Louis Meunier

Analysis I, II MATH254

Contents

1 Logic, Sets, and Functions

1.1 Mathematical Induction & The Naturals

The **natural numbers**, $\mathbb{N} = \{1, 2, 3, \dots\}$, are specified by the 5 **Peano Axioms**:

(1) $1 \in \mathbb{N}$ [1]

(2) every natural number has a successor in $\mathbb{N}$

(3) 1 is not the successor of any natural number

(4) if the successor of x is equal to the successor of y, then x is equal to y [2]

(5) **the axiom of induction**

[1] using 0 instead of 1 is also valid, but we will use 1 here.

[2] axioms (2)-(4) can be equivalently stated in terms of a successor function $s(n)$ more rigorously, but won't here

The **Axiom of Induction** (AI), can be stated in a number of ways.

Axiom 1.1 (AI.i). *Let $S \subseteq \mathbb{N}$ with the properties:*

(a) $1 \in S$

(b) if $n \in S$, *then* $n + 1 \in S$[3]

then $S = \mathbb{N}$.

[3] *(a)* is called the **inductive base**; *(b)* the **inductive step**. All AI restatements are equivalent in having both of these, and only differentiate on their specific values.

Example 1.1. *Prove that, for every* $n \in \mathbb{N}$, $1+2+\cdots+n = \frac{n(n+1)}{2} (\equiv (1))$

Proof (via AI.i). Let S be the subset of $\mathbb{N}$ for which (1) holds; thus, our goal is to show $S = \mathbb{N}$, and we must prove *(a)* and (b) of AI.i.

- by inspection, $1 \in S$ since $1 = \frac{1(1+1)}{2} = 1$, proving *(a)*
- assume $n \in S$; then, $1+2+\cdots+n = \frac{n(n+1)}{2}$ by definition of S. Adding $n+1$ to both sides yields:

$$\begin{aligned}
1+2+\cdots+n+(n+1) &= \frac{n(n+1)}{2} + (n+1) && (1)\\
&= (n+1)(\frac{n}{2}+1) && (2)\\
&= \frac{(n+1)(n+2)}{2} && (3)\\
&= \frac{(n+1)((n+1)+1)}{2} && (4)
\end{aligned}$$

Line (4) is equivalent to statement (1) (substituting n for $n+1$), and thus if $n \in S$, then $n+1 \in S$ and *(b)* holds. Thus, by AI.i, $S = \mathbb{N}$ and $1+2+\cdots+n = \frac{n(n+1)}{2}$ holds $\forall n \in \mathbb{N}$. ∎

Example 1.2. *Prove (by induction), that for every* $n \in \mathbb{N}$, $1^3+2^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Proof. Follows a similar structure to the previous example. Let S be the subset of $\mathbb{N}$ for which the statement holds. $1 \in S$ by inspection (*(a)* holds), and we prove *(b)* by assuming $n \in S$ and showing $n+1 \in S$ (algebraically). Thus, by AI.i, $S = \mathbb{N}$ and the statement holds $\forall n \in \mathbb{N}$. ∎

This can also be proven directly (Gauss' method).

Proof (Gauss' method). Let $A(n) = 1+2+3+\cdots+n$. We can write $2 \cdot A(n) = 1+2+3+\cdots+n+1+2+3+\cdots+n$. Rearranging terms (1 with n, 2 with $n-1$, etc.), we can say $2 \cdot A(n) = (n+1)+(n+1)+\cdots$, where $(n+1)$ is repeated n times; thus, $2 \cdot A(n) = n(n+1)$, and $A(n) = \frac{n(n+1)}{2}$. ∎

Axiom 1.2 (AI.ii). *Let* $S \subseteq \mathbb{N}$ *s.t.*

(a) $m \in S$

(b) $n \in S \implies n+1 \in S$

then $\{m, m+1, m+2, \ldots\} \subseteq S$.

