A version of Goodwillie calculus for non-cubes

Robin Stoll

Born October 3, 1996 in Bonn, Germany

February 19, 2020

Master's Thesis Mathematics

Advisor: Dr. Emanuele Dotto

Second Advisor: Prof. Dr. Stefan Schwede

Mathematical Institute

Mathematisch-Naturwissenschaftliche Fakultät der

Rheinischen Friedrich-Wilhelms-Universität Bonn

Contents

Notation 0.1. Let S be a set and $M \subseteq \mathcal{P}(S)$ a subset of its power set. We write $\mathrm{mc}(S, M)$ for the minimal number of elements of M needed to cover S and set $\mathrm{mc}(S, M) = \infty$ if no such cover exists.

Theorem 0.2. *Let S be a set, $\sigma\colon \check{\mathcal{G}} \to \mathcal{P}(S)$ an injective preshape, and $n \in \mathbb{N}_0$. Assume that $n \leq \mathrm{mc}(S, \check{\mathcal{G}})$. Then a functor $F\colon \mathcal{C} \to \mathcal{D}$ that is $(n-1)$-excisive is also σ-excisive.*

Proof. Throughout the proof let $D\colon \mathcal{P}(S) \to \mathcal{C}$ denote a σ-cocartesian diagram.

First note that the theorem is true if $\check{\mathcal{G}} = \mathcal{P}_{\leq 1}(S)$ (by **??**) or if $\mathrm{mc}(S, \check{\mathcal{G}}) = \infty$. To prove the latter claim take $s \in S \setminus \bigcup_{G \in \check{\mathcal{G}}} G$. Then, for any $A \in \mathcal{P}(S)$, the induced map $\sigma\!\downarrow\! A \to \sigma\!\downarrow\!(A \cup \{s\})$ is an isomorphism. Hence, by **??**, we have that $D(A) \to D(A \cup \{s\})$ is an equivalence, thus, by **??**, that the diagram $F \circ D$ is cartesian for any functor $F\colon \mathcal{C} \to \mathcal{D}$ (we don't even need that it is $(n-1)$-excisive).

We now proceed by induction on ?. Note that this terminates since ?.

Let $\mathcal{M}$ denote the set of maximal elements of $\check{\mathcal{G}}$ and set $\mathcal{Q} \coloneqq \mathcal{P}(\mathcal{M}) \times \mathcal{P}(S)$. Denote by $\mathcal{R} \subseteq \mathcal{P}(\mathcal{M}) \times \check{\mathcal{G}}$ the full subposet spanned by $\mathcal{P}(\mathcal{M}) \times (\check{\mathcal{G}} \setminus \mathcal{M})$ and (A, m) for any $A \in \mathcal{P}(\mathcal{M})$ and $m \in A$. Now let $E' \coloneqq (\mathrm{Res}_\sigma D) \circ \mathrm{pr}_2\colon \mathcal{R} \to \mathcal{C}$ and set $E \coloneqq \mathrm{Lan}_{\mathrm{id} \times \sigma} E'\colon \mathcal{P}(\mathcal{M}) \times \mathcal{P}(S) \to \mathcal{C}$. $\square$

A. The calculus of mates

In this appendix we recall the mate construction as well as a number of lemmas concerning it, which are quite useful when working with adjunctions and natural transformations. Since this is not supposed to be a comprehensive exposition of the topic, we will be brief and only state and give references for the statements we will use. A concise summary of these, and a few more, important statements, though without proofs, can be found in [**GPS**]. A longer exposition with proofs is given (in French) in [**Ayo**].

Notation A.1. Suppose we have, in a (strict) 2-category, a diagram of the form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow\alpha"] & B \arrow[d, "k"] \\ C \arrow[r, "c"] & D \end{tikzcd}$$

and fixed adjunctions $a_! \dashv a$ and $c_! \dashv c$. In this situation we write $\alpha_!$ for the *mate* of α, which is a 2-morphism of the form

$$\begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\nwarrow\alpha_!"] & B \arrow[l, "a_!"'] \arrow[d, "k"] \\ C & D \arrow[l, "c_!"] \end{tikzcd}$$

defined as the composition

$$c_!k \xrightarrow{c_!k\eta_a} c_!kaa_! \xrightarrow{c_!\alpha a_!} c_!cha_! \xrightarrow{\varepsilon_c ha_!} ha_!$$

where η_a and ε_c are the unit respectively counit of the adjunctions $a_! \dashv a$ respectively $c_! \dashv c$ fixed above.

The following is a property of the mate that follows easily from the definitions (and actually characterizes it uniquely).

Lemma A.2. *Let the following be a diagram in a 2-category*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow\alpha"] & B \arrow[d, "k"] \\ C \arrow[r, "c"] & D \end{tikzcd}$$

and $a_! \dashv a$ and $c_! \dashv c$ two fixed adjunctions. Then the following two diagrams commute

$$\begin{tikzcd} k \arrow[r, "k\eta_a"] \arrow[d, "\eta_c k"'] & kaa_! \arrow[d, "\alpha a_!"] \\ cc_!k \arrow[r, "c\alpha_!"] & cha_! \end{tikzcd} \qquad \begin{tikzcd} c_!ka \arrow[r, "\alpha_! a"] \arrow[d, "c_!\alpha"'] & ha_!a \arrow[d, "h\varepsilon_a"] \\ c_!ch \arrow[r, "\varepsilon_c h"] & h \end{tikzcd}$$

where η and ε denote the respective (co)units and $\alpha_!$ is the mate of α.

Proof. This is (the dual of) [**Ayo**]. □

The following two lemmas express a certain functoriality of the mate construction with respect to pasting of squares. A more abstract (and maybe conceptual) way to formulate them is to present the mate construction as an isomorphism of certain double categories. This can be found in [**KS**].

Lemma A.3 (Pasting law I)**.** *Let the following be a diagram in a 2-category and its paste*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "h"'] \arrow[dr, phantom, "\swArrow\alpha"] & B \arrow[r, "b"] \arrow[d, "k"] \arrow[dr, phantom, "\swArrow\beta"] & E \arrow[d, "l"] \\ C \arrow[r, "c"'] & D \arrow[r, "d"'] & F \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "ba"] \arrow[d, "h"'] \arrow[dr, phantom, "\swArrow\alpha*\beta"] & E \arrow[d, "l"] \\ C \arrow[r, "dc"'] & F \end{tikzcd}$$

and $a_! \dashv a$, $b_! \dashv b$, $c_! \dashv c$, and $d_! \dashv d$ four fixed adjunctions. We obtain mates $\alpha_!$ and $\beta_!$ that fit into diagrams of the form

$$\begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\nwArrow\alpha_!"] & B \arrow[l, "a_!"'] \arrow[d, "k"] \arrow[dr, phantom, "\nwArrow\beta_!"] & E \arrow[l, "b_!"'] \arrow[d, "l"] \\ C & D \arrow[l, "c_!"] & F \arrow[l, "d_!"] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\nwArrow\alpha_!*\beta_!"] & E \arrow[l, "a_!b_!"'] \arrow[d, "l"] \\ C & F \arrow[l, "c_!d_!"] \end{tikzcd}$$

and it holds that $\alpha_! * \beta_! = (\alpha * \beta)_!$, where, for the latter mate, we use the adjunctions $a_!b_! \dashv ba$ and $c_!d_! \dashv dc$ given by composing the original ones.

Proof. This is (the dual of) [**Ayo**] (though note that the composition $\alpha_! * \beta_!$ is erroneously written the wrong way around there, and that what is actually proven is the dual version we stated). □

Lemma A.4 (Pasting law II)**.** *Let the following be a diagram in a 2-category and its paste*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "h"'] \arrow[dr, phantom, "\neArrow\alpha"] & B \arrow[r, "b"] \arrow[d, "k"] \arrow[dr, phantom, "\neArrow\beta"] & E \arrow[d, "l"] \\ C \arrow[r, "c"'] & D \arrow[r, "d"'] & F \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "ba"] \arrow[d, "h"'] \arrow[dr, phantom, "\neArrow\beta*\alpha"] & E \arrow[d, "l"] \\ C \arrow[r, "dc"'] & F \end{tikzcd}$$

and $h_! \dashv h$, $k_! \dashv k$, and $l_! \dashv l$ three fixed adjunctions. We obtain mates $\alpha_!$ and $\beta_!$ that fit into diagrams of the form

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[dr, phantom, "\nwArrow\alpha_!"] & B \arrow[r, "b"] \arrow[dr, phantom, "\nwArrow\beta_!"] & E \\ C \arrow[u, "h_!"] \arrow[r, "c"'] & D \arrow[u, "k_!"'] \arrow[r, "d"'] & F \arrow[u, "l_!"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "ba"] \arrow[dr, phantom, "\nwArrow\alpha_!*\beta_!"] & E \\ C \arrow[u, "h_!"] \arrow[r, "dc"'] & F \arrow[u, "l_!"'] \end{tikzcd}$$

and it holds that $\alpha_! * \beta_! = (\beta * \alpha)_!$.

Proof. This is (the dual of) [**Ayo**]. □

Lemma A.5. *Let the following be a diagram in a 2-category*

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow\alpha"] & B \arrow[d, "k"] \\ C \arrow[r, "c"] & D \end{tikzcd}$$

and $a_! \dashv a$ and $c_! \dashv c$ two fixed adjunctions. Furthermore assume that h and k are isomorphisms, and that α is a 2-isomorphism. Then the mate $\alpha_!$ is a 2-isomorphism.

Proof. First note that if $a = c$ (with the same fixed adjunction) and h, k, and α are all identities, then the mate $\alpha_!$ is the identity (by one of the triangle identities). For the general case consider the diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "h"'] \arrow[dr, phantom, "\swarrow\alpha"] & B \arrow[d, "k"] \\ C \arrow[r, "c"] \arrow[d, "h^{-1}"'] \arrow[dr, phantom, "\swarrow\beta"] & D \arrow[d, "k^{-1}"] \\ A \arrow[r, "a"] & B \end{tikzcd}$$

where β is given by

$$k^{-1}c = k^{-1}chh^{-1} \xrightarrow{k^{-1}\alpha^{-1}h^{-1}} k^{-1}kah^{-1} = ah^{-1}$$

and note that the paste $\beta * \alpha$ is the identity. This follows from the diagram

$$\begin{tikzcd} k^{-1}ka \arrow[r, "k^{-1}\alpha"] \arrow[d, equal] & k^{-1}ch \arrow[r, "\beta h"] \arrow[d, equal] & ah^{-1}h \arrow[d, equal] \\ k^{-1}kah^{-1}h \arrow[r, "k^{-1}\alpha h^{-1}h"] \arrow[d, equal] & k^{-1}chh^{-1}h \arrow[r, "k^{-1}\alpha^{-1}h^{-1}h"] & k^{-1}kah^{-1}h \arrow[d, equal] \\ a \arrow[rr, "\mathrm{id}"] & & a \end{tikzcd}$$

being commutative. Thus, by the pasting law for mates, we obtain that $h^{-1}\alpha_!$ has a right inverse (namely $\beta_! k$). In the same way we can show that $\alpha_! k^{-1}$ has a left inverse. Since h^{-1} and k^{-1} are both isomorphisms, this implies that $\alpha_!$ has both a left and a right inverse and thus is a 2-isomorphism. □

Remark A.6. The statement of Lemma A.5 is still true when we only require h and k to be equivalences. Moreover the converse is also true, i.e. if $\alpha_!$ is a 2-isomorphism, then α is as well. (See [**GPS**].)

