Data and Algorithms of the Web

Link Analysis Algorithms
Page Rank

some slides from:
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InfoLab (Stanford University)

Link Analysis Algorithms

- Page Rank
- Hubs and Authorities
- Topic-Specific Page Rank
- Spam Detection Algorithms
- Other interesting topics we won’t cover
  - Detecting duplicates and mirrors
  - Mining for communities

Ranking web pages

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  - www.bernard.com has 10 webpages linking to it

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-> Recursive definition of importance

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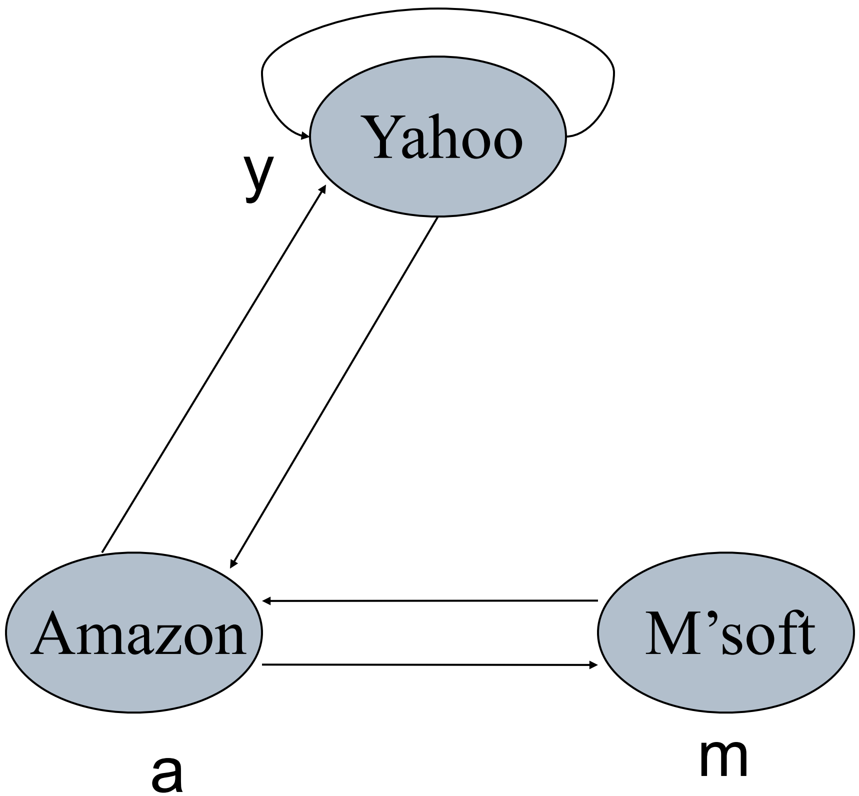
- Each page Q votes for its successors. If page Q with importance x has n successors, each succ. P gets x/n votes
- Page P's own importance is the sum of the votes of its predecessors Q.

Simple “flow” model



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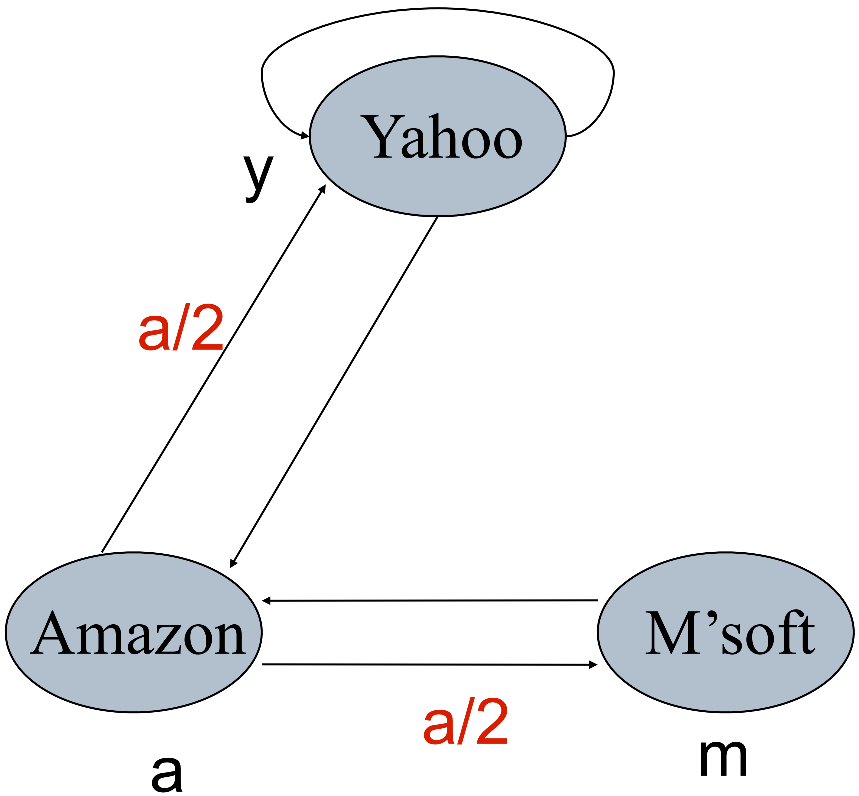

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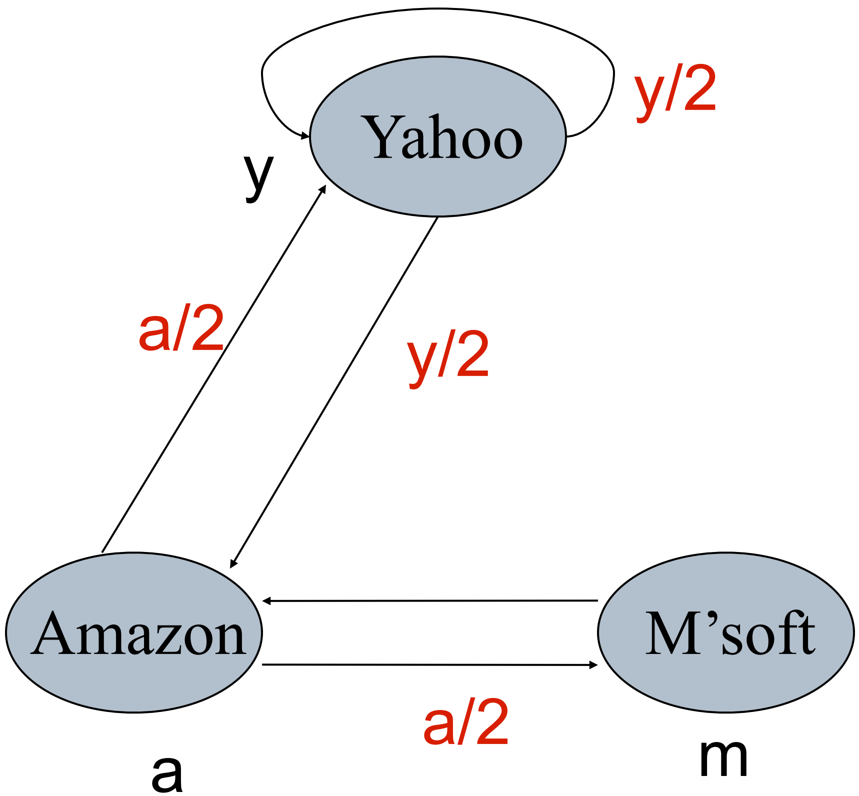

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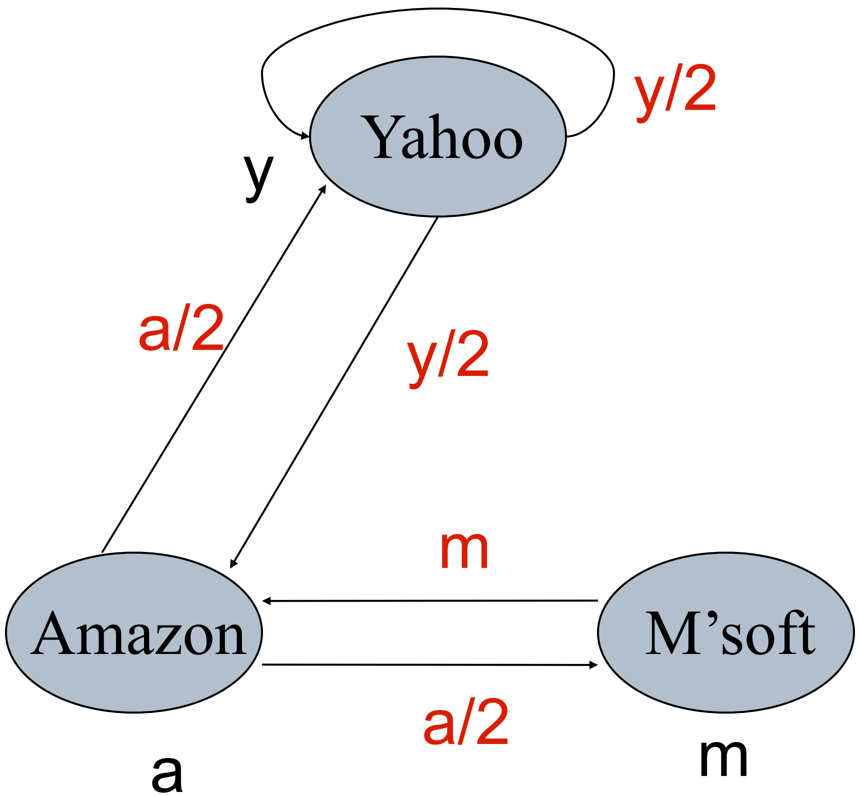

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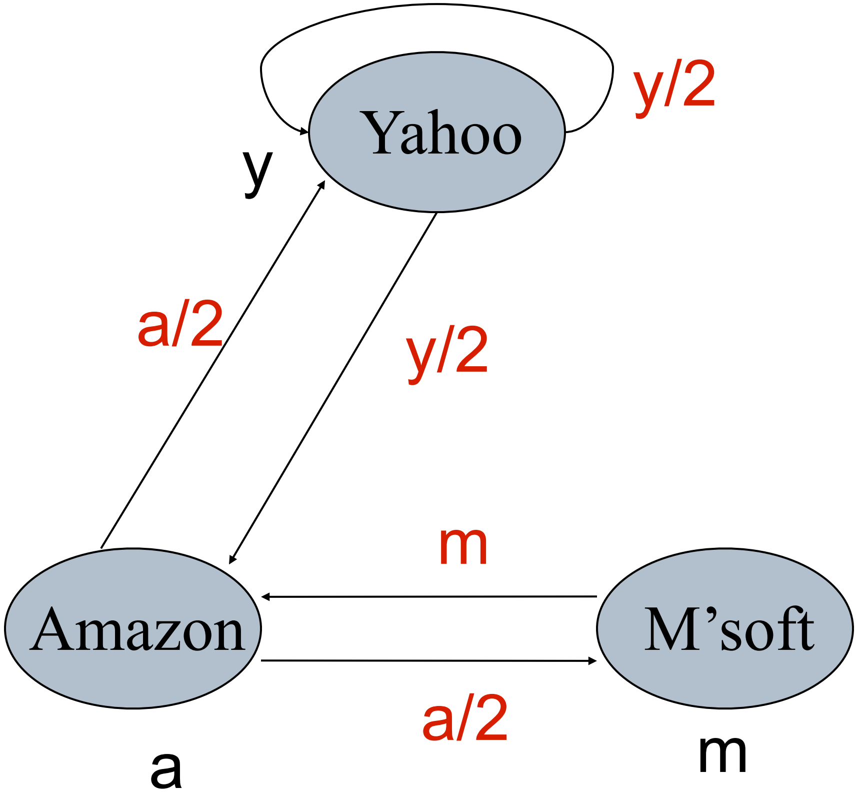

$$y = y/2 + a/2$$

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Solving the flow equations

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- 3 equations, 3 unknowns, no constants
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- Gaussian elimination method works for small examples, but we need a better method for large graphs

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- **M** is a column stochastic matrix
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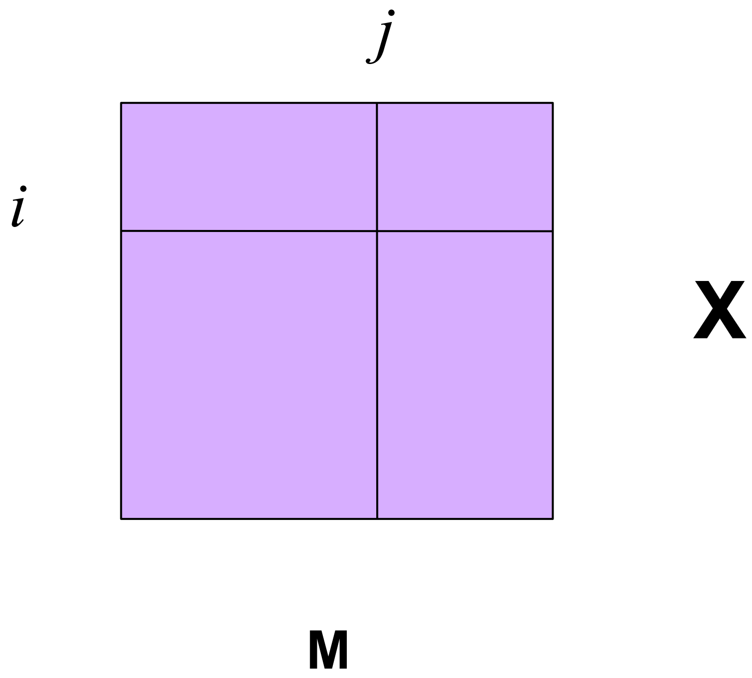
- Let **r** be the rank vector where:
  - r_i is the importance score of page i
  - $|\mathbf{r}| = 1$

