**MATH255** - Honours Analysis 2

Summary of Results

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Complete notes



## 1 Point-Set Topology

**Topology is about abstracting openness. It can typically suffice to consider open, closed sets in $\mathbb{R}$ for intuition, but is obviously not all-general.**

**Definition 1** (Metric Space)**.** A space $X$ equipped with a function $d : X \times X \to [0, \infty)$ is called a metric space and $d$ a metric or distance if

- $d(x, y) = d(y, x) \geqslant 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) + d(y, z) \geqslant d(x, z)$

for any $x, y, z \in X$.

**Definition 2** (Normed Vector Space)**.** A function $||\cdot|| : X \to \mathbb{R}$ defined on a vector space $X$ over $\mathbb{R}$ is a norm if

- $||x|| \geqslant 0$
- $||x|| = 0 \iff x = 0$
- $||c \cdot x|| = |c|\,||x||$
- $||x + y|| \leqslant ||x|| + ||y||$,

for any $x, y \in X, c \in \mathbb{R}$.

*Remark* 1. We can naturally extend this to arbitary fields, but seeing as this is a course in Real Analysis, we won't.

**Proposition 1.** *For a normed vector space* $(X, ||\cdot||)$*,* $d(x, y) := ||x - y||$ *is a metric on* $X$*. We call such a metric the one "induced" by the norm.*

**Definition 3** (Topological Set)**.** A set $X$ is a topological set if we have a collection $\tau$ of subsets of $X$, called open sets, such that

- $\varnothing \in \tau, X \in \tau$
- For $A_\alpha \in \tau$ for $\alpha$ in any $I$ (potentially infinite), $\bigcup_{\alpha \in I} A_\alpha \in \tau$
- For $A_\alpha \in \tau$ for $\alpha \in J$ where $J$ *finite*, then $\bigcap_{\alpha \in J} A_\alpha \in \tau$

ie, arbitrary unions of open sets are open, and finite intersections of open sets are open.

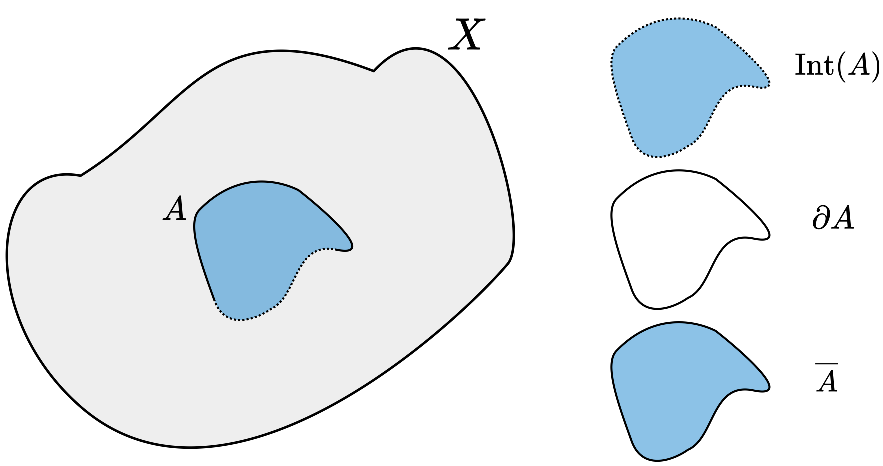
*Remark* 2. Keep $\mathbb{R}$ in mind when initially working with these definitions; for instance, the set $A_n := (0, \frac{1}{n})$ open in $\mathbb{R}$ for any $n \in \mathbb{N}$, but $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$ which is closed.

*Remark* 3. Complemented each of these requirements gives similar definitions for closed sets of $X$.

**Definition 4** (Topology on a Metric Space)**.** A subset $A \subseteq X$ open iff $\forall\, x \in A, \exists r = r(x) \in \mathbb{R}$, where $r(x) > 0$, such that $B(x, r(x)) := \{y \in x : d(x, y) < r(x)\} \subseteq A$. We call such a $B$ an open ball, and $\overline{B}$ a closed ball with the same definition replacing the strict inequality with $\leqslant$.