Remark A.7. Naturally, there is also a dual version of everything we have done here, using right adjoints instead of left adjoints.

B. Basic ∞-categorical facts

This appendix consists of a collection of basic ∞-categorical facts that are used throughout this thesis. They are stated here without proofs, to make it easier to quickly become familiar with the statements. The proofs (or references) can be found in Appendix C. Note that, even though we often only state things for indexing categories (instead of ∞-categories or simplicial sets), this is purely for convenience and there are more general versions of all of these statements.

Lemma B.1. *Let $f\colon I \to J$ and $g\colon J \to K$ be maps of simplicial sets and $\mathcal{C}$ an ∞-category that is both weakly left f-extensible and weakly left g-extensible. Then it is also weakly left $(g \circ f)$-extensible, and we have $\mathrm{Lan}_{g\circ f} \simeq \mathrm{Lan}_g \circ \mathrm{Lan}_f$.*

Lemma B.2. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a fully faithful functor between categories and $\mathcal{C}$ a left f-extensible ∞-category. Then the unit $\mathrm{id} \to \mathrm{Res}_f \mathrm{Lan}_f$ of the adjunction $\mathrm{Lan}_f \dashv \mathrm{Res}_f$ is an equivalence of functors $\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$.*

Lemma B.3. *Let $\mathcal{I}$ be a category and $\mathcal{C}$ an ∞-category. Then $\mathcal{C}$ admits all colimits indexed by $\mathcal{I}$ if and only if, for all diagrams $D\colon \mathcal{I} \to \mathcal{C}$, there is a colimit diagram extending D. In this case a diagram $D\colon \mathcal{I}^{\triangleright} \to \mathcal{C}$ lies in the essential image of $\mathrm{Lan}_{\mathrm{inc}}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C})$ if and only if it is a colimit diagram.*

Lemma B.4. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a functor between categories.*

a) If $\mathcal{J}$ has a terminal object, then it is contractible.

b) The functor f is homotopy terminal if and only if, for each $j \in \mathcal{J}$, the category $j \downarrow f$ is contractible.

c) If f is right adjoint, then it is homotopy terminal.

d) If f is homotopy terminal, then it is a homotopy equivalence.

Lemma B.5. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a homotopy terminal functor between categories and $\mathcal{C}$ an ∞-category that admits colimits indexed both by $\mathcal{I}$ and by $\mathcal{J}$. Then the natural transformation $f_*\colon \mathrm{colim}_{\mathcal{I}} \mathrm{Res}_f \to \mathrm{colim}_{\mathcal{J}}$ of functors $\mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}$ is an equivalence.*

Lemma B.6. *Let $\mathcal{I}$ be a category and $F\colon \mathcal{C} \to \mathcal{D}$ a functor between ∞-categories that admit colimits indexed by $\mathcal{I}$. Then the following conditions are equivalent:*

a) F preserves left Kan extension along the inclusion $\mathrm{inc}\colon \mathcal{I} \to \mathcal{I}^{\triangleright}$.

b) F preserves colimits indexed by $\mathcal{I}$.

c) F sends $\mathcal{I}^{\triangleright}$-indexed colimit diagrams to colimit diagrams.

Lemma B.7. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a functor between categories, $g\colon \mathcal{K} \to \mathcal{L}$ a functor between ∞-categories, and $\mathcal{C}$ a left f-extensible ∞-category. Then $\mathrm{Fun}(\mathcal{L}, \mathcal{C})$ is left f-extensible, and $\mathrm{Res}_g\colon \mathrm{Fun}(\mathcal{L}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{K}, \mathcal{C})$ preserves left Kan extension along f.*

C. Tools for Kan extensions and (co)limits

In this appendix we collect some basic tools for working with Kan extensions and (co)limits in ∞-categories that we need in the rest of this thesis. Note that, even though we often only state things for indexing categories (instead of ∞-categories or simplicial sets), this is purely for convenience and there are more general versions of most of those statements. Generally, if there is a pair of dual statements, we will only give one of them and leave the other implicit.

There is no claim of originality for any of the statements found in this appendix (the correctness of most, if not all, of them should be more or less clear to anyone familiar with the theory); the ones for which a proof is given are merely those for which the author could not find a good reference.

C.1. Kan extensions

Lemma B.1. *Let $f\colon I \to J$ and $g\colon J \to K$ be maps of simplicial sets and $\mathcal{C}$ an ∞-category that is both weakly left f-extensible and weakly left g-extensible. Then it is also weakly left $(g \circ f)$-extensible, and we have $\mathrm{Lan}_{g\circ f} \simeq \mathrm{Lan}_g \circ \mathrm{Lan}_f$.*

Proof. Since adjunctions compose, we have that $\mathrm{Lan}_g \circ \mathrm{Lan}_f$ is a left adjoint of the composition $\mathrm{Res}_f \circ \mathrm{Res}_g = \mathrm{Res}_{g\circ f}$. As adjoints are unique up to isomorphism (in the homotopy 2-category of ∞-categories), we obtain that $\mathrm{Lan}_g \circ \mathrm{Lan}_f \simeq \mathrm{Lan}_{g\circ f}$. □

Lemma B.2. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a fully faithful functor between categories and $\mathcal{C}$ a left f-extensible ∞-category. Then the unit $\mathrm{id} \to \mathrm{Res}_f \mathrm{Lan}_f$ of the adjunction $\mathrm{Lan}_f \dashv \mathrm{Res}_f$ is an equivalence of functors $\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$.*

Proof. This follows from the Beck-Chevalley condition [**RV**], using that by [**RV**] when f is fully faithful a certain square fulfills a condition called exact (here we use that the nerve functor is cosmological by [**RV**], hence preserves fully faithfulness (cf. [**RV**]) since it preserves absolute right and left lifting diagrams by [**RV**]). □

Lemma C.1. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a fully faithful functor between categories and $\mathcal{C}$ a left f-extensible ∞-category. Then $\varepsilon \circ \mathrm{Lan}_f\colon \mathrm{Lan}_f \mathrm{Res}_f \mathrm{Lan}_f \to \mathrm{Lan}_f$ is an equivalence, where ε is the counit of the adjunction $\mathrm{Lan}_f \dashv \mathrm{Res}_f$.*

Proof. Consider the following diagram

$$\begin{tikzcd}
\mathrm{Lan}_f \arrow[r, "\mathrm{id}"] \arrow[d, "\eta"'] & \mathrm{Lan}_f \\
\mathrm{Lan}_f(\mathrm{Res}_f \mathrm{Lan}_f) \arrow[r, equal] & (\mathrm{Lan}_f \mathrm{Res}_f) \mathrm{Lan}_f \arrow[u, "\varepsilon"']
\end{tikzcd}$$

where the vertical maps are given by the unit respectively counit of the adjunction $\mathrm{Lan}_f \dashv \mathrm{Res}_f$. It commutes up to homotopy by one of the triangle identities. Since f is fully faithful, the left vertical morphism is an equivalence. Hence the right vertical map is an equivalence as well. □

Lemma C.2. *Let J be a simplicial set, $f\colon I \to I'$ a map of simplicial sets, and $\mathcal{C}$ a weakly left f-extensible ∞-category. Then there is a homotopy commutative diagram of the form*

$$\begin{tikzcd}
\mathrm{Fun}(I, \mathrm{Fun}(J, \mathcal{C})) \arrow[r, "\mathrm{Lan}_f"] & \mathrm{Fun}(I', \mathrm{Fun}(J, \mathcal{C})) \\
\mathrm{Fun}(I \times J, \mathcal{C}) \arrow[u, "\cong"] \arrow[d, "\cong"'] \arrow[r, "\mathrm{Lan}_{f\times \mathrm{id}}"] & \mathrm{Fun}(I' \times J, \mathcal{C}) \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
\mathrm{Fun}(J, \mathrm{Fun}(I, \mathcal{C})) \arrow[r, "\mathrm{Lan}_f \circ"] & \mathrm{Fun}(J, \mathrm{Fun}(I', \mathcal{C}))
\end{tikzcd}$$

where the vertical maps are the respective currying isomorphisms (in particular, all of these left Kan extensions actually exist).

Proof. Note that $(\mathrm{Lan}_f\ \circ)$ is left adjoint to $(\mathrm{Res}_f\ \circ)$ by Lemma D.5. Since the above diagram with the restrictions, instead of their left adjoints, commutes, we obtain that Res_f and $\mathrm{Res}_{f\times \mathrm{id}}$ actually have left adjoints. Then Lemma A.5 implies the statement. □

Lemma C.3. *Let I and J be simplicial sets, $\mathcal{C}$ an ∞-category that admits colimits indexed by I, and $D\colon I \times J \to \mathcal{C}$ a functor. Denote by $D_I\colon I \to \mathrm{Fun}(J, \mathcal{C})$ and $D_J\colon J \to \mathrm{Fun}(I, \mathcal{C})$ the curried functors. Then $\mathrm{Lan}_{\mathrm{pr}_J} D$, $\mathrm{colim}_I D_I$, and $\mathrm{colim}_I \circ D_J$ exist and are all equivalent in $\mathrm{Fun}(J, \mathcal{C})$.*

Proof. This is a special case of Lemma C.2. □

Lemma C.4. *Let $f\colon \mathcal{I} \to \mathcal{K}$ and $g\colon \mathcal{J} \to \mathcal{K}$ be functors between categories. Consider the natural transformation*

$$\begin{tikzcd}
f \downarrow g \arrow[r, "\mathrm{pr}_{\mathcal{I}}"] \arrow[d, "\mathrm{pr}_{\mathcal{J}}"'] \arrow[dr, phantom, "\swArrow\ \alpha"] & \mathcal{I} \arrow[d, "f"] \\
\mathcal{J} \arrow[r, "g"] & \mathcal{K}
\end{tikzcd}$$

given, at $(i, j, k\colon f(i) \to g(j)) \in f \downarrow g$, by k. Now let $\mathcal{C}$ be a left f-extensible and left $\mathrm{pr}_{\mathcal{J}}$-extensible ∞-category. After applying $\mathrm{Fun}(-, \mathcal{C})$ to the diagram above we get

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{K}, \mathcal{C}) \arrow[r, "\mathrm{Res}_f"] \arrow[d, "\mathrm{Res}_g"'] \arrow[dr, phantom, "\swArrow\ \alpha"] & \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \arrow[d, "\mathrm{Res}_{\mathrm{pr}_{\mathcal{I}}}"] \\
\mathrm{Fun}(\mathcal{J}, \mathcal{C}) \arrow[r, "\mathrm{Res}_{\mathrm{pr}_{\mathcal{J}}}"] & \mathrm{Fun}(f \downarrow g, \mathcal{C})
\end{tikzcd}$$

and taking the mate we obtain a transformation $\alpha_!\colon \mathrm{Lan}_{\mathrm{pr}_{\mathcal{J}}} \mathrm{Res}_{\mathrm{pr}_{\mathcal{I}}} \to \mathrm{Res}_g \mathrm{Lan}_f$. This transformation $\alpha_!$ is an equivalence.

