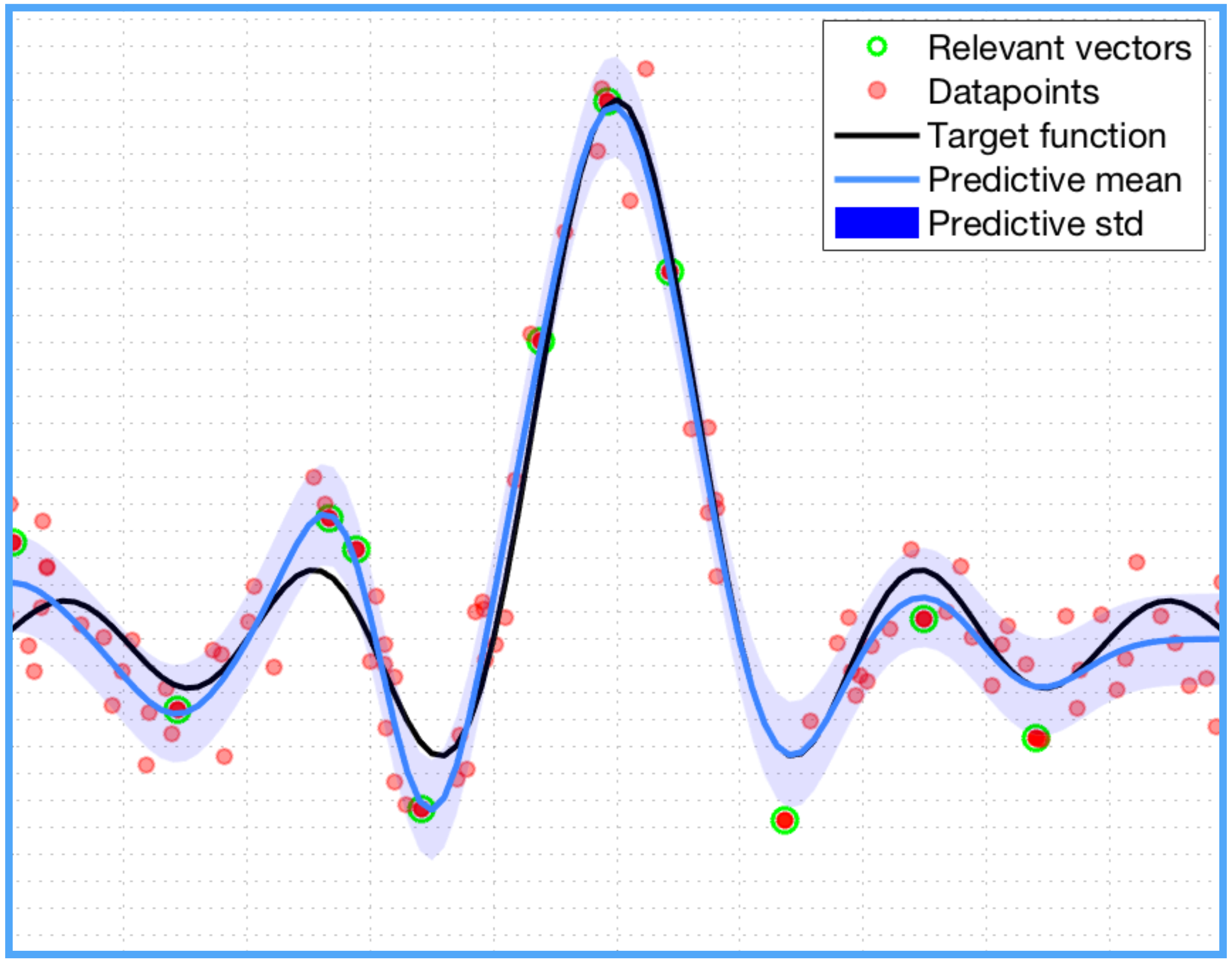
Support Vector Regression vs. Relevance Vector Regression

a sparsity / performance study



L.Faury G.Gallois-Montbrun H.Hendrikx

26/05/2017

Outline

- **Theoretical** reminders on both methods
- Introduction to a **sparse-regression** metric, experimental justification
- Sparse-regression metric based **cross-validation**
- Performance vs. sparsity discussion

- **Regression**

Regression

Learn $f : \mathbb{R}^d \to \mathbb{R}$ thanks to a dataset $\{X, t\} \in (\mathbb{R}^d)^n \times \mathbb{R}^n$

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$$p(t \mid x, w) = \mathcal{N}(t \mid w^T \phi(x), \beta^{-1})$$

the maximum-likelihood estimator (MLE) writes :

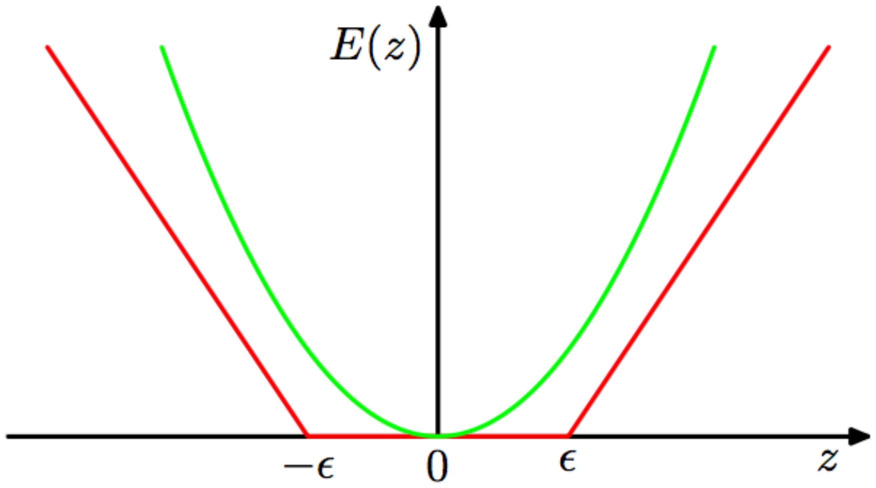
$$\begin{aligned} \hat{w} &= \mathrm{argmax}_w \, p(t \mid X, w) \\ &= \mathrm{argmin}_w \, \frac{1}{2} \sum_{i=1}^{n} \|w^t \phi(x) - t\|^2 \end{aligned}$$

Support Vector Regression

[1] *Vladimir Vapnik*, The nature of statistical learning theory, *1995*

Support Vector Regression

- Introduce the ε-insensitive[(1)] loss-function.



