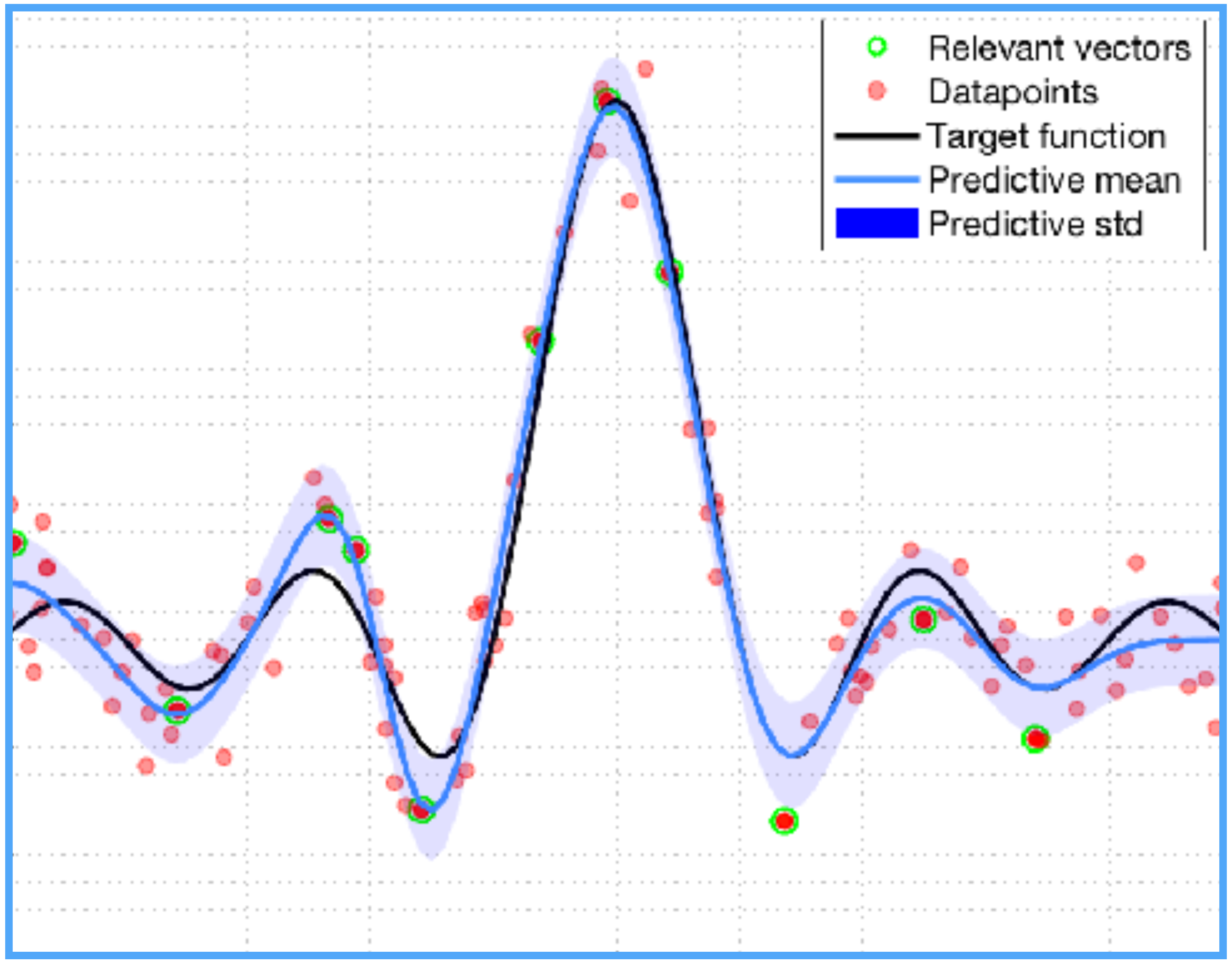
Support Vector Regression vs. Relevance Vector Regression

a sparsity / performance study



L.Faury G.Gallois-Montbrun H.Hendrikx

26/05/2017

Outline

- **Theoretical** reminders on both methods

- Introduction to a **sparse-regression** metric, experimental justification

- Sparse-regression metric based **cross-validation**

- Performance vs. sparsity discussion

Regression

Learn $f\ :\ \mathbb{R}^d \to \mathbb{R}$ thanks to a dataset $\{X, t\} \in (\mathbb{R}^d)^n \times \mathbb{R}^n$

Assuming a Gaussian **conditional p.d.f** around a linear transformation of features :

$$p(t \,|\, x, w) = \mathcal{N}(t \,| w^T \phi(x), \beta^{-1})$$

the maximum-likelihood estimator (MLE) writes :

$$\begin{aligned} \hat{w} &= \mathrm{argmax}_w\, p(t \,|\, X, w) \\ &= \mathrm{argmin}_w\, \frac{1}{2} \sum_{i=1}^{n} \|w^t \phi(x) - t\|^2 \end{aligned}$$

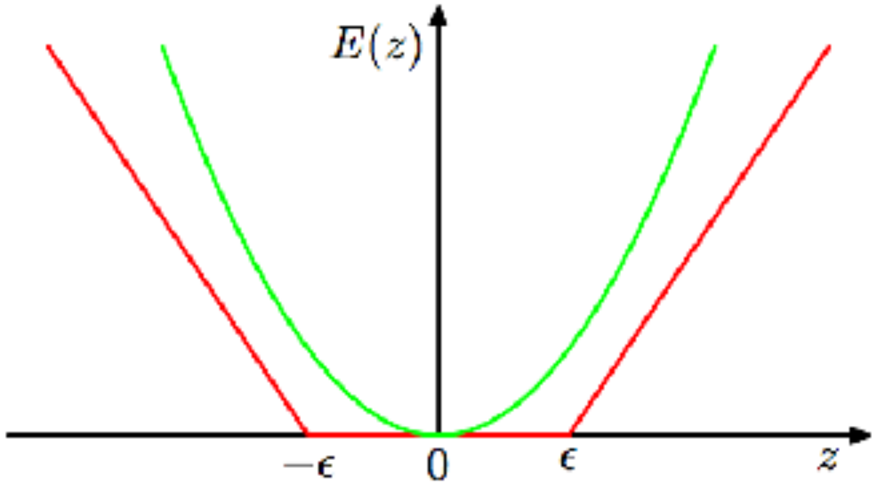
Theoretical Reminders

- ## Support Vector Regression

[1] *Vladimir Vapnik*, The nature of statistical learning theory, *1995*

Support Vector Regression

- Introduce the ε-insensitive[1] loss-function.



