1 Some Exercises in Real Analysis

1.1 (Sequences)

1. Prove the following sequential limits:

$$a)\ \lim_{n\to\infty}\frac{\sin(n)}{n}=0,\quad b)\ \lim_{n\to\infty}\frac{n!}{n^n}=0,\quad c)\ \lim_{n\to\infty}\frac{(1+2+\cdots+n)^2}{n^4}=\frac{1}{4}.$$

Try proving in multiple ways ($N-\varepsilon$, squeeze theorem, etc).
(Hint (for b)): how many times does n appear in "$n!$"? How about "n^n"?)

2. The following sequences are defined recursively. Prove that each converges, and find the limiting value in each case.

$$\begin{aligned}
a)\quad & x_1 := \sqrt{2}, x_{n+1} := \sqrt{2+x_n} \text{ for } n \geq 1\\
b)\quad & x_1 = 2, x_{n+1} := \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\\
c)\quad & x_1 := 1, x_2 := 2, x_{n+1} := \frac{1}{2}(x_n + x_{n-1})\\
d)\quad & x_1 = 1, x_{n+1} = \sin(x_n)
\end{aligned}$$

(Hint: if $\lim_n x_n = L$, then what is $\lim_n x_{n-1}$?)

3. Prove that, for $b\in\mathbb{R}$ with $b>0$, $x_n := \frac{n}{b^n}$ converges to 0 if $b>1$ and properly diverges to $+\infty$ if $b\leq 1$.

4. Prove that

$$x_n := 1+\frac{1}{2^2}+\cdots+\frac{1}{n^2} := \sum_{k=0}^{n}\frac{1}{k^2}$$

converges (don't try to find the limiting value).

5. Let $x_1\in\mathbb{R}$ be nonzero, and define inductively $x_{n+1} := x_n + \frac{1}{x_n}$. Prove that

$$\lim_{n\to\infty} x_n = \begin{cases} +\infty & x_1 > 0\\ -\infty & x_1 < 0\end{cases}.$$

6. Let $\{x_n\}\subset\mathbb{R}$ converge to $x\in\mathbb{R}$, with $x_n>0$ for all n. Prove that both the sequence of *arithmetic means*

$$y_n := \frac{1}{n}(x_1+\cdots+x_n)$$

and the sequence of *geometric means*

$$z_n := (x_1x_2\cdots x_n)^{1/n}$$

converge to x.

1.2 (Functional Limits/Continuity)

1. ★ Let $f:\mathbb{R}\to\mathbb{R}$ satisfy the following:
i. $f(x+y)=f(x)+f(y)$, for all $x,y\in\mathbb{R}$,

ii. f continuous at 0

iii. $f(1) = 1$

Prove that f must be the identity function, i.e. $f(x) = x$ for all $x \in \mathbb{R}$.
(Hint: begin by showing f must be continuous everywhere in $\mathbb{R}$, and use this to prove the claim for $x \in \mathbb{Q}$. Conclude by a density argument.)

2. [Recall that a set $X \subset \mathbb{R}$ is said to be *open* if for every $x \in X$, there exists an $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset X$ (i.e., if a point is in X, so are all of its "neighboring" points for a sufficiently small neighborhood), and that a set $Y \subset \mathbb{R}$ is said to be *closed* if its complement $Y^C = \mathbb{R} \setminus X$ is open.]
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that $X := \{x \in \mathbb{R} \mid f(x) = 0\}$ is a closed subset of $\mathbb{R}$.
(Bonus ★: what can you say about the set $Y := \{x : a \leq f(x) < b\}$ for some real numbers $a < b$? Can you find continuous functions f such that Y open, closed, both, and neither?)