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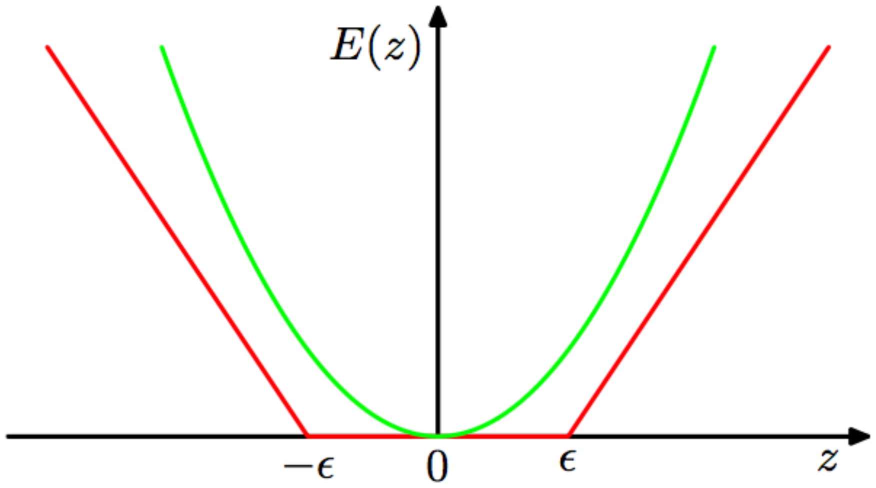
$$min_w \frac{C}{n} \sum_n (\xi_n + \hat{\xi}_n) + \frac{1}{2}\|w\|^2$$

$$\text{s.t} \quad \begin{cases} \xi, \hat{\xi} \geq 0 \\ w^T \phi(x_n) + \xi_n + \varepsilon \geq t_n \\ w^T \phi(x_n) - \hat{\xi}_n - \varepsilon \leq t_n \end{cases}$$



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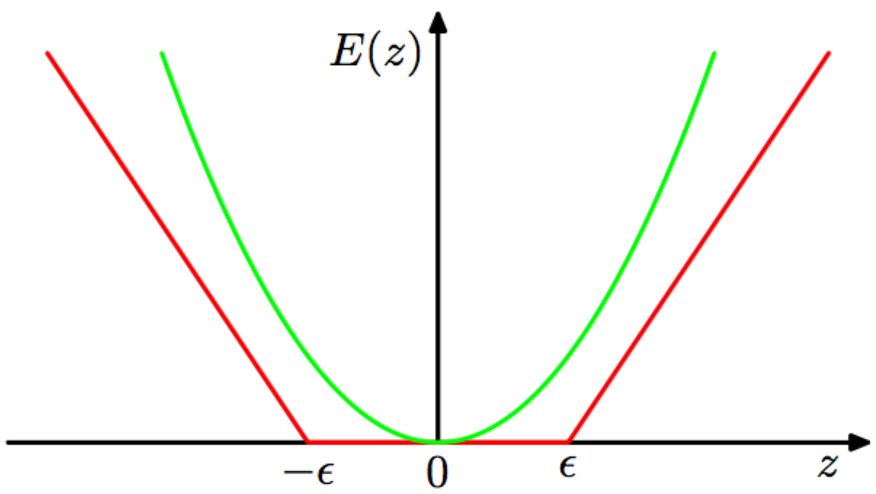

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- Only points outside the ε-tube (**active constraints**) are used for predictions :

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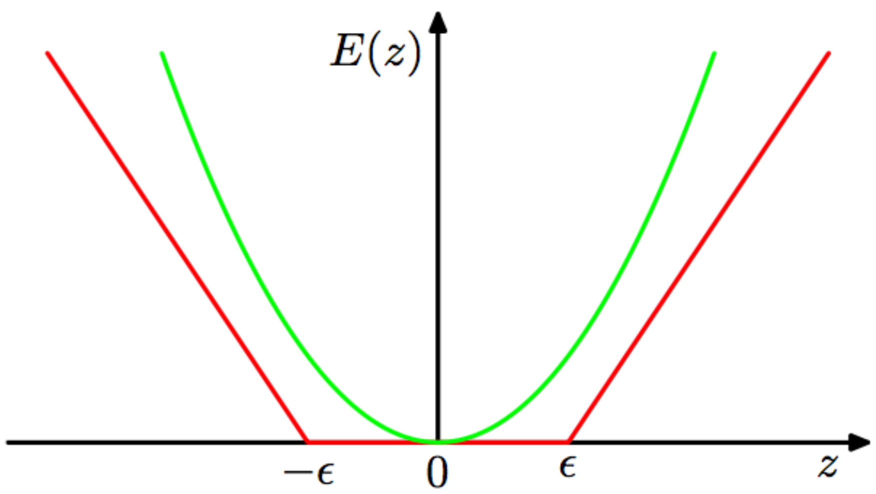

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SVR	RVR

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- **Question** : Compare the tradeoff found between performance (MSE minimization) and sparsity

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Sparsity / Performance

- **Question** : Compare the tradeoff found between performance (MSE minimization) and sparsity

- **Literature**[4] : RVR reaches sparser models with equivalent generalization skills.

- **Initial idea** : Test (**experimentally**) this assertion
  - what is performance ?
  - what do we want with sparsity ?
  - how to measure the tradeoff ?

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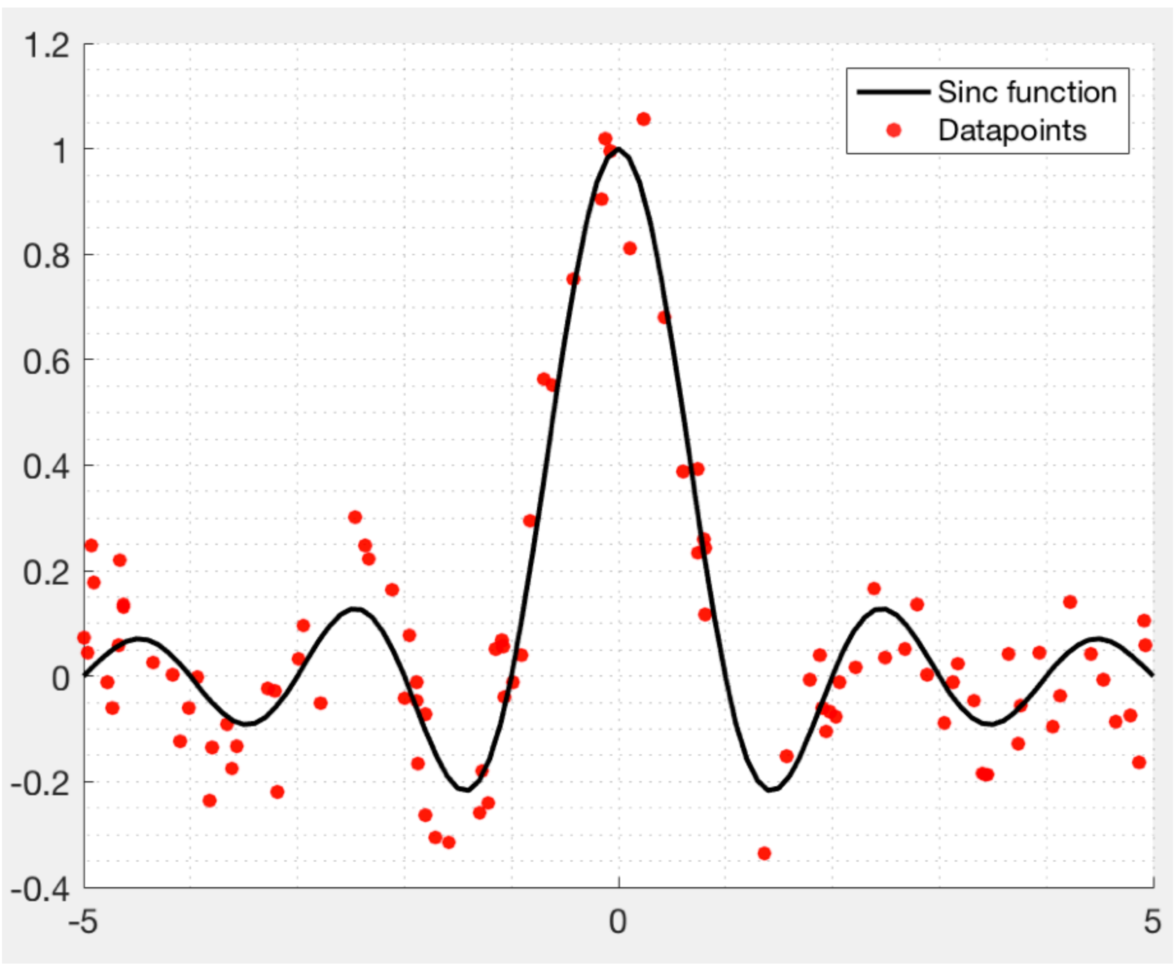
- **Datasets**