Example 1.3. *Using AI.ii, prove that for $n \geq 2$, $n^2 > n+1$*

Proof. Again, very similar to the previous induction examples. Take S to be the subset of $\mathbb{N}$ for which the statement holds. *(a)* of AI.ii holds by inspection (where $m = 2$), and *(b)* holds by assuming $n \in S$ and showing that $n+1 \in S$. Thus, $S = \{2, 3, 4, \dots\}$, and the statement holds $\forall n \geq 2$. ■

Axiom 1.3 (Principle of Complete Induction, AI.iii). *Let $S \subseteq \mathbb{N}$ s.t.*

(a) $1 \in S$

(b) *if* $1, 2, \dots, n-1 \in S$, *then* $n \in S$

then $S = \mathbb{N}$.

Finally, combing AI.ii and AI.iii;

Axiom 1.4 (AI.iv). *Let $S \subseteq \mathbb{N}$ s.t.:*

(a) $m \in S$

(b) *if* $m, m+1, \dots, m+n \in S$, *then* $m+n+1 \in S$

then $\{m, m+1, m+2, \dots\} \subseteq S$.

Theorem 1.1 (Fundamental Theorem of Arithmetic). *Every natural number n can be written as a product of one or more primes.* [4]

[4] 1 is not a prime number

Proof of Theorem 1.1. Let S be the set of all natural numbers that can be written as a product of one or more primes. We will use AI.iv to show $S = \{2, 3, \dots\}$.

- *(a)* holds; 2 is prime and thus $2 \in S$
- suppose that $2, 3, \dots, 2+n \in S$. Consider $2 + (n+1)$:
  - if $2 + (n+1)$ is *prime*, then $2 + (n+1) \in S$, as all primes are products of 1 and themselves and are thus in S by definition.
  - if $2 + (n+1)$ is *not prime*, then it can be written as $2 + (n+1) = a \cdot b$ where $a, b \in \mathbb{N}$, and $1 < a < 2 + (n+1)$ and $1 < b < 2 + (n+1)$. By the definition of S, $a, b \in S$, and can thus be written as the product of primes. Let $a = p_1 \cdot \dots \cdot p_l$ and $b = q_1 \cdot \dots \cdot q_j$, where the p's and q's are prime and $l, j \geq 1$. Then, $a \cdot b$ is a product of primes, and thus so is $2 + (n+1)$. Thus, $2 + (n+1) \in S$, and by AI.iv, $S = \{2, 3, 4, \dots\}$

■

1.2 Extensions: Integers, Rationals, Reals

Consider the set of naturals $\mathbb{N} = \{1, 2, 3, \dots\}$. Adding 0 to $\mathbb{N}$ defines $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We define the **integers** as the set $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, or the set of all positive and negative whole numbers.

Within $\mathbb{Z}$, we can define multiplication, addition and subtraction, with the neturals of 1 and 0, respectively. However, we cannot define division, as we are not guaranteed a quotient in $\mathbb{Z}$. This necessitates the **rationals**, $\mathbb{Q}$. We define

$$\mathbb{Q} = \{\frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}.$$

On $\mathbb{Q}$, we have the familiar operations of multiplication, addition, subtraction and properties of associativity, distributivity, etc. We can also define division, as $\frac{\frac{p}{q}}{\frac{p'}{q'}} = \frac{pq'}{qp'}$.

We can also define a relation $<$ between fractions, such that

- $x < y$ and $y < z \implies x < z$
- $x < y \implies x + z < y + z$

$\mathbb{Q}$, together with its operations and relations above, is called an **ordered field**.

1.2.1 The Insufficiency of the Rationals

We can consider historical reasoning for the extension of $\mathbb{Q}$ to $\mathbb{R}$. Consider a right triangle of legs a, b and hypotenuse c. By the Pythagorean Theorem, $a^2 + b^2 = c^2$. Consider further the case there $a = b = 1$, and thus $c^2 = 2$. Does c exist in $\mathbb{Q}$?

Proposition 1.1. $c^2 = 2$, $c \notin \mathbb{Q}$.