3. Compute the following functional limits using only the ε-δ/ε-M definition:

$$a) \quad \lim_{x \to 1^+} \left(\frac{x}{x-1} \right)$$

$$b) \quad \lim_{x \to \infty} \left(\frac{\sqrt{x+1}}{x} \right)$$

$$c) \quad \lim_{x \to \infty} \frac{\sqrt{x} - x}{\sqrt{x} + x}$$

4. Let $f : \mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. Prove that, for some real number L,

$$\lim_{x \to \infty} f(x) = L \Leftrightarrow \lim_{x \to 0^+} f\left(\frac{1}{x}\right) = L.$$

5. We say a function $f : \mathbb{R} \to \mathbb{R}$ is *Lipschitz continuous* if there exists a constant $K > 0$ such that

$$|f(x) - f(y)| \leq K|x-y|, \quad \forall x, y \in \mathbb{R}.$$

Prove that if f Lipschitz continuous, it is also *uniformly* continuous. Can you find an example of a continuous function that isn't Lipschitz continuous?
(Bonus: prove that if $f : [a,b] \to \mathbb{R}$ is differentiable with continuous derivative, then f is Lipschitz on $[a,b]$)

6. Prove using the δ-ε definition that the following functions are continuous on $\mathbb{R}$:

$$a) f(x) := \frac{1}{1+x^2}, \quad b) f(x) := \sqrt{x^2+1}.$$

7. ★ Let $f : \mathbb{R}_+ \to \mathbb{R}$ be continuous and bounded. Show that for any $T \in \mathbb{R}$, there exists a sequence $\{x_n\} \subset \mathbb{R}_+$ such that $\lim_{n\to\infty} x_n = +\infty$ and

$$\lim_{n \to \infty} [f(x_n + T) - f(x_n)] = 0.$$

(Hint: this one's very hard)

8. Let $f : [0,1] \to \mathbb{R}$ be continuous, with $f(0) = 0 = f(1)$. Show that there must exist a $c \in \left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$.
(Hint: don't reinvent the wheel; turn what you have into a wheel)

9. Let $f : [0,1] \to \mathbb{R}$ continuous, and suppose that for every $x \in [0,1]$, there exists a $y \in [0,1]$ such that

$$|f(y)| \leq \frac{1}{2}|f(x)|.$$

Prove that there exists a $c \in [0,1]$ such that $f(c) = 0$.

10. Prove that the function

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is *not* continuous at $x = 0$, using the sequential criterion for convergence.

11.

2 Some Useful Inequalities to Remember

- **Triangle inequality:** $|x+y| \leq |x| + |y|, \quad \forall x, y \in \mathbb{R}$.

  Proof. If x or y equals 0, or $x + y = 0$, this is clear If x, y both positive, the claim holds with equality. If both negative, then since for any negative number $z < 0$ we have $|z| = -z$, we find that $|x+y| = -(x+y)$ on one side, and $|x| + |y| = -x + -y$ on the other, so equality also holds in this case. Finally, if one of x, y negative and the other positive, wlog $x < 0$ and $y > 0$, then

  $$|x+y| = \begin{cases} y + x = |y| - |x| \leq |y| + |x| & \text{if } y > |x| \\ -y - x - |x| - |y| \leq |y| + |x| & \text{if } y < |x|' \end{cases}$$

  proving the claim. ∎

- **Reverse Triangle Inequality**: $||x| - |y|| \leq |x - y|$

  Proof. By the triangle inequality, we have both bounds

  $$|x| = |x - y + y| \leq |x - y| + |y|, \quad |y| \leq |x - y| + |x|,$$

  from which, subtracting from both sides,

  $$-|x-y| \leq |x| - |y| \leq |x - y| \Rightarrow ||x| - |y|| \leq |x - y|.$$

  ∎

- **AM-GM Inequality (basic)**: $\sqrt{ab} \leq \frac{a+b}{2}, a, b \in \mathbb{R}$ with $a, b \geq 0$.

  Proof. Remark that

  $$\left.\begin{aligned} (a-b)^2 &= a^2 - 2ab + b^2 \\ (a+b)^2 &= a^2 + 2ab + b^2 \end{aligned}\right\} \Rightarrow (a-b)^2 = (a+b)^2 - 4ab.$$

  But also, $(a-b)^2 \geq 0$, so this implies

  $$0 \leq (a+b)^2 - 4ab \Rightarrow 4ab \leq (a+b)^2,$$

  which gives the claimed inequality upon taking square roots of both sides and dividing by two. ∎

- **AM-GM Inequality (extended):** $(a_1 \cdots a_n)^{1/n} \leq \frac{a_1 + \cdots + a_n}{n}$, where $a_i \in \mathbb{R}$ with $a_i \geq 0$.