[5] *T.F. Brooks, D.S. Pope, and A.M. Marcolini.* Airfoil self-noise and prediction. *Technical report NASA. 1989.*

Datasets

Artificial (1d)



Dimension	Points	Support	Noise variance	Outlier
1	100	[-5,5]	0.01	No

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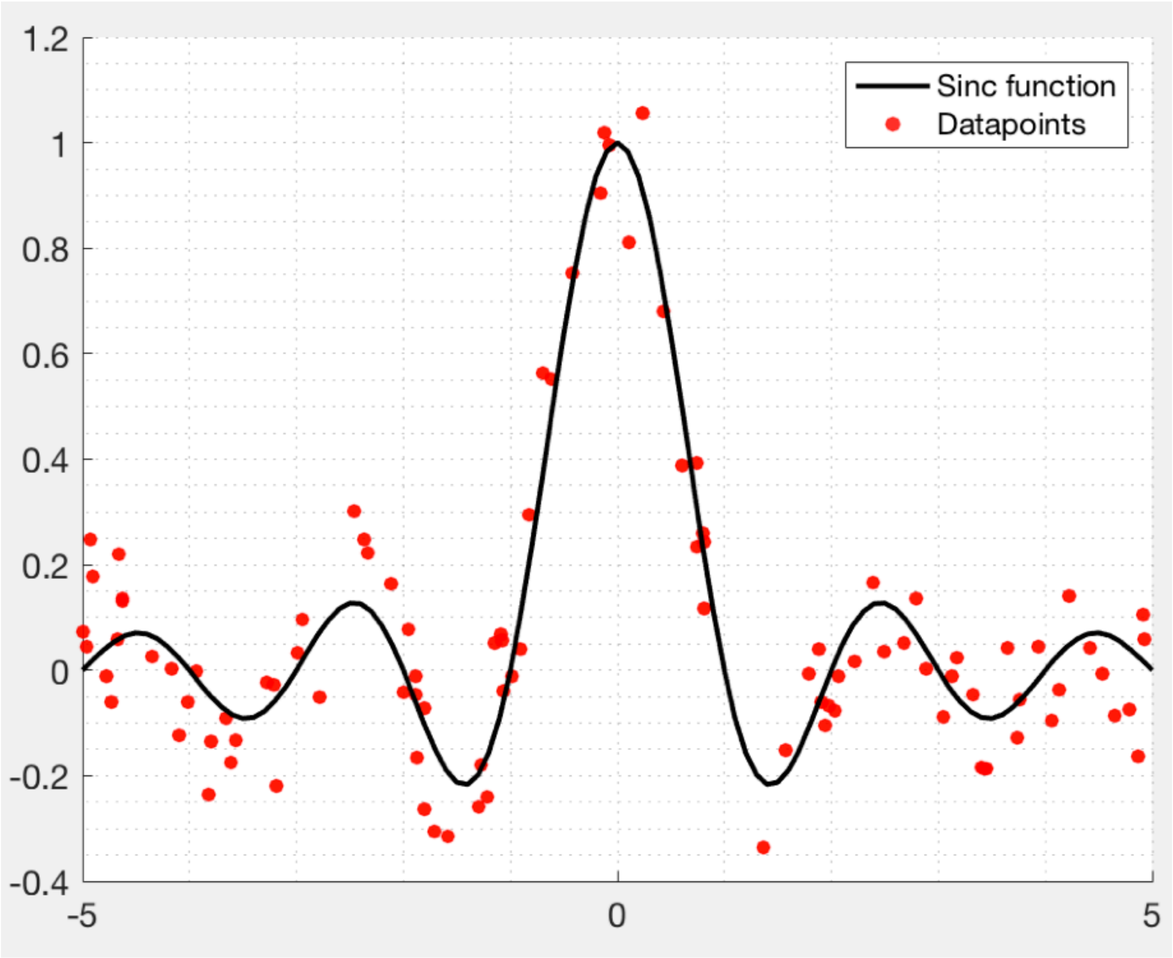

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Real (5d)

- Airfoil Self-Noise Data Set (NASA)[5]

Dimension	Points
5	1503

- Predict sound pressure (dB) according to few features :
  - Eigen frequency
  - Angle of attack
  - Chord Length
  - Free stream
  - Suction side displacement thickness

