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Quantization and Isotropic Submanifolds

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Abstract. We introduce the notion of an isotropic quantum state associated with a Bohr–Sommerfeld manifold in the context of Berezin–Toeplitz quantization of general prequantized symplectic manifolds, and we study its semiclassical properties using the off-diagonal expansion of the Bergman kernel. We then show how these results extend to the case of noncompact orbifolds and give an application to relative Poincaré series in the theory of automorphic forms.

1. Introduction

Let (X, ω) be a compact symplectic manifold of dimension $2n$, and let (L, h^L) be a Hermitian line bundle over X, endowed with a Hermitian connection ∇^L such that its curvature R^L satisfies the following *prequantization condition:*

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L. \tag{1.1}$$

Let J be an almost complex structure on TX compatible with ω, and let g^{TX} be the Riemannian metric on TX induced by ω and J. For any $p \in \mathbb{N}^*$, we denote by L^p the pth tensor power of L, we write Δ^{L^p} for the associated Bochner Laplacian acting on $\mathscr{C}^\infty(X, L^p)$, and consider the *renormalized Bochner Laplacian* given for any $p \in \mathbb{N}^*$ by the formula

$$\Delta^{L^p} - 2\pi np. \tag{1.2}$$

Following [18, (1.7)], it admits a discrete spectrum in $\mathbb{R}$, and there exist constants $\widetilde{C}, C, \mu > 0$ such that for all $p \in \mathbb{N}^*$, it has a finite number of eigenvalues contained in the interval $[-\widetilde{C}, \widetilde{C}]$, whereas all the others are greater than $\mu p - C$. Then for all $p \in \mathbb{N}^*$, we define the finite-dimensional space $\mathscr{H}_p \subset \mathscr{C}^\infty(X, L^p)$ of *almost holomorphic sections* of L^p as the direct sum of the eigenspaces associated with the eigenvalues of the renormalized Bochner Laplacian inside $[-\widetilde{C}, \widetilde{C}]$. As explained in Section 2.1, these spaces satisfy the Riemann–Roch–Hirzebruch formula for $p \in \mathbb{N}^*$ big enough and are a natural generalization of the spaces of holomorphic sections of L^p.

In fact, consider the particular case of integrable J, making (X, J, ω) into a *Kähler manifold,* together with a holomorphic Hermitian line bundle (L, h^L) such that its *Chern connection* ∇^L (its unique Hermitian connection compatible with the holomorphic structure) satisfies (1.1). For any $p \in \mathbb{N}^*$, writing $\overline{\partial}_p$ for the holomorphic $\overline{\partial}$-operator on forms with values in L^p and $\overline{\partial}_p^*$ for its formal adjoint with

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respect to the L^2-Hermitian product, the *Bochner–Kodaira* formula tells us that the operator (1.2) is equal to $2\overline{\partial}_p^*\overline{\partial}_p$. Then by a result of [6, Thm. 1.1], this operator shows a *spectral gap,* so that the eigenvalues inside $[-\widetilde{C}, \widetilde{C}]$ are all equal to 0 for $p \in \mathbb{N}^*$ big enough. The space $\mathscr{H}_p$ of almost holomorphic sections considered above then reduces to the space of *holomorphic sections* of L^p. As explained, for instance, in [34, Section 9.2], these spaces can be thought as the spaces of quantum states of the *holomorphic quantization* of the symplectic manifold (X, ω), seen as a dynamical phase space of classical mechanics. In this context, the integer $p \in \mathbb{N}^*$ represents a *quantum number,* usually inversely proportional to the *Planck constant,* and asymptotic results as p tends to infinity describe the so-called *semiclassical limit,* when the scale gets so large that we recover the laws of classical mechanics as an approximation of the laws of quantum mechanics.

On the other hand, in the framework of geometric quantization associated with a regular Lagrangian fibration on X, the quantum states of X are represented by immersed Lagrangian submanifolds $\iota : \Lambda \hookrightarrow X$ satisfying the property called the *Bohr–Sommerfeld condition,* which asks for the existence of a nonvanishing section $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^* L)$ parallel with respect to $\nabla^{\iota^* L}$ and satisfying $|\zeta(x)|_{\iota^* L} = 1$ for all $x \in \Lambda$ (see, e.g., [30]). We call the data of (Λ, ι, ζ) a *Bohr–Sommerfeld Lagrangian.* The existence of a regular Lagrangian fibration on X being too restrictive, we consider in general singular Lagrangian fibrations, in which we allow the dimension of the fibers to drop on a finite union of submanifolds of positive codimension in X. Removing the condition $\dim \Lambda = n$, we call the data of (Λ, ι, ζ) a *Bohr–Sommerfeld submanifold.* The typical case of a singular Lagrangian fibration is the case of *toric manifolds,* where X is endowed with an effective Hamiltonian action of $\mathbb{T}^n = (S^1)^n$, and the fibers are given by the orbits of this action. For a comparison of holomorphic and real quantizations in this context, see, for example, [4].

In this paper, we use the theory of the generalized Bergman kernel of Ma and Marinescu [26] to study semiclassical properties of Bohr–Sommerfeld submanifolds in the context of the *almost holomorphic quantization* described before. Here the quantization of a Bohr–Sommerfeld submanifold is represented by a sequence $\{s_p \in \mathscr{H}_p\}_{p \in \mathbb{N}^*}$, called an *isotropic state,* defined for any $p \in \mathbb{N}^*$ by the formula

$$s_p = \int_\Lambda P_p(x, \iota(y)) \zeta^p(y) \, dv_\Lambda(y), \tag{1.3}$$

where dv_Λ is the Riemannian volume form of $(\Lambda, \iota^* g^{TX})$, $\zeta^p \in \mathscr{C}^\infty(\Lambda, \iota^* L^p)$ is the pth tensor power of ζ, and $P_p(\cdot, \cdot)$ is the *generalized Bergman kernel,* that is, the Schwartz kernel with respect to dv_X of the orthogonal projection P_p from $\mathscr{C}^\infty(X, L^p)$ to $\mathscr{H}_p$ with respect to the natural L^2-Hermitian product. The expected behavior of a quantum state in the semiclassical limit is to rapidly localize around the corresponding classical object, and we show in Proposition 3.5 that isotropic states indeed concentrate around the associated Bohr–Sommerfeld submanifold as $p \to +\infty$. Furthermore, we establish in Theorem 3.6 the following estimate on the L^2-norm $\|\cdot\|_p$ of these sections as $p \to +\infty$, which is the first main result of this paper, and which we state here in its simplest form.

THEOREM 1.1. *Let* (Λ, ζ, ι) *be a Bohr–Sommerfeld submanifold of* X. *Then there exist* $a_r \in \mathbb{R}, r \in \mathbb{N}$, *such that for any* $k \in \mathbb{N}$ *and as* $p \to +\infty$,

$$\|s_p\|_p^2 = p^{n-\dim\Lambda/2} \sum_{r=0}^{k} p^{-r} a_r + O(p^{n-\dim\Lambda/2-(k+1)}). \tag{1.4}$$

Furthermore, we have $a_0 = 2^{\dim\Lambda/2}\operatorname{Vol}(\Lambda)$, *where* $\operatorname{Vol}(\Lambda) > 0$ *is the Riemannian volume of* $(\Lambda, \iota^* g^{TX})$.

The proof of this theorem uses the off-diagonal expansion expansion of the generalized Bergman kernel as $p \to +\infty$ given in [26, Thm. 1.19], which in fact implies an analogous expansion for the isotropic state s_p around Λ depending on the position of the tangent spaces of Λ with respect to the Riemannian metric g^{TX}, similar to the asymptotic expansion of the G-invariant Bergman kernel of Ma and Zhang [28, Thm. 0.2]. Although we do not state it explicitly, this fact is implicitly used in Section 4, where we study the L^2-Hermitian product $\langle\cdot,\cdot\rangle_p$ of two such sections as $p \to +\infty$. We show that this product tends rapidly to 0 whenever the two associated submanifolds do not intersect, and we establish Theorem 4.4, which is the second main result of this paper, and which we state here in its simplest form, using the notion of *clean intersection* of Definition 4.1.

THEOREM 1.2. *Let* $(\Lambda_1, \iota_1, \zeta_1)$ *and* $(\Lambda_2, \iota_2, \zeta_2)$ *be two Bohr–Sommerfeld submanifolds with clean and connected intersection, and let* $\{s_{j,p}\}_{p\in\mathbb{N}^*}$, $j = 1, 2$, *denote the associated isotropic states. Set* $l = \dim\Lambda_1 \cap \Lambda_2$ *and* $d_j = \dim\Lambda_j$, $j = 1, 2$. *Then there exist* $b_r \in \mathbb{C}, r \in \mathbb{N}$, *such that for any* $k \in \mathbb{N}$ *and as* $p \to +\infty$,

$$\langle s_{1,p}, s_{2,p}\rangle_p = p^{n-(d_1+d_2)/2+l/2}\lambda^p \sum_{r=0}^{k} p^{-r} b_r + O(p^{n-(d_1+d_2)/2+l/2-(k+1)}), \tag{1.5}$$

where $\lambda \in \mathbb{C}$ *is the value of the constant function on* $\Lambda_1 \cap \Lambda_2$ *defined for any* $x \in \Lambda_1 \cap \Lambda_2$ *by* $\lambda(x) = \langle\zeta_1(x), \zeta_2(x)\rangle_L$. *Furthermore, if* $\dim\Lambda_1 = n$, *then*

$$b_0 = 2^{n/2}\int_{\Lambda_1\cap\Lambda_2} \det{}^{-1/2}\left\{\sqrt{-1}\sum_{k=1}^{n-l} h^{TX}(e_k, \nu_i)\,\omega(e_k, \nu_j)\right\}_{i,j=1}^{d_2-l} |dv|_{\Lambda_1\cap\Lambda_2}, \tag{1.6}$$

where $\langle e_i\rangle_{i=1}^{n-l}, \langle\nu_j\rangle_{j=1}^{d_2-l}$ *are local orthonormal frames of the normal bundle of* $\Lambda_1 \cap \Lambda_2$ *in* Λ_1, Λ_2 *respectively, and* $|dv|_{\Lambda_1\cap\Lambda_2}$ *is the Riemannian density on* $\Lambda_1 \cap \Lambda_2$ *induced by* g^{TX}.

The proof of this theorem also gives a formula for the first coefficient (1.6) in the case Λ_1 and Λ_2 are both not Lagrangian, but its geometric meaning is unclear, which is why we do not give it explicitly. Note on the other hand that although the integrand of (1.6) is nowhere vanishing, nothing prevents the whole integral to vanish in general. In any case, this shows that in the semiclassical limit the Hermitian product of two isotropic states is closely related to the geometry of the intersection of the corresponding submanifolds. The left-hand side of (1.5) is called the *intersection product* of $s_{1,p}$ and $s_{2,p}$, and can be thought as the cup product of some Lagrangian intersection theory (see [32] for a discussion on this idea).

To give the most general formulation of Theorems 1.1 and 1.2, we use the theory of Berezin–Toeplitz operators for the generalized Bergman kernel on symplectic manifolds of [19] and consider any J-invariant Riemannian metric g^{TX} on TX and isotropic states taking values in an auxiliary Hermitian vector bundle (E, h^E) with Hermitian connection ∇^E. In the case of nonconnected intersection the expansion (1.5) takes the form of a sum over the connected components. We describe this in Theorems 3.6 and 4.4 in the case of smooth and compact X. When (E, h^E) is the so-called *metaplectic correction,* we recover the setting of metaplectic quantization and Lagrangian submanifolds endowed with half-forms, as explained in Remark 4.5. On the other hand, the case of higher-dimensional (E, h^E) is relevant for the applications to relative Poincaré series in the theory of vector-valued automorphic forms, as explained further. In the same context, note that the metric g^{TX} used in Section 6 is not the metric induced by ω and J, although the difference is rather trivial, as noted in the proof of Theorem 6.3. However, the case of a general Hermitian metric g^{TX} may be useful in the study of relative Poincaré series on more general symmetric spaces.

In Section 5, we explain how the results of Section 3 extend to the case of complete noncompact orbifold (X, g^{TX}) when the immersed isotropic submanifold Λ is compact and (X, J, ω, g^{TX}) is Kähler. As an application to the case where X is the quotient of the Poincaré upper half-plane $\mathbb{H}$ by a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, we derive in Section 6 asymptotic results on relative Poincaré series in the theory of automorphic forms.

In the case of compact Kähler manifold (X, J, ω, g^{TX}) with even $c_1(TX)$, $E = \mathbb{C}$, and $\dim \Lambda_1 = \dim \Lambda_2 = n$, Theorem 4.4 is the main result of Borthwick, Paul, and Uribe [8, Thm. 3.2] with an expansion in half-integer powers of p in [8, (85)] instead of integer powers as in (1.5). This is explained in Remark 4.5, where we translate their use of the formalism of half-forms by taking for E a square root of the canonical bundle of X. In the case where Γ acts freely on $\mathbb{H}$ and where $X = \mathbb{H}/\Gamma$ is compact, the application to relative Poincaré series in Section 6 is the result of [8, Section 4]. In the case where (X, J, ω, g^{TX}) is additionally equipped with an Hamiltonian action of a compact Lie group lifting to (L, h^L, ∇^L), an equivariant version of the results of [8] has been obtained by Debernardi and Paoletti [13]. Semiclassical asymptotics on Lagrangian states have also been obtained by Charles [11] in the case of discrete intersections and in the same particular context as in [8].

The theory of Berezin–Toeplitz operators was first developed by Bordemann, Meinreken, and Schlichenmaier [7] and Schlichenmaier [29] for the Kähler case,

$E = \mathbb{C}$, and $g^{TX}(\cdot,\cdot) = \omega(\cdot, J\cdot)$. The approach of [7; 8; 11], and [13] is based on the work of Boutet de Monvel and Sjöstrand [10] on the Szegö kernel and on the theory of Toeplitz structures developed by Boutet de Monvel and Guillemin [9]. Note that the definitions of Section 3.1 extend in a straightforward way to the case of spin^c *quantization* considered, for example, in [27], and the results of Section 3 and Section 4 certainly hold in this case. If (X, J, ω, g^{TX}) is further endowed with a Hamiltonian action of a compact Lie group G lifting to (L, h^L, ∇^L), (E, h^E, ∇^E) such that $0 \in \mathrm{Lie}(G)^*$ is a regular point of the associated moment map $\mu : X \to \mathrm{Lie}(G)^*$, and if $\iota : \Lambda \to X$ intersects $\mu^{-1}(0)$ cleanly in the sense of Definition 4.1, then we can use the full off-diagonal expansion of the G-invariant Bergman kernel of Ma and Zhang [28, Thm. 0.2, Rem. 0.3] to prove a result analogous to Theorem 3.6 for the G-invariant part of the associated isotropic state.

In the context of relative Poincaré series, Barron (previously Foth) studied in [14] the case of Bohr–Sommerfeld tori in higher-dimensional symmetric spaces. The results of Section 5 can then be used to generalize [14, Section 1.3] to the case of noncompact or orbifold symmetric spaces. In another direction, the results of Sections 3.2 and 4.2 can be applied to study relative Poincaré series associated with isotropic submanifolds in higher-dimensional symmetric spaces. The case of geodesics on some specific compact quotients of the ball has been studied by Barron [5]. On the other hand, Alluhaibi and Barron [1] studied the case of relative Poincaré series associated with some submanifolds of the ball, which are not necessarily isotropic. Note that they consider more generally the case of vector-valued automorphic forms, which corresponds for us to the case of flat Hermitian vector bundle (E, h^E) of arbitrary dimension. Our results can thus also be applied to this case where the underlying submanifold is isotropic.

A final motivation for this work is toward the program initiated by Witten [33] in holomorphic quantization of Chern–Simons theory, showing an asymptotic expansion for Lagrangian states associated with some special Bohr–Sommerfeld Lagrangians inside the moduli space of flat connections on a Riemann surface defined in [20, Prop. 7.2] and [15, Prop. 3.27]. Bohr–Sommerfeld Lagrangians in this context have also been studied by Tyurin [32] and in a more general context of the Abelian Lagrangian Algebraic Geometry program of Gorodentsev and Tyurin [17]. In both cases, it is of particular importance to be able to consider orbifolds.

2. Generalized Bergman Kernels on Symplectic Manifolds

In this section, we set the context and notations, and recall the results of [24; 26], and [19] we will need throughout the paper. We refer to the book [25, Chapters 4–8] as a basic reference for the theory.

2.1. Setting

Let (X,ω) be a compact symplectic manifold of dimension $2n$ with tangent bundle TX, and let (L,h^L) be a Hermitian line bundle over X, together with a Hermitian connection ∇^L satisfying (1.1). Let J be an almost complex structure compatible with ω, and take g^{TX} to be any J-invariant Riemannian metric on TX. We write ∇^{TX} for the associated Levi–Civita connection and $d^X(\cdot,\cdot)$ for the Riemannian distance of (X,g^{TX})

For any Euclidean vector bundle $(\mathscr{E},g^{\mathscr{E}})$, we write $\mathscr{E}_{\mathbb{C}}$ for its complexification and still write $g^{\mathscr{E}}$ for the induced $\mathbb{C}$-bilinear product on $\mathscr{E}_{\mathbb{C}}$. Let us write

$$TX_{\mathbb{C}}=T^{(1,0)}X\oplus T^{(0,1)}X \tag{2.1}$$

for the splitting of $TX_{\mathbb{C}}$ into the eigenspaces of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. Then for any $x\in X$ and $v,w\in T_x^{(1,0)}X$, we define the positive Hermitian endomorphism $\dot{R}_x^L\in\mathrm{End}(T_x^{(1,0)}X)$ by the formula

$$g^{TX}(\dot{R}_x^L v,\overline{w})=R^L(v,\overline{w}). \tag{2.2}$$

We denote by $K_X=\det(T^{*(1,0)}X)$ the canonical line bundle of (X,J), endowed with the Hermitian structure and connection h^{K_X},∇^{K_X} induced by g^{TX},∇^{TX} via (2.1). We will also consider the Riemannian metric g_ω^{TX} on TX defined by the formula

$$g_\omega^{TX}(\cdot,\cdot)=\omega(\cdot,J\cdot) \tag{2.3}$$

and the Hermitian metric h_ω^{TX} on (TX,J) defined by

$$h_\omega^{TX}=g_\omega^{TX}-\sqrt{-1}\omega. \tag{2.4}$$

Note that if $g^{TX}=g_\omega^{TX}$, then $\dot{R}^L=2\pi\mathrm{Id}_{T^{(1,0)}X}$. For any submanifold $Y\subset X$, we write g^{TY},g_ω^{TY} for the Riemannian metrics on Y induced by g^{TX},g_ω^{TX} and $dv_Y,dv_{Y,\omega}$ for the induced Riemannian volume forms. In particular, we have

$$dv_{X,\omega}=\det(\dot{R}^L/2\pi)\,dv_X. \tag{2.5}$$

For any Hermitian vector bundle (E,h^E) over X, we write $\langle\cdot,\cdot\rangle_E$ and $|\cdot|_E$ for the Hermitian product and norm induced by h^E.

Let (E,h^E) be an auxiliary Hermitian vector bundle over X with Hermitian connection ∇^E, and write R^E for the curvature of ∇^E. For any $p\in\mathbb{N}^*$, we write

$$E_p=L^p\otimes E, \tag{2.6}$$

endowed with the Hermitian metric h^{E_p} and connection∇^{E_p} induced by h^L,h^E and ∇^L,∇^E.

DEFINITION 2.1. The *Bochner Laplacian* Δ^{E_p} is the second-order differential operator acting on $\mathscr{C}^\infty(X,E_p)$ by the formula

$$\Delta^{E_p}=-\sum_{j=1}^{2n}[(\nabla_{e_j}^{E_p})^2-\nabla^{E_p}_{\nabla^{TX}_{e_j}e_j}], \tag{2.7}$$

where $\{e_j\}_{j=1}^{2n}$ is any local orthonormal frame of TX with respect to g^{TX}.

For any $p \in \mathbb{N}^*$ and any Hermitian smooth section $\Phi \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the *renormalized Bochner Laplacian* $\Delta_{p,\Phi}$ is the second-order differential operator acting on $\mathscr{C}^\infty(X, E_p)$ by the formula

$$\Delta_{p,\Phi} = \Delta^{E_p} - p\,\mathrm{Tr}[\dot{R}^L] + \Phi. \tag{2.8}$$

From now on, we fix $\Phi \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ and simply write Δ_p for the associated renormalized Bochner Laplacian. In the Kähler case, if $g^{TX} = g^{TX}_\omega$ and if Φ is equal to $-\sqrt{-1}R^E$ contracted with ω, then we recover twice the Kodaira Laplacian of E_p. On the other hand, if $g^{TX} = g^{TX}_\omega$ and $E = \mathbb{C}$, then we recover (1.2).