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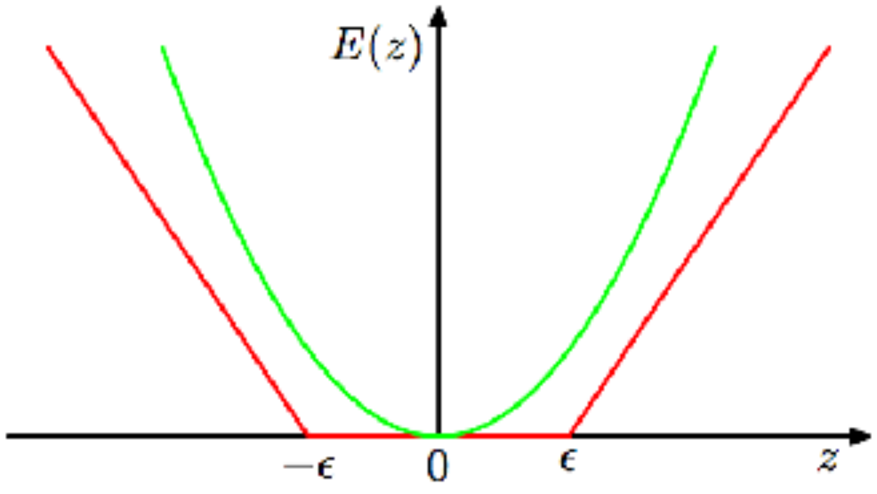
$$min_w \frac{C}{n}\sum_n(\xi_n+\hat{\xi}_n)+\frac{1}{2}\|w\|^2$$

$$\text{s.t} \quad \begin{cases} \xi,\hat{\xi}\geq 0 \\ w^T\phi(x_n)+\xi_n+\varepsilon\geq t_n \\ w^T\phi(x_n)-\hat{\xi}_n-\varepsilon\leq t_n \end{cases}$$



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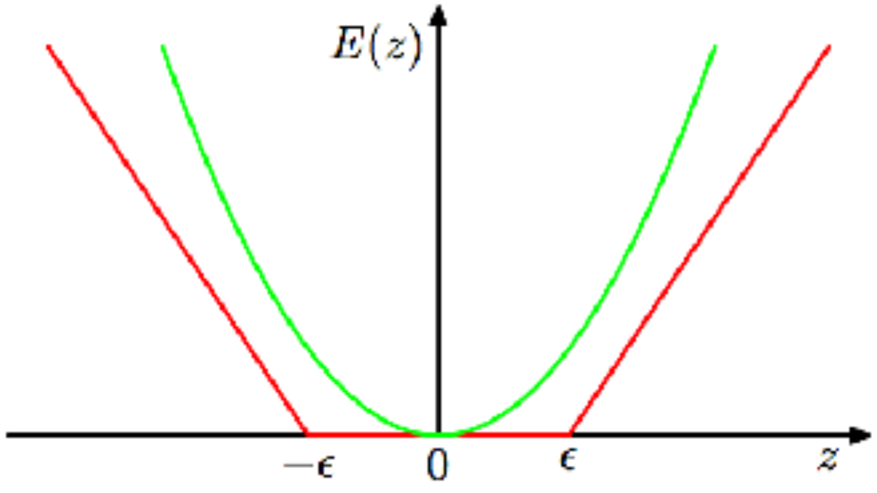

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- Only points outside the ε-tube (**active constraints**) are used for predictions :

$$y(x)=\textstyle\sum_{n\in\mathcal{S}}(a_n-\hat{a}_n)k(x,x_n)\quad\longrightarrow\quad\text{Posterior } \textbf{decision}$$

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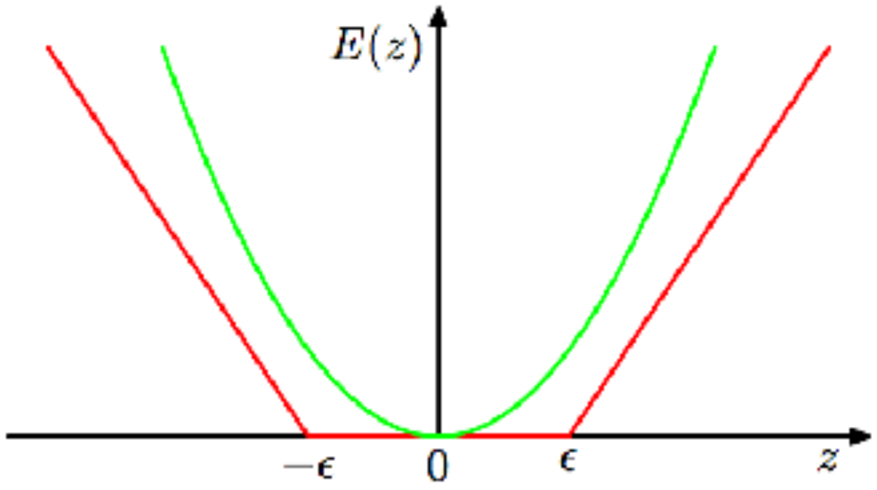

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■Relevance Vector Regression[2]

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Relevance Vector Regression[2]

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Comparison

SVR

RVR

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- Held-out method for hyper-parameters (at least 3)

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- **Training** : SMO(3) (somewhere between linear and quadratic)

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Training : SMO[3] (somewhere between linear and quadratic)	**Training** : cubic complexity
Testing : linear in the SV	**Testing** : linear in the RV

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Goal

- **Sparsity / Performance**

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- **Question** : Compare the tradeoff found between performance (MSE minimization) and sparsity

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- **Literature**[4] : RVR reaches sparser models with equivalent generalization skills.

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Goal

■ Sparsity / Performance

- **Question** : Compare the tradeoff found between performance (MSE minimization) and sparsity

- **Literature**[4] : RVR reaches sparser models with equivalent generalization skills.

- **Initial idea :** Test (**experimentally**) this assertion

  - what is performance ?
  - what do we want with sparsity ?
  - how to measure the tradeoff ?

