Research statement

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Berezin-Toeplitz quantization is a process which associates to a symplectic manifold (X, ω) with some additional structures, a sequence of Hermitian spaces $\{\mathscr{H}_p\}_{p\in\mathbb{N}^*}$. In particular, this process depends on a complex structure J on X, and a way to study the dependence on the quantization on J is to see $\mathscr{H}_p$ as a Hermitian vector bundle over the space of complex structures on X compatible with ω, for all $p \in \mathbb{N}^*$.

1 Setting

Let (X, ω) be a compact symplectic manifold of dimension $2n$, and let (L, h^L) be a Hermitian line bundle endowed with a connection ∇^L such that its curvature R^L satisfies the *prequantization condition* $\omega = \frac{\sqrt{-1}}{2\pi} R^L$. Consider a family of complex structures on X compatible with ω, parametrized by a manifold B. We consider as well an auxiliary Hermitian vector bundle (E, h^E) over $X \times B$, endowed with an Hermitian connection ∇^E.

For any $p \in \mathbb{N}^*$, we denote by $\mathscr{H}_p$ the space of holomorphic sections of $L^p \otimes E$ over X, seen as a Hermitian vector bundle over B, endowed with the L^2-Hermitian product induced by h^L, h^E and ω. Assuming all the data are holomorphic, $\mathscr{H}_p$ is naturally endowed with a holomorphic structure for all $p \in \mathbb{N}^*$. The *holomorphic quantization* of X associated to B is the data of the family of Hermitian vector bundles $\{\mathscr{H}_p\}_{p\in\mathbb{N}^*}$.

2 Variation of complex structure and equivariant index formula

Suppose X endowed with a complex structure J compatible with ω, and let G be a compact Lie group acting holomorphically and isometrically on (X, J, ω), such that this action lifts to (L, h^L, ∇^L). For any $g \in G$, the equivariant index theorem of Atiyah-Segal-Singer gives a formula for the alternated sum of the trace of g on the cohomology groups of L^p in terms of an integral over the fixed points of g (see e.g. [3, Chap.6]). In particular, for p big enough, the cohomology is concentrated in 0 degree, and the action of g is simply the induced action on the holomorphic sections of L^p.

Assume now that G does not preserve any complex structure on X. Taking $B = \mathscr{J}_\omega$, the space of complex structures on X compatible with ω, we can use the Chern connection of $\mathscr{H}_p$ defined in Section 1 to compare the two spaces of holomorphic sections respectively associated with the initial complex structure J and the complex structure g^*J induced by g.

My first goal is to establish a equivariant index formula in this case, which gives back asymptotically the usual formula in the complex case. In order to do this, one needs a local model for the parallel transport operator, analogous to the *Bergman kernel* of the L^2-orthogonal projection

P_p on $\mathscr{H}_p$, in the spirit of Berezin-Toeplitz quantization. In [5], Laurent Charles established such a formula in the case when the fixed points are discrete, and this would generalize it to the case where the fixed points form a positive dimensional manifold.

3 Hitchin connection and Witten's asymptotic expansion conjecture

In [13], Witten introduced a topological invariant of a compact manifold M of dimension 3 which generalizes the Jones polynomial of links, and set a program to compute it effectively. In particular, via methods of quantum field theory, Witten gives in [13, § 2] an asymptotic expansion at the semi-classical limit of this invariant, known as *Witten's asymptotic expansion conjecture*. This invariant is constructed from a moduli space $\mathscr{M}_M$ of flat connections over M, and Witten suggests to reduce the problem to the corresponding space $\mathscr{M}_\Sigma$ of a compact surface Σ, whose geometry is much better known.

In fact, in a number of cases classified in [4, § 5.4], $\mathscr{M}_\Sigma$ has a natural structure of an *orbifold*. In those cases, there exists a natural sympectic form ω_{AB} on $\mathscr{M}_\Sigma$, called the *Atiyah-Bott form*, and a Hermitian line bundle L_{CS} over $\mathscr{M}_\Sigma$, called the *Chern-Simons line bundle*, endowed with a connection staisfying the prequantization condition for ω_{AB}. A complex structure on Σ induces one on $\mathscr{M}_\Sigma$ compatible with ω_{AB}, which depends on Σ only up to the action of a diffeomorphism isotopic to the identity. We can then study the holomorphic quantization of $\mathscr{M}_\Sigma$ as in Section 1 taking for B the space of complex structure on Σ up to isotopy.

In [8], Hitchin shows that for $p \in \mathbb{N}$ big enough, the holomorphic quantization bundle is naturally endowed with a projectively flat connection, called the *Hitchin connection*. We can then apply the method of Section 2 replacing the Chern connection by the Hitchin connection, and g by the symplectomorphism on $\mathscr{M}_\Sigma$ induced by a diffeomorphism $\phi : \Sigma \to \Sigma$. As explained in [1, § 1], this would allow one to solve Witten's asymptotic expansion conjecture in the case M is the mapping torus associated with ϕ.