3 Solutions

3.1 Sequences

SOLUTION 1. *a*) Remember that $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$, thus

$$0 \leq \left|\frac{\sin(n)}{n}\right| \leq \frac{1}{n}.$$

Since $\frac{1}{n}$ converges to zero, then by the squeeze theorem, so does $\frac{\sin(n)}{n}$.
b) We can write

$$0 \leq \frac{n!}{n^n} = \frac{n \cdot (n-1)\cdots(1)}{\underbrace{n \ldots n}_{n \text{ times}}} = \underbrace{\frac{n}{n}}_{=1} \cdot \underbrace{\frac{n-1}{n}}_{\leq 1} \cdot \underbrace{\frac{n-2}{n}}_{\leq 1} \cdots \frac{1}{n} \leq \frac{1}{n},$$

so again $\frac{n!}{n^n}$ converges to zero by squeeze theorem.
c) Remember that $1+2+\cdots+n = \frac{(n)(n+1)}{2}$, so that the sequence may be written

$$\begin{aligned}\frac{(1+2+\cdots+n)^2}{n^4} &= \frac{((n)(n+1))^2}{4n^4} = \frac{n^2\left(n^2+2n+1\right)}{4n^4}\\ &= \frac{n^4+2n^3+n^2}{4n^4} = \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2} \to \frac{1}{4}.\end{aligned}$$

■

SOLUTION 2. *a*) We prove x_n is monotonically increasing and bounded from above, from which case convergence follows by the monotone convergence theorem. We prove both by induction. For increasing, the base case is clear, for $x_2 = \sqrt{2+\sqrt{2}} > \sqrt{2} = x_1$. Supposing this inequality holds for some n, i.e., $x_n \geq x_{n-1}$, we find that

$$x_{n+1} = \sqrt{2+x_n} \overset{\text{induct}}{\geq} \sqrt{2+x_{n-1}} \overset{\text{by def.}}{=} x_n,$$

so we have monotone increasing indeed. For boundedness, we claim $x_n \leq 2$ for all $n \geq 1$. The base case, for $n = 1$, is clear. Supposing $x_{n-1} \leq 2$, then

$$x_n = \sqrt{2+x_{n-1}} \overset{\text{induct}}{\leq} \sqrt{2+2} = 2,$$

so we indeed have boundedness of the whole sequence. Thus, we know x_n converges; let L be its limit. But then, we also know that $x_n = \sqrt{2+x_{n-1}}$, thus

$$L = \lim_n x_n = \lim_n \sqrt{2+x_{n-1}} \overset{\text{cnty of sqrt}}{=} \sqrt{2+L}.$$

Squaring both sides, this means

$$L^2 = L + 2,$$

which one can solve to find two solutions, $L = 2$ and $L = -1$. But x_n is a strictly positive, increasing sequence, so it can't have a negative limit, and thus we conclude $L = 2$.
b) *c*) *d*) ■

SOLUTION 3. ■

SOLUTION 4. We will employ the Cauchy criterion to prove convergence. Let $\varepsilon > 0$ and fix $m > n \geq N$ for some N to be determined later. Then,

$$\begin{aligned}
|x_m - x_n| &= \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{m^2}\right) - \left(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right) \\
&= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{m^2} \\
&\leq \frac{1}{(n+1)n} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{m(m-1)} \\
&= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{m} + \frac{1}{m-1}\right) \\
&= \frac{1}{n} - \frac{1}{m} \leq \frac{1}{N},
\end{aligned}$$

so taking $N \geq \frac{1}{\varepsilon}$ will work to prove convergence.

Remark 3.1: When I originally did this question, I immediately upper-bounded the second line by $\frac{m-n}{(n+1)^2} \leq (m-n)\varepsilon$, which is *not* good enough to prove the desired result.

■

SOLUTION 5. ■

SOLUTION 6. We first prove for y_n. Let $\varepsilon > 0$. Since x_n converges to x, there exists some N_1 such that

$$n \geq N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{2} \quad \text{(i)}.$$

Next, since x_n convergent it must be bounded (check!) so there exists some $M > 0$ such that $|x_n - x| \leq M$ for all $n \in \mathbb{N}$. Finally, the sequence $\frac{1}{n}$ is convergent, so there exists an N_2 such that

$$n \geq N_2 \Rightarrow \frac{1}{n} < \frac{\varepsilon}{2MN_1}. \quad \text{(ii)}$$