Proof of Proposition 1.1. Suppose $c \in \mathbb{Q}$. We can thus write $c = \frac{p}{q}$, where[5] $p, q \in \mathbb{N}$, and p, q share no common divisors, ie they are in "simplest form". Notably, p and q cannot *both* be even (under our initial assumption), as they would then share a divisor of 2. We write

$$\begin{aligned} c &= \frac{p}{q} \\ c^2 = 2 &= \frac{p^2}{q^2} \\ 2q^2 &= p^2 \end{aligned}$$

$p \in \mathbb{N} \implies p^2 \in \mathbb{N}$, and thus p^2, and therefore[6] p, must be divisible by 2 ($\implies$ p even). Therefore, we can write $p = 2p_1, p_1 \in \mathbb{N}$, and thus $2q^2 = (2p_1^2)^2 \implies q^2 = 2p_1^2$. By the same reasoning, q must now be even as well, contradicting our initial assumption that p and q share no common divisors. Thus, $c \notin \mathbb{Q}$. ∎

[5] Note that in the definition of $\mathbb{Q}$, p, q are defined to be in $\mathbb{Z}$; however, as we are using a

1.3 Sets & Set Operations

- $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- $\bigcup_{i=1}^{\infty} A_n = \bigcup_{n \in \mathbb{N}} A_n = \{x : x \in A_n \text{ for some } n \in \mathbb{N}\}$
- $\bigcap_{i=1}^{\infty} A_n = \bigcap_{n \in \mathbb{N}} A_n = \{x : x \in A_n \forall n \in \mathbb{N}\}$
- $A^C = \{x : x \in X \text{ and } x \notin A\}$[7]

[7] X is often omitted if it is clear from context.

Theorem 1.2 (De Morgan's Theorem(s)). *Let A, B be sets. Then,*

$$(a) \qquad (A \cap B)^C = A^C \cup B^C$$

and

$$(b) \qquad (A \cup B)^C = A^C \cap B^C.$$

Proof of Theorem 1.2. (b) *(A similar argument follows…)*

■

Proposition 1.2.

$$(a) \left(\bigcap_{n=1}^{\infty} A_n\right)^C = \bigcup_{n=1}^{\infty} A_n^C$$

$$(b) \left(\bigcup_{n=1}^{\infty} A_n\right)^C = \bigcap_{n=1}^{\infty} A_n^C$$

Proof of Proposition 1.2. Consider Proposition (b). Working from the left-hand side, we have

$$\begin{aligned}
\left(\bigcup_{n=1}^{\infty} A_n\right)^C &= \{x : x \notin \bigcup A_n\} \\
&= \{x : x \notin A_n \forall n \in \mathbb{N}\} \\
&= \bigcap \{x : x \notin A_n\} \\
&= \bigcap A_n^C
\end{aligned}$$

(a) can be logically deduced from this result. Consider the RHS, $\bigcup A_n^C$. Taking the complement:

$$\begin{aligned}
\left(\bigcup A_n^C\right)^C &\overset{\text{via (b)}}{=} \bigcap A_n^{C^C} \\
&= \bigcap A_n
\end{aligned}$$

Taking the complement of both sides, we have $\bigcup A_n^C = \left(\bigcap A_n\right)^C$, proving (a). ■

1.4 Functions

Definition 1.1. *Let A, B be sets. A* function f *is a rule assigned to each $x \in A$ a corresponding unique element $f(x) \in B$. We denote*

$$f : A \to B.$$

Definition 1.2. *The* domain *of a function $f : A \to B$, denoted Dom$(f) = A$. The* range *of f, denoted Ran$(f) = \{f(x) : x \in A\}$. Clearly, Ran$(f) \subseteq B$, though equality is not necessary.*

Example 1.4. *The function $f(x) = \sin x$, $f : \mathbb{R} \to [-1, 1]$. Here, Dom$(f) = \mathbb{R}$, and Ran$(f) =$* $[-1, 1]$.

Example 1.5 (Dirichlet Function). [8] $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \notin \mathbb{Q} \end{cases}$. *Despite not having a true "explicit" formula, so to speak, this is still a valid function (under modern definitions).*

[8] Look up a graph of this function. Its beautiful. It's also interesting to note that its integral is simply 0.