*Remark* 4. While many of the spaces we look at our metric spaces that induce a topology as such, **not all topological spaces are metric spaces**. Indeed, "metrizability" (ie, equipping a topological space $X$ with a metric that respects the open sets) is not a trivial activity.

**Definition 5** (Equivalence of Metrics)**.** We say two metrics on $X$ are equivalent if they admit the same topology; a sufficient condition is that, $\forall\, x \neq y \in X$, $\exists 1 < C < \infty$ such that $\frac{1}{C} < \frac{d_1(x,y)}{d_2(x,y)} < C$, then $d_1, d_2$ equivalent, where $C$ independent of $x, y$.




**Definition 6** (★ Interior, Boundary, Closure)**.** Let $X$-topological space, $A \subseteq X$, $x \in X$.

- If $\exists U$-open s.t. $x \in U \subseteq A$, then we write $x \in \mathrm{Int}(A)$, the interior of $A$.
- If $\exists V$-open s.t. $x \in V \subseteq A^C$, then $x \in \mathrm{Int}(A^C)$.
- If $\forall\, U$-open s.t. $x \in U, U \cap A \neq \varnothing$ and $U \cap A^C \neq \varnothing$, then $x \in \partial A$, the boundary of $A$.

We put $\overline{A} := \mathrm{Int}(A) \cup \partial A$, the closure of $A$. Equivalently, $x \in \overline{A} \iff$ for every open set $U : x \in U, U \cap A \neq \varnothing$. We call $x \in \overline{A}$ the limit points of $A$.

*Remark* 5. The limit point interpretation of the closure can be more intuitive; the points that we can get "arbitrary close to" are the closure. For instance, $\overline{(a, b)} = [a, b] \subseteq \mathbb{R}$ with the standard topology.

**Proposition 2.** *Let $A \subseteq X$-topological space. Then,* $\text{Int}(A)$ *is open, the largest open set contained in $A$, the union of all open sets contained in $A$, and* $\text{Int}(\text{Int}(A)) = \text{Int}(A)$. *Also, $\overline{A}$ closed, the smallest closed set that contains $A$, $\overline{A}$ the intersection of all closed sets that $A$ is contained in, and $\overline{\overline{A}} = \overline{A}$.*

**Corollary 1.** *$A$ open $\iff$ $A = \text{Int}(A)$ and $A$ closed $\iff$ $A = \overline{A}$*

*Remark* 6. Remark that these are not exclusive, nor indeed the only possibilities.

**Definition 7** (Basis)**.** A basis for a topology $X$ with open sets $\tau$ is a collection $B \subseteq \tau$ such that every $U \in \tau$ a union of sets in $B$.

*Remark* 7. Don't think about bases for vector spaces in this regard - there is no "minimality" requirement.

Keep in mind $\{(a, b) : -\infty < a < b < \infty\}$, a basis of topology on $\mathbb{R}$.

**Proposition 3.** *For a metric space $(X, d)$, $\{B(x, r) : x \in X, r > 0\}$ a basis of topology.*

**Definition 8** (Subspace Topology)**.** For a subset $Y \subseteq X$-topological space, we define the subspace topology on $Y$ as $\tau_Y := \{Y \cap U : U \in \tau\}$.

**Definition 9** (★ Continuous)**.** For $X, Y$-topological spaces, a function $f : X \to Y$ is continuous iff $\forall V$-open in $Y$, $f^{-1}(V)$-open in $X$.

*Remark* 8. One can verify that this is consistent with the $\varepsilon - \delta$ definition of continuity for functions on $\mathbb{R}$.

**Theorem 1** (Continuity of Composition)**.** *If $f : X \to Y$, $g : Y \to Z$ continuous, $g \circ f$ continuous.*

*Remark* 9. Note how much easier this is to prove via toplogical spaces than the $\varepsilon - \delta$ definition.

**Definition 10** (Product Space)**.** For an index set $I$ and $X_\alpha, \alpha \in I$, we define $\prod_{\alpha \in I} X_\alpha$ as a product space; $I$ may be finite or infinite.