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- **Test Run**

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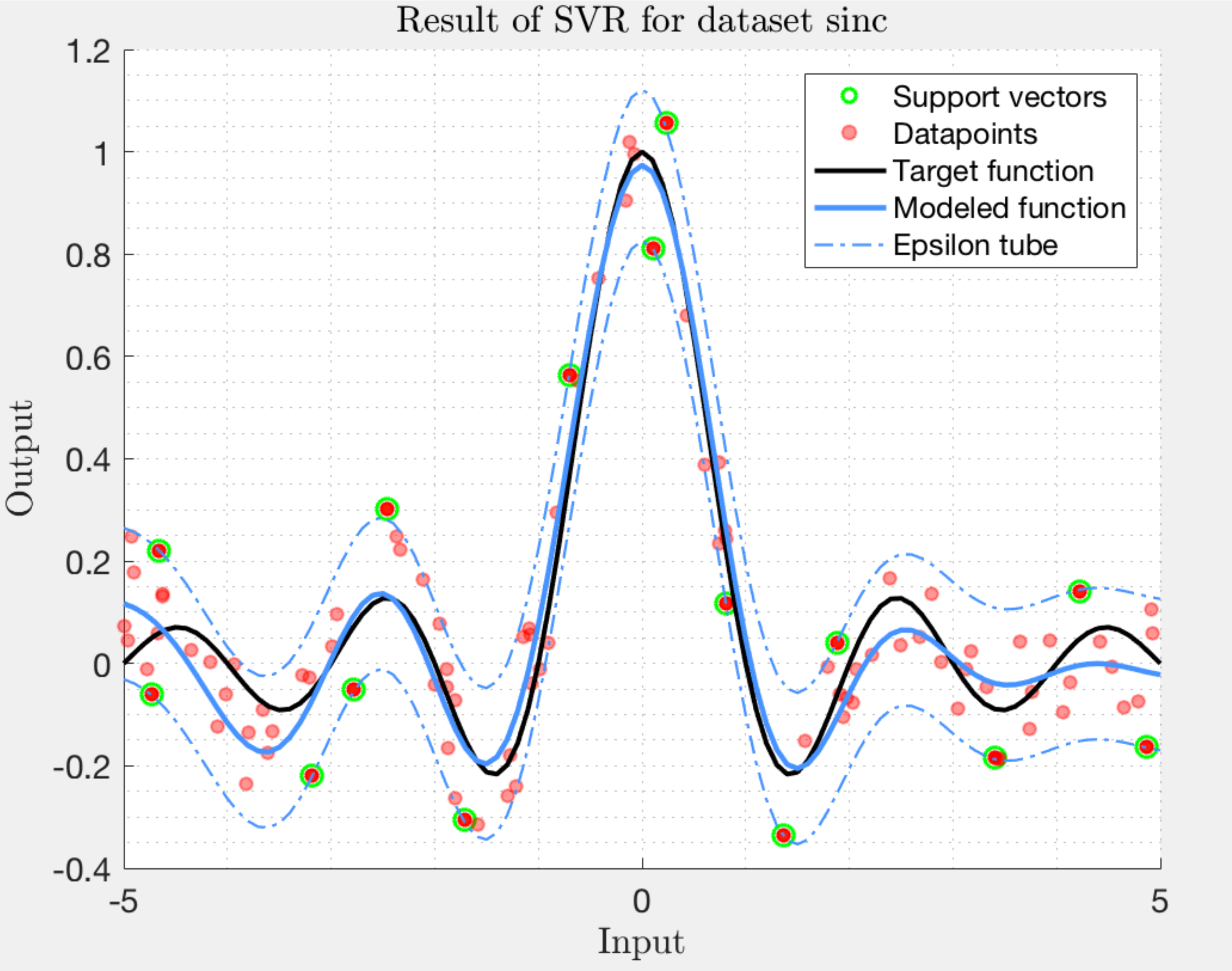

$\nu-$SVR, RBF kernel with:

$$\begin{cases} \nu & = 0.08 \\ C & = 8.5 \\ \sigma & = 1.4 \quad \text{(kernel width)} \end{cases}$$

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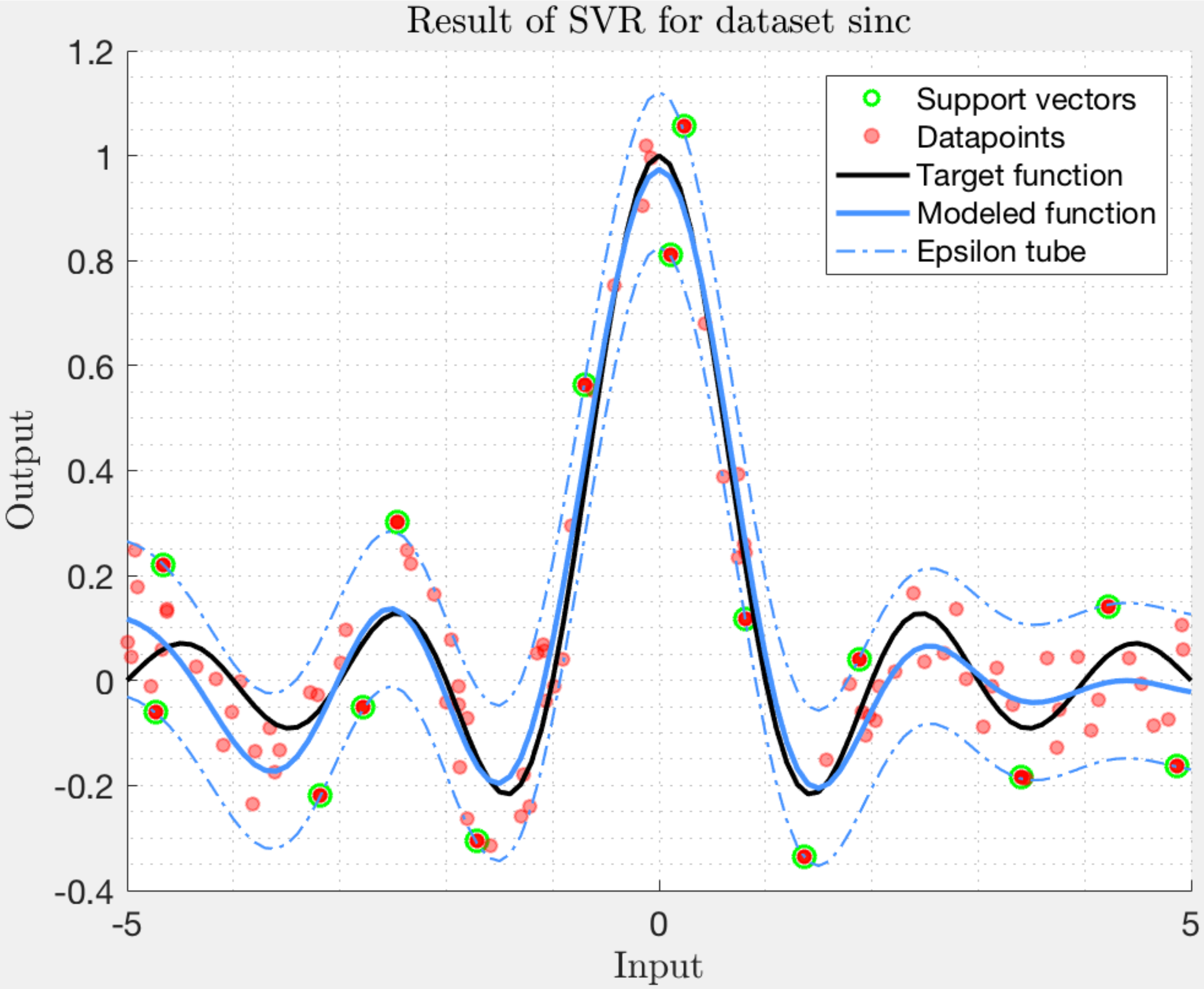