1.4.1 Properties of Functions

Proposition 1.3. *Let $f : A \to B$, $C \subseteq A$, $f(C) = \{f(x) : x \in C\}$. We claim $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.*

Proof. We will prove this by showing (1) $\subseteq$ and (2) $\supseteq$.

(1) $y \in f(C_1 \cup C_2) \implies$ for some $x \in C_1 \cup C_2, y = f(x)$. This means that either for some $x \in C_1, y = f(x)$, or for some $x \in C_2, y = f(x)$. This implies that either $y \in f(C_1)$, or $y \in f(C_2)$, and thus y *must* be in their union, ie $y \in C_1 \cup C_2$.

(2) $y \in f(C_1) \cup f(C_2) \implies y \in f(C_1)$ or $y \in f(C_2)$. This means that for some $x \in C_1, y = f(x)$, or for some $x \in C_2, y = f(x)$. Thus, x *must* be in $C_1 \cup C_2$, and for some $x \in C_1 \cup C_2, y = f(x) \implies y \in f(C_1 \cup C_2)$.

(1) and (2) together imply that $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$. ■

Example 1.6. *Let $A_n = 1, 2, \ldots$ be a sequence of sets. Prove that $f(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f(A_n)$.*

Proof. Let $y \in f(\bigcup_{n=1}^{\infty} A_n)$. This implies that $\exists x \in \bigcup_{n=1}^{\infty} A_n$ s.t. $f(x) = y$. This implies that $x \in A_n$ for some n, and $y \in f(A_n)$ for that same "some" n, and thus y must be in the union of all possible $f(A_n)$, ie $y \in \bigcup f(A_n)$. This shows $\subseteq$, use similar logic for the reverse. ■

Proposition 1.4. $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$ [9]

[9]NB: the reverse is not always true, ie these sets are not always equal; "lack" of equality is more "common" than not.

Proof. $y \in f(C_1 \cap C_2) \implies$ for some $x \in C_1 \cap C_2, y = f(x)$. This implies that for some $x \in C_1, y = f(x)$ **and** for some $x \in C_2, y = f(x)$. Note that this does *not* imply that these x's are the same, ie this reasoning is not reversible as in the previous union case. This implies that $y \in f(C_1)$ and $y \in f(C_2) \implies y \in f(C_1) \cap f(C_2)$. ■

Example 1.7. *Prove that if $A_n, n = 1, 2, \ldots, f(\bigcap_{n=1}^{\infty} A_n) \subseteq \bigcap_{n=1}^{\infty} f(A_n)$.*

Proof (Sketch). Use the same idea as in Example 1.6, but, naturally, with intersections. ■

Example 1.8. *Take $f(x) = \sin x$, $A = \mathbb{R}, B = \mathbb{R}$, and take $C_1 = [0, 2\pi], C_2 = [2\pi, 4\pi]$. Then, $f(C_1) = [-1, 1]$, and $f(C_2) = [-1, 1]$. But $C_1 \cap C_2 = \{2\pi\}$; $f(\{2\pi\}) = \{\sin 2\pi\} = \{0\}$, and thus $f(C_1 \cap C_2) = \{0\}$, while $f(C_1) \cap f(C_2) = [-1, 1]$, as shown in Proposition 1.4.*

Definition 1.3 (Inverse Image of a Set). *Let $f : A \to B$ and $D \subseteq B$. The* inverse image *of D by F is denoted $f^{-1}(D)$*[10] *and is defined as*

$$f^{-1}(D) = \{x \in A : f(x) \in D\}.$$

[10]Note that this is **not** equivalent to the typical definition of an inverse *function*; f^{-1} may not exist

Example 1.9. $A = [0, 2\pi], B = \mathbb{R}, f(x) = \sin x, D = [0, 1]$.

$$f^{-1}(D) = \{x \in A : f(x) \in D\} = \{x \in [0, 2\pi] : \sin(x) \in [0, 1]\} = [0, \pi].$$

Proposition 1.5. *Given function f and sets D_1, D_2,*

(a) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$

(b) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$[11]

[11]Just see next proposition; if you really need convincing, just use 2 rather than ∞ as the upper limit of the unions/intersections and use the same proof.