Proof. This follows from the fact that the second diagram satisfies the Beck-Chevalley condition by [**RV**] as the first one is a so called exact square by [**RV**] (again using that the nerve is a cosmological functor by [**RV**] and thus preserves comma categories by [**RV**]). □

Lemma C.5. *Let $\mathcal{I}$ and $\mathcal{J}$ be categories, $f\colon \mathcal{I} \to \mathcal{J}$ a functor, $\mathcal{C}$ a left f-extensible ∞-category, and $D\colon \mathcal{I} \to \mathcal{C}$ a diagram. Then, for any $j \in \mathcal{J}$, the mate $\vartheta\colon \operatorname{colim}_{f\downarrow j} \operatorname{Res}_{\mathrm{pr}} \to \operatorname{Res}_j \operatorname{Lan}_f$ of the natural transformation in the right diagram (which is the image of the left diagram under $\operatorname{Fun}(-, \mathcal{C})$)*

$$\begin{tikzcd} f\downarrow j \arrow[r, "\mathrm{const}"] \arrow[d, "\mathrm{pr}"'] \arrow[dr, phantom, "\Nearrow \tilde{\rho}_j"] & * \arrow[d, "j"] \\ \mathcal{I} \arrow[r, "f"] & \mathcal{J} \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Fun}(\mathcal{J}, \mathcal{C}) \arrow[r, "\operatorname{Res}_f"] \arrow[d, "\operatorname{Res}_j"'] \arrow[dr, phantom, "\Swarrow \rho_j"] & \operatorname{Fun}(\mathcal{I}, \mathcal{C}) \arrow[d, "\operatorname{Res}_{\mathrm{pr}}"] \\ \mathcal{C} \arrow[r, "\Delta"] & \operatorname{Fun}(f\downarrow j, \mathcal{C}) \end{tikzcd}$$

is an equivalence, where $\tilde{\rho}_j$ is, at $(i, k\colon f(i) \to j)$, just given by k. Furthermore it is natural in j, in the sense that, for a morphism $\kappa\colon j \to j'$ in $\mathcal{J}$, the diagram

$$\begin{tikzcd} \operatorname{colim}_{f\downarrow j} \operatorname{Res}_{\mathrm{pr}_{f\downarrow j}} \arrow[r, "(f\downarrow\kappa)_*"] \arrow[d, "\vartheta"', "\simeq"] & \operatorname{colim}_{f\downarrow j'} \operatorname{Res}_{\mathrm{pr}_{f\downarrow j'}} \arrow[d, "\simeq"', "\vartheta"] \\ \operatorname{Res}_j \operatorname{Lan}_f \arrow[r] & \operatorname{Res}_{j'} \operatorname{Lan}_f \end{tikzcd}$$

commutes up to homotopy.

Proof. That ϑ is an equivalence is a special case of Lemma C.4. For the naturality in j we consider, for a map $\kappa\colon j \to j'$, the two diagrams

$$\begin{tikzcd} f\downarrow j \arrow[r, "f\downarrow\kappa"] \arrow[d] \arrow[dr, phantom, "\Swarrow \mathrm{id}"] & f\downarrow j' \arrow[r, "\mathrm{pr}"] \arrow[d] \arrow[dr, phantom, "\Swarrow \tilde{\rho}_{j'}"] & \mathcal{I} \arrow[d, "f"] \\ * \arrow[r] & * \arrow[r, "j'"] & \mathcal{J} \end{tikzcd} \qquad \begin{tikzcd} f\downarrow j \arrow[r] \arrow[d, "\mathrm{pr}"'] \arrow[dr, phantom, "\Nearrow \tilde{\rho}_j"] & * \arrow[r] \arrow[d, "j"] \arrow[dr, phantom, "\Nearrow \kappa"] & * \arrow[d, "j'"] \\ \mathcal{I} \arrow[r, "f"] & \mathcal{J} \arrow[r, "\mathrm{id}"] & \mathcal{J} \end{tikzcd}$$

for which we note that $\mathrm{id} * \tilde{\rho}_{j'} = \kappa * \tilde{\rho}_j$ by definition of the involved maps. Hence, after applying $\operatorname{Fun}(-, \mathcal{C})$, we obtain, by the pasting laws for mates, that $(\rho_{j'})_! * \mathrm{id}_! = (\mathrm{id} * \rho_{j'})_! = (\kappa * \rho_j)_! = \kappa_! * (\rho_j)_!$ (in the homotopy 2-category of ∞-categories). This is the statement we wanted to show since $\kappa_!$ is the map $\operatorname{Res}_j \to \operatorname{Res}_{j'}$ given by evaluation at κ and $\mathrm{id}_!$ is the map on colimits induced by $f \downarrow \kappa$. □

Lemma C.6. *Let the diagram in the left be a diagram of categories, functors between them, and a natural transformation and the one in the right its image under $\operatorname{Fun}(-, \mathcal{C})$*

$$\begin{tikzcd} \mathcal{I} \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\Swarrow \gamma"] & \mathcal{J} \arrow[d, "d"] \\ \mathcal{K} \arrow[r, "c"] & \mathcal{L} \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Fun}(\mathcal{L}, \mathcal{C}) \arrow[r, "\operatorname{Res}_d"] \arrow[d, "\operatorname{Res}_c"'] \arrow[dr, phantom, "\Swarrow \gamma"] & \operatorname{Fun}(\mathcal{J}, \mathcal{C}) \arrow[d, "\operatorname{Res}_a"] \\ \operatorname{Fun}(\mathcal{K}, \mathcal{C}) \arrow[r, "\operatorname{Res}_b"] & \operatorname{Fun}(\mathcal{I}, \mathcal{C}) \end{tikzcd}$$

where $\mathcal{C}$ is a left b-extensible and left d-extensible ∞-category. Then, for any $k \in \mathcal{K}$, the following diagram commutes up to homotopy

$$\begin{tikzcd}
\operatorname{colim}_{b\downarrow k} \operatorname{Res}_{\mathrm{pr}_{b\downarrow k}} \operatorname{Res}_a \arrow[r, "f_*"] \arrow[d, "\vartheta"', "\simeq"] & \operatorname{colim}_{d\downarrow c(k)} \operatorname{Res}_{\mathrm{pr}_{d\downarrow c(k)}} \arrow[d, "\simeq"', "\vartheta"] \\
\operatorname{Res}_k \operatorname{Lan}_b \operatorname{Res}_a \arrow[r, "\gamma_!"] & \operatorname{Res}_k \operatorname{Res}_c \operatorname{Lan}_d
\end{tikzcd}$$

where $\gamma_!$ is the mate of γ, the maps denoted ϑ are as in Lemma C.5, and f is the functor

$$b\downarrow k \longrightarrow d\downarrow c(k), \quad \Big(i,\ b(i) \xrightarrow{g} k\Big) \longmapsto \Big(a(i),\ d(a(i)) \xrightarrow{\gamma} c(b(i)) \xrightarrow{c(g)} c(k)\Big)$$

acting on morphisms via a.

Proof. Consider the two diagrams

$$\begin{tikzcd}
b\downarrow k \arrow[r, "\mathrm{pr}_{b\downarrow k}"] \arrow[d] \arrow[dr, phantom, "\swarrow \tilde\rho_k"] & \mathcal{I} \arrow[r, "a"] \arrow[d, "b"] \arrow[dr, phantom, "\swarrow \gamma"] & \mathcal{J} \arrow[d, "d"] \\
* \arrow[r, "k"] & \mathcal{K} \arrow[r, "c"] & \mathcal{L}
\end{tikzcd}
\qquad
\begin{tikzcd}
b\downarrow k \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\swarrow \mathrm{id}"] & d\downarrow c(k) \arrow[r, "\mathrm{pr}_{d\downarrow c(k)}"] \arrow[d] \arrow[dr, phantom, "\swarrow \tilde\rho_{c(k)}"] & \mathcal{J} \arrow[d, "d"] \\
* \arrow[r] & * \arrow[r, "c(k)"] & \mathcal{L}
\end{tikzcd}$$

where $\tilde\rho$ is as in Lemma C.5. Note that it follows directly from the definitions that their pastes $\tilde\rho_k * \gamma$ and $\mathrm{id} * \tilde\rho_{c(k)}$ are the same. Applying $\operatorname{Fun}(-, \mathcal{C})$ and using the pasting law for mates yields the desired result. □

C.2. (Co)Limits

Lemma C.7. *Let $\mathcal{C}$ be an ∞-category that admits colimits indexed by a category $\mathcal{I}$. Then the mate $\operatorname{colim}_{\mathcal{I}} \to \operatorname{Res}_{\blacktriangleright} \operatorname{Lan}_{\mathrm{inc}}$ of the diagram on the right (which is the image of the diagram on the left under $\operatorname{Fun}(-, \mathcal{C})$)*

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "\swarrow \alpha"] & \mathcal{I} \arrow[d, "\mathrm{inc}"] \\
* \arrow[r, "\blacktriangleright"] & \mathcal{I}^{\triangleright}
\end{tikzcd}
\qquad
\begin{tikzcd}
\operatorname{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C}) \arrow[r, "\mathrm{Res}_{\mathrm{inc}}"] \arrow[d, "\mathrm{Res}_{\blacktriangleright}"'] \arrow[dr, phantom, "\swarrow \alpha"] & \operatorname{Fun}(\mathcal{I}, \mathcal{C}) \arrow[d, "\mathrm{id}"] \\
\mathcal{C} \arrow[r, "\Delta"] & \operatorname{Fun}(\mathcal{I}, \mathcal{C})
\end{tikzcd}$$

is an equivalence.

Proof. This is a special case of Lemma C.5 since there is an isomorphism $\mathrm{inc}\downarrow \blacktriangleright \cong \mathcal{I}$ over $\mathcal{I}^{\triangleright}$. □

Lemma B.3. *Let $\mathcal{I}$ be a category and $\mathcal{C}$ an ∞-category. Then $\mathcal{C}$ admits all colimits indexed by $\mathcal{I}$ if and only if, for all diagrams $D\colon \mathcal{I} \to \mathcal{C}$, there is a colimit diagram extending D. In this case a diagram $D\colon \mathcal{I}^{\triangleright} \to \mathcal{C}$ lies in the essential image of $\operatorname{Lan}_{\mathrm{inc}}\colon \operatorname{Fun}(\mathcal{I}, \mathcal{C}) \to \operatorname{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C})$ if and only if it is a colimit diagram.*

Proof. The first statement follows from [**RV**]. The second from [**RV**]. □

Lemma B.4. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a functor between categories.*

a) If $\mathcal{J}$ has a terminal object, then it is contractible.

b) The functor f is homotopy terminal if and only if, for each $j \in \mathcal{J}$, the category $j \downarrow f$ is contractible.

c) If f is right adjoint, then it is homotopy terminal.

d) If f is homotopy terminal, then it is a homotopy equivalence.