The *L^2-Hermitian product* $\langle\cdot,\cdot\rangle_p$ on $\mathscr{C}^\infty(X, E_p)$ is given for any $s_1, s_2 \in \mathscr{C}^\infty(X, E_p)$ by the formula

$$\langle s_1, s_2\rangle_p = \int_X \langle s_1(x), s_2(x)\rangle_{E_p}\, dv_X(x). \tag{2.9}$$

Let $\|\cdot\|_p$ be the associated L^2-norm, and let $L^2(X, E_p)$ be the completion of $\mathscr{C}^\infty(X, E_p)$ with respect to $\|\cdot\|_p$. Then Δ_p is a self-adjoint second-order differential operator on $L^2(X, E_p)$ and has a discrete spectrum contained in $\mathbb{R}$. Furthermore, we have the following refinement of [18, Thm. 2a)].

THEOREM 2.2 ([24, Cor. 1.2]). *There exist $\widetilde{C}, C > 0$ such that for all $p \in \mathbb{N}^*$,*

$$\mathrm{Spec}(\Delta_p) \subset [-\widetilde{C}, \widetilde{C}] \cup]2\mu_0 p - C, +\infty[, \tag{2.10}$$

where $\mu_0 = \inf_{x\in X, v\in T^{(1,0)}_x X} R^L_x(v,\overline{v})/g^{TX}_x(v,\overline{v})$.

For any $p \in \mathbb{N}^*$, the *space of almost holomorphic sections* $\mathscr{H}_p \subset L^2(X, E_p)$ of E_p is defined as the direct sum of the eigenspaces of Δ_p associated with the eigenvalues in $[-\widetilde{C}, \widetilde{C}]$. Then by standard elliptic theory we have $\mathscr{H}_p \subset \mathscr{C}^\infty(X, E_p)$ and $\dim \mathscr{H}_p < +\infty$. By [24, Cor.1.2], for any $p \in \mathbb{N}^*$ big enough, the dimension of $\mathscr{H}_p$ is computed by the Riemann–Roch–Hirzebruch formula and is in particular a polynomial of degree n in p. Note that by [26, Cor. 3.3] the eigenvalues in $[-\widetilde{C}, \widetilde{C}]$ are not all equal to 0 in general, and for $p \in \mathbb{N}^*$ big enough, this happens if and only if (X, ω, J) is in fact Kähler.

Let $\pi_j : X \times X \to X$, $j = 1, 2$, denote the first and second projections. For any $p \in \mathbb{N}^*$, we define a vector bundle over $X \times X$ by the formula

$$E_p \boxtimes E_p^* = \pi_1^* E_p \otimes \pi_2^* E_p^*. \tag{2.11}$$

The orthogonal projection $P_p : \mathscr{C}^\infty(X, E_p) \to \mathscr{H}_p$ with respect to (2.9) has smooth Schwartz kernel $P_p(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$ with respect to dv_X, defined for any $s \in \mathscr{C}^\infty(X, E_p)$ and $x \in X$ by

$$(P_p s)(x) = \int_X P_p(x, y) s(y)\, dv_X(y). \tag{2.12}$$

For any $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, we define the *Berezin–Toeplitz quantization* of F as the family $\{T_{F,p}\}_{p\in\mathbb{N}^*}$ of operators acting on $\mathscr{C}^\infty(X, E_p)$ for any $p \in \mathbb{N}^*$ by

$$T_{F,p} = P_p F P_p, \tag{2.13}$$

where F denotes the operator of pointwise application of the endomorphism F. Then $T_{F,p}$ has a smooth Schwartz kernel $T_{F,p}(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$ with respect to dv_X, given for any $x, y \in X$ by

$$T_{F,p}(x, y) = \int_X P_p(x, w)F(w)P_p(w, y)\, dv_X(w). \tag{2.14}$$

For any $\sigma > 0$, we use the notation $O(p^{-\sigma})$ as $p \to +\infty$ in the usual sense with respect to $|\cdot|_E$, uniformly in $x \in X$. The notation $O(p^{-\infty})$ means $O(p^{-\sigma})$ for any $\sigma > 0$. Unless otherwise stated, we also use the convention to sum on free indices appearing twice in a single term.

2.2. *Local Model*

Let $(u, v) := (u_1, \dots, u_n, v_1, \dots, v_n) \in \mathbb{R}^{2n}$ be the canonical symplectic coordinates associated with the standard symplectic form Ω on $\mathbb{R}^{2n}$ given by

$$\Omega = \sum_{j=1}^{n} du_j \wedge dv_j. \tag{2.15}$$

We write $\mathbb{R}^n \times \{0\} = \{(u, 0) \in \mathbb{R}^{2n} | u \in \mathbb{R}^n\}$ and $\{0\} \times \mathbb{R}^n = \{(0, v) \in \mathbb{R}^{2n} | v \in \mathbb{R}^n\}$ for the two canonical oriented Lagrangian subspaces of $(\mathbb{R}^{2n}, \Omega)$ and write $\langle\cdot,\cdot\rangle$ and $|\cdot|$ for the canonical scalar product and norm of $\mathbb{R}^{2n}$. To match with the notations of [26], we write $Z := (u, v) \in \mathbb{R}^{2n}$ and use the same notation for the radial vector field of $\mathbb{R}^{2n}$. For any $\varepsilon > 0$, we denote by $B^{\mathbb{R}^{2n}}(0, \varepsilon)$ the ball with center 0 and radius ε in $\mathbb{R}^{2n}$, and for any linear subspace $\Sigma \subset \mathbb{R}^{2n}$, we write $B^\Sigma(0, \varepsilon) := B^{\mathbb{R}^{2n}}(0, \varepsilon) \cap \Sigma$.

For any $m \in \mathbb{N}$, we write $|\cdot|_{\mathscr{C}^m}$ for the local $\mathscr{C}^m$-norm on local sections of $E_p \boxtimes E_p^*$ over $X \times X$ induced by $h^L, h^E, \nabla^L, \nabla^E$.

PROPOSITION 2.3 ([26, Section 1.1]). *For any $m, k \in \mathbb{N}$, $\varepsilon > 0$, and $\theta \in\,]0, 1[$, there exists $C_{m,k,\theta,\varepsilon} > 0$ such that for all $p \in \mathbb{N}^*$ and $x, x' \in X$ satisfying $d^X(x, x') > \varepsilon p^{-\theta/2}$,*

$$|P_p(x, x')|_{\mathscr{C}^m} \le C_{m,k,\theta,\varepsilon}\, p^{-k}. \tag{2.16}$$

Let us now take $x_0 \in X$, $\varepsilon_0 > 0$, an open neighborhood $V \subset X$ of x_0, and a diffeomorphism

$$\phi_{x_0} : B^{\mathbb{R}^{2n}}(0, \varepsilon_0) \subset \mathbb{R}^{2n} \to V \tag{2.17}$$

sending 0 to x_0, such that its differential at 0 identifies Ω and $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{2n}$ with ω and g^{TX}_ω on $T_{x_0}X$. Let us make such a choice of diffeomorphisms (2.17) for any x_0 in a small open set, smoothly in x_0. We cover X with such open sets and choose $\varepsilon_0 > 0$ that does not depend on $x_0 \in X$. As two Riemannian metrics induce equivalent distances in a continuous way with respect to parameters, there exist $0 < a < b$ such that for any $x_0 \in X$ and $Z, Z' \in B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$,

$$a|Z - Z'| < d^X(\phi_{x_0}(Z), \phi_{x_0}(Z')) < b|Z - Z'|. \tag{2.18}$$

Then by (2.18) we get the following corollary of Proposition 2.3.

COROLLARY 2.4. *For any $\varepsilon > 0$, $m, k \in \mathbb{N}$, and $\theta \in]0, 1[$, there exists $C_{m,k,\theta,\varepsilon} > 0$ such that for all $x_0 \in X$, for all $p \in \mathbb{N}^*$ and for all $Z, Z' \in B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ satisfying $|Z - Z'| > \varepsilon p^{-\theta/2}$,*

$$|P_p(\phi_{x_0}(Z), \phi_{x_0}(Z'))|_{\mathscr{C}^m} \leq C_{m,k,\theta,\varepsilon'} p^{-k}. \tag{2.19}$$

We use the following explicit local model on $\mathbb{R}^{2n}$ for the Bergman kernel, as found in [27, (3.25)] for any $Z, Z' \in \mathbb{R}^{2n}$:

$$\mathscr{P}_{x_0}(Z, Z') = \exp\left(-\frac{\pi}{2}|Z - Z'|^2 - \pi\sqrt{-1}\Omega(Z, Z')\right). \tag{2.20}$$

Note that the difference of (2.20) with [27, (3.25)] comes from the fact that we work with symplectic coordinates $Z \in \mathbb{R}^{2n}$ adapted to ω via (2.17) instead of metric coordinates adapted to g^{TX} via the exponential map as in [27, Section 3.2].

Let dZ be the canonical Lebesgue measure of $\mathbb{R}^{2n}$, and define the smooth function $\kappa_{x_0} \in \mathscr{C}^\infty(B^{\mathbb{R}^{2n}}(0, \varepsilon_0), \mathbb{R})$ such that for any $Z \in B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ in the chart (2.17),

$$dv_X(Z) = \kappa_{x_0}(Z)\, dZ, \quad \text{with } \kappa_{x_0}(0) = \det(\dot{R}^L_{x_0}/2\pi)^{-1}. \tag{2.21}$$

In the chart (2.17), we identify E, L over $B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ with E_{x_0}, L_{x_0} through parallel transport with respect to ∇^E, ∇^L along radial lines of $B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$. For any x_0 in a small open set, we identify L_{x_0} with $\mathbb{C}$ using any unit local frame of L.

For any $f \in \mathscr{C}^\infty(X, E)$, we write $f_{x_0} \in \mathscr{C}^\infty(B^{\mathbb{R}^{2n}}(0, \varepsilon_0), E_{x_0})$ for the restriction of f to $B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ in this trivialization. Similarly, for any smooth kernel $T_p(\cdot, \cdot) \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$, we denote by $T_{p,x_0}(Z, Z') \in \mathrm{End}(E_{x_0})$ its image evaluated at $Z, Z' \in B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ in this trivialization. If $Q(Z, Z')$ is a polynomial in $Z, Z' \in \mathbb{R}^{2n}$, then we write $Q\mathscr{P}_{x_0}(Z, Z') := Q(Z, Z')\mathscr{P}_{x_0}(Z, Z')$.

Recall that we chose a family of charts $\{\phi_{x_0}\}_{x_0 \in W}$ as in (2.17) smoothly in $x_0 \in W$, where W is a small open set of X. Then $P_{p,x_0}(Z, Z')$ can be seen as a smooth section of $\pi^* \mathrm{End}(E)$ over $W \times B^{\mathbb{R}^{2n}}(0, \varepsilon_0) \times B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$ evaluated in $x_0 \in W, Z, Z' \in B^{\mathbb{R}^{2n}}(0, \varepsilon_0)$, where $\pi : W \times B^{\mathbb{R}^{2n}}(0, \varepsilon_0) \times B^{\mathbb{R}^{2n}}(0, \varepsilon_0) \to W$ is the first projection. Let us write $|\cdot|_{\mathscr{C}^m(X)}$ for the local $\mathscr{C}^m$-norm on local sections of $\pi^* \mathrm{End}(E)$ induced by h^E and derivation by $\nabla^{\pi^* \mathrm{End}(E)}$ in the direction of $x_0 \in W$. We are now ready to state the following result, which was first proved in [12, Thm. 4.18'] in the case of the spin^c Dirac operator, and which in the following form comes essentially from [22, Thm. 2.1].

LEMMA 2.5. *For any* $m, k \in \mathbb{N}$, $\varepsilon > 0$, *and* $\delta \in \,]0,1[$, *there exist* $\theta \in \,]0,1[$ *and* $C > 0$ *such that for all* $x_0 \in X$, $p \in \mathbb{N}^*$, *and* $|Z|, |Z'| < \varepsilon p^{-\theta/2}$,

$$\begin{aligned}\Big| p^{-n} P_{p,x_0}(\phi_{x_0}(Z), \phi_{x_0}(Z')) \\ - \sum_{r=0}^{k} p^{-r/2} J_{r,x_0} \mathscr{P}_{x_0}\big(\sqrt{p}Z, \sqrt{p}Z'\big) \kappa_{x_0}^{-1/2}(Z) \kappa_{x_0}^{-1/2}(Z') \Big|_{\mathscr{C}^m(X)} \\ \leq C p^{-(k+1)/2+\delta},\end{aligned} \tag{2.22}$$

where $\{J_{r,x_0}(Z, Z')\}_{r \in \mathbb{N}}$ *is a family of polynomials in* $Z, Z' \in \mathbb{R}^{2n}$ *of the same parity as* r *and with values in* $\mathrm{End}(E_{x_0})$, *depending smoothly on* $x_0 \in X$. *Furthermore, we have*

$$J_{0,x_0}(Z, Z') \equiv \mathrm{Id}_{E_{x_0}}. \tag{2.23}$$

Parallel to Proposition 2.3 and Lemma 2.5, we have the following result on the asymptotic expansion as $p \to +\infty$ of the Berezin–Toeplitz operator (2.13). It was first proved in [27, Lemma 4.6] in the spin^c case and in this form comes essentially from [19, Lemma 3.3].

LEMMA 2.6. *Let* $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. *Then for any* $0 < \varepsilon \leq \varepsilon_0, m, k \in \mathbb{N}$, *and* $\theta \in \,]0,1[$, *there is* $C_{m,k,\theta,\varepsilon} > 0$ *such that for all* $x_0 \in X$, $p \in \mathbb{N}^*$, *and* $Z, Z' \in \mathbb{R}^{2n}$ *such that* $|Z - Z'| > \varepsilon p^{-\theta/2}$,

$$|T_{F,p}(\phi_{x_0}(Z), \phi_{x_0}(Z'))|_{\mathscr{C}^m} \leq C_{m,k,\theta,\varepsilon} p^{-k}. \tag{2.24}$$

Furthermore, for any $m, k \in \mathbb{N}$, $\varepsilon > 0$, *and* $\delta \in]0,1[$, *there are* $C > 0$ *and* $\theta \in]0,1[$ *such that for all* $x_0 \in X$, $p \in \mathbb{N}^*$, $|Z|, |Z'| < \varepsilon p^{-\theta/2}$,

$$\begin{aligned}\Big| p^{-n} T_{F,p,x_0}(\phi_{x_0}(Z), \phi_{x_0}(Z')) \\ - \sum_{r=0}^{k} p^{-r/2} \mathscr{Q}_{r,x_0} \mathscr{P}_{x_0}\big(\sqrt{p}Z, \sqrt{p}Z'\big) \kappa_{x_0}^{-1/2}(Z) \kappa_{x_0}^{-1/2}(Z') \Big|_{\mathscr{C}^m(X)} \\ \leq C p^{-(k+1)/2+\delta},\end{aligned} \tag{2.25}$$

where $\{\mathscr{Q}_{r,x_0}(Z, Z')\}_{r \in \mathbb{N}}$ *is a family of polynomials in* $Z, Z' \in \mathbb{R}^{2n}$ *of the same parity as* r *and with values in* $\mathrm{End}(E_{x_0})$, *depending smoothly on* $x_0 \in X$. *Furthermore, we have*

$$\mathscr{Q}_{0,x_0}(Z, Z') \equiv F_{x_0}. \tag{2.26}$$

2.3. *Gaussian Integrals*

We now recall some well-known facts about Gaussian integrals, which we will use for local computations in the next sections. For any $k \in \mathbb{N}^*$, let $\langle \cdot, \cdot \rangle$ denote the canonical scalar product of $\mathbb{R}^k$. For any positive symmetric matrix C acting on $\mathbb{R}^k$, we recall the following classical formula for the Gaussian integral:

$$\int_{\mathbb{R}^k} \exp(-\pi \langle Z, CZ \rangle)\, dZ = \det{}^{-1/2} C. \tag{2.27}$$

By analytic continuation this formula is still valid when C is a symmetric matrix with complex coefficients, providing the integral is well defined along a path in the space of symmetric matrices joining C with a real positive symmetric matrix. Specifically, for a positive symmetric matrix A and a real symmetric matrix B, we will consider the path

$$\begin{aligned} \gamma : [0,1] &\to \mathrm{GL}_k(\mathbb{C}) \\ t &\mapsto A + t\sqrt{-1}B. \end{aligned} \tag{2.28}$$

Then (2.27) holds for $C = A + \sqrt{-1}B$ with the determination of the square root given by continuation along the image of (2.28) by $\det^{-1} : GL_n(\mathbb{C}) \to \mathbb{C}$. Henceforth we will always use this determination of the square root of the determinant for $C = A + \sqrt{-1}B$ as before.

3. Isotropic States

In this section, we use the context and notations of Section 2. In particular, recall that (X, ω) is a compact symplectic manifold of dimension $2n$ and that the curvature of ∇^L on (L, h^L) over X satisfies (1.1).

3.1. Bohr–Sommerfeld Submanifolds

An immersed submanifold $\iota : \Lambda \to X$ is said to be *isotropic* if $\iota^*\omega = 0$. If in addition $\dim \Lambda = n$, it is said to be *Lagrangian.* Let $\nabla^{\iota^*L}, h^{\iota^*L}$ be the connection and Hermitian metric induced by ∇^L, h^L on the pullback line bundle ι^*L over Λ. Note that by (1.1) the condition $\iota^*\omega = 0$ implies that ∇^{ι^*L} is *flat.* This observation motivates the following definition.

Definition 3.1. A properly immersed oriented isotropic submanifold $\iota : \Lambda \to X$ is said to satisfy the *Bohr–Sommerfeld condition* if there exists a nonvanishing smooth section $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^*L)$ satisfying

$$\nabla^{\iota^*L}\zeta = 0. \tag{3.1}$$

Taking ζ satisfying further $|\zeta(x)|_{\iota^*L} = 1$ for any $x \in \Lambda$, the data of (Λ, ι, ζ) is called a *Bohr–Sommerfeld submanifold* of X, or a *Bohr–Sommerfeld Lagrangian* if in addition $\dim \Lambda = n$.

Note that the properness hypothesis on ι implies that Λ is compact. Furthermore, this definition depends only on the symplectic structure on (X, ω) and the prequantization condition (1.1) on (L, h^L, ∇^L). As ∇^L is Hermitian, up to renormalization, we can always assume that $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^*L)$ satisfying (3.1) is such that $|\zeta(x)|_{\iota^*L} = 1$ for any $x \in \Lambda$. Finally, by the compactness of X the properness hypothesis on ι is equivalent to the compactness of Λ.

REMARK 3.2. As noted before, if $\iota : \Lambda \to X$ is isotropic, then $\nabla^{\iota^* L}$ is flat over Λ and hence determined by its holonomy $\mathrm{hol}_{\iota^* L} : \pi_1(\Lambda) \to S^1 \subset \mathbb{C}$. We can then reformulate (3.1) by saying that $\iota : \Lambda \to X$ satisfies the Bohr–Sommerfeld condition if and only if $\mathrm{hol}_{\iota^* L} = \{1\}$. Now if the order of $\mathrm{hol}_{\iota^* L}$ is finite, then there exists a finite covering $j : \hat{\Lambda} \to \Lambda$ such that $\mathrm{hol}_{j^* \iota^* L} = \{1\}$, so that $\iota \circ j : \hat{\Lambda} \to X$ satisfies the Bohr–Sommerfeld condition. In particular, if there is $k \in \mathbb{N}$ such that $\iota : \Lambda \to X$ satisfies the Bohr–Sommerfeld condition for L^k instead of L, then the order of $\mathrm{hol}_{\iota^* L}$ divides k and thus is finite. Such $\iota : \Lambda \to X$ is called a *Bohr–Sommerfeld submanifold of order k,* and up to finite covering, Definition 3.1 also accounts for these. In the same line of thought, if $\iota : \Lambda \to X$ is not orientable, then we can always work on the orientation double cover of Λ.