Example

Suppose page j links to 3 pages, including i



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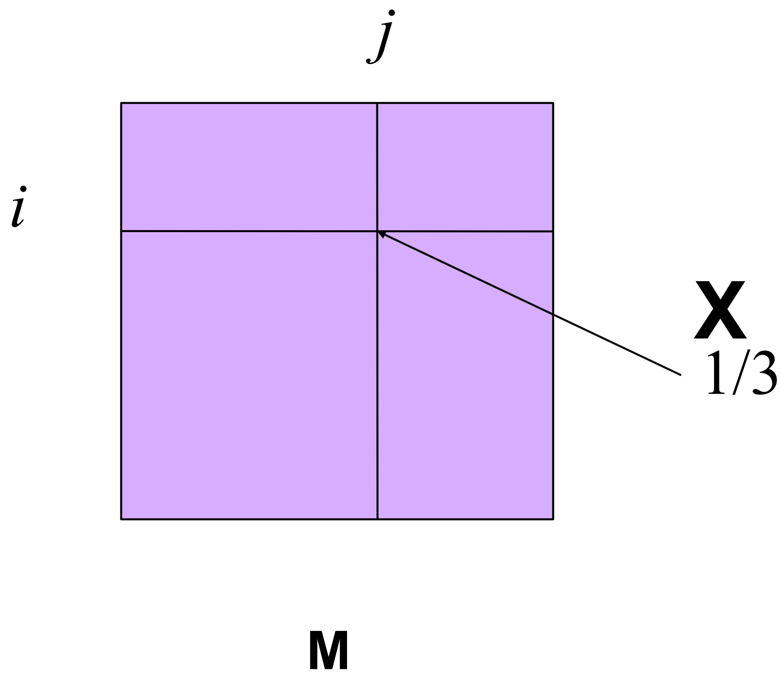

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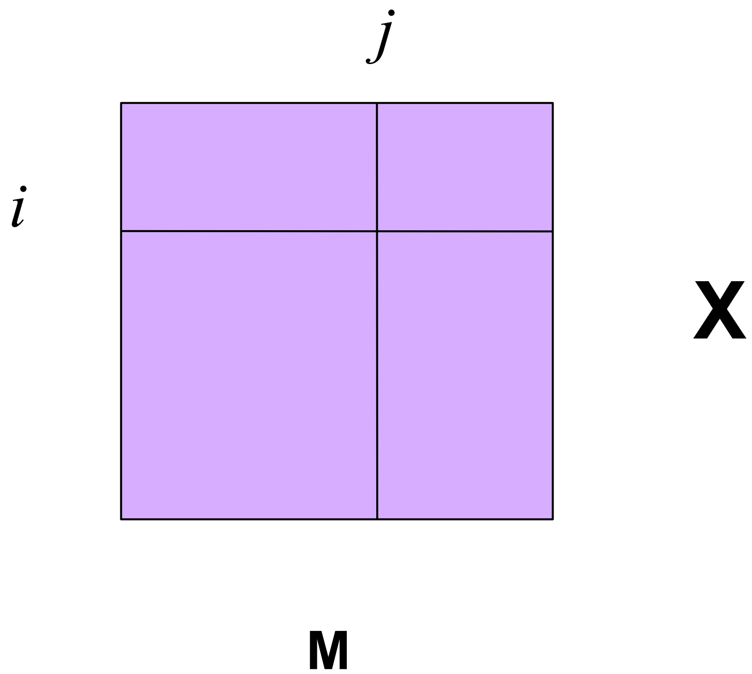

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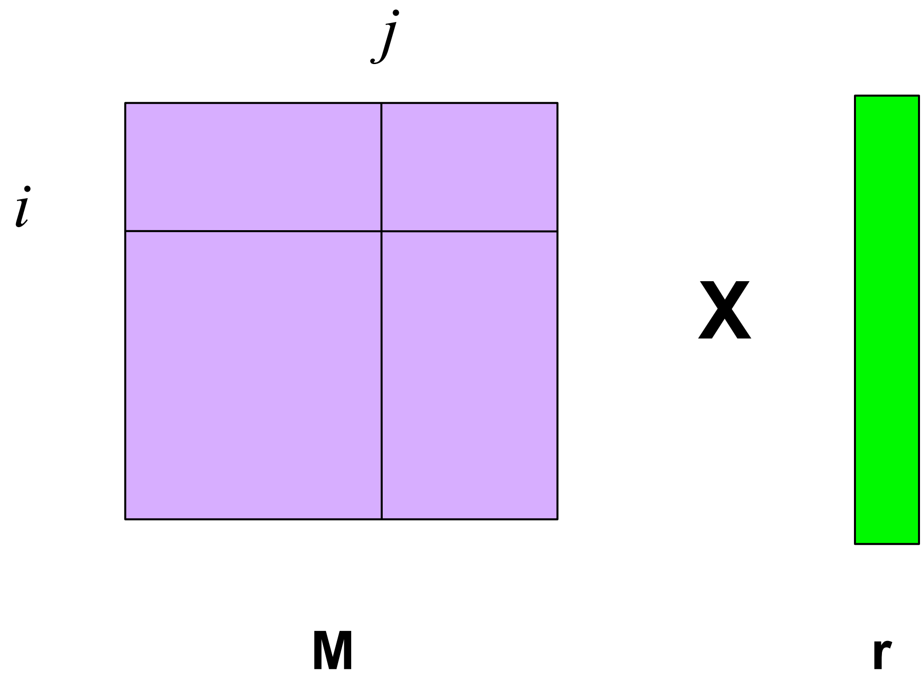

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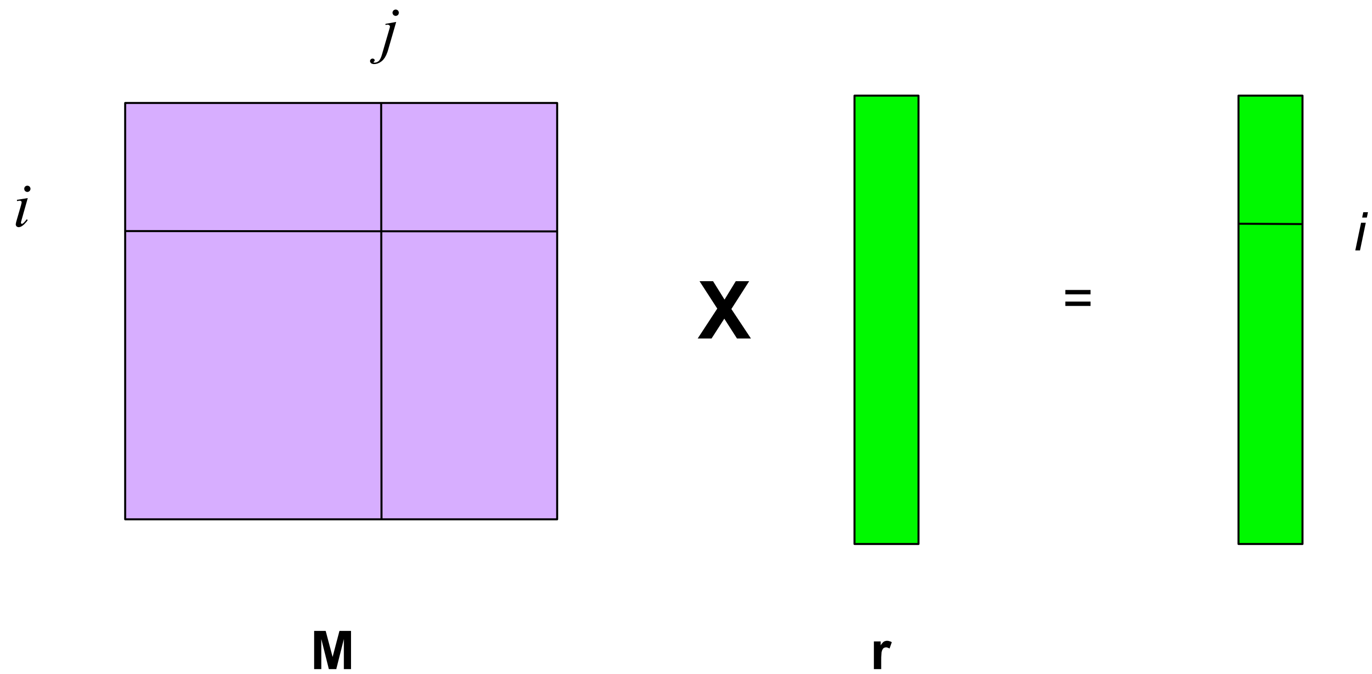

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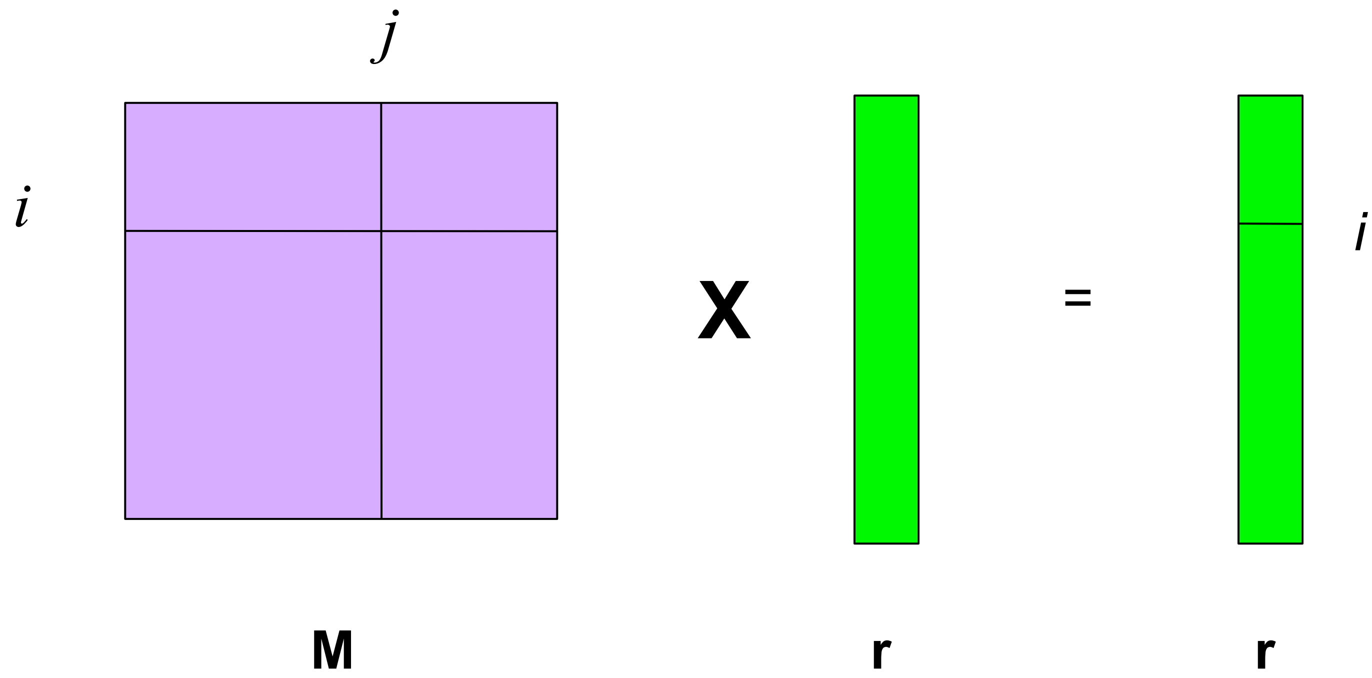

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Eigenvector formulation

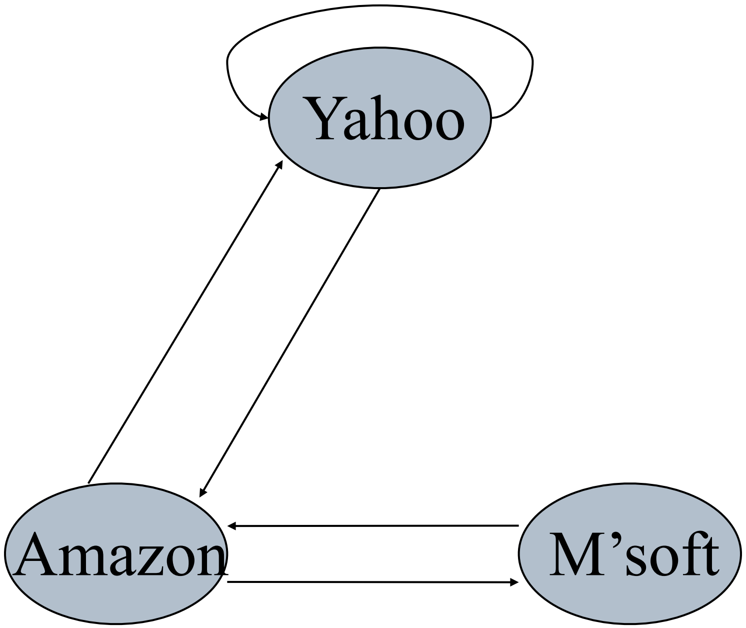
- The system of linear eq. can be written

$$\mathbf{r} = \mathbf{M}\mathbf{r}$$

- So the rank vector is an eigenvector of the stochastic web matrix
  - In fact, its first or principal eigenvector, with corresponding eigenvalue...