**Proposition 4.** *A basis for the product space is given by cyliders of the form $A = \prod_{\alpha \in J} A_\alpha \times \prod \alpha \notin J X_\alpha$ for $A_\alpha$-open in $X_\alpha$, where $J \subseteq I$-finite.*

**Definition 11** (Compact)**.** A set $A \subseteq X$ is compact if every cover has a finite subcover, that is

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha\text{-open} \implies \exists \{\alpha_1, \dots, \alpha_n\} \subseteq I \text{ s.t. } A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

**Proposition 5.** *Closed intervals $[a, b]$ compact in $\mathbb{R}$.*

**Proposition 6.** *$A \subseteq \mathbb{R}^n$ compact $\iff$ closed and bounded.*

**Definition 12** (Connected)**.** $X$ is said to not be connected if $X = U \cup V$ for $U, V$ open, nonempty, disjoint. If $X$ cannot be written as such, $X$ is said to be connected.

**Theorem 2.** *If $X$ connected and $f : X \to Y$, then $f(X)$ connected in $Y$.*

**Proposition 7.** *Intervals in $\mathbb{R}$ are connected.*

**Theorem 3** (Intermediate Value Theorem)**.** *If $X$ connected, $f : X \to \mathbb{R}$ continuous, then $f$ takes intermediate value; if $a = f(x), b = f(y)$ for $x, y \in X$ with $a < b$, then for any $a < c < b$ $\exists z \in X$ s.t. $f(z) = c$.*

**Theorem 4.** *For $X$ compact, $f : X \to Y$ continuous, $f(X)$ compact in $Y$.*

**Proposition 8.** *For $X$ compact and $f : X \to \mathbb{R}$, $f$ attains both max and min on $X$.*

**Definition 13** (Path Connected)**.** A set $A \subseteq X$ is path connected if for any $x, y \in A, \exists f : [a, b] \to X$ continuous such that $f(a) = x, f(b) = y f([a, b]) \subseteq A$.

**Theorem 5.** *Path connected $\implies$ connected.*

*For open sets in $\mathbb{R}^n$, the converse holds too.*

**Definition 14** (Connected Component, Path Component)**.** For $x \in X$, the connected component of $x$ is the largest connected subset of $X$ containing $x$ and the path component of $x$ is the largest path connected subset of $X$ containing $x$.

## 2 Metric Spaces

**We discuss mostly the metric on $\ell_p$ space and notions of completeness, as well as some topological results specific to metric spaces, namely compactness.**

**Definition 15** ($\ell_p$)**.** For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $1 \leqslant p \leqslant +\infty$, we define

$$||x||_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad ||x||_\infty := \max_{i=1}^{n} |x_i|,$$

and similarly, for sequences $x = (x_1, \dots, x_n, \dots)$,

$$||x||_p := \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}, \quad ||x||_\infty := \sup_{i=1}^{\infty} |x_i|,$$

and define $\ell_p := \{x : ||x||_p < +\infty\}$. It can be shown that these are well-defined norms on their respective spaces.

**Theorem 6** (Holder, Minkowski's Inequalities)**.** *For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ and $p, q$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\textit{Holder's:} \qquad \langle x, y\rangle = \left|\sum_{i=1}^{n} x_i y_i\right| \leqslant ||x||_p ||y||_q$$

*and*

$$\textit{Minkowski's:} \qquad ||x + y||_p \leqslant ||x||_p + ||y||_p.$$

*The identical inequalities hold for infinite sequences.*

**Definition 16** (Completeness)**.** We say a metric space is complete if every Cauchy sequence converges to a limit point in the space.

**Proposition 9.** *For $\{x_n\}_{n\in\mathbb{N}}$, $\ell_p$ complete for any $1 \leqslant p \leqslant +\infty$.*

**Proposition 10.** *If $p < q$, $\ell_p \subseteq \ell_q$.*

**Definition 17** (Contraction Mapping)**.** For a metric space $(X, d)$, a function $f : X \to X$ is a contraction mapping if there exists $0 < c < 1$ such that

$$d(f(x), f(y)) \leqslant c \cdot d(x, y)$$

for any $x, y \in X$.

