MATH255 - Honours Analysis 2

Summary of Results

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Complete notes

1 Point-Set Topology

Topology is about abstracting openness. It can typically suffice to consider open, closed sets in $\mathbb{R}$ for intuition, but is obviously not all-general.

Definition 1 (Metric Space)**.** A space X equipped with a function $d : X \times X \to [0, \infty)$ is called a metric space and d a metric or distance if

- $d(x, y) = d(y, x) \geqslant 0$
- $d(x, y) = 0 \iff x = y$
- $d(x, y) + d(y, z) \geqslant d(x, z)$

for any $x, y, z \in X$.

Definition 2 (Normed Vector Space)**.** A function $||\cdot|| : X \to \mathbb{R}$ defined on a vector space X over $\mathbb{R}$ is a norm if

- $||x|| \geqslant 0$
- $||x|| = 0 \iff x = 0$
- $||c \cdot x|| = |c|\,||x||$
- $||x + y|| \leqslant ||x|| + ||y||,$

for any $x, y \in X, c \in \mathbb{R}$.

Remark 1. We can naturally extend this to arbitary fields, but seeing as this is a course in Real Analysis, we won't.

Proposition 1. *For a normed vector space* $(X, ||\cdot||)$*,* $d(x, y) := ||x - y||$ *is a metric on* X*. We call such a metric the one "induced" by the norm.*

Definition 3 (Topological Set)**.** A set X is a topological set if we have a collection τ of subsets of X, called open sets, such that

- $\varnothing \in \tau, X \in \tau$
- For $A_\alpha \in \tau$ for α in any I (potentially infinite), $\bigcup_{\alpha \in I} A_\alpha \in \tau$
- For $A_\alpha \in \tau$ for $\alpha \in J$ where J *finite*, then $\bigcap_{\alpha \in J} A_\alpha \in \tau$

ie, arbitrary unions of open sets are open, and finite intersections of open sets are open.

Remark 2. Keep $\mathbb{R}$ in mind when initially working with these definitions; for instance, the set $A_n := (0, \frac{1}{n})$ open in $\mathbb{R}$ for any $n \in \mathbb{N}$, but $\bigcap_{n \in \mathbb{N}} A_n = \{0\}$ which is closed.

Remark 3. Complemented each of these requirements gives similar definitions for closed sets of X.

Definition 4 (Topology on a Metric Space)**.** A subset $A \subseteq X$ open iff $\forall x \in A, \exists r = r(x) \in \mathbb{R}$, where $r(x) > 0$, such that $B(x, r(x)) := \{y \in x : d(x, y) < r(x)\} \subseteq A$. We call such a B an open ball, and $\overline{B}$ a closed ball with the same definition replacing the strict inequality with $\leqslant$.

Remark 4. While many of the spaces we look at our metric spaces that induce a topology as such, **not all topological spaces are metric spaces**. Indeed, "metrizability" (ie, equipping a topological space X with a metric that respects the open sets) is not a trivial activity.

Definition 5 (Equivalence of Metrics)**.** We say two metrics on X are equivalent if they admit the same topology; a sufficient condition is that, $\forall x \neq y \in X$, $\exists 1 < C < \infty$ such that $\frac{1}{C} < \frac{d_1(x,y)}{d_2(x,y)} < C$, then d_1, d_2 equivalent, where C independent of x, y.

Definition 6 (Interior, Boundary, Closure)**.** Let X-topological space, $A \subseteq X$, $x \in X$.

- If $\exists U$-open s.t. $x \in U \subseteq A$, then we write $x \in \text{Int}(A)$, the interior of A.
- If $\exists V$-open s.t. $x \in V \subseteq A^C$, then $x \in \text{Int}(A^C)$.
- If $\forall U$-open s.t. $x \in U, U \cap A \neq \varnothing$ and $U \cap A^C \neq \varnothing$, then $x \in \partial A$, the boundary of A.