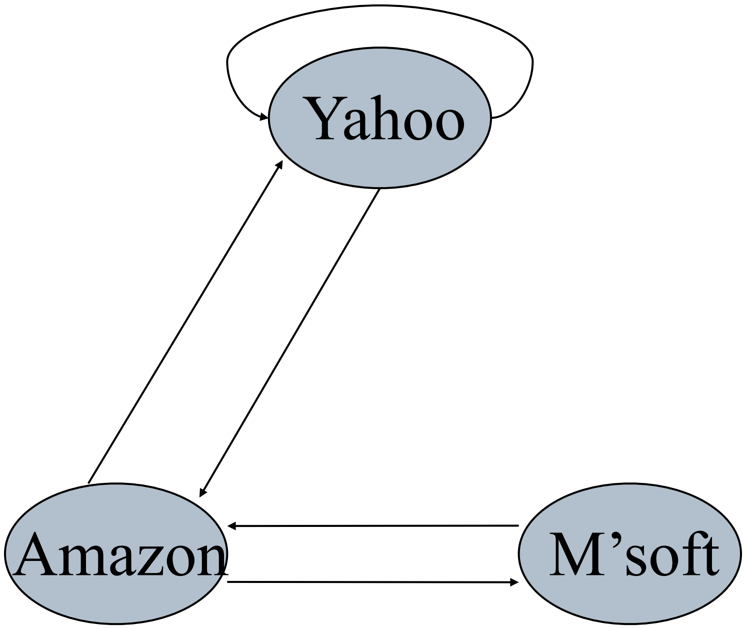
Definition. The vector **x** is an eigenvector of the matrix *A* with eigenvalue λ (lambda) if the following equation holds: $Ax = \lambda x$.

Example



$$\begin{array}{c|ccc|} & y & a & m \\ y & 1/2 & 1/2 & 0 \\ a & 1/2 & 0 & 1 \\ m & 0 & 1/2 & 0 \end{array}$$

Example



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$$\mathbf{r = Mr}$$

$$
\begin{bmatrix} y \\ a \\ m \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} y \\ a \\ m \end{bmatrix}
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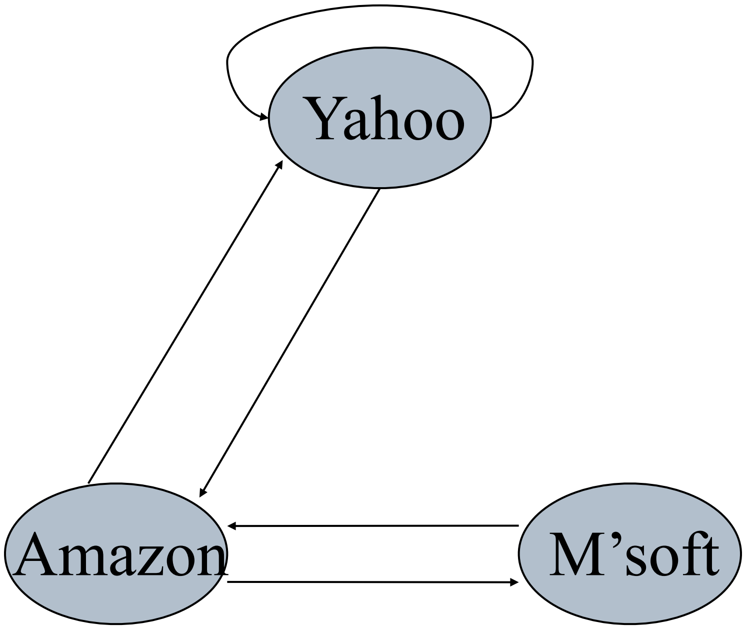

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Power Iteration method

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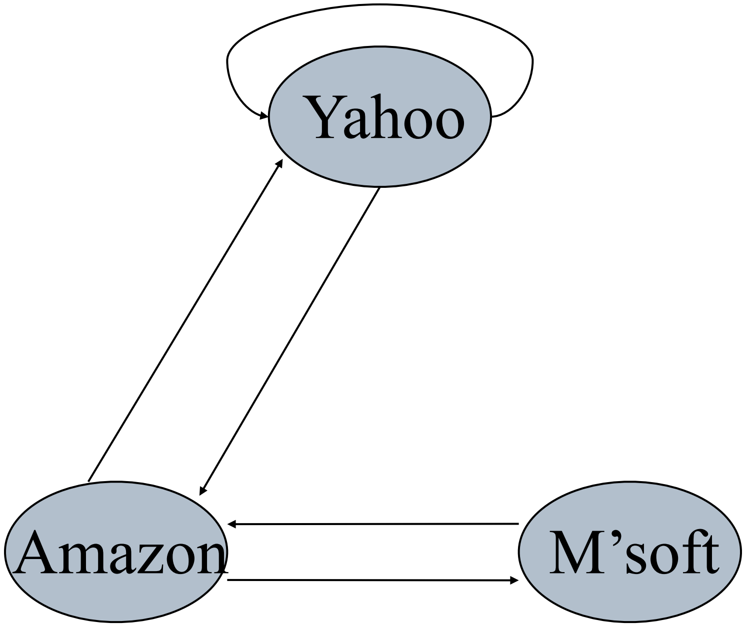
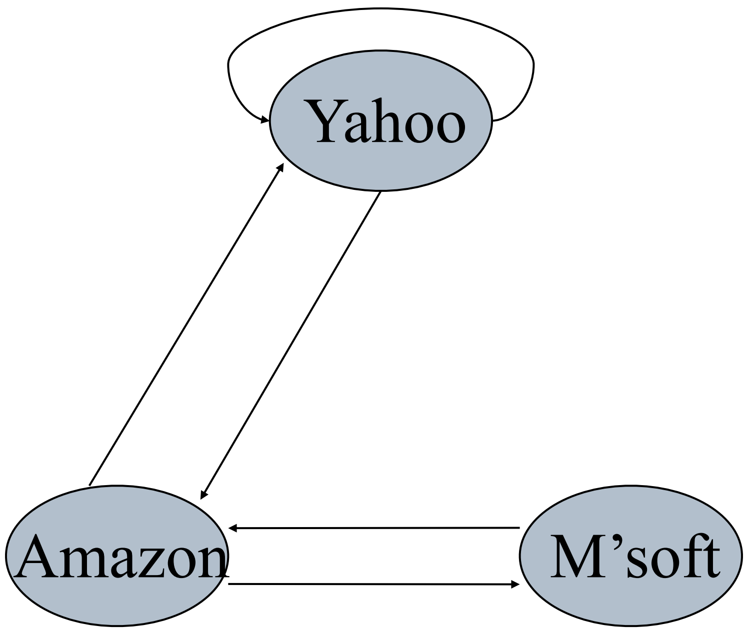
- Iterate: $\mathbf{r}^{k+1} = \mathbf{M}\mathbf{r}^k$
- Stop when $|\mathbf{r}^{k+1} - \mathbf{r}^k|_1 < \varepsilon$
  - $|\mathbf{x}|_1 = \sum_{1\leq i\leq N}|x_i|$ is the L_1 norm
  - Can use any other vector norm e.g., Euclidean

Power Iteration Example



	y	a	m
y	1/2	1/2	0
a	1/2	0	1
m	0	1/2	0

Power Iteration Example



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\hline
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\begin{array}{c} y \\ a \\ m \end{array} =
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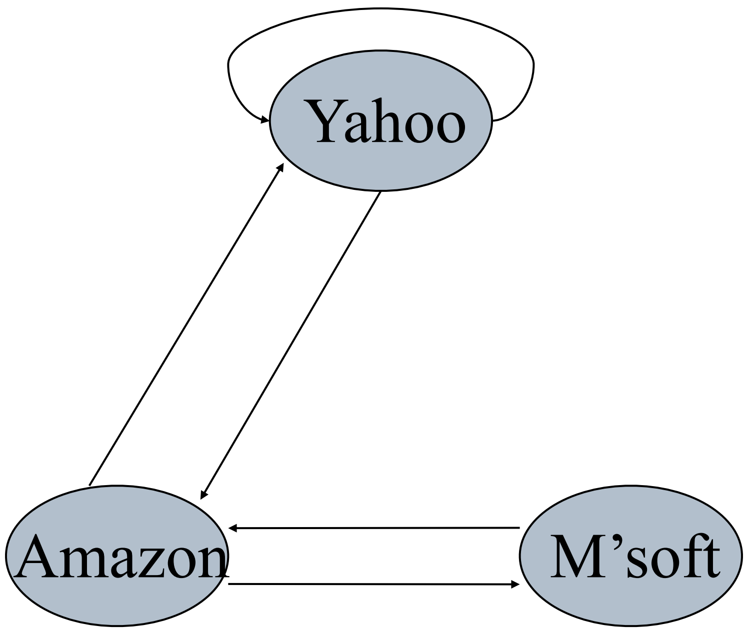

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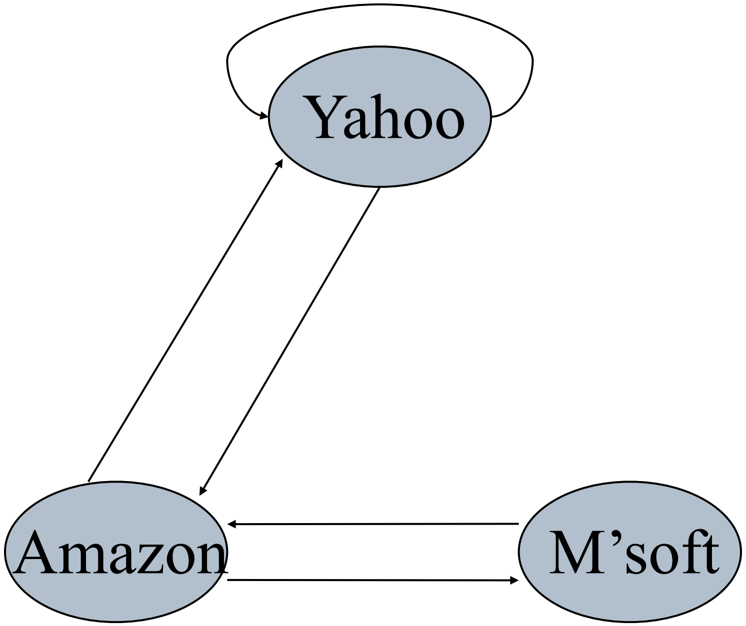

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\begin{array}{c}
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\end{array}
=
\begin{array}{c}
1/3 \\ 1/3 \\ 1/3
\end{array}
\quad
\begin{array}{c}
1/3 \\ 1/2 \\ 1/6
\end{array}
$$

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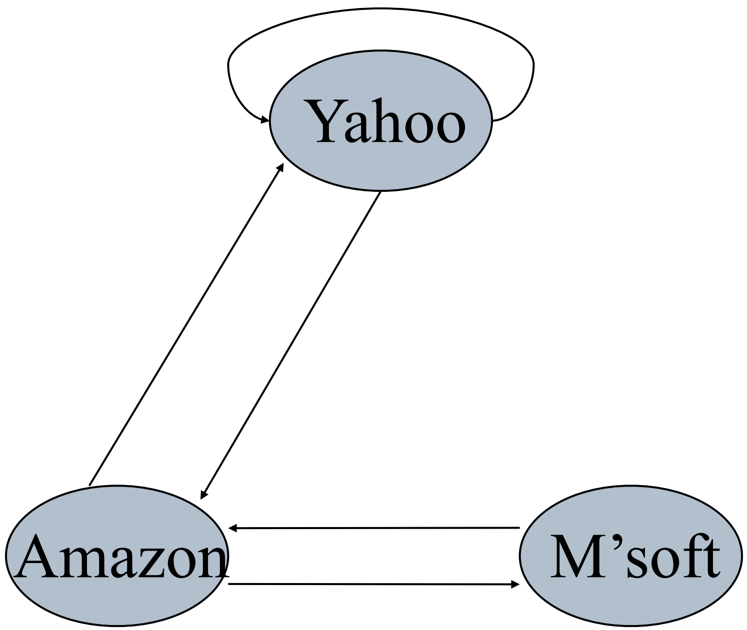