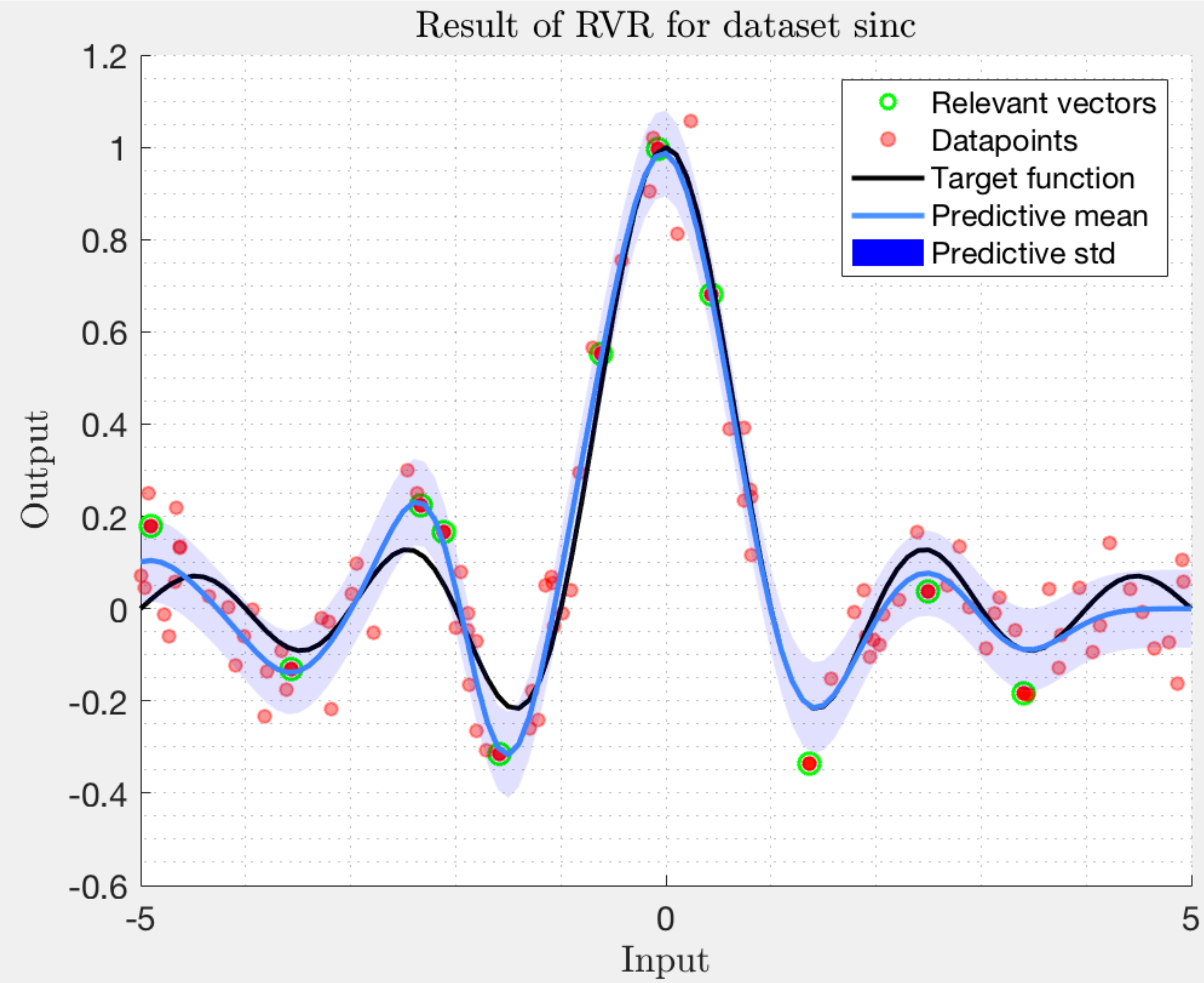

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RVR, RBF kernel with:

$$\sigma = 1$$

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Gaussian likelihood

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- BICSR (BIC for Sparse Regression)

$$\boxed{BICSR = \beta N \cdot MSE + k \log N}$$

- **Experimental Evaluation**

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  - For each method (SVR and RVR) :
    - Cross-validation to find the best hyper-parameters according to BICSR and MSE
    - Compare them with arbitrary models

- **Best hyper-parameters selection**

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- **Example for SVR with BICSR:**

Best hyper-parameters selection

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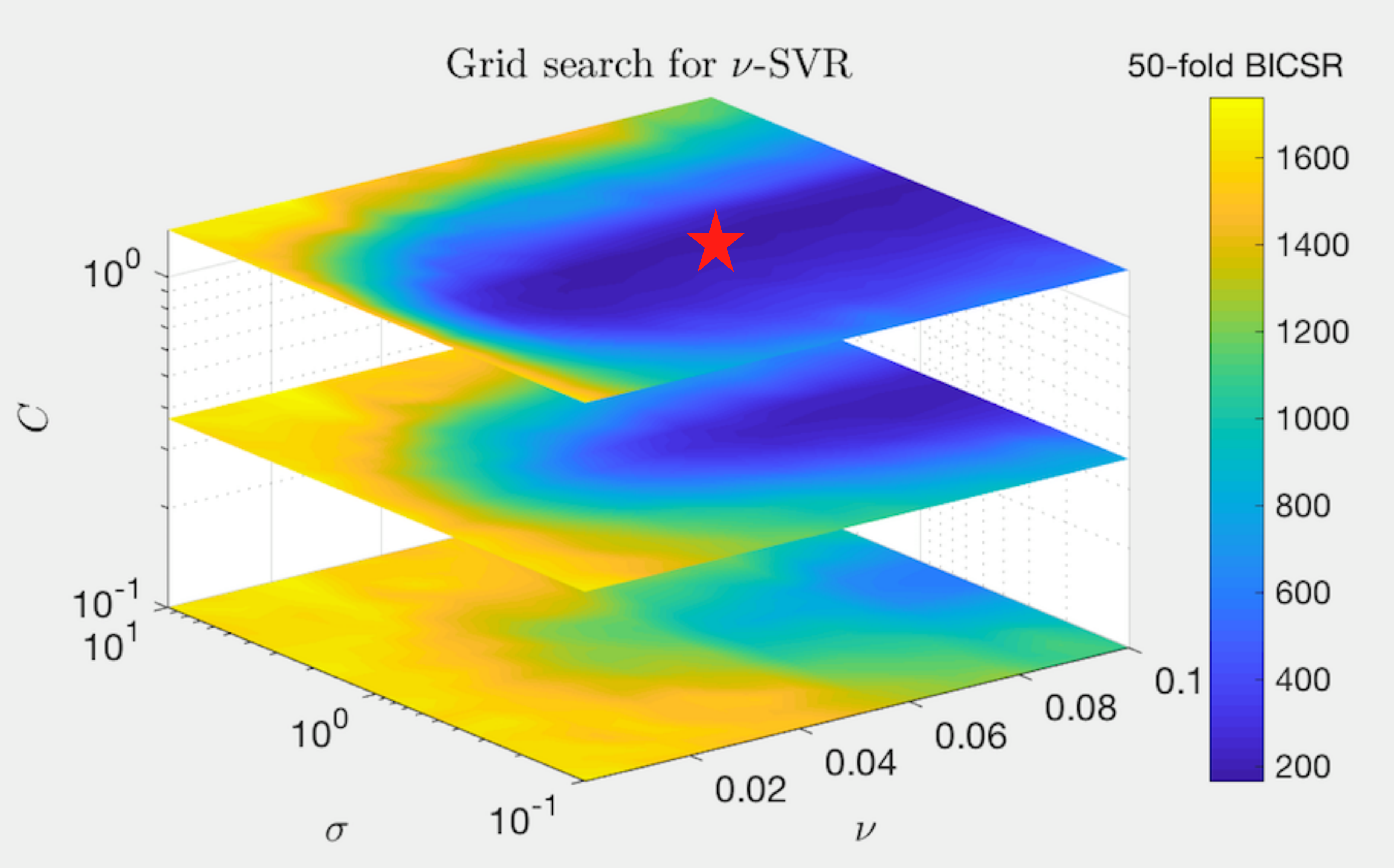


