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Analysis 2 MATH255

Course Outline:

Based on Lectures from Winter, 2024 by Prof. Dmitry Jackobson.

Contents

1 Introduction

1.1 Metric Spaces

↪ **Definition 1.1: Metric Space**

A set X is a *metric space* with distance d if

1. (symmetric) $d(x, y) = d(y, x) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. (triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$

Remark 1.1. *If 1., 3. are satisfied but not 2., d can be called a "pseudo-distance".*

↪ **Definition 1.2: Open Metric Space**

Let (X, d) be a metric space. A subset $A \subseteq X$ is open $\iff \forall x \in A, \exists r = r(x) > 0$ s.t. $B(x, r(x)) \subseteq A$.

↪ **Definition 1.3: Normed Space**

Let X be a vector space over $\mathbb{R}$. The norm on X, denoted $||x|| \in \mathbb{R}$, is a function that satisfies

1. $||x|| \geq 0$
2. $||x|| = 0 \iff x = 0$
3. $||c \cdot x|| = |c| \cdot ||x||$
4. $||x + y|| \leq ||x|| + ||y||$

If X is a normed vector space over $\mathbb{R}$, we can define a distance d on X by $d(x, y) = ||x - y||$.

↪ **Proposition 1.1**

If X is a normed vector space over $\mathbb{R}$, a distance d on X by $d(x, y) = ||x - y||$ makes (X, d) a metric space.

Proof.

1. $d(x, y) = ||x - y|| \geq 0$
2. $d(x, y) = 0 \iff ||x - y|| = 0 \iff x - y = 0 \iff x = y$
3. $d(x, y) + d(y, z) = ||x - y|| + ||y - z|| \geq ||(x - y) + (y - z)|| = ||x - z|| := d(x, z)$

■

⊛ **Example 1.1: L^p distance in $\mathbb{R}^n$**

Let $\overline{x} \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n)$. The L^p norm is defined

$$||x||_p := (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

In the case $p = 2, n = 2$, we simply have the standard Euclidean distance over $\mathbb{R}^2$.

Unit Balls: consider when $||x||_p \leq 1$, over $\mathbb{R}^2$.

- $p = 1 : |x_1| + |x_2| \leq 1$; this forms a "diamond ball" in the plane.
- $p = 2 : \sqrt{|x_1|^2 + |x_2|^2} \leq 1$; this forms a circle of radius 1. Clearly, this surrounds a larger area than in $p = 2$.

A natural question that follows is what happens as $p \to \infty$? Assuming $|x_1| \geq |x_2|$:

$$\begin{aligned} ||x||_p &= (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \\ &= \left[|x_1|^p \left(1 + \left|\frac{x_2}{x_1}\right|^p\right)\right]^{\frac{1}{p}} \\ &= |x_1| \left(1 + \left|\frac{x_2}{x_1}\right|^p\right)^{\frac{1}{p}} \end{aligned}$$

If $|x_1| > |x_2|$, this goes to $|x_1|$. If they are instead equal, then $||x||_p = |x_1| \cdot 2^{\frac{1}{p}} \to |x_1| \cdot 1$ as well. Hence, $\lim_{p\to\infty} ||x||_p = \max\{|x_1|, |x_2|\}$. Thus, the unit ball will approach $\max\{|x_1|, |x_2|\} \leq 1$, that is, the unit square.

↪ **Proposition 1.2**

Let $x \in \mathbb{R}^n$. Then, $||x||_p \to \max\{|x_1|, \ldots, |x_n|\}$ as $p \to \infty$.

Remark 1.2. *This is an extension of the previous example to arbitrary real space; the proof follows nearly identically.*

↪ **Definition 1.4: Convex Set**

Let X be a normed space, and take $x, y \in X$. The line segment from x to y is the set

$$\{t \cdot x + (1 - t) \cdot y : 0 \leq t \leq 1\}.$$

Let $A \subseteq X$. A is *convex* $\iff$ $\forall x, y \in A$, we have that

$$(t \cdot x + (1 - t) \cdot y) \in A \, \forall \, 0 \leq t \leq 1.$$

Remark 1.3. *Think of this as saying "a set is convex iff every point on a line segment connected any two points is in the set".*

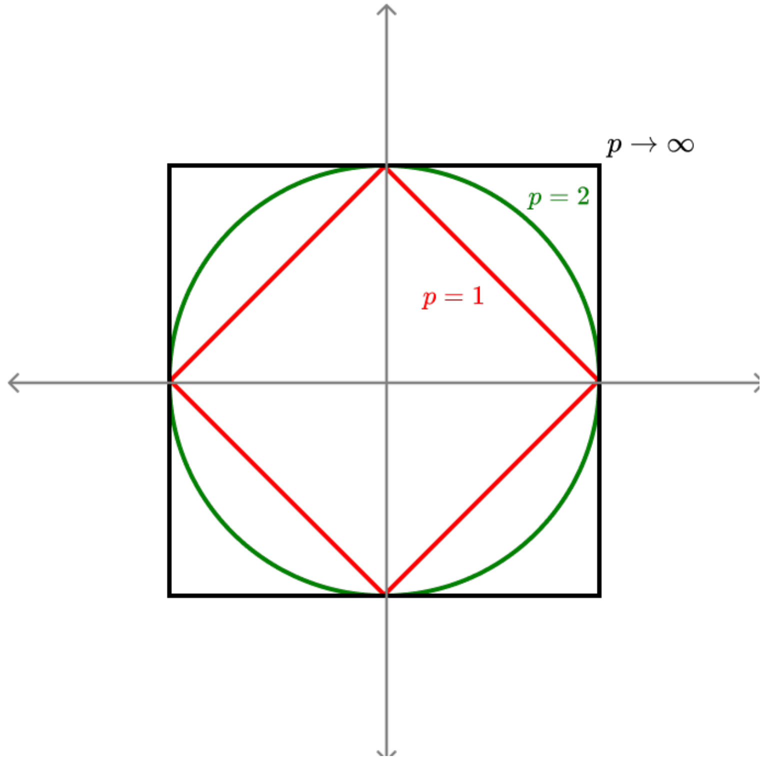

Figure 1: Regions of $\mathbb{R}^2$ where $||x||_p \leq 1$ for various values of p.

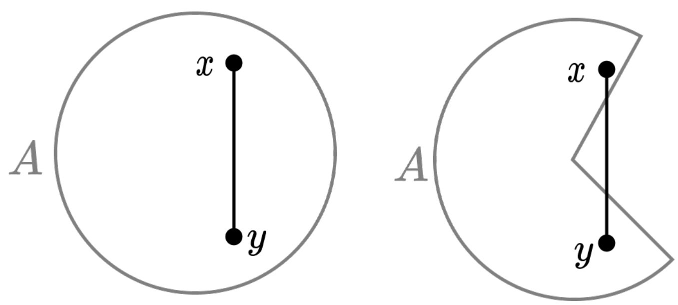

Figure 2: Convex (left) versus not convex (right) sets.

↪ **Definition 1.5:** ℓ_p

The space ℓ_p of sequences is defined as

$$\{x = (x_1, x_2, \ldots, x_n, \ldots) : \sum_{n=1}^{\infty} |x_n|^p < +\infty\} \quad * .$$

Then, $*$ defines the ℓ^p norm on the space of sequences; that is, $||x||_p := (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$.

