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Anticanonically balanced metrics on Fano manifolds

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Abstract
We show that if a Fano manifold has discrete automorphism group and admits a polarized Kähler–Einstein metric, then there exists a sequence of anticanonically balanced metrics converging smoothly to the Kähler–Einstein metric. Our proof is based on a simplification of Donaldson's proof of the analogous result for balanced metrics, replacing a delicate geometric argument by the use of Berezin–Toeplitz quantization. We then apply this result to compute the asymptotics of the optimal rate of convergence to the fixed point of Donaldson's iterations in the anticanonical setting.

Keywords Berezin–Toeplitz quantization · Balanced metrics · Fano manifolds

1 Introduction

A fundamental question in the study of a compact complex manifold X is the existence of a *canonical Riemannian metric*, which reflects its complex geometry in the best possible way. When X comes endowed with an ample holomorphic line bundle L, one should look for such metrics inside the set of *polarized Kähler metrics* induced by positive Hermitian metrics on L. In case X is a *Fano manifold*, so that its *anticanonical line bundle* $K_X^* := \det(T^{(1,0)}X)$ is ample, the ideal candidate for such a canonical Riemannian metric is a polarized *Kähler–Einstein* metric. By a result of Bando and Mabuchi in [2], if such a Kähler–Einstein metric exists, then it is unique. However, finding Kähler–Einstein metrics on Fano manifolds is an extremely difficult problem, and existence is related to deep properties of X as a complex algebraic manifold [13, 43].

A fruitful approach in finding a Kähler–Einstein metric on X, when it exists, is to approximate it by yet another type of canonical metrics, the so-called *anticanonically balanced metrics*, which are associated with a natural sequence of projective embeddings of X. To define them, first recall that a holomorphic line bundle L over a compact complex manifold X is *ample* if it admits a *positive Hermitian metric* $h \in \mathrm{Met}^+(L)$, so that its *Chern curvature*

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$R_h \in \Omega^2(X, \mathbb{C})$ induces a *Kähler form* on X via the formula

$$\omega_h := \frac{\sqrt{-1}}{2\pi} R_h. \tag{1.1}$$

Writing $J \in \mathrm{End}(TX)$ for the complex structure of X, this means that the following formula defines a Riemannian metric on X, called a *polarized Kähler metric*,

$$g_h^{TX} := \omega_h(\cdot, J\cdot). \tag{1.2}$$

Assume now that X is a Fano manifold, so that $L := K_X^*$ is ample, and fix $p \in \mathbb{N}$ big enough. Consider the *Kodaira embedding* of X into the projective space of hyperplanes in the space $H^0(X, L^p)$ of holomorphic sections of the tensor power $L^p := L^{\otimes p}$. Via this embedding, L^p is identified with the restriction of the dual tautological line bundle, and given a Hermitian product $H \in \mathrm{Prod}(H^0(X, L^p))$ on $H^0(X, L^p)$, one gets an induced positive Hermitian metric $\mathrm{FS}(H) \in \mathrm{Met}^+(L^p)$ on L^p, called *Fubini–Study metric*. Conversely, given a positive Hermitian metric $h^p \in \mathrm{Met}^+(L^p)$, one can consider the Hermitian inner product $\mathrm{Hilb}_\nu(h^p) \in \mathrm{Prod}(H^0(X, L^p))$ defined on $s_1, s_2 \in H^0(X, L^p)$ by

$$\langle s_1, s_2\rangle_{\mathrm{Hilb}_\nu(h^p)} := \frac{n_p}{\mathrm{Vol}(d\nu_h)} \int_X \langle s_1(x), s_2(x)\rangle_{h^p}\, d\nu_h(x), \tag{1.3}$$

where $d\nu_h$ is the *anticanonical volume form* induced by $h \in \mathrm{Met}^+(L)$, defined over any contractible open subset $U \subset X$ by the formula

$$d\nu_h := \sqrt{-1}^{n^2} \frac{\theta \wedge \bar{\theta}}{|\theta|^2_{h^{-1}}}, \tag{1.4}$$

for any non-vanishing $\theta \in \mathscr{C}^\infty(U, K_X)$, where h^{-1} denotes the Hermitian metric on K_X induced by $h \in \mathrm{Met}^+(K_X^*)$. A Hermitian metric $h_p \in \mathrm{Met}^+(L^p)$ is called *anticanonically balanced* if it coincides with the Fubini–Study metric induced by the *Hilbert product* (1.3), i.e., if

$$h_p = \mathrm{FS}(\mathrm{Hilb}_\nu(h_p)). \tag{1.5}$$

These metrics have been introduced by Donaldson [17]. Note that the original concept of a *balanced metric*, introduced by Donaldson [15] and which we describe in Example 2.4, uses the *Liouville volume form* $\omega_h^n/n!$ in the Hilbert product (1.3) instead of the anticanonical volume form (1.4). By a result of Berman et al. [4,§ 7], if an anticanonically balanced metric $h_p \in \mathrm{Met}^+(L^p)$ exists, then it is unique up to a multiplicative constant in $\mathrm{Met}^+(L^p)$. On the other hand, a polarized Kähler–Einstein metric is characterized by the property that the associated anticanonical volume form (1.4) coincides with the associated Liouville volume form up to a multiplicative constant.

In Sect. 3, we present a new proof of the following theorem. For any $m \in \mathbb{N}$, let $|\cdot|_{\mathscr{C}^m}$ be a fixed $\mathscr{C}^m$-norm on $\Omega^2(X, \mathbb{R})$.

Theorem 1.1 *Let X be a Fano manifold with discrete automorphism group admitting a polarized Kähler–Einstein metric, and write* $L := K_X^*$*. Then, for any* $m \in \mathbb{N}$*, there exists* $C_m > 0$ *and a sequence of positive Hermitian metrics* $\{h_p \in \mathrm{Met}^+(L^p)\}_{p\in\mathbb{N}}$*, which are anticanonically balanced for all* $p \in \mathbb{N}$ *big enough and such that*

$$\left| \frac{1}{p}\omega_{h_p} - \omega_\infty \right|_{\mathscr{C}^m} \leq \frac{C_m}{p}, \tag{1.6}$$

where $\omega_\infty \in \Omega^2(X,\mathbb{R})$ *is the Kähler form associated with the polarized Kähler–Einstein metric.*

This result has first been announced by Keller [25,Th. 5]. A proof of existence and weak convergence in the sense of currents has first been given by Berman et al. [4,Th. 7.1], and a proof of smooth convergence has then been given by Takahashi [40,Th. 1.3], extending the original proof of Donaldson [15] of the analogous result for the Liouville volume form.

Our proof of Theorem 1.1 also follows the basic strategy of Donaldson's proof, constructing approximately balanced metrics using the asymptotic expansion of the *Bergman kernel* along the diagonal [12, 29, 41, 47] and showing the convergence of the gradient flow of the norm squared of the associated *moment map* close to a zero. However, the most technical part of Donaldson's proof, which consists in estimating the derivative of the moment map from below, has no straightforward analogue in the anticanonical case. In fact, in the original case of Donaldson, the derivative of the moment map has a geometric interpretation, which has been clarified by Phong and Sturm in [34], giving a natural lower bound. By contrast, in the anticanonical case of Theorem 1.1, there are no obvious geometric interpretations for the derivative of the moment map, and adapting [34,Th. 2] is a serious difficulty, which was only overcome recently by Takahashi in [40,Prop. 3.5]. The main novelty of our method is to replace this geometric argument by the use of the asymptotics of the *spectral gap of the Berezin transform* established in [24,Th. 3.1]. More precisely, we use the equivalent asymptotics for the spectral gap of the *Berezin–Toeplitz quantum channel*, recalled in Theorem 2.12, which can be understood as the operation of dequantization followed by quantization of a quantum observable, i.e., the *Berezin–Toeplitz quantization of its Berezin symbol*. This strategy was inspired by the work of Fine in [18], who studied the derivative of the moment map in the original setting of Donaldson, assuming the existence of a balanced metric.

In Sect. 4, we use Theorem 1.1 together with the techniques of [24] and the energy functional of [4,§ 7] to establish the exponential convergence of *Donaldson's iterations* toward the anticanonically balanced metric for each $p \in \mathbb{N}$ big enough, and compute the asymptotics of the optimal rate of convergence as $p \to +\infty$. To explain this result, let us fix $p \in \mathbb{N}$ big enough, and define the anticanonical *Donaldson map* on the space $\mathrm{Prod}(H^0(X,L^p))$ of Hermitian inner products on $H^0(X,L^p)$ by

$$\mathscr{T}_\nu := \mathrm{Hilb}_\nu \circ \mathrm{FS} : \mathrm{Prod}(H^0(X,L^p)) \longrightarrow \mathrm{Prod}(H^0(X,L^p)). \tag{1.7}$$

A fixed point $H \in \mathrm{Prod}(H^0(X,L^p))$ of this map is called an *anticanonically balanced product*. It has been introduced by Donaldson [16, 17] for various different volume forms in the Hilbert product (1.3), and has been used as a dynamical system approximating the corresponding balanced metric, seen as the Fubini–Study metric $\mathrm{FS}(H) \in \mathrm{Met}^+(L^p)$ associated with a fixed point. Our main result in this context is the following, where we use the natural distance on $\mathrm{Prod}(H^0(X,L^p))$ as a symmetric space.

Theorem 1.2 *Let X be a Fano manifold with discrete automorphism group and admitting a polarized Kähler–Einstein metric. Then, for any* $p \in \mathbb{N}$ *big enough, there exists* $\beta_p \in]0,1[$ *such that for any* $H_0 \in \mathrm{Prod}(H^0(X,L^p))$*, there exists an anticanonically balanced product* $H \in \mathrm{Prod}(H^0(X,L^p))$ *and a constant* $C > 0$ *such that for all* $k \in \mathbb{N}$*, we have*

$$\mathrm{dist}\left(\mathscr{T}_\nu^k(H_0)\,,\ H\right) \leq C\beta_p^k. \tag{1.8}$$

Furthermore, the constant $\beta_p \in]0,1[$ *satisfies the following estimate as* $p \to +\infty$*,*

$$\beta_p = 1 - \frac{\lambda_1 - 4\pi}{4\pi p} + O(p^{-2}), \tag{1.9}$$

where $\lambda_1 > 4\pi$ is the first positive eigenvalue of the Riemannian Laplacian associated with the polarized Kähler–Einstein metric acting on $\mathscr{C}^\infty(X, \mathbb{C})$, and this estimate is sharp.

This extends the results of [24,Th. 4.4, Rmk. 4.12] to the anticanonical setting. As explained in Remark 4.8, this confirms a prediction of Donaldson in [17] on the compared rates of convergence of the iterations associated with various notions of balanced products. Note that the *smooth* convergence of the Kähler forms in Theorem 1.1 is necessary to compute the rate of convergence (1.9). On the other hand, the proof of simple convergence in Theorem 1.2 follows from the work of Berman in [3,Prop. 2.9], and is based on the convexity of an appropriate energy functional, which has been established in [4,Lemma 7.2] based on the results of Berndtsson [6, 7] on the positivity of direct images. Note that the exponential convergence of the iterations follows from the estimate (1.9) thanks to the strict lower bound $\lambda_1 > 4\pi$ on the first positive eigenvalue of the Kähler–Einstein Laplacian, which holds under the necessary assumption of discrete automorphism group as a consequence of a classical result of Lichnerowicz [27] and Matsushima [33]. This lower bound plays a fundamental role in the proofs of both Theorems 1.1 and 1.2, in particular in Proposition 3.5 to construct approximately balanced metrics and in Proposition 3.9 via the asymptotics of the spectral gap of the quantum channel. Theorem 1.2 also complements the work of Liu and Ma in [28], who established the convergence of the refined approximations of Donaldson in [17,§ 2.2.1].

The advantage of our proof of Theorem 1.1 is that it can be adapted in a systematic way to various choices of a volume form in the Hilbert product (1.3), leading to the various notions of balanced metrics. In Sect. 2, we give the general setup for an arbitrary *volume map* (2.1) on the space $\mathrm{Met}^+(L)$ of positive Hermitian metrics on an ample holomorphic line bundle L over a compact complex manifold X. This includes in particular the *ν-balanced metrics* on Calabi–Yau manifolds and the *canonically balanced metrics* on manifolds with ample canonical line bundle, introduced by Donaldson [17] and which we describe in Examples 2.5 and 2.6. The proof given in Sect. 3 can readily be adapted to these two cases, which do not need any assumption on the automorphism group and are in fact easier. We present the proof in the case of Fano manifolds only because it is the most delicate one, as the Kähler–Einstein metric does not exist a priori. The smooth convergence of ν-balanced metrics to the polarized *Yau metric* associated with $d\nu$ has been outlined by Donaldson [17,§ 2.2], and then established by Keller [25,Th. 4.2] as a consequence of a result of Wang [45]. The differential of the associated moment map at a ν-balanced embedding has been studied by Keller et al. [26,§ 6.2]. On manifolds with ample canonical line bundle, the uniform convergence of canonically balanced metrics to the polarized Kähler–Einstein metric, which always exists in that case, follows from works of Tsuji [44] and Berndtsson. Our method gives smooth convergence, and also establishes the uniform convergence for anticanonically balanced metrics on Fano manifolds. Finally, our method also applies to the case of coupled Kähler–Einstein metrics considered by Takahashi [40].

The adaptation of our proof for the original notion of balanced metrics requires a refined estimate on the spectral gap of the quantum channel, which we establish in [23,Th. 4.11]. Note that the use of the Kähler–Einstein Laplacian, which is of order two, replaces in the anticanonical setting the use of the *Lichnerowicz operator*, which is of order four, in the original setting of Donaldson. On the other hand, following the works of Berman and Witt Nyström [5] and Takahashi [39], we use in [22] the method of the present paper to handle the case of general automorphism groups, replacing Kähler–Einstein metrics by *Kähler–Ricci solitons*. Finally, we also hope to apply our method to the case of *metaplectically balanced metrics*, giving an approximation of the *Cahen–Gutt moment map* and involving a differential operator of order six, following the program of Futaki and La Fuente-Gravy outlined in [19, 20].

The theory of Berezin–Toeplitz quantization has first been developed by Bordemann, Meinrenken and Schlichenmaier [8], using the work of Boutet de Monvel and Sjöstrand on the Szegö kernel in [11] and the theory of Toeplitz structures of Boutet de Monvel and Guillemin [10]. This paper is based instead on the theory of Ma and Marinescu [31], which uses the off-diagonal asymptotic expansion of the Bergman kernel established by Dai et al. [14,Th. 4.18'] and which holds for an arbitrary volume form in the Hilbert product (1.3). A comprehensive introduction of this theory can be found in the book [30]. The point of view of quantum measurement theory on Berezin–Toeplitz quantization, which we adopt in this paper, has been advocated by Polterovich [35, 36].

2 General setup

In this section, we consider a compact complex manifold X with $\dim_{\mathbb{C}} X = n$ endowed with an ample line bundle L, together with a smooth map

$$\begin{aligned} \nu : \mathrm{Met}^+(L) &\longrightarrow \mathscr{M}(X) \\ h &\longmapsto d\nu_h, \end{aligned} \tag{2.1}$$

from the space $\mathrm{Met}^+(L)$ of positive Hermitian metrics on L to the space $\mathscr{M}(X)$ of smooth volume forms over X. Such a map is called a *volume map*. For any $h \in \mathrm{Met}^+(L)$, we write $\mathrm{Vol}(d\nu_h) > 0$ for the volume of $d\nu_h \in \mathscr{M}(X)$.

For any $h \in \mathrm{Met}^+(L)$ and $p \in \mathbb{N}$, we write $h^p \in \mathrm{Met}^+(L^p)$ for the induced positive Hermitian metric on the p-th tensor power L^p. Conversely, any $h^p \in \mathrm{Met}^+(L^p)$ uniquely determines a positive Hermitian metric $h \in \mathrm{Met}^+(L)$. We write $\mathscr{C}^\infty(X, L^p)$ for the space of smooth sections of L^p and

$$H^0(X, L^p) \subset \mathscr{C}^\infty(X, L^p) \tag{2.2}$$

for the subspace of holomorphic sections of L^p over X. We set

$$n_p := \dim H^0(X, L^p). \tag{2.3}$$

2.1 Balanced metrics

Recall from the classical *Kodaira embedding theorem* that a holomorphic line bundle L is ample if and only if for all $p \in \mathbb{N}$ big enough, the evaluation map $\mathrm{ev}_x : H^0(X, L^p) \to L^p_x$ is surjective for all $x \in X$ and the induced *Kodaira map*

$$\begin{aligned} \mathrm{Kod}_p : X &\longrightarrow \mathbb{P}(H^0(X, L^p)^*), \\ x \; &\longmapsto \{ s \in H^0(X, L^p) \mid s(x) = 0 \} \end{aligned} \tag{2.4}$$

is an embedding. In this section, we fix such a $p \in \mathbb{N}$.