**Theorem 7.** *Let $(X, d)$ be a complete metric space, $f : X \to X$ a contraction. Then, there exist a unique fixed point $z$ of $f$ such that $f(z) = z$; ie $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f \circ f \circ \cdots \circ f(x) = z$ for any $x \in X$.*

**Theorem 8.** *$\ell_p$ complete.*

*Remark* 10. It can be kind of funky to work with sequences in $\ell_p$, since the elements of $\ell_p$ themselves sequences so we have "sequences of sequences".

**Definition 18** (Totally bounded)**.** A metric space $X$ is said to be totally bounded if $\forall\, \varepsilon > 0 \exists x_1, \ldots, x_n \in X, n = n(\varepsilon)$ such that $\bigcup_{i=1}^{n} B(x_i, \varepsilon) = X$.

**Definition 19** (Sequentially compact)**.** A metric space $X$ is said to be sequentially compact if every sequence has a convergent subsequence.

**Theorem 9** (⋆ Equivalent Notions of Compactness in Metric Spaces)**.** *Let $(X, d)$ a metric space. TFAE:*

- *$X$ compact*
- *$X$ complete and totally bounded*
- *$X$ sequentially compact*

*Remark* 11. This is for a metric space, not a general topological space! Hopefully this is clear because some of the requirements necessitate a distance.

## 3 Differentiation

**Definition 20** (Differentiable)**.** $f(x)$ differentiable at $c$ if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, and if so we denote the limit $f'(c)$.

Alternatively, one can view differentiation as a linear map between spaces of differentiable functions.

**Theorem 10.** *Differentiable $\implies$ continuous.*

*Proof.* Short enough to write the full proof; $\lim_{x\to c}(f(x)-f(c)) = \lim_{x\to c}(x-c)\frac{f(x)-f(c)}{x-c} = 0 \cdot f'(c) = 0$. ◻

**Theorem 11** (Caratheodory's)**.** *For $f : I \to \mathbb{R}, c \in I$, $f$ differentiable at $c$ iff $\exists \varphi : I \to \mathbb{R} : \varphi$ continuous at $c$, $f(x) - f(c) = \varphi(x)(x-c)$.*

*Sketch.* Its worth recalling the definition of $\varphi$ for the forward implication,

$$\varphi(x) := \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}.$$

The converse follows by taking limits. ◻

*Remark* 12. While not a particularly enlightening result, used in proofs of the chain rule, etc.

**Theorem 12** (Chain Rule)**.** *Let $f : J \to \mathbb{R}, g : I \to R$ s.t. $f(J) \subseteq I$. If $f(x)$ differentiable at $c$ and $g(y)$ at $f(c)$, $g \circ f$ differentiable at $c$ with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.*

*Sketch.* Apply Caratheodory's to $f$ at $c$ and $g$ at $f(c)$, and compose. ◻

**Theorem 13** (Rolle's)**.** *Let $f : [a,b] \to \mathbb{R}$ continuous. If $f'(x)$ exists on $(a,b)$ and $f(a) = f(b) = 0$, $\exists c \in (a,b) : f'(c) = 0$.*

*Sketch.* If constant function, done. Else, assuming function positive, it obtains a maximum, and thus its derivative 0 at this point. ◻

**Theorem 14** ($\star$ Mean Value)**.** *Let $f$ continuous on $[a,b]$ and differentiable on $(a,b)$. Then, $\exists c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.*

*Sketch.* Let $\phi(x) := f(x) - f(a) - \frac{f(b)-f(a)}{(b-a)}(x-a)$. Then $\phi(a) = \phi(b) = 0$ so applying Rolle's $\exists c \in (a,b) : \varphi'(c) = 0 = f'(x) - \frac{f(b)-f(a)}{b-a}$. The proof is done after rearranging. □

**Proposition 11** (L'Hopital's)**.** *If $f, g : [a,b] \to \mathbb{R}$ with $f(a) = g(a) = 0$, $g(x) \neq 0$ on $a < x < b$, $f, g$ differentiable at $x = 0$ with $g'(a) \neq 0$, then $\lim_{x \to a^+} \frac{f(x)}{g(x)}$ exists and is equal to $\frac{f'(a)}{g'(a)}$.*

*Remark* 13. Other versions exist, but this is certainly one of them.

