# Susceptible-Infected Epidemic in Graphs

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## Plan

# Plan

- Hold the following assumptions for true :
  - Each individual in the population is either *susceptible* or *infected*
  - Once infected, an individual can't recover and tries to infect other individuals
  - An individual can affect any other individual in the population (uniform topology)

- Graph population representation :

$$G = (V, E) \tag{1}$$

with $|V| = n \in \mathbb{N}^*$
Uniform topology $\Rightarrow$ **complete graph**.

■ Propagation :

- Each infected node tries to infect one of its neighbor at the times of a Poisson process of intensity $\lambda > 0$.
- The node's Poisson processes are independent.
- An infected node picks one of its neighbor uniformly at random when trying to infect.

■ Let $X_t$ the number of infected node at time $t > 0$.
$\rightarrow \{X_t\}$ is a **Markov Jump Process** on $\mathbb{R}^+$.

- Markov Jump Process on $E$ :
  - If, $\forall x, y \in E$ :

$$p_{xy}(t) = \mathbb{P}(X_{t+s} = y \,|\, X_s = x) \tag{2}$$

    then

$$\lim_{t \to 0} p_{xx}(t) = 1 \tag{3}$$

  - The process remains at each stage for a strictly positive time with probability 1.
  - We define the jump process's **infinitesimal generator** as, $\forall x \neq y \in E$

$$q_{xy} = \lim_{h \to 0} \frac{p_{xy}(h)}{h} \tag{4}$$

  - It defines the process transitions rates
  - Ex : queuing systems !

- Back to our propagation model :
  - Since :

$$\begin{aligned} p_{x,x-i}(t) &= 0 \quad \forall i > 0 \\ p_{x,x+1}(h) &\propto h \\ p_{x,x+1+i}(h) &\propto h^i \quad \forall i > 0 \end{aligned} \tag{5}$$

  - Only non-zero transition rate is $q_{x,x+1}$
  - If $X_t = x > 0$, the next time before an infection intent is a random variable defined as :

$$\tau_x = \min_{i=1,\ldots,x} \varepsilon_i \quad \text{where } \varepsilon_i \sim Exp(\lambda) \text{ (iid)} \tag{6}$$

Therefore

$$\tau_x \sim Exp(\lambda x) \tag{7}$$

■ Back to our propagation model :

- The transition succeeds if the node picks a suspectible neighbor, which happens with probability $\frac{n-x}{n-1}$ (uniform)
- Therefore :

$$\boxed{q_{x,x+1} = \lambda x \frac{n-x}{n-1}} \tag{8}$$

- Let $X_0 = 1$. The time before $m$ individuals are infected is defined as :

$$T_m = \sum_{x=1}^{m-1} \frac{n-1}{\lambda x(n-x)} \varepsilon_i \tag{9}$$

where

$$\varepsilon_i \overset{i.i.d}{\sim} Exp(1) \tag{10}$$

- Time before complete infection : $T_n$ which first moment is given by :

$$\boxed{\mathbb{E}\left[T_n\right] = \sum_{x=1}^{n} \frac{n-1}{\lambda x(n-x)}} \quad (11)$$

- Large population limit :

$$\mathbb{E}\left[T_n\right] = \frac{2}{\lambda}(\log(n) + \gamma + o(1)) \quad (12)$$

scales with $\log(n)$ !

- Fluctuation : if $S_n = \lambda(T_n - \mathbb{E}\left[T_n\right])$ :

$$\mathbb{P}(S_n \geq t) \leq e^{-\theta t} C_\theta, \quad \forall \theta \in [0, \frac{1}{2}] \quad (13)$$

$\rightarrow$ exponential control around the mean value !

■ Simulation

Hyper-parameters : $n = 30$, $\lambda = 1$.

## Simulation

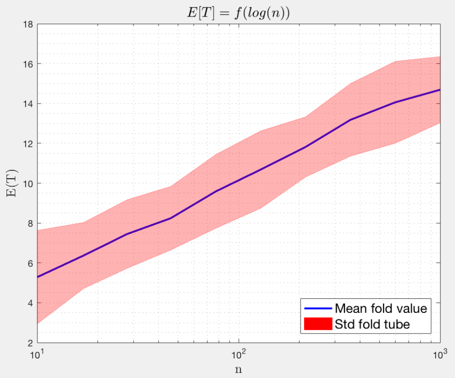


Figure: Mean infection time as a function of $|V|$

$\lambda = 1$ and $\lambda_{MLE} = 0.94$ !

