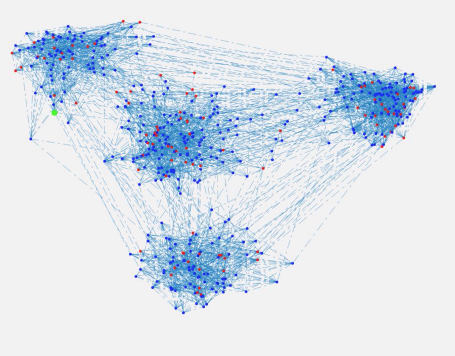

Susceptible-Infected Epidemic in Graphs

Louis Faury
Gallois - Montbrun Grégoire
MICRO506 - Stochastic Methods
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Plan

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1. Theoretical Approach
   - Population Model
   - Propagation Model
   - Time before complete infection
2. Applications
3. Model refinement

- Hold the following assumptions for true :
  - Each individual in the population is either *susceptible* or *infected*
  - Once infected, an individual can't recover and tries to infect other individuals
  - An individual can affect any other individual in the population (uniform topology)

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- Graph population representation :

$$G = (V, E) \tag{1}$$

with $|V| = n \in \mathbb{N}^*$
Uniform topology $\Rightarrow$ **complete graph**.

■ Propagation :

- Each infected node tries to infect one of its neighbor at the times of a Poisson process of intensity $\lambda > 0$.
- The node's Poisson processes are independent.
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- Let X_t the number of infected node at time $t > 0$.
  $\rightarrow \{X_t\}$ is a **Markov Jump Process** on $\mathbb{R}^+$.

■ Markov Jump Process on E :

- If, $\forall x, y \in E$:

$$p_{xy}(t) = \mathbb{P}(X_{t+s} = y \,|\, X_s = x) \tag{2}$$

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- It defines the process transitions rates
- Ex : queuing systems !

- Back to our propagation model :
  - Since :

$$
\begin{aligned}
&p_{x,x-i}(t) = 0 \quad \forall i > 0 \\
&p_{x,x+1}(h) \propto h \\
&p_{x,x+1+i}(h) \propto h^i \quad \forall i > 0
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\tag{5}
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- Only non-zero transition rate is $q_{x,x+1}$
- If $X_t = x > 0$, the next time before an infection intent is a random variable defined as :

$$\tau_x = \min_{i=1,\dots,x} \varepsilon_i \quad \text{where } \varepsilon_i \sim Exp(\lambda) \text{ (iid)} \tag{6}$$

Therefore

$$\tau_x \sim Exp(\lambda x) \tag{7}$$

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  - Let $X_0 = 1$. The time before m individuals are infected is defined as :

$$T_m = \sum_{x=1}^{m-1} \frac{n-1}{\lambda x (n-x)} \varepsilon_i \tag{9}$$

where

$$\varepsilon_i \overset{i.i.d}{\sim} Exp(1) \tag{10}$$

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■ Fluctuation : if $S_n = \lambda(T_n - \mathbb{E}\left[T_n\right])$:

$$\mathbb{P}(S_n \geq t) \leq e^{-\theta t} C_\theta, \quad \forall \theta \in [0, \frac{1}{2}] \tag{13}$$

$\rightarrow$ exponential control around the mean value !

■ Simulation

Hyper-parameters : $n = 30$, $\lambda = 1$.

Simulation

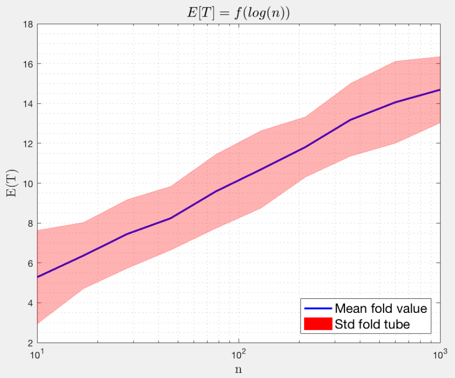

Figure: Mean infection time as a function of $|V|$

$\lambda = 1$ and $\lambda_{MLE} = 0.94$!