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Dataset presentation

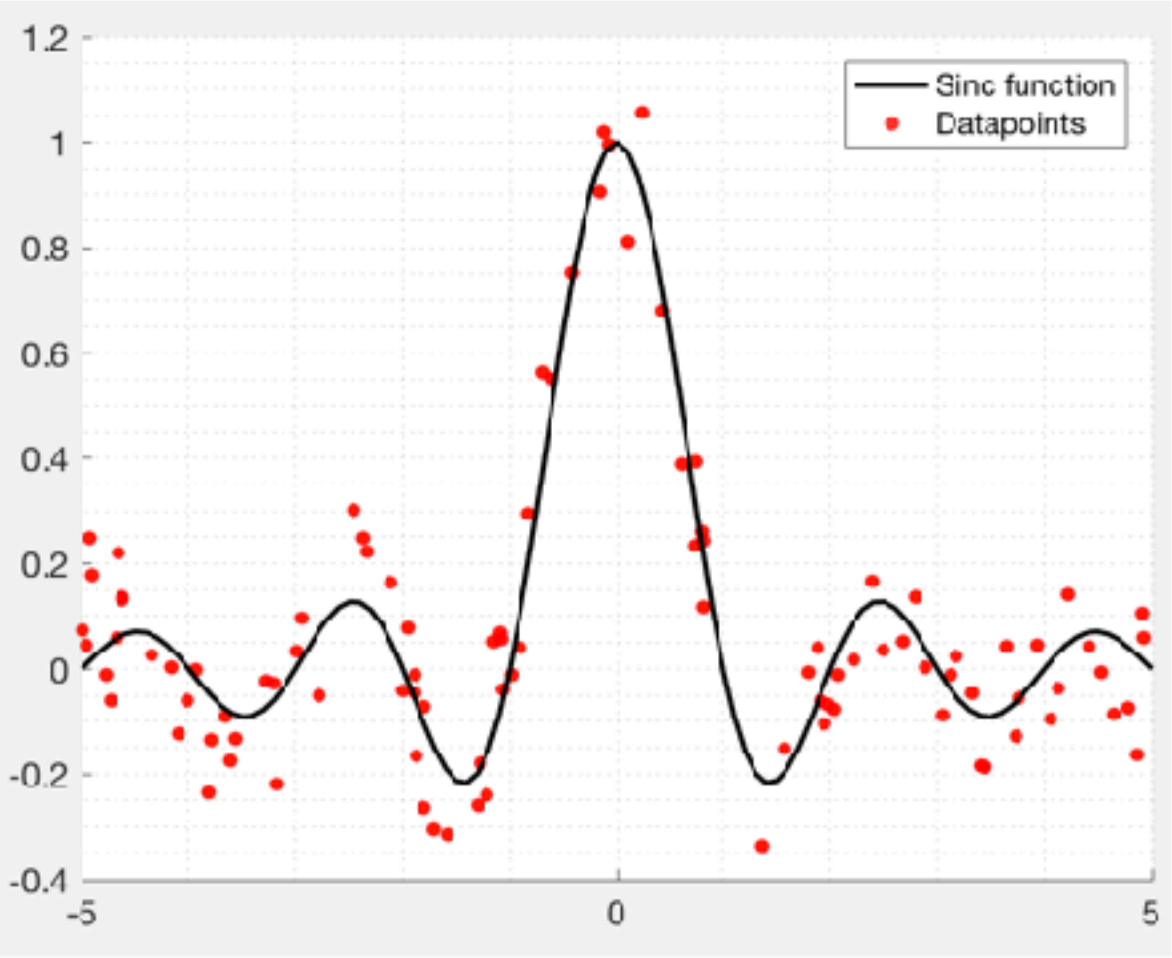
- **Datasets**

[5] *T.F. Brooks, D.S. Pope, and A.M. Marcolini.* Airfoil self-noise and prediction. *Technical report NASA. 1989.*

Dataset presentation

Datasets

Artificial (1d)



Dimension	Points	Support	Noise variance	Outlier
1	100	[-5,5]	0.01	No

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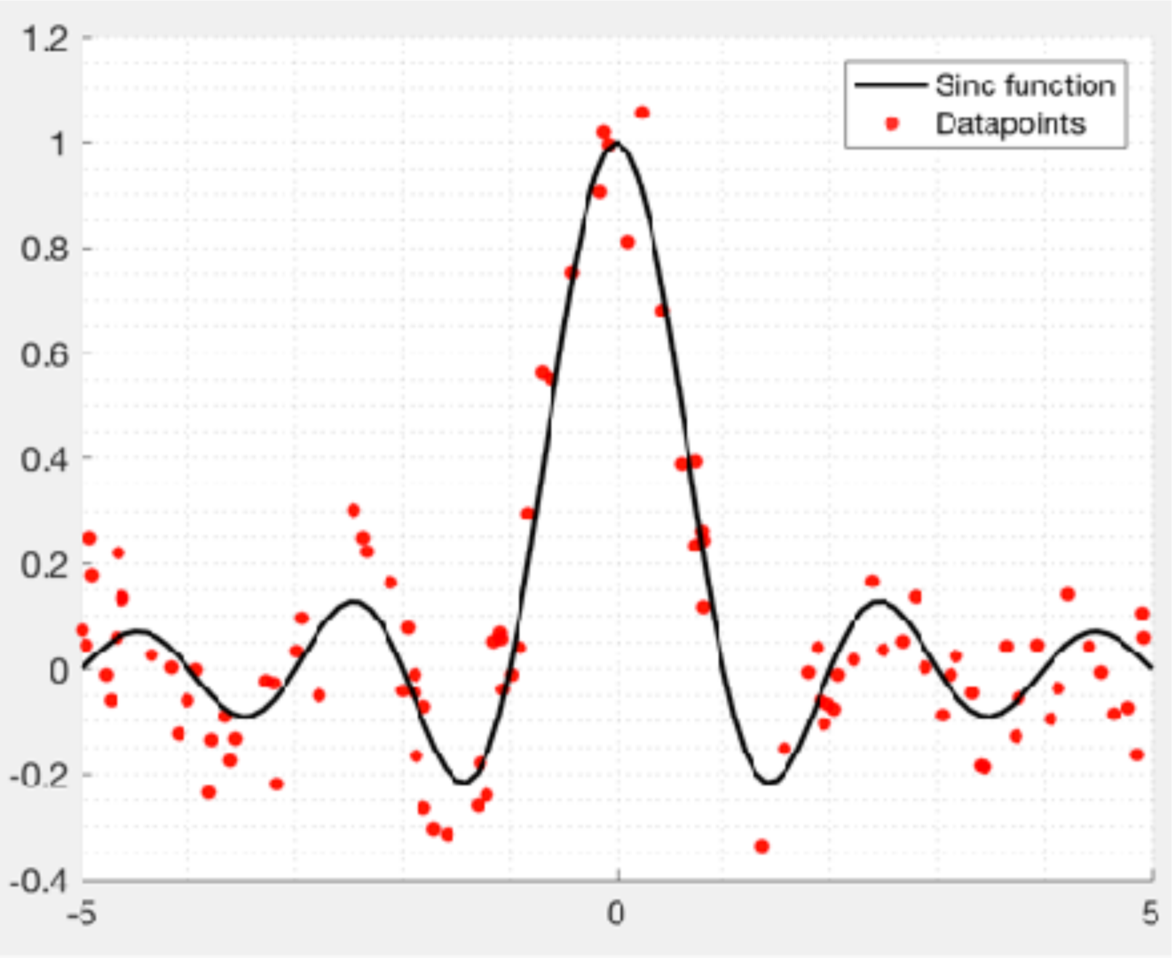

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Real (5d)

- Airfoil Self-Noise Data Set (NASA)[5]

Dimension	Points
5	1503

- Predict sound pressure (dB) according to few features :
  - Eigen frequency
  - Angle of attack
  - Chord Length
  - Free stream
  - Suction side displacement thickness

