

Susceptible-Infected Epidemic in Graphs

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MICRO506 - Stochastic Methods

May 30, 2017

Plan

1 Theoretical Approach

- Population Model
- Propagation Model
- Time before complete infection

2 Applications

- Epidemic Propagation
- Sensor Networks

3 Model refinement

- SIS approach
- Arbitrary graph topology

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■ Hold the following assumptions for true :

- ▶ Each individual in the population is either *susceptible* or *infected*
- ▶ Once infected, an individual can't recover and tries to infect other individuals
- ▶ An individual can affect any other individual in the population (**uniform topology**)

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■ Graph population representation :

$$G = (V, E) \quad (1)$$

with $|V| = n \in \mathbb{N}^*$

Uniform topology \Rightarrow **complete graph**.

■ Propagation :

- ▶ Each infected node tries to infect one of its neighbor at the times of a **Poisson process** of intensity $\lambda > 0$.
- ▶ The node's Poisson processes are independent.
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■ Let X_t the number of infected node at time $t > 0$.
 $\rightarrow \{X_t\}$ is a **Markov Jump Process** on \mathbb{R}^+ .

■ Markov Jump Process on E :

- If, $\forall x, y \in E$:

$$p_{xy}(t) = \mathbb{P}(X_{t+s} = y \mid X_s = x) \quad (2)$$

then

$$\lim_{t \rightarrow 0} p_{xx}(t) = 1 \quad (3)$$

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- ▶ Ex : queuing systems !

■ Back to our propagation model :

► Since :

$$\begin{aligned}p_{x,x-i}(t) &= 0 \quad \forall i > 0 \\p_{x,x+1}(h) &\propto h \\p_{x,x+1+i}(h) &\propto h^i \quad \forall i > 0\end{aligned}\tag{5}$$

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- If $X_t = x > 0$, the next time before an infection intent is a random variable defined as :

$$\tau_x = \min_{i=1,\dots,x} \varepsilon_i \quad \text{where } \varepsilon_i \sim \text{Exp}(\lambda) \text{ (iid)} \tag{6}$$

Therefore

$$\tau_x \sim \text{Exp}(\lambda x) \tag{7}$$

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- ▶ Let $X_0 = 1$. The time before m individuals are infected is defined as :

$$T_m = \sum_{x=1}^{m-1} \frac{n-1}{\lambda x(n-x)} \varepsilon_i \quad (9)$$

where

$$\varepsilon_i \stackrel{i.i.d}{\sim} \text{Exp}(1) \quad (10)$$

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■ Fluctuation : if $S_n = \lambda(T_n - \mathbb{E}[T_n])$:

$$\mathbb{P}(S_n \geq t) \leq e^{-\theta t} C_\theta, \quad \forall \theta \in [0, \frac{1}{2}] \quad (13)$$

→ exponential control around the mean value !

■ Simulation

Hyper-parameters : $n = 30$, $\lambda = 1$.

■ Simulation

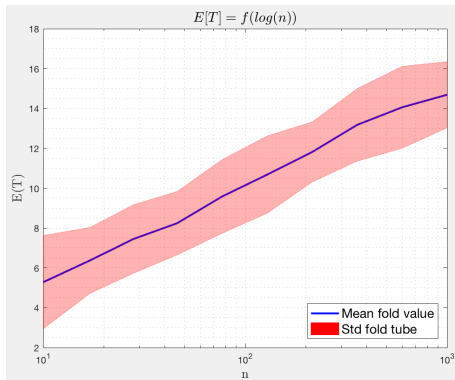


Figure: Mean infection time as a function of $|V|$

$\lambda = 1$ and $\lambda_{MLE} = 0.94$!

■ Simulation

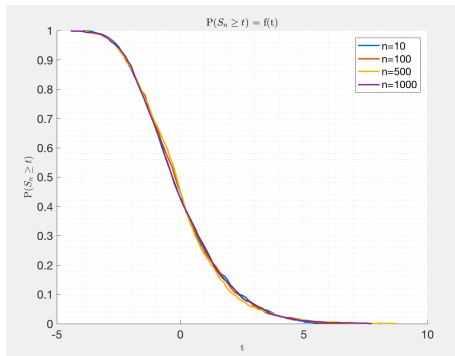


Figure: Fluctuation distribution : $\mathbb{P}(S_n \geq t)$

→ Exponentially bounded tail

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 - ▶ Nodes can't recover from being infected
 - ▶ A complete graph is not realistic
- ▶ What happens if we introduce recovering / dying nodes ?
- ▶ How does the topology of the graph impacts the last results ?

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- ▶ Control over the time before every information has been propagated to every agent
- ▶ Previous results still holds : scales as $\log n$

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■ Critics

- ▶ The SI approach is not realistic !
- ▶ More realistic models : SIS (Susceptible-Infected-Susceptible), SIR (Susceptible-Infected-Removed)
- ▶ Complete topology assumption does not hold (sparser graph, community)

■ SIS (Susceptible-Infected-Susceptible approach) on general topology:

- Let $G = (V, E)$ a graph with adjacency matrix $A \in \mathcal{M}_{|V|}(\mathbb{R})$

$$A = (\delta_{(v_i, v_j) \in E})_{i,j} \quad (14)$$

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- Let :

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- ▶ Then the only non-zero infinitesimal generators are :

$$\begin{cases} q_{x, x+e_i} = \beta \mathbf{1}_{x_i=0} \sum_{j \in V} A_{i,j} x_j \\ q_{x, x-e_i} = \delta x_i \end{cases} \quad (16)$$

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- ▶ **Main absorption result** : If the graph is *finite* and $t > 0$

$$\mathbb{P}(T > t) \leq ne^{t(\beta\rho - \delta)} \quad (18)$$

hence if **$\beta\rho_G \leq \delta$**

$$\begin{aligned} \mathbb{E}[T] &= \int_0^{+\infty} \mathbb{P}(T > t) dt \\ &\leq \frac{\log n + 1}{\delta - \beta\rho_G} \end{aligned} \quad (19)$$

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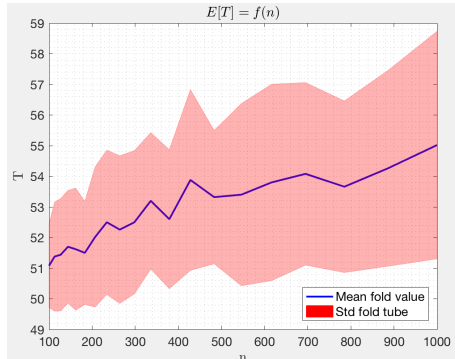


Figure: Mean absorbing time as a function of $|V|$

$$\delta = 0.5, \beta = 0.025, \rho_G \leq 4$$

■ SIS (Susceptible-Infected-Susceptible approach) on general topology:

- More precise and complex results include the *graph's isoperimetric constant* : $\forall m \in \{1, \dots, n-1\}$

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- ▶ Model a *volatile information* propagation on various network topologies
- ▶ Paves the way for SIR model

■ Summary

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- ▶ Propagation of a disease / information along communities
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Thank you for your attention !