**Theorem 15** ($\star$ Taylor's)**.** *Let $f \in C^n([a,b])$ such that $f^{(n+1)}(x)$ exists on $(a,b)$. Let $x_0 \in [a,b]$, then, for any $x \in [a,b]$, $\exists c$ between $x, x_0$ such that*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

**Corollary 2.** *Let $x_0 \in [a,b]$. With the same assumptions as Taylor's (but in a neighborhood of $x_0$), with $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$, then*

- *$n$ even; then $f$ has a local minimum at $x_0$ if $f^{(n)}(x_0) > 0$ and a local max if $f^{(n)}(x_0) < 0$.*
- *$n$ odd; neither.*

## 4 Integration

**Its all just rectangles.**

**Definition 21** (Riemann Integration)**.** Consider an interval $(a,b)$. We call a subdivision $\mathcal{P} := \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ a partition, and $\dot{\mathcal{P}}$ a marked partition if in addition we are given a point $t_i \in (x_i, x_{i+1}]$ for each interval in $\dot{\mathcal{P}}$.

We put $\operatorname{diam}(\mathcal{P}) := \max_{i=1}^{n} |x_i - x_{i-1}|$.

We define the Riemann sum $S(f, \dot{\mathcal{P}}) := \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$, and say that $f$ Riemann integrable on $[a,b]$ if $S(f, \dot{\mathcal{P}}) \to L$ as $\operatorname{diam}(\dot{\mathcal{P}}) \to 0$ for any choice of tag $t_i$, and write $f \in \mathcal{R}([a,b])$

More precisely, if $\forall \varepsilon > 0$, $\exists \delta > 0$ : $\operatorname{diam}(\mathcal{P}) < \delta$, then for any $t_i \in [x_i, x_{i+1}]$, $\left|L - S(f, \dot{\mathcal{P}})\right| < \varepsilon$. We then say the (Riemann) integral of $f$ over $[a,b]$ is $L$ and write $\int_a^b f(x)\,\mathrm{d}x = L$.

**Proposition 12.** *Riemann integrals are unique, linear in $f(x)$, and respect inequalities (if $f \leqslant g$ on $[a,b]$, $\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x$ if both in $\mathcal{R}([a,b])$)*

**Proposition 13** (★)**.** *$f \in \mathcal{R}[a,b] \implies f$ bounded on $[a,b]$*

**Proposition 14** (★ Cauchy Criterion for Integrability)**.** *$f \in \mathcal{R}[a,b] \iff \forall \varepsilon > 0, \exists \delta > 0$ : if $\dot{P}$ and $\dot{Q}$ are tagged partitions of $[a,b]$ s.t. $\operatorname{diam} \dot{P} < \delta$ and $\operatorname{diam} \dot{Q} < \delta$, then $\left|S(f,\dot{P}) - S(f,\dot{Q})\right| < \varepsilon$*

*Remark* 14. Ala Cauchy Sequence.

**Theorem 16** (Squeeze Theorem)**.** *$f \in \mathcal{R}[a,b] \iff \forall \varepsilon > 0, \exists \alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}[a,b] : \alpha_\varepsilon \leqslant f \leqslant \omega_\varepsilon$ and $\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$.*

**Lemma 1.** *Let $J := [c,d] \subseteq [a,b]$ and $\varphi_J(x) := \begin{cases} 1 & x \in J \\ 0 & x \notin J \end{cases}$ be the indicator function of $J$. Then $\varphi_J \in \mathcal{R}[a,b]$ and $\int_a^b \varphi_J = d - c$.*

*Remark* 15. Helpful for "approximations"; follows by linearity, induction that step functions (ie sums of indicator functions times constants) are integrable.