Let $N := \max(N_1, N_2)$. Then, for $n \geq N$, we can split our sequence as follows:

$$
\begin{aligned}
|y_n - x| &= \frac{\left|x_1 + \cdots + x_{N_1} + x_{N_1+1} + \cdots + x_n - nx\right|}{n} \\
&\leq \frac{1}{n}\sum_{i=1}^{N_1}|x_i - x| + \frac{1}{n}\sum_{i=N_1+1}^{n}|x_i - x| \\
&\leq \underbrace{\frac{\varepsilon}{2MN_1}}_{\text{(ii)}} \cdot \underbrace{MN_1}_{\text{boundedness}} + \underbrace{\frac{n-(N_1+1)}{n}\frac{\varepsilon}{2}}_{\text{(i)}} \\
&= \left(1 + \underbrace{\frac{n-(N_1+1)}{n}}_{\leq 1}\right)\frac{\varepsilon}{2} \\
&\leq 2\frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

which proves the claimed convergence. ■

3.2 (Functional Limits/Continuity)

Solution 1. We follow the hint. First, we show f continuous everywhere. Before that, we note that $f(0) = 0$, for

$$f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0,$$

just by using the definition. Fix now $\varepsilon > 0$, and let $\delta > 0$ such that $|x| < \delta \Rightarrow |f(x) - f(0)| = |f(x)| < \varepsilon$, appealing to the assumed continuity of f at 0. Then, for any $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we find

$$|f(x) - f(y)| \overset{\text{definition}}{=} |f(x-y)| < \varepsilon,$$

using the continuity assumption at the origin *(in particular, we have uniform convergence)*. Next, we prove the result for $q \in \mathbb{Q}$. First, remark that for any positive integer n and any $x \in \mathbb{R}$, we have that

$$f(nx) = f((n-1)x + x) = f((n-1)x) + f(x),$$

but we can repeat this work for $f((n-1)x)$, and we deduce that $f(nx) = nf(x)$. By similar reasoning, we can deduce that $f(nx) = nf(x)$ for any negative integer as well. For $q \in \mathbb{Q}$, we can write

$$q = \frac{a}{b},$$

where $a, b \in \mathbb{Z}$. From the previous line of reasoning, we see thus that

$$f\left(\frac{a}{b}\right) = a \cdot f\left(\frac{1}{b}\right).$$

But also,

$$1 = f(1) = f\left(b \cdot \frac{1}{b}\right) = bf\left(\frac{1}{b}\right) \Rightarrow f\left(\frac{1}{b}\right) = \frac{1}{b},$$

from which we conclude

$$f(q) = f\left(\frac{a}{b}\right) = \frac{a}{b} = q$$

indeed. This proves the conclusion for all rational numbers. Now, for any $x \in \mathbb{R}$, let $\{q_n\}$ be a sequence of rational numbers converging to x, which must exist by the density of $\mathbb{Q}$ in $\mathbb{R}$. Then, by the continuity of f we proved above,

$$\lim_n f(q_n) = f(x)$$

on the one hand, but also, since $q_n \in \mathbb{Q}$ for each $\mathbb{N}$,

$$\lim_n f(q_n) = \lim_n q_n = x,$$

from which we conclude $f(x) = x$ for all $x \in \mathbb{R}$, as we aimed to show. ■

Solution 2. We will equivalently show that $X^c = \{f(x) \neq 0\}$ is open. Let $x \in X^c$, and assume wlog that $f(x) > 0$ (if $f(x) < 0$, repeat the proof with $-f$ instead of f), and let $\varepsilon := \frac{f(x)}{2}$, which is strictly positive since $f(x) \neq 0$. Since f continuous, in particular, at x, there exists a $\delta > 0$ such that if $|y - x| < \delta$, then $|f(y) - f(x)| < \varepsilon$. But this means, for such y,

$$\begin{aligned} |f(y) - f(x)| < \varepsilon &\Rightarrow \frac{-f(x)}{2} < f(y) - f(x) < \frac{f(x)}{2} \\ &\Rightarrow \frac{f(x)}{2} < f(y) < \frac{3f(x)}{2}, \end{aligned}$$

which, since $f(x) > 0$, in particular implies that if $|y - x| < \delta$, then $f(y) > 0$. This means that all such $y \in X^c$ as well, i.e. $(x - \delta, x + \delta) \subset X^c$, which is exactly the definition of openness. Thus, X^c open and so X closed. ■