⊛ **Example 1.2:** ℓ_p, $x_n = \frac{1}{n}$

. Let $x_n = \frac{1}{n}$. For which p is $x \in \ell_p$? We have, raising the norm to the power of p for ease:

$$\begin{aligned} ||x||_p^p &= |x_1|^p + |x_2|^p + \cdots + |x_n|^p + \cdots \\ &= 1^p + \left(\frac{1}{2}\right)^p + \cdots < \infty \iff p > 1. \end{aligned}$$

In the case that $p = 1$, this becomes a harmonic sum, which diverges.

⊛ **Example 1.3: L^p space of functions**

Let $f(x)$ be a continuous function. We define the norm of f over an interval $[a, b]$

$$||f||_p = \left[\int_a^b |f(x)|^p \, dx\right]^{\frac{1}{p}}.$$

Remark 1.4. *Triangle inequality for $||x||_p$ or $||f||_p$ is called Minkowski inequality; $||x||_p + ||y||_p \geq ||x + y||_p$. This will be discussed further.*

⊛ **Example 1.4: Distances between sets in $\mathbb{R}^2$**

Let A, B be bounded, closed, "nice" sets in $\mathbb{R}^2$. We define

$$d(A, B) := \text{Area}(A \triangle B),$$

where

$$A \triangle B : (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

It can be shown that this is a "valid" distance.

Remark 1.5. *$\triangle$ denotes the "symmetric difference" of two sets.*

⊛ **Example 1.5: p-adic distance**

Let p be a prime number. Let $x = \frac{a}{b} \in \mathbb{Q}$, and write $x = p^k \cdot \left(\frac{c}{d}\right)$, where c, d are not divisible by p. Then, the p-adic norm is defined $||x||_p := p^{-k}$. It can be shown that this is a norm.

Suppose $p = 2$, $x = 28 = 4 \cdot 7 = 2^2 \cdot 7$. Then, $||28||_2 = 2^{-2} = \frac{1}{4}$; similarly, $||1024||_2 = ||2^{10}||_2 = 2^{-10}$.

More generally, we have that $||2^k||_2 = 2^{-k}$; coversely, $||2^{-k}|| = 2^k$. That is, the closer to 0, the larger the distance, and vice versa, contrary to our notion of Euclidean distance.

↪ **Proposition 1.3**

$||x||_p$ as defined above is a well-defined norm over $\mathbb{Q}$.

Proof. ■

2 Point-Set Topology

2.1 Definitions

↪ **Definition 2.1: Topological space**

A set X is a topological space if we have a collection of subsets τ of X called *open sets* s.t.

1. $\varnothing \in \tau, X \in \tau$
2. Consider $\{A_\alpha\}_{\alpha \in I}$ where A_α an open set for any α; then, $\bigcup_{\alpha \in I} A_\alpha \in \tau$, that is, it is also an open set.
3. If J is a finite set, and A_β open for all $\beta \in J$, then $\bigcap_{\beta \in J} A_\beta \in \tau$ is also open.

In other words, 2.: arbitrary unions of open sets are open, and 3.: finite intersections of open sets are open.

↪ **Definition 2.2: Closed sets**

Closed sets are complements of open sets; hence, axioms for closed sets follow appropriately;

1.* $X, \varnothing$ closed;

2.* B_α closed $\forall\, \alpha \in I \implies \bigcap_{\alpha \in I} B_\alpha$ closed.

3.* B_β closed $\forall\, \beta \in J$, J finite, then $\bigcup_{\beta \in J} B_\beta$ also closed.

↪ Lecture 01; Last Updated: Thu Jan 11 08:35:34 EST 2024

↪ **Definition 2.3: Equivalence of Metrics**

Suppose we have a metric space X with two distances d_1, d_2; will these necessarily admit the same topology?

A sufficient condition is that, if $\forall\, x \neq y \in X, \exists 1 < C < +\infty$ s.t.

$$\frac{1}{C} < \frac{d_1(x,y)}{d_2(x,y)} < C.$$

That is, the distances are equivalent, up to multiplication by a constant.

Indeed, this condition gives that $d_2 < C d_1$ and $d_2 > \frac{d_1}{C}$; this gives

$$B_{d_1}(x, \frac{r}{c}) \subseteq B_{d_2}(x, r) \subseteq B_{d_1}(x, C \cdot r).$$

Hence, d_1, d_2 define the same open/closed sets on X thus admitting the same topologies. We write $d_1 \asymp d_2$.

Remark 2.1. *If $d_1 \asymp d_2$ and $d_2 \asymp d_3$, then also $d_1 \asymp d_3$. Moreover, clearly, $d_1 \asymp d_1$ and $d_1 \asymp d_2 \implies d_2 \asymp d_1$, hence this is a well-defined equivalence relation.*

Hence, its enough to show that $\forall\, 1 < p < +\infty$, we have $||x||_p \asymp ||x||_\infty$ to show that any $||x||_q$ norm are equivalent for all q on $\mathbb{R}^n$.

↪ **Definition 2.4: Interior, Boundary of a Topological Set**

Let X be a topological space, $A \subseteq X$ and let $x \in X$. We have the following possibilities

1. $\exists U$-open : $x \in U \subseteq A$. In this case, we say $x \in$ the *interior* of A, denoted
$$x \in \mathrm{Int}(A).$$
2. $\exists V$-open : $x \in V \subseteq X \setminus A = A^C$. In this case, we write
$$x \in \mathrm{Int}(X^C).$$
3. $\forall\, U$-open : $x \in U$, $U \cap A \neq \varnothing$ AND $U \cap A^C \neq \varnothing$. In this case, we say x is in the *boundary* of A, and denote
$$x \in \partial A.$$

↪ **Definition 2.5: Closure**

$x \in \mathrm{Int}(A)$ or $x \in \partial A$ (that is, $x \in \mathrm{Int}(A) \cup \partial A$) $\iff$ every open set U that contains x intersects A.[1]Such points are called *limit points* of A. The set of all limits points of A is called the *closure* of A, denoted $\overline{A}$.

Remark 2.2. *We have that*
$$\mathrm{Int}(A) \subseteq A \subseteq \overline{A} = \mathrm{Int}(A) \cup \partial A.$$

↪ **Proposition 2.1: Properties of $\mathrm{Int}(A)$**

$\mathrm{Int}(A)$ is *open*, and it is the largest open set contained in A. It is the union of all U-open s.t. $U \subseteq A$. Moreover, we have that
$$\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A).$$

↪ **Proposition 2.2: Properties of $\overline{A}$**

$\overline{A}$ is *closed*; $\overline{A}$ is the smallest closed set that contains A, that is, $\overline{A} = \bigcap B$ where B closed and $A \subseteq B$. We have too that
$$\overline{(\overline{A})} = \overline{A}.$$

[1]"Requires" proof.

↪ **Proposition 2.3**

1. A is open $\iff A = \text{Int}(A)$
2. A is closed $\iff A = \overline{A}$

2.2 Basis

↪ **Definition 2.6: Basis for a Toplogy**

Let τ be a topology on X. Let $\mathcal{B} \subseteq \tau$ be a collection of open sets in X such that every open set is a union of open sets in $\mathcal{B}$.

⊛ **Example 2.1: Example Basis**

$X = \mathbb{R}$, and $\mathcal{B} = \{\text{all open intervals } (a, b) : -\infty < a < b < +\infty\}$.