Proposition 1.6. *Let $A_n, n = 1, 2, 3 \ldots$. Then,*

(a) $f^{-1}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f(A_n)$

(b) $f^{-1}(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} f(A_n)$

Proof. [12]

(a)

$$\begin{aligned} x \in f^{-1}(\bigcup_{n=1}^{\infty} A_n) &\iff f(x) \in \bigcup_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for some } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for some } n \in \mathbb{N} \\ &\iff x \in \bigcup_{n=1}^{\infty} f^{-1}(A_n) \end{aligned}$$

(b)

$$\begin{aligned} x \in f^{-1}(\bigcap_{n=1}^{\infty} A_n) &\iff f(x) \in \bigcap_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for all } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for all } n \in \mathbb{N} \\ &\iff x \in \bigcap_{n=1}^{\infty} f^{-1}(A_n)^{13} \end{aligned}$$

■

[13]This is a "proof by definitions" as I like to call it.

[13]Similar proof can be used to prove Proposition 1.5, less generally.

Remark 1.1. *$f : A \to B$, $A_1 \subseteq A$. Given $f(A_1^C)$ and $f(A_1)^C$, there is **no general relation** between the two.*

For instance, take $A = [0, 6\pi], B = [-1, 2], C = [0, 2\pi]$, and $f(x) = \sin x$. Then, $f(C) = [-1, 1]$, and $f(C^C) = f([-1, 0)) = [-1, 1]$, but $f(C)^C = [-1, 1]^C = (1, 2]$, and $f(C^C) \neq f(C)^C$; in fact, these sets are disjoint.

Proposition 1.7. *Let $f : A \to B$ and let $D \subseteq B$. Then $f^{-1}(D^C) = [f^{-1}(D)]^C$.*

Proof.

$$f^{-1}(D^C) = \{x : f(x) \in D^C\} = \{x : f(x) \notin D\}$$

$$[f^{-1}(D)]^C = [\{x : f(x) \in D\}]^C = \{x : x \notin f^{-1}(D)\} = \{x : f(x) \notin D\}$$

■

1.5 Reals

Axiom 1.5 (Of Completeness)**.** *Any non-empty subset of $\mathbb{R}$ that is bound from above has at least one upper bound (also called the supremum).*

In other words; let $A \subseteq \mathbb{R}$ and suppose A is bounded from above (A has at a least upper bound). Then $\sup(A)$ exists.

Real numbers, algebraically have the same properties as the rationals; we have addition, multiplication, inverse of non-zero real numbers, and we have the relation $<$. All together, $\mathbb{R}$ is an ordered field.

Definition 1.4. *Let $A \subseteq \mathbb{R}$. A number $b \in \mathbb{R}$ is called an* ***upper bound*** *for A if for any $x \in A$, $x \leq B$.*

A number $l \in \mathbb{R}$ is called a ***lower bound*** *for A if for any $x \in A$, $x \geq l$.*

Definition 1.5 (The Least Upper Bound)**.** *Let $A \subseteq \mathbb{R}$. A real number s is called the* ***least upper bound*** *for A if the following holds:*

(a) s is an upper bound for A

(b) if b is any other upper bound for A, then $s \leq b$.

The least upper bound of a set A is unique, *if it exists; if s and s' are two least upper bounds, then by (a), s and s' are upper bound for A, and by (b), $s \leq s'$ and $s' \leq s$, and thus $s = s'$.*

This least upper bound is called the supremum *of A, denoted $\sup(A)$.*

Definition 1.6 (The Greatest Lower Bound)**.** *Let $A \subset \mathbb{R}$. A number $i \in \mathbb{R}$ is called the* ***greatest lower bound*** *for A if the following holds:*

(a) i is a lower bound for A

(b) if l is any other lower bound for A, then $i \geq l$.

