Record of research

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My field of research is differential geometry, in particular Kählerian and symplectic geometry. Specifically, I work on geometric quantization and its semi-classical aspects. The goal of quantization is to associate with a classical phase space, here a symplectic manifold (X, ω), a Hilbert space of quantum states H. This association is supposed to send the algebra of classical observables, here the space $\mathscr{C}^\infty(X)$ equipped with its canonical Poisson structure, to the bounded operators of H. In *Berezin-Toeplitz quantization*, we endow (X, ω) with some additional structures, and the associated Hilbert space is of the form $H = \oplus_{p\in\mathbb{N}}\mathscr{H}_p$. In this context, asymptotic results when p tends to infinity describes the *semi-classical limit*, when the scale gets so large that we recover the laws of classical mechanics from the laws of quantum mechanics.

My work is about a geometric version of Berezin-Toeplitz quantization, developed by Ma and Marinescu, which works for all prequantified symplectic manifolds.

1 Setting

Let (X, ω) be a symplectic manifold of dimension $2n$ with tangent bundle TX, and let (L, h^L) be a Hermitian line bundle over X, together with a Hermitian connection ∇^L such that its curvature R^L satisfies the *prequantization condition*:

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{1.1}$$

Let X be equipped with a Riemannian metric g^{TX} compatible with J, and for any $p \in \mathbb{N}$, write L^p for the p-th tensorial power L. We consider also an auxiliary Hermitian vector bundle (E, h^E) over X, together with a Hermitian connection ∇^E.

Let us consider the case where J comes from a complex structure, so that (X, J, ω) is a Kähler manifold, (L, h^L) is a holomorphic Hermitian line bundle such that its Chern connection ∇^L satisfies (1.1), and (E, h^E) is a holomorphic Hermitian vector bundle. For any $p \in \mathbb{N}$, we denote by $\mathscr{H}_p$ the space of holomorphic sections of $L^p \otimes E$, endowed with the L^2-Hermitian product induced by h^L, h^E and g^{TX}. The *holomorphic quantization* of X is the data of the family of Hermitian spaces $\{\mathscr{H}_p\}_{p\in\mathbb{N}^*}$.

Let $\Omega^{0,\bullet}(X, L^p \otimes E) = \oplus_{q=1}^n \Omega^{0,q}(X, L^p \otimes E)$ denote the Dolbeault complex. We will consider here the two ways of generalizing the Kähler case. The first one is to replace the space of holomorphic sections by the kernel of the *spin^c Dirac operator* acting on $\Omega^{0,\bullet}(X, L^p \otimes E)$, called the *spin^c case* in the sequel, and the second one is to replace the space of holomorphic sections by the space associated with the small eigenvalues of a *renormalized Bochner Laplacian* acting on $\mathscr{C}^\infty(X, L^p \otimes E)$, called the *almost holomorphic case* in the sequel.

From now on, we will always write $\mathscr{H}_p$ for the quantum space associated with $p \in \mathbb{N}^*$ and $\mathscr{C}_p^\infty$ for the space of smooth sections containing it, whatever the case. It will always be endowed with the corresponding L^2-Hermitian product, which we write $\langle\cdot,\cdot\rangle_p$.

2 The first coefficient in the composition of Toeplitz operators

In any case described in Section 1, we can define the operator of orthogonal projection P_p from $\mathscr{C}_p^\infty$ to $\mathscr{H}_p$ with respect to $\langle\cdot,\cdot\rangle_p$, for all $p \in \mathbb{N}*$. The *Berezin-Toeplitz quantization* of $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ is the family of operators $\{T_{f,p}\}_{p\in\mathbb{N}}$ defined by $T_{f,p} = P_p f P_p \in \mathrm{End}(\mathscr{H}_p)$, where f is the pointwise multiplication operator by f. As the dimension of $\mathscr{H}_p$ tends to infinity with p, this operator can be thought of as an approximation of f in the semi-classical limit.