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y		1/3	1/3	5/12	
a	=	1/3	1/2	1/3	
m		1/3	1/6	1/4	

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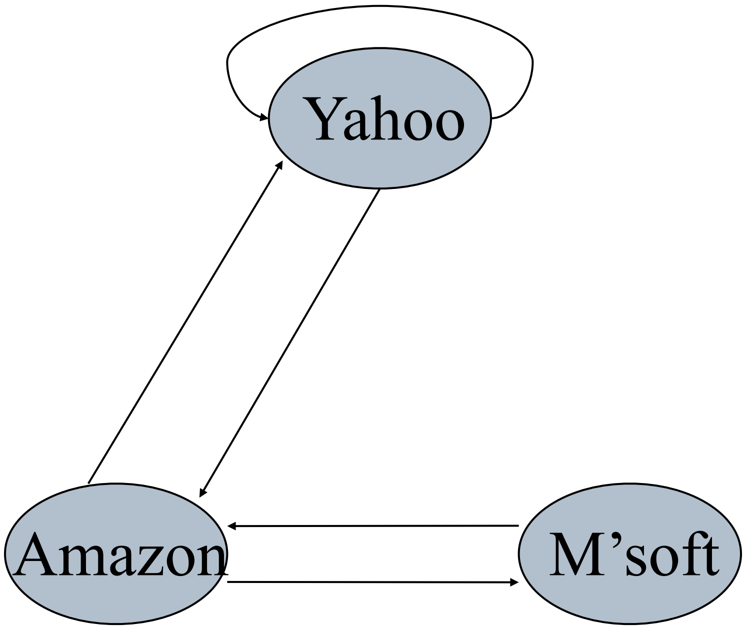

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\hline
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\end{array}
=
\begin{array}{c}
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\end{array}
\quad
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\end{array}
\quad
\begin{array}{c}
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\quad
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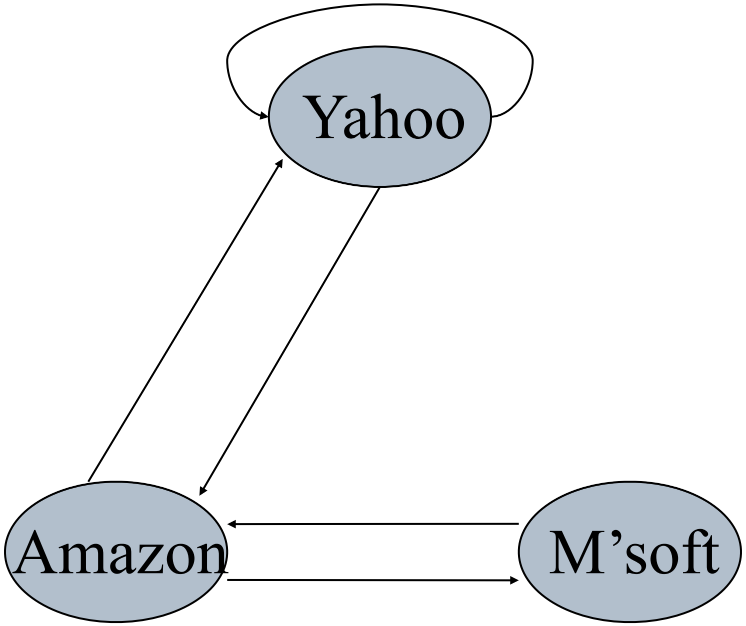

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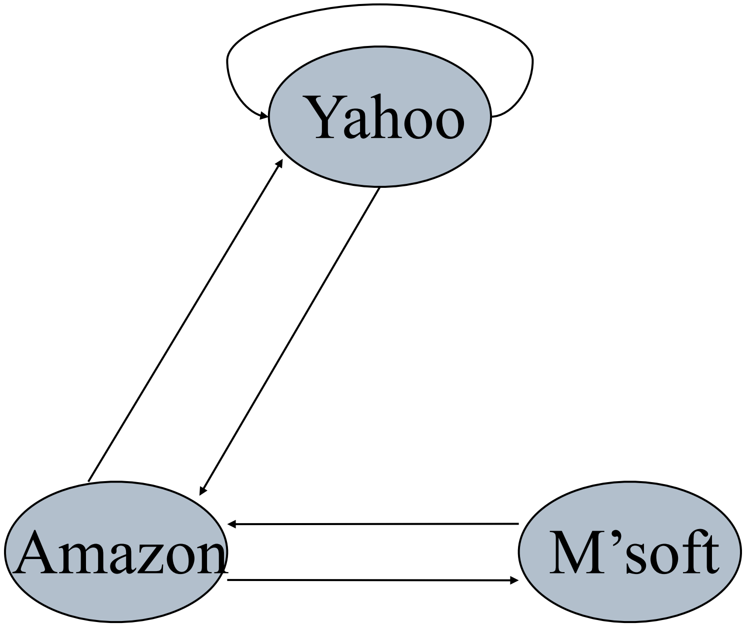

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Random Walk Interpretation

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- Imagine a random web surfer
  - At any time t, surfer is on some page P
  - At time t+1, the surfer follows an outlink from P uniformly at random
  - Ends up on some page Q linked from P
  - Process repeats indefinitely

Random Walk Interpretation

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  - At any time t, surfer is on some page P
  - At time t+1, the surfer follows an outlink from P uniformly at random
  - Ends up on some page Q linked from P
  - Process repeats indefinitely
- Let **p**(t) be a vector whose i^{th} component is the probability that the surfer is at page i at time t
  - **p**(t) is a probability distribution on pages

The stationary distribution

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- Where is the surfer at time t+1?
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- Our rank vector $\mathbf{r}$ satisfies $\mathbf{r} = \mathbf{M}\mathbf{r}$
  - So it is a stationary distribution for the random surfer

Existence and Uniqueness

A central result from the theory of random walks (aka Markov processes):

For graphs that satisfy certain conditions, the stationary distribution is unique and eventually will be reached no matter what the initial probability distribution at time $t = 0$.

Spider traps

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- A group of pages is a spider trap if there are no links from within the group to outside the group
  - Random surfer gets trapped

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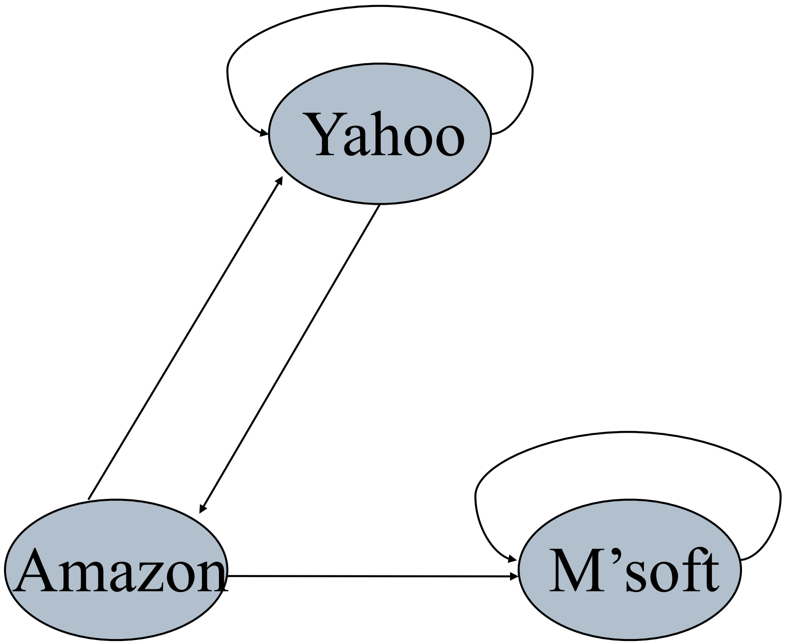
- Spider traps violate the conditions needed for the random walk theorem

Microsoft becomes a spider trap



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\begin{array}{c|ccc|}
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y & 1/2 & 1/2 & 0 \\
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\hline
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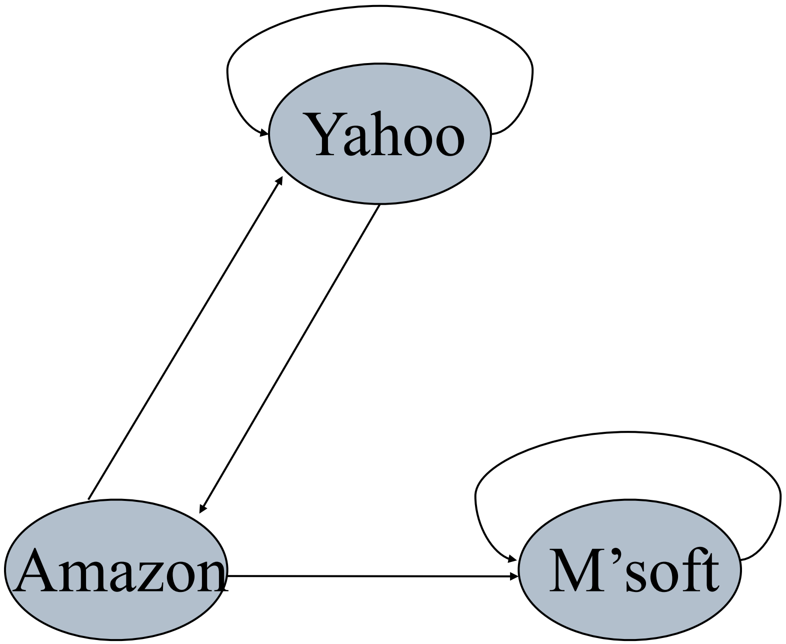

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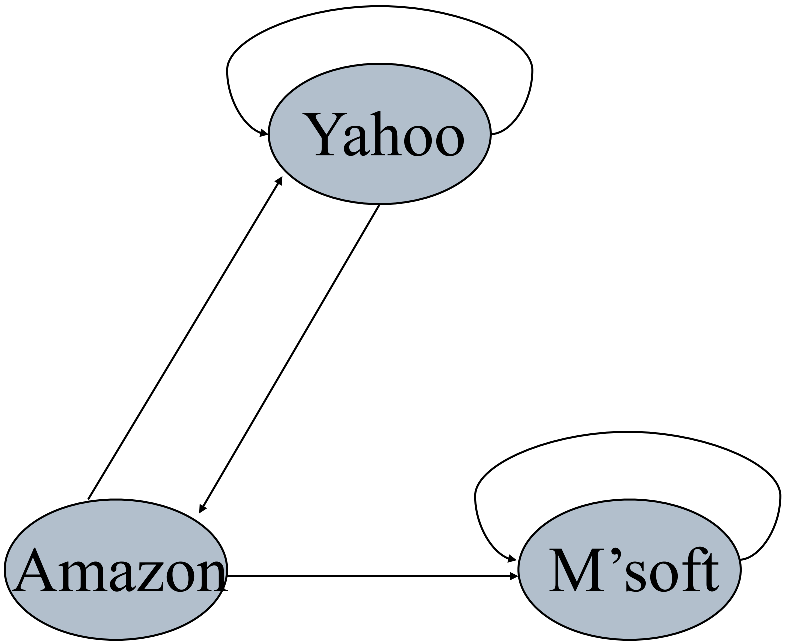

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\hline
\end{array}
$$

$$
\begin{array}{ccccc}
y & & 1/3 & 1/3 & 1/4 \\
a & = & 1/3 & 1/6 & 1/6 \\
m & & 1/3 & 1/2 & 7/12
\end{array}
$$

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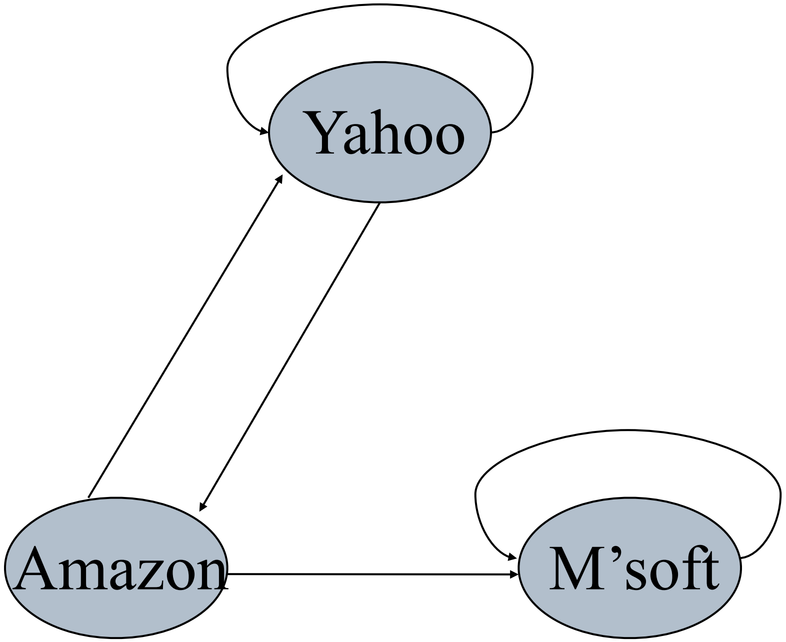