Solution 3. *a*) We claim the given limit diverges properly to $+\infty$. Let $M > 0$, and let $\delta := \frac{1}{M} > 0$. Then, if x is such that $0 < x - 1 < \delta$, then in particular $x > 1$ and $\frac{1}{x-1} > \frac{1}{\delta}$, from which we conclude

$$\frac{x}{x-1} > \frac{1}{\delta} = M,$$

which proves the claim.

b) We claim the given limit is 0. Fix $\varepsilon > 0$. Note that if the function in question was just $\frac{\sqrt{x}}{x}$, our job would be far easier, for this simplifies to $\frac{1}{\sqrt{x}}$, so just taking $M := \frac{1}{\varepsilon^2}$ would do the trick. So, inspired by this work, we can rewrite the function in question:

$$f(x) := \frac{\sqrt{x+1}}{x} = \frac{\sqrt{x}}{x} \cdot \frac{\sqrt{x+1}}{\sqrt{x}} = \frac{1}{\sqrt{x}}\sqrt{1 + \frac{1}{x}}.$$

In particular, we've put all the x dependence into denominators, which should always be a goal when proving something goes to zero. Let now $x > M$ with $M := \frac{2}{\varepsilon^2}$, assuming without loss of generality that $\varepsilon < 1$[1] then

$$f(x) < \frac{1}{\sqrt{M}}\sqrt{1+\frac{1}{M}} = \frac{\varepsilon}{\sqrt{2}}\sqrt{1+\varepsilon^2} < \frac{\varepsilon}{\sqrt{2}}\sqrt{2} = \varepsilon,$$

so the proof follows.

c) Whenever you see ugly functions like this, you should always simplify (in the sense of getting rid of as many x's as possible) before attempting anything. We see that:

$$f(x) := \frac{\sqrt{x}-x}{\sqrt{x}+x} = \frac{-\sqrt{x}+x+2\sqrt{x}}{\sqrt{x}+x} = -1+2\frac{\sqrt{x}}{\sqrt{x}+x} = -1+\frac{2}{1+\sqrt{x}}.$$

In particular, we can see from here that $\lim_{x\to\infty} f(x) = -1$; for $\varepsilon > 0$ (and less than 1, wlog), let $M := \left(\frac{2}{\varepsilon}-1\right)^2$, then for $x > M$,

$$|f(x)-(-1)| = \frac{2}{1+\sqrt{x}} < \frac{2}{1+\sqrt{M}} = \varepsilon,$$

as we needed to show. ■

Solution 4. Assume the first direction. Let $\varepsilon > 0$. By definition, there exists $M > 0$ such that $x > M$ implies $|f(x) - L| < \varepsilon$. Taking $\delta := \frac{1}{M}$, then, this implies that for all y such that $0 < \frac{1}{y} < \frac{1}{\delta}$, (viewing x as $\frac{1}{y}$) we have $\left|f\left(\frac{1}{y}\right) - L\right| < \varepsilon$, which $\lim_{x\to 0^+} f\left(\frac{1}{x}\right) = L$. The inverse implication is very similar. ■

Solution 5. Let $\varepsilon > 0$ and let $\delta := \frac{\varepsilon}{K}$. Then, $|x-y| < \delta$ implies $|f(x) - f(y)| \leq K|x-y| < \frac{K}{K}\varepsilon = \varepsilon$, so f continuous (uniformly). The classic example of a non-Lipschitz but continuous function is $f(x) := x^2$ on $\mathbb{R}$; to see this, it suffices to take $y = 0$. Then, we see that

$$|f(x) - f(y)| = x^2,$$

so any Lipschitz constant would have to be proportional to x, which contradicts the uniformity definition. ■

Solution 6. a) Fix $\varepsilon > 0$, $x \in \mathbb{R}$ and let y such that $|x-y| < \delta$, with δ to be chosen. Note that this implies $y \in (x-\delta, x+\delta)$, so $|y| \leq |x| + \delta$. Then:

[1]This is wlog since what really matters in limit proofs is small ε; if you don't like this, you can instead take

$$M = \max\left(\frac{2}{\varepsilon^2}, 1\right),$$

then in particular $\frac{1}{M} < 1$ and $\frac{1}{\sqrt{M}} < \frac{\varepsilon}{\sqrt{2}}$, then the proof follows identically.