We put $\overline{A} := \text{Int}(A) \cup \partial A$, the closure of A. Equivalently, $x \in \overline{A} \iff$ for every open set $U : x \in U, U \cap A \neq \varnothing$. We call $x \in \overline{A}$ the limit points of A.

Remark 5. The limit point interpretation of the closure can be more intuitive; the points that we can get "arbitrary close to" are the closure. For instance, $\overline{(a, b)} = [a, b] \subseteq \mathbb{R}$ with the standard topology.

Proposition 2. *Let $A \subseteq X$-topological space. Then,* $\text{Int}(A)$ *is open, the largest open set contained in A, the union of all open sets contained in A, and* $\text{Int}(\text{Int}(A)) = \text{Int}(A)$. *Also, $\overline{A}$ closed, the smallest closed set that contains A, $\overline{A}$ the intersection of all closed sets that A is contained in, and $\overline{\overline{A}} = \overline{A}$.*

Corollary 1. *A open $\iff A = \text{Int}(A)$ and A closed $\iff A = \overline{A}$*

Remark 6. Remark that these are not exclusive, nor indeed the only possibilities.

Definition 7 (Basis)**.** A basis for a topology X with open sets τ is a collection $B \subseteq \tau$ such that every $U \in \tau$ a union of sets in B.

Remark 7. Don't think about bases for vector spaces in this regard - there is no "minimality" requirement.

Keep in mind $\{(a, b) : -\infty < a < b < \infty\}$, a basis of topology on $\mathbb{R}$.

Proposition 3. *For a metric space (X, d), $\{B(x, r) : x \in X, r > 0\}$ a basis of topology.*

Definition 8 (Subspace Topology)**.** For a subset $Y \subseteq X$-topological space, we define the subspace topology on Y as $\tau_Y := \{Y \cap U : U \in \tau\}$.

Definition 9 (Continuous)**.** For X, Y-topological spaces, a function $f : X \to Y$ is continuous iff $\forall V$-open in Y, $f^{-1}(V)$-open in X.

Remark 8. One can verify that this is consistent with the $\varepsilon - \delta$ definition of continuity for functions on $\mathbb{R}$.

Theorem 1 (Continuity of Composition)**.** *If $f : X \to Y$, $g : Y \to Z$ continuous, $g \circ f$ continuous.*

Remark 9. Note how much easier this is to prove via toplogical spaces than the $\varepsilon - \delta$ definition.

Definition 10 (Product Space)**.** For an index set I and $X_\alpha, \alpha \in I$, we define $\prod_{\alpha \in I} X_\alpha$ as a product space; I may be finite or infinite.

Proposition 4. *A basis for the product space is given by cyliders of the form $A = \prod_{\alpha \in J} A_\alpha \times \prod \alpha \notin J X_\alpha$ for A_α-open in X_α, where $J \subseteq I$-finite.*

Definition 11 (Compact)**.** A set $A \subseteq X$ is compact if every cover has a finite subcover, that is

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha\text{-open} \implies \exists\{\alpha_1, \ldots, \alpha_n\} \subseteq I \text{ s.t. } A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}.$$

Proposition 5. *Closed intervals* $[a, b]$ *compact in* $\mathbb{R}$*.*

Proposition 6. $A \subseteq \mathbb{R}^n$ *compact* $\iff$ *closed and bounded.*

Definition 12 (Connected)**.** X is said to not be connected if $X = U \cup V$ for U, V open, nonempty, disjoint. If X cannot be written as such, X is said to be connected.

Theorem 2. *If* X *connected and* $f : X \to Y$*, then* $f(X)$ *connected in* Y*.*

Proposition 7. *Intervals in* $\mathbb{R}$ *are connected.*

Theorem 3 (Intermediate Value Theorem)**.** *If* X *connected,* $f : X \to \mathbb{R}$ *continuous, then* f *takes intermediate value; if* $a = f(x), b = f(y)$ *for* $x, y \in X$ *with* $a < b$*, then for any* $a < c < b$ $\exists z \in X$ *s.t.* $f(z) = c$*.*

Theorem 4. *For* X *compact,* $f : X \to Y$ *continuous,* $f(X)$ *compact in* Y*.*

Proposition 8. *For* X *compact and* $f : X \to \mathbb{R}$*,* f *attains both max and min on* X*.*

Definition 13 (Path Connected)**.** A set $A \subseteq X$ is path connected if for any $x, y \in A, \exists f : [a, b] \to X$ continuous such that $f(a) = x, f(b) = y f([a, b]) \subseteq A$.