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$$\begin{array}{ccccccc} y & & 1/3 & 1/3 & 1/4 & 5/24 \\ a & = & 1/3 & 1/6 & 1/6 & 1/8 \\ m & & 1/3 & 1/2 & 7/12 & 2/3 \end{array}$$

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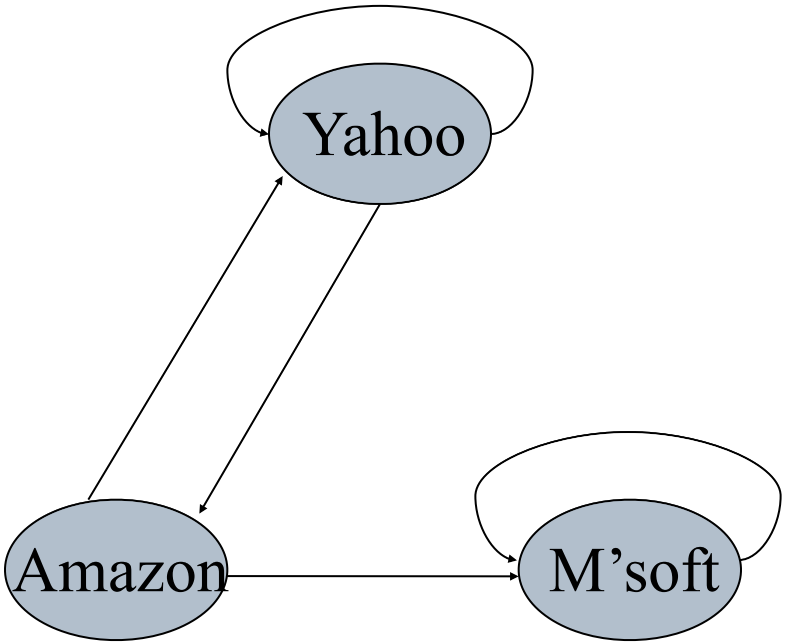

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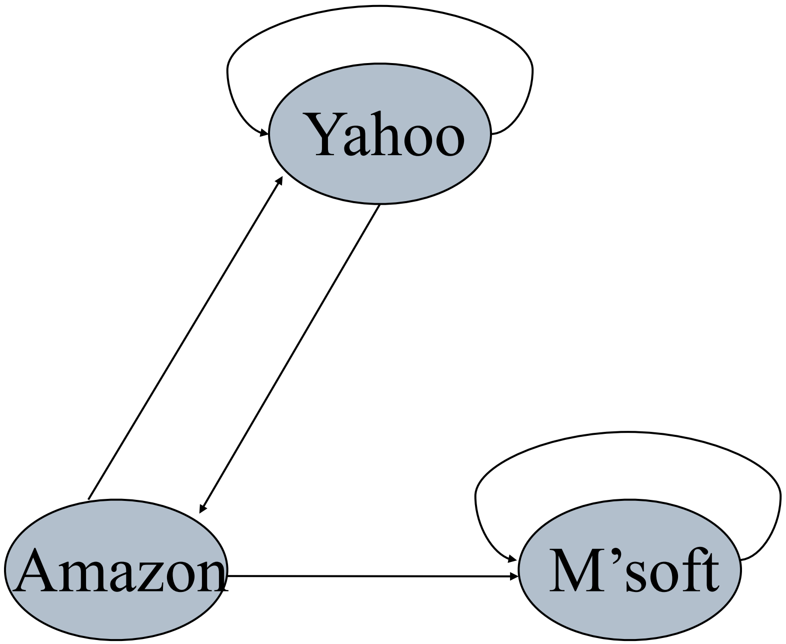

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m & & 1/3 & 1/2 & 7/12 & 2/3 & & 1
\end{array}
$$

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- At each time step, the random surfer has two options:
  - With probability β, follow a link at random
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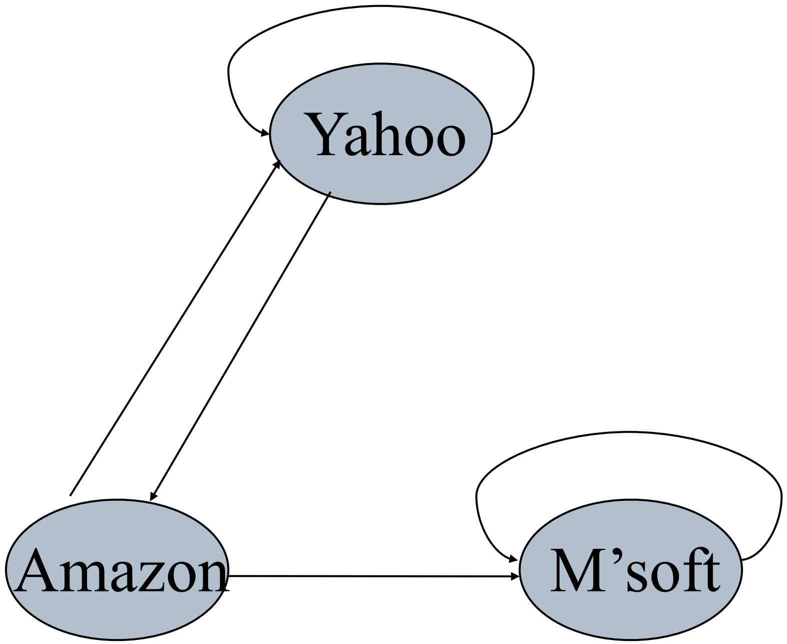
- Surfer will teleport out of spider trap within a few time steps

Random teleports ($\beta = 0.8$)



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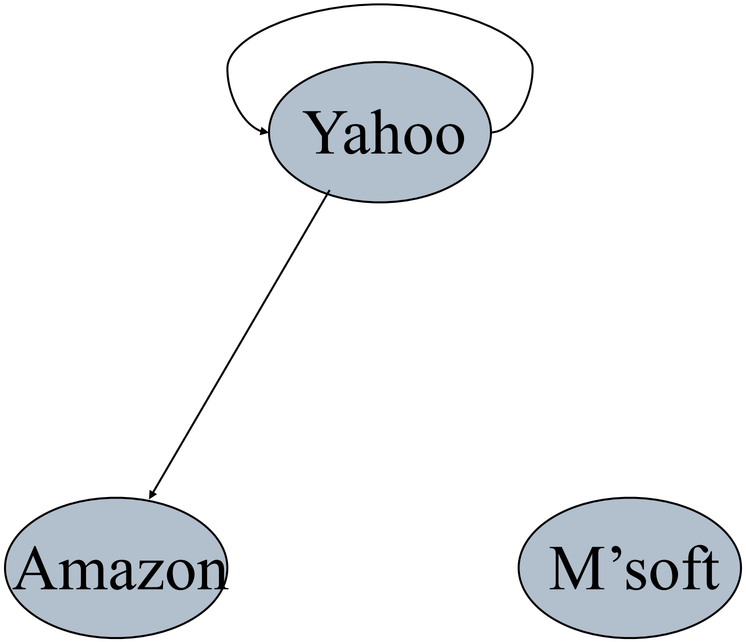

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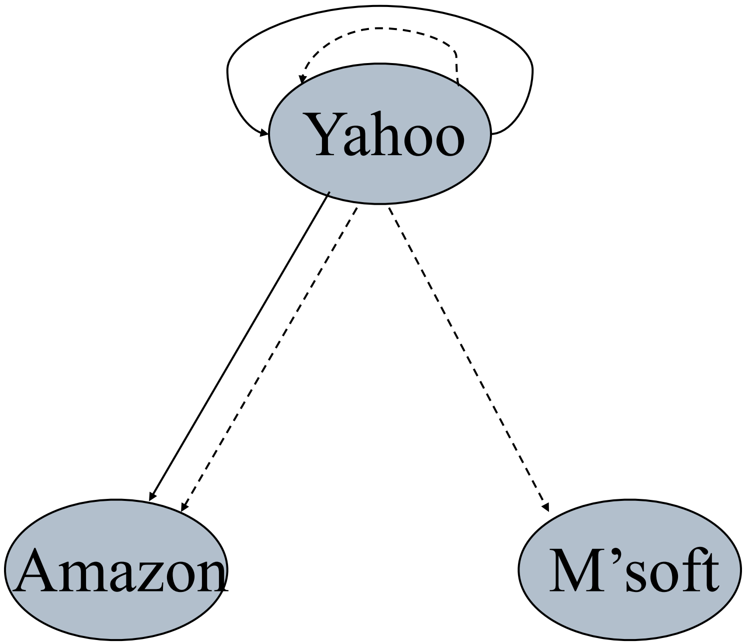

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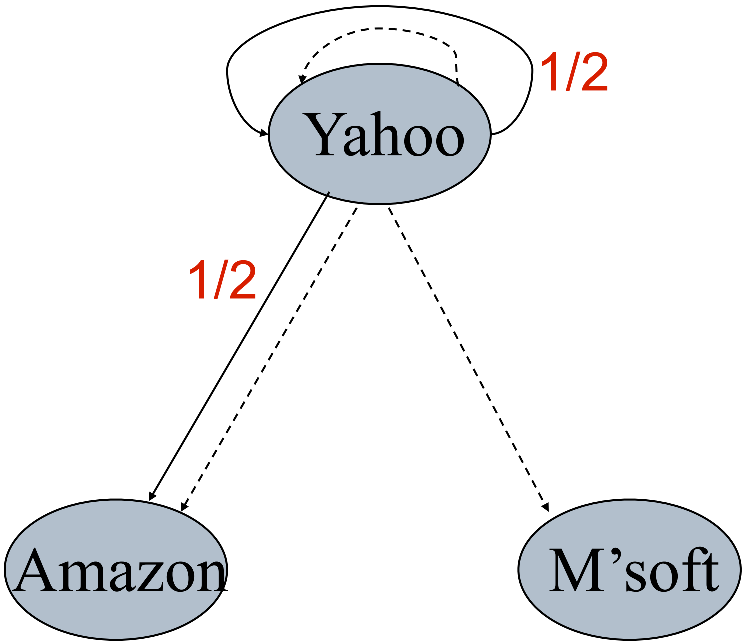

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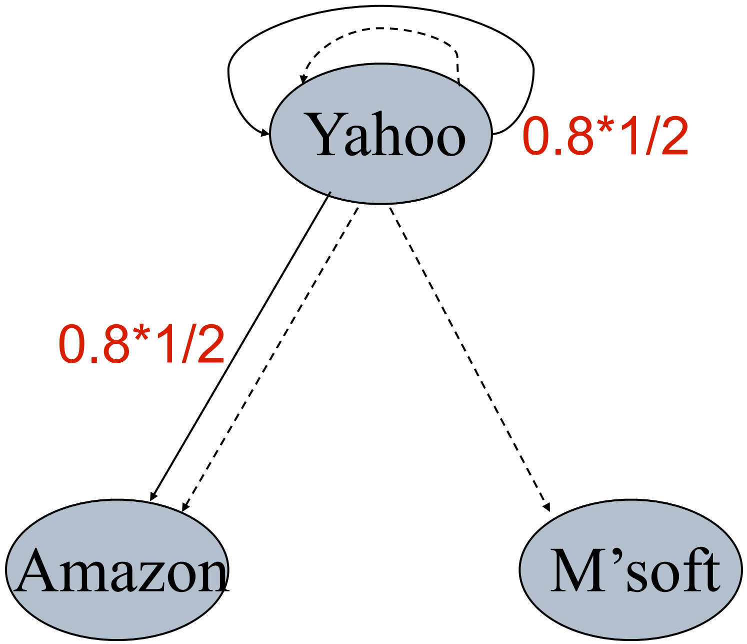

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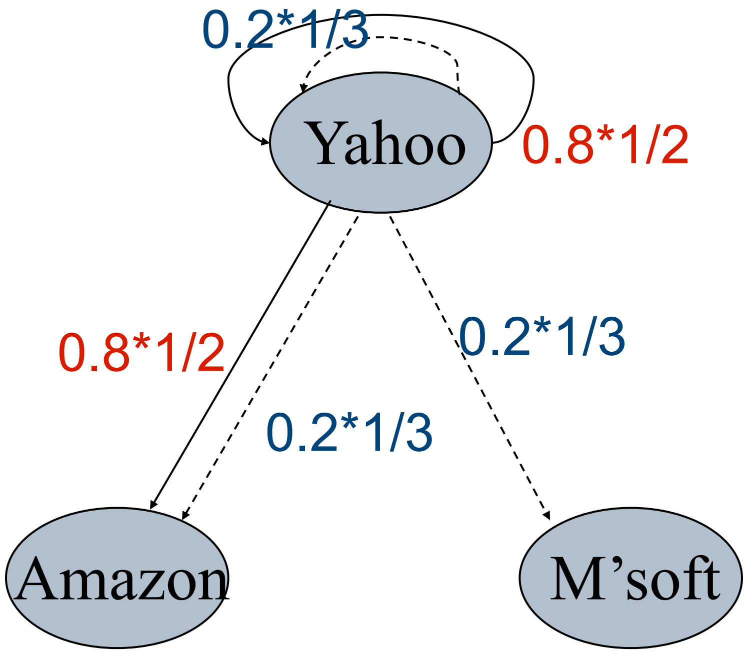