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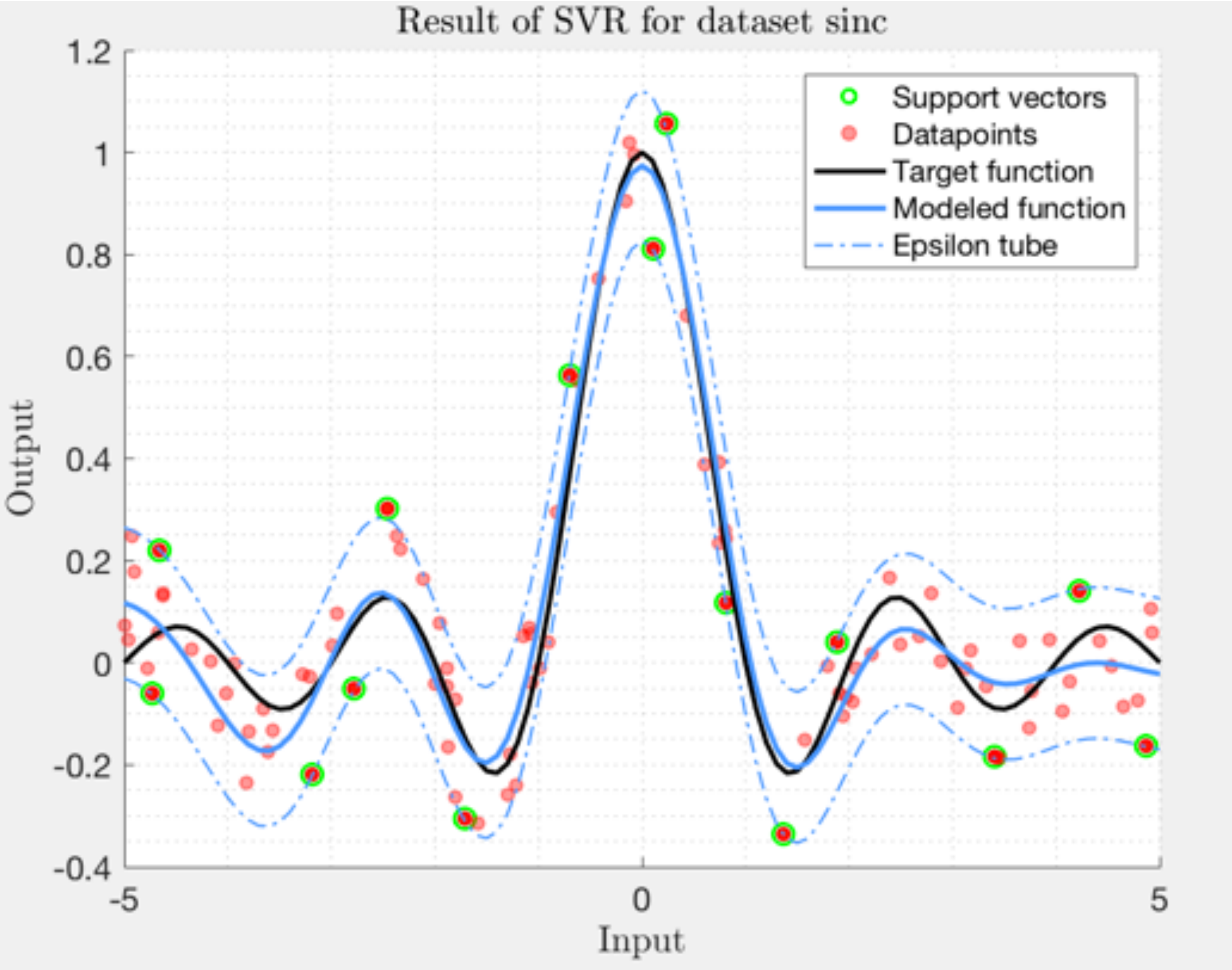

$\nu-$SVR, RBF kernel with:

$$\begin{cases} \nu & = 0.08 \\ C & = 8.5 \\ \sigma & = 1.4 \quad \text{(kernel width)} \end{cases}$$

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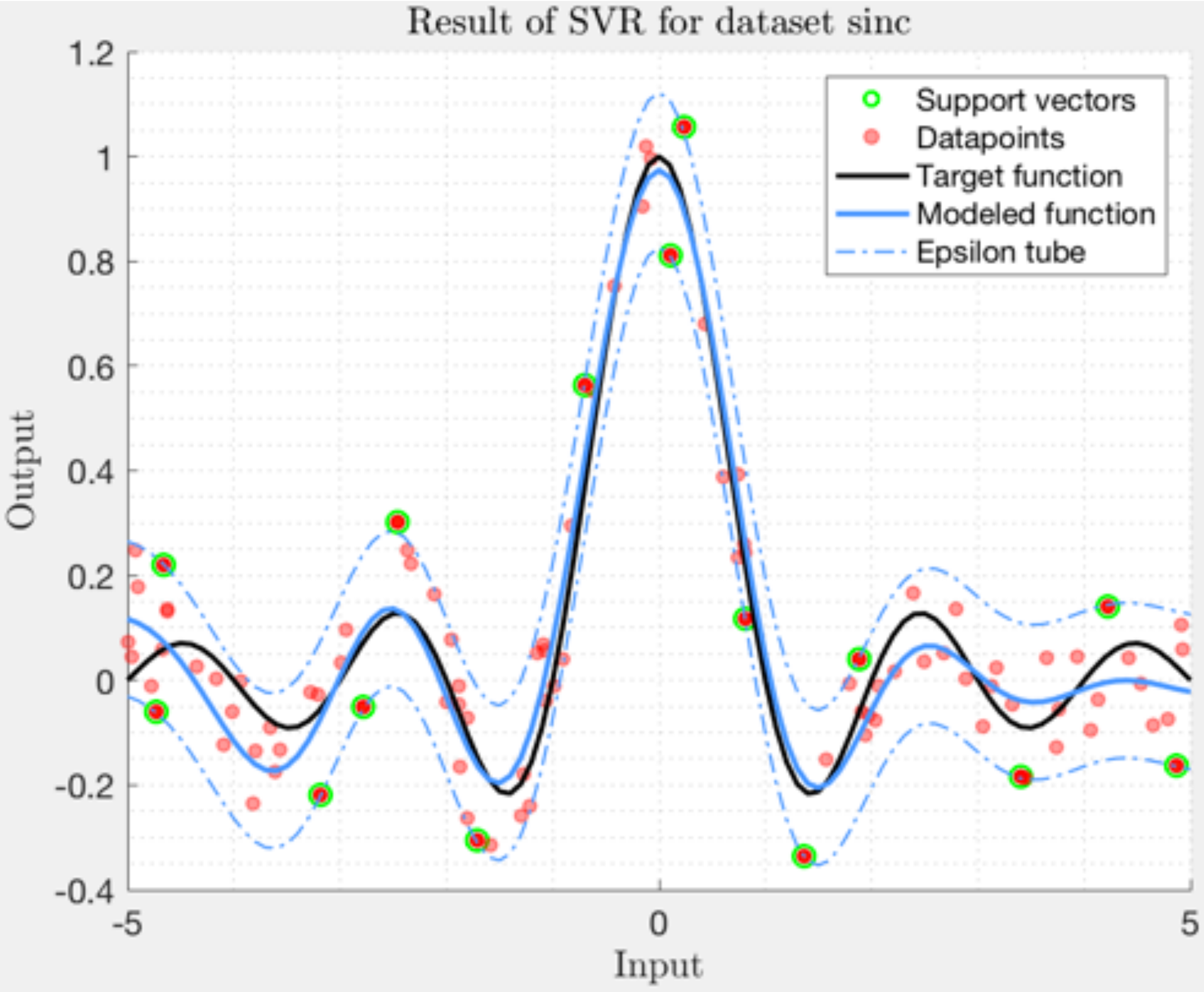