Best hyper-parameters selection

- **Example for SVR with BICSR:**

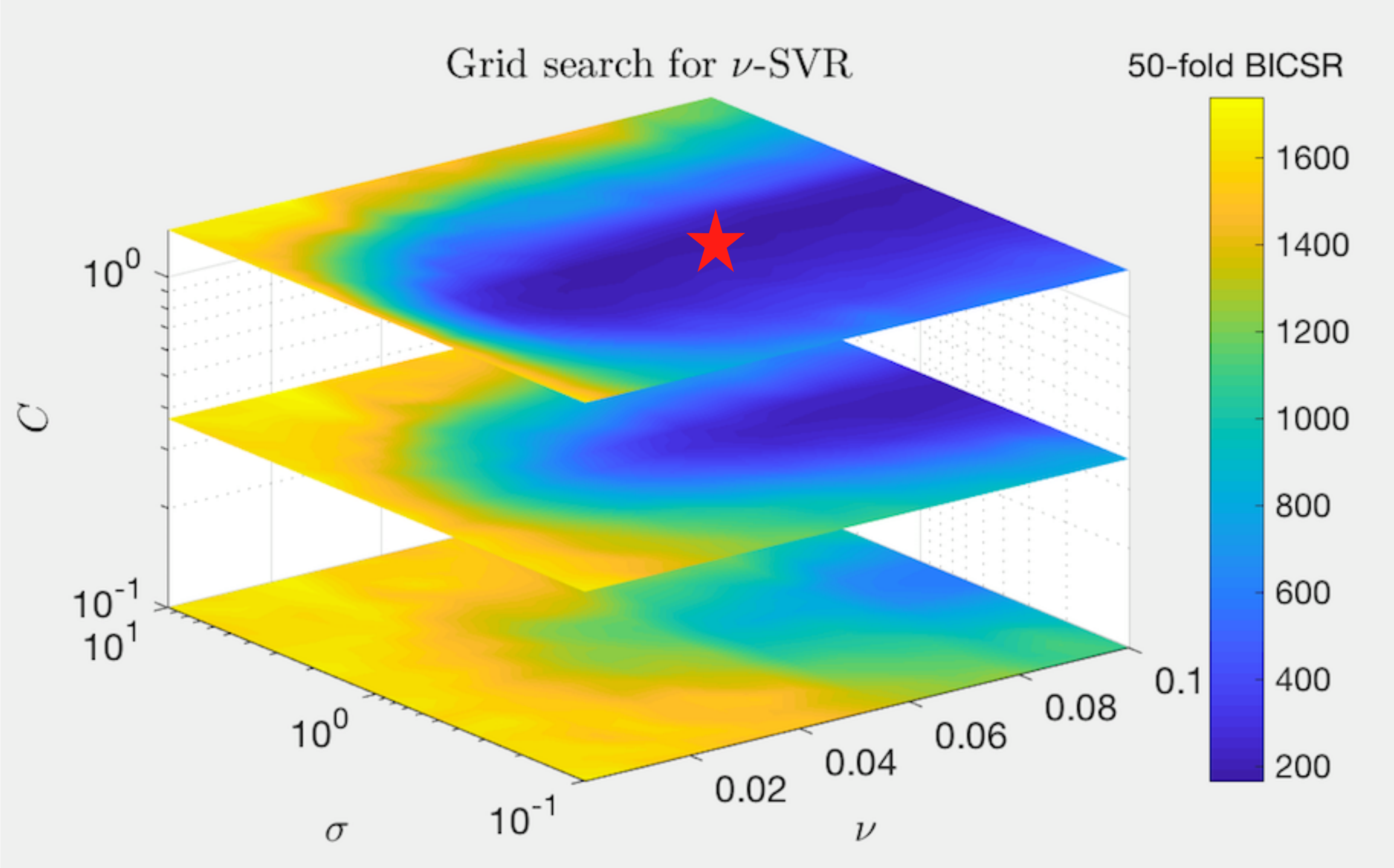
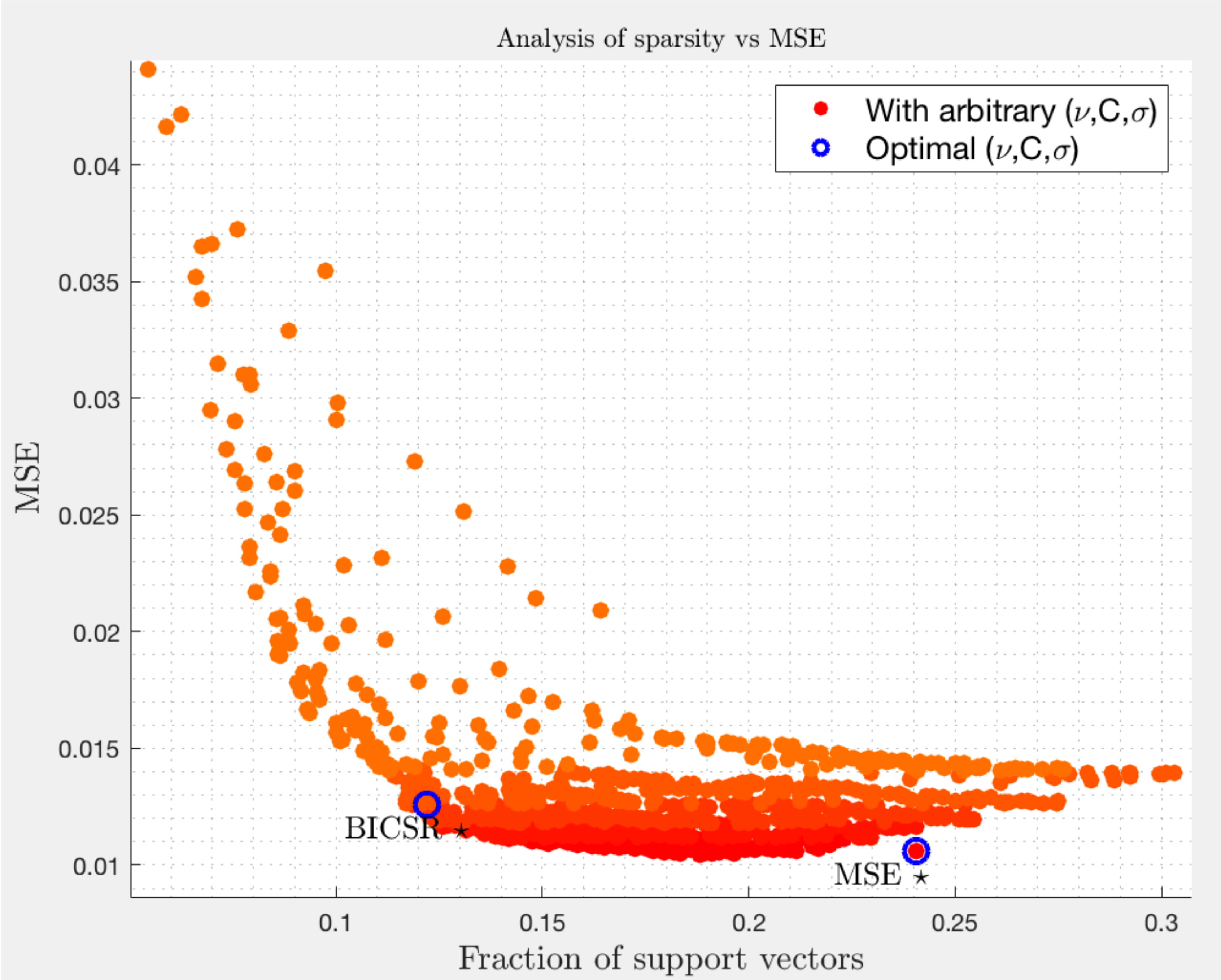