We denote by $\mathrm{Prod}(H^0(X, L^p))$ the space of Hermitian inner products on $H^0(X, L^p)$, and for any $H \in \mathrm{Prod}(H^0(X, L^p))$, we denote by $\mathscr{L}(H^0(X, L^p), H)$ the space of endomorphisms on $H^0(X, L^p)$ which are Hermitian with respect to H. In the following definition, we introduce the basic tools of this paper. Their names will be justified in the next section.

Definition 2.1 The *coherent state projector* associated with $H \in \mathrm{Prod}(H^0(X, L^p))$ is the map

$$\Pi_H : X \longrightarrow \mathscr{L}(H^0(X, L^p), H) \tag{2.5}$$

sending $x \in X$ to the orthogonal projector with respect to H satisfying

$$\operatorname{Ker} \Pi_H(x) = \{ s \in H^0(X, L^p) \mid s(x) = 0 \}. \tag{2.6}$$

The *Berezin symbol* associated with $H \in \operatorname{Prod}(H^0(X, L^p))$ is the map

$$\begin{aligned} \sigma_H : \mathscr{L}(H^0(X, L^p), H) &\longrightarrow \mathscr{C}^\infty(X, \mathbb{R}) \\ A &\longmapsto \operatorname{Tr}[A \Pi_H]. \end{aligned} \tag{2.7}$$

Note that the subspace (2.6) is the hyperplane $\operatorname{Kod}_p(x) \subset H^0(X, L^p)$ given by the Kodaira map (2.4), and the coherent state projector $\Pi_H(x)$ is thus a rank-1 projector, for all $x \in X$.

Recall that L^p is identified with the pullback of the dual *tautological line bundle* over $\mathbb{P}(H^0(X, L^p))$ via the Kodaira map (2.4). Thus, given $H \in \operatorname{Prod}(H^0(X, L^p))$, the induced *Fubini–Study metric* on the dual of the tautological line bundle pulls back to a positive Hermitian metric on L^p. Using the coherent state projector of Definition 2.1, this translates into the following definition.

Definition 2.2 The *Fubini–Study map* is the map

$$\mathrm{FS} : \operatorname{Prod}(H^0(X, L^p)) \longrightarrow \operatorname{Met}^+(L^p), \tag{2.8}$$

sending $H \in \operatorname{Prod}(H^0(X, L^p))$ to the positive Hermitian metric $\mathrm{FS}(H) \in \operatorname{Met}^+(L^p)$ on L^p defined for any $s_1, s_2 \in H^0(X, L^p)$ and $x \in X$ by

$$\langle s_1(x), s_2(x) \rangle_{\mathrm{FS}(H)} := \langle \Pi_H(x)\, s_1,\, s_2 \rangle_H. \tag{2.9}$$

Recall on the other hand the definition (1.3) of the *Hilbert map*

$$\operatorname{Hilb}_\nu : \operatorname{Met}^+(L^p) \longrightarrow \operatorname{Prod}(H^0(X, L^p)), \tag{2.10}$$

which holds for a general volume map (2.1). We are now ready to introduce the main concept of this paper.

Definition 2.3 A Hermitian metric $h^p \in \operatorname{Met}^+(L^p)$ is called *balanced* with respect to $\nu : \operatorname{Met}^+(L) \to \mathscr{M}(X)$ if it satisfies

$$\mathrm{FS} \circ \operatorname{Hilb}_\nu(h^p) = h^p. \tag{2.11}$$

A Hermitian product $H \in \operatorname{Prod}(H^0(X, L^p))$ is called *balanced* with respect to $\nu : \operatorname{Met}^+(L) \to \mathscr{M}(X)$ if it satisfies

$$\operatorname{Hilb}_\nu \circ \mathrm{FS}(H) = H. \tag{2.12}$$

Note that if $H \in \operatorname{Prod}(H^0(X, L^p))$ is a balanced product, then $\mathrm{FS}(H) \in \operatorname{Met}^+(L^p))$ is a balanced metric, and conversely, if $h^p \in \operatorname{Met}^+(L^p)$ is a balanced metric, then $\operatorname{Hilb}_\nu(h^p) \in \operatorname{Prod}(H^0(X, L^p))$ is a balanced product.

Example 2.4 The most fundamental example of a volume map is the *Liouville volume map*

$$\begin{aligned} \nu : \operatorname{Met}^+(L) &\longrightarrow \mathscr{M}(X) \\ h &\longmapsto d\nu_h := \frac{\omega_h^n}{n!}. \end{aligned} \tag{2.13}$$

Note that in that case, the volume $\operatorname{Vol}(X, L) := \operatorname{Vol}(d\nu_h) > 0$ does not depend on $h \in \operatorname{Met}^+(L)$. The analogue of Theorem 1.1 in this context, where the limit metric is a polarized

Kähler metric of *constant scalar curvature*, has been established by Donaldson [15]. The simple convergence of the associated Donaldson iterations as in Sect. 4 has been established by Donaldson [16] and Sano [37,Th. 1.2].

Example 2.5 The simplest example of a volume map is the volume map with a constant value $d\nu \in \mathscr{M}(X)$ not depending on $h \in \mathrm{Met}^+(L)$. Balanced metrics in this context are called ν-balanced metrics, and have first been studied by Bourguignon, Li and Yau [9]. Donaldson apply them in [17] to study the polarized *Yau metric* [46] associated with $d\nu$, which always exists and is defined as the unique polarized Kähler metric such that

$$\frac{\omega_h^n}{n!} = c\, d\nu, \tag{2.14}$$

for some multiplicative constant $c > 0$. This is of specific interest in case X is a *Calabi-Yau manifold,* so that its canonical line bundle K_X is trivial and one can take $d\nu := \sqrt{-1}^{n^2}\,\theta \wedge \overline{\theta}$, where $\theta \in H^0(X, K_X)$ is the unique nowhere vanishing section of K_X up to a multiplicative constant. Then, the polarized Yau metric coincides with the polarized *Ricci-flat metric*. The smooth convergence of the ν-balanced metrics toward the Yau metric as $p \to +\infty$ has been established by Donaldson [17,§ 2.2] and by Keller [25]. In that case, the assumption on the automorphism group is not needed. The simple convergence of the associated Donaldson iterations as in Sect. 4 has been established by Donaldson [17,Prop. 4], and exponential convergence as well as the asymptotics of the optimal rate of convergence have been worked out in [24,Th. 3.1, Rmk. 4.12].

Example 2.6 In case the *canonical line bundle* $L := K_X$ of X is ample, one can consider the *canonical volume map*, sending a positive Hermitian metric $h \in \mathrm{Met}(K_X)$ to the induced volume form defined analogously to (1.4) over any contractible $U \subset X$ via a non-vanishing $\theta \in \mathscr{C}^\infty(U, K_X)$ by

$$d\nu_h := \sqrt{-1}^{n^2}\,\frac{\theta \wedge \overline{\theta}}{|\theta|_h^2}. \tag{2.15}$$

In that case, the polarized Kähler–Einstein metric always exists by a result of Aubin [1] and Yau [46]. The uniform convergence of balanced metrics to the Kähler–Einstein metric as $p \to +\infty$ in this setting has been established by Tsuji [44] and Berndtsson (see also [4,Th. 7.1] for another proof of the convergence in the weak sense of currents). Once again, the assumption on the automorphism group is not needed in that case.

The dual version, when $L := K_X^*$ is ample, uses the *anticanonical volume map* (1.4). Theorem 1.1 on the smooth convergence of the balanced metrics to the polarized Kähler–Einstein metric as $p \to +\infty$ in this setting is the main result of this paper. The exponential convergence of Donaldson's iterations in this context is the result of Theorem 1.2. Note that in this case, and by contrast with the case K_X ample described above, even if we assume that the automorphism group is discrete, Tian showed in [42] that a Kähler–Einstein metric does not exist in general.

2.2 Berezin–Toeplitz quantization

In this section, we fix a positive Hermitian metric $h \in \mathrm{Met}^+(L)$ and assume that $p \in \mathbb{N}$ is big enough so that the Kodaira map (2.4) is well defined and an embedding. We consider the

Hermitian product $L^2(h^p) \in \mathrm{Prod}(H^0(X, L^p))$ defined for any $s_1,\ s_2 \in \mathscr{C}^\infty(X, L^p)$ by

$$\langle s_1, s_2\rangle_{L^2(h^p)} := \int_X \langle s_1(x), s_2(x)\rangle_{h^p}\, \mathrm{d}\nu_h(x). \tag{2.16}$$

We write

$$\mathscr{H}_p := \left(H^0(X, L^p), \langle\cdot,\cdot\rangle_{L^2(h^p)}\right), \tag{2.17}$$

for the associated Hilbert space of holomorphic sections. We write $\mathscr{L}(\mathscr{H}_p)$ for the space of Hermitian endomorphisms of $\mathscr{H}_p$, and

$$\Pi_p : X \longrightarrow \mathscr{L}(\mathscr{H}_p), \tag{2.18}$$

for the associated coherent projector of Definition 2.1. From the point of view of quantum mechanics, this coherent state projector induces a *quantization* of the symplectic manifold (X, ω_h), seen as a classical phase space. A fundamental property in this respect is the following result.

Proposition 2.7 *There exists a unique positive function $\rho_{h^p} \in \mathscr{C}^\infty(X, \mathbb{R})$, called the* Rawnsley (or density of states) function, *such that for any $s_1,\ s_2 \in \mathscr{H}_p$ and $x \in X$, we have*

$$\rho_{h^p}(x)\, \langle \Pi_p(x)s_1, s_2\rangle_{L^2(h^p)} = \langle s_1(x), s_2(x)\rangle_{h^p}. \tag{2.19}$$

In particular, we have

$$\int_X \Pi_p(x)\, \rho_{h^p}(x)\, d\nu_h(x) = \mathrm{Id}_{\mathscr{H}_p}. \tag{2.20}$$

Proof For any $x \in X$, consider the associated evaluation map $\mathrm{ev}_x : \mathscr{H}_p \to L^p_x$, and write $\mathrm{ev}^*_x : L^p_x \to \mathscr{H}_p$ for its dual with respect to h^p and $L^2(h^p)$. Then, for any $s_1,\ s_2 \in \mathscr{H}_p$, we have by definition

$$\langle s_1(x), s_2(x)\rangle_{h^p} = \langle \mathrm{ev}^*_x\, \mathrm{ev}_x\, s_1, s_2\rangle_{L^2(h^p)}. \tag{2.21}$$

By Definition 2.1, the endomorphisms $\mathrm{ev}^*_x\, \mathrm{ev}_x$ and $\Pi_p(x)$ have same kernel in $\mathscr{H}_p$, given by the hyperplane $\mathrm{Kod}_p(x) \subset H^0(X, L^p)$ image of $x \in X$ by the Kodaira map (2.4). As they are both Hermitian, they also have same 1-dimensional image in $\mathscr{H}_p$, so that there exists a unique positive number $\rho_{h^p}(x) > 0$ such that

$$\rho_{h^p}(x)\, \Pi_p(x) = \mathrm{ev}^*_x\, \mathrm{ev}_x\,. \tag{2.22}$$

As they both depend smoothly on $x \in X$, this defines a unique smooth positive function $\rho_{h^p} \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfying formula (2.19). The identity (2.20) then follows by integrating formula (2.19) against $\mathrm{d}\nu_h$ via the definition (2.16) of $L^2(h^p)$. $\square$

The fundamental role played by the Rawnsley function in the study of the balanced metrics of Definition 2.3 comes from the following basic result.

Proposition 2.8 *A positive Hermitian metric $h^p \in \mathrm{Met}^+(L^p)$ is balanced with respect to $\nu : \mathrm{Met}^+(L) \to \mathscr{M}(X)$ if and only if for all $x \in X$, the associated Rawnsley function $\rho_{h^p} \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfies*

$$\rho_{h^p}(x) = \frac{n_p}{\mathrm{Vol}(d\nu_h)}. \tag{2.23}$$

Proof By definition, we have

$$\mathrm{Hilb}_\nu(h^p) = \frac{n_p}{\mathrm{Vol}(d\nu_h)}\, L^2(h^p), \tag{2.24}$$

so that by Definition 2.2 and Proposition 2.7, for all $s_1,\ s_2 \in H^0(X, L^p)$ and $x \in X$ we have

$$\rho_{h^p}(x)\, \langle s_1(x), s_2(x)\rangle_{\mathrm{FS}(\mathrm{Hilb}_\nu(h^p))} = \frac{n_p}{\mathrm{Vol}(d\nu_h)}\, \langle s_1(x), s_2(x)\rangle_{h^p}. \tag{2.25}$$

This gives the result by Definition 2.3 of a balanced metric. □

Proposition 2.7 describes fundamental properties of a *coherent state quantization*, given in our context by the following Definition.

Definition 2.9 The *Berezin–Toeplitz quantization map* is defined by

$$\begin{aligned} T_{h^p} : \mathscr{C}^\infty(X,\mathbb{R}) &\longrightarrow \mathscr{L}(\mathscr{H}_p). \\ f &\longmapsto \int_X f(x)\, \Pi_p(x)\, \rho_{h^p}(x)\, d\nu_h(x) \end{aligned} \tag{2.26}$$

Using Proposition 2.7, we have the following characterization of the Berezin–Toeplitz quantization of $f \in \mathscr{C}^\infty(X,\mathbb{R})$, for all $s_1,\ s_2 \in \mathscr{H}_p$,

$$\begin{aligned} \langle T_{h^p}(f)s_1, s_2\rangle_{L^2(h^p)} &= \int_X f(x)\, \langle \Pi_p(x)s_1, s_2\rangle_{L^2(h^p)}\, \rho_{h^p}(x)\, d\nu_h(x) \\ &= \int_X f(x)\, \langle s_1(x), s_2(x)\rangle_{h^p}\, d\nu_h(x). \end{aligned} \tag{2.27}$$

This shows that Definition 2.9 coincides with the usual definition of Berezin–Toeplitz quantization associated with the volume form $d\nu_h \in \mathscr{M}(X)$, as described in [30,Chap. 7]. In the same way, one readily checks that the Rawnsley function of Proposition 2.7 coincides with the associated *Bergman kernel* along the diagonal, as described in [30,Chap. 4]. We will give in Proposition 2.16 its geometric description as a density of states.

In the context of quantization, the Berezin symbol (2.6) of a quantum observable $A \in \mathscr{L}(\mathscr{H}_p)$ is interpreted as the classical observable given by the expectation value of A at coherent states. This gives rise to the following concept, which will be the main tool of this paper.

Definition 2.10 The *Berezin–Toeplitz quantum channel* is the linear operator

$$\begin{aligned} \mathscr{E}_{h^p} : \mathscr{L}(\mathscr{H}_p) &\longrightarrow \mathscr{L}(\mathscr{H}_p), \\ A &\longmapsto T_{h^p}\left(\sigma_{L^2(h^p)}(A)\right). \end{aligned} \tag{2.28}$$

In the context of quantum measurement theory, the quantum channel describes the effect of a measurement on quantum observables. The basic properties of the Berezin–Toeplitz quantum channel have been studied in [24], based on [8]. They are summarized in the following proposition.

Proposition 2.11 *The Berezin–Toeplitz quantum channel $\mathscr{E}_{h^p}$ is a positive self-adjoint operator on the real Hilbert space $\mathscr{L}(\mathscr{H}_p)$ equipped with the trace norm, and its eigenvalues $\{\gamma_k(h^p)\}_{k=1}^{n_p^2}$ counted with multiplicities satisfy*

$$1 = \gamma_0(h^p) > \gamma_1(h^p) \geq \gamma_2(h^p) \geq \cdots \geq \gamma_{n_p^2}(h^p) > 0, \tag{2.29}$$

where $1 = \gamma_0(h^p)$ is associated with the eigenvector $\mathrm{Id}_{\mathscr{H}_p} \in \mathscr{L}(\mathscr{H}_p)$.