**Theorem 17** (★ Continuous)**.** *$f$ continuous on $[a,b] \implies f \in \mathcal{R}[a,b]$*

*Sketch.* Continuity on a closed interval gives uniform continuity and so a "universal $\delta$"; then, for any partition, take the $x$ such that $f$ attains its minimum and maximum, and define a $\alpha_\varepsilon$, $\omega_\varepsilon$ as the sum of indicator functions taking the minimum, maximum of $f$ respectively on each partition. Then apply the previous theorem and the squeeze theorem. □

**Theorem 18** (Additivity)**.** *$f \in \mathcal{R}[a,b] \iff f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$, and $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$.*

**Theorem 19** (★ Fundamental Theorem of Calculus). *Let $F, f : [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$ a finite set, such that $F$ continuous on $[a,b]$, $F'(x) = f(x) \,\forall\, x \in [a,b] \setminus E$, $f \in \mathcal{R}[a,b]$. Then $\int_a^b f(x) = F(b) - F(a)$. We call F the "primitive" of $f$.*

**Theorem 20.** *For $f \in \mathcal{R}[a,b]$ and any $z \in [a,b]$, put $F(z) := \int_a^z f(x)\,\mathrm{d}X$. Then, $F$ continuous on $[a,b]$.*

**Theorem 21** (★ Fundamental Theorem of Calculus p2). *For $f \in \mathcal{R}[a,b]$ continuous at $c$, then $F(z)$ differentiable at $c$ and $F'(c) = f(c)$.*

**Definition 22** (Lebesgue Measure). We say a set $A \subseteq \mathbb{R}$ has Lebesgue measure 0 iff $\forall\, \varepsilon > 0$, $A$ can be covered by a union of intervals $J_k$ such that $\sum_k |J_k| \leqslant \varepsilon$. We then call $A$ a "null set".

In particular, any countable set is a null set.

**Theorem 22** (★ Lebesgue Integrability Criterion). *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[a,b] \iff$ the set of discontinuities of $f$ has Lebesgue measure 0.*

*Remark* 16. In particular, remark that continuity a stronger requirement than integrability.

**Theorem 23** (Composition). *If $f \in \mathcal{R}[a,b]$, $\varphi : [c,d] \to \mathbb{R}$ continuous and $f([a,b]) \subseteq [c,d]$, then $\varphi \circ f \in \mathcal{R}[a,b]$.*

**Theorem 24** (Integration by Parts). *If $F, G$ differentiable $[a,b]$ with $f := F', g := G'$, and $f, g \in \mathcal{R}[a,b]$, then*

$$\int_a^b f(x)G(x)\,\mathrm{d}x = F(x)G(x)\Big|_a^b - \int_a^b F(x)g(x)\,\mathrm{d}x\,.$$

*Sketch.* Uses additivity and the fundamental theorem of calculus. □

**Theorem 25** (Taylor's Theorem, Remainder's Version). *Suppose $f', f'', \ldots, f^{(n)}$ exist on $[a,b]$ and $f^{(n+1)} \in \mathcal{R}[a,b]$. Then*

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n,$$

*where* $R_n := \frac{1}{n!}\int_a^b f^{(n+1)}(t)(b-t)^n\,\mathrm{d}t$.

## 5 Sequences of Functions

**A good motivation to keep in mind with the "types" of function-sequence convergence is to answer the question: when can we exchange limits of derivatives of functions and derivatives of limits of functions? What about integrals? What about summations (see next section)? Ie, when does $\lim_{n\to\infty} f_n'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\lim_{n\to\infty} f_n(x)$, etc.**

**Definition 23** (Pointwise, Uniform Convergence)**.** We say $f_n \to f$ pointwise on $E$ if $\forall\, x \in E$, $f_n(x) \to f(x)$ as $n \to \infty$.

We say $f_n \to f$ uniformly on $E$ if $\forall\, \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall\, n \geqslant N, x \in E$, $|f_n(x) - f(x)| < \varepsilon$.

*Remark* 17. Pointwise doesn't care about the "rate of convergence"; as long as each point converges eventually, we're good. Uniform convergence needs all points to converge "at the same rate" (so to speak).

A good example to keep in mind is $f_n := \begin{cases} 2nx & 0 \leqslant x \leqslant \frac{1}{2n} \\ 0 & x > \frac{1}{2n} \end{cases}$ on $[0,1]$, which converges pointwise to 0 but not uniformly.

**Proposition 15.** *Uniform* $\implies$ *pointwise convergence.*

**Theorem 26.** *The metric space of continuous functions* $C([a,b])$ *complete with respect to* $d_\infty(f,g) := \sup_{x\in[a,b]} |f(x) - g(x)|$.