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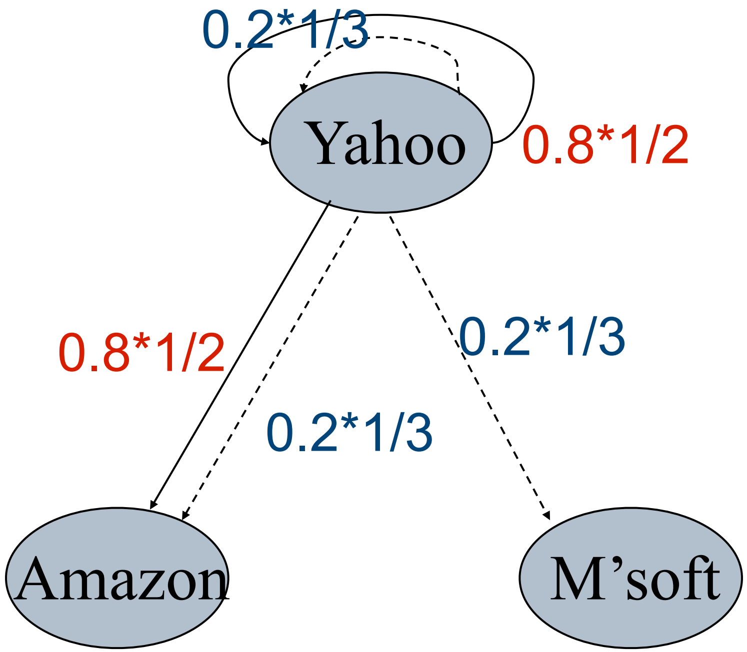

$$
\begin{array}{c} \\ y \\ a \\ m \end{array}
\begin{array}{c} y \\ \boxed{\begin{array}{c} 1/2 \\ 1/2 \\ 0 \end{array}} \end{array}
\longrightarrow 0.8* \begin{array}{c} y \\ \boxed{\begin{array}{c} 1/2 \\ 1/2 \\ 0 \end{array}} \end{array}
+ 0.2* \begin{array}{c} y \\ \boxed{\begin{array}{c} 1/3 \\ 1/3 \\ 1/3 \end{array}} \end{array}
$$

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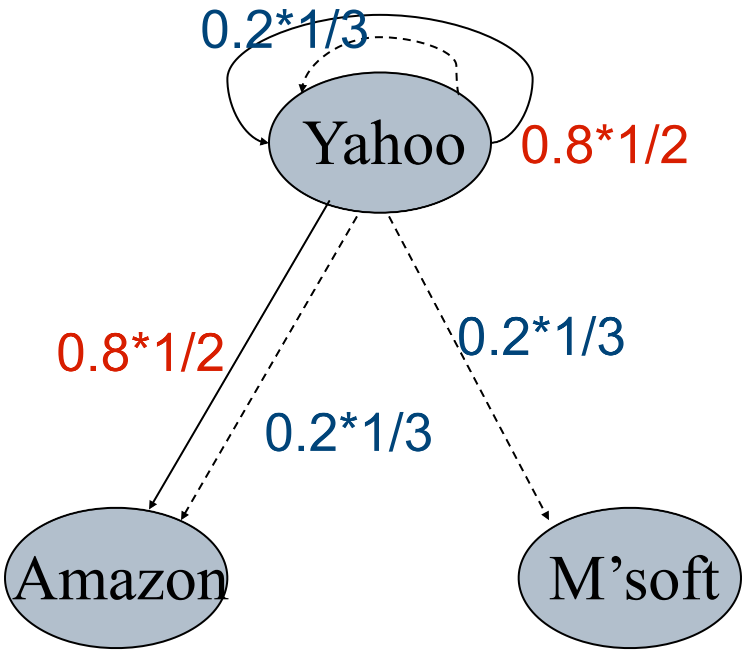

$$\begin{array}{c} \\ y \\ a \\ m \end{array} \begin{array}{c} y \\ \boxed{\begin{array}{c} 1/2 \\ 1/2 \\ 0 \end{array}} \end{array} \longrightarrow 0.8* \begin{array}{c} y \\ \boxed{\begin{array}{c} 1/2 \\ 1/2 \\ 0 \end{array}} \end{array} + 0.2* \begin{array}{c} y \\ \boxed{\begin{array}{c} 1/3 \\ 1/3 \\ 1/3 \end{array}} \end{array}$$

$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

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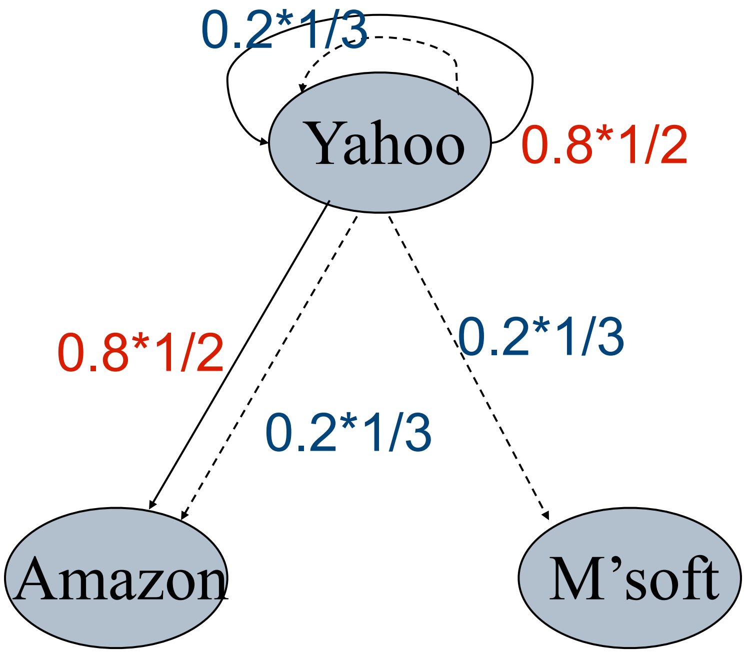

$$\begin{array}{cc} & y \\ \begin{array}{c} y \\ a \\ m \end{array} & \boxed{\begin{array}{c} 1/2 \\ 1/2 \\ 0 \end{array}} \end{array} \longrightarrow 0.8* \begin{array}{c} y \\ \boxed{\begin{array}{c} 1/2 \\ 1/2 \\ 0 \end{array}} \end{array} + 0.2* \begin{array}{c} y \\ \boxed{\begin{array}{c} 1/3 \\ 1/3 \\ 1/3 \end{array}} \end{array}$$

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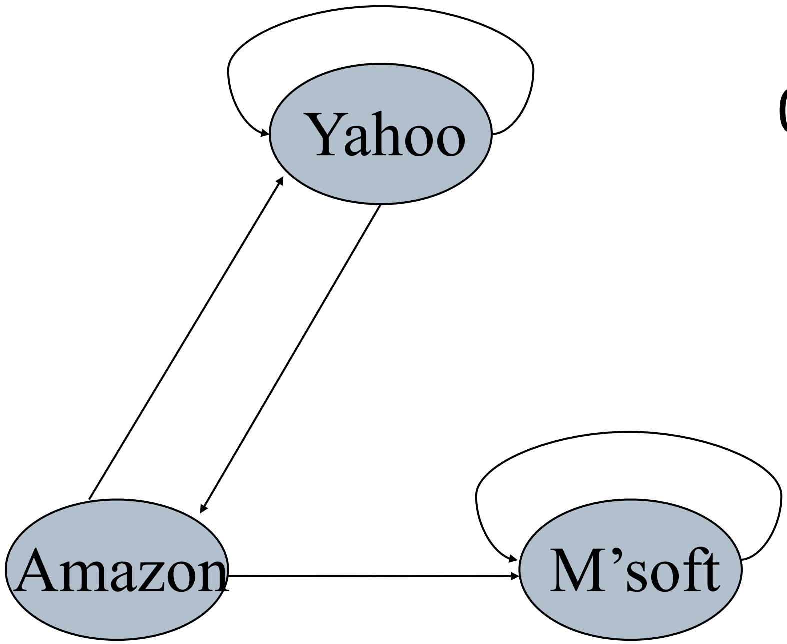
$$\begin{array}{c} y \\ a \\ m \end{array} \boxed{\begin{array}{ccc} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{array}}$$

Random teleports ($\beta = 0.8$)



$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

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$$\begin{array}{c} y \\ a \\ m \end{array} = \begin{array}{c} 1/3 \\ 1/3 \\ 1/3 \end{array}$$

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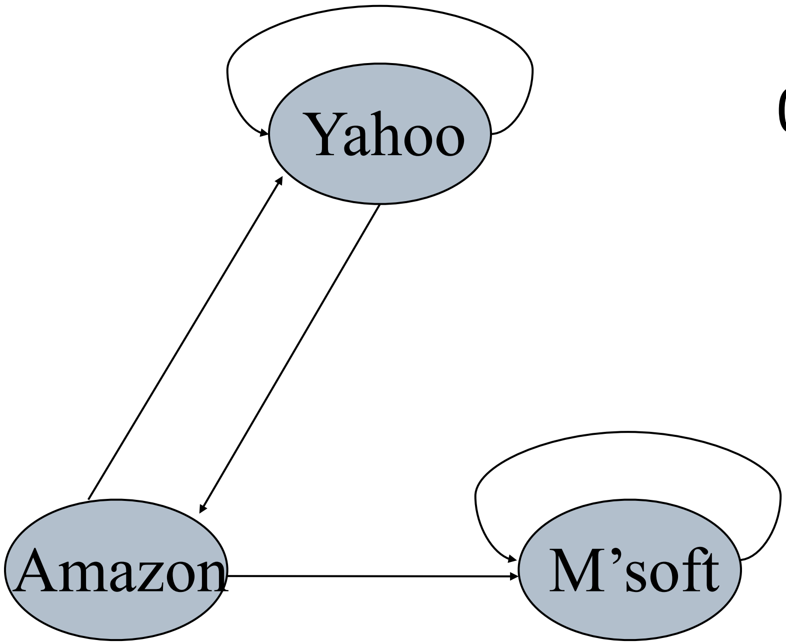

$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\begin{array}{c} y \\ a \\ m \end{array} \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}$$

$$\begin{array}{ccccc} y & & 1/3 & 1/3 \\ a & = & 1/3 & 0.20 \\ m & & 1/3 & 0.47 \end{array}$$

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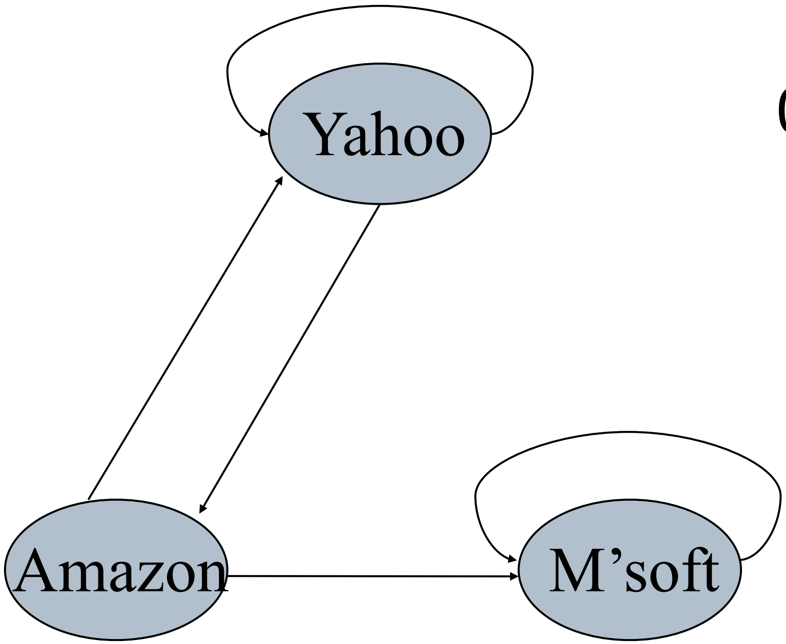

$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\begin{array}{c} y \\ a \\ m \end{array} \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}$$

$$\begin{array}{ccccc} y & & 1/3 & 1/3 & 0.27 \\ a & = & 1/3 & 0.20 & 0.20 \\ m & & 1/3 & 0.47 & 0.52 \end{array}$$

