

Susceptible-Infected Epidemic in Graphs

Louis Faury

Gallois - Montbrun Grégoire

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Plan

1 Theoretical Approach

- Population Model
- Propagation Model
- Time before complete infection

2 Applications

- Epidemic Propagation
- Sensor Networks

3 Model refinement

- SIS approach
- Arbitrary graph topology

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- 1 Theoretical Approach
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■ Hold the following assumptions for true :

- ▶ Each individual in the population is either *susceptible* or *infected*
- ▶ Once infected, an individual can't recover and tries to infect other individuals
- ▶ An individual can affect any other individual in the population (**uniform topology**)

■ Graph population representation :

$$G = (V, E) \quad (1)$$

with $|V| = n \in \mathbb{N}^*$

Uniform topology \Rightarrow **complete graph**.

■ Propagation :

- ▶ Each infected node tries to infect one of its neighbor at the times of a **Poisson process** of intensity $\lambda > 0$.
- ▶ The node's Poisson processes are independent.
- ▶ An infected node picks one of its neighbor uniformly at random when trying to infect.

■ Let X_t the number of infected node at time $t > 0$.

→ $\{X_t\}$ is a **Markov Jump Process** on \mathbb{R}^+ .

■ Markov Jump Process on E :

- ▶ If, $\forall x, y \in E$:

$$p_{xy}(t) = \mathbb{P}(X_{t+s} = y \mid X_s = x) \quad (2)$$

then

$$\lim_{t \rightarrow 0} p_{xx}(t) = 1 \quad (3)$$

- ▶ The process remains at each stage for a strictly positive time with probability 1.
- ▶ We define the jump process's **infinitesimal generator** as, $\forall x \neq y \in E$

$$q_{xy} = \lim_{h \rightarrow 0} \frac{p_{xy}(h)}{h} \quad (4)$$

- ▶ It defines the process **transitions rates**
- ▶ Ex : queuing systems !

■ Back to our propagation model :

- Since :

$$\begin{aligned} p_{x,x-i}(t) &= 0 \quad \forall i > 0 \\ p_{x,x+1}(h) &\propto h \\ p_{x,x+1+i}(h) &\propto h^i \quad \forall i > 0 \end{aligned} \tag{5}$$

- Only non-zero transition rate is $q_{x,x+1}$
- If $X_t = x > 0$, the next time before an infection intent is a random variable defined as :

$$\tau_x = \min_{i=1,\dots,x} \varepsilon_i \quad \text{where } \varepsilon_i \sim \text{Exp}(\lambda) \text{ (iid)} \tag{6}$$

Therefore

$$\tau_x \sim \text{Exp}(\lambda x) \tag{7}$$

■ Back to our propagation model :

- ▶ The transition succeeds if the node picks a susceptible neighbor, which happens with probability $\frac{n-x}{n-1}$ (uniform)
- ▶ Therefore :

$$q_{x,x+1} = \lambda x \frac{n-x}{n-1} \quad (8)$$

- ▶ Let $X_0 = 1$. The time before m individuals are infected is defined as :

$$T_m = \sum_{x=1}^{m-1} \frac{n-1}{\lambda x(n-x)} \varepsilon_i \quad (9)$$

where

$$\varepsilon_i \stackrel{i.i.d}{\sim} \text{Exp}(1) \quad (10)$$

■ Time before complete infection : T_n which first moment is given by :

$$\mathbb{E}[T_n] = \sum_{x=1}^n \frac{n-1}{\lambda x(n-x)} \quad (11)$$

■ Large population limit :

$$\mathbb{E}[T_n] = \frac{2}{\lambda}(\log(n) + \gamma + o(1)) \quad (12)$$

scales with $\log(n)$!

■ Fluctuation : if $S_n = \lambda(T_n - \mathbb{E}[T_n])$:

$$\mathbb{P}(S_n \geq t) \leq e^{-\theta t} C_\theta, \quad \forall \theta \in [0, \frac{1}{2}] \quad (13)$$

→ exponential control around the mean value !

■ Simulation

Hyper-parameters : $n = 30$, $\lambda = 1$.

■ Simulation

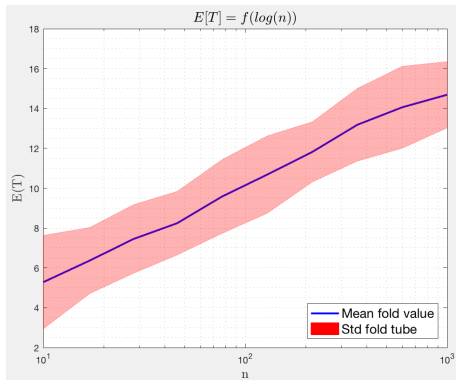


Figure: Mean infection time as a function of $|V|$

$\lambda = 1$ and $\lambda_{MLE} = 0.94$!