Let us now set some notations. We write ι^L, ι^E, and ι_p for the natural maps covering $\iota : \Lambda \to X$ on the respective total spaces of L, E, and E_p for any $p \in \mathbb{N}^*$. If ζ is any section of $\iota^* L$, then we write ζ^p for the pth power of ζ defined as a section of $\iota^* L^p$. If additionally f is a section of $\iota^* E$, then we write $\zeta^p f$ for the induced tensor product in $\iota^* E_p$.

From now on we fix an almost complex structure J on TX compatible with ω, an auxiliary Hermitian vector bundle (E, h^E) with Hermitian connection ∇^E, and a J-invariant Riemannian metric g^{TX} on TX. We write $g^{T\Lambda} := \iota^* g^{TX}$ for the induced Riemannian metric on $T\Lambda$ and dv_Λ for the Riemannian volume form of $(\Lambda, g^{T\Lambda})$. Recall that Λ is compact by hypothesis.

DEFINITION 3.3. The *isotropic state* associated with a Bohr-Sommerfeld manifold (Λ, ι, ζ) and $f \in \mathscr{C}^\infty(\Lambda, \iota^* E)$ is the family of sections $\{s_{f,p} \in \mathscr{H}_p\}_{p \in \mathbb{N}^*}$ defined for any $x \in X$ by the formula

$$s_{f,p}(x) = \int_\Lambda P_p(x, \iota(y)) \iota_p . \zeta^p f(y)\, dv_\Lambda(y). \tag{3.2}$$

As ι is locally an embedding, when working locally, we will often omit the mention of ι, considering locally Λ as a submanifold of X. With this convention, equation (3.2) becomes

$$s_{f,p}(x) = \int_\Lambda P_p(x, y) \zeta^p f(y)\, dv_\Lambda(y). \tag{3.3}$$

We list the basic properties of isotropic states in the following proposition, which holds for any $p \in \mathbb{N}^*$.

PROPOSITION 3.4. *For any $f_1, f_2 \in \mathscr{C}^\infty(\Lambda, \iota^* E)$, we have the following additivity property*:

$$s_{f_1 + f_2, p} = s_{f_1, p} + s_{f_2, p}. \tag{3.4}$$

For any $s \in \mathscr{H}_p$, we have the following reproducing property:

$$\langle s, s_{f,p} \rangle_p = \int_\Lambda \langle s(\iota(x)), \iota_p . \zeta^p f(x) \rangle_{E_p}\, dv_\Lambda(x). \tag{3.5}$$

For any $f \in \mathscr{C}^\infty(\Lambda, \iota^* E)$ *and* $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, *the action of* $T_{F,p}$ *on* $s_{f,p}$ *is given for any* $x \in X$ *by the formula*

$$T_{F,p} s_{f,p} = \int_\Lambda T_{F,p}(x, \iota(y)) \iota_p . \zeta^p f(y)\, dv_\Lambda(y). \tag{3.6}$$

Proof. First, the additivity property (3.4) is obvious from (3.2). Next, recall that P_p is self-adjoint with respect to $\langle \cdot, \cdot \rangle_p$ for any $p \in \mathbb{N}^*$ and restricts to the identity of $\mathscr{H}_p$. Then using (2.12), (3.3), and the Fubini theorem, for any $s \in \mathscr{H}_p$, we compute

$$\begin{aligned}
\langle s, s_{f,p} \rangle_p &= \int_X \left\langle s(y), \int_\Lambda P_p(y, \iota(x)) \iota_p . \zeta^p f(x)\, dv_\Lambda(x) \right\rangle_{E_p} dv_X(y) \\
&= \int_\Lambda \left\langle \int_X P_p(\iota(x), y) s(y)\, dv_X(y), \iota_p . \zeta^p f(x) \right\rangle_{E_p} dv_\Lambda(x) \\
&= \int_\Lambda \langle s(\iota(x)), \iota_p . \zeta^p f(x) \rangle_{E_p}\, dv_\Lambda(x).
\end{aligned} \tag{3.7}$$

The reproducing property (3.5) follows from (3.7). Finally, from (2.13) we get for any $f \in \mathscr{C}^\infty(\Lambda, \iota^* E)$ and $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ that $T_{F,p} s_{f,p} = P_p F s_{f,p}$. Then by (2.14), (3.2), and the Fubini theorem, for any $x \in X$, we get

$$\begin{aligned}
(T_{F,p} s_{f,p})(x) &= \int_X \int_\Lambda P_p(x, w) F(w) P_p(w, \iota(y)) \iota_p . \zeta^p f(y)\, dv_\Lambda(y)\, dv_X(w) \\
&= \int_\Lambda T_{F,p}(w, \iota(y)) \iota_p . \zeta^p f(y)\, dv_\Lambda(y).
\end{aligned} \tag{3.8}$$

From (3.8) we get (3.6). □

3.2. *Asymptotic Expansion of Isotropic States*

In this section, we establish the first semiclassical properties of isotropic states. In particular, we show that the L^2-norm of an isotropic state admits an asymptotic expansion as $p \to +\infty$, and we compute the highest order term.

For any $p \in \mathbb{N}^*$, we write $|\cdot|_{E_p}$ for the norm on E_p induced by h^L and h^E. In the following proposition, we show how an isotropic state concentrates around the image of the associated isotropic submanifold as $p \to +\infty$.

PROPOSITION 3.5. *Let* $f \in \mathscr{C}^\infty(\Lambda, \iota^* E)$. *For any closed subset* $K \subset X$ *such that* $K \cap \iota(\Lambda) = \emptyset$ *and for any* $k \in \mathbb{N}$, *there exists* $C_k > 0$ *such that for all* $x \in K$ *and* $p \in \mathbb{N}^*$,

$$|s_{f,p}(x)|_{E_p} < C_k p^{-k}. \tag{3.9}$$

Proof. This is a direct consequence of Proposition 2.3 and formula (3.2). □

Recall that for any $p \in \mathbb{N}^*$, we write $\| \cdot \|_p$ for the norm on $\mathscr{C}^\infty(X, E_p)$ induced by $\langle \cdot, \cdot \rangle_p$, and we write $|\cdot|_{\iota^* E}$ for the norm on $\iota^* E$ over Λ induced by h^E. The rest of the section is devoted to the proof of the following theorem.

THEOREM 3.6. *Let $f \in \mathscr{C}^\infty(\Lambda, \iota^* E)$. Then there exist $a_r \in \mathbb{R}$, $r \in \mathbb{N}$, such that for any $k \in \mathbb{N}$ and as $p \to +\infty$,*

$$\|s_{f,p}\|_p^2 = p^{n-\dim\Lambda/2} \sum_{r=0}^{k} p^{-r} a_r + O(p^{n-\dim\Lambda/2-(k+1)}) \tag{3.10}$$

with the first coefficient $a_0 \in \mathbb{R}$ given by

$$a_0 = 2^{\dim\Lambda/2} \int_\Lambda |f|^2_{\iota^* E} \det(\dot{R}^L_{x_0}/2\pi) \frac{dv_\Lambda}{dv_{\Lambda,\omega}} \, dv_\Lambda. \tag{3.11}$$

Additionally, for any $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, the product $\langle T_{F,p} s_{f,p}, s_{f,p}\rangle_p$ satisfies the expansion of (3.10) with $a_r \in \mathbb{C}$, $r \in \mathbb{N}$, and

$$a_0 = 2^{\dim\Lambda/2} \int_\Lambda \langle Ff, f\rangle_{\iota^* E} \det(\dot{R}^L_{x_0}/2\pi) \frac{dv_\Lambda}{dv_{\Lambda,\omega}} \, dv_\Lambda. \tag{3.12}$$

Proof. Note first that the reproducing property (3.5) gives

$$\|s_{f,p}\|_p^2 = \int_\Lambda \langle s_{f,p}(\iota(x)), \zeta^p f(x)\rangle_{E_p} \, dv_\Lambda(x). \tag{3.13}$$

Using (3.13), we are reduced to evaluate $s_{f,p}$ on the image of $\iota : \Lambda \to X$. Then let $x_0 \in X$ be in the image of ι. As $\iota : \Lambda \to X$ is an immersion, there is an integer $m \in \mathbb{N}$ such that for any small enough connected neighborhood V of x_0 in X, there are m disjoint connected open sets $U_1, \dots, U_m \subset \Lambda$ such that $\iota^{-1}(V) = \bigcup_{j=1}^m U_j$. Using Proposition 2.3, we can localize the problem as $p \to +\infty$ in the following way:

$$\begin{aligned} s_{f,p}(x_0) &= \int_\Lambda P_p(x_0, \iota(x)) \zeta^p f(x) \, dv_\Lambda(x) \\ &= \sum_{j=1}^m \int_{U_j} P_p(x_0, \iota(x)) \zeta^p f(x) \, dv_\Lambda(x) + O(p^{-\infty}). \end{aligned} \tag{3.14}$$

In view of (3.10), (3.13), and (3.14), we can assume that f has compact support around $\bigcup_{j=1}^m U_j$. Using (3.4) and (3.13), we are further reduced to the case where f has compact support around one of the U_j for some j. As $U := U_j$ is embedded in X through ι, we can consider U as a submanifold of X, and (3.14) translates to

$$s_{f,p}(x_0) = \int_U P_p(x_0, x) \zeta^p f(x) \, dv_\Lambda(x) + O(p^{-\infty}). \tag{3.15}$$

By the definition of U as a submanifold of X, we can take $\phi_{x_0} : B^{\mathbb{R}^{2n}}(0, \varepsilon) \to V$ with $\varepsilon > 0$ and $V \subset X$ as in (2.17) such that ϕ_{x_0} identifies $U \subset V$ with $B^\Sigma(0, \varepsilon)$, where Σ is a vector subspace of $\mathbb{R}^{2n}$. Then Σ is an isotropic subspace of $(\mathbb{R}^{2n}, \Omega)$. We identify E and L over $B^{\mathbb{R}^{2n}}(0, \varepsilon)$ with E_{x_0} and L_{x_0} as in Section 2.2. In particular, we use the unitary vector $\zeta(x_0)$ to identify L_{x_0} with $\mathbb{C}$, where the section $\zeta \in \mathscr{C}^\infty(\Lambda, \iota^* L)$ is associated with (Λ, ι, ζ) as in Definition 3.1. As ζ is parallel with respect to $\nabla^{\iota^* L}$ along Λ, it is identified with $1 \in \mathbb{C}$ over $B^\Sigma(0, \varepsilon)$ in this trivialization. Let du be the Lebesgue measure of Σ, and define the function $h \in \mathscr{C}^\infty(B^\Sigma(0,\varepsilon),\mathbb{R})$ for all $u \in B^\Sigma(0,\varepsilon)$ by

$$dv_\Lambda(u) = h(u)\,du, \quad \text{with } h(0) = (dv_\Lambda/dv_{\Lambda,\omega})(x_0). \tag{3.16}$$

Using Corollary 2.4, Lemma 2.5, and (2.21), for any $\delta \in]0,1[$, we get $\theta \in]0,1[$ such that as $p \to +\infty$,

$$\begin{aligned}
&\langle s_{f,p}(x_0), \zeta^p f(x_0)\rangle_{E_p}\\
&= \int_{B^\Sigma(0,\varepsilon p^{-\theta/2})} \langle P_p(x_0,\phi_{x_0}(u))\zeta^p f(\phi_{x_0}(u)), \zeta^p f(x_0)\rangle_{E_p}\, dv_\Lambda(u) + O(p^{-\infty})\\
&= p^n \int_{B^\Sigma(0,\varepsilon p^{-\theta/2})} \sum_{r=0}^k p^{-r/2}\big\langle J_{r,x_0}\mathscr{P}_{x_0}\big(0,\sqrt{p}u\big) f_{x_0}(u), f(x_0)\big\rangle_E\\
&\quad \times \kappa_{x_0}^{-1/2}(u)\kappa_{x_0}^{-1/2}(0)\, dv_\Lambda(u)\\
&\quad + p^n \int_{B^\Sigma(0,\varepsilon p^{-\theta/2})} O(p^{-(k+1)/2+\delta})\, dv_\Lambda(u) + O(p^{-\infty})\\
&= p^n \int_{B^\Sigma(0,\varepsilon p^{-\theta/2})} \det(\dot{R}^L_{x_0}/2\pi)^{1/2}\\
&\quad \times \sum_{r=0}^k p^{-r/2}\big\langle J_{r,x_0}\mathscr{P}_{x_0}\big(0,\sqrt{p}u\big) f_{x_0}(u), f(x_0)\big\rangle_E\\
&\quad \times \kappa_{x_0}^{-1/2}(u) h(u)\, du + p^n p^{-\theta \dim\Lambda/2} O(p^{-(k+1)/2+\delta}).
\end{aligned} \tag{3.17}$$

Let us write $g_{x_0} = h\kappa_{x_0}^{1/2} f_{x_0} \in \mathscr{C}^\infty(B^\Sigma(0,\varepsilon), E_{x_0})$. Then from (2.21) and (3.16) we get the following Taylor expansion in $u \in \mathbb{R}^n$ up to order $k \in \mathbb{N}$:

$$\begin{aligned}
g_{x_0}(u) &= (h\kappa_{x_0}^{-1/2} f_{x_0})(0) + \sum_{1\le|\alpha|\le k} \frac{\partial^{|\alpha|} g_{x_0}}{\partial u^\alpha}\frac{u^\alpha}{\alpha!} + O(|u|^{k+1})\\
&= f(x_0)(dv_\Lambda/dv_{\Lambda,\omega})(x_0)\det(\dot{R}^L_{x_0}/2\pi)^{1/2}\\
&\quad + \sum_{1\le|\alpha|\le k} p^{-\alpha/2}\frac{\partial^{|\alpha|} g_{x_0}}{\partial u^\alpha}\frac{(\sqrt{p}u)^\alpha}{\alpha!}\\
&\quad + p^{-(k+1)/2} O\big(\big|\sqrt{p}u\big|^{k+1}\big).
\end{aligned} \tag{3.18}$$

On the other hand, recall from Lemma 2.5 that $J_{r,x_0}(0,\sqrt{p}u) \in \mathrm{End}(E_{x_0})$ is a polynomial in $\sqrt{p}u$ of the same parity as $r \in \mathbb{N}$. Let M_k be the supremum of the degree of J_{r,x_0} for $1 \le r \le k$, and write $\delta' = \delta + (M_k + k + 1 + d)(1-\theta)/2$. From (3.17) and (3.18) we deduce the existence of a sequence $\{G_r\}_{r\in\mathbb{N}}$ of polynomials in one variable of $\mathbb{R}^n$ of the same parity as r, with values in $\mathbb{C}$ and with

$$G_0 \equiv |f(x_0)|_E^2 \frac{dv_\Lambda}{dv_{\Lambda,\omega}}(x_0)\det(\dot{R}^L_{x_0}/2\pi) \tag{3.19}$$

such that, as $p \to +\infty$,

$$\begin{aligned}
&\langle s_{f,p}(x_0), \zeta^p f(x_0)\rangle_{E_p} \\
&\quad = p^n \sum_{r=0}^{k} p^{-r/2} \int_{B^\Sigma(0,\varepsilon p^{-\theta/2})} G_r\big(\sqrt{p}u\big)\mathscr{P}_{x_0}\big(0,\sqrt{p}u\big)\,du \\
&\qquad + O(p^{n-(\dim\Lambda+k+1)/2+\delta'}) \\
&\quad = p^{n-\dim\Lambda/2} \sum_{r=0}^{k} p^{-r/2} \int_{B^\Sigma(0,\varepsilon p^{(1-\theta)/2})} G_r(u)\mathscr{P}_{x_0}(0,u)\,du \\
&\qquad + O(p^{n-(\dim\Lambda+k+1)/2+\delta'}).
\end{aligned} \tag{3.20}$$

Recall from (2.20) that

$$\mathscr{P}_{x_0}(0,u) = \exp\left(-\frac{\pi}{2}|u|^2\right), \tag{3.21}$$

so that as $1-\theta > 0$, the integral of $\mathscr{P}_{x_0}(0,u)$ over $\mathbb{R}^n \backslash B^\Sigma(0,\varepsilon p^{(1-\theta)/2})$ with respect to u decreases exponentially as $p \to +\infty$, and then we deduce from (3.20) that

$$\begin{aligned}
&\langle s_{f,p}(x_0), \zeta^p f(x_0)\rangle_{E_p} \\
&\quad = p^{n-d/2} \sum_{r=0}^{k} p^{-r/2} \int_\Sigma G_r(u)\mathscr{P}_{x_0}(0,u)\,du \\
&\qquad \times + O(p^{n-(\dim\Lambda+k+1)/2+\delta'}).
\end{aligned} \tag{3.22}$$

As G_r is of the same parity as r, we immediately deduce from (3.21) that for any $m \in \mathbb{N}$,

$$\int_\Sigma G_{2m+1}(u)\mathscr{P}_{x_0}(0,u)\,du = 0. \tag{3.23}$$

Finally, from (3.19) and (3.21) we get the following formula for the highest order term of (3.22):

$$\begin{aligned}
&\int_\Sigma G_0(u)\mathscr{P}_{x_0}(0,u)\,du \\
&\quad = |f(x_0)|_E^2 (dv_\Lambda/dv_{\Lambda,\omega})(x_0)\det(\dot{R}^L_{x_0}/2\pi)\int_\Sigma \exp\left(-\frac{\pi}{2}|u|^2\right)du \\
&\quad = 2^{\dim\Lambda/2}|f(x_0)|_E^2 (dv_\Lambda/dv_{\Lambda,\omega})(x_0)\det(\dot{R}^L_{x_0}/2\pi).
\end{aligned} \tag{3.24}$$

Then recalling that all the estimates above are uniform in $x_0 \in \iota(\Lambda)$, by (3.13), (3.23), and (2.5) it suffices to integrate (3.22) and (3.24) over $x_0 \in \iota(\Lambda)$ with respect to dv_Λ to get (3.10) and (3.11).

Using property (3.6), the proof of the asymptotic expansion as $p \to +\infty$ of $\langle T_{F,p}s_{f,p}, s_{f,p}\rangle_p$ is completely analogous to the proof of the asymptotic expansion of $\|s_p\|_p$, simply replacing the polynomials J_{r,x_0} of Lemma 2.5 by the polynomials $\mathcal{Q}_{r,x_0}$ of Lemma 2.6 in the previous computations. This achieves the proof of Theorem 3.6. □

4. Isotropic Intersections

Let us consider two Bohr–Sommerfeld submanifolds $(\Lambda_j, \iota_j, \zeta_j)$ together with $f_j \in \mathscr{C}^\infty(\Lambda_j, \iota_j^* E)$ for $j = 1, 2$, and set $d_j = \dim \Lambda_j$. In this section, we establish the existence of an asymptotic expansion as $p \to +\infty$ of the Hermitian product $\langle s_{f_1,p}, s_{f_2,p} \rangle_p$ of the two associated isotropic states, and we compute the highest order term, which depends only on the geometry of the intersection. Note that the case $\{s_{f_1,p}\}_{p\in\mathbb{N}^*} = \{s_{f_2,p}\}_{p\in\mathbb{N}^*}$ is precisely the result of Theorem 3.6.

We need the following regularity assumption, which we will use throughout the section.

Definition 4.1. We say that two proper immersions $\iota_j : \Lambda_j \to X$, $j = 1, 2$, are intersecting *cleanly* if for any $x \in \iota_1(\Lambda_1) \cap \iota_2(\Lambda_2)$ and $y_j \in \Lambda_j$ such that $\iota_1(y_1) = \iota_2(y_2) = x$, there exist neighborhoods $U_j \subset \Lambda_j$ of y_j such that the intersection $\iota_1(U_1) \cap \iota_2(U_2)$ is a submanifold of X satisfying

$$T_x\iota_1(U_1) \cap T_x\iota_2(U_2) = T_x(\iota_1(U_1) \cap \iota_2(U_2)) .$$

The *intersection* of two immersions $\iota_1 : \Lambda_1 \to X$ and $\iota_2 : \Lambda_2 \to X$ over X is defined as their fibered product, which we write $\Lambda_1 \cap \Lambda_2$ and which comes with two immersions $j_i : \Lambda_1 \cap \Lambda_2 \to \Lambda_i$, $i = 1, 2$, such that $\iota_1 \circ j_1 = \iota_2 \circ j_2$ and universal for this property. Under the assumption of Definition 4.1, this fibered product has a natural smooth structure. In fact, consider smooth atlases $\mathscr{U}_1, \mathscr{U}_2$ of Λ_1, Λ_2, respectively, such that for any $U_j \in \mathscr{U}_j$, $j = 1, 2$, the immersion ι_j restricted to U_j is an embedding satisfying the assumption of Definition 4.1 as soon as the intersection is nonempty. Then we can define an atlas of $\Lambda_1 \cap \Lambda_2$ as the set of all intersections $U_1 \cap U_2$ for all $U_1 \in \mathscr{U}_1$ and $U_2 \in \mathscr{U}_2$, with transition maps induced by those of $\mathscr{U}_1$ and $\mathscr{U}_2$.