$$
\begin{aligned}
|f(x)-f(y)| &= \left|\frac{1}{1+x^2}-\frac{1}{1+y^2}\right| \\
&= \frac{|y^2-x^2|}{(1+x^2)(1+y^2)} \\
&= \frac{|(y-x)(y+x)|}{(1+x^2)(1+y^2)} \\
&\leq |y-x||y+x| \\
&< \delta(|y|+|x|) < \delta(\delta+2|x|).
\end{aligned}
$$

If $x=0$, then we can just take $\delta=\sqrt{\varepsilon}$. Else, $|x|>0$, so that with the choice

$$\delta := \delta(x,\varepsilon) = \min\left\{2|x|, \frac{\varepsilon}{4|x|}\right\} > 0,$$

we have that $\delta < 2|x|$ and $\delta < \frac{\varepsilon}{4|x|}$, so that continuing our work above we get

$$|f(x)-f(y)| < \delta(\delta+2|x|) < \frac{\varepsilon}{4|x|}(4|x|) = \varepsilon,$$

as needed.

b) Let $\varepsilon>0, x\in\mathbb{R}$ and $y\in\mathbb{R}$ such that $|x-y|<\delta$ with $\delta>0$ to be chosen. By the reverse triangle inequality,

$$|f(x)-f(y)| = \left|\sqrt{x^2+1}-\sqrt{y^2+1}\right| \leq \sqrt{|x^2-y^2|} = \sqrt{|x-y|}\sqrt{|x+y|} \leq \delta^{\frac{1}{2}}\sqrt{2|x|+\delta}.$$

So, if $x=0$, $\delta=\varepsilon$ will work. Else, we can take

$$\delta := \min\left\{2|x|, \frac{\varepsilon^2}{4|x|}\right\} > 0,$$

then continuing form our work above,

$$|f(x)-f(y)| \leq \delta^{\frac{1}{2}}\sqrt{2|x|+\delta} \leq \delta^{\frac{1}{2}}2|x|^{\frac{1}{2}} < \varepsilon,$$

as we needed to show. ■

Solution 7. This is tricky. One way you can interpret the result is that, if f a continuous and bounded function, then it must be "eventually periodic" with arbitrary period T; i.e., we can find some properly diverging sequence of $\{x_n\}$ such that $f(x_n+T)-f(x_n)$ converges to zero.

With this interpretation, we fix $T\in\mathbb{R}\setminus\{0\}$ (if $T=0$ we're done) and let $g(x) := f(x+T) - f(x)$ for convenience. Consider the following three possibilities:

(i) f is identically zero outside of some bounded set, or f decays to zero as $|x|\to\infty$, i.e. $\lim_{x\to+\infty} f(x) = 0$; in either case, we can take $x_n := n$ and conclude (why?).

(ii) g changes sign infinitely often for sufficiently large x, i.e. for all $n\in\mathbb{N}$, there exists a

$y_n > n$ such that $g(y_n) > 0 (< 0)$ iff $g(n) < 0 (> 0)$. In this case, we can construct an unbounded, increasing sequence $\{y_n\} \subset \mathbb{R}$ such that $\{g(y_n)\}$ alternates sign, i.e. if $g(y_1) > 0$, then $g(y_2) < 0$, $g(y_3) > 0$, etc. Since f continuous, then by the intermediate value theorem, then we can find a sequence $\{x_n\}$ such that:
- $x_n \in (y_n, y_{n+1})$ for each $n \in \mathbb{N}$
- $g(x_n) = 0$ for all $n \in \mathbb{N}$

Moreover, this first condition implies $\lim_n x_n \geq \lim_n y_n = \infty$, so $\{x_n\}$ also diverges to infinity. The second condition moreover gives that $f(x_n + T) - f(x_n) = 0$ identically, so the claim is proven.