Proof By Definitions 2.1 and 2.9, for any $A,\ B \in \mathscr{L}(\mathscr{H}_p)$, we have

$$\mathrm{Tr}\,[A\,\mathscr{E}_{h^p}\,(B)] = \int_X \mathrm{Tr}[A\Pi_p(x)]\;\mathrm{Tr}[B\Pi_p(x)]\,\rho_{h^p}(x)\,\mathrm{d}\nu_h(x), \tag{2.30}$$

so that as $\rho_{h^p} > 0$ by definition, the quantum channel $\mathscr{E}_{h^p}$ is positive and self-adjoint for the trace norm on $\mathscr{L}(\mathscr{H}_p)$. Furthermore, as $\mathrm{Tr}[\Pi_p(x)] = 1$ for all $x \in X$, we see from Proposition 2.7 that $\mathscr{E}_{h^p}(\mathrm{Id}_{\mathscr{H}_p}) = \mathrm{Id}_{\mathscr{H}_p}$. The injectivity of $\mathscr{E}_{h^p}$ and the fact that $\gamma_1(h^p) < 1$ follow from the results of [24,Ex. 4.1, Props. 4.7, 4.8]. □

The positive number $\gamma := 1 - \gamma_1(h^p) > 0$ is called the *spectral gap* of the quantum channel, and it measures the loss of information associated with repeated quantum measurements. The following estimate on its *semiclassical limit* as $p \to +\infty$ is central to this paper.

Theorem 2.12 [24,Th. 3.1, Rmk. 3.12] *There exists a constant $C > 0$ such that for all $p \in \mathbb{N}$, we have*

$$\left|1 - \gamma_1(h^p) - \frac{\lambda_1(h)}{4\pi p}\right| \leq \frac{C}{p^2}, \tag{2.31}$$

where $\lambda_1(h) > 0$ is the first positive eigenvalue of the Riemannian Laplacian of (X, g_h^{TX}) acting on $\mathscr{C}^\infty(X, \mathbb{C})$.

Moreover, there exists $l \in \mathbb{N}$ such that for any bounded subset $K \subset \mathrm{Met}^+(L)$ in $\mathscr{C}^l$-norm over which the volume map (2.1) is bounded from below, the constant $C > 0$ can be chosen uniformly in $h \in K$.

The uniformity in the metric is not explicitly stated in [24,Th. 3.1], but as noted in [24,Rmk. 4.9], it readily follows from the uniformity in the metric of the estimates on the Bergman kernel of [14,Th. 4.18'].

Furthermore, as explained in [24,Rmk. 3.12], the case of a general volume form $\mathrm{d}\nu_h \in \mathscr{M}(X)$ follows from a trick due to Ma and Marinescu in [30,§ 4.1.9]. This trick is based on the fact that the L^2-product (2.16) coincides with the L^2-product associated with the Liouville form $\omega_h^n/n!$ and the Hermitian metric $h^p \otimes h^E$ on $L^p \otimes E$, where $E = \mathbb{C}$ is the trivial line bundle and $h^E \in \mathrm{Met}^+(E)$ is defined by $|1|_{h^E}^2\,\omega_h^n/n! := \mathrm{d}\nu_h$. This implies in particular that the Rawnsley function $\widetilde{\rho}_{h^p} \in \mathscr{C}^\infty(X, \mathbb{R})$ associated with $\omega_h^n/n!$ and $h^p \otimes h^E$ as above satisfies

$$\rho_{h^p}\,\mathrm{d}\nu_h = \widetilde{\rho}_{h^p}\,\frac{\omega_h^n}{n!}. \tag{2.32}$$

This gives the following version of a classical result on the asymptotics as $p \to +\infty$ of the Rawnsley function, which is the other crucial estimate needed in this paper.

Theorem 2.13 [14,Th. 1.3] *There exist functions $b_r(h) \in \mathscr{C}^\infty(X, \mathbb{R})$ for all $r \in \mathbb{N}$ such that for any $m,\ k \in \mathbb{N}$, there exists $C_{m,k} > 0$ such that for all $p \in \mathbb{N}$ big enough,*

$$\left|\rho_{h^p} - p^n \sum_{r=0}^{k-1} \frac{1}{p^r} b_r(h)\right|_{\mathscr{C}^m} \leq \frac{C_{m,k}}{p^k}, \tag{2.33}$$

Furthermore, the functions $b_r(h) \in \mathscr{C}^\infty(X, \mathbb{R})$, $r \in \mathbb{N}$, depend polynomially on $h \in \mathrm{Met}^+(L)$ and its successive derivatives along X, and the function $b_0(h) \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfies the identity

$$b_0(h)\,d\nu_h = \frac{\omega_h^n}{n!}. \tag{2.34}$$

Finally, for each $m, k \in \mathbb{N}$, *there exists* $l \in \mathbb{N}$ *such that for any bounded subset* $K \subset \mathrm{Met}^+(L)$ *in* $\mathscr{C}^l$*-norm over which the volume map* (2.1) *is bounded from below, the constant* $C_{m,k} > 0$ *can be chosen uniformly in* $h \in K$.

In particular, using Proposition 2.7 and the fact that $\mathrm{Tr}[\Pi_p] = 1$, Theorem 2.13 implies that the dimension of $\mathscr{H}_p$ satisfies the following estimate as $p \to +\infty$,

$$n_p = \mathrm{Tr}[\mathrm{Id}_{\mathscr{H}_p}] = \int_X \rho_{h^p}(x)\, \mathrm{d}\nu_h(x) = p^n \,\mathrm{Vol}(X, L) + O(p^{n-1}), \tag{2.35}$$

where $\mathrm{Vol}(X, L) > 0$ is the volume of the Liouville volume map (2.13), which does not depend on $h \in \mathrm{Met}^+(L)$.

2.3 Moment map

In this section, we fix $p \in \mathbb{N}$ big enough so that the Kodaira map (2.4) is well defined and an embedding, and we consider the space $\mathscr{B}(H^0(X, L^p))$ of bases of $H^0(X, L^p)$. For any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$, we write $H_\mathbf{s} \in \mathrm{Prod}(H^0(X, L^p))$ for the Hermitian product for which it is an orthonormal basis, and write $h_\mathbf{s} \in \mathrm{Met}^+(L)$ for the positive Hermitian metric defined through Definition 2.2 by the formula

$$h_\mathbf{s}^p := \mathrm{FS}(H_\mathbf{s}) \in \mathrm{Met}^+(L^p). \tag{2.36}$$

Write $\mathrm{Herm}(\mathbb{C}^{n_p})$ for the space of Hermitian matrices of $\mathbb{C}^{n_p}$. The following central tool in the study of balanced metrics has been introduced by Donaldson [15, 17] in his moment map picture for the study of canonical Kähler metrics.

Definition 2.14 The *moment map* associated with $\nu : \mathrm{Met}^+(L) \to \mathscr{M}(X)$ is the map

$$\mu_\nu : \mathscr{B}(H^0(X, L^p)) \longrightarrow \mathrm{Herm}(\mathbb{C}^{n_p}) \tag{2.37}$$

defined for all $\mathbf{s} = \{s_j\}_{j=1}^{n_p} \in \mathscr{B}(H^0(X, L^p))$ by the formula

$$\mu_\nu(\mathbf{s}) := \left(\int_X \langle s_j(x), s_k(x) \rangle_{h_\mathbf{s}^p} \, \mathrm{d}\nu_{h_\mathbf{s}}(x) \right)_{j,k=1}^{n_p} - \frac{\mathrm{Vol}(\mathrm{d}\nu_{h_\mathbf{s}})}{n_p} \mathrm{Id}_{\mathbb{C}^{n_p}}. \tag{2.38}$$

The fundamental role of this moment map in the study of the balanced products of Definition 2.3 comes from the following basic result.

Proposition 2.15 *For any* $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$, *the induced Hermitian product* $H_\mathbf{s} \in \mathrm{Prod}(H^0(X, L^p))$ *is balanced with respect to* $\nu : \mathrm{Met}^+(L) \to \mathscr{M}(X)$ *if and only if*

$$\mu_\nu(\mathbf{s}) = 0. \tag{2.39}$$

Proof Comparing Definition 2.2 and formula (1.3) with Definition 2.14 and formula (2.36), we see that $\mathbf{s} = \{s_j\}_{j=1}^{n_p} \in \mathscr{B}(H^0(X, L^p))$ satisfies $\mu_\nu(\mathbf{s}) = 0$ if and only if

$$\left(\langle s_j, s_k \rangle_{\mathrm{Hilb}_\nu(\mathrm{FS}(H_\mathbf{s}))} \right)_{j,k=1}^{n_p} = \mathrm{Id}_{\mathbb{C}^{n_p}}, \tag{2.40}$$

i.e., if and only if $\mathbf{s} = \{s_j\}_{j=1}^{n_p}$ is an orthonormal basis for $\mathrm{Hilb}_\nu(\mathrm{FS}(H_\mathbf{s})) \in \mathrm{Prod}(H^0(X, L^p))$. But this property characterizes $H_\mathbf{s} \in \mathrm{Prod}(H^0(X, L^p))$, so that $\mu_\nu(\mathbf{s}) = 0$ if and only if

$$\mathrm{Hilb}_\nu(\mathrm{FS}(H_\mathbf{s})) = H_\mathbf{s}, \tag{2.41}$$

which is Definition 2.3 of a balanced product. □

In the following proposition, we give useful characterizations for the Fubini–Study metric of Definition 2.2 and the Rawnsley function of Proposition 2.7 in terms of bases of $H^0(X, L^p)$, recovering their familiar descriptions in this context.

Proposition 2.16 *For any* $h^p \in \mathrm{Met}^+(L^p)$*, the associated Rawnsley function* $\rho_{h^p} \in \mathscr{C}^\infty(X, L^p)$ *is given for any* $x \in X$ *by the formula*

$$\rho_{h^p}(x) = \sum_{j=1}^{n_p} |s_j(x)|^2_{h^p}, \tag{2.42}$$

where $\{s_j\}_{j=1}^{n_p} \in \mathscr{B}(H^0(X, L^p))$ *is an orthonormal basis for* $L^2(h^p)$.

For any basis $\mathbf{s} = \{s_j\}_{j=1}^{n_p} \in \mathscr{B}(H^0(X, L^p))$*, the induced Fubini–Study metric* $h^p_{\mathbf{s}} \in \mathrm{Met}^+(L^p)$ *is characterized by the following formula, for any* $x \in X$*,*

$$\sum_{j=1}^{n_p} |s_j(x)|^2_{h^p_{\mathbf{s}}} = 1. \tag{2.43}$$

In particular, we have $\mathrm{Tr}[\mu_\nu(\mathbf{s})] = 0$ *for all* $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$.

Proof By Proposition 2.7, if $\{s_j\}_{j=1}^{n_p} \in \mathscr{B}(H^0(X, L^p))$ is an orthonormal basis for $L^2(h^p)$, then we have

$$\sum_{j=1}^{n_p} |s_j|^2_{h^p} = \sum_{j=1}^{n_p} \rho_{h^p} \langle \Pi_p s_j, s_j \rangle_{L^2(h^p)} = \rho_{h^p}\, \mathrm{Tr}[\Pi_p] = \rho_{h^p}, \tag{2.44}$$

which shows formula (2.42). On the other hand, any $\mathbf{s} = \{s_j\}_{j=1}^{n_p} \in \mathscr{B}(H^0(X, L^p))$ is by definition an orthonormal basis for $H_{\mathbf{s}} \in \mathrm{Prod}(H^0(X, L^p))$, so that by Definition 2.2 we get

$$\sum_{j=1}^{n_p} |s_j|^2_{h^p_{\mathbf{s}}} = \sum_{j=1}^{n_p} \langle \Pi_{H_{\mathbf{s}}} s_j, s_j \rangle_{H_{\mathbf{s}}} = \mathrm{Tr}[\Pi_{H_{\mathbf{s}}}] = 1, \tag{2.45}$$

which clearly characterizes $h^p_{\mathbf{s}} \in \mathrm{Met}^+(L^p)$. By Definition 2.14, this readily implies $\mathrm{Tr}[\mu_\nu(\mathbf{s})] = 0$. □

Recall Definition 2.1 for the Berezin symbol associated with a Hermitian product $H \in \mathrm{Prod}(H^0(X, L^p))$.

Proposition 2.17 *For any* $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$ *and* $B \in \mathscr{L}(H^0(X, L^p), H_{\mathbf{s}})$*, we have*

$$\sigma_{H_{\mathbf{s}}}(e^{2B})\, h^p_{e^B \mathbf{s}} = h^p_{\mathbf{s}}. \tag{2.46}$$

Proof By Definitions 2.1 and 2.2, for any $B \in \mathscr{L}(H^0(X, L^p), H_{\mathbf{s}})$ and writing $\mathbf{s} = \{s_j\}_{j=1}^{n_p}$, we have

$$\begin{aligned} \sigma_{H_{\mathbf{s}}}(e^{2B}) &= \mathrm{Tr}[e^B \Pi_{H_{\mathbf{s}}} e^B] \\ &= \sum_{j=1}^{n_p} \langle \Pi_{H_{\mathbf{s}}} e^B s_j, e^B s_j \rangle_{H_{\mathbf{s}}} = \sum_{j=1}^{n_p} \left| e^B s_j \right|^2_{h^p_{\mathbf{s}}}. \end{aligned} \tag{2.47}$$

As $\{e^B s_j\}_{j=1}^{n_p}$ is an orthonormal basis for $H_{e^B \mathbf{s}} \in \mathrm{Prod}(H^0(X, L^p))$ by definition, this shows the result by the characterization of the Fubini–Study metric given in Proposition 2.16. □

Consider now the free and transitive action of $\mathrm{GL}(\mathbb{C}^{n_p})$ on $\mathscr{B}(H^0(X, L^p))$ via the formula

$$G.\mathbf{s} := \left\{\sum_{k=1}^{n_p} G_{jk} s_k\right\}_{j=1}^{n_p}, \tag{2.48}$$

for any $G = (G_{jk})_{j,k=1}^{n_p} \in \mathrm{GL}(\mathbb{C}^{n_p})$ and $\mathbf{s} = \{s_j\}_{j=1}^{n_p} \in \mathscr{B}(H^0(X, L^p))$. By derivation, this induces a canonical identification of tangent spaces

$$T_{\mathbf{s}}\mathscr{B}(H^0(X, L^p)) \simeq \mathrm{End}(\mathbb{C}^{n_p}), \tag{2.49}$$

making $\mathscr{B}(H^0(X, L^p))$ into a complete Riemannian manifold via the Hermitian product defined on $A, B \in \mathrm{End}(\mathbb{C}^{n_p})$ by the formula

$$\langle A, B\rangle_{tr} := \mathrm{Tr}[AB^*]. \tag{2.50}$$

Restricting to Hermitian matrices $\mathrm{Herm}(\mathbb{C}^{n_p}) \subset \mathrm{End}(\mathbb{C}^{n_p})$, this induces for all $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$ an isometry

$$\mathrm{Herm}(\mathbb{C}^{n_p}) \simeq \mathscr{L}(H^0(X, L^p), H_{\mathbf{s}}). \tag{2.51}$$

The unitary group $U(n_p) \subset \mathrm{GL}(\mathbb{C}^{n_p})$ acts by isometries on $\mathscr{B}(H^0(X, L^p))$, and the quotient map

$$\begin{aligned} \mathscr{B}(H^0(X, L^p)) &\longrightarrow \mathrm{Prod}(H^0(X, L^p)) \\ \mathbf{s} &\longmapsto H_{\mathbf{s}} \end{aligned} \tag{2.52}$$

makes in turn $\mathrm{Prod}(H^0(X, L^p))$ into a complete Riemannian manifold, whose geodesics are of the form

$$t \longmapsto H_{e^{tA}\mathbf{s}} \in \mathrm{Prod}(H^0(X, L^p)),\ t \in \mathbb{R}, \tag{2.53}$$

for all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})$.

We will write $\Pi_{\mathbf{s}} : X \to \mathrm{Herm}(\mathbb{C}^{n_p})$ and $\sigma_{\mathbf{s}} : \mathrm{Herm}(\mathbb{C}^{n_p}) \to \mathscr{C}^\infty(X, \mathbb{R})$ for the coherent state projector and the Berezin symbol of Definition 2.1 associated with $H_{\mathbf{s}} \in \mathrm{Prod}(H^0(X, L^p))$ under the identification (2.51) induced by any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$. In these notations, we have the following comparison formula for the Berezin symbols associated with two different bases in the corresponding identifications.