↪ **Proposition 2.4**

Let $\mathcal{B}$ be a collection of open sets in X. Then, $\mathcal{B}$ is a basis $\iff$

1. $\forall x \in X, \exists U$-open $\in \mathcal{B}$ s.t. $x \in U$.
2. If $U_1 \in \mathcal{B}$ and $U_2 \in \mathcal{B}$, and $x \in U_1 \cap U_2$, then $\exists U_3 \in \mathcal{B}$ s.t. $x \in U_3 \subseteq U_1 \cap U_2$.

⊛ **Example 2.2**

Consider $X = \mathbb{R}$. Requirement 1. follows from taking $U = (x - \varepsilon, x + \varepsilon)$ for any $\varepsilon > 0$. For 2., suppose $x \in (a, b) \cap (c, d) =: U_1 \cap U_2$. Let $U_3 = (\max\{a, c\}, \min\{b, d\})$; then, we have that $U_3 \subseteq U_1 \cap U_2$, while clearly $x \in U_3$.

↪ **Proposition 2.5**

In a metric space, a basis for a topology is a collection of open balls,

$$\{B(x, r) : x \in X, r > 0\} = \{\{y \in X : d(x, y) < r\} : x \in X, r > 0\}.$$

Proof. We prove via proposition 2.4. Property 1. holds clearly; $x \in B(x, \varepsilon)$-open $\subseteq \mathcal{B}$.

For property 2., let $x \in B(y_1, r_1) \cap B(y_2, r_2)$, that is, $d(x, y_1) < r_1$ and $d(x, y_2) < r_2$. Let

$$\delta := \min\{r_1 - d(x, y_1), r_2 - d(x, y_2)\}.$$

We claim that $B(x, \delta) \subseteq U_1 \cap U_2$.

Let $z \in B(x, \delta)$. Then,

$$d(z, y_1) \overset{\triangle\neq}{\leq} d(z, x) + d(x, y_1) < \delta + d(x, y_1) \leq r_1 - d(x, y_1) + d(x, y_1) = r_1,$$

hence, as $d(z, y_1) < r_1 \implies z \in B(y_1, r_1) = U_1$. Replacing each occurrence of y_1, r_1 with y_2, r_2 respectively gives identically that $z \in B(y_2, r_2) = U_2$. Hence, we have that $B(x, \delta) \subseteq U_1 \cap U_2$ and 2. holds. ■

2.3 Subspaces

↪ **Definition 2.7**

Let X be a topological space and let $Y \subseteq X$. We define the subspace topology on Y:

1. Open sets in $Y = \{Y \cap \text{ open sets in } X\}$

↪ **Proposition 2.6: Consequences of Subspace Topologies**

Suppose $\mathcal{B}$ is a basis for a topology in X. Then, $\{U \cap Y : U \in \mathcal{B}\}$ forms a basis for the subspace $Y \subseteq X$.

Suppose X a metric space. Then, Y is also a metric space, with the same distance.

↪ **Proposition 2.7**

Let $Y \subseteq X$- a metric space. Then, the metric space topology for (Y, d) is the same as the subspace topology.

Proof. (Sketch) A basis for the open sets in X can be written $\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)$; hence

$$Y \cap (\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)) = \bigcup_{\alpha \in I} (Y \cap B(x_\alpha, r_\alpha))$$

is an open set topology for Y. ■

↪ **Lemma 2.1**

Let $A \subseteq X$-open, $B \subseteq A$; B-open in subspace topology for $A \iff B$-open in X.

↪ **Lemma 2.2**

Let $Y \subseteq X$, $A \subseteq Y$. Then, $\overline{A}$ in $Y = Y \cap \overline{A}$ in X. We can denote this

$$\overline{A}_Y = \overline{A}_X \cap Y.$$

2.4 Continuous Functions

↪ **Definition 2.8: Continuous Function**

Let X, Y be topological spaces. Let $f : X \to Y$. f is *continuous* $\iff$ $\forall$ open $V \in Y$, $f^{-1}(V)$-open in X.

↪ **Proposition 2.8**

This definition is consistent with the normal ε-δ definition on the real line.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$, continuous; that is, $\forall \varepsilon > 0$, $\forall x \in \mathbb{R} \exists \delta > 0$ s.t. $|x_1 - x| < \delta$, then $|f(x_1) - f(x)| < \varepsilon$.

Let $V \subseteq \mathbb{R}$ open. Let $y \in V$. Then, $\exists \varepsilon : (y - \varepsilon, y + \varepsilon) \subseteq V$. Let $y = f(x)$, hence $y \in f^{-1}(V)$. Now, if $d(x, x_1) < \delta$, we have that $d(f(x_1), f(x)) < \varepsilon$ (by continuity of f), hence $f(x_1) \in (y - \varepsilon, y + \varepsilon) \subseteq V$; moreover, $(x - \delta, x + \delta) \subseteq f{-}1(V)$, thus $f^{-1}(V)$ is open as required.

The inverse of this proof follows identically. ■

↪ *Lecture 02; Last Updated: Thu Jan 11 08:52:09 EST 2024*

↪ **Proposition 2.9**

Suppose $\mathcal{B}$ forms a basis of topology for Y. Then, $f : X \to Y$ is continuous if $f^{-1}(U)$ open $\forall U \in \mathcal{B}$.

Proof. If U-open set in Y, then $\exists I$-index set and a collection of open sets $\{A_\alpha\}_{\alpha \in I}$, $A_\alpha \in \mathcal{B}$, s.t. $U = \cup_{\alpha \in I} A_\alpha$. Then, we have

$$f^{-1}(U) = f^{-1}(\cup_{\alpha \in I}(A_\alpha)) = \cup_{\alpha \in I} \underbrace{f^{-1}(A_\alpha)}$$

Hence, if each $f^{-1}(A_\alpha)$ open, then $\cup_{\alpha \in I} f^{-1}(A_\alpha)$ open; hence it suffices to check if $f^{-1}(U) \forall U$-open in V is open to see if f continuous. ■

↪ **Theorem 2.1: Continuity of Composition**

If $f : X \to Y$ continuous and $g : Y \to Z$ continuous, then $g \circ f$ continuous as well.

Proof. Let U-open in Z. Then

$$(g \circ f)^{-1}(U) = \underbrace{f^{-1}(\underbrace{g^{-1}(U)}_{\text{open in } Y})}_{\text{open in } X}$$

■

↪ **Proposition 2.10**

If $f : X \to Y$ continuous and $A \subseteq X$, A has subspace topology, then $f|_A : A \to Y$ is also continuous.[2]

Proof. Let U-open in Y. Then

$$(f|_A)^{-1}(U) = \underbrace{f^{-1}(U)}_{\text{open}} \cap \underbrace{A}_{\text{open}}$$

By the definition of subspace topology, this is an open set and hence $f|_A$ is continuous. ■

2.5 Product Spaces

↪ **Definition 2.9: Finite Product Spaces**

Let $X_1, \ldots, X_n$ be topological spaces. We define

$$(X_1 \times X_2 \times \cdots \times X_n),$$

and aim to define a *product topology*; a basis of which consists of cylinder sets.

↪ **Definition 2.10: Cylinder Set**

A *cylinder set* has the form

$$A_1 \times A_2 \times \cdots \times A_n$$

where each A_j-open in X_j.

⊛ **Example 2.3**

Given an open interval $(a_1, b_1), (a_2, b_2) \subset \mathbb{R}$, the set $(a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$ is a basis for the topology on $\mathbb{R}^2$.

