Susceptible-Infected Epidemic in Graphs

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Plan

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■ Hold the following assumptions for true :

- Each individual in the population is either *susceptible* or *infected*
- Once infected, an individual can't recover and tries to infect other individuals
- An individual can affect any other individual in the population (uniform topology)

■ Graph population representation :

$$G = (V, E) \tag{1}$$

with $|V| = n \in \mathbb{N}^*$
Uniform topology $\Rightarrow$ **complete graph**.

- Propagation :
  - Each infected node tries to infect one of its neighbor at the times of a Poisson process of intensity $\lambda > 0$.
  - The node's Poisson processes are independent.
  - An infected node picks one of its neighbor uniformly at random when trying to infect.

- Let X_t the number of infected node at time $t > 0$.
  $\rightarrow \{X_t\}$ is a **Markov Jump Process** on $\mathbb{R}^+$.

- Markov Jump Process on E :
  - If, $\forall x, y \in E$:

$$p_{xy}(t) = \mathbb{P}(X_{t+s} = y \,|\, X_s = x) \tag{2}$$

then

$$\lim_{t \to 0} p_{xx}(t) = 1 \tag{3}$$

  - The process remains at each stage for a strictly positive time with probability 1.
  - We define the jump process's **infinitesimal generator** as, $\forall x \neq y \in E$

$$q_{xy} = \lim_{h \to 0} \frac{p_{xy}(h)}{h} \tag{4}$$

  - It defines the process transitions rates
  - Ex : queuing systems !

- Back to our propagation model :
  - Since :

$$\begin{aligned} p_{x,x-i}(t) &= 0 \quad \forall i > 0 \\ p_{x,x+1}(h) &\propto h \\ p_{x,x+1+i}(h) &\propto h^i \quad \forall i > 0 \end{aligned} \tag{5}$$

  - Only non-zero transition rate is $q_{x,x+1}$
  - If $X_t = x > 0$, the next time before an infection intent is a random variable defined as :

$$\tau_x = \min_{i=1,\dots,x} \varepsilon_i \quad \text{where } \varepsilon_i \sim Exp(\lambda) \text{ (iid)} \tag{6}$$

Therefore

$$\tau_x \sim Exp(\lambda x) \tag{7}$$

- Back to our propagation model :
  - The transition succeeds if the node picks a suspectible neighbor, which happens with probability $\frac{n-x}{n-1}$ (uniform)
  - Therefore :

$$\boxed{q_{x,x+1} = \lambda x \frac{n-x}{n-1}} \tag{8}$$

  - Let $X_0 = 1$. The time before m individuals are infected is defined as :

$$T_m = \sum_{x=1}^{m-1} \frac{n-1}{\lambda x(n-x)} \varepsilon_i \tag{9}$$

where

$$\varepsilon_i \overset{i.i.d}{\sim} Exp(1) \tag{10}$$

■ Time before complete infection : T_n which first moment is given by :

$$\boxed{\mathbb{E}\left[T_n\right] = \sum_{x=1}^{n} \frac{n-1}{\lambda x(n-x)}} \tag{11}$$

■ Large population limit :

$$\mathbb{E}\left[T_n\right] = \frac{2}{\lambda}(2\log(n) + 2\gamma + o(1)) \tag{12}$$

scales with $\log(n)$!

■ Fluctuation : if $S_n = \lambda(T_n - \mathbb{E}\left[T_n\right])$:

$$\mathbb{P}(S_n \geq t) \leq e^{-\theta t} C_\theta, \quad \forall \theta \in [0, \frac{1}{2}] \tag{13}$$

$\rightarrow$ exponential control around the mean value !

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