Proposition 2.18 *For any $A, B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ and $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$, we have*

$$\sigma_{e^B s}(A) = \sigma_s(e^{2B})^{-1}\,\sigma_s(e^B A e^B). \tag{2.54}$$

Proof Write $\mathbf{s} =: \{s_j\}_{j=1}^{n_p}$ and $e^B\mathbf{s} =: \{\widetilde{s}_j\}_{j=1}^{n_p}$, so that by definition (2.48) of the action and writing $e^B := (G_{jk})_{j,k=1}^{n_p}$, we have $\widetilde{s}_j = \sum_{k=1}^{n_p} G_{jk}s_k$ for all $1 \le j \le n_p$. Then, using Definition 2.2, Proposition 2.17 and the fact that $e^B \in \mathrm{GL}(\mathbb{C}^{n_p})$ is Hermitian, we get

$$\begin{aligned} \sigma_{e^B\mathbf{s}}(A) = \sum_{j,k=1}^{n_p} \langle A_{jk}\widetilde{s}_k, \widetilde{s}_j\rangle_{h^p_{e^B\mathbf{s}}} &= \sigma_{\mathbf{s}}(e^{2B})^{-1} \sum_{j,k,l,m=1}^{n_p} \langle A_{jk}G_{kl}s_l, G_{jm}s_m\rangle_{h^p_{\mathbf{s}}} \\ &= \sigma_{\mathbf{s}}(e^{2B})^{-1} \sum_{l,m=1}^{n_p} \left(e^B A e^B\right)_{ml} \langle s_l, s_m\rangle_{h^p_{\mathbf{s}}}. \end{aligned} \tag{2.55}$$

This implies the result by Definitions 2.1 and 2.2. $\square$

3 Anticanonically balanced metrics

In this section, we consider the general setup of Sect. 2 in the particular case when X is a Fano manifold, meaning that its anticanonical line bundle $K_X^* := \det(T^{(1,0)}X)$ is ample. We take $L := K_X^*$ and consider the anticanonical volume map $\nu : \mathrm{Met}(K_X^*) \longrightarrow \mathscr{M}(X)$ defined by formula (1.4).

3.1 Kähler–Einstein metrics and anticanonical volume map

A Kähler form $\omega \in \Omega^2(X, \mathbb{R})$ on a compact complex manifold X induces a natural Hermitian metric $h_\omega \in \mathrm{Met}(K_X^*)$, defined using the anticanonical volume form (1.4) by the formula

$$\frac{\omega^n}{n!} = d\nu_{h_\omega}. \tag{3.1}$$

Conversely, a positive Hermitian metric $h \in \mathrm{Met}^+(K_X^*)$ induces a Kähler form $\omega_h \in \Omega^2(X, \mathbb{R})$ as in (1.1), but ω_{h_ω} do not coincide with ω in general. This motivates the following important notion of Kähler geometry.

Definition 3.1 A positive Hermitian metric $h \in \mathrm{Met}^+(K_X^*)$ is called *Kähler–Einstein* if there exists a constant $c > 0$ such that the associated Kähler form ω_h satisfies

$$\frac{\omega_h^n}{n!} = c\, d\nu_h. \tag{3.2}$$

The associated polarized Kähler metric g_h^{TX} is then called a *Kähler–Einstein metric*.

Let us recall some basic facts about such Kähler–Einstein metrics, which can be found for instance in [38,Chap. 3–4]. First of all, for a positive Hermitian metric $h \in \mathrm{Met}^+(K_X^*)$ and in our convention (1.1) for the associated Kähler form $\omega_h \in \Omega^2(X, \mathbb{R})$, the Kähler–Einstein condition of Definition 3.1 is equivalent to the identity

$$\omega_h = \frac{1}{2\pi}\,\mathrm{Ric}(g_h^{TX}), \tag{3.3}$$

where $\mathrm{Ric}(g_h^{TX}) \in \Omega^2(X, \mathbb{R})$ is the *Ricci form* of (X, J, g_h^{TX}). This implies that the *scalar curvature* $\mathrm{scal}(g_h^{TX})$ of (X, g_h^{TX}) is constant, given by

$$\mathrm{scal}(g_h^{TX}) = 4\pi n. \tag{3.4}$$

We then have the following classical result of Lichnerowicz and Matsushima, in a form which can be found in [21,Chap. 3] and which will be a key input in our proof of Theorem 1.1. Write $\mathrm{Aut}(X)$ for the automorphism group of X as a complex manifold.

Theorem 3.2 [27, 33] *Assume that* $\mathrm{Aut}(X)$ *is discrete, and let* $h_\infty \in \mathrm{Met}^+(K_X^*)$ *be Kähler–Einstein. Then, the first positive eigenvalue* $\lambda_1(h_\infty) > 0$ *of the Riemannian Laplacian* Δ_{h_∞} *of* $(X, g_{h_\infty}^{TX})$ *acting on* $\mathscr{C}^\infty(X, \mathbb{C})$ *satisfies*

$$\lambda_1(h_\infty) > 4\pi. \tag{3.5}$$

We will often need the following variation formula for the anticanonical volume form (1.4).

Proposition 3.3 *The anticanonical volume form* (1.4) *satisfies the following formula, for any* $f \in \mathscr{C}^\infty(X,\mathbb{R})$ *and* $h \in \mathrm{Met}(K_X^*)$,

$$d\nu_{e^f h} = e^f \, d\nu_h. \tag{3.6}$$

Proof If $h^{-1} \in \mathrm{Met}(K_X)$ denotes the Hermitian metric induced by $h \in \mathrm{Met}(K_X^*)$, then for any $f \in \mathscr{C}^\infty(X,\mathbb{R})$ and $h \in \mathrm{Met}(K_X^*)$, we have $(e^f h)^{-1} = e^{-f} h^{-1}$. This readily implies the result by formula (1.4). □

Remark 3.4 Let $L := K_X^*$ be ample, and recall from Sect. 2.3 that we write $h_{\mathbf{s}} \in \mathrm{Met}^+(L)$ for the positive Hermitian metric induced by the Fubini–Study metric of $H_{\mathbf{s}}$, for any $\mathbf{s} = \{s_j\}_{j=1}^{n_p} \in \mathscr{B}(H^0(X, L^p))$. Restricted to such metrics, the anticanonical volume form (1.4) admits a metric-independent characterization. In fact, using Proposition 2.16, one computes

$$\mathrm{d}\nu_{h_{\mathbf{s}}} = \left(\sum_{j=1}^{n_p} s_j \otimes \bar{s}_j\right)^{-1/p}, \tag{3.7}$$

where the expression inside the parentheses in the last line is to be considered as a positive section of $L^p \otimes \overline{L}^p$ equipped with its natural $\mathbb{R}_+$-structure, so that its inverse p-th root defines a smooth form. These volume forms have been introduced by Donaldson in [17,§ 2.2.2] to approximate numerically Kähler–Einstein metrics on Fano manifolds, a program for which Theorems 1.1 and 1.2 provide a rigorous basis.

3.2 Approximately balanced metrics

Let X be a Fano manifold with $\mathrm{Aut}(X)$ discrete and admitting a Kähler–Einstein metric $h_\infty \in \mathrm{Met}^+(K_X^*)$. In this section, we consider the setting of Sect. 2 with $L := K_X^*$ for the anticanonical volume map (1.4).

The proof of the following result is parallel to the proof of the analogous result of Donaldson [15,Th. 26] in the case of Example 2.4, replacing the positivity of the Lichnerowicz operator by Theorem 3.2. All the local $\mathscr{C}^m$-norms are taken with respect to the fixed Kähler–Einstein metric.

Proposition 3.5 *There exists a sequence of functions* $f_r \in \mathscr{C}^\infty(X,\mathbb{R})$, $r \in \mathbb{N}$, *such that for every* $k,\ m \in \mathbb{N}$, *there exists a constant* $C_{k,m} > 0$ *such that all* $p \in \mathbb{N}$ *big enough, the positive Hermitian metric*

$$h_k(p) := \exp\left(\sum_{r=1}^{k-1} \frac{1}{p^r} f_r\right) h_\infty \in \mathrm{Met}^+(L^p), \tag{3.8}$$

have associated Rawnsley function $\rho_{h_p^k(p)} \in \mathscr{C}^\infty(X,\mathbb{R})$ *satisfying*

$$\left| \rho_{h_k^p(p)} - \frac{n_p}{\mathrm{Vol}(d\nu_{h_k(p)})} \right|_{\mathscr{C}^m} \leq C_{k,m}\, p^{n-k}. \tag{3.9}$$

Proof First note by Definition 3.1 of the Kähler–Einstein metric $h_\infty \in \mathrm{Met}^+(L)$ that the coefficient $b_0(h_\infty) \in \mathscr{C}^\infty(X,\mathbb{R})$ of Theorem 2.13 is constant. This implies the result for $k = 1$.

Let us write Δ_{h_∞} for the Riemannian Laplacian of $(X, g^{TX}_{h_\infty})$ acting on $\mathscr{C}^\infty(X, \mathbb{C})$. Using Proposition 3.3 and a classical formula in Kähler geometry, for any $f \in \mathscr{C}^\infty(X, \mathbb{R})$ we get

$$\frac{\partial}{\partial t}\Big|_{t=0} \frac{\omega^n_{e^{tf}h_\infty}}{\mathrm{d}\nu_{e^{tf}h_\infty}} = \left(\frac{1}{4\pi}\Delta_{h_\infty} f - f\right) \frac{\omega^n_{h_\infty}}{\mathrm{d}\nu_{h_\infty}}. \tag{3.10}$$

Recall by Definition 3.1 that the Riemannian volume form of $(X, g^{TX}_{h_\infty})$ is a constant multiple of $\mathrm{d}\nu_{h_\infty}$. Then, Theorem 3.2 shows that the restriction of the operator $\left(\frac{1}{4\pi}\Delta_{h_\infty} - 1\right)$ admits an inverse on the orthogonal of the constant functions inside $L^2(X, \mathbb{C})$, so that for any function $f \in \mathscr{C}^\infty(X, \mathbb{R})$, there exists a function $\widetilde{f} \in \mathscr{C}^\infty(X, \mathbb{R})$ satisfying

$$f - \int_X f \, \frac{\mathrm{d}\nu_{h_\infty}}{\mathrm{Vol}(\mathrm{d}\nu_{h_\infty})} = \widetilde{f} - \frac{1}{4\pi}\Delta_{h_\infty}\widetilde{f}. \tag{3.11}$$

Take $f := b_1(h_\infty) \in \mathscr{C}^\infty(X, \mathbb{R})$ in (3.11), and consider the Rawnsley function $\rho_{h_1^p(p)} \in \mathscr{C}^\infty(X, \mathbb{R})$ associated with the metric $h_1(p) := e^{\widetilde{f}/p} h_\infty \in \mathrm{Met}^+(L)$. As $h_1(p) \to h_\infty$ smoothly as $p \to +\infty$ by construction, we can use the uniformity in Theorem 2.13 to replace h_∞ by $h_1(p)$ in the expansion (2.33). As the coefficients in the expansion are polynomials in the derivatives of $h_1(p) \in \mathrm{Met}^+(L)$, we can take the Taylor expansion as $p \to +\infty$ of formula (2.34) to get from formulas (3.10) and (3.11) the following expansion as $p \to +\infty$ in $\mathscr{C}^m$-norm for all $m \in \mathbb{N}$,

$$\begin{aligned} b_0(h_1(p)) &+ p^{-1} b_1(h_1(p)) \\ &= b_0(h_\infty) + p^{-1}\left(\frac{1}{4\pi}\Delta_{h_\infty}\widetilde{f} - \widetilde{f}\right) + p^{-1} b_1(h_\infty) + O(p^{-2}) \\ &= b_0(h_\infty) + p^{-1} \int_X b_1(h_\infty) \, \frac{\mathrm{d}\nu_{h_\infty}}{\mathrm{Vol}(\mathrm{d}\nu_{h_\infty})} + O(p^{-2}). \end{aligned} \tag{3.12}$$

As $b_0(h_\infty)$ is constant by assumption, this implies that there exists a constant $C_p > 0$ for all $p \in \mathbb{N}$ such that as $p \to +\infty$ in $\mathscr{C}^m$-norm for any $m \in \mathbb{N}$, we have

$$\rho_{h_1^p(p)} = C_p + O(p^{n-2}), \tag{3.13}$$

and the constant $C_p > 0$ is determined up to order $O(p^{-2})$ by taking the integral of both sides against $\mathrm{d}\nu_{h_1(p)}$ and using formula (2.35). This implies the result for $k = 2$.

Let us assume now that the result holds for some $k \in \mathbb{N}$, so that we have Hermitian metrics $h_k(p) \in \mathrm{Met}^+(L)$ as in (3.8) with associated Rawnsley function satisfying the asymptotic expansion (3.9) as $p \to +\infty$. As $h_k(p) \to h_\infty$ smoothly as $p \to +\infty$ by construction, we can again apply Theorem 2.13 to $\rho_{h_k^p(p)}$, and taking the Taylor expansion as $p \to +\infty$ of the coefficients $b_r(h_k(p))$ for all $1 \leq r \leq k+1$, we get a sequence of functions $b'_r \in \mathscr{C}^\infty(X, \mathbb{R})$ for $1 \leq r \leq k$, not depending on $p \in \mathbb{N}$, such that the asymptotic expansion (2.33) holds for these functions. Furthermore, for every $r \leq k-1$, the function b'_r is constant over X by assumption. We then take

$$h_{k+1}(p) := e^{f_k/p^k} h_k(p) \in \mathrm{Met}^+(L^p) \tag{3.14}$$

for all $p \in \mathbb{N}$, where the function $f_k \in \mathscr{C}^\infty(X, \mathbb{R})$ is constructed as the function $\widetilde{f}$ of formula (3.11) for $f := b'_k$. One can then repeat the process above to get the result for $k+1$, which gives the result for general $k \in \mathbb{N}$ by induction. □

Let us now consider orthonormal bases $\mathbf{s}_k(p) \in \mathscr{B}(H^0(X, L^p))$ for the L^2-products $L^2(h_k^p(p)) \in \mathrm{Prod}(H^0(X, L^p))$ induced by the Hermitian metrics of Proposition 3.5, for all

$k \in \mathbb{N}$ and all $p \in \mathbb{N}$ big enough. Then, under the identification (2.51) and by Propositions 2.16 and 2.17, for any $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ we have

$$h^p_{e^B \mathbf{s}_k(p)} = \sigma_{\mathbf{s}_k(p)} \left(e^{2B}\right)^{-1} \rho^{-1}_{h^p_k(p)} h^p_k(p). \tag{3.15}$$

The following Lemma is essentially due to Donaldson [15,Prop. 27 (1)], and we prove it here under our conventions for convenience.

Lemma 3.6 *For any $k,\ k_0,\ m \in \mathbb{N}$ with $k_0 > n + 1 + m/2$, there exists $C > 0$ such that for all $p \in \mathbb{N}$ big enough, we have*

$$\left|\omega_{e^B \mathbf{s}_k(p)} - \omega_\infty\right|_{\mathscr{C}^m} \leq \frac{C}{p}, \tag{3.16}$$

where $\omega_{e^B \mathbf{s}_k(p)}$ is the Kähler form induced by $h_{e^B \mathbf{s}_k(p)} \in \mathrm{Met}^+(L)$ for all $p \in \mathbb{N}$ and ω_∞ is the Kähler form induced by the Kähler–Einstein metric $h_\infty \in \mathrm{Met}^+(L)$.