## Simulation

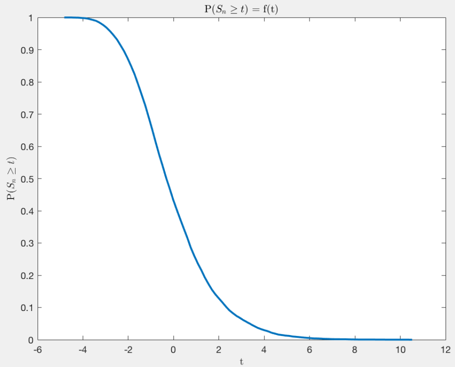


Figure: Fluctuation distribution : $\mathbb{P}(S_n \geq t)$

$\rightarrow$ Exponentially bounded tail !

# Plan

■ Epidemic Propagation :

- Modelisation of simple epidemics in population
- Simplistic model :
  - Nodes can't recover from being infected
  - A complete graph is not realistic
- What happens if we introduce recovering / dying nodes ?
- How does the topology of the graph impacts the last results ?

- Sensor Network Modelisation (All to All propagation) :
  - Modelling a network of sensors / agents
  - Each agent tries to pass a bit of information to all others
  - Control over the time before every information has been propagated to every agent
  - Previous results still holds : scales as $\log n$

## Plan

## Critics

- The SI approach is not realistic !
- More realistic models : SIS (Susceptible-Infected-Susceptible), SIR (Susceptible-Infected-Removed)
- Complete topology assumption does not hold (sparser graph, community)

- SIS (Susceptible-Infected-Susceptible approach) on general topology:
  - Let $G = (V, E)$ a graph with adjacency matrix $A \in \mathcal{M}_{|V|}(\mathbb{R})$

$$A = (\delta_{(v_i, v_j) \in E})_{i,j} \tag{14}$$

  - $\{X(t)\}_t \in \{0,1\}^{|V|}$ our Markovian jump process.
  - Let :

$$\begin{aligned} \beta \quad & : \text{ infection rate} \\ \delta \quad & : \text{ remission rate} \\ e_i \quad & = (\delta_{ji})_{j \in \{1,\dots,V\}} \end{aligned} \tag{15}$$

  - Then the only non-zero infinitesimal generators are :

$$\begin{cases} q_{x, x+e_i} = \beta \mathbb{1}_{x_i = 0} \sum_{j \in V} A_{i,j} x_j \\ q_{x, x-e_i} = \delta x_i \end{cases} \tag{16}$$

■ Simulation :

Hyper-Parameters : $\delta = 0.5$, $\beta = 0.25$

■ SIS (Susceptible-Infected-Susceptible approach) on general topology:

- $0^V$ is an absorbant state $\leftarrow$ time for absorption ?
- Spectral radius of the graph :

$$\rho_G = \max_{\lambda \in Sp(A)} |\lambda| \tag{17}$$

- **Main absorption result** : If the graph is *finite* and $t > 0$

$$\mathbb{P}(T > t) \leq n e^{t(\beta\rho - \delta)} \tag{18}$$

hence if $\boldsymbol{\beta\rho \leq \delta}$

$$\boxed{\begin{aligned} \mathbb{E}\left[T\right] &= \int_0^{+\infty} \mathbb{P}(T > t)dt \\ &\leq \frac{\log n + 1}{\delta - \beta\rho} \end{aligned}} \tag{19}$$

- SIS (Susceptible-Infected-Susceptible approach) on general topology:
  - More precise and complex results include the *graph's isoperimetric constant* : $\forall m \in \{1, \dots, n-1\}$

$$\eta_m(G) = \inf_{S \subset V, |S| \leq m} \left\{ \frac{|E(S, \bar{S})|}{|S|} \right\} \tag{20}$$

  - Derive results of control for many different topologies (complete, hypercube, Erdos-Renyi's,...)
  - Model the volatile information propagation on various network topologies
  - Paves the way for SIR model