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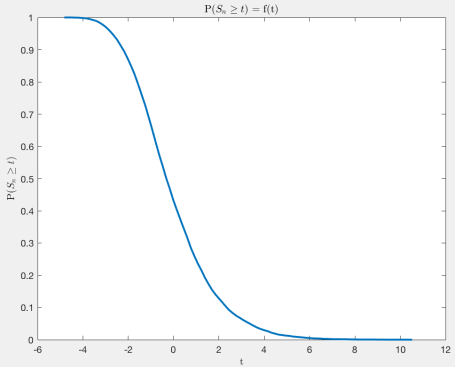

Figure: Fluctuation distribution : $\mathbb{P}(S_n \geq t)$

→ Exponentially bounded tail

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- What happens if we introduce recovering / dying nodes ?
- How does the topology of the graph impacts the last results ?

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- Control over the time before every information has been propagated to every agent
- Previous results still holds : scales as $\log n$

Plan

Critics

- The SI approach is not realistic !
- More realistic models : SIS (Susceptible-Infected-Susceptible), SIR (Susceptible-Infected-Removed)
- Complete topology assumption does not hold (sparser graph, community)

- SIS (Susceptible-Infected-Susceptible approach) on general topology:
  - Let $G = (V, E)$ a graph with adjacency matrix $A \in \mathcal{M}_{|V|}(\mathbb{R})$

$$A = (\delta_{(v_i, v_j) \in E})_{i,j} \tag{14}$$

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- Let :

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  - Then the only non-zero infinitesimal generators are :

$$\begin{cases} q_{x,x+e_i} = \beta \mathbb{1}_{x_i=0} \sum_{j \in V} A_{i,j} x_j \\ q_{x,x-e_i} = \delta x_i \end{cases} \tag{16}$$

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  - **Main absorption result** : If the graph is *finite* and $t > 0$

$$\mathbb{P}(T > t) \leq ne^{t(\beta\rho - \delta)} \tag{18}$$

hence if $\boldsymbol{\beta\rho_G \leq \delta}$

$$\boxed{\begin{aligned} \mathbb{E}\left[T\right] &= \int_0^{+\infty} \mathbb{P}(T > t)dt \\ &\leq \frac{\log n + 1}{\delta - \beta\rho_G} \end{aligned}} \tag{19}$$

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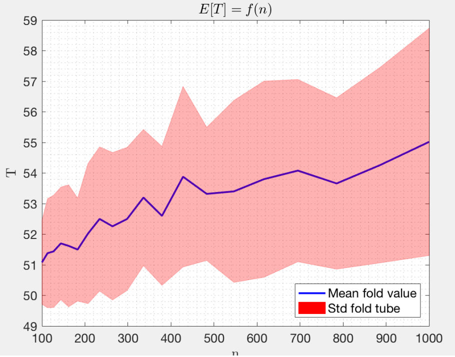

Figure: Mean absorbing time as a function of $|V|$

$$\delta = 0.5,\ \beta = 0.025,\ \rho_G \leq 4$$

■ SIS (Susceptible-Infected-Susceptible approach) on general topology:

- More precise and complex results include the *graph's isoperimetric constant* : $\forall m \in \{1, \ldots, n-1\}$

$$\eta_m(G) = \inf_{S \subset V, |S| \leq m} \left\{ \frac{|E(S, \bar{S})|}{|S|} \right\} \tag{20}$$

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- Derive results of control for many different topologies (complete, hypercube, Erdos-Renyi's,. . .)

- Model a *volatile information* propagation on various network topologies

- Paves the way for SIR model

Summary

- Stochastic Process and Markov Jump Process on Graph
- Propagation of a disease / information along communities
- Control over mean propagation time and fluctuation around that mean

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Thank you for your attention !