Proof Fix $k \in \mathbb{N}$, and note from Proposition 3.3 that $\mathrm{Vol}(d\nu_{h_k(p)}) \to \mathrm{Vol}(d\nu_{h_\infty})$ as $p \to +\infty$. Using Proposition 3.5 and the estimate (2.35) for the dimension, we know that there is a constant $C > 0$ such that for all $p \in \mathbb{N}$, we have

$$\begin{aligned} &\left|\rho^{-1}_{h^p_k(p)} - \frac{\mathrm{Vol}(d\nu_{h_k(p)})}{n_p}\right|_{\mathscr{C}^0} \\ &\quad = \rho^{-1}_{h^p_k(p)} \frac{\mathrm{Vol}(d\nu_{h_k(p)})}{n_p} \left|\rho_{h^p_k(p)} - \frac{n_p}{\mathrm{Vol}(d\nu_{h_k(p)})}\right|_{\mathscr{C}^0} \leq C p^{-k-n}, \end{aligned} \tag{3.17}$$

so that by induction on the number $m \in \mathbb{N}$ of successive derivatives of $\rho^{-1}_{h^p_k(p)}$ and using Proposition 3.5 up to $m \in \mathbb{N}$, we get constants $C_m > 0$ such that for all $p \in \mathbb{N}$,

$$\left|\rho^{-1}_{h^p_k(p)} - \frac{\mathrm{Vol}(d\nu_{h_k(p)})}{n_p}\right|_{\mathscr{C}^m} \leq C_m p^{-k-n}. \tag{3.18}$$

On the other hand, using the Sobolev embedding theorem as in [32,Lemma 2], we get for any $m \in \mathbb{N}$ and $h \in \mathrm{Met}^+(L^p)$ a constant $C_m > 0$ such that for all $p \in \mathbb{N}$ and any holomorphic section $s \in H^0(X, L^p)$, we have

$$|s|_{\mathscr{C}^m(h^p)} \leq C_m\, p^{\frac{n+m}{2}} \|s\|_{L^2(h^p)}, \tag{3.19}$$

where $|\cdot|_{\mathscr{C}^m(h^p)}$ denotes the $\mathscr{C}^m$-norm with respect to the Chern connection of (L^p, h^p). Using formula (3.8), this inequality readily extends to the approximately balanced metrics $h^p_k(p) \in \mathrm{Met}^+(L^p)$.

Writing now $\mathbf{s}_k(p) = \{s_j\}_{j=1}^{n_p}$, Propositions 2.16 and 2.17 show that for all $A = (A_{jk})_{j,k=1}^{n_p} \in \mathrm{Herm}(\mathbb{C}^{n_p})$, we have

$$\sigma_{\mathbf{s}_k(p)}(A) = \rho^{-1}_{h^p_k(p)} \sum_{j,k=1}^{n_p} A_{jk} \langle s_k, s_j \rangle_{h^p_k(p)}. \tag{3.20}$$

Then, using the estimates (2.35), (3.19) and (3.18) together with Cauchy–Schwarz inequality on the trace norm, we get for all $m \in \mathbb{N}$ constants $C,\ C',\ C'' > 0$ such that for all $A \in$

$\mathrm{Herm}(\mathbb{C}^{n_p})$ and all $p \in \mathbb{N}$, we have

$$\begin{aligned} |\sigma_{\mathbf{s}_k(p)}(A)|_{\mathscr{C}^m} &\leq \left|\rho^{-1}_{h_k^p(p)}\right|_{\mathscr{C}^m} \sum_{j,k=1}^{n_p} \left|A_{jk}\langle s_k, s_j\rangle_{h_k(p)}\right|_{\mathscr{C}^m} \\ &\leq \left(\frac{\mathrm{Vol}(d\nu_{h_k(p)})}{n_p} + Cp^{-n-k}\right) C' p^{n+\frac{m}{2}} \|A\|_{tr}\, n_p \\ &\leq C'' p^{n+\frac{m}{2}} \|A\|_{tr}. \end{aligned} \tag{3.21}$$

This implies in particular that for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{tr} \leq C^{-1}p^{-k_0}$, we have

$$\left|\sigma_{\mathbf{s}_k(p)}(e^{2B}) - 1\right|_{\mathscr{C}^m} = \left|\sigma_{\mathbf{s}_k(p)}(e^{2B} - \mathrm{Id})\right|_{\mathscr{C}^m} \leq Cp^{n+\frac{m}{2}-k_0}. \tag{3.22}$$

Now by formula (3.15) and classical properties of the Kähler form (1.1), we have

$$\begin{aligned} \omega_{e^B\mathbf{s}_k(p)} &= \omega_{h_k(p)} - \frac{\sqrt{-1}}{2\pi p}\bar\partial\partial \log \sigma_{\mathbf{s}_k(p)}\left(e^{2B}\right) - \frac{\sqrt{-1}}{2\pi p}\bar\partial\partial \log \rho_{h_k^p(p)} \\ &= \omega_{h_k(p)} - \frac{\sqrt{-1}}{2\pi p}\bar\partial\partial \log\left(1 + \sigma_{\mathbf{s}_k(p)}\left(e^{2B} - \mathrm{Id}\right)\right) \\ &\quad - \frac{\sqrt{-1}}{2\pi p}\bar\partial\partial \log\left(1 + \left(\frac{\mathrm{Vol}(d\nu_{h_k(p)})}{n_p}\rho_{h_k^p(p)} - 1\right)\right), \end{aligned} \tag{3.23}$$

which by Proposition 3.5 and formula (3.22) implies that for any $k,\ k_0,\ m \in \mathbb{N}$ with $k_0 > n + m/2$, there exists $C > 0$ such for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{tr} \leq C^{-1}p^{-k_0}$, we have

$$\left|\omega_{e^B\mathbf{s}_k(p)} - \omega_{h_k(p)}\right|_{\mathscr{C}^{m-2}} \leq \frac{C}{p}. \tag{3.24}$$

By formula (3.8) for $h_k(p)$ and the corresponding formula for $\omega_{h_k(p)}$ as in (3.23), this implies the result. □

In the case $m = 0$, the estimate (3.22) admits an elementary improvement. In fact, Definition 2.1 together with Cauchy–Schwarz inequality and the fact that $\|\Pi_{\mathbf{s}}\|_{tr} = 1$ implies that for any $\varepsilon > 0$ small enough, there is $C > 0$ such that for any $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{tr} \leq \varepsilon$ and any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$, we have

$$|\sigma_{\mathbf{s}}(e^{2B}) - 1|_{\mathscr{C}^0} \leq C\|B\|_{tr}. \tag{3.25}$$

This inequality will be used repeatedly in all the sequel.

One of the technical differences of our situation compared to the classical situation of Example 2.4 is the fact that the volumes of the anticanonical volume map depend on the positive Hermitian metric. To control these volumes, we will use the following Lemma, where for any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$, we write $d\nu_{\mathbf{s}}$ for the anticanonical volume form (1.4) associated with $h_{\mathbf{s}} \in \mathrm{Met}^+(L)$.

Lemma 3.7 *For any $k_0,\ k \in \mathbb{N}$ with $k \geq k_0$, there exists a constant $C > 0$ such that for all $p \in \mathbb{N}$ and any $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{tr} \leq C^{-1}p^{-k_0}$, we have*

$$\left|\frac{d\nu_{e^B\mathbf{s}_k(p)}}{d\nu_{h_k(p)}} - \frac{\mathrm{Vol}(d\nu_{e^B\mathbf{s}_k(p)})}{\mathrm{Vol}(d\nu_{h_k(p)})}\right|_{\mathscr{C}^0} \leq Cp^{-k_0-1}, \tag{3.26}$$

and $C^{-1} < \mathrm{Vol}(d\nu_{e^B\mathbf{s}_k(p)}) < C$.

Proof Fix $k,\ k_0 \in \mathbb{N}$ with $k \geq k_0$. Using Proposition 3.3, for any $p \in \mathbb{N}$ and $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$, formula (3.15) gives

$$\log \frac{\mathrm{d}\nu_{e^B \mathbf{s}_k(p)}}{\mathrm{d}\nu_{h_k(p)}} = -\frac{1}{p}\log \rho_{h_k^p(p)} - \frac{1}{p}\log \sigma_{\mathbf{s}_k(p)}(e^{2B}). \tag{3.27}$$

Then, using Proposition 3.5 and formula (3.25), we get a constant $C > 0$ such that for all $p \in \mathbb{N}$ and all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{tr} \leq C^{-1}p^{-k_0}$, we have

$$\begin{aligned}
&\left|\log \frac{\mathrm{d}\nu_{e^B \mathbf{s}_k(p)}}{\mathrm{d}\nu_{h_k(p)}} - \frac{1}{p}\log \frac{\mathrm{Vol}(\mathrm{d}\nu_{h_k(p)})}{n_p}\right|_{\mathscr{C}^0} \\
&= \frac{1}{p}\left|\log\left(1 + \left(\frac{\mathrm{Vol}(\mathrm{d}\nu_{h_k(p)})}{n_p}\rho_{h_k^p(p)} - 1\right)\right) + \log\left(1 + \sigma_{\mathbf{s}_k(p)}\left(e^{2B} - \mathrm{Id}\right)\right)\right|_{\mathscr{C}^0} \\
&\leq Cp^{-k_0-1}.
\end{aligned} \tag{3.28}$$

Note that we used the asymptotic expansion (2.35) for the dimension and that $\mathrm{Vol}(\mathrm{d}\nu_{h_k(p)}) \to \mathrm{Vol}(\mathrm{d}\nu_h)$ as $p \to +\infty$, which also shows that $\frac{1}{p}\log\frac{\mathrm{Vol}(\mathrm{d}\nu_{h_k(p)})}{n_p} \to 0$ as $p \to +\infty$. In other words, there exist constants $V_p > 0$ satisfying $V_p \to 1$ as $p \to +\infty$ such that for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{tr} \leq C^{-1}p^{-k_0}$, we have

$$\left|\frac{\mathrm{d}\nu_{e^B \mathbf{s}_k(p)}}{\mathrm{d}\nu_{h_k(p)}} - V_p\right|_{\mathscr{C}^0} \leq Cp^{-k_0-1}. \tag{3.29}$$

Taking the integral of both sides against $\mathrm{d}\nu_{h_k(p)}$, we see that there is $C > 0$ such that the constants $V_p > 0$ for all $p \in \mathbb{N}$ satisfy

$$\left|V_p - \frac{\mathrm{Vol}(\mathrm{d}\nu_{e^B \mathbf{s}_k(p)})}{\mathrm{Vol}(\mathrm{d}\nu_{h_k(p)})}\right| < Cp^{-k_0-1}. \tag{3.30}$$

This gives the result. □

3.3 Convergence of the balanced metrics

In this section, we consider a Fano manifold X endowed with $L := K_X^*$, and work in the setting the anticanonical volume map $\nu : \mathrm{Met}(K_X^*) \longrightarrow \mathscr{M}(X)$ defined by formula (1.4).

The goal of this section is to establish Theorem 1.1. The proof is based on the following fundamental link between the Berezin–Toeplitz quantum channel of Definition 2.10 associated with an anticanonically balanced metric and the derivative of the moment map of Definition 2.14 at the corresponding anticanonically balanced product. For any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$ and $A \in \mathrm{Herm}(\mathbb{C}^{n_p})$, write

$$D_{\mathbf{s}}\mu_\nu(A) := \frac{\partial}{\partial t}\Big|_{t=0} \mu_\nu(e^{tA}\mathbf{s}). \tag{3.31}$$

To simplify notations, we will write $\mathrm{d}\nu_{\mathbf{s}} \in \mathscr{M}(X)$ for the anticanonical volume form (1.4) associated with $h_{\mathbf{s}} \in \mathrm{Met}^+(L)$.

Proposition 3.8 *Assume that $h^p \in \mathrm{Met}^+(L^p)$ is balanced with respect to the anticanonical volume form (1.4), and let $\mathbf{s}_p \in \mathscr{B}(H^0(X, L^p))$ be orthonormal with respect to $L^2(h^p)$. Then, for all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\mathrm{Tr}[A] = 0$ and all $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$, we have*

$$\frac{n_p}{2\,\mathrm{Vol}(d\nu_{s_p})}\mathrm{Tr}[A\, D_{s_p}\mu_\nu(A)] = \mathrm{Tr}[A^2] - \left(1 + \frac{1}{p}\right)\mathrm{Tr}[A\mathscr{E}_{h^p}(A)]. \tag{3.32}$$

Proof Let us first compute $D_{\mathbf{s}}\mu_\nu(A) \in \mathrm{Herm}(\mathbb{C}^{n_p})$, for general $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$ and $A \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\mathrm{Tr}[A] = 0$. First recall from Proposition 2.17 that

$$\frac{\partial}{\partial t}\Big|_{t=0} h^p_{e^{tA}\mathbf{s}} = -2\sigma_{\mathbf{s}}(A)\, h^p_{\mathbf{s}}. \tag{3.33}$$

Recall also that $\Pi_{\mathbf{s}} : X \to \mathrm{Herm}(\mathbb{C}^{n_p})$ denotes the coherent state projector of Definition 2.1 associated with $H_{\mathbf{s}} \in \mathrm{Prod}(H^0(X, L^p))$ under the identification (2.51) induced by any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$. Writing $\mathbf{s} =: \{s_j\}_{j=1}^{n_p}$, Definition 2.2 implies that for all $x \in X$, we have

$$\Pi_{\mathbf{s}}(x) = \left(\langle s_j(x), s_k(x)\rangle_{h^p_{\mathbf{s}}}\right)_{j,k=1}^{n_p}. \tag{3.34}$$

Then, by Definition 2.14 and Proposition 3.3, we compute

$$\begin{aligned} D_{\mathbf{s}}\mu_\nu(A) &= \left(\int_X \frac{\partial}{\partial t}\Big|_{t=0} \langle e^{tA}s_j, e^{tA}s_k\rangle_{h^p_{e^{tA}\mathbf{s}}}\, d\nu_{\mathbf{s}}\right)_{j,k=1}^{n_p} \\ &\quad + \left(\int_X \langle s_j, s_k\rangle_{h^p_{\mathbf{s}}} \frac{\partial}{\partial t}\Big|_{t=0} d\nu_{e^{tA}\mathbf{s}}\right)_{j,k=1}^{n_p} - \left(\frac{\partial}{\partial t}\Big|_{t=0} \frac{\mathrm{Vol}(d\nu_{e^{tA}\mathbf{s}})}{n_p}\right)\mathrm{Id}_{\mathbb{C}^{n_p}} \\ &= \int_X (A\Pi_{\mathbf{s}} + \Pi_{\mathbf{s}}A - 2\sigma_{\mathbf{s}}(A)\Pi_{\mathbf{s}})\, d\nu_{\mathbf{s}} - \frac{2}{p}\int_X \sigma_{\mathbf{s}}(A)\Pi_{\mathbf{s}}\, d\nu_{\mathbf{s}} \\ &\quad - \left(\frac{2}{pn_p}\int_X \sigma_{\mathbf{s}}(A)\, d\nu_{\mathbf{s}}\right)\mathrm{Id}_{\mathbb{C}^{n_p}}, \end{aligned} \tag{3.35}$$

so that using Definition 2.1 and the fact that $\mathrm{Tr}[A] = 0$, we get

$$\frac{1}{2}\mathrm{Tr}[A\, D_{\mathbf{s}}\mu_\nu(A)] = \int_X \sigma_{\mathbf{s}}(A^2)\, d\nu_{\mathbf{s}} - \left(1 + \frac{1}{p}\right)\int_X \sigma_{\mathbf{s}}(A)^2\, d\nu_{\mathbf{s}}. \tag{3.36}$$

On the other hand, for any $h \in \mathrm{Met}(L^p)^+$ and letting $\mathbf{s}_p \in \mathscr{B}(H^0(X, L^p))$ be orthonormal with respect to $L^2(h^p)$, by Definition 2.1, Proposition 2.7 and formula (2.30) for the quantum channel of Definition 2.10, we have

$$\begin{aligned} \mathrm{Tr}[A^2] &= \int_X \sigma_{\mathbf{s}_p}(A^2)\, \rho_{h^p}\, d\nu_h, \\ \mathrm{Tr}\left[A\, \mathscr{E}_{h^p}(A)\right] &= \int_X \sigma_{\mathbf{s}_p}(A)^2\, \rho_{h^p}\, d\nu_h. \end{aligned} \tag{3.37}$$

Then, comparing formulas (3.36) and (3.37) with $h \in \mathrm{Met}(L^p)^+$ balanced with respect to the anticanonical volume form (1.4), so that $h^p = h_{\mathbf{s}_p}$, and using Proposition 2.8, we get the result. □

Consider now the setting of the previous section, so that $\mathrm{Aut}(X)$ is discrete and X admits a Kähler–Einstein metric $h \in \mathrm{Met}^+(K_X^*)$. For any $k \in \mathbb{N}$ and $p \in \mathbb{N}$ big enough, let $\mathbf{s}_k(p) \in \mathscr{B}(H^0(X, L^p))$ be orthonormal bases for the L^2-products $L^2(h_k^p(p)) \in \mathrm{Prod}(H^0(X, L^p))$ induced by Proposition 3.5. The key part of the proof of Theorem 1.1 is the following result, giving a lower bound for the derivative of the moment map at the approximately balanced bases. It is based on the asymptotics of Theorem 2.12 on the spectral gap of the quantum channel $\mathscr{E}_{h_k^p(p)}$ associated with $h_k^p(p)$, which allow us to bypass the difficult geometric argument in the proofs of Donaldson [15] and Phong and Sturm [34] of the analogous result for Example 2.4.

