Susceptible-Infected Epidemic in Graphs

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Plan

1 Theoretical Approach
- Population Model
- Propagation Model
- Time before complete infection

2 Applications
- Epidemic Propagation
- Sensor Networks

3 Model refinement
- SIS approach
- Arbitrary graph topology

Plan

■ Hold the following assumptions for true :

▶ Each individual in the population is either *susceptible* or *infected*

▶ Once infected, an individual can't recover and tries to infect other individuals

▶ An individual can affect any other individual in the population (uniform topology)

■ Graph population representation :

$$G = (V, E) \tag{1}$$

with $|V| = n \in \mathbb{N}^*$
Uniform topology $\Rightarrow$ **complete graph**.

■ Propagation :

- ▶ Each infected node tries to infect one of its neighbor at the times of a Poisson process of intensity $\lambda > 0$.
- ▶ The node's Poisson processes are independent.
- ▶ An infected node picks one of its neighbor uniformly at random when trying to infect.

■ Let X_t the number of infected node at time $t > 0$.
$\rightarrow \{X_t\}$ is a **Markov Jump Process** on $\mathbb{R}^+$.

■ Markov Jump Process on E :

  ▶ If, $\forall x, y \in E$:

$$p_{xy}(t) = \mathbb{P}(X_{t+s} = y \mid X_s = x) \tag{2}$$

  then

$$\lim_{t \to 0} p_{xx}(t) = 1 \tag{3}$$

  ▶ The process remains at each stage for a strictly positive time with probability 1.

  ▶ We define the jump process's **infinitesimal generator** as, $\forall x \neq y \in E$

$$q_{xy} = \lim_{h \to 0} \frac{p_{xy}(h)}{h} \tag{4}$$

  ▶ It defines the process transitions rates

  ▶ Ex : queuing systems !

■ Back to our propagation model :

▶ Since :

$$p_{x,x-i}(t) = 0 \quad \forall i > 0$$
$$p_{x,x+1}(h) \propto h \tag{5}$$
$$p_{x,x+1+i}(h) \propto h^i \quad \forall i > 0$$

▶ Only non-zero transition rate is $q_{x,x+1}$

▶ If $X_t = x > 0$, the next time before an infection intent is a random variable defined as :

$$\tau_x = \min_{i=1,\dots,x} \varepsilon_i \quad \text{where } \varepsilon_i \sim Exp(\lambda) \text{ (iid)} \tag{6}$$

Therefore

$$\tau_x \sim Exp(\lambda x) \tag{7}$$

■ Back to our propagation model :

▶ The transition succeeds if the node picks a suspectible neighbor, which happens with probability $\frac{n-x}{n-1}$ (uniform)

▶ Therefore :

$$q_{x,x+1} = \lambda x \frac{n-x}{n-1} \tag{8}$$

▶ Let $X_0 = 1$. The time before m individuals are infected is defined as :

$$T_m = \sum_{x=1}^{m-1} \frac{n-1}{\lambda x(n-x)} \varepsilon_i \tag{9}$$

where

$$\varepsilon_i \stackrel{i.i.d}{\sim} Exp(1) \tag{10}$$

■ Time before complete infection : T_n which first moment is given by :

$$\mathbb{E}\left[T_n\right] = \sum_{x=1}^{n} \frac{n-1}{\lambda x(n-x)} \tag{11}$$

■ Large population limit :

$$\mathbb{E}\left[T_n\right] = \frac{2}{\lambda}(2\log(n) + 2\gamma + o(1)) \tag{12}$$

scales with $\log(n)$!

■ Fluctuation : if $S_n = \lambda(T_n - \mathbb{E}\left[T_n\right])$:

$$\mathbb{P}(S_n \geq t) \leq e^{-\theta t} C_\theta, \quad \forall \theta \in [0, \frac{1}{2}] \tag{13}$$

$\rightarrow$ exponential control around the mean value !

TODO : simulation explanation & results

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■ Epidemic Propagation :

- ▶ Modelisation of simple epidemics in population

- ▶ Simplistic model :
    - ▶ Nodes can't recover from being infected
    - ▶ A complete graph is not realistic

- ▶ What happens if we introduce recovering / dying nodes ?

- ▶ How does the topology of the graph impacts the last results ?

■ Sensor Network Modelisation (All to All propagation) :

▶ Modelling a network of sensors / agents

▶ Each agent tries to pass a bit of information to all others

▶ Control over the time before every information has been propagated to every agent

▶ Previous results still holds : scales as log n

Plan

■ Critics

- ▶ The SI approach is not realistic !

- ▶ More realistic models : SIS (Susceptible-Infected-Susceptible), SIR (Susceptible-Infected-Removed)

- ▶ Complete topology assumption does not hold (sparser graph, community)

■ SIS (Susceptible-Infected-Susceptible approach) on general topology:

▶ Let $G = (V, E)$ a graph with adjacency matrix $A \in \mathcal{M}_{|V|}(\mathbb{R})$

$$A = (\delta_{(v_i, v_j) \in E})_{i,j} \tag{14}$$

▶ $\{X(t)\}_t \in \{0, 1\}^{|V|}$ our Markovian jump process.

▶ Let :

$$
\begin{aligned}
\beta &\quad : \text{infection rate} \\
\delta &\quad : \text{remission rate} \\
e_i &\quad = (\delta_{ji})_{j \in \{1, \dots, V\}}
\end{aligned}
\tag{15}
$$

▶ Then the only non-zero infinitesimal generators are :

$$
\begin{cases}
q_{x, x+e_i} = \beta \mathbb{1}_{x_i = 0} \sum_{j \in V} A_{i,j} x_j \\[2mm]
q_{x, x-e_i} = \delta x_i
\end{cases}
\tag{16}
$$

■ SIS (Susceptible-Infected-Susceptible approach) on general topology:

▶ 0^V is an absorbant state ← time for absorption ?

▶ Spectral radius of the graph :

$$\rho_G = \max_{\lambda \in Sp(A)} |\lambda| \tag{17}$$

▶ **Main absorption result** : If the graph is *finite* and $t > 0$

$$\mathbb{P}(T > t) \leq ne^{t(\beta\rho - \delta)} \tag{18}$$

hence if $\boldsymbol{\beta\rho \leq \delta}$

$$\begin{aligned} \mathbb{E}[T] &= \int_0^{+\infty} \mathbb{P}(T > t)dt \\ &\leq \frac{\log n + 1}{\delta - \beta\rho} \end{aligned} \tag{19}$$

■ SIS (Susceptible-Infected-Susceptible approach) on general topology:

▶ More precise and complex results include the *graph's isoperimetric constant* : $\forall m \in \{1, \ldots, n-1\}$

$$\eta_m(G) = \inf_{S \subset V, |S| \leq m} \left\{ \frac{|E(S, \bar{S})|}{|S|} \right\} \tag{20}$$

▶ Derive results of control for many different topologies (complete, hypercube, Erdos-Renyi's,...)

▶ Model the volatile information propagation on various network topologies

▶ Paves the way for SIR model