Note that this definition of intersection is local and reduces to the usual one in the case of embeddings. For that reason, we can readily reduce to the usual definition of a clean intersection when working locally. A typical situation where this general definition is needed is in the natural case where $\iota_1 : \Lambda_1 \to X$ and $\iota_2 : \Lambda_2 \to X$ are Bohr–Sommerfeld submanifolds of respective orders $k_1 \in \mathbb{N}^*$ and $k_2 \in \mathbb{N}^*$ in the sense Remark 3.2, with k_1 and k_2 prime with each other.

4.1. Asymptotic Expansion of Discrete Intersections

In this section, we deal with the case of discrete intersections. We first consider the easy case where the intersection is empty.

Proposition 4.2. *Suppose that* $\Lambda_1 \cap \Lambda_2 = \emptyset$, *and let* $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$. *Then for any* $k \in \mathbb{N}$, *there exists* $C_k > 0$ *such that for all* $p \in \mathbb{N}^*$,

$$|\langle T_{F,p} s_{f_1,p}, s_{f_2,p} \rangle_p| < C_k p^{-k}. \tag{4.1}$$

Proof. Using the reproducing property (3.5), for any $p \in \mathbb{N}^*$, we get

$$\langle T_{F,p} s_{f_1,p}, s_{f_2,p} \rangle_p = \int_\Lambda \langle T_{F,p} s_{f_1,p}(\iota_2(x)), \zeta_2^p f_2(x) \rangle_{E_p} dv_{\Lambda_2}(x). \tag{4.2}$$

In particular, as Λ_2 is compact by hypothesis, we can choose $K = \iota_2(\Lambda_2)$ in Proposition 3.5, and we deduce (4.1) from (4.2). $\square$

In view of Proposition 4.2, from now on we assume that $\Lambda_1 \cap \Lambda_2$ is not empty. In the statement of the following theorem, the immersions $j_i : \Lambda_1 \cap \Lambda_2 \to \Lambda_i$ and $\iota_i : \Lambda_i \to X$, $i = 1, 2$, are implicit, and we omit to mention them for simplicity.

Theorem 4.3. *Suppose that* $(\Lambda_1, \iota_1, \zeta_1)$ *and* $(\Lambda_2, \iota_2, \zeta_2)$ *intersect cleanly and that their intersection* $\Lambda_1 \cap \Lambda_2$ *in the sense above is discrete. Set* $m = \#\Lambda_1 \cap \Lambda_2$ *and write* $\Lambda_1 \cap \Lambda_2 = \{x_1, \dots, x_m\}$. *Then for any* $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, *there exist* $b_{q,r} \in \mathbb{C}$, $r \in \mathbb{N}$, $1 \le q \le m$, *such that for any* $k \in \mathbb{N}$ *and as* $p \to +\infty$,

$$\begin{aligned}\langle T_{F,p} s_{f_1,p}, s_{f_2,p}\rangle_p \\ = p^{n-(d_1+d_2)/2} \sum_{q=1}^m \lambda_q^p \sum_{r=0}^k p^{-r} b_{q,r} + O(p^{n-(d_1+d_2)/2-(k+1)}),\end{aligned} \tag{4.3}$$

where $\lambda_q = \langle \zeta_1(x_q), \zeta_2(x_q)\rangle_L$. *Furthermore, if* $\dim \Lambda_1 = n$, *then we have*

$$\begin{aligned} b_{q,0} = 2^{n/2} \langle F_{x_q} f_1(x_q), f_2(x_q)\rangle_{x_q} \det(\dot{R}^L_{x_q}/2\pi) \frac{dv_{\Lambda_1}}{dv_{\Lambda_1,\omega}} \frac{dv_{\Lambda_2}}{dv_{\Lambda_2,\omega}}(x_q) \\ \times \det{}^{-1/2}\left\{\sqrt{-1}\sum_{k=1}^n h^{TX}_\omega(e_k, \nu_i)\omega(e_k, \nu_j)\right\}_{i,j=1}^{d_2}, \end{aligned} \tag{4.4}$$

where $\langle e_i\rangle_{i=1}^n$, $\langle \nu_j\rangle_{j=1}^{d_2}$ *are oriented orthonormal bases for* g^{TX}_ω *of the tangent spaces of* Λ_1, Λ_2 *in* X *at* x_q, *and the square root of the determinant is determined by* (2.28).

Proof. We will prove Theorem 4.3 for $F = \mathrm{Id}_E$ (so that $T_{F,p} = P_p$), the proof of the general case being totally analogous by Lemma 2.6 and property (3.6). First, using the reproducing property (3.5), for any $p \in \mathbb{N}^*$, we get

$$\langle s_{f_1,p}, s_{f_2,p}\rangle_p = \int_\Lambda \langle s_{f_1,p}(\iota_2(x)), \zeta_2^p f_2(x)\rangle_{E_p} \, dv_{\Lambda_2}(x). \tag{4.5}$$

Then we can reproduce the argument in the proof of Proposition 4.2, using Proposition 3.5 to reduce the proof to the case of f_2 with compact support in a given neighborhood of $\iota_2^{-1}(\iota_1(\Lambda_1) \cap \iota_2(\Lambda_2))$, which is a finite set by assumption. Symmetrically, using the reproducing property of $s_{f_1,p}$ instead of $s_{f_2,p}$, we can assume further that f_1 has compact support in a given neighborhood of $\iota_1^{-1}(\iota_1(\Lambda_1) \cap \iota_2(\Lambda_2))$. By the additivity property (3.4) and (4.3), we are further reduced to the case of f_i with compact support in a neighborhood of a single point $y_i \in \Lambda_i$ for $i = 1, 2$. Using Proposition 3.5, we are finally reduced to the case $\iota_1(y_1) = \iota_2(y_2)$. Set $x_0 := \iota_1(y_1) = \iota_2(y_2) \in X$.

Let $U_j \subset \Lambda_j$ be as in Definition 4.1, intersecting cleanly at $x_0 \in X$ only. In particular, using Definition 3.3 of an isotropic state, equation (4.5) becomes, as

$p \to +\infty$,

$$\begin{aligned}\langle s_{1,p}, s_{2,p}\rangle_p &= \int_{U_2} \langle s_{f_1,p}(x), \zeta_2^p f_2(x)\rangle_{E_p}\, dv_{\Lambda_2}(x) + O(p^{-\infty}) \\ &= \int_{U_2}\int_{U_1} \langle P_p(x,y)\zeta_1^p f_1(y), \zeta_2^p f_2(x)\rangle_{E_p}\, dv_{\Lambda_1}(y)\, dv_{\Lambda_2}(x) \\ &\quad + O(p^{-\infty}).\end{aligned} \tag{4.6}$$

By definition of U_1 as a submanifold of X, we can consider a chart as in (2.17) in which U_1 is identified with $B^{\Sigma_1}(0,\varepsilon)$ for some $\varepsilon > 0$, where Σ_1 is an isotropic space of $(\mathbb{R}^{2n}, \Omega)$. In this chart, we consider a projection $\pi_{1,2} : \mathbb{R}^{2n} \to \Sigma_2$ preserving Σ_1, where Σ_2 it the tangent space to U_2 at x_0 in this chart, and use it to identify U_2 with $B^{\Sigma_2}(0,\varepsilon)$. In that way, we can construct $\phi_{x_0} : B^{\mathbb{R}^{2n}}(0,\varepsilon) \to V$, with $\varepsilon > 0$ and $V \subset X$ as in (2.17) such that $V \cup \Lambda_j = U_j$, such that φ_{x_0} identifies U_j with $B^{\Sigma_j}(0,\varepsilon)$ for any $j = 1, 2$, where Σ_1 and Σ_2 are isotropic subspaces of $(\mathbb{R}^{2n}, \Omega)$. As U_1 and U_2 intersect cleanly at x_0 only, we have $\Sigma_1 \cap \Sigma_2 = \{0\}$. We identify E and L over $B^{\mathbb{R}^{2n}}(0,\varepsilon)$ with E_{x_0} and L_{x_0} as in Section 2.2 and use the unitary vector $\zeta_1(x_0)$ to identify L_{x_0} with $\mathbb{C}$. Then ζ_1 is identified with $1 \in \mathbb{C}$ over $B^{\Sigma_1}(0,\varepsilon)$ in this trivialization. As ζ_2 is parallel with respect to $\nabla^{\iota_2^* L}$ over U_2, it is identified with $\bar\lambda \in \mathbb{C}$ over $B^{\Sigma_2}(0,\varepsilon)$, where $\lambda = \langle \zeta_1(x_0), \zeta_2(x_0)\rangle_L$.

Then, as $p \to +\infty$, equation (4.6) becomes

$$\begin{aligned}\langle s_{1,p}, s_{2,p}\rangle_p = \lambda^p \int_{B^{\Sigma_2}(0,\varepsilon)}\int_{B^{\Sigma_1}(0,\varepsilon)} &\langle P_p(\phi_{x_0}(Z), \phi_{x_0}(Z')) f_{1,x_0}(Z'), f_{2,x_0}(Z)\rangle_E \\ &\times dv_{\Lambda_1}(Z')\, dv_{\Lambda_2}(Z) + O(p^{-\infty}).\end{aligned} \tag{4.7}$$

Let du and dw be the Lebesgue measures of Σ_1 and Σ_2, respectively. For any $j = 1, 2$, define the functions $h_j \in \mathscr{C}^\infty(B^{\Sigma_j}(0,\varepsilon), \mathbb{R})$ in the chart (2.17) for any $u \in B^{\Sigma_1}(0,\varepsilon)$ and $w \in B^{\Sigma_2}(0,\varepsilon)$ by

$$dv_{\Lambda_1}(u) = h_1(u)\, du \quad \text{and} \quad dv_{\Lambda_2}(w) = h_2(w)\, dw \tag{4.8}$$

with $h_j(0) = (dv_{\Lambda_j}/dv_{\Lambda_j,\omega})(x_0)$ for $j = 1, 2$. Recalling (2.18) and the fact that $|\lambda^p| = 1$ for all $p \in \mathbb{N}^*$, we can use Corollary 2.4 and Lemma 2.5 to get $\theta \in \,]0,1[$ for any $k \in \mathbb{N}$ and $\delta \in \,]0,1[$ such that, as $p \to +\infty$, equation (4.7) becomes

$$\begin{aligned}&\langle s_{1,p}, s_{2,p}\rangle_p \\ &\quad= \lambda^p \int_{B^{\Sigma_2}(0,\varepsilon p^{-\theta/2})}\int_{B^{\Sigma_1}(0,\varepsilon p^{-\theta/2})} \\ &\qquad \langle P_p(\phi_{x_0}(Z), \phi_{x_0}(Z')) f_{1,x_0}(Z'), f_{2,x_0}(Z)\rangle_E\, dv_{\Lambda_1}(Z')\, dv_{\Lambda_2}(Z) \\ &\qquad + O(p^{-\infty}) \\ &\quad= \lambda^p p^n \sum_{r=0}^k p^{-r/2} \int_{B^{\Sigma_2}(0,\varepsilon p^{-\theta/2})}\int_{B^{\Sigma_1}(0,\varepsilon p^{-\theta/2})} \\ &\qquad \big\langle J_{r,x_0}\mathscr{P}_{x_0}\big(\sqrt{p}Z, \sqrt{p}Z'\big) f_{1,x_0}(Z'), f_{2,x_0}(Z)\big\rangle_E \\ &\qquad \times \kappa_{x_0}^{-1/2}(Z')\kappa_{x_0}^{-1/2}(Z)\, dv_{\Lambda_1}(Z')\, dv_{\Lambda_2}(Z)\end{aligned}$$

$$
\begin{aligned}
&+ p^n \int_{B^{\Sigma_2}(0,\varepsilon p^{-\theta/2})} \int_{B^{\Sigma_2}(0,\varepsilon p^{-\theta/2})} O(p^{-(k+1)/2+\delta})\, dv_{\Lambda_1}(Z')\, dv_{\Lambda_2}(Z) \\
&+ O(p^{-\infty}) \\
= {} & \lambda^p p^n \sum_{r=0}^{k} p^{-r/2} \int_{B^{\Sigma_2}(0,\varepsilon p^{-\theta/2})} \int_{B^{\Sigma_1}(0,\varepsilon p^{-\theta/2})} \\
&\big\langle J_{r,x_0} \mathscr{P}_{x_0}\big(\sqrt{p}w, \sqrt{p}u\big) f_1(u), f_2(w)\big\rangle_E \\
&\times \kappa_{x_0}^{-1/2}(u)\kappa_{x_0}^{-1/2}(w) h_1(u) h_2(w)\, du\, dw \\
&+ p^n p^{-(d_1+d_2)\theta/2} O(p^{-(k+1)/2+\delta}).
\end{aligned} \tag{4.9}
$$

Consider now the Taylor expansion up to order $k \in \mathbb{N}$ of $g_j = h_j \kappa_{x_0}^{-1/2} f_{j,x_0}$ for $j = 1, 2$ as in (3.18). By Lemma 2.5 and formula (2.21), following the proof of Theorem 3.6, we get $\delta' > 0$ and a sequence $\{G_r\}_{r\in\mathbb{N}}$ of polynomials in two variables of $\mathbb{R}^{2n}$ with values in $\mathbb{C}$ of the same parity as r with

$$
G_0 \equiv \langle f_1(x_0), f_2(x_0)\rangle_E \frac{dv_{\Lambda_1}}{dv_{\Lambda_1,\omega}} \frac{dv_{\Lambda_2}}{dv_{\Lambda_2,\omega}}(x_0) \det(\dot{R}^L_{x_0}/2\pi) \tag{4.10}
$$

such that, as $p \to +\infty$, equation (4.9) becomes

$$
\begin{aligned}
\langle s_{1,p}, s_{2,p}\rangle_p = \lambda^p p^{n-(d_1+d_2)/2} \sum_{r=1}^{k} p^{-r/2} \int_{B^{\Sigma_2}(0,\varepsilon p^{(1-\theta)/2})} \int_{B^{\Sigma_1}(0,\varepsilon p^{(1-\theta)/2})} \\
G_r \mathscr{P}_{x_0}(w,u)\, du\, dw + O(p^{n-(d_1+d_2+k+1)/2+\delta'}).
\end{aligned} \tag{4.11}
$$

As $\Sigma_1 \cap \Sigma_2 = \{0\}$, from (2.20) we get that

$$
|\mathscr{P}_{x_0}(w,u)| \le \exp(C(|u| + |w|)) \tag{4.12}
$$

for some $C > 0$ and all $w \in \Sigma_1$ and $u \in \Sigma_2$. In particular, as $1-\theta > 0$, its integral in $u \in \Sigma_1 \backslash B^{\Sigma_1}(0, \varepsilon p^{(1-\theta)/2})$ and $w \in \Sigma_2 \backslash B^{\Sigma_2}(0, \varepsilon p^{(1-\theta)/2})$ decreases exponentially and uniformly as $p \to +\infty$. Equation (4.11) then becomes

$$
\begin{aligned}
&\langle s_{1,p}, s_{2,p}\rangle_p \\
&= \lambda^p \sum_{r=1}^{k} p^{-r/2} \int_{\Sigma_2} \int_{\Sigma_1} G_r \mathscr{P}_{x_0}(w,u)\, du\, dw + O(p^{-(k+1)/2+\delta'}).
\end{aligned} \tag{4.13}
$$

Let us now evaluate the integrals in (4.13). Up to linear symplectic transformation, the canonical symplectic basis $\{e_j, f_j\}_{j=1}^n$ of $(\mathbb{R}^{2n}, \Omega)$ can be chosen such that $\Sigma_1 = \langle e_1, \dots, e_{d_1}\rangle$ as an oriented isotropic subspace. Let $\nu_1, \dots, \nu_{d_2} \in \Sigma_2$ form an oriented orthonormal basis of Σ_2 for the metric induced by $\langle\cdot,\cdot\rangle$. Consider the matrices A and B given by

$$
\begin{aligned}
A &= (a_i^j)_{1\le i\le n, 1\le j\le d_2} \quad \text{with } a_i^j = \Omega(e_i, \nu_j), \\
B &= (b_i^j)_{1\le i\le n, 1\le j\le d_2} \quad \text{with } b_i^j = \langle e_i, \nu_j\rangle.
\end{aligned} \tag{4.14}
$$

As $\Omega(e_i, \nu_j) = \langle f_i, \nu_j \rangle$ for all $1 \leq i \leq n, 1 \leq j \leq d_2$, we know that for any $1 \leq j \leq d_2$,

$$\nu_j = \sum_{i=1}^{n} b_i^j e_i + \sum_{i=1}^{n} a_i^j f_i. \tag{4.15}$$

Let us write $dt := dt_1 \dots dt_{d_2}$ for the Lebesgue measure of $\mathbb{R}^{d_2}$, and let φ be any measurable function with compact support on $\mathbb{R}^{2n}$. Setting $w = t_i \nu_i$ for any $w \in \Sigma_2$, integration of φ along Σ_2 for its Lebesgue measure dw becomes

$$\int_{\Sigma_2} \varphi(w)\, dw = \int_{\mathbb{R}^{d_2}} \varphi\left(\sum_{j=1}^{d_2} t_j \nu_j \right) dt. \tag{4.16}$$

Let us use the convention of Section 2.1, summing i from 1 to d_1 and k, j from 1 to d_2 whenever they appear as free indices. From the explicit expression (2.20), taking Fourier transform and performing a change of variables, we compute

$$\begin{aligned}
&\int_{\Sigma_2} \int_{\Sigma_1} G_r(w,u) \mathscr{P}_{x_0}(w,u)\, du\, dw \\
&\quad = \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} G_r(t_j \nu_j, u_i e_i) \mathscr{P}_{x_0}(t_j \nu_j, u_i e_i)\, du\, dt \\
&\quad = \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} G_r(t_j \nu_j, u_i e_i) \exp\left(-\frac{\pi}{2} \sum_{i=d_1+1}^{n} ((t_j b_i^j)^2 + (t_j a_i^j)^2) \right) \\
&\qquad \times \exp\left(-\frac{\pi}{2} \sum_{i=1}^{d_1} \big((u_i - t_j b_i^j)^2 + (t_j a_i^j)^2 + 2\sqrt{-1} u_i t_j a_i^j \big) \right) du\, dt \\
&\quad = \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} G_r(t_j \nu_j, (u_i + t_j b_i^j) e_i) \exp\left(-\frac{\pi}{2} \sum_{i=d_1+1}^{n} ((t_j b_i^j)^2 + (t_j a_i^j)^2) \right) \\
&\qquad \times \exp\left(-\frac{\pi}{2} \sum_{i=1}^{d_1} \big(u_i^2 + (t_j a_i^j)^2 + 2\sqrt{-1} u_i t_j a_i^j + 2\sqrt{-1} t_k b_i^k a_i^j t_j \big) \right) du\, dt \\
&\quad = 2^{d_2/2} \int_{\mathbb{R}^{d_2}} \widetilde{G}_r(t) \exp\left(-\frac{\pi}{2} \sum_{i=d_1+1}^{n} ((t_j b_i^j)^2 + (t_j a_i^j)^2) \right) \\
&\qquad \times \exp\left(-\pi \sum_{i=1}^{d_1} \big((t_j a_i^j)^2 + \sqrt{-1} t_k b_i^k a_i^j t_j \big) \right) dt,
\end{aligned} \tag{4.17}$$

where $\widetilde{G}_r(t)$ are polynomials in $t \in \mathbb{R}^{d_1}$ of the same parity as r. Using the fact that $\Sigma_1 \cap \Sigma_2 = \{0\}$, we get the convergence of the integral in (4.17), and as the integrand is of the same parity as r, the integral vanishes if r is odd. Together with (4.13), this proves (4.3).