↪ **Definition 2.11: Projection**

Let $X_1 \times X_2 \times \cdots X_n =: X$. The *projection* $\pi_j : X \to X_j$ maps $(x_1, \ldots, x_n) \to x_j \in X_j$.

Remark 2.3. *One can show π_j continuous.*

↪ **Definition 2.12: Coordinate Function**

[2]We denote $f|_A$ as the restriction of the domain of f to A.

Given a function $f : Y \to X_1 \times \cdots X_n = (x_1(y), x_2(y), \ldots, x_n(y))$. The *coordinate function* is

$$f_j = \pi_j \circ f; \quad f_j = x_j(y).$$

↪ **Proposition 2.11**

$f : Y \to X = X_1 \times \cdots \times X_n$ continuous $\iff$ $f_j : Y \to X_j$ continuous.

Proof. Its enough to show that $\forall U \in \mathcal{B}$-basis for X-product space, $f^{-1}(U)$-open in Y. Take $U = A_1 \times \cdots A_n$-open. Then, we claim that

$$f^{-1}(U) = f^{-1}(A_1 \times \cdots \times A_n) = f_1^{-1}(A_1) \cap f_2^{-1}(A_2) \cap \cdots \cap f_n^{-1}(A_n). \quad \star$$

If this holds, then as each f_i continuous (being a composition of continuous functions) and each A_i open in X_i, then each $f_i^{-1}(A_i)$ open in Y and hence $\star$, being the finite intersection of open sets in Y, is itself open in Y.

■

⊛ **Example 2.4: Fourier Transform: Motivation for Infinite Product Toplogies**

Let $f \in C([0, 2\pi])$ is real-valued. We write the nth Fourier coefficients

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, \mathrm{d}x \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos(nx) \, \mathrm{d}x - i \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin(nx) \, \mathrm{d}x \, . \end{aligned}$$

And the Fourier transform of f as the infinite product

$$f(x) \mapsto (\ldots, \hat{f}(-n), \hat{f}(-n+1), \ldots, \hat{f}(-1), \hat{f}(0), \hat{f}(1), \cdots, \hat{f}(n), \ldots) \in \prod_{n \in \mathbb{Z}} (\mathbb{C})_n.$$

Hence, this is an (countably, as indexed by integers) infinite product space.

Now, let $f : \mathbb{R} \to \mathbb{R}$. Suppose $f(x) \to 0$ “fast enough” as $|x| \to \infty$ and f continuous. Then, we can define the Fourier coefficients

$$\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-itx} \, \mathrm{d}x \, ,$$

where $t \in \mathbb{R}$. We then have the transform

$$f \mapsto \{\hat{f}(t)\}_{t \in \mathbb{R}}.$$

In this case, our index set is $\mathbb{R}$ is (uncountably) infinite.

↪ **Definition 2.13: Product Topology/Cylinder Sets for ∞ Products**

Let $X = \prod_{\alpha \in I} X_\alpha$. Then, a basis for X is given by cylinder sets of the form $A = \prod_{\alpha \in I} A_\alpha$ where A_α-open in X_α, AND $A_\alpha = X_\alpha$ except for finitely many indices α.

That is, there exists a finite set $J = (\alpha_1, \ldots, \alpha_k) \subseteq I$, such that we can write $A = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$ (where A_α open in X_α).

↪ **Proposition 2.12**

Given $f : Y \to \prod_{\alpha \in I} X_\alpha = X$, then (taking $f_\alpha = \pi_\alpha \circ f$ as before) we have that f is continuous in $X \iff f_\alpha : Y \to X_\alpha$ continuous in $X_\alpha \, \forall \, \alpha \in I$.

Remark 2.4. *Extension of proposition 2.11 to infinite product space.*

Proof. Write $U = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$. Then,

$$f^{-1}(U) = \bigcap_{\alpha \in J} f_\alpha^{-1}(A_\alpha)$$

which is open in Y, hence f continuous. ■

Remark 2.5. *The intersection of the entire spaces give no restriction.*

↪ Lecture 03; Last Updated: Fri Jan 19 11:49:27 EST 2024

2.6 Metrizability

↪ **Proposition 2.13**

Different metrics can define the same topology.

⊛ **Example 2.5**

1. Different ℓ_p metrics in $\mathbb{R}^n$ (PSET 1)

2. Let (X, d) be a metric space. Then,

$$\tilde{d}(x, y) := \frac{d(x, y)}{d(x, y) + 1}$$

is also a metric (the first two axioms are trivial), and defines the same topology. Note, moreover, that $\tilde{d}(x, y) \leq 1 \, \forall \, x, y$; this distance is bounded, and can often be more convenient to work with in particular contexts.

↪ **Question 2.1**

Suppose (X_k, d_k) are metric spaces $\forall\, k \geq 1$. Then, we can define the product topology τ on

$$X := \prod_{k=1}^{\infty} X_k.$$

Does the product topology τ come from a metric? That is, is τ *metrizable*?

Remark 2.6. *There do indeed exist examples of non-metrizable topological spaces; this question is indeed well-founded.*

Answer. Let $\underline{x} = (x_1, x_2, \ldots, x_n, \ldots), \underline{y} = (y_1, y_2, \ldots, y_n, \ldots) \in \prod_{k=1}^{\infty}$ (where $x_i, y_i \in X_i$) be infinite sequences of elements. Then, for each metric space X_k take the metric

$$\tilde{d}_k(x_k, y_k) = \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)}$$

(as in the example above). Then, we define

$$D(\underline{x}, \underline{y}) = \sum_{k=1}^{\infty} \frac{\tilde{d}_k(x_k, y_k)}{2^k},$$

noting that $D(\underline{x}, \underline{y}) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ (by our construction, "normalizing" each metric), hence this is a valid, *converging* metric (which wouldn't otherwise be guaranteed if we didn't normalize the metrics). It remains to show whether this metric omits the same topology as τ. ■

2.7 Compactness, Connectedness

↪ **Definition 2.14: Compact**

A set A in a topological space is said to be *compact* if every cover has a finite subcover. That is, if

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha - \text{open},$$

then $\exists \{\alpha_1, \ldots, \alpha_n \in I\}$ such that $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

↪ **Proposition 2.14**

A closed interval $[a, b]$ is compact.

Proof. If[3] $a = b$, this is clear. Suppose $a < b$, and let $[a, b] \subseteq \bigcup_{i \in I} U_i =: \mathcal{U}$ be an arbitrary cover. Then, we proceed in the following steps:

1. **Claim:** Given $x \in [a, b], x \neq b, \exists y \in [a, b]$ s.t. $[x, y]$ has a finite subcover.

[3] This proof is adapted from that of Theorem 27.1 in Munkre's Topology, an identical theorem but applied to more general ordered topologies.

Let $x \in [a, b]$, $x \neq b$. Then, $\exists U_\alpha \in \mathcal{U} : x \in U_\alpha$. Since U_α open, and $x \neq b$, we further have that $\exists c \in [a, b]$ s.t. $[x, c) \subseteq U_\alpha$.

Now, let $y \in (x, c)$; then, the interval $[x, y] \subseteq [x, c) \subseteq U_\alpha$, that is, $[x, y]$ has a finite subcover.