Proposition 3.9 *For any $k,\ k_0 \in \mathbb{N}$ with $k \geq k_0 > n+1$, there exists a constant $\varepsilon > 0$ such that for all $p \in \mathbb{N}$ big enough, for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{tr} \leq \varepsilon p^{-k_0}$ and all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\mathrm{Tr}[A] = 0$, we have*

$$\frac{n_p}{\mathrm{Vol}(d\nu_{e^B\mathbf{s}_k(p)})} \mathrm{Tr}[A\, D_{e^B\mathbf{s}_k(p)}\mu_\nu(A)] \geq \frac{\varepsilon}{p}\, \|A\|_{tr}^2. \tag{3.38}$$

Proof The proof consists of an approximate version of Proposition 3.8, whose proof will be used in a crucial way. We will use the following inequality, which holds for any triple of Hermitian matrices $A,\ B,\ G \in \mathrm{Herm}(\mathbb{C}^{n_p})$ as a consequence of Cauchy–Schwarz inequality,

$$|\mathrm{Tr}[ABG]| \leq \|A\|_{tr}\|B\|_{tr}\|G\|_{op}. \tag{3.39}$$

By Definition 2.1 and the fact that $\|\Pi_{\mathbf{s}}\|_{tr} = \|\Pi_{\mathbf{s}}\|_{op} = 1$, this shows that for all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})$ and all $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$,

$$|\sigma_{\mathbf{s}}(A)|_{\mathscr{C}^0} \leq \|A\|_{tr} \quad \text{ and } \quad |\sigma_{\mathbf{s}}(A^2)|_{\mathscr{C}^0} \leq \|A\|_{tr}^2. \tag{3.40}$$

Using Proposition 2.18, the submultiplicativity of the operator norm and the fact that $\|B\|_{op} \leq \|B\|_{tr}$ for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$, the inequality (3.39) also shows that that for any $\varepsilon > 0$, there is a constant $C > 0$ such that for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{tr} \leq \varepsilon p^{-k_0}$ and all $p \in \mathbb{N}$, we have

$$\begin{aligned} |\sigma_{e^B\mathbf{s}}(A)^2 - \sigma_{\mathbf{s}}(A)^2|_{\mathscr{C}^0} &\leq 2\|A\|_{tr} \left|\sigma_{\mathbf{s}}(e^{2B})^{-1}\sigma_{\mathbf{s}}(e^B A e^B) - \sigma_{\mathbf{s}}(A)\right|_{\mathscr{C}^0} \\ &\leq Cp^{-k_0}\|A\|_{tr}^2, \end{aligned} \tag{3.41}$$

and in the same way,

$$\begin{aligned} |\sigma_{e^B\mathbf{s}}(A^2) - \sigma_{\mathbf{s}}(A^2)|_{\mathscr{C}^0} &= \left|\sigma_{\mathbf{s}}(e^{2B})^{-1}\sigma_{\mathbf{s}}(e^B A^2 e^B) - \sigma_{\mathbf{s}}(A^2)\right|_{\mathscr{C}^0} \\ &\leq Cp^{-k_0}\|A\|_{tr}^2. \end{aligned} \tag{3.42}$$

Consider the operator S_p acting on $A \in \mathrm{Herm}(\mathbb{C}^{n_p})$ by

$$S_p(A) := A - \left(1 + \frac{1}{p}\right)\mathscr{E}_{h_k^p(p)}(A). \tag{3.43}$$

Assume now $k \geq k_0 > n$, and recall that $\mathbf{s}_k(p) \in \mathscr{B}(H^0(X, L^p))$ is an orthonormal basis for $L^2(h_k^p(p))$, for all $p \in \mathbb{N}$ big enough. Then, plugging $\mathbf{s} = e^B\mathbf{s}_k(p)$ into (3.36) and comparing with (3.37) for $h_k(p)$, we can use Proposition 3.5 and Lemma 3.7 together with (3.40), (3.41) and (2.35), to get a constant $C > 0$ such that for all $p \in \mathbb{N}$ big enough, for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{tr} \leq C^{-1}p^{-k_0}$ and for all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\mathrm{Tr}[A] = 0$, we have

$$\begin{aligned} &\left|\frac{n_p}{2\,\mathrm{Vol}(\mathrm{d}\nu_{e^B\mathbf{s}_k(p)})}\mathrm{Tr}[A\, D_{e^B\mathbf{s}_k(p)}\,\mu_\nu(A)] - \mathrm{Tr}[A\, S_p(A)]\right| \\ &\leq \int_X \left|\sigma_{e^B\mathbf{s}_k(p)}(A^2)\frac{n_p}{\mathrm{Vol}(\mathrm{d}\nu_{e^B\mathbf{s}_k(p)})}\frac{\mathrm{d}\nu_{e^B\mathbf{s}_k(p)}}{\mathrm{d}\nu_{h_k(p)}} - \sigma_{\mathbf{s}_k(p)}(A^2)\rho_{h_k^p(p)}\right| \mathrm{d}\nu_{h_k(p)} \\ &\quad + \int_X \left|\sigma_{e^B\mathbf{s}_k(p)}(A)^2\frac{n_p}{\mathrm{Vol}(\mathrm{d}\nu_{e^B\mathbf{s}_k(p)})}\frac{\mathrm{d}\nu_{e^B\mathbf{s}_k(p)}}{\mathrm{d}\nu_{h_k(p)}} - \sigma_{\mathbf{s}_k(p)}(A)^2\rho_{h_k^p(p)}\right| \mathrm{d}\nu_{h_k(p)} \\ &\leq C\, p^{n-k_0}\|A\|_{tr}^2. \end{aligned} \tag{3.44}$$

Recall that for any $h \in \mathrm{Met}^+(L)$, we write $\lambda_1(h) > 0$ for the first positive eigenvalue of the Riemannian Laplacian of (X, g_h^{TX}) acting on $\mathscr{C}^\infty(X, \mathbb{C})$. Then, formula (3.8) shows that there exists a constant $C > 0$ such that for all $p \in \mathbb{N}$, we have

$$|\lambda_1(h_k(p)) - \lambda_1(h_\infty)| \leq C/p. \tag{3.45}$$

Using the uniformity in Theorem 2.12, this gives a constant $C > 0$ such that for all $p \in \mathbb{N}$,

$$\begin{aligned} \mathrm{Tr}[A\, S_p(A)] &\geq \|A\|_{tr}^2 - \left(1 + \frac{1}{p}\right)\left(1 - \frac{\lambda_1(h_\infty)}{4\pi p} + Cp^{-2}\right)\|A\|_{tr}^2 \\ &\geq \left(\frac{\lambda_1(h_\infty) - 4\pi}{4\pi p} - Cp^{-2}\left(1 + \frac{1}{p}\right)\right)\|A\|_{tr}^2 . \end{aligned} \tag{3.46}$$

Using Theorem 3.2 and assuming $k \geq k_0 > n + 1$, we get from the estimates (3.44) and (3.46) a constant $\varepsilon > 0$ such that for all $p \in \mathbb{N}$ big enough, for all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B\|_{tr} < \varepsilon p^{-k_0}$ and all $A \in \mathrm{Herm}(\mathbb{C}^n)$ with $\mathrm{Tr}[A] = 0$, we have

$$\frac{n_p}{\mathrm{Vol}(d\nu_{e^B \mathbf{s}_k(p)})}\, \mathrm{Tr}[A\, D_{e^B \mathbf{s}_k(p)}\, \mu_\nu(A)] \geq \frac{\varepsilon}{p}\|A\|_{tr}^2 . \tag{3.47}$$

This gives the result. □

In the following result, we show that the moment map Lemma of Donaldson in [15,Prop. 17] is valid in our setting, although we do not exhibit any associated Kähler structure.

Proposition 3.10 *Fix $p \in \mathbb{N}$ and assume that there exist $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$ and $\lambda, \delta > 0$ such that*

(1) $\lambda\, \|\mu_\nu(\mathbf{s})\|_{tr} < \delta$;
(2) $\lambda\, \mathrm{Tr}[A\, D_{e^B \mathbf{s}}\mu_\nu(A)] \geq \|A\|_{tr}^2$ *for all* $A \in \mathrm{Herm}(\mathbb{C}^{n_p})$ *such that* $\mathrm{Tr}[A] = 0$*and all* $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ *such that* $\|B\|_{tr} \leq \delta$.

Then, there exists $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ *with* $\|B\|_{tr} \leq \delta$ *and* $\mu_\nu(e^B \mathbf{s}) = 0$.

Proof First note that for any unitary endomorphism $U \in U(n_p)$ and any $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$, Definition 2.14 shows that $\mu_\nu(U\mathbf{s}) = U\mu_\nu(\mathbf{s})\, U^*$. Thus, for any $A, B \in \mathrm{Herm}(\mathbb{C}^{n_p})$, one computes that

$$\begin{aligned} \mathrm{Tr}[A\, D_{U\mathbf{s}}\mu_\nu(A)] &= \frac{\partial}{\partial t}\Big|_{t=0} \mathrm{Tr}[A\, \mu_\nu(e^{tA} U\mathbf{s})] \\ &= \mathrm{Tr}[U^* A U\, D_{\mathbf{s}}\mu_\nu(U^* A U)]. \end{aligned} \tag{3.48}$$

In particular, assumption (2) is equivalent to

(2′) $\lambda\, \mathrm{Tr}[A\, D_{Ue^B \mathbf{s}}\mu_\nu(A)] \geq \|A\|_{tr}^2$ for all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})$ such that $\mathrm{Tr}[A] = 0$, all $U \in U(n_p)$ and all $B \in \mathrm{Herm}(\mathbb{C}^{n_p})$ such that $\|B\|_{tr} \leq \delta$.

Let us now consider $\mu_\nu : \mathscr{B}(H^0(X, L^p)) \to \mathrm{Herm}(\mathbb{C}^{n_p})$ as a vector field on $\mathscr{B}(H^0(X, L^p))$ via the identification (2.49). Let $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$ be such that assumptions (1) and (2) are satisfied, and let $\mathbf{s}_t \in \mathscr{B}(H^0(X, L^p))$ for all $t > 0$ be the solution of the ODE

$$\begin{cases} \frac{\partial}{\partial t}\mathbf{s}_t = -\mu_\nu(\mathbf{s}_t) & \text{for all} \quad t \geq 0, \\ \mathbf{s}_0 = \mathbf{s}. \end{cases} \tag{3.49}$$

If $\mu_\nu(\mathbf{s}) = 0$, then the result is trivially satisfied, so that we can assume $\mu_\nu(\mathbf{s}) \neq 0$, in which case $\mu_\nu(\mathbf{s}_t) \neq 0$ for all $t \geq 0$. Let $t_0 \geq 0$ be such that there exist $U_t \in U(n_p)$ and

$B_t \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B_t\|_{tr} \leq \delta$ such that $\mathbf{s}_t = U_t e^{B_t}\mathbf{s}$ for all $t \in [0, t_0]$. Using assumption (2′) and recalling that $\mathrm{Tr}[\mu_\nu] = 0$ by Proposition 2.16, for all $t \in [0, t_0]$ we have

$$-\lambda \frac{\partial}{\partial t}\|\mu_\nu(\mathbf{s}_t)\|_{tr}^2 = 2\lambda\,\mathrm{Tr}[\mu_\nu(\mathbf{s}_t)\,D_{\mathbf{s}_t}\mu_\nu(\mu_\nu(\mathbf{s}_t))] \geq 2\|\mu_\nu(\mathbf{s}_t)\|_{tr}^2. \tag{3.50}$$

By derivation of the square, this implies $\lambda\frac{\partial}{\partial t}\|\mu_\nu(\mathbf{s}_t)\|_{tr} \leq -\|\mu_\nu(\mathbf{s}_t)\|_{tr}$ for all $t \in [0, t_0]$, so that using Grönwall's lemma with initial condition (1) and the fact that $\mu_\nu(\mathbf{s}_t) = U_t\mu_\nu(e^{B_t}\mathbf{s})\,U_t^*$, we get

$$\|\mu_\nu(e^{B_t}\mathbf{s})\|_{tr} = \|\mu_\nu(\mathbf{s}_t)\|_{tr} \leq e^{-t/\lambda}\,\|\mu_\nu(\mathbf{s}_0)\|_{tr} < \frac{\delta}{\lambda}\,e^{-t/\lambda}. \tag{3.51}$$

Let us now consider $\mathrm{Prod}(H^0(X, L^p))$ as a symmetric space via the quotient map (2.52), and recall that the geodesics are the image of the 1-parameter groups of the action of $\mathrm{GL}(\mathbb{C}^{n_p})$ as in formula (2.53). Then, by Eq. (3.49), the Riemannian length $L(t_0) \geq 0$ of the path $\{t \mapsto H_{\mathbf{s}_t}\}_{t\in[0,t_0]} \subset \mathrm{Prod}(H^0(X, L^p))$ satisfies

$$L(t_0) = \int_0^{t_0}\|\mu_\nu(\mathbf{s}_t)\|_{tr}\,dt < \frac{\delta}{\lambda}\int_0^{+\infty} e^{-t/\lambda}\,dt = \delta. \tag{3.52}$$

This means that there exists $\varepsilon > 0$ such that all points of $\{t \mapsto H_{\mathbf{s}_t}\}_{t\in[0,t_0+\varepsilon]}$ can be joined by a geodesic of length strictly less than δ, i.e., that for each $t \in [0, t_0 + \varepsilon]$, there exists $B_t \in \mathrm{Herm}(\mathbb{C}^{n_p})$ with $\|B_t\|_{tr} \leq \delta$ such that $H_{\mathbf{s}_t} = H_{e^{B_t}\mathbf{s}}$, so that there exists $U_t \in U(n_p)$ such that $\mathbf{s}_t = U_t e^{B_t}\mathbf{s}$. Thus, $I := \{t_0 \geq 0 \,|\, L(t_0) < \delta\}$ is non-empty, open and closed in $[0, +\infty[$, so that $I = [0, +\infty[$. In particular, the path $\{t \mapsto H_{\mathbf{s}_t}\}_{t>0}$ has total Riemannian length strictly less than δ, so that it converges to a limit point $H_{e^{B_\infty}\mathbf{s}} \in \mathrm{Prod}(H^0(X, L^p))$ by completeness, with $B_\infty \in \mathrm{Herm}(\mathbb{C}^{n_p})$ satisfying $\|B_\infty\|_{tr} \leq \delta$. Finally, inequality (3.51) for all $t > 0$ implies

$$\|\mu_\nu(e^{B_\infty}\mathbf{s})\|_{tr} = \lim_{t\to+\infty}\|\mu_\nu(e^{B_t}\mathbf{s})\|_{tr} = 0. \tag{3.53}$$

This gives the result. □

With all these prerequisites in hand, we can finally give the proof of Theorem 1.1.