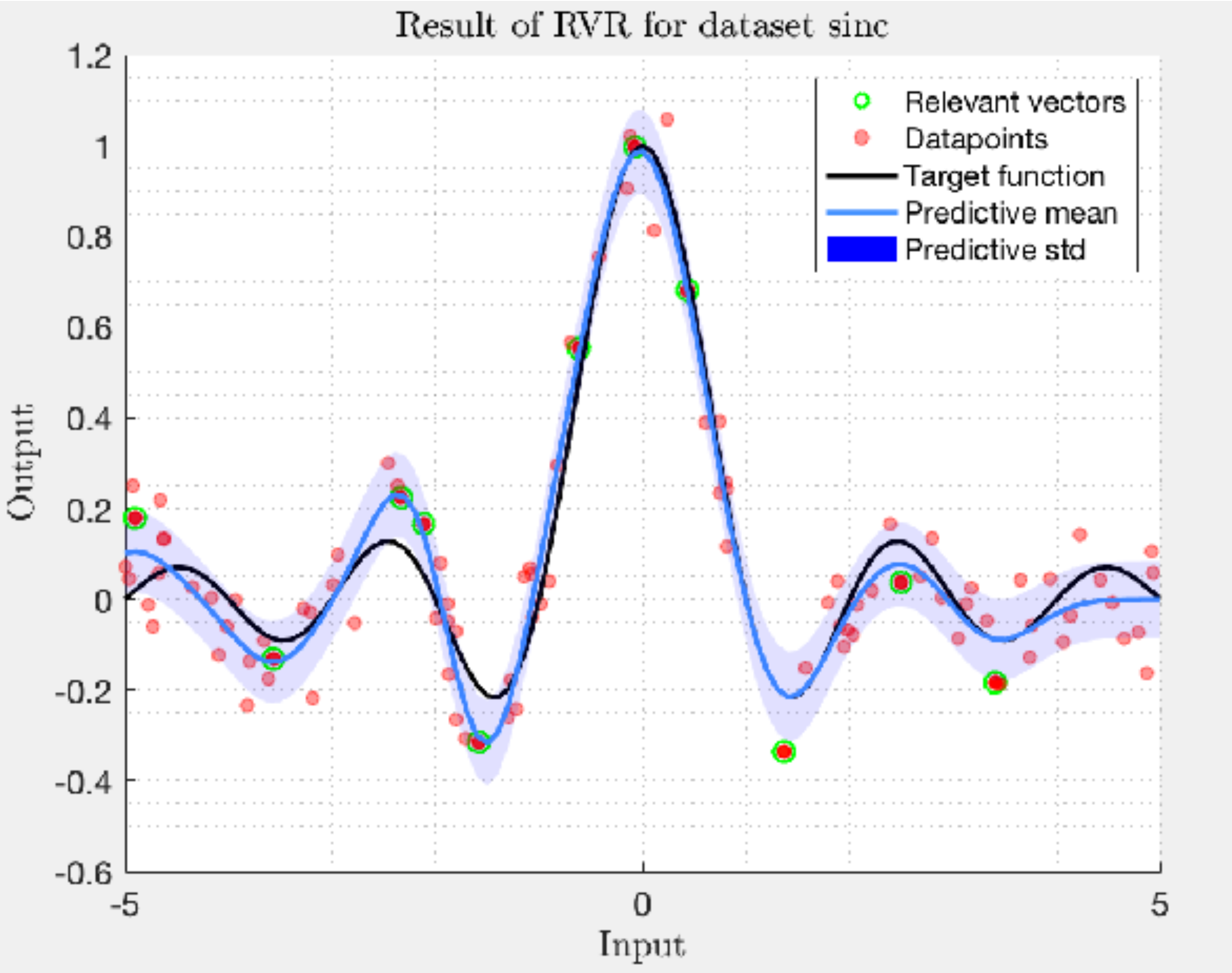

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RVR, RBF kernel with:

$$\sigma = 1$$

Sparse Regression Metric

- **Intuition**

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Intuition

- **Goal**: Maximize performance while penalizing complexity

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$$BIC = -2 \underbrace{\log \mathcal{L}(x)}_{\text{likelihood}} + \underbrace{M \log N}_{\text{model complexity}}$$

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- Adaptation to regression :

$$-\log \mathcal{L}(x) = \beta^{-1} N \cdot MSE(x,t)$$

Gaussian likelihood

$$M \overset{\triangle}{\sim} |SV| = k$$

complexity = number of support vectors

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- BICSR (BIC for Sparse Regression)

$$\boxed{BICSR = \beta^{-1} N \cdot MSE + k \log N}$$

Sparse Regression Metric

- **Experimental Evaluation**

Experimental Evaluation

- **Goal**: Evaluate the tradeoff found by the BICSR metric

Experimental Evaluation

- **Goal**: Evaluate the tradeoff found by the BICSR metric

- For each method (SVR and RVR) :
  - Cross-validation to find the best hyper-parameters according to BICSR and MSE
  - Compare them with arbitrary models