If i exists, it is called the infimum *of A and is denoted $i = \inf(A)$, and is unique by the same argument used for $\sup(A)$.*

Proposition 1.8. *Let $A \subseteq \mathbb{R}$ and let s be an upper bound for A. Then $s = \sup(A)$ iff for any $\varepsilon > 0$, there exists $x \in A$ s.t. $s - \varepsilon < x$.*

Proof. We have two statements:

I. $s = \sup(A)$;

II. For any $\epsilon > 0$, $\exists x \in A$ s.t. $s - \epsilon < x$;

and we desire to show that I $\iff$ II.

- I $\implies$ II: Let $\epsilon > 0$. Then, since $s = \sup(A)$, $s - \epsilon$ *cannot* be an upper bound for A (as s is the least upper bound, and thus $s - \epsilon < s$ cannot be an upper bound at all). Thus, there exists $x \in A$ such that $s - \epsilon < x$, and thus if I holds, II must hold.

- II $\implies$ I: suppose that this does not hold, ie II holds for an upper bound s for A, but $s \neq \sup(A)$. Then, there exists some upper bound b of A s.t. $b < s$. Take $\epsilon = s - b$. $\epsilon > 0$, and since II holds, there exists $x \in A$ such that $s - \epsilon < x$. But since $s - \epsilon = b$ and thus $b < x$, then b cannot be an upper bound for A, contradicting our initial condition. So, if II $\implies$ I does *not* hold, we have a "impossibility", ie a value b which is an upper bound for A which cannot be an upper bound, and thus II $\implies$ I.

■

Proposition 1.9. *Let $A \subseteq \mathbb{R}$ and let i be a lower bound for A. Then $i = \inf(A) \iff$ for every $\epsilon > 0$ there exists $x \in A$ s.t. $x < i + \epsilon$.*[14]

[14]Use similar argument to proof of previous proposition.

Remark 1.2. *Axiom 1.5 can also be expressed in terms of infimum. Define $-A = \{-x : x \in A\}$. Then, if b is an upper bound for A, then $b \geq x \forall x \in A$, then $-b \leq -x \forall x \in A$, ie -b is a lower bound of $-A$. Similarly, if l is a lower bound for A, $-l$ is an upper bound for $-A$.*

Thus, if A is bounded from above, then

$$-\sup(A) = \inf(-A),$$

and if A is bounded from below,

$$-\inf(A) = \sup(-A).$$

Axiom 1.6 (AC (infimum))**.** *Let $A \subseteq \mathbb{R}$; if A bounded from below, $\inf(A)$ exists.*

Definition 1.7 (max, min)**.** *Let $A \subseteq \mathbb{R}$. An $M \in A$ is called a* maximum *of A if for any $x \in A$, $x \leq M$. M is an upper bound for A,* ***but also*** *$M \in A$.*

If M exists, then $M = \sup(A)$; M is an upper bound, and if b any other upper bound, then $b \geq M$, because $M \in A$, and thus $M = \sup(A)$.

NB: $M = \max(A)$ ***need not*** *exist, while $\sup(A)$ must exist. Consider $A = [0, 1)$; $\sup(A) = 1$, but there exists no $\max(A)$.*

The same logic exists for the existence of minimum vs infimum (consider $(0, 1)$, with no maximum nor minimum).

Theorem 1.3 (Nested interval property of $\mathbb{R}$). *Let $I_n = [a_n, b_n] = \{x : a_n \leq x \leq b_n\}, n = 1, 2, 3 \ldots$ be an infinite sequence of bounded, closed intervals s.t.*

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots I_n \subseteq I_{n+1} \subseteq \ldots$$

Then, $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$ (note that this does not *hold in $\mathbb{Q}$).*

Proof. We have $I_n = [a_n, b_n], I_{n+1} = [a_{n+1}, b_{n+1}], \ldots$. And the inclusion $I_n \subseteq I_{n+1}$. $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n, \forall n \geq 1$. So, the sequence a_n (left-end) is increasing, and the sequence b_n (right-end) is decreasing.