Proof. The first statement is clear (one can explicitly construct the contraction). The latter three statements follow, in order, from [**LurHTT**], [**RV**], and [**LurHTT**]. □

Lemma C.8. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a homotopy terminal functor between categories and $\mathcal{C}$ an ∞-category. Then $\mathcal{C}$ admits colimits indexed by $\mathcal{I}$ if and only if it admits colimits indexed by $\mathcal{J}$.*

Proof. This follows from Lemma B.3 and [**LurHTT**]. □

Lemma B.5. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a homotopy terminal functor between categories and $\mathcal{C}$ an ∞-category that admits colimits indexed both by $\mathcal{I}$ and by $\mathcal{J}$. Then the natural transformation $f_*\colon \operatorname{colim}_{\mathcal{I}} \operatorname{Res}_f \to \operatorname{colim}_{\mathcal{J}}$ of functors $\operatorname{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}$ is an equivalence.*

Proof. By Lemma C.6, the mate $\mathrm{id}_!\colon \operatorname{Lan}_{\mathrm{inc}_{\mathcal{I}}} \operatorname{Res}_f \to \operatorname{Res}_{f^\triangleright} \operatorname{Lan}_{\mathrm{inc}_{\mathcal{J}}}$ of the natural transformation in the diagram on the right (which is the image of the diagram on the left under $\operatorname{Fun}(-, \mathcal{C})$)

$$\begin{tikzcd} \mathcal{I} \arrow[r, "f"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\swArrow}"] & \mathcal{J} \arrow[d, "\mathrm{inc}"] \\ \mathcal{I}^\triangleright \arrow[r, "f^\triangleright"] & \mathcal{J}^\triangleright \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Fun}(\mathcal{J}^\triangleright, \mathcal{C}) \arrow[r, "\mathrm{Res}_{\mathrm{inc}}"] \arrow[d, "\mathrm{Res}_{f^\triangleright}"'] \arrow[dr, phantom, "\overset{\mathrm{id}}{\swArrow}"] & \operatorname{Fun}(\mathcal{J}, \mathcal{C}) \arrow[d, "\mathrm{Res}_f"] \\ \operatorname{Fun}(\mathcal{I}^\triangleright, \mathcal{C}) \arrow[r, "\mathrm{Res}_{\mathrm{inc}}"] & \operatorname{Fun}(\mathcal{I}, \mathcal{C}) \end{tikzcd}$$

is given, at the cocone point, by f_*. Hence it is enough to prove that $\mathrm{id}_!$ is an equivalence. This mate is given by the composition

$$\begin{aligned} \operatorname{Lan}_{\mathrm{inc}_{\mathcal{I}}} \operatorname{Res}_f \xrightarrow{\eta} {} & \operatorname{Lan}_{\mathrm{inc}_{\mathcal{I}}} \operatorname{Res}_f \operatorname{Res}_{\mathrm{inc}_{\mathcal{J}}} \operatorname{Lan}_{\mathrm{inc}_{\mathcal{J}}} \\ & \qquad\qquad \| \\ & \operatorname{Lan}_{\mathrm{inc}_{\mathcal{I}}} \operatorname{Res}_{\mathrm{inc}_{\mathcal{I}}} \operatorname{Res}_{f^\triangleright} \operatorname{Lan}_{\mathrm{inc}_{\mathcal{J}}} \xrightarrow{\varepsilon} \operatorname{Res}_{f^\triangleright} \operatorname{Lan}_{\mathrm{inc}_{\mathcal{J}}} \end{aligned}$$

of the unit η of the adjunction $\operatorname{Lan}_{\mathrm{inc}_{\mathcal{J}}} \dashv \operatorname{Res}_{\mathrm{inc}_{\mathcal{J}}}$ and the counit ε of the adjunction $\operatorname{Lan}_{\mathrm{inc}_{\mathcal{I}}} \dashv \operatorname{Res}_{\mathrm{inc}_{\mathcal{I}}}$. Since $\mathrm{inc}_{\mathcal{J}}$ is fully faithful, the map η is an equivalence. So we only need to show that $\varepsilon \circ \operatorname{Res}_{f^\triangleright} \circ \operatorname{Lan}_{\mathrm{inc}_{\mathcal{J}}}$ is an equivalence. Let $D\colon \mathcal{I} \to \mathcal{C}$ be a diagram. By Lemma B.3 and assumption the diagram $(\operatorname{Res}_{f^\triangleright} \circ \operatorname{Lan}_{\mathrm{inc}_{\mathcal{J}}})(D)$ is a colimit diagram. But ε applied to a colimit diagram is an equivalence by Lemma C.1 and again Lemma B.3. □

Lemma C.9. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a functor between categories and $\mathcal{C}$ an ∞-category that admits colimits indexed both by $\mathcal{I}$ and by $\mathcal{J}$. Then the following diagram in* $\mathrm{Fun}(\mathrm{Fun}(\mathcal{J}^{\triangleright}, \mathcal{C}), \mathcal{C})$ *commutes up to homotopy*

$$\begin{tikzcd}
\operatorname{colim}_{\mathcal{I}} \operatorname{Res}_f \operatorname{Res}_{\mathrm{inc}_{\mathcal{J}}} \arrow[r, "f_*"] \arrow[d, equal] & \operatorname{colim}_{\mathcal{J}} \operatorname{Res}_{\mathrm{inc}_{\mathcal{J}}} \arrow[dd] \\
\operatorname{colim}_{\mathcal{I}} \operatorname{Res}_{\mathrm{inc}_{\mathcal{I}}} \operatorname{Res}_{f^{\triangleright}} \arrow[d] & \\
\operatorname{Res}_{\blacktriangleright} \operatorname{Res}_{f^{\triangleright}} \arrow[r, equal] & \operatorname{Res}_{\blacktriangleright}
\end{tikzcd}$$

where the vertical maps are the canonical maps out of the colimit.

Proof. Consider the two diagrams

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "\mathrm{inc}"] \arrow[d] \arrow[dr, phantom, "\xi_{\mathcal{I}} \swArrow"] & \mathcal{I}^{\triangleright} \arrow[r, "f^{\triangleright}"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\mathrm{id}_1 \swArrow"] & \mathcal{J}^{\triangleright} \arrow[d, "\mathrm{id}"] \\
* \arrow[r, "\blacktriangleright"] & \mathcal{I}^{\triangleright} \arrow[r, "f^{\triangleright}"] & \mathcal{J}^{\triangleright}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{I} \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\mathrm{id}_2 \swArrow"] & \mathcal{J} \arrow[r, "\mathrm{inc}"] \arrow[d] \arrow[dr, phantom, "\xi_{\mathcal{J}} \swArrow"] & \mathcal{J}^{\triangleright} \arrow[d, "\mathrm{id}"] \\
* \arrow[r] & * \arrow[r, "\blacktriangleright"] & \mathcal{J}^{\triangleright}
\end{tikzcd}$$

where $\xi_{\mathcal{I}}$ and $\xi_{\mathcal{J}}$ are as in **??**, and note that their pastes agree, i.e. $\xi_{\mathcal{I}} * \mathrm{id}_1 = \mathrm{id}_2 * \xi_{\mathcal{J}}$. Applying $\mathrm{Fun}(-, \mathcal{C})$ and using the pasting law for mates yields $(\mathrm{id}_1)_! * (\xi_{\mathcal{I}})_! \simeq (\xi_{\mathcal{J}})_! * (\mathrm{id}_2)_!$. This is what we wanted to show since $(\mathrm{id}_2)_! = f_*$ and $(\mathrm{id}_1)_!$ is the identity. □

Lemma C.10. *Let $\mathcal{I}$ be a category with a terminal object $*$, $\mathcal{C}$ an ∞-category, and $D\colon \mathcal{I}^{\triangleright} \to \mathcal{C}$ a diagram such that D applied to the unique morphism $* \to \blacktriangleright$ is an equivalence. Then the canonical morphism $\operatorname{colim}_{\mathcal{I}} D|_{\mathcal{I}} \to D(\blacktriangleright)$ is an equivalence as well.*

Proof. First note that $\mathcal{C}$ admits colimits indexed by $\mathcal{I}$ by Lemma C.8. Applying Lemma C.9 to the functor $\mathrm{const}_*\colon * \to \mathcal{I}$, we obtain a diagram

$$\begin{tikzcd}
\operatorname{colim}_* D(*) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \operatorname{colim}_{\mathcal{I}} D|_{\mathcal{I}} \arrow[d] \\
D(\blacktriangleright) \arrow[r, equal] & D(\blacktriangleright)
\end{tikzcd}$$

where the top horizontal morphism is an equivalence since const_* is homotopy terminal, and the left vertical morphism is an equivalence by **??**. □

Lemma C.11. *Let $\mathcal{I}$ be a category, $\mathcal{C}$ an ∞-category that admits colimits indexed by $\mathcal{I}$, and $D\colon \mathcal{I}^{\triangleright} \to \mathcal{C}$ a diagram. Then the canonical map $(\operatorname{colim}_{\mathcal{I}} \operatorname{Res}_{\mathrm{inc}})(D) \to \operatorname{Res}_{\blacktriangleright}(D)$ is an equivalence if and only if the counit $\operatorname{Lan}_{\mathrm{inc}} \operatorname{Res}_{\mathrm{inc}} \to \mathrm{id}$ of the adjunction $\operatorname{Lan}_{\mathrm{inc}} \dashv \operatorname{Res}_{\mathrm{inc}}$ is an equivalence at D.*

Proof. The mate of the natural transformation on the right (which is the image under $\mathrm{Fun}(-, \mathcal{C})$ of the natural transformation on the left)

$$\begin{tikzcd} \mathcal{I} \arrow[r, "\mathrm{inc}"] \arrow[d, "\mathrm{inc}"'] \arrow[dr, phantom, "\swarrow \mathrm{id}"] & \mathcal{I}^{\triangleright} \arrow[d, "\mathrm{id}"] \\ \mathcal{I}^{\triangleright} \arrow[r, "\mathrm{id}"] & \mathcal{I}^{\triangleright} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C}) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow \mathrm{id}"] & \mathrm{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C}) \arrow[d, "\mathrm{Res}_{\mathrm{inc}}"] \\ \mathrm{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C}) \arrow[r, "\mathrm{Res}_{\mathrm{inc}}"] & \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \end{tikzcd}$$

is precisely the counit $\mathrm{Lan}_{\mathrm{inc}} \mathrm{Res}_{\mathrm{inc}} \to \mathrm{id}$. Hence, by Lemma C.6, it is an equivalence at D if and only if, for all $k \in \mathcal{I}^{\triangleright}$, the map

$$\mathrm{colim}_{\mathrm{inc}\downarrow k} \mathrm{Res}_{\mathrm{pr}_{\mathrm{inc}\downarrow k}} \mathrm{Res}_{\mathrm{inc}} = \mathrm{colim}_{\mathrm{inc}\downarrow k} \mathrm{Res}_{f_k} \mathrm{Res}_{\mathrm{pr}_{\mathrm{id}\downarrow k}} \xrightarrow{(f_k)_*} \mathrm{colim}_{\mathrm{id}\downarrow k} \mathrm{Res}_{\mathrm{pr}_{\mathrm{id}\downarrow k}}$$

is an equivalence at D, where $f_k \colon \mathrm{inc} \downarrow k \to \mathrm{id} \downarrow k$ is the canonical inclusion. When k is not $\blacktriangleright$, then f_k is an isomorphism and $(f_k)_*$ is an equivalence. When k is $\blacktriangleright$, then f_k is just (isomorphic to) inc, and $\mathrm{pr}_{\mathrm{id}\downarrow k}$ is an isomorphism. So it is enough to show that $\mathrm{inc}_* \colon \mathrm{colim}_{\mathcal{I}} \mathrm{Res}_{\mathrm{inc}} \to \mathrm{colim}_{\mathcal{I}^{\triangleright}}$ is an equivalence at D if and only if the canonical map $(\mathrm{colim}_{\mathcal{I}} \mathrm{Res}_{\mathrm{inc}})(D) \to \mathrm{Res}_{\blacktriangleright}(D)$ is an equivalence. This follows from Lemma C.9 by considering the diagram $D' \colon (\mathcal{I}^{\triangleright})^{\triangleright} \to \mathcal{C}$ obtained from D by pulling back along the functor $(\mathcal{I}^{\triangleright})^{\triangleright} \to \mathcal{I}^{\triangleright}$ that is the identity on $\mathcal{I}^{\triangleright}$ and sends the new cocone point to the old one (here we also use Lemma C.10 to see that canonical map $(\mathrm{colim}_{\mathcal{I}^{\triangleright}} \mathrm{Res}_{\mathrm{inc}_{\mathcal{I}^{\triangleright}}})(D') \to \mathrm{Res}_{\blacktriangleright}(D')$ is an equivalence). $\square$