Sparse Regression Metric

- **Best hyper-parameters selection**

Sparse Regression Metric

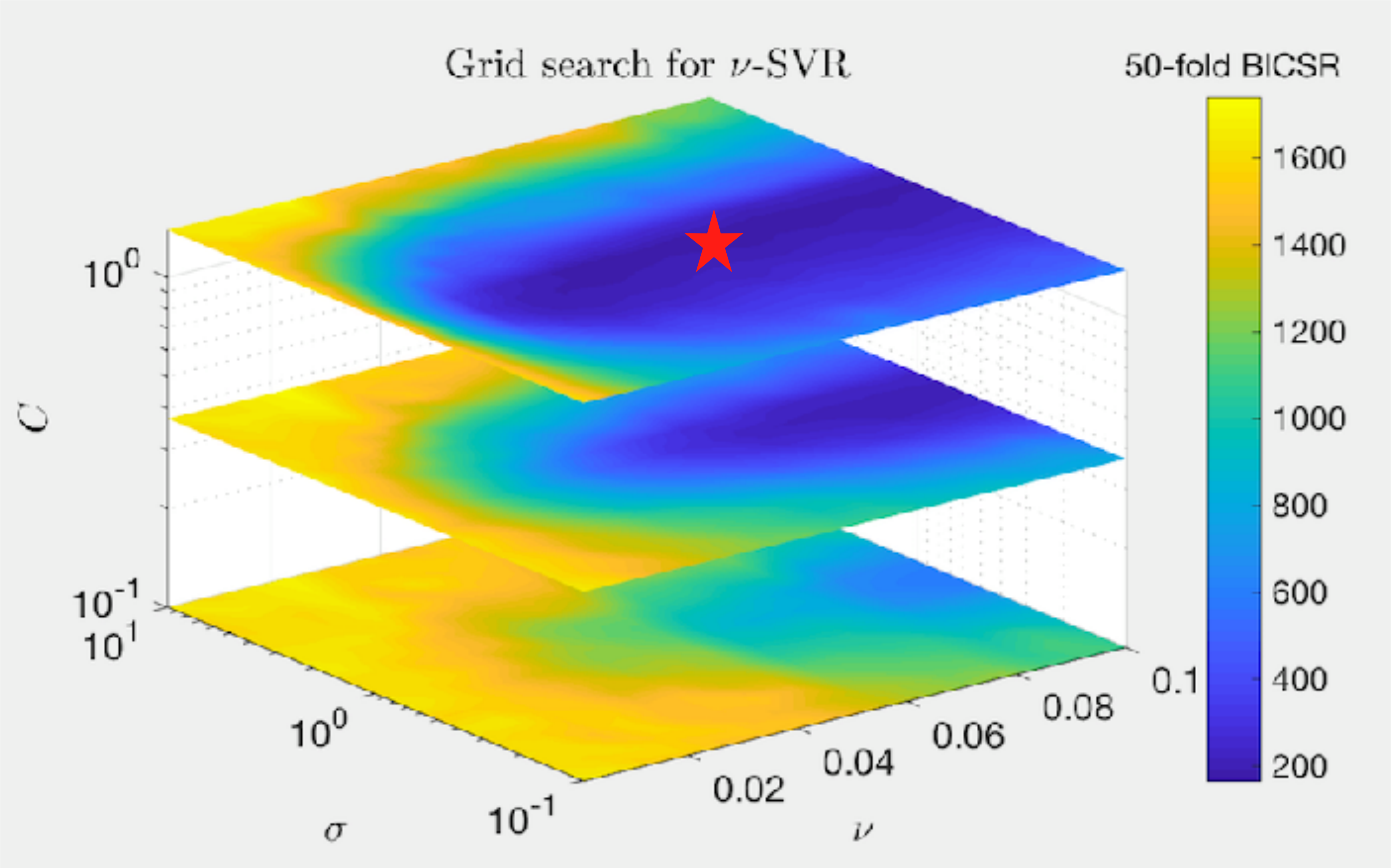
■ Best hyper-parameters selection

- **Example for SVR with BICSR:**

Sparse Regression Metric

Best hyper-parameters selection

- **Example for SVR with BICSR:**



Sparse Regression Metric

■ Best hyper-parameters selection

- **Example for SVR with BICSR:**

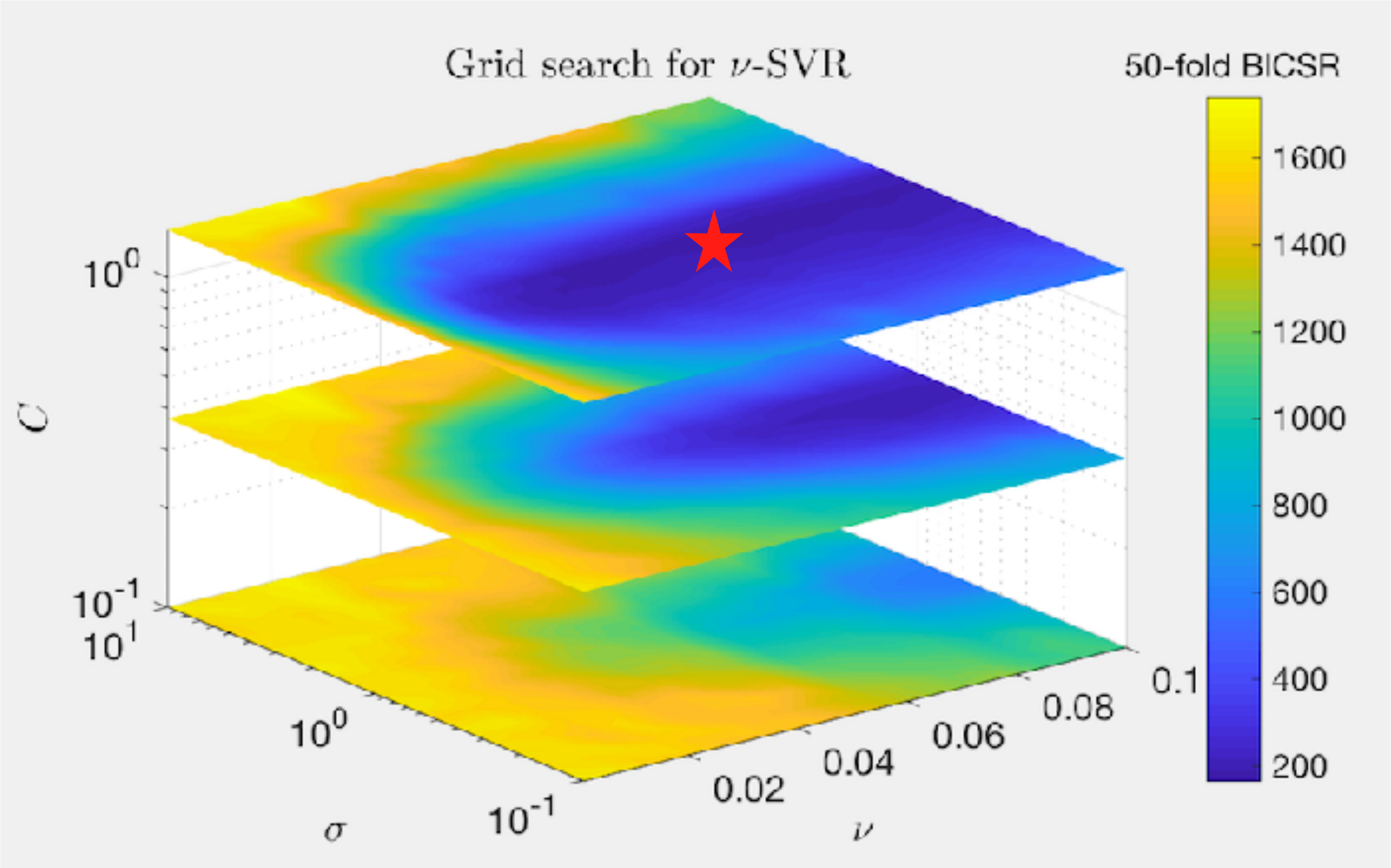
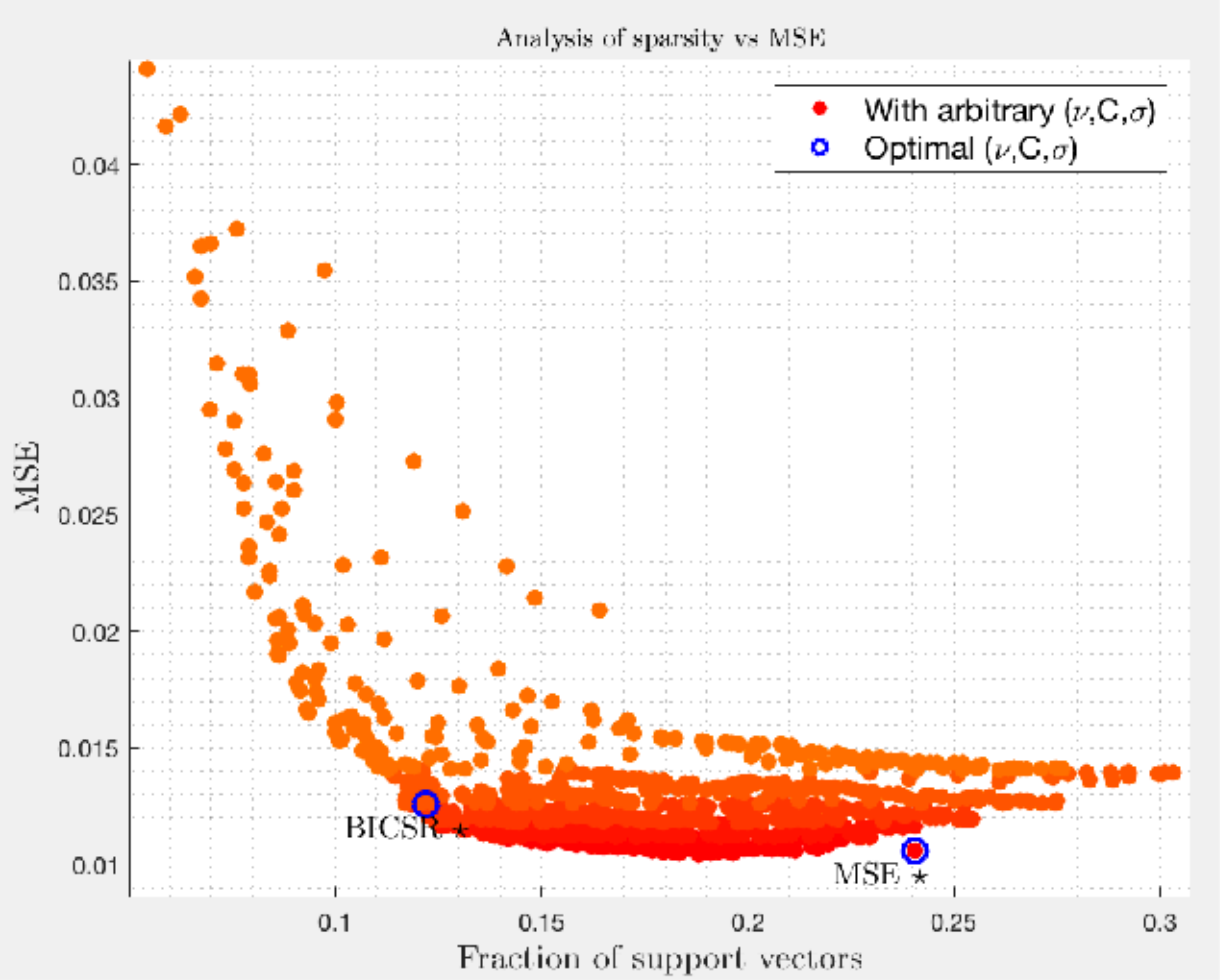


