

If we define the *flux quantum* as

$$\Phi_0 = \frac{2\pi\hbar}{e}, \tag{28}$$

then

$$N = \frac{L_x L_y B}{\Phi_0}, \tag{29}$$

or

$$n_d = \frac{B}{\Phi_0}. \tag{30}$$

3.2 Quantization of Transverse Resistivity

Let's revisit Figure 2 now. We expect the behavior of ρ_{xx} and ρ_{xy} to be determined by Eq. (10), and for $B \ll 1$T this appears to be the case. For large B, we see:

$$\rho_{xy} = \frac{2\pi\hbar}{e^2\nu} \quad \nu \in \mathbf{Z},$$

while ρ_{xx} vanishes everywhere except points where ρ_{xy} changes. Solving for the density of electrons in Eq. (10) gives us

$$n_d = \frac{B}{e\rho_{xy}}.$$

Plugging in the quantized values of ρ_{xy} found at high B gives the classical expectation of n_d:

$$n_d = B\nu\frac{e}{2\pi\hbar} = \frac{B\nu}{\Phi_0}. \tag{31}$$

Comparing to Eq. (30), this result differs only by multiplication of an integer ν. This has the following interpretation: when ρ_{xy} equals the step value indexed by an integer ν, the density of states is that of ν filled Landau Levels.
Furthermore, in the low temperature limit $k_BT \ll \hbar\omega_B$, the energy required to enter the next highest Landau Level is very large, leaving these states unoccupied. When a small electric field is present, it still is not enough to excite the electrons into motion. Consequently, $\tau \to \infty$, and so $\rho_{xx} \to 0$.
Let's look at what happens to the system when we add an electric field in the x direction ($\mathbf{E} = E\hat{\mathbf{x}}$). This corresponds to adding an electric potential $\phi = -Ex$ to the Hamiltonian:

$$H = \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right) + eEx \tag{32}$$

We can complete the square to again write H as the Hamiltonian of a displaced simple harmonic oscillator:

$$\begin{aligned} H &= \frac{1}{2m}\left(p_x^2 + (p_y + eBx)^2\right) + eEx \\ &= \frac{p_x^2}{2m} + (\frac{m\omega_B^2}{2})\left[x^2 + 2(l_B^2 + \frac{eE}{m\omega_B^2})x + (l_B^2 k)^2\right] \\ &= \frac{p_x^2}{2m} + (\frac{m\omega_B^2}{2})\left(x + l_B^2 k + \frac{eE}{m\omega_B^2}\right)^2 - l_B^2 keE - \frac{e^2E^2}{2m\omega_B^2} \end{aligned} \tag{33}$$

Since the eigenstates are unaffected by the addition of the rightmost two terms in Eq. (33), the solutions are just translated versions of Eq. (24):

$$\psi(x,y) = \psi_{n,k}(x + \frac{eE}{m\omega_B^2}). \tag{34}$$

The energies are now k-dependent, lifting the degeneracy:

$$E = \hbar\omega_B(n + \frac{1}{2}) - l_B^2 keE - \frac{e^2E^2}{2m\omega_B^2}. \tag{35}$$

Now, the current $\mathbf{I} = -e\dot{\mathbf{x}}$, where

$$\dot{\mathbf{x}} = \frac{\mathbf{p} + e\mathbf{A}}{m}. \tag{36}$$

Quantum mechanically,

$$\mathbf{I} = -\frac{e}{m}\sum_{\text{filled states}} \langle\psi_{n,k}| - i\hbar\nabla + e\mathbf{A}\,|\psi_{n,k}\rangle\,, \tag{37}$$

where we sum over all filled Landau Levels and allowed k values within them. The current in the x direction is

$$I_x = -\frac{e}{m}\sum_{n=1}^{\nu}\sum_{k}\langle\psi_{n,k}| - i\hbar\frac{\partial}{\partial x}\,|\psi_{n,k}\rangle\,.$$

Since our wavefunctions are of the simple harmonic oscillator form in the x direction, $\langle\psi_{n,k}|\, p_x\, |\psi_{n,k}\rangle = 0$, and therefore

$$I_x = 0 \rightarrow J_x = 0. \tag{38}$$

As for the current in the y direction,

$$\begin{aligned}
I_y &= -\frac{e}{m}\sum_{n=1}^{\nu}\sum_k \langle\psi_{n,k}|-i\hbar\frac{\partial}{\partial y} + eBx\,|\psi_{n,k}\rangle \\
&= -\frac{e}{m}\sum_{n=1}^{\nu}\sum_k \langle\psi_{n,k}|\,\hbar k + eBx\,|\psi_{n,k}\rangle \\
&= -\frac{e}{m}\sum_{n=1}^{\nu}\sum_k \hbar k + eB\,\langle\psi_{n,k}|\,x\,|\psi_{n,k}\rangle \\
&= -\frac{e}{m}\sum_{n=1}^{\nu}\sum_k \hbar k + eB(-\hbar k/eB - mE/eB^2) \\
&= e\nu\sum_k \frac{E}{B} \\
&= \frac{e\nu E}{B}\left(\frac{L_x L_y B}{\Phi_0}\right)
\end{aligned}$$

Dividing through by the area of the surface,

$$J_y = \frac{e\nu E}{\Phi_0} \tag{39}$$

We now know J_x and J_y with $\mathbf{E} = E\hat{\mathbf{x}}$. In matrix form,

$$\begin{bmatrix} 0 \\ \frac{e\nu E}{\Phi_0} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} E \\ 0 \end{bmatrix}. \tag{40}$$

The longitudinal and transverse conductivities are given by

$$\sigma_{xx} = 0 \quad \textit{and} \quad \sigma_{xy} = \frac{e\nu}{\Phi_0} = \frac{e^2\nu}{2\pi\hbar}, \tag{41}$$

respectively. Inverting the matrix, we find the longitudinal and transverse resistivities:

$$\rho_{xx} = 0 \quad \textit{and} \quad \rho_{xy} = -\frac{\Phi_0}{e\nu} = -\frac{2\pi\hbar}{e^2\nu}. \tag{42}$$

This is the *Integer Quantum Hall Effect*: in the presence of a magnetic field and electric field, the transverse resitivity takes on quantized values proportional to Φ_0, while the longitudinal resistivity vanishes.

4 Important Points

- The Classical Hall Effect breaks down at large enough applied B and low temperatures.
- An electron confined to the xy plane and subject to a uniform magnetic field in the z direction

takes on the energy spectrum of a harmonic oscillator with spacing proportional to B. These are called Landau Levels, and they are largely degenerate.

- Adding a small electric field lifts the degeneracy, but does not excite states into higher levels.
- For large enough B, the classical expectation of n_d at the step value of ρ_{xy} indexed by ν is equal to the electron density of ν filled Landau Levels.
- At low temperatures, the longitudinal resistivity vanishes due to a large energy gap between levels and symmetry of the wavefunction in the longitudinal direction.
- The quantization of ρ_{xy} is a direct consequence of the discrete Landau Levels.

Acknowledgement

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D. Tong, "Lectures on The Quantum Hall Effect", http://www.damtp.cam.ac.uk/user/tong/qhe.html