2. Define $C := \{y \in [a, b] : y > a, [a, y] \text{ has a finite subcover}\}$. We note that

   - $C \neq \varnothing$; taking $x = a$ in Step 1. above, we have that $\exists y \in [a, b]$ such that $[a, y]$ has a finite step cover, so this $y \in C$.
   - C bounded; by construction, $\forall\, y \in C, a < y \leq c$.

   Thus, we can validly define $c := \sup C$, noting that $a < c \leq b$. Ultimately, we wish to prove that $c = b$, completing the proof that $[a, b]$ has a finite subcover.

3. **Claim:** $c \in C$.

   Let $U_\beta \in \mathcal{U} : c \in U_\beta$. Then, by the openness of U_β, $\exists d \in [a, b]$ s.t. $(d, c] \subseteq U_\beta$.

   Supposing $c \notin C$, then $\exists z \in C$ such that $z \in (d, c)$; if one did not exist, then this would imply that d was a smaller upper bound that c, a contradiction. Thus, $[z, c] \subseteq (d, c] \subseteq U_\beta$.

   Moreover, we have that, given $z \in C$, $[a, z]$ has a finite subcover; call it $U_z \subseteq \mathcal{U}$. This gives, then:

   $$[a, c] = [a, z] \cup [z, c] \subseteq U_z \cup U_\beta.$$

   But this is a finite subcover of $[a, c]$, contradicting the fact that $c \notin C$. We conclude, then, that $c \in C$ after all.

4. **Claim:** $c = b$.

   Suppose not; then, since we have $c \leq b$, then assume $c < b$. Then, applying Step 1. with $x = c$ (which we can do, by our assumption of $c \neq b$!), then we have that $\exists y > c$ s.t. $[c, y]$ has a finite subcover, call this $U_y \subseteq \mathcal{U}$.

   Moreover, we had $c \in C$, hence $[a, c]$ has a finite subcover, call this $U_c \subseteq \mathcal{U}$.

   Then, this gives us that
   $$[a, y] = [a, c] \cup [c, y] \subseteq U_c \cup U_y,$$
   that is, $[a, y]$ has a finite subcover, and so $y \in C$. But recall that $y > c$; hence, this a contradiction to c being the least upper bound of C. We conclude that $c = b$, and thus $[a, b]$ has a finite subcover, and is thus compact.

■

Remark 2.7. *A similar proof shows that $[a, b]$ is* connected*; we cannot cover it by two disjoint open sets.*

↪ **Theorem 2.2: On Compactness**

Let $A \subseteq \mathbb{R}^n$. Then, A compact $\iff$ A closed and bounded.

↪ **Proposition 2.15**

If X, Y are compact topological spaces, then $X \times Y$ is compact.

Remark 2.8. *By induction, if $X_1, \ldots, X_n$ compact, so is $\prod_{i=1}^n X_i$.*

↪ **Proposition 2.16**

A closed subset of a compact topological space is compact in the subspace topology.

Proof. (Of theorem 2.2)

($\Longleftarrow$) If $A \subseteq \mathbb{R}^n$ closed and bounded, then $A \subseteq [-R, +R]^n$ for some $R > 0$ (it is contained in some "n-cube"). Then, we have that $[-R, R]$ is compact, by proposition 2.14, proposition 2.15, and proposition 2.16, A itself compact.

($\Longrightarrow$) Suppose $A \subseteq \mathbb{R}^n$ is compact. Then, $\bigcup_{x \in A} B(x, \varepsilon)$ for some $\varepsilon > 0$ is an open cover of A. As A compact, there must exist a finite subcover of this cover, $A \subseteq \bigcup_{i=1}^N B(x_i, r_i)$. Let $R := \max_{i=1}^N (||x_i|| + r_i)$. Then, $A \subseteq \overline{B(0, R)}$, that is, A is bounded.

Now, suppose x is a limit point of A. Then, any neighborhood of x contains a point in A, so $\forall\, r > 0$, $B(x, r) \cap A \neq \varnothing$, and so $\overline{B}(x, r)$ also contains a point of A for any $r > 0$.

Now, suppose $x \notin A$ (looking for a contradiction). Then,

$$U := \bigcup_{r > 0} U_r := \bigcup_{r > 0} (\mathbb{R}^n \setminus \overline{B(x, r)}) = \mathbb{R}^n \setminus \{x\}$$

is an open cover for the set A. A being compact implies that U has an finite subcover such that $A \subset U_{r_1} \cup U_{r_2} \cup \cdots \cup U_{r_N}$. Let $r_0 = \min_{i=1}^N r_i$. Then, $A \subseteq U_{r_0}$, and $A \cap B(x, r_0) = \varnothing$; but this is a contradiction to the definition of a limit point, hence any limit point x is contained in A and A is thus closed by definition. ■

↪ **Proposition 2.17**

Compact $\Longrightarrow$ sequentially compact; that is, every sequence in a compact set has a convergent subsequence.

↪ Lecture 04; Last Updated: Wed Jan 24 21:27:59 EST 2024

↪ **Definition 2.15: Connected**

A topological space X is *not connected* if $X = U \cup V$ for two open, nonempty, disjoint sets U, V.

If this does not hold, X is said to be *connected.*

A set $A \subseteq X$ is not connected if A is not connected in the subspace topology $\iff$ $A =\subseteq U \cup V$, for U, V-open in X, $(U \cap A) \neq \varnothing$, $(V \cap A) \neq \varnothing$ and $U \cap V = \varnothing$.

↪ **Theorem 2.3**

Let X be a connected topological space. Let $f : X \to Y$ be a continuous. Then, $f(X)$ is also connected.

Proof. Suppose, seeking a contradiction, that X is connected, but $f(X)$ is not. Then, we can write $f(X) \subseteq Y$ as $f(X) \subseteq U \cup V$, such that U, V open in Y and $U \cap V = \varnothing$. Then,

$$(U \cap f(X)) \cap (V \cap f(X)) = \varnothing.$$

We also have that

$$X \subseteq \underbrace{f^{-1}(U)}_{\text{open in } X, \neq \varnothing} \cup \underbrace{f^{-1}(V)}_{\text{open in } X, \neq \varnothing}.$$

$f^{-1}(U) \cap f^{-1}(V) = \varnothing$ (that is, they are disjoint) by our assumption; this is a contradiction to the connectedness of X, as we are able to write it as a subset of two disjoint open sets. Hence, $f(X)$ is indeed connected. ■

↪ **Lemma 2.3**

Any interval $(a, b), [a, b], [a, b), \ldots, \subseteq \mathbb{R}$ is connected.

Proof. ■

↪ **Theorem 2.4: "Intermediate Value Theorem"**

Suppose X is connected and $f : X \to \mathbb{R}$ is a continuous function. Then, f takes intermediate values.

More precisely, let $a = f(x), b = f(y)$ for $x, y \in X$. Assume $a < b$. Then, $\forall\, a < c < b, \exists z \in X$ s.t. $f(z) = c$.

Proof. Suppose, seeking a contradiction, that $\exists c : a < c < b$ s.t. $c \notin f(X)$ (that is, there exists an intermediate value that is "not reached" by the function).

Let $U = (-\infty, c)$ and $V = (c, +\infty)$; note that these are disjoint open sets. Then, we have that

$$X = f^{-1}(U) \cup f^{-1}(V),$$

by our assumption of $c \notin f(X)$. But this gives that X is not connected, as the union of two open (by continuity), disjoint, nonempty ($f(x) = a \in U \implies x \in f^{-1}(U)$, and $f(y) = b \in V \implies y \in f^{-1}(V)$) sets, a contradiction. ■

↪ **Theorem 2.5**

Suppose X is compact, Y-topological space, $f : X \to Y$ is a continuous function. Then, $f(X)$ is also compact.