Proof of Theorem 1.1 First note by Proposition 2.7 and formula (3.34) that for any $k \in \mathbb{N}$ and $p \in \mathbb{N}$ big enough, the value of the moment map of Definition 2.14 at the orthonormal basis $\mathbf{s}_k(p) \in \mathscr{B}(H^0(X, L^p))$ for $L^2(h_k^p(p))$ satisfies the following formula ,

$$\frac{n_p}{\mathrm{Vol}(d\nu_{\mathbf{s}_k(p)})}\mu_\nu(\mathbf{s}_k(p)) = \int_X \Pi_{\mathbf{s}_k(p)}\left(\frac{n_p}{\mathrm{Vol}(d\nu_{\mathbf{s}_k(p)})}\frac{d\nu_{\mathbf{s}_k(p)}}{d\nu_{h_k(p)}} - \rho_{h_k^p(p)}\right)d\nu_{h_k(p)}. \tag{3.54}$$

Thus, using Proposition 3.5 and Lemma 3.7 together with the estimate (2.35) for the dimension and the fact that $\|\Pi_\mathbf{s}\|_{tr} = 1$ for all $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$, we get a constant $C > 0$ such that for all $p \in \mathbb{N}$, we have

$$\begin{aligned}&\frac{n_p}{\mathrm{Vol}(d\nu_{\mathbf{s}_k(p)})}\|\mu_\nu(\mathbf{s}_k(p))\|_{tr}\\ &\qquad\leq \mathrm{Vol}(d\nu_{h_k(p)})\left|\frac{n_p}{\mathrm{Vol}(d\nu_{\mathbf{s}_k(p)})}\frac{d\nu_{\mathbf{s}_k(p)}}{d\nu_{h_k(p)}} - \rho_{h_k^p(p)}\right|_{\mathscr{C}^0} \leq Cp^{n-k}.\end{aligned} \tag{3.55}$$

Thus, taking $k_0 > n + 1$, we can then choose $k > k_0 + n + 1$, and Proposition 3.9 shows that Proposition 3.10 applies for $p \in \mathbb{N}$ big enough and $\mathbf{s} = \mathbf{s}_k(p)$, with

$$\lambda = \frac{p}{\varepsilon}\frac{n_p}{\mathrm{Vol}(d\nu_{\mathbf{s}_k(p)})} \quad \text{and} \quad \delta = \frac{C}{\varepsilon}p^{n+1-k}. \tag{3.56}$$

This gives a sequence of Hermitian endomorphisms $B_p \in \mathrm{Herm}(\mathbb{C}^{n_p})$, $p \in \mathbb{N}$, with $\|B_p\|_{tr} \leq \varepsilon p^{-k_0}$ such that $\mu_\nu(e^{B_p}\mathbf{s}_k(p)) = 0$ for all $p \in \mathbb{N}$ big enough. By Proposition 2.15, the Hermitian metrics $h_p := h^p_{e^B\mathbf{s}_k(p)} \in \mathrm{Met}^+(L^p)$ are then anticanonically balanced for all $p \in \mathbb{N}$ big enough, and the associated Kähler forms satisfy

$$\omega_{h_p} = p\,\omega_{e^B\mathbf{s}_k(p)}, \tag{3.57}$$

where $\omega_{e^B\mathbf{s}_k(p)}$ is induced by $h_{e^B\mathbf{s}_k(p)}$. If we also chose $k_0 > n + 1 + m/2$ for some $m \in \mathbb{N}$, Lemma 3.6 shows the $\mathscr{C}^m$-convergence (1.6) to the Kähler–Einstein form ω_∞. This establishes Theorem 1.1. $\square$

4 Donaldson's iterations toward anticanonically balanced metrics

In this section, we consider a Fano manifold X, together with its anticanonical line bundle $L := K_X^*$ and the associated anticanonical volume map (1.4). We will apply Theorem 1.1 to establish the exponential convergence of the associated Donaldson's iterations and compute the optimal rate of convergence.

4.1 Donaldson map

Our goal is to study the following dynamical system on the space $\mathrm{Prod}(H^0(X, L^p))$ of Hermitian inner products on $H^0(X, L^p)$. To this end, recall Definition 2.2 for the Fubini–Study map $\mathrm{FS} : \mathrm{Prod}(H^0(X, L^p)) \to \mathrm{Met}^+(L^p)$.

Definition 4.1 For any $p \in \mathbb{N}$ big enough, the associated anticanonical *Donaldson map* is defined by

$$\mathscr{T}_\nu := \mathrm{Hilb}_\nu \circ \mathrm{FS} : \mathrm{Prod}(H^0(X, L^p)) \longrightarrow \mathrm{Prod}(H^0(X, L^p)), \tag{4.1}$$

where $\mathrm{Hilb}_\nu : \mathrm{Met}^+(L^p) \to \mathrm{Prod}(H^0(X, L^p))$ is the *anticanonical Hilbert map* defined by (1.3) using the anticanonical volume form (1.4).

By construction, the balanced products of Definition 2.3 coincide with the fixed points of the Donaldson map. Using formula (2.9) for the Fubini–Study metric and writing $h^p_H := \mathrm{FS}(H) \in \mathrm{Met}^+(L^p)$ for any $H \in \mathrm{Prod}(H^0(X, L^p))$, we get the explicit description

$$\mathscr{T}_\nu(H) = \frac{n_p}{\mathrm{Vol}(d\nu_{h_H})} \int_X \langle \Pi_H(x)\,\cdot, \cdot\rangle_H \, d\nu_{h_H}(x), \tag{4.2}$$

For any $h^p \in \mathrm{Met}^+(L^p)$ and $H \in \mathrm{Prod}(H^0(X, L^p))$, consider the natural identifications

$$\begin{aligned} \mathscr{C}^\infty(X, \mathbb{R}) &\xrightarrow{\sim} T_{h^p}\,\mathrm{Met}^+(L^p) \\ f &\longmapsto \frac{\partial}{\partial t}\Big|_{t=0} e^{tf}\, h^p, \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \mathscr{L}(H^0(X, L^p), H) &\xrightarrow{\sim} T_H\,\mathrm{Prod}(H^0(X, L^p)) \\ A &\longmapsto \frac{\partial}{\partial t}\Big|_{t=0} \langle e^{tA}\cdot, \cdot\rangle_H. \end{aligned} \tag{4.4}$$

In the notations of Sect. 2.3, if $s \in \mathscr{B}(H^0(X, L^p))$ is such that $H = H_{\mathbf{s}}$, then for any $A \in \mathscr{L}(H^0(X, L^p), H)$ we have

$$H_{e^A \mathbf{s}} = H(e^{-2A} \cdot, \cdot). \tag{4.5}$$

In particular, the identification (4.4) differs from the identification (2.51) induced by the quotient map (2.52) by a factor of -2.

Recall now Definitions 2.1 and 2.9.

Proposition 4.2 *The derivative of the anticanonical Hilbert map at $h^p \in \mathrm{Met}^+(L^p)$ is given by*

$$\begin{aligned} D_{h^p} \mathrm{Hilb}_\nu : \mathscr{C}^\infty(X, \mathbb{R}) &\longrightarrow \mathscr{L}(H^0(X, L^p), \mathrm{Hilb}_\nu(h^p)), \\ f &\longmapsto \left(1 + \frac{1}{p}\right) T_{h^p}(f) - \frac{1}{p}\left(\int_X f \frac{d\nu_h}{\mathrm{Vol}(d\nu_h)}\right) \mathrm{Id}. \end{aligned} \tag{4.6}$$

The derivative of the Fubini–Study map at $H \in \mathrm{Prod}(H^0(X, L^p))$ is given by

$$\begin{aligned} D_H \mathrm{FS} : \mathscr{L}(H^0(X, L^p), H) &\longrightarrow \mathscr{C}^\infty(X, \mathbb{R}), \\ A &\longmapsto \sigma_H(A). \end{aligned} \tag{4.7}$$

Proof For any $f \in \mathscr{C}^\infty(X, \mathbb{R})$ and $t \in \mathbb{R}$, set

$$h_t^p := e^{tf} h^p \in \mathrm{Met}^+(L^p). \tag{4.8}$$

Then, for any $s_1, s_2 \in H^0(X, L^p)$, using Proposition 3.3 and the fact that $T_{h^p}(1) = \mathrm{Id}$ by formula (2.27), we compute

$$\begin{aligned} &\frac{\partial}{\partial t}\Big|_{t=0} \langle s_1, s_2 \rangle_{\mathrm{Hilb}_\nu(h_t^p)} \\ &= \frac{n_p}{\mathrm{Vol}(d\nu_h)}\left(\int_X \frac{\partial}{\partial t}\Big|_{t=0} \langle s_1, s_2\rangle_{h_t^p} \, d\nu_h + \int_X \langle s_1, s_2\rangle_{h^p} \frac{\partial}{\partial t}\Big|_{t=0} d\nu_{h_t}\right) \\ &\quad + \left(\frac{\partial}{\partial t}\Big|_{t=0} \frac{n_p}{\mathrm{Vol}(d\nu_{h_t})}\right) \int_X \langle s_1, s_2\rangle_{h_t^p} \, d\nu_h \\ &= \frac{n_p}{\mathrm{Vol}(d\nu_{h^p})} \int_X \left(f + \frac{1}{p} f - \frac{1}{p}\int_X f \frac{d\nu_h}{\mathrm{Vol}(d\nu_h)}\right) \langle s_1, s_2\rangle_{h^p} \, d\nu_h \\ &= \frac{n_p}{\mathrm{Vol}(d\nu_{h^p})} \left\langle \left(\left(1 + \frac{1}{p}\right) T_{h^p}(f) - \frac{1}{p}\int_X f \frac{d\nu_h}{\mathrm{Vol}(d\nu_h)}\right) s_1, s_2 \right\rangle_{L^2(h^p)}. \end{aligned} \tag{4.9}$$

Using formula (2.24), this proves the first statement (4.6).

On the other hand, in the identifications (4.3), (4.4) and using formula (4.5), the second statement (4.7) is a consequence of Proposition 2.17. □

Let $H \in \mathrm{Prod}(H^0(X, L^p))$ be an anticanonically balanced product, and consider the setting of Sect. 2.2 for the anticanonically balanced metric $h_H^p := \mathrm{FS}(H) \in \mathrm{Met}^+(L^p)$. Then, in particular, Definition 2.3 of a balanced product implies that $\mathscr{L}(H^0(X, L^p), H)$ and $\mathscr{L}(\mathscr{H}_p)$ coincide as real Hilbert spaces. Via the identification (4.4), Proposition 4.2 implies the following formula for the derivative of the anticanonical Donaldson map at a fixed point.

Corollary 4.3 *The differential of the anticanonical Donaldson map at a fixed point $H \in \mathrm{Prod}(H^0(X, L^p))$ is given by the following formula, for all $A \in \mathscr{L}(\mathscr{H}_p)$,*

$$D_H \mathscr{T}_\nu(A) = \left(1 + \frac{1}{p}\right) \mathscr{E}_{h_H^p}(A) - \frac{1}{p} \frac{\mathrm{Tr}[A]}{n_p} \mathrm{Id}_{\mathscr{H}_p}. \tag{4.10}$$

Proof Definition 2.3 of a balanced product implies that H coincides with $L^2(h^p_H)$ up to a multiplicative constant, and Definition 2.1 then shows that the Berezin symbol maps $\sigma_{L^2(h^p_H)}$ and σ_H coincide. Using Propositions 2.7 and 2.8 for the balanced metric h^p_H, we then get for all $A \in \mathscr{L}(\mathscr{H}_p)$,

$$\frac{n_p}{\mathrm{Vol}(\mathrm{d}\nu_{h_H})}\left(\int_X \sigma_H(A)\,\mathrm{d}\nu_{h_H}\right) = \mathrm{Tr}[A]. \tag{4.11}$$

Then, using Definitions 2.10 and 4.1, the result follows from Proposition 4.2 and formula (4.7). □

4.2 Energy functional

In this section, we consider a Fano manifold X with $\mathrm{Aut}(X)$ is discrete, and show that if the anticanonical Donaldson map of Definition 4.1 admits a fixed point, then its iterations converge to this fixed point, which is unique up to a multiplicative constant. The results in this section are essentially a combination of results of Berman [3] and Berman et al. [4]. We gather them here as they play a central role in the proof of Theorem 1.2 given in the next section.

Recall that we write $L := K_X^*$ for the anticanonical line bundle of X, and let us introduce the energy functional $E : \mathrm{Met}^+(L^p) \to \mathbb{R}$ defined for any $h^p \in \mathrm{Met}^+(L^p)$ by

$$E(h^p) := -\log \mathrm{Vol}(\mathrm{d}\nu_h). \tag{4.12}$$

It has been considered in [4,§ 6.3] as a replacement of the *Aubin–Yau functional* in the anticanonical setting. Its key property in our context is the following Lemma of Berman [3,Lemma 2.6], for which we give a proof as it is quite elementary.

Lemma 4.4 *For any $h^p \in \mathrm{Met}^+(L^p)$, we have*

$$E(\mathrm{FS}\circ \mathrm{Hilb}_\nu(h^p)) \le E(h^p). \tag{4.13}$$

Proof Let us first show that $E : \mathrm{Met}^+(L^p) \to \mathbb{R}$ is concave along paths in $\mathrm{Met}^+(L^p)$ of the form

$$t \longmapsto h^p_t := e^{-tf} h^p,\ t \in \mathbb{R}, \tag{4.14}$$

for any $f \in \mathscr{C}^\infty(X,\mathbb{R})$ such that $e^{-f}h^p$ is positive. By Proposition 3.3, for any $t \in \mathbb{R}$ we have

$$\frac{\partial}{\partial t} E(h^p_t) = \int_X f \,\frac{\mathrm{d}\nu_{h_t}}{\mathrm{Vol}(\mathrm{d}\nu_{h_t})}, \tag{4.15}$$

so that using the Cauchy–Schwarz inequality, we get

$$\frac{\partial^2}{\partial t^2} E(h^p_t) = \left(\int_X f \,\frac{\mathrm{d}\nu_{h_t}}{\mathrm{Vol}(\mathrm{d}\nu_{h_t})}\right)^2 - \int_X f^2 \,\frac{\mathrm{d}\nu_{h_t}}{\mathrm{Vol}(\mathrm{d}\nu_{h_t})} \le 0. \tag{4.16}$$

Recall the setting of Sect. 2.2 for $h^p \in \mathrm{Met}^+(L^p)$, and let us take

$$f := \frac{1}{p}\log\left(\frac{\mathrm{Vol}(\mathrm{d}\nu_h)}{n_p}\,\rho_{h^p}\right), \tag{4.17}$$

so that $h_1^p = \mathrm{FS} \circ \mathrm{Hilb}_\nu(h^p)$ by Proposition 2.7, and consider the smooth function $\Phi : \mathbb{R} \to \mathbb{R}$ defined for any $t \in \mathbb{R}$ by

$$\Phi(t) := E(h_t^p) - E(h_0^p) - t \int_X f \, \frac{\mathrm{d}\nu_h}{\mathrm{Vol}(\mathrm{d}\nu_h)}. \tag{4.18}$$

Then, this function satisfies $f(0) = f'(0) = 0$ by formula (4.15) and is concave by formula (4.16), so that in particular $f(1) \leq 0$ and

$$E(\mathrm{FS} \circ \mathrm{Hilb}_\nu(h^p)) - E(h^p) \leq \int_X f \, \frac{\mathrm{d}\nu_h}{\mathrm{Vol}(\mathrm{d}\nu_h)}. \tag{4.19}$$

Now using formula (2.35) for the dimension, the concavity of the logarithm implies

$$\int_X f \, \frac{\mathrm{d}\nu_h}{\mathrm{Vol}(\mathrm{d}\nu_h)} \leq \frac{1}{p} \log\left(\frac{1}{n_p} \int_X \rho_{h^p} \, \mathrm{d}\nu_h\right) = 0. \tag{4.20}$$

This shows the result. □

From now on, we fix a base point $H_0 \in \mathrm{Prod}(H^0(X, L^p))$, and identify any $H \in \mathrm{Prod}(H^0(X, L^p))$ with a Hermitian endomorphism $H \in \mathscr{L}(H^0(X, L^p), H_0)$ via the formula

$$H = H_0(H\cdot, \cdot). \tag{4.21}$$

Recall that $\mathrm{Prod}(H^0(X, L^p))$ is endowed with a natural structure of a symmetric space via the quotient map (2.52), with geodesics given by formula (2.53). The following result is a consequence of the results of [6, 7] on positivity of direct images.