Let us now compute the first coefficient of (4.13) in the case $\dim \Lambda_1 = n$. From (4.17) we get

$$\int_{\Sigma_2}\int_{\Sigma_1} \mathscr{P}_{x_0}(u,w)\,du\,dw$$
$$= 2^{n/2}\int_{\mathbb{R}^{d_2}} \exp\Big(-\pi \sum_{i=1}^{n}\big((t_j a_i^j)^2 + \sqrt{-1}t_k b_i^k a_i^j t_j\big)\Big)\,dt. \tag{4.18}$$

As $\langle \nu_1, \dots, \nu_{d_2}\rangle$ is the basis of an isotropic submanifold, we get $\omega(\nu_j, \nu_k) = 0$ for all $1 \le j, k \le d_2$, which is equivalent by (4.15) to the fact that $B^T A$ is symmetric. Summing i from 1 to n, the matrix $(a_i^k a_i^j + \sqrt{-1}b_i^k a_i^j)_{k,j=1}^{d_2} = A^T A + \sqrt{-1}B^T A$ is symmetric, and its real part $A^T A$ is strictly positive as A has maximal rank. Thus from (4.18), using (2.27), we get

$$\int_{\Sigma_2}\int_{\Sigma_1} \mathscr{P}_{x_0}(u,w)\,du\,dw = 2^{n/2}\det{}^{-1/2}\big(\sqrt{-1}\big(B - \sqrt{-1}A\big)^T A\big). \tag{4.19}$$

Then formula (4.4) for the first coefficient follows from (2.4), (2.5), (4.10), (4.18) and (4.14). □

4.2. Asymptotic Expansion of Clean Intersections

In this section, we deal with the case of general clean intersection in the sense of Definition 4.1. The main difference with Theorem 4.3 is that the coefficients of the expansion are now given as integrals over the fixed point set. The main additional difficulty is to show that we can in fact split the integral between an integral over the fixed point set and an integral over transversal slices and then integrate the expansion of Lemma 2.5 over the transversal slices following the proof of Theorem 4.3.

As in Section 4.1, the immersions $\iota_i : \Lambda_i \to X$ and $j_i : \Lambda_1 \cap \Lambda_2 \to \Lambda_i$, $i = 1, 2$, are implicit in the statement of the following theorem, and we omit to mention them for simplicity.

THEOREM 4.4. *Suppose that $(\Lambda_1, \iota_1, \zeta_1)$ and $(\Lambda_2, \iota_2, \zeta_2)$ intersect cleanly. Let $\Lambda_1 \cap \Lambda_2 = \bigcup_{q=1}^m Y_m$ be the decomposition into connected components of their intersection in the sense above, and set $l_q = \dim Y_q$. Then for any $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, there exist $b_{q,r} \in \mathbb{C}$, $r \in \mathbb{N}$, $1 \le q \le m$, such that for any $k \in \mathbb{N}$ and as $p \to +\infty$,*

$$\langle T_{F,p} s_{f_1,p}, s_{f_2,p}\rangle_p$$
$$= \sum_{q=1}^{m} p^{n-(d_1+d_2)/2+l_q/2}\lambda_q^p \sum_{r=0}^{k} p^{-r} b_{q,r}$$
$$+ O(p^{n-(d_1+d_2)/2+l_q/2-(k+1)}), \tag{4.20}$$

where $\lambda_q \in \mathbb{C}$ is the value of the constant function on Y_q defined for any $x \in Y_q$ by $\lambda_q(x) = \langle \zeta_1(x), \zeta_2(x)\rangle_L$. If $\dim \Lambda_1 = n$, then we have

$$b_{q,0} = 2^{n/2} \int_{Y_q} \langle F f_1(x), f_2(x)\rangle_E \det{}^{1/2}(\dot{R}^L/2\pi) \frac{dv_{\Lambda_2}}{dv_{\Lambda_2,\omega}}(x)$$
$$\times \det{}^{-1/2}\left\{\sqrt{-1}\sum_{k=1}^{n-l_q} h_\omega^{TX}(e_k, \nu_i)\omega(e_k, \nu_j)\right\}_{i,j=1}^{d_2-l_q}(x)|dv|_{Y_q,\omega}(x), \quad (4.21)$$

where $\langle e_i\rangle_{i=1}^{n-l_q}$, $\langle \nu_j\rangle_{j=1}^{d_2-l_q}$ are local orthonormal frames of the normal bundle of Y_q inside Λ_1, Λ_2 with respect to $g_\omega^{T\Lambda_1}$, $g_\omega^{T\Lambda_2}$, and $|dv|_{Y_q,\omega}$ is the Riemannian density of $(Y_q, g_\omega^{TY_q})$. The square root of the determinant is determined by (2.28).

Proof. Let us set $F = \mathrm{Id}_E$, the proof of the general case being totally analogous by Lemma 2.6 and (3.6). Using Proposition 3.5, (3.4), and (4.20), we can assume that $\Lambda_1 \cap \Lambda_2$ has a unique connected component Y and that f_j, $j = 1, 2$, have compact supports in a given open set. Then the following computations are local on Y, and we may assume that Y is oriented and embedded in Λ_2 by $j_2 : Y \to \Lambda_2$. We further omit the mention of j_2. We set $l = \dim Y$.

Let N be the *normal bundle* of Y inside Λ_2, identified with the orthogonal complement of TY in $(T\Lambda_2, g_\omega^{T\Lambda_2})$, and let g_ω^N be the induced metric on N. Let $\varepsilon > 0$ be such that the exponential map $\exp_\omega^{\Lambda_2}$ of $(\Lambda_2, g_\omega^{T\Lambda_2})$ restricted to $B^N(0,\varepsilon) := \{w \in N | |w|_{g^N} < \varepsilon\}$ is a diffeomorphism on its image. With Y embedded in N as its zero section, the differential $d\exp_{\omega,x}^{\Lambda_2} : T_xY \oplus N_x \to T_x\Lambda_2$ is the identity map for any $x \in Y$, and $\exp_\omega^{\Lambda_2}(B^N(0,\varepsilon))$ is a tubular neighborhood of Y in Λ_2.

Let dw be an Euclidean volume form on the fibers of (N, g_ω^N) such that the volume form $dw dv_{Y,\omega}$ on the total space of N is compatible with the orientation of X. Let $h_2 \in \mathscr{C}^\infty(B^N(0,\varepsilon), \mathbb{R})$ be defined for any $x \in Y$ and $w \in N_x$ with $|w|_{g^N_{\omega,x}} < \varepsilon$ via the exponential map by

$$dv_{\Lambda_2}(x, w) = h_2(x, w)\, dw\, dv_{Y,\omega}(x). \quad (4.22)$$

Then $h_2(x, 0) = (dv_{\Lambda_2}/dv_{\Lambda_2,\omega})(x)$. Let $I(f_1, f_2) \in \mathscr{C}^\infty(B^N(0,\varepsilon), \mathbb{C})$ at $x \in Y$ and $w \in N_x$ with $|w|_{g^N_{\omega,x}} < \varepsilon$ be defined by the formula

$$I(f_1, f_2)(x, w) = \int_{\Lambda_1} \langle P_p((x, w), \iota_1(y))\iota_{1,p}.\zeta_1^p f_1(y), \zeta_2^p f_2(x_0, w)\rangle_{E_p}$$
$$\times h_2(x_0, w)\, dv_{\Lambda_1}(y). \quad (4.23)$$

Using (3.2), (3.4), (3.5), and Proposition 3.5, from (4.22) and (4.23) we get

$$\langle s_{1,p}, s_{2,p}\rangle_p = \int_{\Lambda_2}\int_{\Lambda_1} \langle P_p(\iota_2(x), \iota_1(y))\iota_{1,p}.\zeta_1^p f_1(y), \iota_{2,p}.\zeta_2^p f_2(x)\rangle_{E_p}$$
$$\times dv_{\Lambda_1}(y)\, dv_{\Lambda_2}(x)$$

$$= \int_{\exp_\omega^{\Lambda_2}(B^N(0,\varepsilon))} \int_{\Lambda_1} \langle P_p(x, \iota_1(y))\iota_{1,p}.\zeta_1^p f_1(y), \zeta_2^p f_2(x)\rangle_{E_p} \times dv_{\Lambda_1}(y)\, dv_{\Lambda_2}(x) + O(p^{-\infty})$$

$$= \int_{x\in Y} \int_{B^{N_x}(0,\varepsilon)} I(f_1, f_2)(x, w)\, dw\, dv_{Y,\omega}(x) + O(p^{-\infty}). \quad (4.24)$$

Fix now $x_0 \in Y$. Take $\varepsilon > 0, U \subset \Lambda_1$, and let $\phi_{x_0}^{\Lambda_1} : B^{\mathbb{R}^{d_1}}(0,\varepsilon) \to U$ be a diffeomorphism sending 0 to x_0 and such that its differential at 0 identifies $\langle\cdot,\cdot\rangle$ with $g_\omega^{T\Lambda_1}$. As $\exp_\omega^{\Lambda_2}(B^{N_{x_0}}(0,\varepsilon))$ and Λ_1 intersect cleanly at x_0 only, following the proof of Theorem 4.3, for $\varepsilon > 0$ small enough, we can extend the union map $\exp_\omega^{\Lambda_2} \cup \phi_{x_0}^{\Lambda_1} : B^{N_{x_0}}(0,\varepsilon) \cup B^{\mathbb{R}^n}(0,\varepsilon) \to X$ to a diffeomorphism $\phi_{x_0} : B^{\mathbb{R}^{2n}}(0,\varepsilon) \to V$ as in (2.17), identifying U with $B^\Sigma(0,\varepsilon)$, where Σ is an isotropic subspace of $(\mathbb{R}^{2n}, \Omega)$, and where the fiber $(N_{x_0}, g^N_{\omega,x_0})$ is seen as an Euclidean subspace of $(\mathbb{R}^{2n}, \langle\cdot,\cdot\rangle)$.

Let us identify E and L over $B^{\mathbb{R}^{2n}}(0,\varepsilon)$ with E_{x_0} and L_{x_0} as in Section 2.2 and use $\zeta_1(x_0)$ to identify L_{x_0} with $\mathbb{C}$. Then ζ_1, ζ_2 are identified with $1, \overline{\lambda} \in \mathbb{C}$ over $B^{\mathbb{R}^{2n}}(0,\varepsilon)$, where $\lambda = \langle\zeta_1(x_0), \zeta_2(x_0)\rangle_L$. Let du be the Lebesgue measure of Σ, and let $h_1 \in \mathscr{C}^\infty(B^\Sigma(0,\varepsilon), \mathbb{R})$ be such that for $u \in B^\Sigma(0,\varepsilon)$,

$$dv_{\Lambda_1}(u) = h_1(u)\, du \quad \text{with } h_2(0) = (dv_{\Lambda_1}/dv_{\Lambda_1,\omega})(x_0). \quad (4.25)$$

By Corollary 2.4 and Lemma 2.5, for any $k \in \mathbb{N}$ and $\delta \in]0,1[$, we get $\theta \in]0,1[$ such that, as $p \to +\infty$,

$$\begin{aligned}
&\int_{B^{N_{x_0}}(0,\varepsilon)} I(f_1, f_2)(x_0, w)\, dw \\
&\quad = \int_{B^{N_{x_0}}(0,\varepsilon)} \int_{B^\Sigma(0,\varepsilon)} \langle P_p(w,u)\zeta_1^p f_1(u), \zeta_2^p f_2(w)\rangle_{E_p} h_2(x_0, w)\, dv_{\Lambda_1}(u)\, dw \\
&\quad = \int_{B^{N_{x_0}}(0,\varepsilon p^{-\theta/2})} \int_{B^\Sigma(0,\varepsilon p^{-\theta/2})} \langle P_p(w,u)\zeta_1^p f_1(u), \zeta_2^p f_2(w)\rangle_{E_p} \\
&\quad\quad \times h_2(x_0, w) h_1(u)\, du\, dw + O(p^{-\infty}) \\
&\quad = \lambda^p p^n \sum_{r=0}^k p^{-r/2} \int_{B^{N_{x_0}}(0,\varepsilon p^{-\theta/2})} \int_{B^\Sigma(0,\varepsilon p^{-\theta/2})} \\
&\quad\quad \times \left\langle J_{r,x_0} \mathscr{P}_{x_0}\big(\sqrt{p}w, \sqrt{p}u\big) f_{1,x_0}(u), f_{2,x_0}(w)\right\rangle_E \\
&\quad\quad \times \kappa_{x_0}^{-1/2}(w) \kappa_{x_0}^{-1/2}(u) h_2(x_0, w) h_1(u)\, du\, dw \\
&\quad\quad + p^{n-(d_1+d_2)/2+l/2} O(p^{-(k+1)/2+\delta}). \quad (4.26)
\end{aligned}$$

Consider now the Taylor expansions up to order $k \in \mathbb{N}$ of $h_j \kappa_{x_0}^{-1/2} f_{j,x_0}$ for $j = 1, 2$ as in (3.18). As in the proof of Theorem 4.3, we get $\delta' > 0$ and a sequence $\{F_{x_0,r}\}_{r\in\mathbb{N}}$ of polynomials in two variables of $\mathbb{R}^{2n}$ with values in $\mathbb{C}$ of the same parity as r and with

$$F_{x_0,0}(Z,Z') = \langle f_1(x_0), f_2(x_0)\rangle_E \frac{dv_{\Lambda_1}}{dv_{\Lambda_1,\omega}} \frac{dv_{\Lambda_2}}{dv_{\Lambda_2,\omega}}(x_0)\det(\dot{R}^L_{x_0}/2\pi) \tag{4.27}$$

such that, as $p \to +\infty$, equation (4.11) becomes

$$\begin{aligned}\int_{B^{N_{x_0}}(0,\varepsilon)} & I(f_1,f_2)(x_0,w)\,dw \\ &= p^{n-(d_1+d_2)/2+l/2}\lambda^p \sum_{r=0}^{k} p^{-r/2}\int_{N_{x_0}}\int_\Sigma F_{x_0,r}(w,u)\mathscr{P}_{x_0}(w,u)\,du\,dw \\ &\quad + p^{n-(d_1+d_2)/2+l/2} O(p^{-k+1/2+\delta'}).\end{aligned} \tag{4.28}$$

Thus writing

$$b_r(x_0) = \int_{N_{x_0}}\int_\Sigma F_{x_0,r}(w,u)\mathscr{P}_{x_0}(w,u)\,du\,dw \tag{4.29}$$

and recalling that the estimates are uniform in $x_0 \in Y$, from (4.23), (4.24), and (4.28) we get

$$\begin{aligned}\langle s_{1,p}, s_{2,p}\rangle_p = p^{n-(d_1+d_2)/2+l/2}\lambda^p &\sum_{r=0}^{k} p^{-r/2}\int_Y b_r(x)\,dv_Y(x) \\ &+ p^{n-(d_1+d_2)/2+l/2} O(p^{-(k+1)/2}).\end{aligned} \tag{4.30}$$

Now we can use (4.17) to compute (4.29) in general, and the argument of parity holds in the same way, so that the coefficients b_r defined in (4.29) for $r \in \mathbb{N}$ vanish identically for odd r. By (4.30) this gives (4.20).

Assume now that $\dim \Lambda_1 = n$, and let us compute

$$\begin{aligned}b_0(x_0) = \frac{dv_{\Lambda_1}}{dv_{\Lambda_1,\omega}} \frac{dv_{\Lambda_2}}{dv_{\Lambda_2,\omega}}&(x_0)\det(\dot{R}^L_{x_0}/2\pi)\langle f_1(x_0), f_2(x_0)\rangle_E \\ \times \int_{N_{x_0}}\int_\Sigma &\mathscr{P}_{x_0}(w,u)\,du\,dw.\end{aligned} \tag{4.31}$$

In the same way as in the proof of Theorem 4.3, we can take the canonical symplectic basis $\{e_j, f_j\}_{j=1}^n$ of $(\mathbb{R}^{2n}, \Omega)$ such that $\Sigma = \mathbb{R}^n \times \{0\}$ and such that $\langle e_{n-l+1}, \ldots, e_n\rangle$ is an oriented orthonormal basis of $(T_{x_0}Y, g^{TY}_\omega)$ in the identification of $\mathbb{R}^{2n}$ with $T_{x_0}X$ via $d\phi_{x_0}$. Let $\nu_1, \ldots, \nu_{d_2-l} \in N_{x_0}$ be such that

$$\langle \nu_1, \ldots, \nu_{d_2-l}, e_{n-l+1}, \ldots, e_n\rangle$$

is an oriented orthonormal basis of the isotropic subspace $\Sigma_2 := N_{x_0} \oplus T_{x_0}Y$. Then for $1 \le i \le d_2 - l,\ n-l+1 \le j \le n$, we have $\langle \nu_i, f_j\rangle = -\omega(\nu_i, e_j) = 0$. Thus setting

$$\begin{aligned} A &= (a_i^j)_{1\le i\le n-l, 1\le j\le d_2-l} \quad \text{with } a_i^j = \omega(e_i, \nu_j), \\ B &= (b_i^j)_{1\le i\le n-l, 1\le j\le d_2-l} \quad \text{with } b_i^j = \langle e_i, \nu_j\rangle, \end{aligned} \tag{4.32}$$

we get for all $1 \leq j \leq d_2 - l$,

$$\nu_j = \sum_{i=1}^{n-l} b_i^j e_i + \sum_{i=1}^{n-l} a_i^j f_i. \tag{4.33}$$

Write $dt := dt_1 \dots dt_{d_2-l}$ for the Lebesgue measure on $\mathbb{R}^{d_2-l}$. Using the summation convention of Section 2.1 with i from 1 to $n-l$ and j,k from 1 to d_2-l whenever they appear as free indices, we get

$$\begin{aligned}
&\int_{N_{x_0}} \int_\Sigma \mathscr{P}_{x_0}(w,u)\,du\,dw \\
&\quad = \int_{\mathbb{R}^{d_2-l}} \int_{\mathbb{R}^n} \mathscr{P}_{x_0}(t_j\nu_j, u_i e_i)\,du\,dt \\
&\quad = \int_{\mathbb{R}^{d_2-l}} \int_{\mathbb{R}^n} \exp\Big(-\frac{\pi}{2} \sum_{i=n-l+1}^{n} u_i^2\Big) \\
&\qquad \times \exp\Big(-\frac{\pi}{2} \sum_{i=1}^{n-l} \big((u_i - t_j b_i^j)^2 + (t_j a_i^j)^2 + 2\sqrt{-1} u_i t_j a_i^j\big)\Big)\,du\,dt \\
&\quad = 2^{l/2} \int_{\mathbb{R}^{d_2-l}} \int_{\mathbb{R}^{n-l}} \exp\Big(-\frac{\pi}{2} \sum_{i=1}^{n-l} u_i^2 + (t_j a_i^j)^2 \\
&\qquad \times +2\sqrt{-1} u_i t_j a_i^j + 2\sqrt{-1} t_k b_i^k a_i^j t_j\Big)\,du_1 \dots du_{n-l}\,dt \\
&\quad = 2^{n/2} \int_{\mathbb{R}^{d_2-l}} \exp\Big(-\frac{\pi}{2} \sum_{j=1}^{n-l} \big(2(t_j a_i^j)^2 + 2\sqrt{-1} t_k b_i^k a_i^j t_j\big)\Big)\,dt.
\end{aligned} \tag{4.34}$$

As $\omega(\nu_j, \nu_k) = 0$ for all $1 \leq j,k \leq d_2 - l$, we know from (4.33) that the matrix $B^T A$ is symmetric. Then as in (4.19), we get

$$\int_{N_{x_0}} \int_\Sigma \mathscr{P}_{x_0}(w,u)\,du\,dw = 2^{n/2} \det{}^{-1/2}\big(\sqrt{-1} A\big(B - \sqrt{-1} A\big)\big). \tag{4.35}$$

Using the explicit definition of A and B above, from (4.35), (2.4), and (2.5) we get (4.21). $\square$

REMARK 4.5. Suppose that the first Chern class $c_1(TX)$ of (TX, J) is even in $H^2(X, \mathbb{Z})$. Then there exists a complex line bundle $K_X^{1/2}$ over X such that its second tensor power is equal to the canonical line bundle K_X of X. The choice of $K_X^{1/2}$ does not depend on J compatible with ω and is called a *metaplectic structure* on (X, ω). Note that such a choice is not unique in general. Now if $\iota : \Lambda \to X$ is an immersed Lagrangian submanifold, then $\iota^* K_X$ is canonically isomorphic to $\det(T^*\Lambda_\mathbb{C})$ over Λ, and we call $\iota^* K_X^{1/2}$ the *half-form bundle* of Λ. We endow $K_X^{1/2}$ with the Hermitian structure induced by $h_\omega^{K_X}$ as in Section 2.1.