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(X) \subseteq Y$, that is,

$$f(X) \subseteq \bigcup_{\alpha \in I} U_\alpha \implies X \subseteq f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(U_\alpha) =: \bigcup_{\alpha \in I} V_\alpha - \text{open}.$$

Then, this is an open cover of X; X is compact, thus there exists a finite subcover, that is, indices $\{\alpha_1, \ldots, \alpha_n\} \subseteq I$ such that $X = \bigcup_{i=1}^n V_{\alpha_i}$. Thus,

$$f(X) \subseteq \bigcup_{i=1}^n U_{\alpha_i},$$

which is a finite subcover of $f(X)$. Thus, $f(X)$ is compact. ■

Remark 2.9. *Recall the "extreme value theorem": let $f : [a, b] \to \mathbb{R}$ a continuous function; then, a minimum and maximum is obtained for $f(x)$ on this interval for values in this interval.*

↪ **Theorem 2.6**

Let X compact, and $f : X \to \mathbb{R}$ a continuous function. Then,

$$\max_{x \in X} f(x) \text{ and } \min_{x \in X} f(x)$$

are both attained.

Proof. $f(X) \subseteq \mathbb{R}$ is compact by theorem 2.5, and so by theorem 2.2, $f(X)$ is closed and bounded. Let, then, $m = \inf f(X)$ and $M = \sup f(X)$; these necessarily exist, since $f(X)$ is bounded. Both m and M are limit points of $f(X)$. But $f(X)$ is closed, and hence contains all of its limit points, and thus $m \in f(X)$ and $M \in f(X)$, and thus $\exists y_m : f(y_m) = m$ and $y_M : f(y_M) = M$. ■

↪ **Definition 2.16: Path Connected**

A set $A \subseteq X$ is called *path connected* if $\forall x, y \in A$, $\exists f : [a, b] \to X$, continuous, s.t. $f(a) = x, f(b) = y$ and $f([a, b]) \subseteq A$.

The set $\{f(t) : a \leq t \leq b\}$ is called a *path* from x to y.

↪ **Theorem 2.7: Path connected $\implies$ connected**

If $A \subseteq X$ is path connected, then A is connected.

Proof. Suppose, seeking a contradiction, that A is path connected, but not connected. Then, we can write $A \subseteq U \cup V$, for open, disjoint, nonempty subsets $U, V \subseteq X$.

Let $x \in U \cap A$ and $y \in V \cap A$. Then, $\exists f : [a, b] \to A$ s.t. $f(a) = x, f(b) = y$, and $f([a, b]) \subseteq A$, by the path connectedness of A. Then,

$$[a, b] \subseteq f^{-1}(A) \subseteq \underbrace{f^{-1}(U \cap A)}_{\text{open}} \cup \underbrace{f^{-1}(V \cap A)}_{\text{open}} =: \underbrace{U_1}_{a \in} \cup \underbrace{U_2}_{b \in},$$

that is, $[a, b]$ is contained in a union of open, nonempty, disjoint sets, contradicting $[a, b]$ the connectedness of $[a, b]$ by lemma 2.3. Thus, A is connected. ■

Remark 2.10. *A counterexample to the opposite side of the implication is the Topologist's sine curve, the set*

$$\{(x, \sin\left(\frac{1}{x}\right)) : x \in (0, 1]\} \cup \{0\} \times [-1, 1].$$

This set is connected in $\mathbb{R}^2$, but is not path connected.

↪ **Proposition 2.18**

For open sets in $\mathbb{R}^n$, path connected $\iff$ connected.

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2.8 Path Components, Connected Components

Remark 2.11. *Remark that if a metric space X is not connected, then we can write $X = U \cup V$ where U, V are open, nonempty and disjoint. It follows, then, that $U = V^C$ (and vice versa) and hence U, V are both open and closed.*

↪ **Definition 2.17: Connected Component**

A connected component of $x \in X$ is the largest connected subset of X that contains x.

⊛ **Example 2.6**

Let $X = (0, 1) \cup (1, 2)$. Here, we have two connected components, $(0, 1)$ and $(1, 2)$

⊛ **Example 2.7: Middle Thirds Cantor Set**

Let $C_0 := [0, 1]$, and given C_n, define $C_{n+1} := \frac{1}{3}(C_n \cup (2 + C_n))$ for $n \geq 0$. C_∞ is totally disconnected.

↪ **Definition 2.18: Path Component**

A *path component* $P(x)$ of $x \in X$ is the largest path connected subset of X that contains x.

↪ **Proposition 2.19**

$P(x) = \{x \in X : \exists \text{ conintuous path } \gamma : [0, 1] \to X : \gamma(0) = x, \gamma(1) = y\}$.

Remark 2.12. *Where we "start" a path does not matter. We write $x \sim y$ if $\exists \gamma$ from x to y; this is an equivalence relation on the elements of X.*

Remark 2.13. *The choice of $[0, 1]$ here is arbitrary; any closed interval is homeomorphic.*

↪ **Lemma 2.4**

If $P(x) \cap P(y) \neq \varnothing$, then $P(x) = P(y)$.

Proof. $P(x) \cap P(y) \neq \varnothing \implies \exists z : x \sim z \wedge y \sim z \implies x \sim y$. ■

↪ **Lemma 2.5**

If $A \subseteq X$ is connected, then $\overline{A}$ is also connected.

↪ **Lemma 2.6**

Suppose $A \subseteq X$ is both open and closed. Then, if $C \subseteq X$ is connected and $C \cap A \neq \varnothing$, then $C \subseteq A$.

Proof. If A is both open and closed, then $C \cap A$ is both open and closed in C. If $C \cap A^C \neq \varnothing$, then this is also open and closed in C. Hence, we can write $C = (C \cap A) \cup (C \cap A^C)$, that is, a disjoint union of two nonempty open sets, contradicting the connectedness of C. Hence, $C \cap A^C = \varnothing$, and so $C \subseteq A$. ■

↪ **Proposition 2.20**

Let $\{C_\alpha\}_{\alpha \in I}$ be a collection of nonempty connected subspaces of X s.t. $\forall\, \alpha, \beta \in I, C_\alpha \cap C_\beta \neq \varnothing$. Then, $\cup_{\alpha \in I} C_\alpha$ is connected.

↪ **Proposition 2.21**

Suppose each $x \in X$ has a path-connected neighborhood. Then, the path components in X are the same as the connected components in X.

2.8.1 Cantor Staircase Function

↪ **Definition 2.19: An Explicit Definition**

Let $x \in C : x = 0.a_1 a_2 a_3 \dots$ (base 3), ie $a_j = \begin{cases} 0 \\ 2 \end{cases}$. Define

$$f(x) = \begin{cases} \sum \frac{a_j/2}{2^j} & x \in C \\ \text{extend by continuity} & x \notin C. \end{cases}$$

That is, if $x \notin C$, set $f(y) = \sup_{x \in C, x < y} f(x) = \inf_{x \in C, x > y} f(x)$.

↪ **Definition 2.20: Complement Definition**

To construct the complement of the Cantor set, begin with $[0, 1]$ and at a step n, we remove 2^n open intervals from this interval. $f(x)$ will be constant on each of these intervals with values $\frac{k}{2^n}$ where k odd and $0 < k < 2^n$. Extend by continuity to all $x \in C$.