■ Simulation

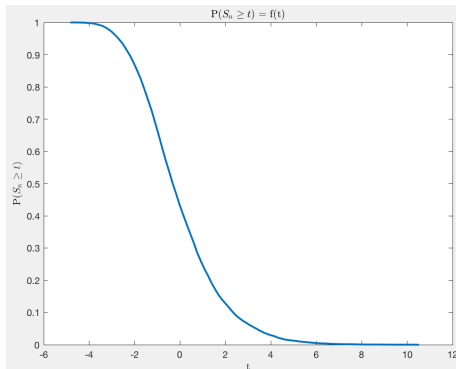


Figure: Fluctuation distribution : $\mathbb{P}(S_n \geq t)$

→ Exponentially bounded tail

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■ Epidemic Propagation :

- ▶ Modelisation of simple epidemics in population
- ▶ Simplistic model :
 - ▶ Nodes can't recover from being infected
 - ▶ A complete graph is not realistic
- ▶ What happens if we introduce recovering / dying nodes ?
- ▶ How does the topology of the graph impacts the last results ?

■ Sensor Network Modelisation (All to All propagation) :

- ▶ Modelling a network of sensors / agents
- ▶ Each agent tries to pass a bit of information to all others
- ▶ Control over the time before every information has been propagated to every agent
- ▶ Previous results still holds : scales as $\log n$

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■ Critics

- ▶ The SI approach is not realistic !
- ▶ More realistic models : SIS (Susceptible-Infected-Susceptible), SIR (Susceptible-Infected-Removed)
- ▶ Complete topology assumption does not hold (sparser graph, community)

■ SIS (Susceptible-Infected-Susceptible approach) on general topology:

- ▶ Let $G = (V, E)$ a graph with adjacency matrix $A \in \mathcal{M}_{|V|}(\mathbb{R})$

$$A = (\delta_{(v_i, v_j) \in E})_{i,j} \quad (14)$$

- ▶ $\{X(t)\}_t \in \{0, 1\}^{|V|}$ our Markovian jump process.
- ▶ Let :

$$\begin{aligned} \beta &: \text{infection rate} \\ \delta &: \text{remission rate} \\ e_i &= (\delta_{ji})_{j \in \{1, \dots, V\}} \end{aligned} \quad (15)$$

- ▶ Then the only non-zero infinitesimal generators are :

$$\begin{cases} q_{x, x+e_i} = \beta \mathbf{1}_{x_i=0} \sum_{j \in V} A_{i,j} x_j \\ q_{x, x-e_i} = \delta x_i \end{cases} \quad (16)$$

■ Simulation :

Hyper-Parameters : $\delta = 0.5$, $\beta = 0.25$, $\rho_G \simeq 4$

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■ SIS (Susceptible-Infected-Susceptible approach) on general topology:

- ▶ 0^V is an **absorbant state** \leftarrow time for absorption ?
- ▶ Spectral radius of the graph :

$$\rho_G = \max_{\lambda \in Sp(A)} |\lambda| \quad (17)$$

- ▶ **Main absorption result** : If the graph is *finite* and $t > 0$

$$\mathbb{P}(T > t) \leq ne^{t(\beta\rho - \delta)} \quad (18)$$

hence if $\beta\rho_G \leq \delta$

$$\begin{aligned} \mathbb{E}[T] &= \int_0^{+\infty} \mathbb{P}(T > t) dt \\ &\leq \frac{\log n + 1}{\delta - \beta\rho_G} \end{aligned} \quad (19)$$

Simulation

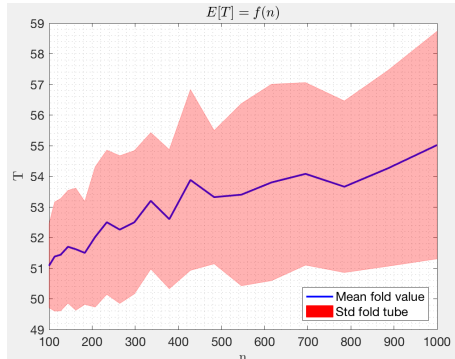


Figure: Mean absorbing time as a function of $|V|$

$$\delta = 0.5, \beta = 0.025, \rho_G \leq 4$$

■ SIS (Susceptible-Infected-Susceptible approach) on general topology:

- ▶ More precise and complex results include the *graph's isoperimetric constant* : $\forall m \in \{1, \dots, n-1\}$

$$\eta_m(G) = \inf_{S \subset V, |S| \leq m} \left\{ \frac{|E(S, \bar{S})|}{|S|} \right\} \quad (20)$$

- ▶ Derive results of control for many different topologies (complete, hypercube, Erdos-Renyi's, ...)
- ▶ Model a *volatile information* propagation on various network topologies
- ▶ Paves the way for SIR model

■ Summary

- ▶ Stochastic Process and Markov Jump Process on Graph
- ▶ Propagation of a disease / information along communities
- ▶ Control over mean propagation time and fluctuation around that mean

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Thank you for your attention !