Susceptible-Infected Epidemic in Graphs

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Plan

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1 Theoretical Approach
- Population Model
- Propagation Model
- Time before complete infection

2 Applications

3 Model refinement

■ Hold the following assumptions for true :

▶ Each individual in the population is either *susceptible* or *infected*

▶ Once infected, an individual can't recover and tries to infect other individuals

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■ Graph population representation :

$$G = (V, E) \tag{1}$$

with $|V| = n \in \mathbb{N}^*$
Uniform topology $\Rightarrow$ **complete graph**.

■ Propagation :

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■ Let X_t the number of infected node at time $t > 0$.
    $\rightarrow \{X_t\}$ is a **Markov Jump Process** on $\mathbb{R}^+$.

■ Markov Jump Process on E :

▶ If, $\forall x, y \in E$:

$$p_{xy}(t) = \mathbb{P}(X_{t+s} = y \mid X_s = x) \tag{2}$$

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▶ Ex : queuing systems !

■ Back to our propagation model :

▶ Since :

$$p_{x,x-i}(t) = 0 \quad \forall i > 0$$
$$p_{x,x+1}(h) \propto h \tag{5}$$
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▶ If $X_t = x > 0$, the next time before an infection intent is a random variable defined as :

$$\tau_x = \min_{i=1,\ldots,x} \varepsilon_i \quad \text{where } \varepsilon_i \sim Exp(\lambda) \text{ (iid)} \qquad (6)$$

Therefore

$$\tau_x \sim Exp(\lambda x) \qquad (7)$$

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▶ Let $X_0 = 1$. The time before m individuals are infected is defined as :

$$T_m = \sum_{x=1}^{m-1} \frac{n-1}{\lambda x(n-x)} \varepsilon_i \tag{9}$$

where

$$\varepsilon_i \overset{i.i.d}{\sim} Exp(1) \tag{10}$$

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■ Fluctuation : if $S_n = \lambda(T_n - \mathbb{E}\left[T_n\right])$:

$$\mathbb{P}(S_n \geq t) \leq e^{-\theta t} C_\theta, \quad \forall \theta \in [0, \frac{1}{2}] \qquad (13)$$

$\rightarrow$ exponential control around the mean value !

■ Simulation

Hyper-parameters : $n = 30$, $\lambda = 1$.
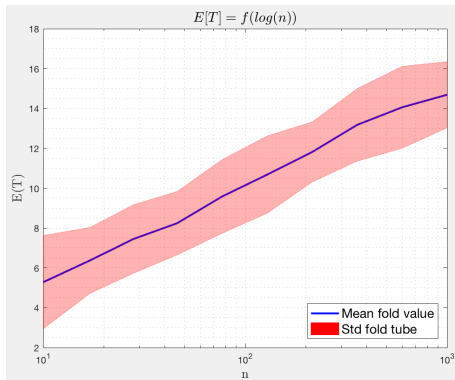
■ Simulation

Figure: Mean infection time as a function of $|V|$

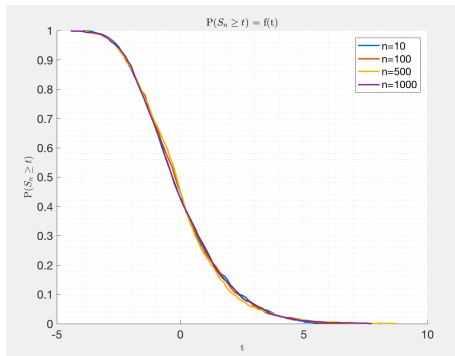$\lambda = 1$ and $\lambda_{MLE} = 0.94$!

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Figure: Fluctuation distribution : $\mathbb{P}(S_n \geq t)$

$\rightarrow$ Exponentially bounded tail

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- ▶ What happens if we introduce recovering / dying nodes ?

- ▶ How does the topology of the graph impacts the last results ?

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▶ Control over the time before every information has been propagated to every agent

▶ Previous results still holds : scales as log n

Plan

■ Critics

▶ The SI approach is not realistic !

▶ More realistic models : SIS (Susceptible-Infected-Susceptible), SIR (Susceptible-Infected-Removed)

▶ Complete topology assumption does not hold (sparser graph, community)

■ SIS (Susceptible-Infected-Susceptible approach) on general topology:
  ▶ Let $G = (V, E)$ a graph with adjacency matrix $A \in \mathcal{M}_{|V|}(\mathbb{R})$

$$A = (\delta_{(v_i, v_j) \in E})_{i,j} \tag{14}$$

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▶ Let :

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\begin{aligned}
\beta & \quad : \text{infection rate} \\
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  ▶ Then the only non-zero infinitesimal generators are :

$$\begin{cases} q_{x, x+e_i} = \beta \mathbb{1}_{x_i = 0} \displaystyle\sum_{j \in V} A_{i,j} x_j \\[2mm] q_{x, x-e_i} = \delta x_i \end{cases} \tag{16}$$

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  - **Main absorption result** : If the graph is *finite* and $t > 0$

$$\mathbb{P}(T > t) \leq ne^{t(\beta\rho - \delta)} \qquad (18)$$

  hence if $\beta\rho_G \leq \delta$

$$\boxed{\begin{aligned} \mathbb{E}\left[T\right] &= \int_0^{+\infty} \mathbb{P}(T > t)dt \\ &\leq \frac{\log n + 1}{\delta - \beta\rho_G} \end{aligned}} \qquad (19)$$

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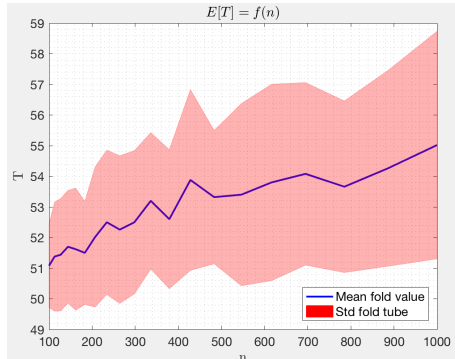
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Figure: Mean absorbing time as a function of $|V|$

$$\delta = 0.5, \ \beta = 0.025, \ \rho_G \leq 4$$

■ SIS (Susceptible-Infected-Susceptible approach) on general topology:

▶ More precise and complex results include the *graph's isoperimetric constant* : $\forall m \in \{1, \ldots, n-1\}$

$$\eta_m(G) = \inf_{S \subset V, |S| \leq m} \left\{ \frac{|E(S, \bar{S})|}{|S|} \right\} \tag{20}$$

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▶ Derive results of control for many different topologies (complete, hypercube, Erdos-Renyi's,...)

▶ Model a *volatile information* propagation on various network topologies

▶ Paves the way for SIR model

■ Summary

- ▶ Stochastic Process and Markov Jump Process on Graph

- ▶ Propagation of a disease / information along communities

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Thank you for your attention !