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Algebra 2 MATH251

Course Outline:

Based on Lectures from Winter, 2024 by Prof. Anush Tserunyan.

Contents

1 Introduction

Remark 1.1. *This course is about vector spaces and linear transformations between them; a vector space involves multiplication by* scalars, *where the scalars come from some field. We recall first examples of fields, then vector spaces, as a motivation, before presenting a formal definition.*

1.1 Definitions

Remark 1.2. *Much of this is recall from Algebra 1.*

⊛ **Example 1.1: Examples of Fields**

1. $\mathbb{Q}$; the field of rational numbers.
2. $\mathbb{R}$; the field of real numbers; $\mathbb{Q} \subseteq \mathbb{R}$.
3. $\mathbb{C}$; the field of complex numbers; $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.
4. $\mathbb{F}_p \equiv \mathbb{Z}/p\mathbb{Z} \equiv \{0, 1, \ldots, p-1\}; the (unique) field of \text{p} elements, where \text{p} prime.$[a]
   (a) $p = 2$; $\mathbb{F}_2 \equiv \{0, 1\}$.
   (b) $p = 3$; $\mathbb{F}_3 \equiv \{0, 1, 2\}$.
   (c) $\cdots$

[a]where $a +_p b :=$ remainder of $\frac{a+b}{p}$, $a \cdot_p b :=$ remainder of $\frac{a \cdot b}{p}$.

Remark 1.3. *Throughout the course, we will denote an abstract field as* $\mathbb{F}$.

⊛ **Example 1.2: Examples of Vector Spaces**

1. $\mathbb{R}^3 := \{(x, y, z) : x, y, z \in \mathbb{R}\}$. We can add elements in $\mathbb{R}^3$, and multiply them by real scalars.
2. $\mathbb{F}^n := \underbrace{\mathbb{F} \times \mathbb{F} \times \cdots \mathbb{F}}_{n \text{ times}} := \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{F}\}$, where $n \in \mathbb{N}$[1]; this is a generalization of the previous example, where we took $n = 3, \mathbb{F} = \mathbb{R}$. Operations follow identically; addition:

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and, taking a scalar $\lambda \in \mathbb{F}$, multiplication:

$$\lambda \cdot (a_1, a_2, \ldots, a_n) := (\lambda \cdot a_1, \lambda \cdot a_2, \ldots, \lambda \cdot a_n).$$

We refer to these elements $(a_1, \cdots, a_n)$ as *vectors* in $\mathbb{F}^n$; the vector for which

$a_i = 0 \, \forall \, i$ is the 0 *vector*, and is the additive identity, making $\mathbb{F}^n$ an abelian group under addition, that admits multiplication by scalars from $\mathbb{F}$.

3. $C(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : f \text{ continuous}\}$. Here, we have the constant zero function as our additive identity ($x \mapsto 0 \, \forall \, x$), and addition/scalar multiplication of two continuous real functions are continuous.

4. $\mathbb{F}[t] := \{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n : a_i \in \mathbb{F} \, \forall \, i, n \in \mathbb{N}\}$, ie, the set of all polynomials in t with coefficients from $\mathbb{F}$. Here, we can add two polynomials;

$$(a_0 + a_1 t + \cdots + a_n t^n) + (b_0 + b_1 t + \cdots + b_m t^m) := \sum_{i=0}^{\max\{n,m\}} (a_i + b_i) t^i,$$

(where we "take" undefined a_i/b_i's as 0; that is, if $m > n$, then $a_{m-n}, a_{m-n+1}, \ldots, a_m$ are taken to be 0). Scalar multiplication is defined

$$\lambda \cdot (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) := \lambda a_0 + \lambda a_1 t + \lambda a_2 t^2 + \cdots + \lambda a_n t^n.$$

Here, the zero polynomial is simply 0 (that is, $a_i = 0 \, \forall \, i$).

[1]Where we take $0 \in \mathbb{N}$, for sake of consistency. Moreover, by convention, we define $\mathbb{F}^0$ (that is, when $n = 0$) to be $\{0\}$; the trivial vector space.

↪ **Definition 1.1: Vector Space**

A *vector space* V over a field $\mathbb{F}$ is an *abelian group* with an operation denoted $+$ (or $+_V$) and identity element[2] denoted 0_V, equipped with *scalar multiplication* for each scalar $\lambda \in \mathbb{F}$ satisfying the following axioms:

1. $1 \cdot v = v$ for $1 \in \mathbb{F}$, $\forall \, v \in V$.
2. $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta) v$, $\forall \, \alpha, \beta \in \mathbb{F}, v \in V$.
3. $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$, $\forall \, \alpha, \beta \in \mathbb{F}, v \in V$.
4. $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$, $\forall \, \alpha \in \mathbb{F}, u, v \in V$.

We refer to elements $v \in V$ as *vectors*.

[2]The "zero vector".

↪ **Proposition 1.1**

For a vector space V over a field $\mathbb{F}$, the following holds:

1. $0 \cdot v = 0_V$, $\forall \, v \in V$.

Proof. 1. $0 = 0 + 0$ (by definition in $\mathbb{F}$) $\implies 0 \cdot v = (0 + 0) \cdot v \overset{\text{axiom 3.}}{\implies} 0 \cdot v = 0 \cdot v + 0 \cdot v \overset{\text{group, inverses exist}}{\implies} (0 \cdot v) + (0 \cdot v)^{-1} = (0 \cdot v) + (0 \cdot v)^{-1} + 0 \cdot v \implies 0_V = 0 \cdot v$.

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↪ *Fri Jan 5 15:15:17 EST 2024*