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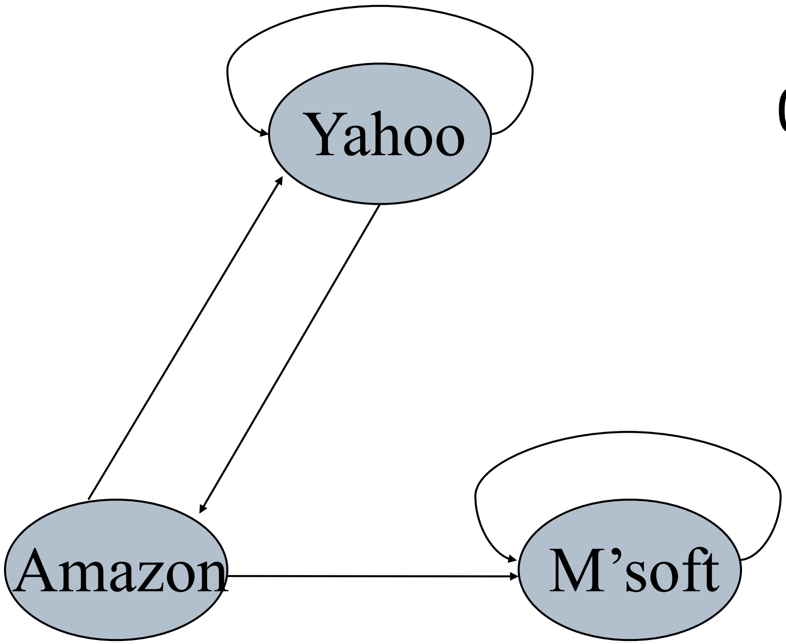
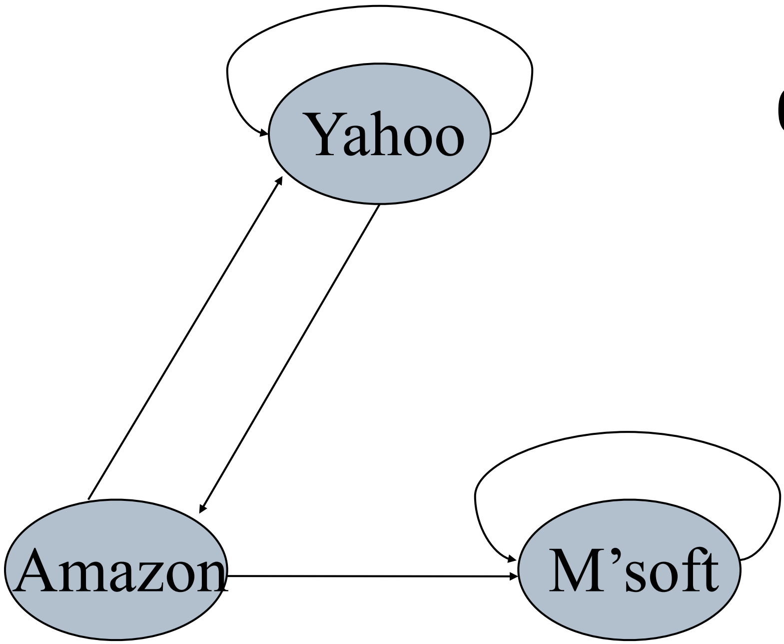


$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\begin{array}{c} y \\ a \\ m \end{array} \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}$$

$$\begin{array}{c} y \\ a \\ m \end{array} = \begin{array}{cccc} 1/3 & 1/3 & 0.27 & 0.258 \\ 1/3 & 0.20 & 0.20 & 0.178 \\ 1/3 & 0.47 & 0.52 & 0.562 \end{array}$$

Random teleports ($\beta = 0.8$)



$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\begin{array}{c} y \\ a \\ m \end{array} \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}$$

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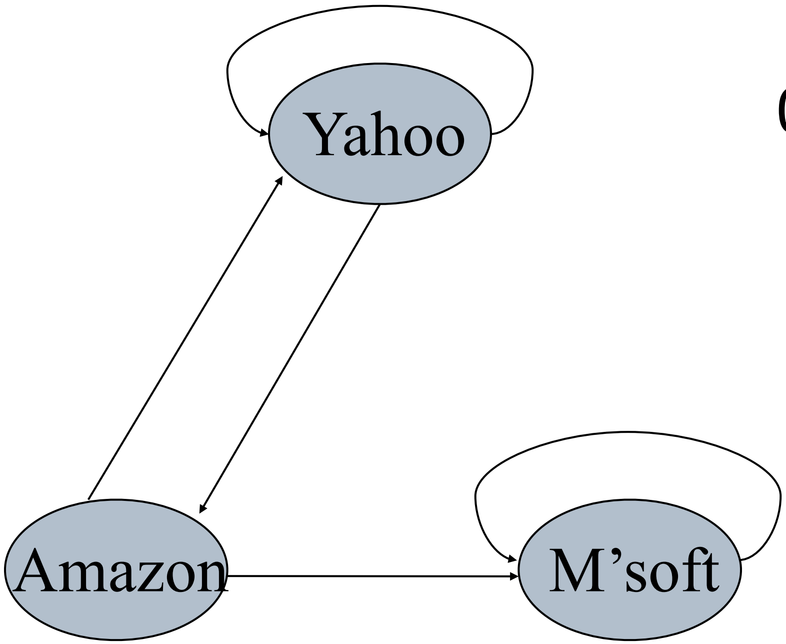

$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\begin{array}{c} y \\ a \\ m \end{array} \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}$$

$$\begin{array}{c} y \\ a \\ m \end{array} = \begin{array}{c} 1/3 \\ 1/3 \\ 1/3 \end{array} \quad \begin{array}{c} 1/3 \\ 0.20 \\ 0.47 \end{array} \quad \begin{array}{c} 0.27 \\ 0.20 \\ 0.52 \end{array} \quad \begin{array}{c} 0.258 \\ 0.178 \\ 0.562 \end{array} \quad \dots \quad \begin{array}{c} 7/33 \\ 5/33 \\ 21/33 \end{array}$$

Page Rank

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- Construct the NxN matrix **A** as follows
  - $A_{ij} = \beta M_{ij} + (1-\beta)/N$

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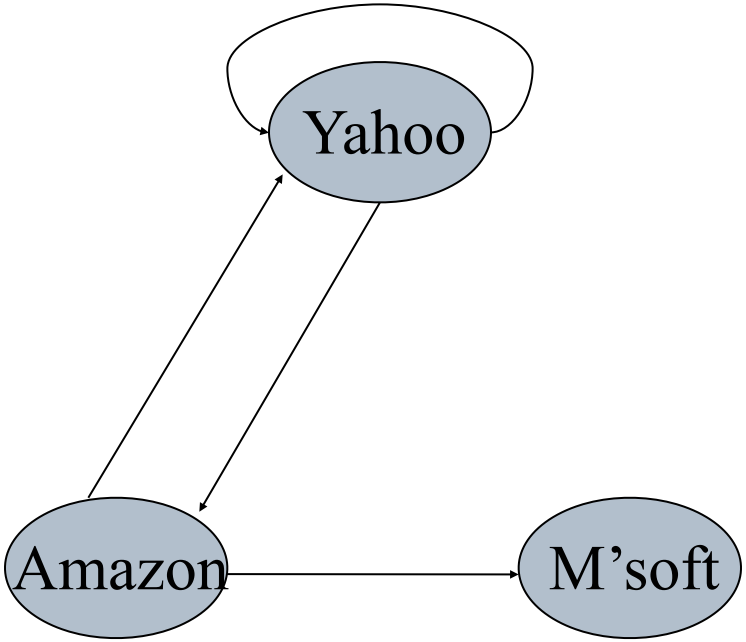
  - satisfying **r** = **Ar**
- Equivalently, **r** is the stationary distribution of the random walk with teleports

Dead ends

- The description of the PageRank algorithm is essentially complete. Minor problem with "dead ends".
- Pages with no outlinks are "dead ends" for the random surfer -> Nowhere to go in the next step.
- Our algorithm so far is not well-defined when the number of successors k=0 (we would have 1/0!).

Microsoft becomes a dead end

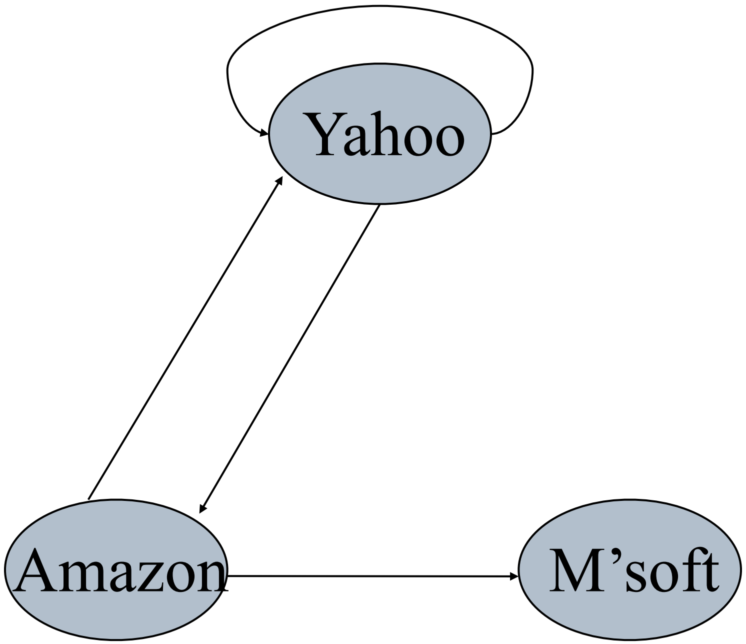


$$0.8\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix} + 0.2\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\begin{array}{c} y \\ a \\ m \end{array}\begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 1/15 \end{bmatrix}$$

$$\begin{matrix} y \\ a \\ m \end{matrix} = \begin{matrix} 1/3 \\ 1/3 \\ 1/3 \end{matrix}$$

Microsoft becomes a dead end



$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

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Non-stochastic!

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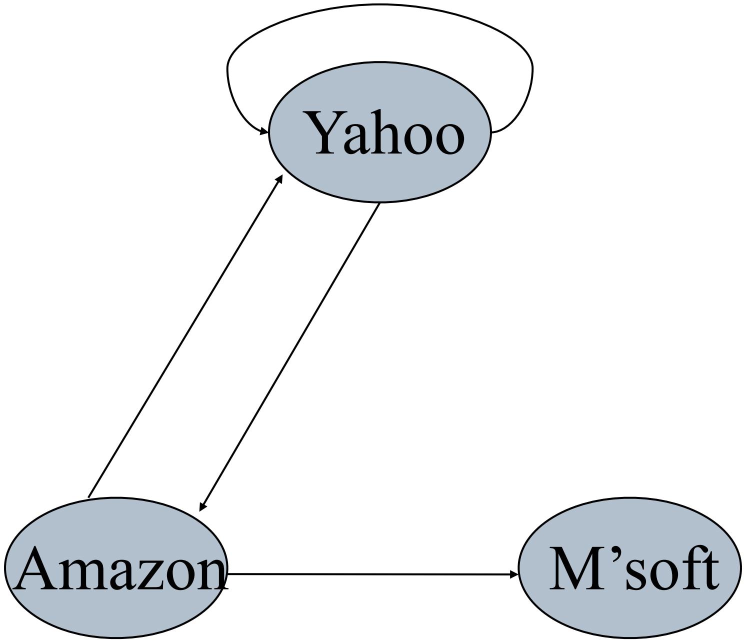

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. . .

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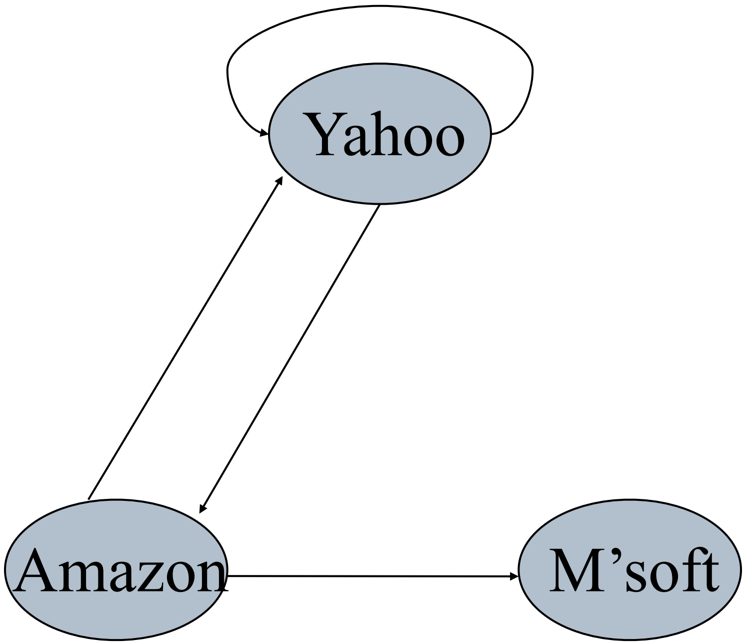

$$0.8\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{bmatrix} + 0.2\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\begin{array}{c} y \\ a \\ m \end{array}\begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 1/15 \end{bmatrix}$$

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- Teleport
  - Follow random teleport links with probability 1.0 from dead-ends
  - Adjust matrix accordingly

Dealing with dead-ends

- Teleport
  - Follow random teleport links with probability 1.0 from dead-ends
  - Adjust matrix accordingly
- More efficient: prune and propagate
  - Preprocess the graph to eliminate dead-ends
  - Might require multiple passes
  - Compute page rank on reduced graph
  - Approximate values for deadends by propagating values from reduced graph

Efficiency issues

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- Key step is matrix-vector multiplication
  - $\mathbf{r}^{new} = \mathbf{A}\mathbf{r}^{old}$

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- Easy if we have enough main memory to hold $\mathbf{A}$, $\mathbf{r}^{old}$, $\mathbf{r}^{new}$
- Say N = 1 billion pages
  - Matrix A has N^2 entries
    - 10^{18} is a large number!