We also have that for any $n, k \geq 1$, $a_n \leq b_k$. We see this by considering two cases:

- Case 1: $n \leq k$, then $a_n \leq a_k$ (as a_n is increasing), and thus $a_n \leq a_k \leq b_k$.
- Case 2: $n > k$, then $a_n \leq b_n \leq b_k$ (again, as b_n is decreasing).

Let $A = \{a_n : n \in \mathbb{N}\}$. Then, A is bounded from above by *any* b_k (as in our inequality we showed above). Let $x = \sup(A)$, which must exist by Axiom 1.5.

Note that as a result, $x \geq a_n$ for all n, and for all k, $x \leq b_k$, as x is the lowest upper bound and must be $\leq$ all other upper bounds, and so for all $n \geq 1$, $a_n \leq x \leq b_n$, ie $x \in I_n \forall n \geq 1$, and thus $x \in \bigcap_{n=1}^{\infty} I_n$ and so $\bigcap_{n=1}^{\infty} \neq \varnothing$. ■

Remark 1.3. *The proof above emphasized the left-end points; it can equivalently be proven via the right-end points, and using $y = \inf(\{b_n : n \in \mathbb{N}\}) = \inf(B)$, rather than $\sup(A)$, and showing that $y \in \bigcap I_n$.*

Remark 1.4. *Note too that, if $x = \sup(A)$ and $y = \sup(B)$, then $x, y \in \bigcap_{n=1}^{\infty} I_n$; in fact, $\bigcap_{n=1}^{\infty} I_n = [x, y]$.*

Remark 1.5. *The intervals I_n* must *be closed; if not, eg $I_n = (0, \frac{1}{n})$, then $\bigcap_{n=1}^{\infty} I_n = \varnothing$.*

1.6 Density of Rationals in Reals

Proposition 1.10. *(a) For any $x \in \mathbb{R}$, there exists a natural number n s.t. $n > x$.*

(b) For any $y \in \mathbb{R}$ satisfying $y > 0$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < y$.

Remark 1.6. *(b) follows from (a) by taking $x = \frac{1}{y}$ in (a), then $\exists n \in \mathbb{N}$ s.t. $n > \frac{1}{y} \implies \frac{1}{n} < y$, and thus we need only prove (a).*

2 Appendix

2.1 Tutorials

2.1.1 Tutorial I (Sept 13)

1. We say n odd if $\exists k, n = 2k+1$. Prove that the product of two odds is odd.

   Proof. Take two odd integers, $n_1 = 2k+1$ and $n_2 = 2j+1$. The product $n_1 \times n_2 = (2k+1)(2j+1) = 4kj + 2(k+j) + 1$. We have, then

   $$\underbrace{4kj + 2(k+j)}_{\text{even}} + 1.$$

   Even + odd = odd, thus odd. ■

2. **Proof by Contrapositive:** $P \implies Q \equiv \neg Q \implies \neg P$. Let $q \in \mathbb{Q}$. Prove: If $x \in \mathbb{R} \setminus \mathbb{Q}$, then $q + x$ is irrational.

   Proof (contrapositive). Let $q + x$ be rational. The sum of rationals is rational, and thus $q, x \in \mathbb{Q}$, and thus $x \notin \mathbb{R} \setminus \mathbb{Q}$. ■

3. **Proofs by Induction**

   (a) Prove that $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

   . Let P_n be the statement that $1^3 + \cdots = \left(\frac{n(n+1)}{2}\right)^2$. P_0 holds as $1 = \frac{(1)(2)}{2}^2 = 1$. Let P_n hold:

   $$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

   Adding $(n+1)^3$ to both sides:

   $$1^3 + 2^3 + \cdots + n^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

   Focusing on the RHS:

   $$\begin{aligned}\left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 &= (n+1)^2\left(\frac{n^2}{4} + (n+1)\right) \\ &= (n+1)^2\left(\frac{n^2+4n+4}{4}\right) \\ &= (n+1)^2\left(\frac{(n+2)^2}{4}\right) \\ &= \left(\frac{(n+1)(n+2)}{2}\right)^2 \qquad \equiv P_{n+1}\end{aligned}$$

Thus, by AI, P_n holds for all $n \in \mathbb{N}$. ■

(b) We have an 8×8 checker board. We remove the top-left and bottom-right squares. Prove that the remaining board cannot be covered by 2×1 dominoes.