In the spin^c case, Ma and Marinescu established in [6, Th.1.1] the following estimate, holding for any $f,\ g \in \mathscr{C}^\infty(X, \mathrm{End}(E)),\ k \in \mathbb{N}$ and all $p \in \mathbb{N}^*$,

$$\left\| T_{f,p}T_{g,p} - \sum_{r=0}^{\infty} p^{-r} T_{C_r(f,g),p} \right\| \leq \frac{c_k}{p^k}, \tag{2.1}$$

where $\|\cdot\|$ denote the operator norm on $\mathscr{H}_p$, where $c_k > 0$ for all $k \in \mathbb{N}$ and C_r are bidifferential operators, with $C_0(f,g) = fg$. Moreover, in the case $f, g \in \mathscr{C}^\infty(X)$, they showed that $C_1(f,g) - C_1(g,f) = \sqrt{-1}\{f,g\}$, where $\{.,.\}$ denotes the Poisson bracket associated to the symplectic form $2\pi\omega$. Berezin-Toeplitz theory in the Kähler case, for E trivial and $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$, was initially developped by Bordemann, Meinreken and Schlichenmaier in [1] and Schlichenmaier in [8], where they establish the asymptotic formula (2.1) for $f, g \in \mathscr{C}^\infty(X, \mathbb{C})$.

Let $\nabla^{1,0}$ and $\nabla^{0,1}$ be the holomorphic and anti-holomorphic part of the connection on $\mathrm{End}(E)$ induced by ∇^E, and let $\langle .,. \rangle$ be the pairing induced by g^{TX} on $T^*X \otimes \mathrm{End}(E)$ with values in $\mathrm{End}(E)$. My first result is a formula for $C_1(f,g)$ in the spin^c case for $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$.

Theorem 2.1. *[4, Th. 1.1] Assume $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$. Then for any $f,\ g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, we have the following formula for $C_1(f,g)$ in* (2.1)*,*

$$C_1(f,g) = -\frac{1}{2\pi}\langle \nabla^{1,0} f, \nabla^{0,1} g \rangle. \tag{2.2}$$

In the Kähler case, formula (2.2) has been computed in [7, Th.0.3]. In [5, Th.1.2], we prove together with Lu, Ma and Marinescu that Berezin-Toeplitz theory holds in the almost holomorphic case as well, and I computed the coefficient $C_1(f,g)$ adapting the method of [4].

Theorem 2.2. *[5, Th. 1.2, Th. 1.3] For any $f,\ g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the asymptotic expansion* (2.1) *for the product $T_{f,p}T_{g,p}$ of their respective Berezin-Toeplitz quantization hold in the almost holomorphic case. Furthermore, the corresponding coefficient $C_1(f,g)$ from* (2.1) *satisfies* (2.2).

3 Quantization of isotropic submanifolds and their intersection product

In the framework of geometric quantization associated with a real regular polarization, the quantum states of X are represented by immersed Lagrangian submanifolds $\iota : \Lambda \to X$ satisfying a property called the *Bohr-Sommerfeld condition*. In [2], Borthwick, Paul and Uribe study the semi-classical properties of these Bohr-Sommerfeld Lagrangians in the context of holomorphic quantization, in the case X is compact Kähler and metaplectic equipped with $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$. They use these results to show the non-vanishing of some relative Poincaré series associated with remarkable curves on hyperbolic compact Riemann surfaces.

In general, one needs to consider singular fibrations, in which case Λ might only be isotropic, so that its dimension is not necessarily equal to n. Furthermore, it is useful to consider proper immersions $\iota : \Lambda \to X$ instead of embeddings. We thus set the following definition.

Definition 3.1. An immersion $\iota : \Lambda \to X$ such that $\iota^*\omega = 0$ is said to satisfy the *Bohr-Sommerfeld condition* if there exists $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^*L)$ such that $\nabla^{\iota^*L}\zeta = 0$ identically.

In [3], I study the semi-classical properties of these Bohr-Sommerfeld submanifolds in the almost holomorphic case. Here, the quantization of a Bohr-Sommerfeld submanifold is represented by a sequence $\{s_p \in \mathscr{H}_p\}_{p\in\mathbb{N}^*}$, called an *isotropic state* (or *Lagrangian state* if $\dim \Lambda = n$), defined for any $p \in \mathbb{N}^*$ by the formula

$$s_p = \int_\Lambda P_p(x, \iota(y))\zeta^p(y)dv_X(y), \tag{3.1}$$

where dv_X is the Riemannian volume form on X, $\zeta^p \in \mathscr{C}^\infty(X, L^p)$ is the p-th tensor power of a parallel and unitary $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^*L)$ as in Definition 3.1 and $P_p(\cdot,\cdot)$ is the smooth Schwartz kernel of the orthogonal projection P_p defined in Section 2.