Remark 2.14. *Wikipedia's explanation of this is far better than whatever this definition is trying to say.*

↪ Lecture 06; Last Updated: Tue Jan 23 11:03:35 EST 2024

3 L^p Spaces

3.1 Review of ℓ^p Norms

Remark 3.1. *Recall that for $1 \leq p \leq +\infty$, we define for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the norm*

$$||x||_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}, \quad ||x||_\infty = \max_{i=1}^{n} |x_i| \, .$$

Similarly, for infinite vector spaces, we had, for $x = (x_1, \ldots, x_n, \ldots)$, the norm

$$||x||_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}}, \quad ||x||_\infty = \sup_{i \geq 1} |x_i| \, .$$

Here, we define

$$\ell_p := \{x = (x_1, \ldots, x_n) : ||x||_p < +\infty\}.$$

3.2 ℓ^p Norms, Hölder-Minkowski Inequalities

↪ **Definition 3.1: Hölder Conjugates**

For $1 \leq p, q \leq +\infty$, we say that p, q are said to be *Hölder conjugates* if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Remark 3.2. *We refer to these simply as "conjugates" throughout as no other conception of conjugate numbers will be discussed.*

Further, we take by convention $\frac{1}{\infty} = 0$.

↪ **Proposition 3.1: Hölder's Inequality**

Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Suppose $p, q : 1 \leq p, q \leq +\infty$ are conjugate. Then,

$$\langle x, y \rangle_{\mathbb{R}^n} := \left|\sum_{i=1}^{n} x_i y_i\right| \leq ||x||_p \cdot ||y||_q$$

⊛ **Example 3.1**

For the case $p = 1$ or ∞ (functionally, the same case):

↪ **Lemma 3.1**

Let p, q be conjugates, and $x, y \geq 0$. Then,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Remark 3.3. *If the inequality holds, then, for some $t > 0$, let $\tilde{x} = t^{\frac{1}{p}} \cdot x, \tilde{y} = t^{\frac{1}{q}} y$. Substituting x for $\tilde{x}$ and y for $\tilde{y}$, we have*

$$\begin{aligned} &\textit{LHS: } \tilde{x}\tilde{y} = t^{\frac{1}{p}} x \cdot t^{\frac{1}{q}} y = t^{\frac{1}{p}+\frac{1}{q}} \cdot xy = xy \\ &\textit{RHS: } \cdots = t(\frac{x^p}{p} + \frac{y^q}{q}.) \end{aligned}$$

That is, we have

$$t \cdot xy \leq t \left(\frac{x^p}{p} + \frac{y^q}{q} \right),$$

hence, the inequality is preserved under multiplication by a positive scalar; moreover, the original inequality holds iff this "scaled" version holds. Hence, choosing t such that $\tilde{y} = 1$ (let $t = \left(\frac{1}{y}\right)^q$), it suffices to prove the lemma for $y = 1$.

Proof. If $x = 0$ or $y = 0$, then the entire LHS becomes 0 and we are done; assume $x, y > 0$; by the previous remark, assume wlog $y = 1$. Then, we have

$$\begin{aligned} x \cdot y \leq \frac{x^p}{p} + \frac{y^q}{q} &\iff x \cdot 1 \leq \frac{x^p}{p} + \frac{1}{q} \\ &\iff \frac{x^p}{p} - x + \frac{1}{q} =: f(x) \geq 0. \end{aligned}$$

Taking the derivative, we have

$$f'(x) = \frac{\cancel{p} x^{p-1}}{\cancel{p}} - 1 = x^{p-1} - 1$$

$$p > 1 \implies p - 1 > 0 \implies \begin{cases} f'(x) > 0 & \forall x > 1 \\ f'(x) = 0 & x = 0 \\ f'(x) < 0 & \forall 0 < x < 1 \end{cases}$$

Hence, $x = 1$ is a local minimum of the function, and thus $f(x) \geq f(1) \forall 0 < x \leq 1$. But $f(1) = \frac{1^p}{p} - 1 + \frac{1}{q} = 1 - 1 = 0$, hence $f(x) \geq 0 \forall x \geq 0$, as desired, and the inequality holds. ■

Proof. Assume $||x||_p = ||y||_q = 1$. Then,

$$\begin{aligned}
\left|\sum_{i=1}^n x_i y_i\right| &\leq \sum_{i=1}^n |x_i y_i| && \textit{(by triangle inequality)} \\
&\leq \sum_{i=1}^n \left|\frac{x_i^p}{p} + \frac{y_i^q}{q}\right| && \textit{(by lemma 3.1)} \\
&= \frac{1}{p}\left(\sum_{i=1}^n |x_i|^p\right) + \frac{1}{q}\left(\sum_{i=1}^n |y_i|^q\right) \\
&= \frac{1}{p}||x||_p^p + \frac{1}{q}||y||_q^q && \textit{(by staring)} \\
&= \frac{1}{p}\cdot 1^p + \frac{1}{q}\cdot 1^1 = \frac{1}{p} + \frac{1}{q} = 1 && \textit{(by assumption)} \\
&= ||x||_p \cdot ||y||_q,
\end{aligned}$$

and the proposition holds, in the special case $||x||_p = ||y||_q = 1$.

If $||x||_p = 0$ or $||y||_q = 0$, then $x_1 = \cdots = x_n = 0$ or $y_1 = \cdots = y_n = 0$, resp, then we'd have ($||x||_p = 0$ case)

$$0 \cdot y_1 + \cdots + 0 \cdot y_n \leq 0,$$

which clearly holds.

Assume, then, $||x||_p > 0, ||y||_q > 0$. Let $\tilde{x} := \frac{x}{||x||_p}, \tilde{y} := \frac{y}{||y||_q}$. Then,

$$||\tilde{x}||_p^p = \frac{\left(\sum_{i=1}^n |x_i|^p\right)}{||x||_p^p} = \frac{||x||_p^p}{||x||_p^p} = 1 \implies ||\tilde{x}||_p = 1.$$

The same case holds for $\tilde{y}$, hence $||\tilde{y}||_q = 1$; that is, we have "rescaled" both vectors. Hence, we can use the case we proved above for when the norms were identically 1 on $\tilde{x}, \tilde{y}$. We have:

$$\left|\sum_{i=1}^n \tilde{x}_i \tilde{y}_i\right| \leq 1$$

But by definition of $\tilde{x}, \tilde{y}$, we have

$$\left|\sum_{i=1}^n \tilde{x}_i \tilde{y}_i\right| = \left|\frac{1}{||x||_p ||y||_q}\sum_{i=1}^n x_i y_i\right| \leq 1 \implies \left|\sum_{i=1}^n x_i y_i\right| \leq ||x||_p \cdot ||y||_q,$$

and the proof is complete. ■

$\hookrightarrow$ **Proposition 3.2: Minkowski Inequality**

Let $1 \leq p \leq \infty$, $x, y \in \mathbb{R}^n$. Then,

$$||x + y||_p \leq ||x||_p + ||y||_p.$$

Remark 3.4. *This is just the triangle inequality for ℓ_p norms.*

Proof. The cases $p = 1, \infty$ are left as an exercise.