Proposition 4.5 [4,Lemma 7.2] *Assume that* $\mathrm{Aut}(X)$ *is discrete. Then, the functional* $\Psi : \mathrm{Prod}(H^0(X, L^p)) \to \mathbb{R}$ *defined for all* $H \in \mathrm{Prod}(H^0(X, L^p))$ *by*

$$\Psi(H) = E(\mathrm{FS}(H)) + \frac{1}{p} \frac{\log \det H}{n_p} \tag{4.22}$$

is convex along geodesics of $\mathrm{Prod}(H^0(X, L^p))$*, and strictly convex when the geodesic is not generated by a multiple of the identity.*

The fundamental role of the energy functional (4.22) in finding anticanonically balanced products comes from the following identity, which follows from Proposition 2.17 as in the proof of Proposition 3.8 for all $\mathbf{s} \in \mathscr{B}(H^0(X, L^p))$ and all $A \in \mathrm{Herm}(\mathbb{C}^{n_p})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Psi(H_{e^{tA}\mathbf{s}}) = -\frac{2}{p \, \mathrm{Vol}(\mathrm{d}\nu_{\mathbf{s}})} \mathrm{Tr}[\mu_\nu(\mathbf{s})A]. \tag{4.23}$$

By Proposition 2.15, this implies in particular that critical points of Ψ coincide with anticanonically balanced products, and Proposition 4.5 shows that they are unique up to a multiplicative constant. This also implies the following result on the iterations of Donaldson's map, due to Berman [3,Th. 4.14]. It essentially follows the proof of Donaldson [17,Prop. 4], and we give it here as it will be used in the next section.

Proposition 4.6 *Assume that* $\mathrm{Aut}(X)$ *is discrete, and let* $p \in \mathbb{N}$ *be such that an anticanonically balanced product exists. Then, for any* $H_0 \in \mathrm{Prod}(H^0(X, L^p))$*, there exists an anticanonically balanced product* $H \in \mathrm{Prod}(H^0(X, L^p))$ *such that*

$$\mathscr{T}_\nu^k(H_0) \xrightarrow{k \to +\infty} H. \tag{4.24}$$

Proof Let us first show that for any $H \in \mathrm{Prod}(H^0(X, L^p))$, we have $\Psi(\mathscr{T}_\nu(H)) \leq \Psi(H)$. By formula (4.2) and via the identification (4.21), as Π_H is rank-1 we have

$$\mathrm{Tr}\left[\frac{\mathscr{T}_\nu(H)H^{-1}}{n_p}\right] = \frac{1}{\mathrm{Vol}(d\nu_{h_H})}\int_X \mathrm{Tr}[\Pi_H]\, d\nu_{h_H} = 1, \tag{4.25}$$

so that by concavity of the logarithm,

$$\begin{aligned} \frac{\log\det \mathscr{T}_\nu(H)}{n_p} - \frac{\log\det H}{n_p} &= \frac{\log\det\left(\mathscr{T}_\nu(H)H^{-1}\right)}{n_p} \\ &\leq \log \mathrm{Tr}\left[\frac{\mathscr{T}_\nu(H)H^{-1}}{n_p}\right] = 0, \end{aligned} \tag{4.26}$$

with equality if and only if $\mathscr{T}_\nu(H) = H$. On the other hand, using Lemma 4.4 we get $E(\mathscr{T}_\nu(\mathrm{FS}(H))) \leq E(\mathrm{FS}(H))$, so that $\Psi(\mathscr{T}_\nu(H)) \leq \Psi(H)$, for all $H \in \mathrm{Prod}(H^0(X, L^p))$.

Now by Proposition 4.5 and identity (4.23), the existence of a balanced product implies that the functional Ψ is bounded from below, so that in particular, the decreasing sequence $\{\Psi(\mathscr{T}_\nu^r(H_0))\}_{r\in\mathbb{N}}$ converges to its lower bound. As the Donaldson map $\mathscr{T}_\nu$ decreases both terms of (4.22) separately, this implies that the decreasing sequence $\{\log\det(\mathscr{T}_\nu^r(H_0))\}_{r\in\mathbb{N}}$ is also bounded from below, so that $\{\det(\mathscr{T}_\nu^r(H_0))\}_{r\in\mathbb{N}}$ is bounded in $]0, +\infty[$ and

$$\frac{1}{n}\log\det\left(\mathscr{T}_\nu^{r+1}(H_0)\mathscr{T}_\nu^r(H_0)^{-1}\right) \longrightarrow 0 \quad \text{as} \quad r \to +\infty\,. \tag{4.27}$$

Again by Proposition 4.5 and identity (4.23), the existence of a balanced product implies that the functional Ψ is proper over any subset of $\mathrm{Prod}(H^0(X, L^p))$ with bounded determinant. We thus get that the sequence $\{\mathscr{T}_\nu^r(H_0)\}_{r\in\mathbb{N}}$ admits an accumulation point $H_p \in \mathrm{Prod}(H^0(X, L^p))$. On the other hand, the equality case in formula (4.25) and formula (4.27) implies

$$\mathscr{T}_\nu^{r+1}(H_0)\mathscr{T}_\nu^r(H_0)^{-1} \longrightarrow \mathrm{Id}, \quad \text{as} \quad r \to +\infty\,. \tag{4.28}$$

We thus get that $H \in \mathrm{Prod}(H^0(X, L^p))$ is the unique accumulation point, and satisfies $\mathscr{T}_\nu(H_p) = H$. This concludes the proof. ◻

4.3 Exponential convergence of Donaldson's iterations

This section is dedicated to the proof of Theorem 1.2. It follows the argument of the analogous result in [24,Th. 4.4] for the constant volume map of Example 2.5.

Consider the setting of Sect. 2.2 for an anticanonically balanced metric $h^p \in \mathrm{Met}^+(L^p)$, so that $H := L^2(h^p) \in \mathrm{Prod}(H^0(X, L^p))$ is an anticanonically balanced product.

Recall that if $H \in \mathrm{Prod}(H^0(X, L^p))$ is an anticanonically balanced product, then we have $\mathscr{L}(H^0(X, L^p), H) = \mathscr{L}(\mathscr{H}_p)$ as real Hilbert spaces for the trace norm. Write $D_H\mathscr{T}_\nu : \mathscr{L}(\mathscr{H}_p) \to \mathscr{L}(\mathscr{H}_p)$ for the differential of the Donaldson map at H in the identification (4.4).

Lemma 4.7 *Let X be a Fano manifold with* $\mathrm{Aut}(X)$ *discrete admitting a polarized Kähler–Einstein metric, and let $\{H_p \in \mathrm{Prod}(H^0(X, L^p))\}_{p\in\mathbb{N}}$ be a sequence of anticanonically balanced products for all $p \in \mathbb{N}$ big enough.*

Then, $D_{H_p}\mathscr{T}_\nu$ is an injective self-adjoint operator acting on $\mathscr{L}(\mathscr{H}_p)$ satisfying $D_{H_p}\mathscr{T}_\nu(\mathrm{Id}) = \mathrm{Id}$. *Furthermore, the highest eigenvalue $\gamma_1(H_p) \in \mathbb{R}$ of its restriction to* *the subspace of traceless matrices satisfies the following estimate as* $p \to +\infty$*,*

$$\gamma_1(H_p) = 1 - \frac{\lambda_1 - 4\pi}{4\pi p} + O(p^{-2}), \tag{4.29}$$

where $\lambda_1 > 0$ *is the first positive eigenvalue of the Riemannian Laplacian associated with the polarized Kähler–Einstein metric acting on* $\mathscr{C}^\infty(X, \mathbb{C})$*.*

Proof Recall from Proposition 2.11 that the quantum channel of Definition 2.10 is a self-adjoint operator acting on $\mathscr{L}(\mathscr{H}_p)$, so that by Corollary 4.3, the differential $D_{H_p}\mathscr{T}_\nu$ is self-adjoint and satisfies $D_{H_p}\mathscr{T}_\nu(\mathrm{Id}) = \mathrm{Id}$. In particular, it preserves the orthogonal of the identity, i.e., the space of traceless endomorphisms, and Corollary 4.3 implies that for all $A \in \mathscr{L}(\mathscr{H}_p)$ with $\mathrm{Tr}[A] = 0$, we have

$$D_{H_p}\mathscr{T}_\nu(A) = \left(1 + \frac{1}{p}\right) \mathscr{E}_{\mathrm{FS}(H_p)}(A). \tag{4.30}$$

Then, Proposition 2.11 implies that $D_{H_p}\mathscr{T}_\nu$ is injective and positive as an operator acting on $\mathscr{L}(\mathscr{H}_p)$.

To establish formula (4.29), recall from Proposition 4.5 and identity (4.23) that if $\mathrm{Prod}(H^0(X, L^p))$ contains an anticanonically balanced product, then it is unique up to a multiplicative constant. Furthermore, Definition 4.1 shows that $\mathscr{T}_\nu(cH_p) = c\mathscr{T}_\nu(H_p)$ for every $c > 0$, so that the spectrum of $D_{H_p}\mathscr{T}_\nu$ does not depend on the chosen anticanonically balanced product. Using Theorem 1.1, to compute the estimate (4.29), we can then assume that $H_p := L^2(h(p))$ for each $p \in \mathbb{N}$, where $\{h(p) \in \mathrm{Met}^+(L)\}_{p\in\mathbb{N}}$ is a sequence of positive Hermitian metrics converging to the Kähler–Einstein metric $h_\infty \in \mathrm{Met}^+(L)$. The statement is then an immediate consequence of the uniformity in Theorem 2.12, as in the proof of Proposition 3.9. □

Recall now that $\mathrm{Prod}(H^0(X, L^p))$ admits a natural structure of a symmetric space via the quotient map (2.52), and write $\mathrm{dist}(\cdot, \cdot)$ for the associated distance. Using Lemma 4.7 and the geometric input of the previous section, we can now give the proof of Theorem 1.2 following [24,Th. 4.4].

Proof of Theorem 1.2 Fix $p \in \mathbb{N}$ such that an anticanonically balanced product exists by Theorem 1.1, and fix any $H_0 \in \mathrm{Prod}(H^0(X, L^p))$. By Proposition 4.6, there exists an anticanonically balanced product $H_p \in \mathrm{Prod}(H^0(X, L^p))$ such that

$$\mathscr{T}_\nu^k(H_0) \longrightarrow H_p, \quad \text{as } k \to +\infty\,. \tag{4.31}$$

Then, up to enlarging the constant $C > 0$ in (1.8), we can assume that H_0 belongs to any fixed neighborhood $U \subset \mathrm{Prod}(H^0(X, L^p))$ of H_p. Consider H_p as a base point metric as in (4.21), so that any $H \in \mathrm{Prod}(H^0(X, L^p))$ is identified with an Hermitian endomorphism $H \in \mathscr{L}(\mathscr{H}_p)$ via the formula $H := H_p(H\cdot, \cdot)$. Take a neighborhood $U \subset \mathrm{Prod}(H^0(X, L^p))$ such that there is a diffeomorphism

$$\begin{aligned} U &\longrightarrow V \times I \\ H &\longmapsto \left(\frac{H}{\det(H)}, \det(H)\right), \end{aligned} \tag{4.32}$$

where $I \subset \mathbb{R}$ is a neighborhood of $1 \in \mathbb{R}$ and V is a neighborhood of $H_p \simeq \mathrm{Id}_{\mathscr{H}_p}$ in the space of positive Hermitian endomorphisms of determinant 1 acting on $\mathscr{H}_p$. In particular, the tangent space $T_{H_p}V$ is naturally identified with the space of traceless endomorphisms in $\mathscr{L}(\mathscr{H}_p)$. Then, for any $H \in V$, the map

$$H \longmapsto \frac{\mathscr{T}_\nu(H)}{\det(\mathscr{T}_\nu(H))} \tag{4.33}$$

fixes H_p, and its differential acts on traceless Hermitian endomorphisms in $\mathscr{L}(\mathscr{H}_p)$ by

$$D_{H_p}\mathscr{T}_\nu - \mathrm{Tr}[D_{H_p}\mathscr{T}_\nu]\,\mathrm{Id}_{\mathscr{L}(\mathscr{H}_p)}\ . \tag{4.34}$$

By Lemma 4.7, it is a self-adjoint operator with eigenvalues contained in $]0,1[\subset \mathbb{R}$, which implies in particular that the map (4.33) is a local diffeomorphism around H_p in V. Furthermore, by the classical Hartman–Grobman theorem, the map (4.33) is conjugate by a local homeomorphism to its linearization at H_p. In particular, taking $\beta_p\ \in]0,1[$ as the largest eigenvalue of (4.33), we get a constant $C > 0$ such that for all $k \in \mathbb{N}$,

$$\mathrm{dist}\left(\frac{\mathscr{T}_\nu^k(H_0)}{\det(\mathscr{T}_\nu^k(H_0))}, H_p\right) \leq C\beta_p^k\ . \tag{4.35}$$

In view of (4.32), we see that to get the exponential convergence (1.8) from (4.35), we need to show that there is a constant $C > 0$ such that for all $k \in \mathbb{N}$, we have

$$\left|\det \mathscr{T}_\nu^k(H_0) - 1\right| < C\beta_p^k\ . \tag{4.36}$$

To this end recall that the functional $\Psi : \mathrm{Prod}(H^0(X, L^p)) \to \mathbb{R}$ of Proposition 4.5 is decreasing under iterations of $\mathscr{T}_\nu$ and invariant with respect to the action of $\mathbb{R}_+$ by multiplication. By (4.35) and the differentiability of Ψ, there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$, we have

$$0 \leq \Psi(\mathscr{T}_\nu^k(H_0)) - \Psi(H_p) \leq C\beta_p^r\ . \tag{4.37}$$

A both terms appearing in the definition (4.22) of Ψ are decreasing by Lemma 4.4 and formula (4.26), respectively, we deduce in particular that for all $k \in \mathbb{N}$ big enough,

$$0 \leq \log\det(\mathscr{T}_\nu^k(H_0)) \leq C\beta_p^k\ , \tag{4.38}$$

from which (4.36) follows. This completes the proof of the exponential convergence (1.8). The asymptotic expansion (1.9) is then immediate consequence of Lemma 4.7, and the fact that it is sharp follows from the fact that (4.33) is conjugate to its linearization (4.34) by a local homeomorphism. □

Remark 4.8 Consider a general compact complex manifold X equipped with an ample line bundle L, and consider a volume map equal to a constant value $d\nu \in \mathscr{M}(X)$ as in Example 2.5. Then, the asymptotics of the optimal rate of convergence (1.9) for the associated Donaldson map have been computed in [24,Th. 3.1, Rmk. 4.12], and are given by the following estimate as $p \to +\infty$

$$\beta_p = 1 - \frac{\lambda_1}{4\pi p} + O(p^{-2}), \tag{4.39}$$

where $\lambda_1 > 0$ is the first eigenvalue of the polarized Yau metric associated with $d\nu$. Then, if X is a Fano manifold with $L := K_X^*$ and if $d\nu \in \mathscr{M}(X)$ is a Kähler–Einstein volume form as in Definition 3.1, Theorem 1.2 shows that the iterations of the Donaldson map associated with the constant volume map converge faster than the iterations associated with the anticanonical Donaldson map of Definition 4.1, as soon as $p \in \mathbb{N}$ is big enough. This behavior was predicted numerically by Donaldson [17,§ 2.2.2]. Note that the iterations of the Donaldson map for the constant volume map are of no practical interest to approximate Kähler–Einstein metrics, as one would need to know the Kähler–Einstein volume form a priori. By contrast, in case X is a Calabi-Yau manifold, the relevant volume form $d\nu \in \mathscr{M}(X)$ is purely determined by the complex geometry of the manifold, and the iterations of the Donaldson map in this case can be used to approximate numerically the polarized Ricci-flat metric.

On the other hand, the methods of this paper also apply to manifolds with $L := K_X$ ample and the canonical volume map of Example 2.6, giving the following estimate as $p \to +\infty$ for the rate of convergence (1.9),

$$\beta_p = 1 - \frac{\lambda_1 + 4\pi}{4\pi p} + O(p^{-2}), \tag{4.40}$$

where $\lambda_1 > 0$ is the first positive eigenvalue of the Kähler–Einstein Laplacian acting on $\mathscr{C}^\infty(X, \mathbb{C})$. We then see that the iterations associated with the canonical volume map converge faster than both previous examples when $p \in \mathbb{N}$ is large enough. Note that the existence of the Kähler–Einstein metric in this case is the easiest case of the celebrated theorem of Yau [46], as shown by Aubin [1].

Finally, using the methods of this paper and a refined estimate on the spectral gap of the quantum channel, it is showed in [23,Th. 1.5] that the rate of convergence of iterations for the Liouville volume map of Example 2.4 as $p \to +\infty$ satisfies

$$\beta_p = 1 + O(p^{-2}), \tag{4.41}$$

which also confirms a prediction of Donaldson [17,§ 2.2]. Theorem 1.2 thus shows that the convergence of the iterations of Donaldson's map is much faster in the anticanonical case than in the Liouville case, when $p \in \mathbb{N}$ is taken big enough.

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