Figure : 50-fold cross-validation (0.75 training/test ratio)

Tradeoff evaluation (artificial dataset)

- **Example for SVR** (50 fold, 75 training/test ratio)**:**



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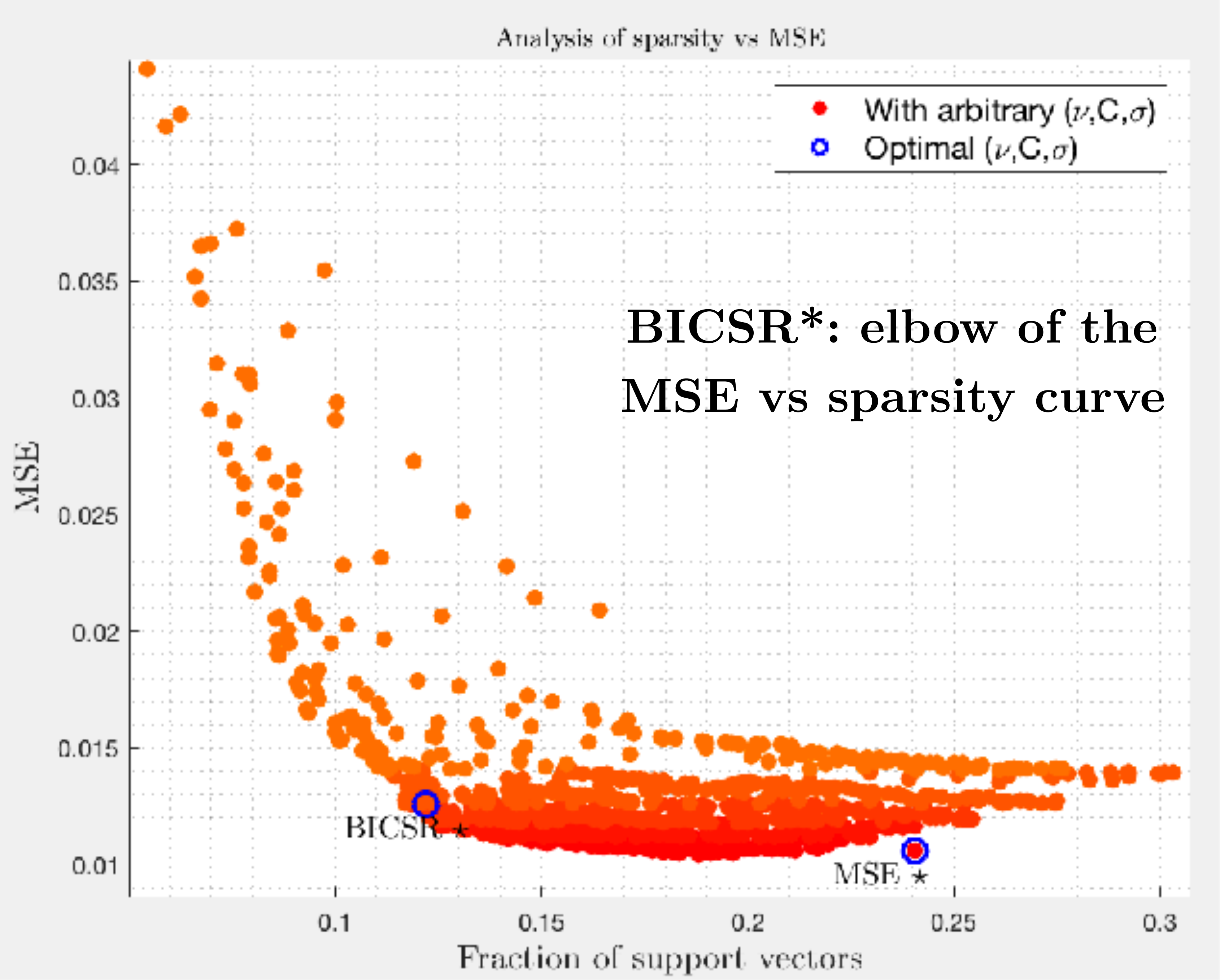

BICSR*: elbow of the MSE vs sparsity curve

Sparse Regression Metric

■ **Tradeoff evaluation** (artificial dataset)