(iii) Finally, if neither of the two previous conditions are satisfied, then we know g must eventually be strictly positive or negative, i.e. there must exist some sufficiently large $M > 0$ such that $x \geq M \Rightarrow g(x) > 0$ or $g(x) < 0$ (indeed, if we couldn't find such an M, two cases would be possible: either g identically zero beyond some sufficiently large value of x, in which case we enter case (i), or g alternates sign infinitely often for large x, in which case we are in case (ii)).
Suppose first $g(x) > 0$ for $x \geq M$. This implies $f(x+T) > f(x)$; if $T > 0$, we can inductively argue then that $f(x + nT) > f(x)$ for all $n \in \mathbb{N}$. So, if we define the sequence $x_n := M + (n-1)T$, we conclude that $f(x_n) > f(x_{n-1})$ for all $n \in \mathbb{N}$. But then, $\{f(x_n)\}$ a bounded (by assumption, f bounded) sequence which monotonically increases, so by the Monotone Convergence Theorem, we know $\lim_n f(x_n)$ exists. In particular, this implies

$$\lim_n [f(x_n + T) - f(x_n)] = \lim_n f(x_{n+1}) - \lim_n f(x_n) = 0,$$

using our definition of x_n, which implies $x_{n+1} = x_n + T$; so, we are done in this case. If on the other hand $T < 0$, then we similarly define $x_n := M - (n-1)T$, and conclude similarly. Finally, if instead $g(x) < 0$ for $x \geq M$, the same sequence gives rise to a monotonically decreasing sequence, and the conclusion is the same. ■

Solution 8. Let $g(x) := f(x) - f\left(x + \frac{1}{2}\right)$ for $x \in \left[0, \frac{1}{2}\right]$. Then, $g(0) = -f\left(\frac{1}{2}\right)$ and $g\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right)$; in particular, g must change sign on the interval $\left[0, \frac{1}{2}\right]$, for $g(0) = -g\left(\frac{1}{2}\right)$. Thus, there must exist a c for which $g(c) = 0$, which, unravelling the definition of g, proves the assertion. ■

Solution 9. Let $x_1 \in [0,1]$ arbitrary. Let $x_2 \in [0,1]$ such that $|f(x_2)| \leq \frac{1}{2}|f(x_1)|$, which exists by hypothesis. Repeat this inductively, defining a sequence of $\{x_n\} \subset [0,1]$ such that $|f(x_{n+1})| \leq \frac{1}{2}|f(x_n)|$ for each $n \geq 1$. In particular, $\{x_n\}$ a bounded sequence of real numbers, and thus by the Bolzano-Weierstrauss Theorem, there exists a point $c \in [0,1]$ and a subsequence $\left\{x_{n_k}\right\} \subset \{x_n\}$ such that x_{n_k} converges to c. By continuity, $f\left(x_{n_k}\right)$ converges to $f(c)$. In addition, we see that, applying the hypothesis inductively,

$$|f(x_n)| \leq \frac{1}{2}|f(x_{n-1})| \leq \left(\frac{1}{2}\right)^2 |f(x_{n-2})| \leq \cdots \leq \left(\frac{1}{2}\right)^{n-1} |f(x_1)|.$$

Applying this bound to the subsequence, we conclude

$$\left|f\left(x_{n_k}\right)\right| \leq \left(\frac{1}{2}\right)^{n_k-1} |f(x_1)|,$$

and, taking limits on both sides, we find

$$|f(c)| \leq |f(x_1)| \lim_k \left(\frac{1}{2}\right)^{n_k-1} = 0,$$

hence $f(c) = 0$, as we aimed to find. ■

Solution 10. To disprove continuity, we need to create a sequence $\{x_n\}$ that converges to zero, but for which $f(x_n)$ does not converge to $f(0) = 0$. Graphically, one can see that $\sin\left(\frac{1}{x}\right)$ oscillates increasing wildly near the origin; one expects $\sin\left(\frac{1}{x}\right)$ to hit 1, for instance, infinitely often as $x \to 0$. Concretely, consider the sequence

$$x_n := \frac{1}{\frac{\pi}{2} + 2\pi n}.$$

One sees that $\lim_n x_n = 0$ (check), but since $x_n > 0$ for all n,

$$f(x_n) = \sin\left(\frac{\pi}{2} + 2\pi n\right) = 1$$

for all n. So, $\lim_n f(x_n) = 1 \neq 0 = f(0)$, so f cannot be continuous at zero. *(In fact, we for any $y \in [-1, 1]$, we can take a sequence of the form*

$$x_n := \frac{1}{c + 2\pi n}$$

with $c := \arcsin(y)$*; then* $\lim_n f(x_n) = y$.*)* ■