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$\mathbf{r} = \beta\mathbf{Mr} + [(1-\beta)/N]_N$

where $[x]_N$ is a vector with N entries equal to x

Sparse matrix formulation

Sparse matrix formulation

- We can rearrange the page rank equation:
  - $\mathbf{r} = \beta\mathbf{M}\mathbf{r} + [(1-\beta)/N]_N$
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- **M** is a sparse matrix!
  - 10 links per node, approx 10N entries
- So in each iteration, we need to:
  - Compute $\mathbf{r}^{new} = \beta\mathbf{M}\mathbf{r}^{old}$
  - Add a constant value $(1-\beta)/N$ to each entry in $\mathbf{r}^{new}$

Sparse matrix encoding

- Encode sparse matrix using only nonzero entries
  - Space proportional roughly to number of links
  - say 10N, or 4*10*1 billion = 40GB
  - still won't fit in memory, but will fit on disk

source node	destination node
0	1
0	5
2	17

PageRank: summary

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- Remove iteratively dead ends from G

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PageRank: summary

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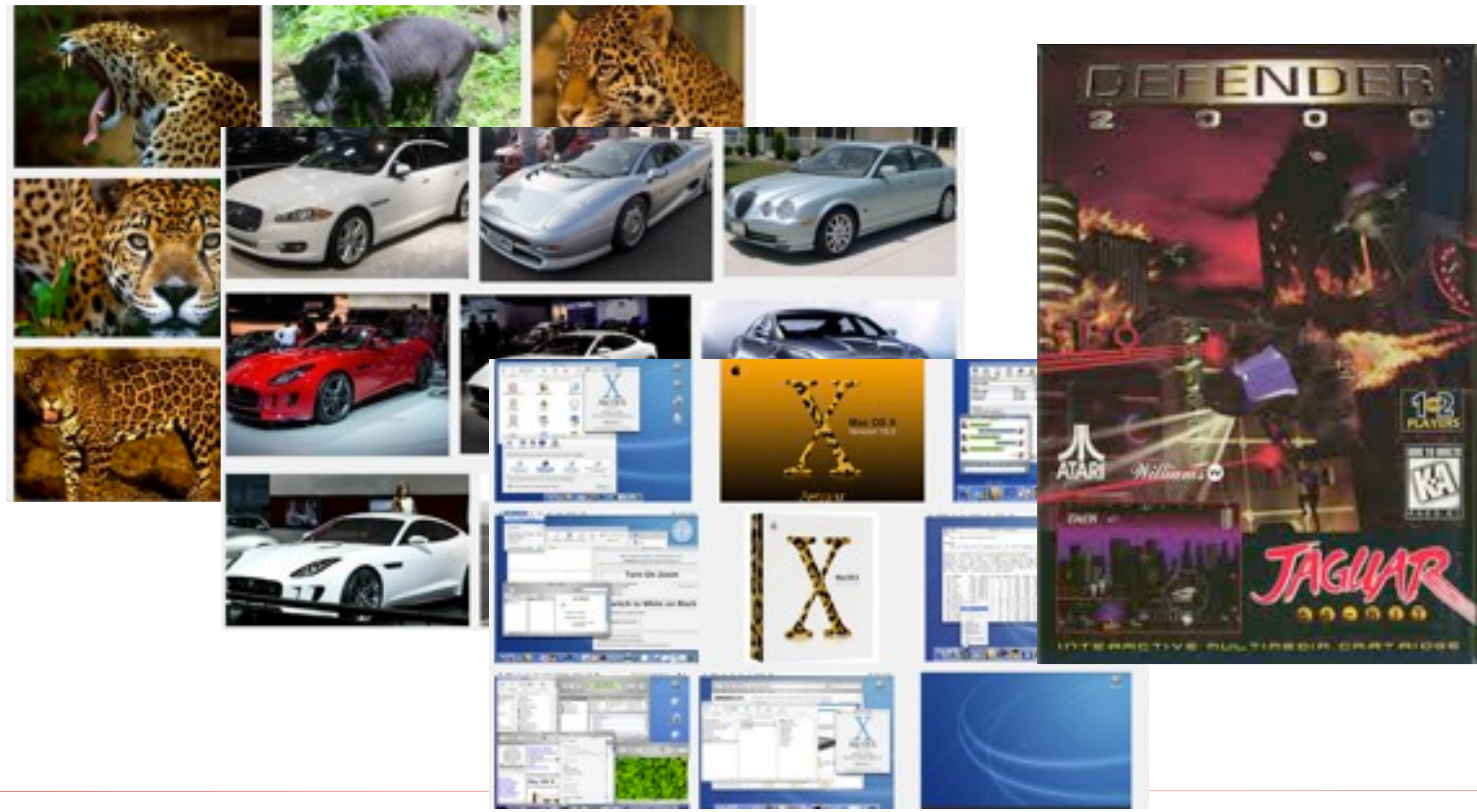
- Build stochastic matrix M_G (M for short)
- Initialize: $\mathbf{r}^0 = [1/N,\ldots.,1/N]^T$
- Iterate:
  - $\mathbf{r}^{\mathbf{k+1}} = \beta\mathbf{M}\mathbf{r}^{\mathbf{k}} + [(1-\beta)/N]_N$
  - Stop when $|\mathbf{r}^{k+1} - \mathbf{r}^k|_1 < \varepsilon$

Queries might be ambiguous...

- Suppose a user googles “jaguar”. What should we show to the user?



Private Page Rank Vector

- If we know that a user is interested in cars then we should show cars first.
- Ideally, each user should have his own private rank vector...
- ...but this is not feasible!

Personalized/Topic-Sensitive PR

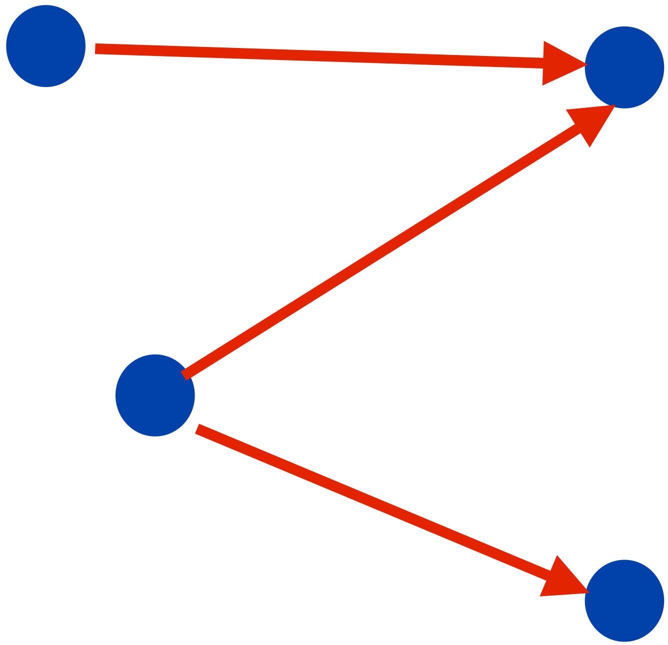
- Select a set of topics (e.g. 16 topics from the Open Directory DMOZ) and run a topic-sensitive version of PageRank (pages on that topic are ranked higher).

- If from previous queries we know that a user is interested in sports we visualize results according to that PR vector.

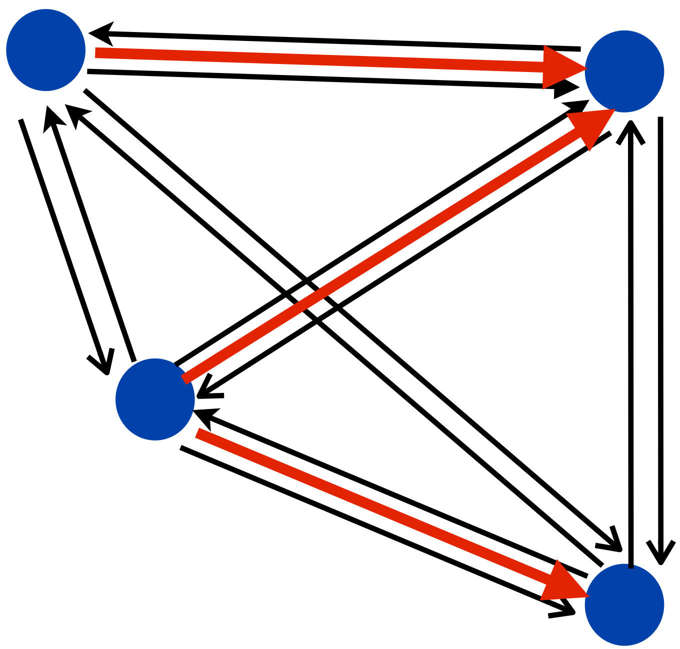
Personalized Page Rank

- For each topic (say sport) we select a set of pages S dealing with that topic.
- Pages that are linked to S are likely to be related to sport, while those that are far away from S are less related...
- ...

PR - Unbiased Random Walk



PR - Unbiased Random Walk

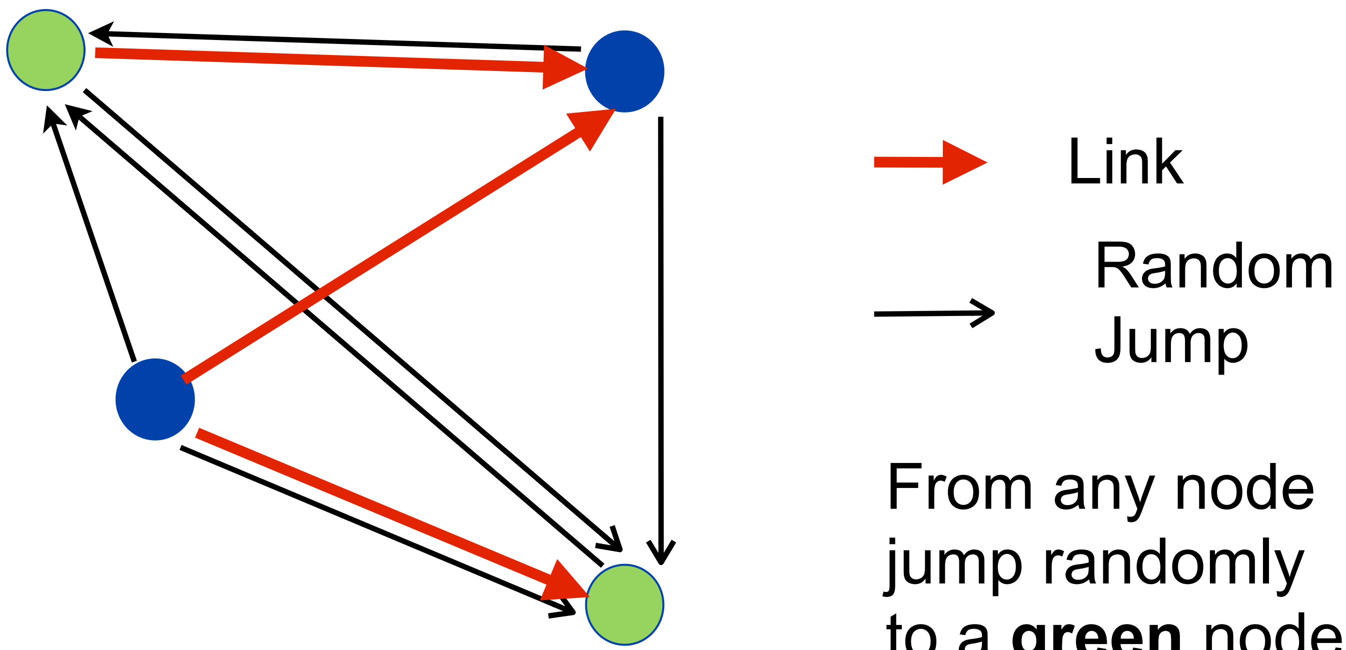


Link

Random Jump

From any node jump randomly to **any** node

Personalized PR - Biased Random Walk



From any node jump randomly to a **green** node

Personalized PR (formally)

- Let S be the vector of pages dealing with the selected topic.

- Let $\mathbf{e_S}$ be the indicator vector of S, that is $\mathbf{e_S}[p]=1$ if page p is in S, 0 otherwise.

- Equation $\mathbf{r} = \beta\mathbf{M}\mathbf{r} + [(1-\beta)/N]$ becomes
  $\mathbf{r} = \beta\mathbf{M}\mathbf{r} + (1-\beta)\mathbf{e_S}/|S|$

Issues

- How to decide which topics a user is interested in?
  - Allow users to select a topic from a menu.
  - Infer topics according to the queries issued by the users and pages he/she visited.
  - Use the information provided by user in FB.

- How to determine the topics of a set of pages? -> Clustering...