- **Can we do better (different penalization) ?**

$$BICSR = \beta^{-1} N \cdot MSE + \boldsymbol{k} \log \boldsymbol{N}$$

Sparse Regression Metric

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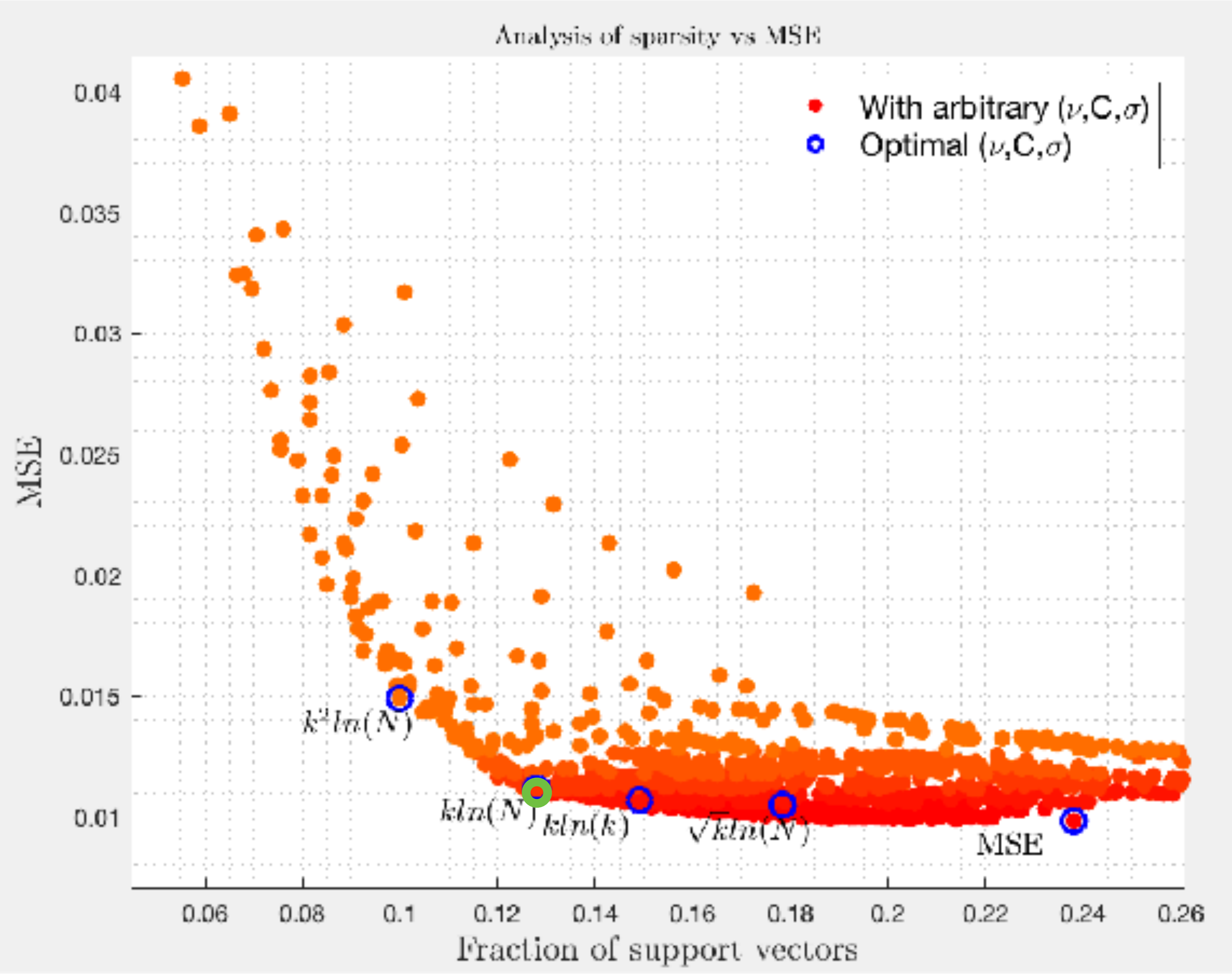
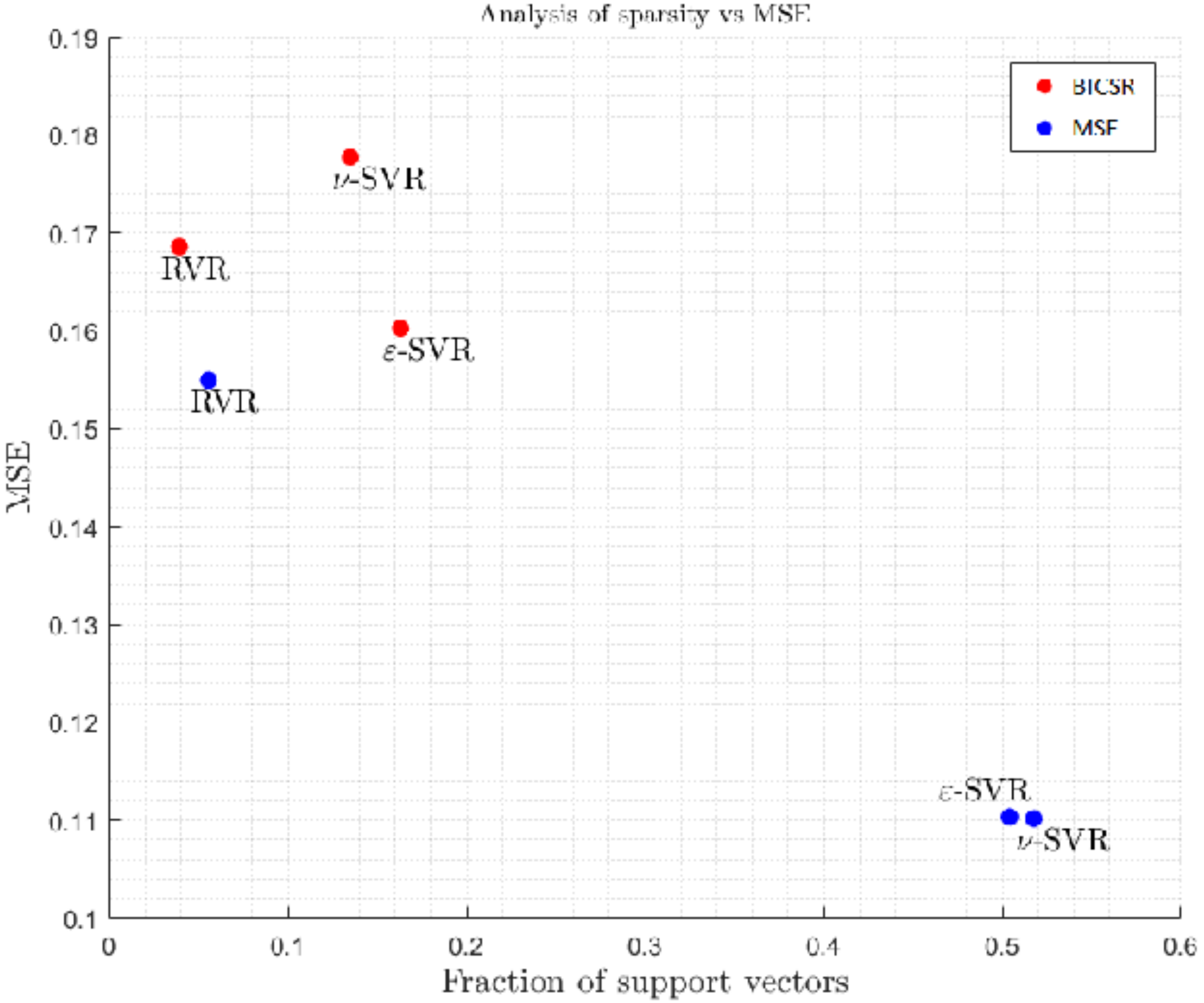


Model Comparaison (real dataset)



Conclusions

- BICSR seems to be a well-behaved sparse-regression metric (tradeoff between sparsity and performance)

- Even without sparsity penalization, RVR finds a fairly good compromise
  - most suited for fast predictions !

- SVR can be tuned to achieve either high sparsity or high regression performance

Other aspects :

- Behavior far from data
- Training cost
- Decision theory for predictions (predictive distribution)
- ..

Thank you for your attention !