Theorem 5. *Path connected* $\implies$ *connected.*

For open sets in $\mathbb{R}^n$*, the converse holds too.*

Definition 14 (Connected Component, Path Component)**.** For $x \in X$, the connected component of x is the largest connected subset of X containing x and the path component of x is the largest path connected subset of X containing x.

2 Metric Spaces

We discuss mostly the metric on ℓ_p space and notions of completeness, as well as some topological results specific to metric spaces, namely compactness.

Definition 15 (ℓ_p)**.** For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $1 \leqslant p \leqslant +\infty$, we define

$$||x||_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad ||x||_\infty := \max_{i=1}^{n} |x_i|,$$

and similarly, for sequences $x = (x_1, \dots, x_n, \dots)$,

$$||x||_p := \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}, \quad ||x||_\infty := \sup_{i=1}^{\infty} |x_i|,$$

and define $\ell_p := \{x : ||x||_p < +\infty\}$. It can be shown that these are well-defined norms on their respective spaces.

Theorem 6 (Holder, Minkowski's Inequalities)**.** *For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ and p, q such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\textit{Holder's:} \qquad \langle x, y\rangle = \left|\sum_{i=1}^{n} x_i y_i\right| \leqslant ||x||_p ||y||_q$$

and

$$\textit{Minkowski's:} \qquad ||x + y||_p \leqslant ||x||_p + ||y||_p.$$

The identical inequalities hold for infinite sequences.

Definition 16 (Completeness)**.** We say a metric space is complete if every Cauchy sequence converges to a limit point in the space.

Proposition 9. *For $\{x_n\}_{n\in\mathbb{N}}$, ℓ_p complete for any $1 \leqslant p \leqslant +\infty$.*

Proposition 10. *If $p < q$, $\ell_p \subseteq \ell_q$.*

Definition 17 (Contraction Mapping)**.** For a metric space (X, d), a function $f : X \to X$ is a contraction mapping if there exists $0 < c < 1$ such that

$$d(f(x), f(y)) \leqslant c \cdot d(x, y)$$

for any $x, y \in X$.

Theorem 7. *Let (X, d) be a complete metric space, $f : X \to X$ a contraction. Then, there exist a unique fixed point z of f such that $f(z) = z$; ie $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f \circ f \circ \cdots \circ f(x) = z$ for any $x \in X$.*

Theorem 8. *ℓ_p complete.*

Remark 10. It can be kind of funky to work with sequences in ℓ_p, since the elements of ℓ_p themselves sequences so we have "sequences of sequences".

Definition 18 (Totally bounded)**.** A metric space X is said to be totally bounded if $\forall\, \varepsilon > 0 \exists x_1, \ldots, x_n \in X, n = n(\varepsilon)$ such that $\bigcup_{i=1}^n B(x_i, \varepsilon) = X$.

Definition 19 (Sequentially compact)**.** A metric space X is said to be sequentially compact if every sequence has a convergent subsequence.

Theorem 9 (★ Equivalent Notions of Compactness in Metric Spaces)**.** *Let (X, d) a metric space. TFAE:*

- *X compact*
- *X complete and totally bounded*
- *X sequentially compact*

Remark 11. This is for a metric space, not a general topological space! Hopefully this is clear because some of the requirements necessitate a distance.

3 Differentiation

Definition 20 (Differentiable)**.** $f(x)$ differentiable at c if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, and if so we denote the limit $f'(c)$.

Theorem 10. *Differentiable $\implies$ continuous.*

4 Integration

5 Sequences of Functions

6 Infinite Series