My second goal is to apply the results of Section 2 to this problem, in the case when the fixed points of the action induced by ϕ on $\mathscr{M}_\Sigma$ are non-degenerate. Parallel to the situation of Section 2, Laurent Charles established the case of discrete fixed points, and this would generalize this result to the case where the fixed points form a positive dimensional manifold.

Another aspect of the strategy suggested by Witten in [13, § 3] consist in establishing gluing formulas: if M is a compact manifold of dimension 3 with boundary $\partial M = \Sigma$, then there is a restriction morphism $r : \mathscr{M}_M \to \mathscr{M}_\Sigma$. In [10, Prop. 7.2] and [7, Prop. 3.27], it is shown that the image of r is a Bohr-Sommerfeld Lagrangian submanifold, and [9, Th. 4.5] computes the asymptotic expansion of Witten's invariant of the manifold obtained by gluing two manifolds M_1 and M_2 along their common boundary Σ. Another of my goals is to develop this program from the results of [9].

4 Quantization of special Lagrangian submanifolds

In the framework of geometric quantization associated with a real regular polarization, the quantum states of X are represented by immersed Lagrangian submanifolds $\iota : \Lambda \to X$ satisfying a property called the *Bohr-Sommerfeld condition*, which asks for the connection induced by ∇^L on $L|_\Lambda$ to be trivial. In general, one needs to consider singular fibrations, in which case Λ might only be isotropic, so that its dimension is not necessarily equal to n.

In the case of (X, L) *toric*, that is endowed with an effective Hamiltonian action of the torus $\mathbb{T}^n = (S^1)^n$ preserving all the datas of Section 1, we can consider the singular fibration induced by the moment map $\mu : X \to \Delta$ from X to its *Delzant polytope* Δ. The fibres are then Lagrangian tori outside $D = \mu^{-1}(\partial\Delta)$, and the Bohr-Sommerfeld fibres of μ in the sense of [9, Def. 3.1] are in finite number. Furthermore, the isotropic states associated with the Bohr-Sommerfeld fibres of this application form an orthogonal basis of $\mathscr{H}_p$. It is a case where the correpondance between real quantization and holomorphic quantization holds perfectly, and this is far to be so in general.

Assume now $L = K_X^*$, where K_X^* denotes the dual of the canonical bundle of X. On $X\backslash D \simeq (\mathbb{C}^*)^n$, the standard holomorphic volume form induces a complex-valued volume form with constant argument on the orbits of the usual action of $\mathbb{T}^n$ on $(\mathbb{C}^*)^n$. Furthermore, these orbits form a fibration on $X\backslash D$. More generally, if X is a smooth Fano variety and if D is an effective anti-canonical divisor, then there exists an holomorphic volume form Ω on $X\backslash D$, which generalizes the standard holomorphic form above.

Definition 4.1. A Lagrangian submanifold $\Lambda \subset X\backslash D$ is *special* if the argument of $\Omega|_\Lambda$ is constant.

We can then consider fibrations in Lagrangian submanifolds of $X\backslash D$, which corresponds in the toric case to the fibres of the moment map $\mu : X \to \Delta$ restricted to $X\backslash D$.

My third goal is to compare holomorphic quantization and real quantization in this case, considering the Lagrangian states of my paper [9, Def. 3.3] associated with special Lagrangian submanifolds satisfying the Bohr-Sommerfeld condition. For instance, my results in [9] show that such states do not vanish for p big enough, and tend to be linearly independant as p tens to infinity, approaching the situation of the toric case. We can then compare the dimension of $\mathscr{H}_p$ with the number of Bohr-Sommerfeld special Lagrangians, as in [12], and this is a tool to approach special Lagrangians fibrations by finite dimensional objects, in the spirit of Donaldson's work [6]. This gives a starting point for a moment map formulation of this problem, as in [6]

Following [2], we can start with the example of $X = \mathbb{CP}^2$ and D non-singular cubic, generalizing the case of the toric fibration on $\mathbb{CP}^2$. We can also consider a Calabi-Yau manifold X, on which there exists a holomorphic volume form Ω and in which case we choose $D = \emptyset$.

This approach build a bridge between geometric quantization and mirror symmetry, via the Strominger-Yau-Zaslow conjecture announced in [11]. In this setting, we can study the isotropic states associated with singular Bohr-Sommerfeld fibres of the fibration, that is corresponding to submanifolds of D obtained as degenerations of special Lagrangian submanifolds, and it is natural to conjecture a link with the *instanton corrections* of [11].

References

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