Lemma C.12. *Let I and J be simplicial sets, $\mathcal{C}$ an ∞-category that admits colimits indexed by J, and $f \colon I \to \mathrm{Fun}(J^{\triangleright}, \mathcal{C})$ a functor. Denote by $g \colon J^{\triangleright} \to \mathrm{Fun}(I, \mathcal{C})$ the functor obtained from f via currying. Then there is a homotopy commutative diagram of the form*

$$\begin{tikzcd} \mathrm{Res}_{\blacktriangleright} \circ f \arrow[d, equal] & \mathrm{colim}_J \circ \mathrm{Res}_{\mathrm{inc}} \circ f \arrow[l] \arrow[d, "\simeq"] \\ \mathrm{Res}_{\blacktriangleright} g & \mathrm{colim}_J (\mathrm{Res}_{\mathrm{inc}} g) \arrow[l] \end{tikzcd} \tag{C.1}$$

where the horizontal morphisms are the canonical maps from the colimit.

Proof. Consider the two diagrams

$$\begin{tikzcd} \mathrm{Fun}(J^{\triangleright}, \mathrm{Fun}(I, \mathcal{C})) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{Res}_{\blacktriangleright}"'] \arrow[dr, phantom, "\swarrow \mathrm{id}_1"] & \mathrm{Fun}(J^{\triangleright}, \mathrm{Fun}(I, \mathcal{C})) \arrow[d, "\mathrm{Res}_{\mathrm{inc}}"] \\ \mathrm{Fun}(I, \mathcal{C}) \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\swarrow \mathrm{id}_2"] & \mathrm{Fun}(J, \mathrm{Fun}(I, \mathcal{C})) \arrow[d, "\cong"] \\ \mathrm{Fun}(I, \mathcal{C}) \arrow[r, "\Delta \circ"] & \mathrm{Fun}(I, \mathrm{Fun}(J, \mathcal{C})) \end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Fun}(J^{\triangleright}, \mathrm{Fun}(I, \mathcal{C})) \arrow[r, "\mathrm{id}"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\swarrow \mathrm{id}_3"] & \mathrm{Fun}(J^{\triangleright}, \mathrm{Fun}(I, \mathcal{C})) \arrow[d, "\cong"] \\
\mathrm{Fun}(I, \mathrm{Fun}(J^{\triangleright}, \mathcal{C})) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{Res}_{\blacktriangleright} \circ"'] \arrow[dr, phantom, "\swarrow \mathrm{id}_4"] & \mathrm{Fun}(I, \mathrm{Fun}(J^{\triangleright}, \mathcal{C})) \arrow[d, "\mathrm{Res}_{\mathrm{inc}} \circ"] \\
\mathrm{Fun}(I, \mathcal{C}) \arrow[r, "\Delta \circ"] & \mathrm{Fun}(I, \mathrm{Fun}(J, \mathcal{C}))
\end{tikzcd}$$

and note that their pastes agree. Now the mate of id_1 gives the lower horizontal map in diagram (C.1), the mate of id_4 the upper horizontal map (where we use the adjunction $(\mathrm{colim}_J \circ) \dashv (\Delta \circ)$ obtained from Lemma D.5), the mate of id_2 the right equivalence (using Lemma A.5), and the mate of id_3 the left identity. An application of the pasting law for mates yields the desired statement. □

Lemma C.13. *Let $\mathcal{I}$ be a contractible category, $\mathcal{C}$ an ∞-category that admits colimits indexed by $\mathcal{I}$, and $D\colon \mathcal{I} \to \mathcal{C}$ a diagram such that, for all morphisms k of $\mathcal{I}$, the induced map $D(k)$ is an equivalence. Then, for all $i \in \mathcal{I}$, the structure map $D(i) \to \mathrm{colim}_{\mathcal{I}} D$ is an equivalence.*

Proof. We will show that the composition

$$\mathrm{Fun}(\mathcal{I}, \mathcal{C}) \xrightarrow{\mathrm{Lan}_{\mathrm{inc}}} \mathrm{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C}) \xrightarrow{\mathrm{Res}_{t_i}} \mathrm{Fun}\big(\Delta^1, \mathcal{C}\big)$$

sends D to an equivalence, where $t_i\colon \Delta^1 \to \mathcal{I}^{\triangleright}$ is as in **??**, i.e. the functor representing the unique morphism $i \to \blacktriangleright$. Note that $\mathrm{Lan}_{\mathrm{inc}} D$ is a colimit diagram indexed by $\mathcal{I}^{\triangleright}$ that sends any morphism in $\mathcal{I}$ to an equivalence. Hence, by [**LurHTT**] (together with [**LurHTT**]), the diagram $\mathrm{Lan}_{\mathrm{inc}} D$ sends every morphism of $\mathcal{I}^{\triangleright}$ to an equivalence, which implies the claim. □

C.3. Preservation of Kan extensions and (co)limits

Lemma B.7. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a functor between categories, $g\colon \mathcal{K} \to \mathcal{L}$ a functor between ∞-categories, and $\mathcal{C}$ a left f-extensible ∞-category. Then $\mathrm{Fun}(\mathcal{L}, \mathcal{C})$ is left f-extensible, and $\mathrm{Res}_g\colon \mathrm{Fun}(\mathcal{L}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{K}, \mathcal{C})$ preserves left Kan extension along f.*

Proof. That $\mathrm{Fun}(\mathcal{L}, \mathcal{C})$ is weakly left f-extensible when $\mathcal{C}$ is so was part of Lemma C.2. This also implies the corresponding statement for left f-extensible since this was defined as certain colimits existing which in turn was defined via weakly const-extensible.

For the second part we want that the mate of the transformation id_2 in the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{J}\times\mathcal{L},\mathcal{C}) \arrow[r, "\mathrm{Res}_{f\times\mathrm{id}}"] \arrow[d, "\cong"'] \arrow[ddd, bend right, "\mathrm{Res}_{\mathrm{id}\times g}"'] \arrow[dr, phantom, "\swarrow \mathrm{id}_1"] & \mathrm{Fun}(\mathcal{I}\times\mathcal{L},\mathcal{C}) \arrow[d, "\cong"] \arrow[ddd, bend left, "\mathrm{Res}_{\mathrm{id}\times g}"] \\
\mathrm{Fun}(\mathcal{J},\mathrm{Fun}(\mathcal{L},\mathcal{C})) \arrow[r, "\mathrm{Res}_f"] \arrow[d, "\mathrm{Res}_g\circ"'] \arrow[dr, phantom, "\swarrow \mathrm{id}_2"] & \mathrm{Fun}(\mathcal{I},\mathrm{Fun}(\mathcal{L},\mathcal{C})) \arrow[d, "\mathrm{Res}_g\circ"] \\
\mathrm{Fun}(\mathcal{J},\mathrm{Fun}(\mathcal{K},\mathcal{D})) \arrow[r, "\mathrm{Res}_f"] \arrow[d, "\cong"'] \arrow[dr, phantom, "\swarrow \mathrm{id}_3"] & \mathrm{Fun}(\mathcal{I},\mathrm{Fun}(\mathcal{K},\mathcal{D})) \arrow[d, "\cong"] \\
\mathrm{Fun}(\mathcal{J}\times\mathcal{K},\mathcal{D}) \arrow[r, "\mathrm{Res}_{f\times\mathrm{id}}"] & \mathrm{Fun}(\mathcal{I}\times\mathcal{K},\mathcal{D})
\end{tikzcd}$$

is an equivalence. For this note that the mates of id_1 and id_3 are equivalences by Lemma A.5 and that the paste of all three transformations is just $\mathrm{Fun}(-,\mathcal{C})$ applied to the transformation

$$\begin{tikzcd}
\mathcal{I}\times\mathcal{K} \arrow[r, "\mathrm{id}\times g"] \arrow[d, "f\times\mathrm{id}"'] \arrow[dr, phantom, "\swarrow \mathrm{id}"] & \mathcal{I}\times\mathcal{L} \arrow[d, "f\times\mathrm{id}"] \\
\mathcal{J}\times\mathcal{K} \arrow[r, "\mathrm{id}\times g"] & \mathcal{J}\times\mathcal{L}
\end{tikzcd}$$

which is a so called exact square by [**RV**]. Hence, again by the Beck-Chevalley condition [**RV**] (which we can use since the argument in **??** applies to $f\times\mathrm{id}$ as its slice ∞-categories admit homotopy terminal functors from those of f), the mate of this paste is an equivalence. Now the pasting law for mates implies that the mate of id_2 is an equivalence which we wanted to show. □

Lemma C.14. *Let $\mathcal{K}$ be an ∞-category, $f\colon \mathcal{I}\to\mathcal{J}$ a functor between categories, $\mathcal{C}$ and $\mathcal{D}$ two left f-extensible ∞-categories, and $F\colon \mathcal{C}\to\mathrm{Fun}(\mathcal{K},\mathcal{D})$ a functor. Then F preserves left Kan extension along f if and only if, for all $k\in\mathcal{K}$, the functor $\mathrm{Res}_k\circ F\colon \mathcal{C}\to\mathcal{D}$ preserves left Kan extension along f.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{J},\mathcal{C}) \arrow[r, "\mathrm{Res}_f"] \arrow[d, "F\circ"'] \arrow[dr, phantom, "\swarrow \mathrm{id}_1"] & \mathrm{Fun}(\mathcal{I},\mathcal{C}) \arrow[d, "F\circ"] \\
\mathrm{Fun}(\mathcal{J},\mathrm{Fun}(\mathcal{K},\mathcal{D})) \arrow[r, "\mathrm{Res}_f"] \arrow[d, "\mathrm{Res}_k\circ"'] \arrow[dr, phantom, "\swarrow \mathrm{id}_2"] & \mathrm{Fun}(\mathcal{I},\mathrm{Fun}(\mathcal{K},\mathcal{D})) \arrow[d, "\mathrm{Res}_k\circ"] \\
\mathrm{Fun}(\mathcal{J},\mathcal{D}) \arrow[r, "\mathrm{Res}_f"] & \mathrm{Fun}(\mathcal{I},\mathcal{D})
\end{tikzcd}$$

and note that $(\mathrm{id}_1)_! * (\mathrm{id}_2)_! \simeq (\mathrm{id}_2 * \mathrm{id}_1)_!$ by the pasting law for mates. Since Res_k preserves left Kan extension along f, the mate $(\mathrm{id}_2)_!$ is an equivalence. Now, noting that $(\mathrm{Res}_k \circ) \circ (\mathrm{id}_1)_!$ is the other part of the composition in the paste $(\mathrm{id}_1)_! * (\mathrm{id}_2)_!$, we obtain that $(\mathrm{Res}_k \circ) \circ (\mathrm{id}_1)_!$ is an equivalence if and only if $(\mathrm{id}_2 * \mathrm{id}_1)_!$ is an equivalence. As the former being true for all k is equivalent to F preserving left Kan extension along f, and the latter is the definition of $\mathrm{Res}_k \circ F$ preserving left Kan extension along f, this implies the claim. $\square$

Lemma C.15. *Let $\mathcal{I}$ and $\mathcal{J}$ be ∞-categories, $\mathcal{K}$ a category, and $\mathcal{C}$ and $\mathcal{D}$ two ∞-categories that admit colimits indexed by $\mathcal{K}$.*

a) Let $g\colon \mathcal{C} \to \mathcal{D}$ be a functor that preserves colimits indexed by $\mathcal{K}$. Then the induced functor $(g\,\circ)\colon \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{D})$ preserves colimits indexed by $\mathcal{K}$.

b) The functor $h\colon \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{Fun}(\mathcal{I}, \mathcal{J}), \mathrm{Fun}(\mathcal{I}, \mathcal{C}))$ given by $f \mapsto (f\,\circ)$ preserves colimits indexed by $\mathcal{K}$.

