MATH255 - Algebra 2

A sample writing.

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Contents

1 Inner Product Spaces

1.1 *Projections and Cauchy-Schwartz.*

↳ **Definition 1.1** (Orthogonal): Let V be an inner product space. We say $u, v \in V$ are *orthogonal* and write $u \perp v$ if $\langle u, v \rangle = 0$.

⊛ **Example 1.1**: In $\mathbb{R}^3$ equipped with the dot product, $(1, 0, -1) \perp (1, 0, 1)$.

↳ **Theorem 1.1** (Pythagorean Theorem): For an inner product space V and $u, v \in V$, if $u \perp v$ then

$$\|u\|^2 + \|v\|^2 = \|u + v\|^2.$$

In particular, $\|u\|, \|v\| \leq \|u + v\|$.

Proof. $\|u + v\|^2 = \langle u + v, u + v \rangle = \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle = \|u\|^2 + \|v\|^2$ ■

↳ **Definition 1.2** (Projection): For $v \in V$ and $u \in V$ a unit vector, put $\mathrm{proj}_{u(v)} := \langle v, u \rangle \cdot u$.

↳ **Proposition 1.1**: Let $u \in V$ a unit vector. For each $v \in V$, $v - \mathrm{proj}_{u(v)} \perp u$.

Proof. Clear. ■

↳ **Corollary 1.1**: For each $v \in V$, $\left\|\mathrm{proj}_{u(v)}\right\| \leq \|v\|$.

↳ **Theorem 1.2**: Let V be an inner product space and $x, y \in V$.

a) (Cauchy-Banyakovski-Schwartz Inequality) $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$.
b) (Triangle Inequality) $\|x + y\| \leq \|x\| + \|y\|$.

Proof.

a) If $\|y\| = 0$, then $y = 0_V$ and $0 \leq 0$ and we are done. Hence, supposing $\|y\| \neq 0$, divide both sides by $\|y\|$:

$$\left\langle x, \|y\|^{-1} \cdot y \right\rangle \leq \|x\|,$$

i.e., we need only to prove that $|\langle x, y \rangle| \leq \|x\|$, where u a unit. But notice

$$|\langle x, u \rangle| = \|\langle x, u \rangle \cdot u\| = \left\|\mathrm{proj}_{u(x)}\right\| \leq \|x\|,$$

by ↳ Corollary 1.1.

a) Squaring the LHS, we have

$$
\begin{aligned}
\|x+y\|^2 = \langle x+y, x+y\rangle &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\
&\leq \|x\|^2 + \|y\|^2 + 2|\langle x,y\rangle| \\
&\underbrace{\leq}_{\text{by a)}} \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| = (\|x\| + \|y\|)^2.
\end{aligned}
$$

■