I show in [3, § 3] that these states rapidly concentrate around the image of ι as p tends to infinity, and that their norm satisfy the following asymptotics.

Theorem 3.2. *Set $d = \dim \Lambda$. There exist $b_r \in \mathbb{R}$, $r \in \mathbb{N}$, such that for any $k \in \mathbb{N}$ and as $p \to +\infty$,*

$$\|\xi_p\|_p^2 = p^{n-\frac{d}{2}} \sum_{r=0}^{k} p^{-r} b_r + O(p^{n-\frac{d}{2}-(k+1)}), \tag{3.2}$$

where $b_0 = 2^{d/2}\operatorname{Vol}(\Lambda)$.

In particular, s_p does not vanish identically for p big enough. Furthermore, I study the L^2-Hermitian product $\langle\cdot,\cdot\rangle_p$ of two such sections. I prove that this product decreases rapidly if the corresponding submanifolds do not intersect, and I establish the following asymptotics, described here in its simplest form.

Theorem 3.3. *Let $(\Lambda_1, \iota_1, \zeta_1)$ and $(\Lambda_2, \iota_2, \zeta_2)$ be two embedded Bohr-Sommerfeld submanifolds with clean and connected intersection, and let $\{s_{j,p}\}_{p\in\mathbb{N}^*}$, $j = 1, 2$, denote the associated isotropic states. Set $l = \dim \Lambda_1 \cap \Lambda_2$ and $d_j = \dim \Lambda_j$, $j = 1, 2$. Then there exist $b_r \in \mathbb{C}$, $r \in \mathbb{N}$, such that for any $k \in \mathbb{N}$ and as $p \to +\infty$,*

$$\langle s_{1,p}, s_{2,p}\rangle_p = p^{n-\frac{d_1+d_2}{2}+\frac{l}{2}} \lambda^p \sum_{r=0}^{k} p^{-r} b_r + O(p^{n-\frac{d_1+d_2}{2}+\frac{l}{2}-(k+1)}), \tag{3.3}$$

where $\lambda \in \mathbb{C}$ is the value of the constant function on $\Lambda_1 \cap \Lambda_2$ defined for any $x \in \Lambda_1 \cap \Lambda_2$ by $\lambda(x) = \langle\zeta_1(x), \zeta_2(x)\rangle_L$. Furthermore, if $\dim \Lambda_1 = n$, the following formula holds,

$$b_0 = 2^{n/2} \int_{\Lambda_1\cap\Lambda_2} \det{}^{-\frac{1}{2}} \left\{ \sqrt{-1} \sum_{k=1}^{n-l} h^{TX}(e_k, \nu_i)\omega(e_k, \nu_j) \right\}_{i,j=1}^{d_2-l} |dv|_{\Lambda_1\cap\Lambda_2}, \tag{3.4}$$

where $\langle e_i\rangle_{i=1}^{n-l}, \langle\nu_j\rangle_{j=1}^{d_2-l}$ are local orthonormal frames of the normal bundles of $\Lambda_1 \cap \Lambda_2$ in Λ_1, Λ_2 respectively, and $|dv|_{\Lambda_1\cap\Lambda_2}$ is the Riemannian density on $\Lambda_1 \cap \Lambda_2$ induced by g^{TX}.

We thus see that in the semi-classical limit, the Hermitian product of two isotropic states is closely related to the geometry of the intersection of the corresponding submanifolds, and we call it the *intersection product* of the isotropic states.

Furthermore, I generalize Theorem 3.2 and Theorem 3.3 to the case when L^p is twisted by (E, h^E, ∇^E), where we compose one of the two states with a Toeplitz operator defined in Section 2, for any J-invariant metric g^{TX} and for X non-compact and orbifold.

Finally, as an application of Theorem 3.2, I show that some relative Poincaré series associated to hyperbolic geodesic of a general hyperbolic surface does not vanish. In the case X smooth and compact, this is the result of [2, § 4]. In the case of (X, J, ω) compact Kähler and metaplectic, for E trivial and in the language of half-forms, Theorem 3.3 is the principal result of [2, Th. 3.2], with an expansion in $p^{-1/2}$ instead of p^{-1}.

References

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