**Theorem 27** (★ Interchange of Limits)**.** *Let* $J \subseteq \mathbb{R}$ *be a bounded interval such that* $\exists x_0 \in J$ : $f_n(x_0) \to f(x_0)$. *Suppose* $f_n'(x) \to g(x)$ *uniformly on* $J$. *Then,* $\exists f : f_n(x) \to f(x)$ *uniformly on* $J$, $f(x)$ *differentiable on* $J$, *and moreover* $f_n'(x) = g(x)\,\forall\, x \in J$.

**Theorem 28** (★ Interchange of Integrals)**.** *Let* $f_n \in \mathcal{R}[a,b]$, $f_n \to f$ *uniformly on* $[a,b]$. *Then* $f \in \mathcal{R}[a,b]$ *and* $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$

**Theorem 29** (Bounded Convergence)**.** *Let $f_n \in \mathcal{R}[a,b]$, $f_n \to f \in \mathcal{R}[a,b]$ (not necessarily uniform). Suppose $\exists B > 0$ s.t. $|f_n(x)| \leqslant B \, \forall \, x \in [a,b]$ and $\forall \, n \in \mathbb{N}$, then $\int_a^b f_n \to \int_a^b f$ as $n \to \infty$.*

*Remark* 18. This provides a weaker condition, but equivalent result as the previous theorem, although remark now that we need the limit function itself to be in $\mathcal{R}[a,b]$, which was a result, not a necessity, of the previous theorem. In general, uniform continuity very strong, but leads to helpful results.

**Theorem 30** (Dimi's)**.** *If $f_n \in C([a,b])$, $f_n(x)$ monotone (as a sequence), and $f_n \to f \in C([a,b])$, then $f_n \to f$ uniformly.*

## 6 Infinite Series

**Definition 24** (Covergence of Series)**.** Let $\{x_j\} \in X$-normed vector space over $\mathbb{R}$. We say $\sum_{j=1}^{\infty} x_j$ converges absolutely iff $\sum_{j=1}^{\infty} ||x_j|| < +\infty$. In particular, if $X = \mathbb{R}$, then $|| \cdot || = |\cdot|$.

We say $\sum_{j=1}^{\infty} x_j$ converges conditionally if $\sum_{j=1}^{\infty} x_j < +\infty$, but $\sum_{j=1}^{\infty} ||x_j|| = +\infty$.

**Proposition 16.** *Any rearrangement of an absolutely convergent series gives the same sum. Conversely, the order of summation of a conditionally convergent summation can be rearranged such as to equal any real number.*

**Proposition 17** (Absolute Convergence Tests)**.**

- ***Comparison Test:*** *let $x_n, y_n$ be nonzero real sequences and $r := \lim \left|\frac{x_n}{y_n}\right|$. If such a limit exists, then if*
  - *(a) $r \neq 0$, $\sum_n x_n$ absolutely convergent $\iff$ $\sum_n y_n$ absolutely convergent.*
  - *(b) $r = 0$, $\sum_n y_n$ absoltuely convergent $\implies$ $\sum_n x_n$ absolteuly convergent.*
- ***Root Test:*** *if $\exists r < 1$ s.t. $|x_n|^{1/n} \leqslant r \, \forall \, n \geqslant K$-sufficiently large, then $\sum_{n=K}^{\infty} |x_n|$ converges. Conversely, if$|x_n|^{1/n} \geqslant 1$ for $n \geqslant K$-sufficiently large, $\sum_n x_n$ diverges.*
- ***Ratio Test:*** *if $x_n \neq 0$ and $\exists 0 < r < 1$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \leqslant r$ for $n \geqslant K$ sufficiently large, $\sum_n x_n$ absolutely convergent. Conversely, if $\left|\frac{x_{n+1}}{x_n}\right| \geqslant 1$ for $n \geqslant K$ sufficiently large, then $\sum_n x_n$ diverges.*

- ***Integral Test:*** *if $f(x) \geqslant 0$ non-increasing/non-decreasing function of $x \geqslant 1$, $\sum_{k=1}^{\infty} f(k)$ converges iff $\lim_{k\to\infty} \int_1^k f(x)\,\mathrm{d}x$ finite.*