Assume $1 < p < \infty$. Then,

$$\begin{aligned} ||x+y||_p^p = \sum_{j=1}^n |x_j + y_j|^p &= \sum_{j=1}^n |x_j + y_j| \, |x_j + y_j|^{p-1} \\ &\leq \sum_{j=1}^{\infty} (|x_j| + |y_j|) \cdot |x_j + y_j|^{p-1} \\ &= \underbrace{\sum_{j=1}^n |x_j| \cdot |x_j + y_j|^{p-1}}_{:=A} + \overbrace{\sum_{j=1}^n |y_j| \cdot |x_j + y_j|^{p-1}}^{:=B} \quad \circledast \end{aligned}$$

Let $\vec{u} = (|x_1|, \cdots, |x_n|)$ and $\vec{v} = (|x_1 + y_1|^{p-1}, \cdots, |x_n + y_n|^{p-1})$, then, $A = \vec{u} \cdot \vec{v} = \langle \vec{u}, \vec{v} \rangle_{\mathbb{R}^n}$. We have

$$\begin{aligned} ||\vec{u}||_p &= \left(\sum_{i=1}^n (|x_i|^p) \right)^{\frac{1}{p}} = ||x||_p \\ ||\vec{v}||_q &= \left(\sum_{i=1}^n \left(|x_i + y_i|^{p-1} \right)^q \right)^{\frac{1}{q}} \\ &= \left[\sum_{i=1}^n \left(|x_i + y_i|^{p-1} \right)^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}} \\ &= \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\ &= ||x+y||_p^{p-1} \end{aligned}$$

where the second-to-last line follows from p, q being conjugate, hence $q = \frac{p}{p-1}$. Thus, by Hölder's Inequality, we have that

$$A = \langle \vec{u}, \vec{v} \rangle \leq ||u||_p \cdot ||v||_q = ||x||_p \cdot ||x+y||_p^{p-1}.$$

By a similar construction, we can show that

$$B \leq ||y||_p \cdot ||x+y||_p^{p-1}.$$

Thus, returning to our original inequality in $\circledast$, we have

$$\begin{aligned} ||x+y||_p^p &\leq A + B \\ &\leq ||x||_p \cdot ||x+y||_p^{p-1} + ||y||_p \cdot ||x+y||_p^{p-1} \\ &\implies ||x+y||_p \leq ||x||_p + ||y||_p, \end{aligned}$$

and the proof is complete. ■

↪ *Lecture 07; Last Updated: Tue Jan 30 12:54:59 EST 2024*

3.3 An Aside on Complete Metric Spaces

↪ **Theorem 3.1**

The sequence of centers of balls with monotonically decreasing radii is a Cauchy sequence in X.

Proof. Let $\varepsilon > 0$ and let $N : \forall j > N, r_j < \varepsilon$. Then,

$$d(x_j, x_k) < r_{\min(j,k)} = r_j$$

■

↪ **Definition 3.2: Complete Metric Space**

A metric space is complete if every Cauchy sequence converges to a limit in that space.

⊛ **Example 3.2: Examples of Complete Metric Spaces**

1. $\mathbb{R}$, p-adic integers $(\mathbb{Z}_p)$/rationals$(\mathbb{Q}_p)$.
2. $\ell_p = \{x = (x_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |x_i|^p < +\infty\}, 1 \leq p \leq +\infty$
3. $\ell_\infty = \{x = (x_i) : \sup_{i=1}^{\infty} |x_i| < +\infty\}$.

↪ **Proposition 3.3**

Hölder's Inequality and Minkowski Inequality inequalities hold for infinite sequences. that is,

1. if $x = (x_i) \in \ell_p$ and $y = (y_i) \in \ell_q$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left| \sum_{i=1}^{\infty} x_i y_i \right| \leq ||x_i||_{\ell_p} ||y_i||_{\ell_q}.$$

2. if $x, y \in \ell_p$, then

$$||x + y||_p \leq ||x||_p + ||y||_p.$$

Remark 3.5. *2. gives the triangle inequality for the $||x||_p$ norm on ℓ_p.*

Moreover,

$$\begin{aligned} ||c \cdot x||_p &= ||(c_1 x_1, \ldots, c_n x_n, \ldots)||_p \\ &= \left(\sum_{i=1}^{\infty} |c x_i|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty} |c|^p |x_i|^p \right)^{\frac{1}{p}} \\ &= (|c|^p)^{\frac{1}{p}} ||x||_p = c \cdot ||x||_p \end{aligned}$$

Proof. (of 2.) If $x, y \in \ell_p$, we have that $\sum_{i=1}^{\infty} |x_i|^p < +\infty, \sum_{i=1}^{\infty} |y_i|^p < +\infty$, so $\exists N > 0 : \sum_{i=N+1}^{\infty} |x_i|^p < \varepsilon, \sum_{i=N+1}^{\infty} |y_i|^p < \varepsilon$. Let $x_i^{(n)} = (x_1, \dots, x_n, 0, 0, \dots)$ be (x) truncated after n (finite) coordinates. This gives

$$||(x_i + y_i)^{(n)}||_p \leq ||x_i^{(n)}||_p + ||y_i^{(n)}||_p \leq ||x||_p + ||y||_p$$

by Minkowski on finite spaces. Taking $n \to \infty$ (ie, "detruncating"), we have $(x + y) \in \ell_p$, and thus $||x + y||_p \leq ||x||_p + ||y||_p$.

1. left as an exercise. ■

↪ **Proposition 3.4**

Let $1 \leq p \leq +\infty$, and $||x||_\infty = \sup_{i=1}^{\infty} |x_i| = A < +\infty, ||y||_\infty = \sup_{i=1}^{\infty} |y_i| = B < +\infty$. Then, the triangle inequality $||x + y||_\infty \leq ||x||_\infty + ||y||_\infty$ holds.

Proof. We have

$$\sup_{i=1}^{\infty} |x_i + y_i| \leq \sup_{i=1}^{\infty} (|x_i| + |y_i|) \leq \sup_{i=1}^{\infty} |x_i| + \sup_{i=1}^{\infty} |y_i| = ||x||_\infty + ||y||_\infty.$$

■

↪ **Proposition 3.5**

$||x||_\infty := \sup_{i=1}^{\infty} |x_i|$ is a well-defined norm on ℓ_∞.

Proof. The triangle inequality is prove in proposition 3.4. The remainder of the requirements are left as an exercise. ■

↪ **Proposition 3.6**

$\ell_p \subseteq \ell_q$ if $p < q$.

Proof. Let $x \in \ell_p$. If $\sum_{i=1}^{\infty} |x_i|^p < +\infty$, then $\exists N : \forall i \geq N, |x_i| \leq 1$. Then,

$$\begin{aligned} \sum_{i \geq N} |x_i|^q \leq \sum_{i \geq N} |x_i|^p < \infty \\ \implies \sum_{i=1}^{\infty} |x_i|^q < +\infty \implies x \in \ell_q \\ \implies \ell_p \subseteq \ell_q \end{aligned}$$

■

↪ **Theorem 3.2: ℓ_p complete**

The space ℓ_p is complete for all $1 \leq p \leq +\infty$.

Equivalently, if $(x^1), (x^2), \dots, (x^n)$ is a Cauchy sequence in ℓ^p, $\exists y \in \ell^p$ s.t. $x^n \to y$ as $n \to \infty$.

Proof. ■

$\hookrightarrow$ *Lecture 08; Last Updated: Tue Jan 30 09:50:44 EST 2024*