Consider now the setting of Theorem 4.4 with $\dim \Lambda_1 = \dim \Lambda_2 = n$ and $g^{TX} = g_\omega^{TX}$. Via the isomorphism above, we define the *angle* of $\iota_j : \Lambda_j \to X$

for $j = 1, 2$, as a function on any connected component Y of their intersection by the formula

$$\det\{\Lambda_1, \Lambda_2\} = h_\omega^{K_X}(dv_{\Lambda_1}, dv_{\Lambda_2})^{-1} = \det\{h_\omega^{TX}(e_i, \nu_j)\}_{i,j=1}^{n-l}.$$

On the other hand, following [8, Lemma 3.1], we can construct a sesquilinear pairing $\# : \iota_1^* K_X^{1/2}|_Y \times \iota_2^* K_X^{1/2}|_Y \to \det(T^*Y_{\mathbb{C}})$ over Y, depending only on the metaplectic structure of (X, ω), which at any $x \in Y$ takes two square roots $dv_{\Lambda_j,x}^{1/2}$ of $dv_{\Lambda_j,x}$ for $j = 1, 2$ to

$$dv_{\Lambda_1,x}^{1/2} \# dv_{\Lambda_2,x}^{1/2} = \det{}^{-1/2}\{\omega(e_i, \nu_j)\}_{i,j=1}^{n-l}\, dv_{Y,x} \tag{4.36}$$

for an Euclidean volume form $dv_{Y,x}$ of (T_xY, g_x^{TY}) and some coherent choice of square root induced by $dv_{\Lambda_1,x}^{1/2}, dv_{\Lambda_2,x}^{1/2}$, and $dv_{Y,x}$. Then taking $E = K_X^{1/2}$, Theorem 4.4 gives the following formula for b_0 on Y as in (4.21):

$$b_0 = 2^{(n-l)/2} e^{-\sqrt{-1}\pi(n-l)/2} \int_Y \det\{\Lambda_1, \Lambda_2\}^{-1} f_1 \# f_2. \tag{4.37}$$

In the particular case of Kähler (X, J, ω), this formula can be compared with that appearing in [8, Prop. 3.16]. In particular, they get $\det\{\Lambda_1, \Lambda_2\}^{-1/2}$ instead of $\det\{\Lambda_1, \Lambda_2\}^{-1}$ as in (4.37). This discrepancy is due to the fact that even though they use half-forms, their Lagrangian states do not take values in $L^p \otimes K_X^{1/2}$, but in L^p. Note that without metaplectic structure on (X, ω), only the product of the square root of (4.36) with (4.36) makes sense in general (see [31] for related results).

Finally, note that the assumption $\dim \Lambda_1 = n$ for formula (4.21) of the first coefficient of the expansion was used in the proof of Theorem 4.4 to compute the elegant formula (4.35). Without this assumption, we still get an integral of the form (4.34) by following the method of the proof, and the classical formulas of Section 2.3 for the Gaussian integral can be used to compute it explicitly. We also get a formula in terms of the symplectic form, the Riemannian metric, and local frames via definition (4.32) of the coefficients appearing inside the Gaussian function.

5. Extensions to Noncompact Manifolds and Orbifolds

In this section, we show how we can adapt the results of the previous sections in the case of noncompact manifolds and orbifolds. We will work for simplicity in the case of Kähler (X, J, ω) and $g^{TX} = g_\omega^{TX}$. Then as underlined in Introduction, the renormalized Bochner Laplacian (2.8) reduces to the Kodaira Laplacian on sections.

Note further that the existence of an expansion of the form (2.25) is a straightforward consequence of the existence of an expansion as in [27, (4.9)].

5.1. Noncompact Case

Let (X, J, ω, g^{TX}) be a complete Kähler manifold with $\omega(\cdot,\cdot) = g^{TX}(J\cdot,\cdot)$, let (L, h^L) be a holomorphic line Hermitian bundle over X with Chern connection ∇^L satisfying (1.1), and let (E, h^E) be an auxiliary holomorphic Hermitian bundle with Chern connection ∇^E. For any $p \in \mathbb{N}^*$, let $H^0_{(2)}(X, E_p)$ denote the space of holomorphic sections of $E_p = L^p \otimes E$ that are square integrable with respect to the L^2-Hermitian product defined as in (2.9). Let P_p denote the orthogonal projection from the space of L^2-sections of E_p onto $H^0_{(2)}(X, E_p)$ with respect to this product. Then as noticed in [25, Rem.1.4.3], P_p has a smooth Schwartz kernel $P_p(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$ with respect to the Riemannian volume form dv_X of (X, g^{TX}), and $P_p(\cdot,\cdot)$ is square integrable and holomorphic with respect to its first variable.

Let us write $R^{\det}$ for the curvature of the Chern connection of K_X^*. Then we have the following result.

THEOREM 5.1 ([27, Thm. 5.2, 5.3]). *Suppose that there exists $C > 0$ such that for all $x \in X$ and $v \in T_xX$, the following inequality holds in the sense of endomorphisms of E*:

$$\sqrt{-1}(R^{\det}\mathrm{Id}_E + R^E)(v, Jv) > -C\omega(v, Jv)\mathrm{Id}_E. \tag{5.1}$$

Then for any compact set $K \subset X$, Proposition 2.3 holds uniformly for any $x, x' \in K$, and Lemma 2.5 holds uniformly for any $x_0 \in X$.

If $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ has a compact support, then Lemma 2.6 holds uniformly for any $x_0 \in X$.

From now on we suppose that (5.1) is verified for X. Then Definition 3.1 still makes sense in this context, provided that Λ is compact. Precisely, for a Bohr–Sommerfeld manifold (Λ, ι, ζ) as in Definition 3.1 with compact Λ and for $f \in \mathscr{C}^\infty(\Lambda, \iota^* E)$, we define the associated isotropic state $\{s_{f,p}\}_{p\in\mathbb{N}}$ in the same way as in (3.2) for any $p \in \mathbb{N}^*$ and $x \in X$ by the formula

$$s_{f,p}(x) = \int_\Lambda P_p(x, \iota(y))\iota_p.\zeta^p f(y)\, dv_\Lambda(y). \tag{5.2}$$

Then as Λ is compact, we get that $s_{f,p} \in H^0_{(2)}(X, E_p)$. Furthermore, we have the following analogue of Proposition 3.4.

LEMMA 5.2. *Suppose that (X, J, ω, g^{TX}) is a complete Kähler manifold satisfying (5.1), and let (Λ, ι, ζ) be a compact Bohr–Sommerfeld submanifold of X. Then for any $s \in H^0_{(2)}(X, E_p)$, we have the following reproducing property*:

$$\langle s, s_{f,p}\rangle_p = \int_\Lambda \langle s(\iota(x)), \iota_p.\zeta^p f(x)\rangle_{E_p}\, dv_\Lambda(y). \tag{5.3}$$

Furthermore, for any $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ with compact support, property (3.6) *holds.*

Proof. As Λ is compact, we can repeat the computations of (3.7), so that (5.3) holds. As $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ has a compact support, we can repeat in the same way the computations of (3.8), and (3.6) also holds in this context. □

With these preliminaries, we can state the following generalization of the results of Section 3.2, Section 4.1, and Section 4.2.

THEOREM 5.3. *Suppose that (X, J, ω, g^{TX}) is a complete Kähler manifold satisfying (5.1). If (Λ, ι, ζ) is a compact Bohr–Sommerfeld submanifold of (X, ω), then Theorem 3.6 holds.*

Furthermore, if $(\Lambda_j, \iota_j, \zeta_j)$, $j = 1, 2$, are two compact Bohr–Sommerfeld submanifolds of (X, ω) intersecting cleanly, then Theorem 4.4 holds.

Proof. Let $(\Lambda_j, \iota_j, \zeta_j)$, $j = 1, 2$, be two compact Bohr–Sommerfeld submanifolds of X, and consider $f_j \in \mathscr{C}^\infty(X, \iota_j^* E)$, $j = 1, 2$. By Theorem 5.1 we know that Proposition 3.5 is still true uniformly in any compact set $K \subset X$. Furthermore, using (5.2) and (5.3) and omitting the immersions, for any $p \in \mathbb{N}^*$, we get

$$\begin{aligned}
&\langle s_{f_1,p}, s_{f_2,p}\rangle_p \\
&\quad = \int_{\Lambda_2} \langle s_{f_1,p}(x), \zeta_2^p f_2(x)\rangle_{E_p}\, dv_{\Lambda_2}(x) \\
&\quad = \int_{\Lambda_2}\int_{\Lambda_1} \langle P_p(x,y)\zeta_1^p f_1(y), \zeta_2^p f_2(x)\rangle_{E_p}\, dv_{\Lambda_1}(y)\, dv_{\Lambda_2}(x). \qquad (5.4)
\end{aligned}$$

Then we can choose a compact set K in Theorem 5.1 containing $\iota(\Lambda_1) \cup \iota(\Lambda_2)$, and the proof of Theorem 5.3 goes along the lines of the proofs of Theorem 3.6, Theorem 4.3, and Theorem 4.4. By the second part of Lemma 5.2 the case of $\langle T_{F,p} s_{f_1,p}, s_{f_2,p}\rangle_p$ such that $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ has compact support is strictly analogous. □

5.2. *Orbifold Case*

In this section, we consider a complete Kähler orbifold (X, J, ω, g^{TX}) satisfying (5.1), a proper holomorphic Hermitian orbifold line (L, h^L) bundle over X with Chern connection ∇^L satisfying (1.1), and a proper holomorphic Hermitian orbifold vector bundle (E, h^E) over X endowed with its Chern connection ∇^E. To give a precise meaning to these notions, we first state some notations and definitions from [25, Section 5.4].

DEFINITION 5.4. Let $\mathscr{M}$ be the category whose objects are the pairs (M, G) with M a smooth connected manifold and G a finite group acting effectively on M and whose morphisms $\Phi : (M, G) \to (M', G')$ are families of open embeddings $\varphi : M \to M'$ satisfying:

- For each $\varphi \in \Phi$, there is an injective group homomorphism $\lambda_\varphi : G \to G'$ such that φ is λ_φ-equivariant.
- For $g \in G'$ and $\varphi \in \Phi$, define $g\varphi : M \to M'$ by the formula $(g\varphi)(x) = g\varphi(x)$ for any $x \in M$. If $(g\varphi)(M) \cap \varphi(M) \neq \emptyset$, then $g \in \lambda_\varphi(G)$.

- For $\varphi \in \Phi$, we have $\Phi = \{g\varphi | g \in G'\}$.

DEFINITION 5.5. Let X be a paracompact Hausdorff space, and let $\mathscr{U}_X$ be a covering of X consisting of connected open subsets satisfying the condition

$$\begin{aligned}&\text{For any } U, U' \in \mathscr{U}_X \text{ and } x \in U \cap U',\\ &\text{there is } U'' \in \mathscr{U}_X \text{ such that } x \in U'' \subset U \cap U'.\end{aligned} \tag{5.5}$$

An *orbifold structure* $\mathscr{V}_X$ on X consists of the following data:

- For any $U \in \mathscr{U}_X$, an object $(G_U, \widetilde{U})$ of $\mathscr{M}$ and a ramified covering $\tau_U : \widetilde{U} \to U$ that is G_U-invariant and induces a homeomorphism $U \simeq \widetilde{U}/G_U$.
- For any $U, V \in \mathscr{U}_X$ such that $U \subset V$, a morphism $\Phi_{VU} : (G_U, \widetilde{U}) \to (G_V, \widetilde{V})$ of $\mathscr{M}$ that covers the inclusion $U \subset V$ and satisfies $\Phi_{WU} = \Phi_{WV} \circ \Phi_{VU}$ for any $U, V, W \in \mathscr{U}_X$ with $U \subset V \subset W$.

If $\mathscr{U}'_X$ is a refinement of $\mathscr{U}_X$ satisfying condition (5.5), then there is an orbifold structure $\mathscr{V}'_X$ associated with $\mathscr{U}'_X$ such that $\mathscr{V}_X \cup \mathscr{V}'_X$ is again an orbifold structure. We then say that $\mathscr{V}_X$ and $\mathscr{V}'_X$ are *equivalent*. An equivalence class is called an *orbifold structure* on X. In particular, we can suppose that $\mathscr{U}_X$ is arbitrarily fine. We further always consider the unique maximal representative in the equivalence class.

In the above definitions, we can replace the objects of $\mathscr{M}$ by manifolds with specified structures together with a group and morphisms preserving these structures. In the case in hand, by structure we mean an orientation, a Riemannian metric, a symplectic structure, and an almost-complex structure or a complex structure. Furthermore, we can realize Cartesian products of orbifolds in an obvious way.

Let $(X, \mathscr{V}_X)$ be an orbifold. For each $x \in X$, up to refinement of $\mathscr{V}_X$, there exists $U_x \in \mathscr{U}_X$ containing x and $\widetilde{x} \in \widetilde{U}$, $\tau_U(\widetilde{x}) = x$, such that $\widetilde{x}$ is a fixed point of G_U. Then by the second axiom of Definition 5.4 such a group is unique up to isomorphism, and we denote it by G_x^X. If $|G_x^X| = 1$, then X has a smooth structure in a neighborhood of x, and we call such x a *smooth point* of X. If $|G_x^X| > 1$, then we call such x a *singular point* of X. We denote $X_{\text{sing}} = \{x \in X \| G_x^X| > 1\}$ the *singular set* of X, and $X_{\text{reg}} = \{x \in X \| G_x^X| = 1\}$ the *regular set* of X. We further denote by $\widetilde{x} \in \widetilde{U}$ a lift of $x \in U \in \mathscr{U}_X$.

The next definitions are adaptations of the notions of orbifold embedding and submersion from [23, Defs. 1.6 and 1.7].

DEFINITION 5.6. An *orbifold immersion* $I : (Y, \mathscr{V}_Y) \to (X, \mathscr{V}_X)$ is a continuous map $\iota : Y \to X$ such that for any $V \in \mathscr{U}_X$ and any connected component $U \in \mathscr{U}_Y$ of $\iota^{-1}(V)$, there is a family I_{UV} of immersions $\iota_{UV} : \widetilde{U} \to \widetilde{V}$ covering ι together with surjective group homomorphisms $\lambda_{UV} : G_V \to G_U$ such that ι_{UV} is λ_{UV}-equivariant. Furthermore, the families I_{UV} satisfy $I_{UV} = \{g\iota_{UV} | g \in G_U\}$ and are compatible with the orbifold structures in the obvious sense. In that case, we define the *stabilizer* of V in U by $K_{UV} = \operatorname{Ker} \lambda_{UV}$. Then $m_{X,Y} := |K_{UV}|$ is locally constant on Y and is called the *relative multiplicity* on Y.

A *singular immersion* $\hat{I}$ from a smooth manifold Y to an orbifold $(X, \mathscr{V}_X)$ is a continuous map $\iota : Y \to X$ together with immersions $\widetilde{\iota}_V : U \to \widetilde{V}$ covering ι for any $V \in \mathscr{U}_X$ such that $g.\iota(U)$ intersects $\iota(U)$ cleanly in the sense of Definition 4.1 for all $g \in G_V$. In that case, we define the stabilizer of U in V by the subgroup $K_{UV} \subset G_V$ fixing each point of $\widetilde{\iota}_V(U)$, and the *relative multiplicity* $m_{X,Y} = |K_{UV}|$ is again locally constant on Y.

An *orbifold submersion* $P : (M, \mathscr{V}_M) \to (X, \mathscr{V}_X)$ is the data of a continuous map $\pi : M \to X$ such that $\pi(U) \in \mathscr{U}_X$ for any $U \in \mathscr{U}_M$, together with submersions $\pi_U : \widetilde{U} \to \widetilde{\pi(U)}$ covering π and surjective group homomorphisms $\lambda_U : G_U \to G_{\pi(U)}$ for any $U \in \mathscr{U}_X$ making π_U be λ_U-equivariant. Furthermore, we assume compatibility with the orbifold structures in the obvious sense.

Note that any $x \in X$ can be seen as an immersed orbifold with $m_{X,x} = |G_x|$. In both definitions of an immersion above, if $\iota^{-1}(X_{\text{sing}})$ has a strictly positive measure for the density induced by any Riemannian metric, then G_V fixes $\iota(U)$, and $m_{X,Y}$ is strictly positive. The intersection of two orbifold immersions is still defined as in Definition 4.1 to be their fibered product over X, which gets a natural orbifold structure making all maps into orbifold immersions.

Finally, note that we can easily combine the definitions above to get the notion of a singular orbifold immersion, and the results of this section hold in this case as well. For simplicity and clarity, we will keep both notions separated from each other.

Definition 5.7. An *orbifold vector bundle* E over X is an orbifold submersion $P : (E, \mathscr{V}_E) \to (X, \mathscr{V}_X)$ such that for any $U \in \mathscr{U}_X$, the open set $E_U := \pi^{-1}(U)$ belongs to $\mathscr{U}_E$ and $\pi_{E_U} : \widetilde{E}_U \to \widetilde{U}$ is a G_{E_U}-equivariant vector bundle. Furthermore, we suppose that the inclusions $\Phi_{E_V E_U}$ covering Φ_{VU} are equivariant vector bundle maps for any $U, V \in \mathscr{U}_X$ such that $U \subset V$.

If G_{E_U} acts effectively on $\widetilde{U}$ for all $U \in \mathscr{U}_X$, that is, the group morphisms $\lambda_{E_U} : G_{E_U} \to G_U$ associated with P as in Definition 5.6 are isomorphisms, then we say that E is *proper.*

Then we can define the proper tangent orbifold bundle TX and the proper cotangent orbifold bundle T^*X over any orbifold $(X, \mathscr{V}_X)$ in the obvious way. We can as well form tensor products of vector bundles by taking the tensor products locally over each orbifold chart, and we easily check that this operation preserves properness. If E is a proper orbifold bundle over X and if $\Psi : (X, \mathscr{V}_X) \to (Y, \mathscr{V}_Y)$ is any of the orbifold maps of Definition 5.6, then we can pullback E to Y by Ψ in the obvious way, and we write Ψ^*E for the pullback orbifold vector bundle, which is still proper.

We define a distance on X for any $x, y \in X$ by

$$d(x, y) = \inf_{\gamma} \left\{ \sum_j \int_{t_{j-1}}^{t_{j+1}} \left| \frac{\partial}{\partial t} \widetilde{\gamma}_j(t) \right| dt \,|\, \gamma : [0, 1] \to X, \gamma(0) = x, \gamma(1) = y, \right.$$

$$\text{such that there exist } t_0 = 0 < t_1 < \cdots < t_k = 1, \gamma([t_{j-1}, t_j]) \subset U_j,$$

$$\left.\begin{aligned}&U_j \in \mathscr{U}_X,\ \text{and a smooth map } \widetilde{\gamma}_j : [t_{j-1}, t_j] \to \widetilde{U}_j\\ &\text{that covers } \gamma|_{[t_{j-1},t_j]}\end{aligned}\right\}. \tag{5.6}$$

Let $E \to X$ be an orbifold vector bundle. An orbifold section $s : X \to E$ is called *smooth* if for each $U \in \mathscr{U}_X$, the restriction of s to U is covered by a G_U^E-equivariant smooth section $\widetilde{s}_U : \widetilde{U} \to \widetilde{E}_U$. In the same way, if X is a complex orbifold and E is a holomorphic orbifold vector bundle, then we say that s is holomorphic if it is locally covered by holomorphic sections. The space of smooth (resp., holomorphic) sections of E is denoted by $\mathscr{C}^\infty(X, E)$ (resp., $H^0(X, E)$).

If X is oriented and α is a smooth section of the exterior product orbifold bundle $\Lambda(T^*X)$ with support in $U \in \mathscr{U}$, then we define

$$\int_X \alpha = \frac{1}{|G_U|}\int_{\widetilde{U}} \widetilde{\alpha}_U, \tag{5.7}$$

where $\widetilde{\alpha}_U$ is an invariant section covering α over $\widetilde{U}$. We extend this definition for general α using a partition of unity. In particular, if X is oriented and Riemannian, then there is an induced Riemannian volume form dv_X on X, so that we can integrate functions.

Let now (X, J, ω) be a Kähler orbifold. As we can verify locally, for any Hermitian holomorphic proper orbifold bundle over X, its Chern connection is well-defined and unique. Let (L, h^L) be a proper holomorphic Hermitian orbifold line bundle over X such that its Chern connection satisfies (1.1). We write g^{TX} for the Riemannian metric on X satisfying (2.3) and dv_X for the associated Riemannian volume form. Let (E, h^E) be an auxiliary proper holomorphic Hermitian orbifold vector bundle over X.