Proof. The first statement follows from Lemma C.14 since, for all $i \in \mathcal{I}$, it holds that $\mathrm{Res}_i \circ (g\,\circ) = g \circ \mathrm{Res}_i$ and both functors in the latter composition preserve colimits indexed by $\mathcal{K}$. The second statement follows from the same lemma by noting that, for any $f \in \mathrm{Fun}(\mathcal{I}, \mathcal{J})$, the functor $\mathrm{Res}_{\{f\}} \circ h = \mathrm{Res}_f$ preserves colimits indexed by $\mathcal{K}$. $\square$

Lemma C.16. *Let $f\colon \mathcal{I} \to \mathcal{J}$ be a functor between categories, $\mathcal{C}$ and $\mathcal{D}$ two left f-extensible ∞-categories, and $F\colon \mathcal{C} \to \mathcal{D}$ a functor that preserves colimits indexed by $f \downarrow j$ for all $j \in \mathcal{J}$. Then F preserves left Kan extensions along f.*

Proof. Consider the two diagrams

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{J},\mathcal{C}) \arrow[r, "\mathrm{Res}_f"] \arrow[d, "F\circ"'] \arrow[dr, phantom, "\Swarrow \mathrm{id}_1"] & \mathrm{Fun}(\mathcal{I},\mathcal{C}) \arrow[d, "F\circ"] \\
\mathrm{Fun}(\mathcal{J},\mathcal{D}) \arrow[r, "\mathrm{Res}_f"] \arrow[d, "\mathrm{Res}_j"'] \arrow[dr, phantom, "\Swarrow \rho_j"] & \mathrm{Fun}(\mathcal{I},\mathcal{D}) \arrow[d, "\mathrm{Res}_{\mathrm{pr}}"] \\
\mathcal{D} \arrow[r, "\Delta"] & \mathrm{Fun}(f\downarrow j,\mathcal{D})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Fun}(\mathcal{J},\mathcal{C}) \arrow[r, "\mathrm{Res}_f"] \arrow[d, "\mathrm{Res}_j"'] \arrow[dr, phantom, "\Swarrow \rho_j"] & \mathrm{Fun}(\mathcal{I},\mathcal{C}) \arrow[d, "\mathrm{Res}_{\mathrm{pr}}"] \\
\mathcal{C} \arrow[r, "\Delta"] \arrow[d, "F"'] \arrow[dr, phantom, "\Swarrow \mathrm{id}_2"] & \mathrm{Fun}(f\downarrow j,\mathcal{C}) \arrow[d, "F\circ"] \\
\mathcal{D} \arrow[r, "\Delta"] & \mathrm{Fun}(f\downarrow j,\mathcal{D})
\end{tikzcd}$$

(where ρ_j is as in Lemma C.5) and note that their pastes agree. We want to show that $(\mathrm{id}_1)_!\colon \mathrm{Lan}_f \circ (F\,\circ) \to (F\,\circ) \circ \mathrm{Lan}_f$ is an equivalence. For this it is enough to show that $\mathrm{Res}_j \circ (\mathrm{id}_1)_!$ is an equivalence for all $j \in \mathcal{J}$. This is one of the transformations that is composed in the paste $(\mathrm{id}_1)_! * (\rho_j)_!$, which is homotopic to $(\mathrm{id}_2)_! * (\rho_j)_!$ by the pasting law for mates. Since $(\rho_j)_!$ is an equivalence by Lemma C.5 and $(\mathrm{id}_2)_!$ is one by assumption, this finishes the proof. $\square$

Lemma B.6. *Let $\mathcal{I}$ be a category and $F\colon \mathcal{C} \to \mathcal{D}$ a functor between ∞-categories that admit colimits indexed by $\mathcal{I}$. Then the following conditions are equivalent:*

a) F preserves left Kan extension along the inclusion $\mathrm{inc}\colon \mathcal{I} \to \mathcal{I}^{\triangleright}$.

b) F preserves colimits indexed by $\mathcal{I}$.

c) F sends $\mathcal{I}^{\triangleright}$-indexed colimit diagrams to colimit diagrams.

Proof. By Lemma C.16, if the functor F preserves colimits indexed by $\mathcal{I}$, then it also preserves left Kan extension along inc. The proof of the same lemma also shows that if F preserves left Kan extension along inc, then it preserves the colimits of all diagrams $\mathcal{I} \to \mathcal{C}$ that lie in the essential image of $\mathrm{Res}_{\mathrm{inc}}\colon \mathrm{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$. But, as inc is fully faithful, we have $\mathrm{Res}_{\mathrm{inc}} \mathrm{Lan}_{\mathrm{inc}} \simeq \mathrm{id}$ and thus all diagrams lie in the essential image of $\mathrm{Res}_{\mathrm{inc}}$. This shows the equivalence of the first two conditions.

Now note that, by definition, the functor F preserves left Kan extension along inc if and only if the natural transformation

$$\begin{tikzcd}
\mathrm{Lan}_{\mathrm{inc}} \circ (F\,\circ) \arrow[r, "\eta"] & \mathrm{Lan}_{\mathrm{inc}} \circ (F\,\circ) \circ \mathrm{Res}_{\mathrm{inc}} \circ \mathrm{Lan}_{\mathrm{inc}} \arrow[d, phantom, "\parallel"] & \\
& \mathrm{Lan}_{\mathrm{inc}} \circ \mathrm{Res}_{\mathrm{inc}} \circ (F\,\circ) \circ \mathrm{Lan}_{\mathrm{inc}} \arrow[r, "\varepsilon"] & (F\,\circ) \circ \mathrm{Lan}_{\mathrm{inc}}
\end{tikzcd}$$

is an equivalence, where η and ε are the unit respectively counit of the adjunction $\mathrm{Lan}_{\mathrm{inc}} \dashv \mathrm{Res}_{\mathrm{inc}}$. Since η is an equivalence (as inc is fully faithful), this is equivalent to ε being an equivalence on any diagram in the essential image of $(F\,\circ) \circ \mathrm{Lan}_{\mathrm{inc}}$. By Lemmas C.1 and B.3, this is equivalent to F sending $\mathcal{I}^{\triangleright}$-indexed colimit diagrams to colimit diagrams. $\square$

Lemma C.17. *Let I, J, and K be simplicial sets, $f\colon I \to J$ a map, and $\mathcal{C}$ a weakly left f-extensible ∞-category that admits colimits indexed by K. Then the functor* $\mathrm{Lan}_f\colon \mathrm{Fun}(I, \mathcal{C}) \to \mathrm{Fun}(J, \mathcal{C})$ *preserves colimits indexed by K.*

Proof. By [**LurHTT**] left adjoints preserve colimits. Noting that Lan_f is left adjoint, this implies the statement. $\square$

Lemma C.18. *Let $\mathcal{I}$ and $\mathcal{J}$ be two categories and $\mathcal{C}$ an ∞-category that admits colimits indexed by $\mathcal{I}$ and limits indexed by $\mathcal{J}$. Then* $\mathrm{colim}_{\mathcal{I}}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$ *preserves limits indexed by $\mathcal{J}$ if and only if* $\lim_{\mathcal{J}}\colon \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}$ *preserves colimits indexed by $\mathcal{I}$.*

Proof. We show that, if $\mathrm{colim}_{\mathcal{I}}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$ preserves limits indexed by $\mathcal{J}$, then $\lim_{\mathcal{J}}\colon \mathrm{Fun}(\mathcal{J}, \mathcal{C}) \to \mathcal{C}$ preserves colimits indexed by $\mathcal{I}$. The other direction follows dually.

By Lemmas B.6, C.7, and C.14, our assumption implies that $\mathrm{Lan}_{\mathrm{inc}_{\mathcal{I}}}$ preserves right Kan extension along $\mathrm{inc}_{\mathcal{J}}$. Hence there is an equivalence

$$(\mathrm{Lan}_{\mathrm{inc}_{\mathcal{I}}}\circ) \circ \mathrm{Ran}_{\mathrm{inc}_{\mathcal{J}}} \simeq \mathrm{Ran}_{\mathrm{inc}_{\mathcal{J}}} \circ (\mathrm{Lan}_{\mathrm{inc}_{\mathcal{I}}}\circ)$$

of functors $\mathrm{Fun}(\mathcal{J}, \mathrm{Fun}(\mathcal{I}, \mathcal{C})) \longrightarrow \mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathrm{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C}))$. This transforms, through a few applications of Lemma C.2, to an equivalence

$$\mathrm{Lan}_{\mathrm{inc}_{\mathcal{I}}} \circ (\mathrm{Ran}_{\mathrm{inc}_{\mathcal{J}}}\circ) \simeq (\mathrm{Ran}_{\mathrm{inc}_{\mathcal{J}}}\circ) \circ \mathrm{Lan}_{\mathrm{inc}_{\mathcal{I}}}$$

of functors $\mathrm{Fun}(\mathcal{I}, \mathrm{Fun}(\mathcal{J}, \mathcal{C})) \longrightarrow \mathrm{Fun}(\mathcal{I}^{\triangleright}, \mathrm{Fun}(\mathcal{J}^{\triangleleft}, \mathcal{C}))$. This becomes, after postcomposing with $(\mathrm{Res}_{\blacktriangleleft_{\mathcal{J}}} \circ)$, an equivalence

$$\mathrm{Lan}_{\mathrm{inc}_{\mathcal{I}}} \circ (\lim_{\mathcal{J}} \circ) \simeq (\lim_{\mathcal{J}} \circ) \circ \mathrm{Lan}_{\mathrm{inc}_{\mathcal{I}}}$$

of functors $\mathrm{Fun}(\mathcal{I}, \mathrm{Fun}(\mathcal{J}, \mathcal{C})) \longrightarrow \mathrm{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C})$ (using Lemma C.7 and that restrictions preserve left Kan extensions). Thus $\lim_{\mathcal{J}}$ sends colimit diagrams indexed by $\mathcal{I}^{\triangleright}$ to colimit diagrams, which we wanted to show. □