Figure : 50-fold cross-validation (0.75 training/test ratio)

Tradeoff evaluation (artificial dataset)

- **Example for SVR** (50 fold, 75 training/test ratio)**:**



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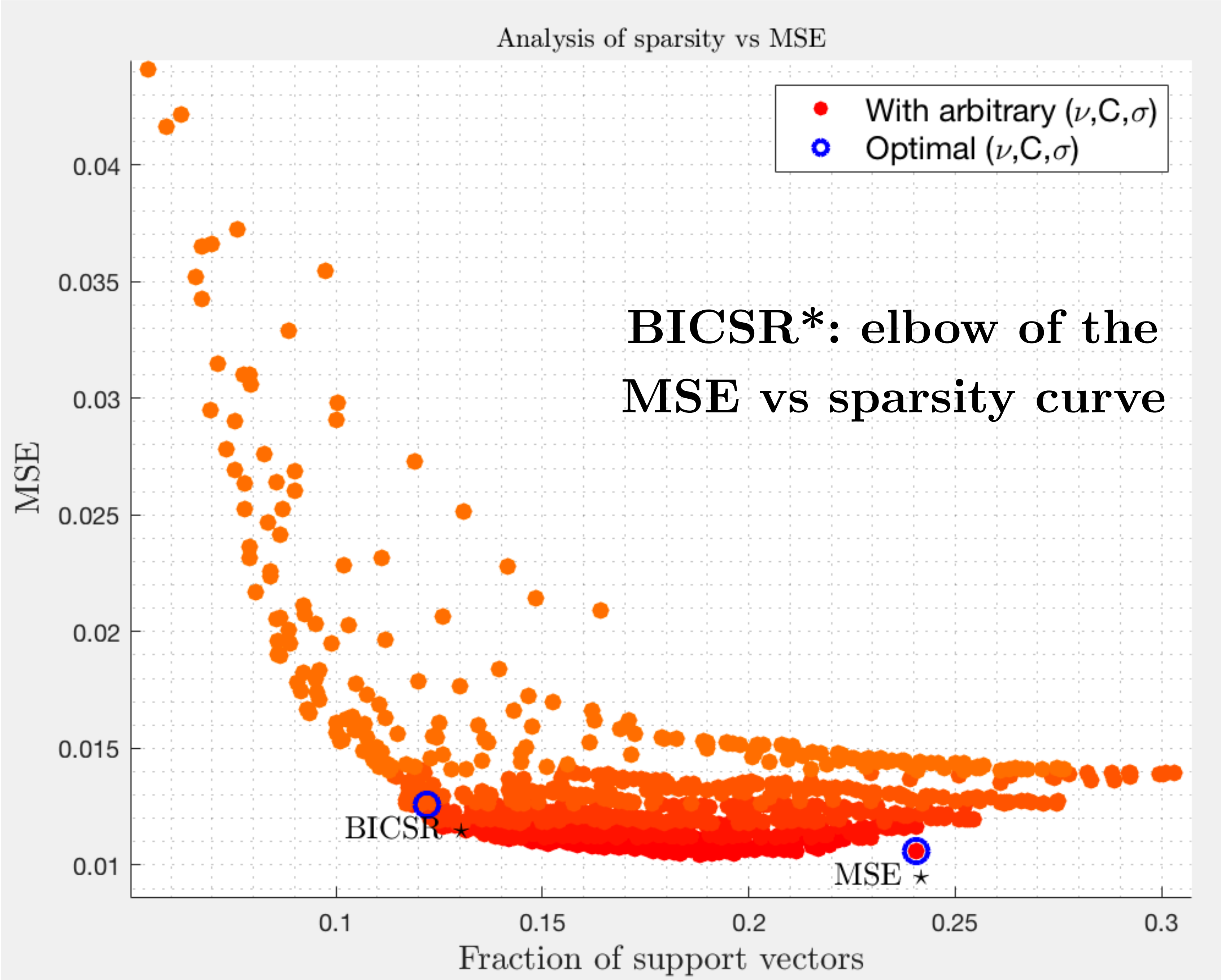

Tradeoff evaluation (artificial dataset)