* ***Raube's Test:*** *let $x_n \neq 0$.*

  *(a) If $\exists a > 1$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \leqslant 1 - \frac{1}{n}\,\forall\, n \geqslant K$-sufficiently large, then $\sum_n x_n$ converges absolutely.*

  *(b) If $\exists a \leqslant 1$ s.t. $\left|\frac{x_{n+1}}{x_n}\right| \geqslant 1 - \frac{1}{n}\,\forall\, n \geqslant K$-sufficiently large, $\sum_n x_n$ does not converge absolutely.*

*Remark* 19. Proofs of these tests aren't really important (Dima-speaking), but knowing the conditions in which they apply is.

**Proposition 18** (Tests for Non-Absolute Convergence)**.**
- ***Alternating Series:*** *if $x > 0, x_{n+1} \leqslant x_n$ such that $\lim_{n\to\infty} x_n = 0$, then $\sum_n (-1)^n x_n$ converges.*

- ***Dirichlet's Test:*** *if $x_n$ decreasing with limit $0$, and the partial sum $s_n := y_1 + \cdots + y_n$ is bounded, then $\sum_n x_n y_n$ converges.*

- ***Abel's Test:*** *let $x_n$ convergent and monotone, and suppose $\sum_n y_n$ converges. Then $\sum_n x_n y_n$ also converges.*

**Definition 25** (Convergence of Series of Functions)**.** We say a series $\sum_n f_n(x)$ converges absolutely to some $g(x)$ on $E$ if $\sum_n |f_n(x)|$ converges for all $x \in E$.

We say that the convergence is uniform if it is uniform for any $x \in E$, ie $\forall\, \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $\forall\, n \geqslant N, x \in E, |g(x) - \sum_n f_n(x)| < \varepsilon$.

**Proposition 19** (Interchanging Integrals and Summations)**.** *Suppose for $f_n : [a,b] \to \mathbb{R}$, $\sum_n f_n(x) \to g(x)$ uniformly and $f_n \in \mathcal{R}[a,b]$. Then $\int_a^b g(x) = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,\mathrm{d}x$.*

**Proposition 20** (Interchanging Derivatives and Summations)**.** *Let $f_n : [a,b] \to \mathbb{R}, f_n' \exists$, $\sum_n f(x)$ converges for some $[a,b]$ and $\sum_n f_n'(x)$ converges uniformly. Then, there exists some $g : [a,b] \to \mathbb{R}$ such that $\sum_n f_n \to g$ uniformly, $g$ differentiable, and $g'(x) = \sum_n f_n'(x)$, all on $[a,b]$.*

**Theorem 31** (★ Cauchy Criterion of Series). *$f_n(x) : D \to \mathbb{R}$ converges uniformly on $D$ iff $\forall\, \varepsilon > 0, \exists N$ s.t. $\forall\, m, n \geqslant N, \sum_{i=n+1}^{m} f_i(x) < \varepsilon \,\forall\, x \in D$.*

**Proposition 21** (Weierstrass M-Test). *If $|f_n(x)| \leqslant M_n \,\forall\, x \in D \subseteq \mathbb{R}$ and $\sum_n M_n < +\infty$, then $\sum_n f_n(x)$ converges uniformly on $D$.*

**Definition 26** (Power Series). A function of the form $f(x) := \sum_{n=0}^{\infty} a_n (x-c)^n$ is said to be a power series centered at $c$.

Put $\rho := \limsup_{n\to\infty} \sqrt[n]{|a_n|}$, and put

$$R := \begin{cases} \frac{1}{\rho} & 0 < \rho < +\infty \\ 0 & \rho = +\infty \\ \infty & \rho = 0 \end{cases}.$$

We call $R$ the radius of convergence of $f$.

**Theorem 32** (★ Cauchy-Hadamard). *Let $R$ be the radius of converges of $f$. Then, $f(x)$ converges if $|x-c| < R$, and diverges if $|x-c| > R$.*

*Sketch.* Apply the root test to the definition of $R$. □

*Remark* 20. If $|x-c| = R$, the theorem is inconclusive, and we need to manually check.