Lemma C.19. *Let $\mathcal{I}$ be a category, i an object of $\mathcal{I}$, $\mathcal{C}$ and $\mathcal{D}$ two ∞-categories that admit colimits indexed by $\mathcal{I}$, $F\colon \mathcal{C} \to \mathcal{D}$ a functor that preserves colimits indexed by $\mathcal{I}$, and $D\colon \mathcal{I} \to \mathcal{C}$ a diagram. Then F applied to the structure map $D(i) \to \mathrm{colim}_{\mathcal{I}} D$ is an equivalence if and only if the structure map $(F \circ D)(i) \to \mathrm{colim}_{\mathcal{I}} (F \circ D)$ is.*

Proof. This follows from Lemma B.6 and **??**. □

Lemma C.20. *Let $f\colon I \to J$ be a map of simplicial sets and $F\colon \mathcal{C} \to \mathcal{D}$ a functor between weakly left f-extensible ∞-categories. Then the diagram*

$$\begin{tikzcd}
(F\circ) \arrow[r, "\eta^{\mathcal{D}}"] \arrow[d, "\eta^{\mathcal{C}}"'] & \mathrm{Res}_f \circ \mathrm{Lan}_f \circ (F\circ) \arrow[d, "\chi"] \\
(F\circ) \circ \mathrm{Res}_f \circ \mathrm{Lan}_f \arrow[r, equal] & \mathrm{Res}_f \circ (F\circ) \circ \mathrm{Lan}_f
\end{tikzcd}$$

commutes up to homotopy, where χ is as in **??**, *and $\eta^{\mathcal{C}}$ and $\eta^{\mathcal{D}}$ are the units of the adjunctions $\mathrm{Lan}_f \dashv \mathrm{Res}_f$ with the respective target categories.*

Proof. This is a special case of Lemma A.2. □

Lemma C.21. *Let $\mathcal{I}$ be a category, $F\colon \mathcal{C} \to \mathcal{D}$ a functor between ∞-categories that admit limits indexed by $\mathcal{I}$, and denote by $\mathrm{inc}\colon \mathcal{I} \to \mathcal{I}^{\triangleleft}$ the inclusion. Then the following diagram in $\mathrm{Fun}(\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{C}), \mathcal{D})$ commutes up to homotopy*

$$\begin{tikzcd}
F \circ \mathrm{Res}_{\blacktriangleleft} \arrow[r] \arrow[d, equal] & F \circ \lim_{\mathcal{I}} \circ \mathrm{Res}_{\mathrm{inc}} \arrow[r, "\chi"] & \lim_{\mathcal{I}} \circ (F\circ) \circ \mathrm{Res}_{\mathrm{inc}} \arrow[d, equal] \\
\mathrm{Res}_{\blacktriangleleft} \circ (F\circ) \arrow[rr] & & \lim_{\mathcal{I}} \circ \mathrm{Res}_{\mathrm{inc}} \circ (F\circ)
\end{tikzcd}$$

where the upper left and the bottom horizontal morphism are given by the respective canonical map to the limit, and χ is as in **??**.

Proof. Consider the two diagrams

$$
\begin{tikzcd}
\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{C}) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{Res}_{\blacktriangleleft}"'] \arrow[dr, phantom, "\overset{\xi}{\Nearrow}"] & \mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{C}) \arrow[d, "\mathrm{Res}_{\mathrm{inc}}"] \\
\mathcal{C} \arrow[r, "\Delta"] \arrow[d, "F"'] \arrow[dr, phantom, "\overset{\mathrm{id}_1}{\Nearrow}"] & \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \arrow[d, "F\circ"] \\
\mathcal{D} \arrow[r, "\Delta"] & \mathrm{Fun}(\mathcal{I}, \mathcal{D})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Fun}(\mathcal{I}^{\triangleright}, \mathcal{C}) \arrow[r, "\mathrm{id}"] \arrow[d, "F\circ"'] \arrow[dr, phantom, "\overset{\mathrm{id}_2}{\Nearrow}"] & \mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{C}) \arrow[d, "F\circ"] \\
\mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{D}) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{Res}_{\blacktriangleleft}"'] \arrow[dr, phantom, "\overset{\xi}{\Nearrow}"] & \mathrm{Fun}(\mathcal{I}^{\triangleleft}, \mathcal{D}) \arrow[d, "\mathrm{Res}_{\mathrm{inc}}"] \\
\mathcal{D} \arrow[r, "\Delta"] & \mathrm{Fun}(\mathcal{I}, \mathcal{D})
\end{tikzcd}
$$

where ξ is as in **??**, and note that their pastes agree. Now the pasting law for mates implies the desired statement since the mates of the transformations labeled ξ are the canonical maps to the limit, the mate of id_1 is χ, and the mate of id_2 is the identity. □

D. Generalities

D.1. about posets

Lemma D.1. *Let $\mathcal{I}$ be a poset, $\mathcal{C}$ a category, and $f\colon \mathcal{I} \to \mathcal{C}$ a full functor. Then f is injective.*

Proof. Assume that there exist $i \neq i' \in \mathcal{I}$ such that $f(i) = f(i')$. Since f is full there must be both a map $i \to i'$ and a map $i' \to i$ being mapped to $\mathrm{id}_{f(i)}$ by f. This contradicts the definition of a poset. □

Lemma D.2. *Let $f\colon \mathcal{I} \to \mathcal{J}$ and $g\colon \mathcal{J} \to \mathcal{K}$ be functors between categories such that $\mathcal{J}$ is a poset and $g \circ f$ is full. Then f is full.*

Proof. This follows from the fact that, when a surjective map of sets factors over a set with at most one element, the first map in this factorization is also surjective. □

Lemma D.3. *Let $\mathcal{I}$ and $\mathcal{J}$ be posets. Assume that both $\mathcal{I}$ and $\mathcal{J}$ have initial objects $\varnothing_{\mathcal{I}}$ respectively $\varnothing_{\mathcal{J}}$ and that $\mathcal{I}$ has a terminal object $*_{\mathcal{I}} \neq \varnothing_{\mathcal{I}}$. Then $(\mathcal{I} \times \mathcal{J})_{>\varnothing}$ is contractible.*

Proof. Since $\mathcal{J}$ has an initial object and is thus contractible, it is enough to show that there is an adjoint pair of functors between $(\mathcal{I} \times \mathcal{J})_{>\varnothing}$ and $\mathcal{J}$ as this implies that they are homotopy equivalent.

To this end, let $l\colon (\mathcal{I} \times \mathcal{J})_{>\varnothing} \to \mathcal{J}$ be given by the projection, i.e. $l(i,j) = j$, and $r\colon \mathcal{J} \to (\mathcal{I} \times \mathcal{J})_{>\varnothing}$ by $r(j) = (*_{\mathcal{I}}, j)$. Note that r is well-defined as, by assumption, we have $*_{\mathcal{I}} \neq \varnothing_{\mathcal{I}}$. To check that l is indeed left adjoint to r, we need to prove that, for all $(i,j) \in (\mathcal{I} \times \mathcal{J})_{>\varnothing}$ and $j' \in \mathcal{J}$, we have $j = l(i,j) \leq j'$ if and only if $(i,j) \leq r(j') = (*_{\mathcal{I}}, j')$. This is true by the assumption of $*_{\mathcal{I}}$ being terminal in $\mathcal{I}$. □

Lemma D.4. *Let $f\colon \mathcal{S} \to \mathcal{T}$ and $f'\colon \mathcal{S}' \to \mathcal{T}$ be maps of posets where $\mathcal{S}, \mathcal{S}' \in \mathbf{Pos}_{\varnothing}$ and $\mathcal{T} \in \mathbf{Pos}_{\amalg}$. Assume that $f^{-1}(\varnothing_{\mathcal{T}}) = \{\varnothing_{\mathcal{S}}\}$ and $(f')^{-1}(\varnothing_{\mathcal{T}}) = \{\varnothing_{\mathcal{S}'}\}$ and that for all $t \in \mathcal{T}_{>\varnothing}$ one of the posets $f \downarrow t$ and $f' \downarrow t$ has a terminal object which is different from the initial object (in particular this is fulfilled if $f = \mathrm{id}_{\mathcal{T}}$). Then $p\colon (\mathcal{S} \times \mathcal{S}')_{>\varnothing} \to \mathcal{T}_{>\varnothing}$ given by $(s, s') \mapsto f(s) \amalg f'(s')$ is homotopy initial.*

Proof. We need that, for all $t \in \mathcal{T}_{>\varnothing}$, the category $p \downarrow t$ is contractible. This comma category can be identified with the full subposet

$$\{(s, s') \in (\mathcal{S} \times \mathcal{S}')_{>\varnothing} \mid f(s) \leq t \text{ and } f'(s') \leq t\} \subseteq (\mathcal{S} \times \mathcal{S}')_{>\varnothing}$$

using the universal property of the coproduct. This, in turn, is isomorphic to the category $((f \downarrow t) \times (f' \downarrow t))_{>\varnothing}$ which is contractible by Lemma D.3. Here, we use that, by our assumptions both $f \downarrow t$ and $f' \downarrow t$ have an initial object ($\varnothing_{\mathcal{S}}$ respectively $\varnothing_{\mathcal{S}'}$) and one of them has a terminal object different from the initial object. □

D.2. about ∞-categories

Lemma D.5. *Let $l\colon \mathcal{C} \to \mathcal{D}$ and $r\colon \mathcal{D} \to \mathcal{C}$ be two functors between ∞-categories such that l is left adjoint to r with unit η and counit ε. Then, for any simplicial set K, the functor $(l\,\circ)\colon \mathrm{Fun}(K, \mathcal{C}) \to \mathrm{Fun}(K, \mathcal{D})$ is left adjoint to $(r\,\circ)\colon \mathrm{Fun}(K, \mathcal{D}) \to \mathrm{Fun}(K, \mathcal{C})$ with unit $(\eta\,\circ)$ and counit $(\varepsilon\,\circ)$.*

Proof. This is [**RV**]. □

Lemma D.6. *Let the following be a pullback square in the 1-category of simplicial sets*

$$\begin{tikzcd} K \arrow[r] \arrow[d, "f"'] & \mathcal{D} \arrow[d, "g"] \\ \mathcal{E} \arrow[r] & \mathcal{C} \end{tikzcd}$$

where $\mathcal{D}$ and $\mathcal{E}$ are ∞-categories and $\mathcal{C}$ is a category. Then K is an ∞-category.