- **Can we do better (different penalization) ?**

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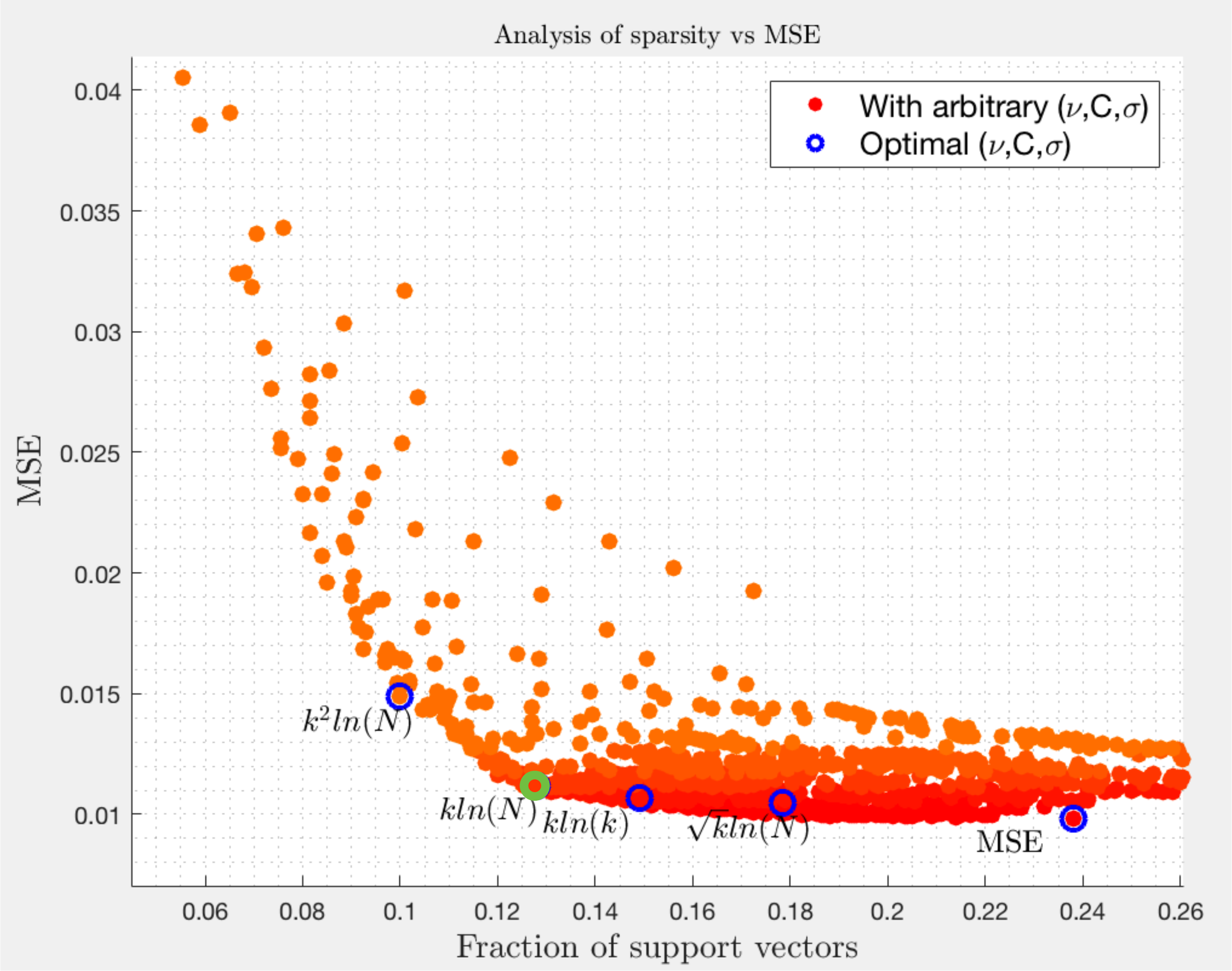
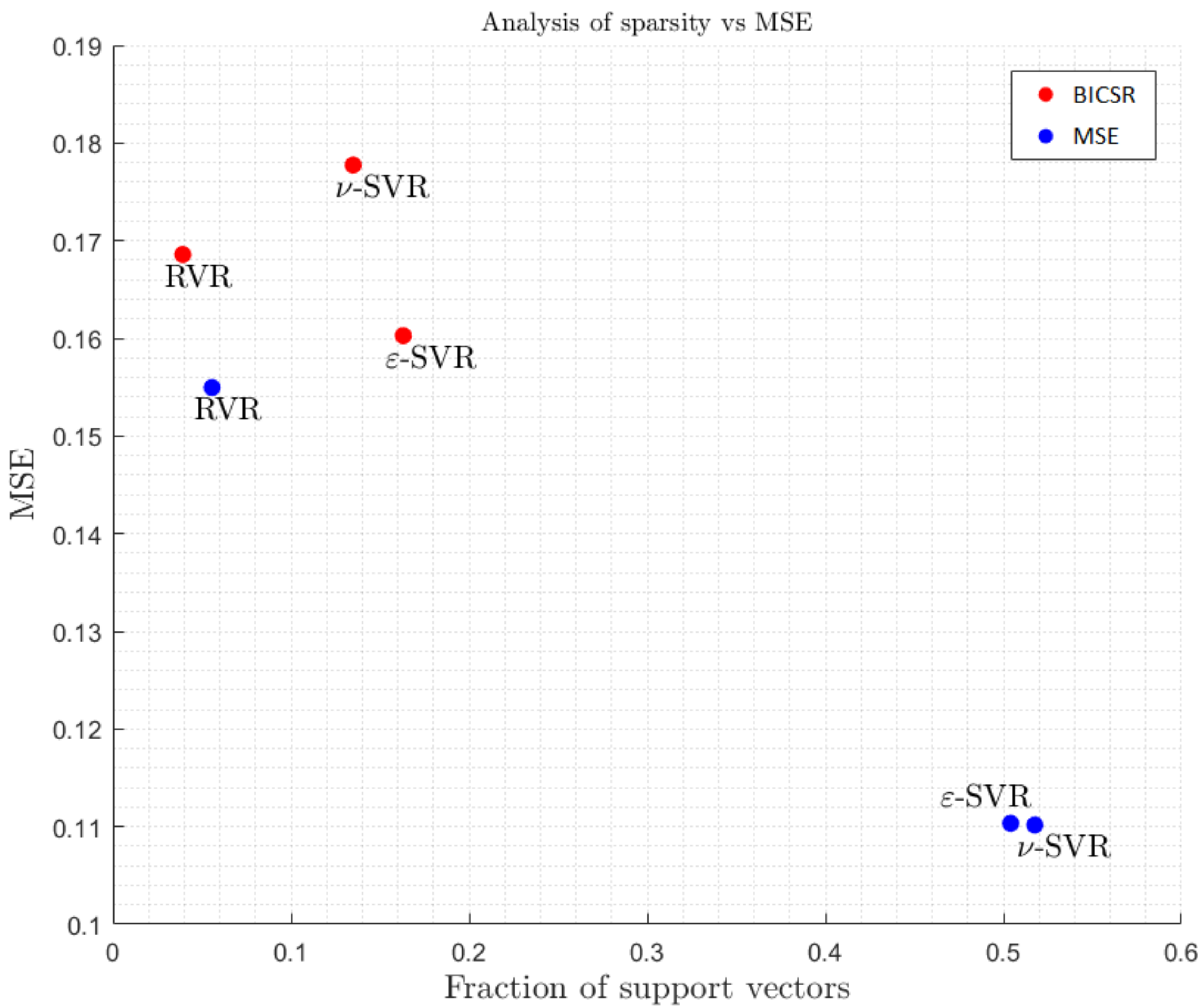


Model Comparaison (real dataset)



Conclusions

- BICSR seems to be a well-behaved sparse-regression metric

  (tradeoff between sparsity and performance)

- Even without sparsity penalization, RVR finds a fairly good compromise

  - most suited for fast predictions !

- SVR can be tuned to achieve either high sparsity or high regression performance

Other aspects :

- Behavior far from data
- Training cost
- Decision theory for predictions (predictive distribution)
- ..

Thank you for your attention !