Proof. Note that every domino must cover a black square and a white square. However, the board is missing 2 white squares (say). Thus, there are 62 squares (32 black, 30 white), and we would need *exactly* 31 dominos (62/2). Each requires 1 black, 1 white tile, and thus we will run out of white squares before we reach our 31 dominos, and thus we cannot cover the board. ■

(c) Take F_n to represent the nth Fibonacci number. Let $\varphi = \frac{1+\sqrt{5}}{2}$. Show that $F_n > \varphi^{n-2} \forall n \geq 3$.

Proof. Let P_n represent the "truth" of the given statement. $P_3 : F_3 = F_2 + F_1 = 1 + 1 = 2$. $\varphi^1 = \varphi$; clearly $2 > \frac{1+\sqrt{5}}{2}$. Note that we should also prove P_4, P_5 for use in our induction.
$P_4 : (\frac{1+\sqrt{5}}{2})^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} < 3$.
$P_5 : (\frac{1+\sqrt{5}2}{)}^3 \cdots < 5$
Take P_{n-1}, P_n to hold, ie $F_{n-1} > \varphi^{n-3}$ and $F_n > \varphi^{n-2}$.

$$\begin{aligned} F_{n+1} = F_n + F_{n-1} &> \varphi^{n-2} + \varphi^{n-3} \\ &= \varphi^{n-3}(\underbrace{\varphi + 1}_{=\varphi^2}) \\ &= \varphi^{n-1}, \end{aligned}$$

as desired, Noting that $\varphi + 1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{1+\sqrt{5}+2}{2} = \ldots \varphi^2$. ■

(d) $a_1 = 1, a_2 = 8, a_n = a_{n-1} + 2a_{n-2}$. Prove $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$.

Proof. $a_1 = 1 = 3 \cdot 2^0 + 2(-1)^1 = 3 - 2 = 1$ $a_2 = 8 = 3 \cdot 2^1 + 2(-1)^2 = 6 + 2 = 8$ So, P_1, P_2 holds. Assume P_n, P_{n+1} holds. Then, we have $a_n = 3 \cdot 2^{n-1} + 2(-1)^n$ and so:

$$\begin{aligned} a_{n+1} &= 3 \cdot 2^{n-1} + 2(-1)^n + 2 \cdot \left(3 \cdot 2^{n-2} + 2(-1)^{n-1}\right) \\ &= \cdots = 3 \cdot 2^n + 2(-1)^{n+1} \end{aligned}$$

Thus, proven. ■

4. Show $A \setminus (B \setminus A) = A$.

Proof. Let $x \in A \setminus (B \setminus A)$. x must be in A, but not $B \setminus A$. Thus, x is in A, but not in B. Thus, LHS $\subseteq$ RHS.

Let $x \in A$. Thus, $x \notin B \setminus A$, and thus $x \in A \setminus (B \setminus A)$, and so $A \subseteq A \setminus (B \setminus A)$. Thus, LHS = RHS.

■

5. $A_n = \{nk : k \in \mathbb{N}\}, n \geq 2$. Find $\bigcup_{n=2}^{\infty} An \bigcap_{n=2}^{\infty} A_n$.

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$$\bigcup_{n=2}^{\infty} A_n = \bigcup \{2k, 3k, 4k, \dots\} = \{n : n \geq 2, n \in \mathbb{N}\} = \mathbb{N} \setminus \{1\}$$

$$\bigcap_{n=2}^{\infty} A_n = \varnothing \textit{ consider just } n = 2, n = 3 \text{ cases...}$$

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