Proof. By [**LurHTT**], the functor g is an inner fibration. But then f is also an inner fibration since they are stable under pullbacks. Since $\mathcal{E}$ is an ∞-category, the constant map $c\colon \mathcal{E} \to *$ is also an inner fibration. Since inner fibrations are closed under composition, the constant map $K \to *$ is thus also an inner fibration and hence K an ∞-category. □

Lemma D.7. *Let $\mathcal{C}$ be an ∞-category, $\mathcal{D}$ a category, and $f\colon \mathcal{D} \to \mathrm{h}\mathcal{C}$ a functor. Furthermore, let $\mathcal{E}$ be a pullback (in the 1-category of simplicial sets) as in the following diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "g"] \arrow[d, "p"'] & \mathcal{C} \arrow[d, "\pi_{\mathcal{C}}"] \\ \mathcal{D} \arrow[r, "f"] & \mathrm{h}\mathcal{C} \end{tikzcd}$$

where $\pi_{\mathcal{C}}$ denotes the canonical functor to the homotopy category. Then $\mathcal{E}$ is an ∞-category, there is a unique isomorphism $\mathcal{D} \cong \mathrm{h}\mathcal{E}$ under $\mathcal{E}$, and, for two objects $E, E' \in \mathcal{E}$ and morphism $d\colon p(E) \to p(E')$ in $\mathcal{D}$, the functor g induces an equivalence from the path component of $\mathrm{Map}_{\mathcal{E}}(E, E')$ over d (there is only one such component by the identification $\mathcal{D} \cong \mathrm{h}\mathcal{E}$) to the path component of $\mathrm{Map}_{\mathcal{C}}(g(E), g(E'))$ over $f(d)$.

Proof. That $\mathcal{E}$ is an ∞-category follows directly from Lemma D.6. Furthermore note that, since $\pi_{\mathcal{C}}$ is a bijection on objects, the map p is as well, i.e. we can identify objects of $\mathcal{E}$ with objects of $\mathcal{D}$. Now, by the universal property of the pullback the functor g induces, for any morphism $d\colon D \to D'$ in $\mathcal{D}$, an isomorphism from the simplicial subset of $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{E}}(D, D')$ lying over d to the simplicial subset of $\mathrm{Hom}^{\mathrm{R}}_{\mathcal{C}}(f(D), f(D'))$ lying over $f(d)$ (cf. [**LurHTT**] for the definition of $\mathrm{Hom}^{\mathrm{R}}$). This shows the last statement. To obtain the identification $\mathcal{D} \cong \mathrm{h}\mathcal{E}$, note that what we have already shown implies that the part of $\mathrm{Map}_{\mathcal{E}}(D, D')$ lying over d is path-connected and that these parts are, for different morphisms in $\mathcal{D}$, disjoint path-components that cover the whole space. □

Lemma D.8. *Let $\mathcal{C}$ be an ∞-category. Then the canonical map $\pi_{\mathcal{C}}\colon \mathcal{C} \to \mathrm{h}\mathcal{C}$ to its homotopy category is a categorical fibration.*

Proof. By [**LurHTT**] the statement is equivalent to $\pi_{\mathcal{C}}$ being an inner fibration such that for every equivalence $f\colon D \to D'$ in $\mathrm{h}\mathcal{C}$ and $C \in \mathcal{C}$ with $\pi_{\mathcal{C}}(C) = D$ there exists an equivalence $g\colon C \to C'$ in $\mathcal{C}$ such that $\pi_{\mathcal{C}}(g) = f$. That it is an inner fibration follows directly from [**LurHTT**] and the other property is clear from the definition of the homotopy category. □

Lemma D.9. *Denote by S the simplicial set obtained from the directed graph with vertices $\mathbb{N}_0$ and an edge $n \to n+1$ for every $n \in \mathbb{N}_0$, and by $i\colon S \to \mathbb{N}_0$ the canonical inclusion of simplicial sets. Then, for every ∞-category $\mathcal{C}$, the restriction $\mathrm{Res}_i\colon \mathrm{Fun}(\mathbb{N}_0, \mathcal{C}) \to \mathrm{Fun}(S, \mathcal{C})$ is a trivial Kan fibration.*

In particular, for every functor $f\colon S \to \mathcal{C}$, there is an essentially unique functor $g\colon \mathbb{N}_0 \to \mathcal{C}$ such that $g \circ i = f$. Less formally: to specify a sequential diagram in $\mathcal{C}$ it is enough to specify a sequence of composable morphisms $(D_n \to D_{n+1})_{n\in\mathbb{N}_0}$ in $\mathcal{C}$.

Proof. This is [**LurK**] applied to the directed graph used to define S. □

Lemma D.10. *Let $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ be two functors between ∞-categories such that there is an adjunction $L \dashv R$ with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to R \circ L$. Then, for all $c \in \mathcal{C}$ and $d \in \mathcal{D}$, the composition*

$$\mathrm{Map}_{\mathcal{D}}(L(c), d) \xrightarrow{R} \mathrm{Map}_{\mathcal{C}}(RL(c), R(d)) \xrightarrow{\circ\eta(c)} \mathrm{Map}_{\mathcal{C}}(c, R(d))$$

is an equivalence.

Proof. We claim that $(\varepsilon(d)\,\circ) \circ L$ is a quasi-inverse, where $\varepsilon\colon L \circ R \to \mathrm{id}_{\mathcal{D}}$ is the counit of the above adjunction $L \dashv R$. To see that it is a left inverse consider the following homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{D}}(L(c), d) \arrow[d, "R"'] \arrow[dr, "LR"] \arrow[drr, bend left, "\circ\varepsilon(L(c))"] & & \\
\mathrm{Map}_{\mathcal{C}}(RL(c), R(d)) \arrow[r, "L"] \arrow[d, "\circ\eta(c)"'] & \mathrm{Map}_{\mathcal{D}}(LRL(c), LR(d)) \arrow[r, "\varepsilon(d)\circ"] \arrow[d, "\circ L(\eta(c))"'] & \mathrm{Map}_{\mathcal{D}}(LRL(c), d) \arrow[d, "\circ L(\eta(c))"] \\
\mathrm{Map}_{\mathcal{C}}(c, R(d)) \arrow[r, "L"] & \mathrm{Map}_{\mathcal{D}}(L(c), LR(d)) \arrow[r, "\varepsilon(d)\circ"] & \mathrm{Map}_{\mathcal{D}}(L(c), d)
\end{tikzcd}$$

and note that the composition along the right side of the diagram is homotopic to the identity by one of the triangle identities. Analogously one can show that it is also a right inverse. □

D.3. about cartesian diagrams

Lemma D.11. *Let $\mathcal{I}$ and $\mathcal{J}$ be categories that have initial objects, $f\colon \mathcal{I} \to \mathcal{J}$ an initial object preserving functor, and $\mathcal{C}$ an ∞-category that admits limits indexed by both $\mathcal{I}_{>\varnothing}$ and by $\mathcal{J}_{>\varnothing}$. Furthermore, assume that f restricts to a functor $\mathcal{I}_{>\varnothing} \to \mathcal{J}_{>\varnothing}$ and that this functor is homotopy initial.*

Then a diagram $D\colon \mathcal{J} \to \mathcal{C}$ is cartesian if and only if $D \circ f\colon \mathcal{I} \to \mathcal{C}$ is cartesian.

Proof. We have, by (the dual of) Lemma C.9, a homotopy commutative diagram

$$\begin{tikzcd} D(\varnothing_{\mathcal{J}}) \arrow[r, equal] \arrow[d] & (D \circ f)(\varnothing_{\mathcal{I}}) \arrow[d] \\ \lim\limits_{\mathcal{J}_{>\varnothing}} D|_{\mathcal{J}_{>\varnothing}} \arrow[r, "\simeq", "f^*"'] & \lim\limits_{\mathcal{I}_{>\varnothing}} (D \circ f)|_{\mathcal{I}_{>\varnothing}} \end{tikzcd}$$

in which the bottom horizontal map is an equivalence by assumption. Hence, the left vertical map is an equivalence if and only if the right vertical map is an equivalence, which we wanted to show. □

Lemma D.12. *Let $\mathcal{I}$ and $\mathcal{J}$ be categories such that $\mathcal{J}$ has an initial object, $\mathcal{C}$ an ∞-category that admits limits indexed by $\mathcal{J}_{>\varnothing}$, and $D\colon \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ a functor. Denote by $D_{\mathcal{I}}\colon \mathcal{I} \to \mathrm{Fun}(\mathcal{J}, \mathcal{C})$ and $D_{\mathcal{J}}\colon \mathcal{J} \to \mathrm{Fun}(\mathcal{I}, \mathcal{C})$ the curried functors. Furthermore assume that $D_{\mathcal{I}}(i) = \mathrm{Res}_i \circ D_{\mathcal{J}}\colon \mathcal{J} \to \mathcal{C}$ is cartesian for all $i \in \mathcal{I}$.*

a) If $\mathcal{C}$ admits limits indexed by $\mathcal{I}$, then $D_{\mathcal{J}}$, $\lim_{\mathcal{I}} \circ D_{\mathcal{J}}$, and $\lim_{\mathcal{I}} D_{\mathcal{I}}$ are all cartesian.

b) If $\mathcal{C}$ admits colimits indexed by $\mathcal{I}$ and the functor $\mathrm{colim}_{\mathcal{I}}\colon \mathrm{Fun}(\mathcal{I}, \mathcal{C}) \to \mathcal{C}$ preserves limits indexed by $\mathcal{J}_{>\varnothing}$, then $\mathrm{colim}_{\mathcal{I}} \circ D_{\mathcal{J}}$ and $\mathrm{colim}_{\mathcal{I}} D_{\mathcal{I}}$ are cartesian.

Proof. We first show that $D_{\mathcal{J}}$ is cartesian, i.e. that the canonical map $D_{\mathcal{J}}(\varnothing) \to (\lim_{\mathcal{J}_{>\varnothing}} \mathrm{Res}_{\mathcal{J}_{>\varnothing}})(D_{\mathcal{J}})$ is an equivalence. For this it is enough that its restriction to i is an equivalence for all $i \in \mathcal{I}$, which follows by assumption and Lemma C.21 since Res_i preserves limits.

Now, if $\mathcal{C}$ admits limits indexed by $\mathcal{I}$, then $\lim_{\mathcal{I}} \circ D_{\mathcal{J}}$ is also cartesian by again Lemma C.21 since $\lim_{\mathcal{I}}$ preserves limits by Lemma C.17. This also implies that $\lim_{\mathcal{I}} D_{\mathcal{I}}$ is cartesian by Lemma C.3. The statement about colimits can be shown in the same way since we only used that $\lim_{\mathcal{I}}$ preserves limits indexed by $\mathcal{J}_{>\varnothing}$. □

Lemma D.13. *Let $\mathcal{I}$ and $\mathcal{J}$ be categories with initial objects and $\mathcal{C}$ an ∞-category that admits limits indexed by $\mathcal{J}_{>\varnothing}$. Furthermore, let $D\colon \mathcal{I} \times \mathcal{J} \to \mathcal{C}$ be a diagram such that, for each $i \in \mathcal{I}$, the restriction $D|_{\{i\} \times \mathcal{J}}\colon \mathcal{J} \to \mathcal{C}$ is cartesian. Then D is a limit diagram.*

Proof. We consider the inclusions

$$\mathcal{I} \times (\mathcal{J}_{>\varnothing}) \xrightarrow{\iota} (\mathcal{I} \times \mathcal{J})_{>\varnothing} \xrightarrow{\kappa} \mathcal{I} \times \mathcal{J}.$$

By assumption and [**LurHTT**], the functor D is a right Kan extension of $\mathrm{Res}_{\kappa \circ \iota}$ along $\kappa \circ \iota$ in the sense of [**LurHTT**]. In the same way we also obtain that $\mathrm{Res}_{\kappa} D$ is a right Kan extension of $\mathrm{Res}_{\kappa \circ \iota} D$ along ι. Then, by [**LurHTT**], the diagram D is a right Kan extension of $\mathrm{Res}_{\kappa} D$ along κ, i.e. D is a limit diagram. □