We define the L^2-Hermitian product associated with all the previous data on $\mathscr{C}^\infty(X, E_p)$ by formula (2.9), and the *Bergman kernel* is the Schwartz kernel $P_p(\cdot,\cdot) \in \mathscr{C}^\infty(X \times X, E_p \boxtimes E_p^*)$ with respect to dv_X of the orthogonal projection P_p from $\mathscr{C}^\infty(X, E_p)$ to $H^0_{(2)}(X, E_p)$ as in (2.12). For any $V \in \mathscr{U}_X$ and $p \in \mathbb{N}^*$, let $\widetilde{P}_p(\cdot,\cdot) \in \mathscr{C}^\infty(\widetilde{V} \times \widetilde{V}, \widetilde{E}_{p,V} \boxtimes \widetilde{E}^*_{p,V})$ be the $G_V \times G_V$-invariant lift of $P_p(\cdot,\cdot) \in \mathscr{C}^\infty(V \times V, E_p \boxtimes E_p^*)$. More generally, for any object on $V \in \mathscr{U}_X$, we add a superscript $\widetilde{\ }$ to denote the corresponding object on $\widetilde{V}$.

For any $m \in \mathbb{N}$, let $|\cdot|_{\mathscr{C}^m}$ denote the local $\mathscr{C}^m$-norm induced by h^L, h^E, and ∇^L, ∇^E on local sections of $E_p \boxtimes E_p^*$ over $X \times X$. The following result is the version of Lemma 2.5 for orbifolds. It uses the fact, noticed in [23], that the finite propagation speed of the wave equation holds on orbifolds.

PROPOSITION 5.8 ([27, Section 6.2], [25, Rem. 5.4.12b)]). *Proposition 2.3 holds in the case of complete Kähler orbifold (X, J, ω, g^{TX}) satisfying (5.1). Moreover, for any $V \in \mathscr{U}_X$, there exists a section $F(\widetilde{D}_p)(\cdot,\cdot) \in \mathscr{C}^\infty(\widetilde{V} \times \widetilde{V}, \widetilde{E}_{p,V} \boxtimes \widetilde{E}^*_{p,V})$ satisfying the following properties*:

For any $\widetilde{x}, \widetilde{y} \in \widetilde{V}$ and $g \in G_V$,

$$(g, 1)F(\widetilde{D}_p)(g^{-1}\widetilde{x}, \widetilde{y}) = (1, g^{-1})F(\widetilde{D}_p)(\widetilde{x}, g\widetilde{y}). \tag{5.8}$$

For any $m, l \in \mathbb{N}$, *there is* $C_{m,l} > 0$ *such that for all* $\widetilde{x}, \widetilde{y} \in \widetilde{V}$ *and* $p \in \mathbb{N}^*$,

$$\left| \widetilde{P}_p(\widetilde{x}, \widetilde{y}) - \sum_{g \in G_U} (1, g^{-1}) F(\widetilde{D}_p)(\widetilde{x}, g\widetilde{y}) \right|_{\mathscr{C}^m} \leq C_{m,l}\, p^{-l}. \tag{5.9}$$

$F(\widetilde{D}_p)(\cdot, \cdot)$ *satisfies the expansion of Lemma* 2.5 *at any* $x_0 \in \widetilde{V}$.

With all these prerequisites in hand, Definition 3.1 still makes sense in this context replacing the immersion ι by an orbifold immersion or singular immersion I as in Definition 5.6. In the second case, we talk about a *singular* Bohr–Sommerfeld submanifold. In any case, if Λ is compact, then the associated isotropic state as in (3.2) is well defined, and Proposition 3.4 still holds. We will use the additivity property (3.4) to assume that the section f of Definition 3.3 has a compact support in some given open set $U \in \mathscr{U}_\Lambda$.

THEOREM 5.9. *Let* (X, J, ω, g^{TX}) *be a complete Kähler orbifold satisfying* (2.3), *let* (L, h^L) *be a holomorphic Hermitian proper orbifold line bundle such that the curvature of its Chern connection satisfies* (1.1), *and let* (E, h^E) *be a holomorphic Hermitian proper orbifold vector bundle. Suppose that* (X, J, ω, g^{TX}) *satisfies* (5.1).

If (Λ, I, ζ) *is a compact Bohr–Sommerfeld submanifold of* X *and if the endomorphism* $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ *has a compact support, then Theorem* 3.6 *holds with the following formula for the first coefficient of* (3.12):

$$b_0 = 2^{d/2} m_{X,\Lambda} \int_\Lambda \langle Ff, f \rangle_{\iota^* E}\, dv_\Lambda. \tag{5.10}$$

If $(\Lambda_j, I_j, \zeta_j)$, $j = 1, 2$, *are two compact Bohr–Sommerfeld submanifolds of* X *intersecting cleanly and if* $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ *has a compact support, then the expansion of Theorem* 4.4 *holds. If* $\dim \Lambda_1 = n$, *then the first coefficients* $b_{q,0}$ *of* (4.20) *satisfy formula* (4.4) *multiplied by*

$$m_{X,\Lambda_2} / m_{\Lambda_1, Y_q}. \tag{5.11}$$

Finally, the above holds for compact singular Bohr–Sommerfeld submanifolds of X, *provided that their intersection locus is away from the singular set.*

Proof. Let (Λ, I, ζ) be a compact Bohr–Sommerfeld submanifold, let $U \in \mathscr{U}_\Lambda$ be a connected component of $\iota^{-1}(V)$ for $V \in \mathscr{U}_X$ sufficiently small and take $f \in \mathscr{C}^\infty(\Lambda, I^* E)$ to have compact support in U. Then using (5.7) and (5.9), for any $\widetilde{x} \in \widetilde{V}$, we have, as $p \to +\infty$,

$$\begin{aligned}
\widetilde{s}_{f,p}(\widetilde{x}) &= \frac{1}{|G_U|} \int_{\widetilde{U}} \widetilde{P}_p(\widetilde{x}, \iota_{UV}(\widetilde{y})) \iota_{p,UV} . \widetilde{f}\widetilde{\zeta}^p(\widetilde{y})\, dv_{\widetilde{U}}(\widetilde{y}) \\
&= \frac{1}{|G_U|} \int_{\widetilde{U}} \sum_{g \in G_V} (1, g^{-1}) F(\widetilde{D}_p)(\widetilde{x}, g\iota_{UV}(\widetilde{y})) \iota_{p,UV} . \widetilde{f}\widetilde{\zeta}^p(\widetilde{y})\, dv_{\widetilde{U}}(\widetilde{y}) \\
&\quad + O(p^{-\infty})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|G_U|} \int_{\widetilde{U}} \sum_{g \in G_V} F(\widetilde{D}_p)(\widetilde{x}, \iota_{UV}(\widetilde{y}))\iota_{p,UV}.(g.\widetilde{f}\widetilde{\zeta}^p(g^{-1}\widetilde{y}))\, dv_{\widetilde{U}}(\widetilde{y}) \\
&\quad + O(p^{-\infty}) \\
&= \frac{|G_V|}{|G_U|} \int_{\widetilde{U}} F(\widetilde{D}_p)(\widetilde{x}, \iota_{UV}(\widetilde{y}))\iota_{p,UV}.\widetilde{f}\widetilde{\zeta}^p(\widetilde{y})\, dv_{\widetilde{U}}(\widetilde{y}) \\
&\quad + O(p^{-\infty}).
\end{aligned} \tag{5.12}
$$

Here $\iota_{UV} : \widetilde{U} \to \widetilde{V}$ is any member of the family of maps in I_{UV}. Now by Definition 5.6 we have $|G_V|/|G_U| = m_{X,\Lambda}$. By Proposition 5.8 $F(\widetilde{D}_p)(\cdot,\cdot)$ satisfies the expansion of Lemma 2.5 at any $x_0 \in \widetilde{V}$, so that we can follow the proof of Theorem 3.6 to deduce from (5.12) an asymptotic expansion as $p \to +\infty$ of the form (3.10) for the norm of $s_{f,p}$ with highest coefficient given by (5.10) in the case $F = \mathrm{Id}_E$.

For any $j = 1, 2$, let $(\Lambda_j, I_j, \zeta_j)$ be compact Bohr–Sommerfeld submanifolds, and let $f_j \in \mathscr{C}^\infty(\Lambda, I^*E)$ have compact supports in a sufficiently small open set $U_j \in \mathscr{U}_\Lambda$, a connected component of $\iota^{-1}(V)$ for some $V \in \mathscr{U}_X$. Then as the reproducing property (3.5) still holds, analogously to (4.6) and (5.12), using (5.7) and (5.9) and omitting the immersion maps, we have, as $p \to +\infty$,

$$
\begin{aligned}
&\langle s_{1,p}, s_{2,p}\rangle_p \\
&\quad = \frac{1}{|G_{U_2}|} \int_{\widetilde{U}_2} \langle \widetilde{\zeta}_{f_1,p}(\widetilde{x}), \widetilde{\zeta}_2^p \widetilde{f}_2(\widetilde{x})\rangle_{E_p}\, dv_{\widetilde{U}_2}(\widetilde{x}) \\
&\quad = \frac{1}{|G_{U_1}|}\frac{1}{|G_{U_2}|} \int_{\widetilde{U}_2}\int_{\widetilde{U}_1} \langle \widetilde{P}_p(\widetilde{x},\widetilde{y})\widetilde{\zeta}_1^p \widetilde{f}_1(\widetilde{y}), \widetilde{\zeta}_2^p \widetilde{f}_2(\widetilde{x})\rangle_{E_p}\, dv_{\widetilde{U}_1}(\widetilde{y})\, dv_{\widetilde{U}_2}(\widetilde{x}) \\
&\quad = \frac{|G_V|}{|G_{U_1}||G_{U_2}|} \int_{\widetilde{U}_2}\int_{\widetilde{U}_1} \langle F(\widetilde{D}_p)(\widetilde{x},\widetilde{y})\widetilde{\zeta}_1^p \widetilde{f}_1(\widetilde{y}), \widetilde{\zeta}_2^p \widetilde{f}_2(\widetilde{x})\rangle_{E_p}\, dv_{\widetilde{U}_1}(\widetilde{y})\, dv_{\widetilde{U}_2}(\widetilde{x}) \\
&\quad\quad + O(p^{-\infty}).
\end{aligned} \tag{5.13}
$$

By Definition 5.6 we have $m_{X,\Lambda_2} = |G_V|/|G_{U_2}|$, thus $m_{\Lambda_1,y} = |G_y^{\Lambda_1}| = |G_{U_1}|$ for U_1 small enough. For discrete intersection, take $y \in \iota_2^{-1}(\iota_1(\Lambda_1) \cap \iota_2(\Lambda_2))$ and a small enough neighborhood $V \in \mathscr{U}_X$ of $\iota_1(y) \in X$ to get (5.11) in the case $F = \mathrm{Id}_E$ and discrete intersections.

Recall Definition 4.1. Let now $\widetilde{W}$ be the lift of some open set $W \in \mathscr{U}_Y$, where Y is the connected component of $\Lambda_1 \cap \Lambda_2$ such that its image by j_1 intersects the support of f_1, and set $l = \dim Y$. In the case of clean intersection, we can follow the proof of Theorem 4.4 until (4.30) to get an asymptotic expansion of the form (4.20) and get from (5.13) a sequence $b_r \in \mathscr{C}^\infty(Y, \mathbb{C})$, $r \in \mathbb{N}$, such that, as $p \to +\infty$,

$$
\begin{aligned}
&\langle s_{1,p}, s_{2,p}\rangle_p \\
&\quad = \frac{|G_V|}{|G_{U_1}||G_{U_2}|} p^{l/2}\lambda^p \sum_{r=0}^{k} p^{-r/2} \int_{\widetilde{W}} \widetilde{b}_r(\widetilde{x})\, dv_{\widetilde{W}}(\widetilde{x}) + O(p^{l/2-(k+1)/2})
\end{aligned}
$$

$$= \frac{|G_V|}{|G_{U_2}|}\frac{|G_W|}{|G_{U_1}|}p^{l/2}\lambda^p \sum_{r=0}^{k} p^{-r/2}\int_W b_r(x)\,dv_Y(x) + O(p^{l/2-(k+1)/2})$$

$$= \frac{m_{X,\Lambda_2}}{m_{\Lambda_1,Y}}p^{l/2}\lambda^p \sum_{r=0}^{k} p^{-r/2}\int_W b_r(x)\,dv_Y(x) + O(p^{l/2-(k+1)/2}). \quad (5.14)$$

Then we can follow the proof of Theorem 4.4 to get (5.11) in the case $F = \mathrm{Id}_E$. Now for the general case, if $F \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ has a compact support, then we can define its Berezin–Toeplitz quantization by (2.13), and it is shown in [27, Lemma 6.10] that it satisfies Lemma 2.6 as well. Furthermore, formula (3.6) holds in the same way.

Finally, let us consider the case of singular Bohr–Sommerfeld submanifolds. Following (5.12)–(5.14), it suffices to prove the case $m_{X,Y} = 1$, and as we assumed the intersection locus away from the singular set, we need only to prove the analogue of (3.10) and suppose that f has a compact support in some $U \in \mathscr{U}_X$.

First, recall that the reproducing property gives

$$\begin{aligned}\|s_{f,p}\|_p^2 &= \int_\Lambda \langle s_{f,p}(\iota(x)), \iota_p.\zeta^p f(x)\rangle_{E_p}\,dv_\Lambda(x)\\
&= \int_U\int_U \langle \widetilde{P}_p(\widetilde{\iota}_V(x), \widetilde{\iota}_V(y))\widetilde{\iota}_p.\widetilde{\zeta}^p\widetilde{f}(y), \widetilde{\iota}_p.\widetilde{\zeta}^p\widetilde{f}(x)\rangle_{E_p}\,dv_\Lambda(y)\,dv_\Lambda(x)\\
&= \sum_{g\in G_V}\int_U\int_U \langle F(\widetilde{D}_p)(\widetilde{\iota}_V(x), g\widetilde{\iota}_V(y))g.\widetilde{\iota}_p.\widetilde{\zeta}^p\widetilde{f}(y), \widetilde{\iota}_p.\widetilde{\zeta}^p\widetilde{f}(x)\rangle_{E_p}\\
&\quad \times dv_\Lambda(y)\,dv_\Lambda(x) + O(p^{-\infty}). \end{aligned} \quad (5.15)$$

Now, as G_V acts on $\widetilde{V}$ preserving all the structures, by Definition 5.6 the immersion $g\widetilde{\iota}_V$ is an isotropic immersion intersecting $\widetilde{\iota}_V$ cleanly for any $g \in G_V$. As $F(\widetilde{D}_p)(\cdot,\cdot)$ satisfies the expansion of Lemma 2.5, we can apply Theorem 4.4 to compute each term of the last line of (5.15). Then we have an asymptotic expansion of the form (3.12).

To compute the first-order term, note that if $g\widetilde{\iota}_V$ and $\widetilde{\iota}_V$ do not coincide, then the highest order of the corresponding expansion (3.12) is strictly smaller than $n/2$. Thus we need only to consider the subgroup of G_V fixing the image of ι, which contains at least the identity element of G_V. Summing the contributions of all the elements of this subgroup, by (4.21) we get a function $b_U \in \mathscr{C}^\infty(U, \mathbb{C})$, depending on f only locally, such that the highest order term of (5.15) is given by integration of b_U along U. Now, as $\iota^{-1}(X_{\mathrm{sing}})$ is of measure 0, we can pick a sequence $U_n \subset U, n \in \mathbb{N}$, of open sets in $\mathscr{U}_\Lambda$ containing $\iota^{-1}(X_{\mathrm{sing}})$ and whose measure tends to 0. Then we can repeat (5.15) replacing U by U_n and use (5.10) on the regular part of V to get the following formula for the highest order term for all $n \in \mathbb{N}$;

$$b_0 = 2^{d/2}\int_{\Lambda\setminus U_n}\langle Ff(x), f(x)\rangle_{\iota^*E}\,dv_\Lambda(x) + \int_{U_n} b_U(x)\,dv_\Lambda(x). \quad (5.16)$$

As the second term can be made arbitrarily small, we can take the limit of (5.16) as n tends to infinity, so that formula (4.21) holds for singular Bohr–Sommerfeld submanifolds. □

6. Application to Relative Poincaré Series

In this section, we apply the results of the previous section in the case of quotients of the hyperbolic plane $\mathbb{H}$ by a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$. In that case the Bergman kernel admits an explicit global formula given in Proposition 6.5 as a sum over Γ, realizing it as a *Poincaré series.* In Proposition 6.6, we show that then the isotropic states associated with remarkable curves over $\mathbb{H}/\Gamma$ can be expressed as *relative Poincaré series,* where the sum is over a quotient of Γ instead. The main result of this section is Theorem 6.3, which is an explicit version of Theorem 1.1 in this setting, and which shows that such relative Poincaré series do not vanish as soon as their weight as *holomorphic cusp forms* is large enough.

Recall that the special linear group

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\} \tag{6.1}$$

acts on the Poincaré upper half-plane $\mathbb{H} = \{z = x + \sqrt{-1}y \in \mathbb{C} | y > 0\}$ by the formula

$$g.z = \frac{az + b}{cz + d}. \tag{6.2}$$

The induced action of g on the canonical holomorphic vector field $\partial/\partial z$ over $\mathbb{H}$ is given by $g.\partial/\partial z = (cz + d)^{-2}\partial/\partial z$, so that the dual action on the canonical line bundle $K_{\mathbb{H}} = T^{*(1,0)}\mathbb{H}$ over $\mathbb{H}$ is given on the canonical section dz by

$$g.dz = (cz + d)^2\, dz =: j(g, z)^2\, dz. \tag{6.3}$$

Let $g^{T\mathbb{H}}$ be the *hyperbolic metric* on $\mathbb{H}$ defined by the formula

$$g^{T\mathbb{H}} = \frac{dx^2 + dy^2}{y^2}, \tag{6.4}$$

so that it is invariant by the action of $\mathrm{SL}_2(\mathbb{R})$. The associated Kähler metric $\omega_{\mathbb{H}}$ satisfies

$$\omega_{\mathbb{H}} = \frac{\sqrt{-1}}{2}\frac{dz \wedge d\bar{z}}{y^2}. \tag{6.5}$$

Let us write $|\cdot|_{K_{\mathbb{H}}}$ for the $\mathrm{SL}_2(\mathbb{R})$-invariant Hermitian norm on $K_{\mathbb{H}}$ given by

$$|dz|^2_{K_{\mathbb{H}}} = y^2. \tag{6.6}$$

Note that it differs from the norm induced by $g^{T\mathbb{H}}$ from a constant factor $\sqrt{2}$. Then the curvature $R^{K_{\mathbb{H}}}$ of the Chern connection of $(K_{\mathbb{H}}, h^{K_{\mathbb{H}}})$ satisfies $\sqrt{-1}R^{K_{\mathbb{H}}} = \omega_{\mathbb{H}}$, so that $R^{K_{\mathbb{H}}}$ satisfies condition (1.1) for the renormalized Kähler form $\omega_{\mathbb{H}}/2\pi$. As $R^{\det} = -R^{K_{\mathbb{H}}}$ is proportional to $\sqrt{-1}\omega_{\mathbb{H}}$, we easily see that $K_{\mathbb{H}}$ satisfies (5.1).

Now if Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, then the quotient $X := \mathbb{H}/\Gamma$ has an induced structure of a Kähler orbifold, and its canonical line bundle K_X is the quotient of $K_{\mathbb{H}}$ by the induced action (6.3). We denote by g^{TX} and ω_X the quotient metric and quotient Kähler form on X, respectively, and we endow K_X with the Hermitian metric h^{K_X} induced by (6.6). Then (1.1) holds for K_X up to a factor 2π as before, and it satisfies (5.1) as well. Therefore, taking $L = K_X$ and $E = \mathbb{C}$, we are precisely in the context of the previous sections for the renormalized Kähler form $\omega = \omega_X/2\pi$ with $g^{TX}_\omega = g^{TX}/2\pi$.

Recall that a smooth path $\gamma : [0,l] \to X, l > 0$, is said to be a *closed loop* if it induces a (singular) immersion $\widetilde{\gamma} : S^1 \to X$ by identification of 0 with l. The following lemma describes the class of (singular) Bohr–Sommerfeld submanifolds we will be interested in.

LEMMA 6.1. *For $l > 0$, let $\gamma : [0,l] \to X$ be a closed loop in X parameterized by arclength with respect to g^{TX}, and suppose that the holonomy of K_X along γ with respect to $\nabla^{K_{\mathbb{H}}}$ is trivial. Then the immersion $\widetilde{\gamma} : S^1 \to X$, obtained from γ by identification of* 0 *and l, satisfies the Bohr–Sommerfeld condition of Definition 3.1.*

Proof. As ω_X is a 2-form, any smooth map $f : S^1 \to X$ satisfies $f^*\omega = 0$. Thus as $\dim X = 2$, any immersion $\iota : S^1 \to X$ is Lagrangian. By Remark 3.2 it satisfies the Bohr–Sommerfeld condition if and only if the holonomy of the pullback connection is trivial, which is exactly the hypothesis of Lemma 6.1 by Remark 3.2. □

In any case, such a path $\gamma : [0,l] \to X, l > 0$, is called a *Bohr–Sommerfeld curve.* The orientation on $\widetilde{\gamma} : S^1 \to X$ is determined by the canonical vector field ∂_t on $[0,l]$. Following Remark 3.2, if $\gamma : [0,l] \to X, l > 0$, is a smooth closed loop such that its holonomy is a kth root of unity for some $k \in \mathbb{N}$, then we can take a cover of degree k of this loop to get a Bohr–Sommerfeld curve $\gamma_k : [0,kl] \to X$.

Note that as X is a complex orbifold with $\dim_{\mathbb{C}} X = 1$ and as Γ acts on $\mathbb{H}$ holomorphically, the singular set X_{sing} is necessarily a discrete set. By Definition 5.6, as S^1 is a manifold, the stabilizer of $\widetilde{\gamma}$ is necessarily trivial in any case.

COROLLARY 6.2. *A closed geodesic loop $\gamma : [0,l] \to X, l > 0$ parameterized by arclength is a Bohr–Sommerfeld curve.*

Proof. Recall that $K_X = T^{*(1,0)}X$ is equipped with Hermitian metric and connection h^{K_X}, ∇^{K_X} induced by g^{TX}, ∇^{TX} via (2.1). For any $t \in [0,l]$, let $\dot{\gamma}_t \in T_{\gamma(t)}X$ denote the vector tangent to the curve $\gamma : [0,l] \to X$, inducing $\dot{\gamma}_t^{(0,1)} \in T^{(0,1)}X$ via (2.1). We write $\dot{\gamma}_t^{(0,1),*} \in K_{X,\gamma(t)}$ for its metric dual. As $\gamma : [0,l] \to X$ is geodesic, we know that $\nabla^{TX}_{\dot{\gamma}}\dot{\gamma} = 0$, so that $\nabla^{K_X}_{\dot{\gamma}}\dot{\gamma}^{(0,1),*} = 0$, which precisely means that $\widetilde{\gamma} : S^1 \to X$ satisfies the Bohr–Sommerfeld condition with associated section $\gamma^{(0,1),*} \in \mathscr{C}^\infty(S^1, \widetilde{\gamma}^*K_X)$.

Now if X is an orbifold and if $z \in X$ is a singular point of X, then its associated group G^X_z preserves the Riemannian structure and sends a geodesic through z to another geodesic through z, which intersect transversally by unicity of the geodesics. Thus $\gamma : [0,l] \to X$ satisfies the definition of a singular immersion as in Definition 5.6. □

Let $\gamma : [0,l] \to X, l > 0$, be a Bohr–Sommerfeld curve together with a unitary flat section $\zeta \in \mathscr{C}^\infty([0,l], \gamma^* K_X)$, inducing a (possibly singular) Bohr–Sommerfeld submanifold $(S^1, \widetilde{\gamma}, \zeta)$ as before. For any $p \in \mathbb{N}^*$, we define $s_{\gamma,p} \in H^0_{(2)}(X, K_X^p)$ by

$$s_{\gamma,p}(x) = \int_0^l P_p^X(x, \gamma(t))\gamma_p.\zeta^p(t)\, dt \tag{6.7}$$

for any $x \in X$, where $P_p^X(\cdot,\cdot)$ is the Bergman kernel with respect to dv_X of the orthogonal projection on $H^0_{(2)}(X, K_X^p)$. Then $s_{\gamma,p}$ is precisely the Lagrangian state associated with $(S^1, \widetilde{\gamma}, \zeta)$ and $f = 1$ in the sense of Definition 3.3.

Then we can apply Theorems 5.3 and 5.9 to get the following specialization of (3.11) and (4.4), where we adopt the convention that $\sqrt{-a} = \sqrt{-1}\sqrt{a}$ if $a > 0$.

THEOREM 6.3. *Let $\gamma : [0,l] \to X, l > 0$, be a Bohr–Sommerfeld curve, and let $\{s_{\gamma,p}\}_{p\in\mathbb{N}^*}$ be as in (6.7). Then*

$$\|s_{\gamma,p}\|^2_{L^2} = \left(\frac{p}{\pi}\right)^{1/2} l + O(p^{-1/2}). \tag{6.8}$$

Furthermore, if γ_1 and γ_2 are two Bohr–Sommerfeld curves intersecting cleanly away from the singular set, then we get

$$\langle s_{\gamma_1,p}, s_{\gamma_2,p}\rangle = \sqrt{2} \sum_{z\in\gamma_1\cap\gamma_2} \sum_{\substack{t_1,t_2>0,\\ \gamma_1(t_1)=\gamma_2(t_2)=z}} \lambda^p_{t_1,t_2} \frac{e^{\sqrt{-1}(\theta_z/2-\pi/4)}}{\sqrt{\sin(\theta_z)}} + O(p^{-1}), \tag{6.9}$$

where $\theta_z \in \,]0, 2\pi[$ is the oriented angle from γ_1 to γ_2 at z, and where for all $t_1, t_2 > 0$ such that $\gamma_1(t_1) = \gamma_2(t_2)$, we define $\lambda_{t_1,t_2} = \langle\zeta_1(t_1), \zeta_2(t_2)\rangle_{K_X}$.

Proof. In the case of smooth and compact X, (6.8) and (6.9) are standard computations from (3.24) and (4.4). We will indicate how to modify directly the argument to get the case $g^{TX} = 2\pi g^{TX}_\omega$ from the case $g^{TX} = g^{TX}_\omega$ in all generality.

For any $p \in \mathbb{N}^*$, let us write $P_{p,\omega}$ for the orthogonal projection to $H^0_{(2)}(X, K_X^p)$ with respect to the L^2-Hermitian product induced by g^{TX}_ω. Then $P_{p,\omega} = P_p^X$, but $dv_{X,\omega} = dv_X/2\pi$, so that the associated Bergman kernel with respect to $dv_{X,\omega}$ satisfies $P_{p,\omega}(\cdot,\cdot) = 2\pi P_p^X(\cdot,\cdot)$. On the other hand, the Riemannian volume form dt_ω on $[0, L[$ induced by g^{TX}_ω satisfies $dt_\omega = dt/\sqrt{2\pi}$. Thus, writing $\{s_{\omega,\gamma,p}\}_{p\in\mathbb{N}^*}$ for the Lagrangian state obtained replacing g^{TX} by g^{TX}_ω, from (3.2) we get that $s_{\omega,\gamma,p} = \sqrt{2\pi} s_{\gamma,p}$ for any $p \in \mathbb{N}^*$.

Consider now two Bohr–Sommerfeld curves γ_1 and γ_2. Following the above notations, for any $p \in \mathbb{N}^*$, we get

$$\langle s_{\gamma_1,p}, s_{\gamma_2,p}\rangle_p = \frac{1}{2\pi}\int_X \langle s_{\omega,\gamma_1,p}, s_{\omega,\gamma_2,p}\rangle_{K_X^p}\, dv_X = \langle s_{\omega,\gamma_1,p}, s_{\omega,\gamma_2,p}\rangle_{\omega,p}, \tag{6.10}$$

where $\langle\cdot,\cdot\rangle_{\omega,p}$ denotes the L^2-Hermitian product with respect to g_ω^{TX}. Noticing finally that $\mathrm{Vol}_\omega(\gamma) = l/\sqrt{2\pi}$ for any $\gamma : [0,l] \to X, l > 0$, parameterized by arclength with respect to g^{TX}, we recover (6.8) and (6.9) as in the case of smooth and compact X. □

In the case where X is a compact Riemann surface, so that in particular Γ acts freely on $\mathbb{H}$, Theorem 6.3 is a result of [8, Thm. 4.4], where (6.8) and (6.9) are shown with a weaker error term. As shown in Proposition 6.6, formulas (6.8) and (6.9) are especially interesting in the case of curves $\gamma : \mathbb{R} \to \mathbb{H}$ such that there exist $l > 0$ and $g_0 \in \Gamma$ satisfying $g_0.\gamma(t) = \gamma(t+l)$ for any $t \in \mathbb{R}$. We say that γ is *associated with* g_0.

In particular, if γ is a closed geodesic, then γ is associated with a *hyperbolic* element $g_0 \in \Gamma$, that is, satisfying $\mathrm{Tr}(g_0) > 2$, unique up to conjugation. Closed geodesics belong to a larger class of hyperbolic curves called *hypercycles.*

If $g_0 \in \Gamma$ is *parabolic,* that is, satisfying $\mathrm{Tr}(g_0) = 2$, then its action has no fixed points in $\mathbb{H}$, and it occurs in Γ only in the case of X noncompact. The most interesting associated curves in that case are the so-called *horocycles,* which are isometric to a horizontal line in $\mathbb{H}$.

If $g_0 \in \Gamma$ is *elliptic,* that is, satisfying $\mathrm{Tr}(g_0) < 2$, then g_0 fixes a unique point $z \in \mathbb{H}$, which descends to a singular point of X. The most interesting associated curves in that case are circles with center at the fixed point of g_0 in $\mathbb{H}$. Note that Γ acts freely on $\mathbb{H}$ if and only if it contains no elliptic elements.

Our next goal is to explicitly identify the Lagrangian states associated with such curves. Let $\mathscr{F}$ be a measurable fundamental domain of Γ in $\mathbb{H}$. Through the natural identification $\mathscr{C}^\infty(X, K_X) \simeq \mathscr{C}^\infty(\mathbb{H}, K_{\mathbb{H}})^\Gamma$ and trivializing $K_{\mathbb{H}}$ using its canonical section dz, from (6.3) we have the following natural identification for any $p \in \mathbb{N}^*$:

$$H^0_{(2)}(X, K_X^p) \simeq \Big\{ f \in \mathscr{C}^\infty(\mathbb{H}) \;\Big|\; f \text{ holomorphic}, \\ f(g.z) = f(z) j(g,z)^{2p}, \int_{\mathscr{F}} |f(z)|^2 y^{2p-2}\, dxdy < \infty \Big\}. \quad (6.11)$$

We will implicitly use this identification throughout the rest of this section.

Remark 6.4. Assume that $\mathrm{Vol}(X) < +\infty$, that is, Γ is a *Fuchsian group of the first kind.* As explained in [2; 3, Section 6], then the space $H^0_{(2)}(X, K_X^p)$ is identified through the identification (6.11) with the space $\mathrm{S}_{2p}(\Gamma)$ of *holomorphic cusp forms of weight* $2p$ with the space of holomorphic functions on $\mathbb{H}$ satisfying the equivariance property of (6.11) and vanishing at infinity. Such spaces are of particular interest in arithmetic.

The following result is classical and follows, for instance, from [16, Prop. I.5.3, II.1].

PROPOSITION 6.5. *Under the identifications above, for any $p \in \mathbb{N}^*$, the Bergman kernel of $H^0_{(2)}(\mathbb{H}, K^p_{\mathbb{H}})$ satisfies the formula*

$$P^{\mathbb{H}}_p(z, w) = \frac{2p-1}{4\pi}\left(\frac{2\sqrt{-1}}{z-\overline{w}}\right)^{2p} dz^p\, d\overline{w}^p \tag{6.12}$$

for any $z, w \in \mathbb{H}$, where $d\overline{w} \in \overline{K}_{\mathbb{H},w} \simeq K^*_{\mathbb{H},w}$ denotes the dual of $dw \in K_{\mathbb{H},w}$ with respect to the metric. Furthermore, for any $\widetilde{w} \in \mathbb{H}$ descending to $w \in X$ in the quotient, we have

$$P^X_p(z, w) = \sum_{g\in\Gamma} j(g, z)^{-2p}\, P^{\mathbb{H}}_p(g.z, \widetilde{w}) \tag{6.13}$$

through identification (6.11) in $z \in \mathbb{H}$, where the convergence of the right-hand side is absolute and uniform for $z, \widetilde{w}$ in any compact set of $\mathbb{H}$.

Series (6.13) is an example of *Poincaré series* and is a standard method to construct functions in $\mathrm{S}_{2p}(\Gamma)$ as in Remark 6.4. A fundamental problem of the theory of cusp forms is deciding whether a given series vanishes identically or not.

If $\Gamma_0 \subset \Gamma$ is a subgroup of Γ, then we write $\Gamma_0\backslash\Gamma$ for the set of equivalence classes $[g] := \{g_0 g \in \Gamma | g_0 \in \Gamma_0\}$ for all $g \in \Gamma$. Recall that if g_0 is hyperbolic or parabolic, then it generates a free group $\Gamma_0 \subset \Gamma$, whereas if g_0 is elliptic, then it generates a cyclic subgroup $\Gamma_0 \subset \Gamma$.

Using Proposition 6.5 and a classical unfolding technique, we get explicit formulas for the Lagrangian states associated with remarkable curves. This is described in the next result.

PROPOSITION 6.6. *Let $g_0 \in \Gamma$, and let $\gamma : \mathbb{R} \to \mathbb{H}$ be a smooth curve on $\mathbb{H}$ parameterized by arclength, together with a unitary flat section $\zeta \in \gamma^* K_{\mathbb{H}}$ such that there is $l > 0$ satisfying $g_0.\gamma(t) = \gamma(t+l)$ and $g_0.\zeta(t) = \zeta(t+l)$ for all $t \in \mathbb{R}$. Write $\Gamma_0 \subset \Gamma$ for the subgroup generated by g_0.*

If g_0 is hyperbolic or parabolic, then the Lagrangian state $\{s_{\gamma,p}\}_{p\in\mathbb{N}^*}$ associated with γ is given through (6.11) and for any $p \in \mathbb{N}^*$ by

$$s_{\gamma,p}(z) = \frac{2p-1}{4\pi} \sum_{[g]\in\Gamma_0\backslash\Gamma} j(g, z)^{-2p} \times \int_{-\infty}^{+\infty} \left(\frac{2\sqrt{-1}}{g.z - \overline{\gamma(t)}}\right)^{2p} \langle \zeta(t), d\gamma(t)\rangle_{K_X}\, dt. \tag{6.14}$$

If g_0 is elliptic, then letting $n \in \mathbb{N}$ be the order of Γ_0, the Lagrangian state $\{s_{\gamma,p}\}_{p\in\mathbb{N}^*}$ is given through (6.11) and for any $p \in \mathbb{N}^*$ by

$$s_{\gamma,p}(z) = \frac{2p-1}{4\pi} \sum_{[g]\in\Gamma_0\backslash\Gamma} j(g, z)^{-2p} \times \int_{0}^{n} \left(\frac{2\sqrt{-1}}{g.z - \overline{\gamma(t)}}\right)^{2p} \langle \zeta(t), d\gamma(t)\rangle_{K_X}\, dt. \tag{6.15}$$

The convergences of the series in (6.14) *and* (6.15) *are absolute and uniform for* z *in any compact set of* $\mathbb{H}$.

Proof. Recall that $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by holomorphic isometries and that the induced action on K_X preserves h^{K_X}. This implies in particular that the Bergman kernel of $H^0_{(2)}(X, K_X)$ is invariant by $\mathrm{SL}_2(\mathbb{R})$. Using (6.3), we have for any $w \in \mathbb{H}$, $g \in \mathrm{SL}_2(\mathbb{R})$, and $\zeta \in K_{\mathbb{H},w}$,

$$j(g,z)^{-2p} P_p^{\mathbb{H}}(g.z, w)\zeta = P_p^{\mathbb{H}}(z, g^{-1}.w)g^{-1}.\zeta \tag{6.16}$$

through identification (6.11) in $z \in \mathbb{H}$. On the other hand, for any $g, h \in \mathrm{SL}_2(\mathbb{R})$ and $w \in \mathbb{H}$, the cocycle formula $j(gh, w) = j(g, h.w)j(h, w)$ holds by definition. Consider hyperbolic or parabolic $g_0 \in \Gamma$, and let $l > 0$ be the smallest positive number satisfying $g_0.\gamma(t) = \gamma(t + l)$ and $g_0.\zeta(t) = \zeta(t + l)$ for all $t \in \mathbb{R}$. Then from (6.7) and from the uniform convergence of (6.13) we get

$$\begin{aligned} s_{\gamma,p}(z) &= \int_\gamma \sum_{g\in\Gamma} j(g,z)^{-2p} P_p^{\mathbb{H}}(g.z, \gamma(t))\zeta(t)\,dt \\ &= \sum_{[g]\in\Gamma_0\backslash\Gamma} \sum_{n\in\mathbb{Z}} j(g_0^n g, z)^{-2p} \int_0^l P_p^{\mathbb{H}}(g_0^n g.z, \gamma(t))\zeta(t)\,dt \\ &= \sum_{[g]\in\Gamma_0\backslash\Gamma} j(g,z)^{-2p} \sum_{n\in\mathbb{Z}} \int_0^l P_p^{\mathbb{H}}(g.z, g_0^{-n}.\gamma(t))g_0^{-n}.\zeta(t)\,dt \\ &= \sum_{[g]\in\Gamma_0\backslash\Gamma} j(g,z)^{-2p} \sum_{n\in\mathbb{Z}} \int_{-nl}^{-(n+1)l} P_p^{\mathbb{H}}(g.z, \gamma(t))\zeta(t)\,dt \\ &= \sum_{[g]\in\Gamma_0\backslash\Gamma} j(g,z)^{-2p} \int_{-\infty}^{+\infty} P_p^{\mathbb{H}}(g.z, \gamma(t))\zeta(t)\,dt, \end{aligned} \tag{6.17}$$

and we conclude by (6.12). Note that the sums in (6.17) do not depend on the choice of the representatives $g \in \Gamma$ of any $[g] \in \Gamma_0\backslash\Gamma$. The elliptic case (6.15) is strictly analogous. □

The series (6.14) and (6.15) are called *relative Poincaré series.* We can now state our main theorem, which is a consequence of Theorem 6.3.

THEOREM 6.7. *If* $\gamma : \mathbb{R} \to \mathbb{H}$ *satisfying the hypotheses of Proposition* 6.6 *descends to a Bohr–Sommerfeld curve, then there is* $p_0 \in \mathbb{N}$ *such that the associated series* (6.14) *or* (6.15) *do not vanish identically for* $p > p_0$. *This holds in particular if* $\gamma : \mathbb{R} \to \mathbb{H}$ *is a closed geodesic.*

Proof. By (6.8) we know that there is $p_0 \in \mathbb{N}$ such that $s_{\gamma,p}$ is nonvanishing for $p \geq p_0$, so that we can conclude by Corollary 6.2 and Proposition 6.6. □

In general, there are simple numerical criterions for horocycles, circles, and hypercycles to satisfy the Bohr–Sommerfeld condition, and the integral in sums (6.14) and (6.15) can be computed explicitly using Proposition 6.5 and elementary complex analysis. In particular, as computed in [8, Thm. 4.11], if $g_0 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a hyperbolic element of Γ, then the series (6.14) for a closed geodesic γ associated with g_0 takes the form

$$s_{\gamma,p}(z) = C_p \sum_{[g]\in\Gamma_0\backslash\Gamma} j(g,z)^{-2p}(c(g.z)^2 + (d-a)(g.z) - b)^{-p} \tag{6.18}$$

with explicit nonvanishing constant $C_p \in \mathbb{C}$ for all $p \in \mathbb{N}^*$, and we recover up to normalization the relative Poincaré series associated with closed hyperbolic geodesics by Katok [21, Section 1]. Furthermore, from Theorem 6.3 we get a formula for the highest order term as $p \to +\infty$ of the intersection product of two closed geodesics, recovering a result of [21, Thm. 3]. As shown in [21, Thm. 1], if Γ is a Fuchsian group of the first kind, then the series associated with the primitive hyperbolic elements of Γ as before generate the whole space $S_{2p}(\Gamma)$.

Finally, note that there are many discrete subgroups $\Gamma \subset SL_2(\mathbb{R})$ of interest containing elliptic points and leading to noncompact quotients $X = \mathbb{H}/\Gamma$, even in the case where Γ is Fuchsian of the first kind. The most famous examples are the classical modular curves.

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