# **MATH458** - Differential Geometry

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### Contents

# §1 Some Review

We will work in $\mathbb{R}^n$, usually with $n = 2, 3$. For vectors $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in \mathbb{R}^n$, we denote the dot product

$$v \cdot w = \sum_{i=1}^{n} v_i w_i.$$

More generally, an *inner product* on $\mathbb{R}^n$ is any function $b : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that is symmetric, bilinear and positive definite. For instance, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear and invertible $b_T(v, w) := T(v) \cdot T(w)$ a new inner product. In fact, it turns out every inner product on $\mathbb{R}^n$ is of this form; this implies that every inner product is just a coordinate-change away from the dot product.

We will say a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is *orthogonal* if it is inner product preserving, i.e. $T(v) \cdot T(w) = v \cdot w$ for every $v, w \in \mathbb{R}^n$.

**Exercise 1.1**: Show that $T$ is inner product preserving iff it is norm preserving ($\|Tv\| = \|v\|$) iff it is distance preserving ($\|T(v - w)\| = \|v - w\|$).

**Exercise 1.2**: Show that if $T$ orthogonal, it is a bijection with determinant $\pm 1$.

We say $T : \mathbb{R}^n \to \mathbb{R}^n$, linear, is *orientation preserving* if $\det(T) > 0$.

↪**Definition 1.1** (Rigid Motion): A function $M : \mathbb{R}^n \to \mathbb{R}^n$ is called a *rigid motion* if there exists an $a \in \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ orthogonal and orientation preserving such that

$$M(v) = a + Tv, \quad \forall v \in \mathbb{R}^n.$$

We view the space $\mathbb{E}^n$ as $\mathbb{R}^n$ equipped with the Euclidean distance, which we'll denote $d_{\mathbb{E}}$ or $d$ if no confusion arises, *up to rigid motions*. In practice, this means working in $\mathbb{E}^n$ has no distinguished origin point or coordinate axes. However, also in practice, we will make the identification $\mathbb{E}^n \simeq \mathbb{R}^n$ by picking an origin and axes, as we will see.

However, working in $\mathbb{E}^n$, abstractly, still preserves orientation and distance, since these are both preserved under rigid motions.

For $r > 0$ and $\rho \in \mathbb{E}^n$, we write $\mathbb{D}_r(\rho)$ for the open unit disk, and $\mathbb{D}^n := \mathbb{D}_1(0) \subset \mathbb{R}^n$.

↪**Theorem 1.1** (Heine-Borel): $C \subset \mathbb{E}^n$ compact iff closed and bounded.

**Exercise 1.3**: Let $r' > r > 0$ and $\rho \in \mathbb{E}^n$. Let $f : \mathbb{D}_{r'}(\rho) \to \mathbb{E}^n$ be continuous. Show that $f|_{\mathbb{D}_r(\rho)}$ uniformly continuous.

We'll denote the derivative of a function $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$ at a point $a$ by $D_a f : \mathbb{R}^n \to \mathbb{R}^m$, which is represented by the Jacobian $m \times n$ matrix $J(f)_a = \left( \frac{\partial f}{\partial x_1}|_a, \dots, \frac{\partial f}{\partial x_n}|_a \right)$.

↪**Definition 1.2**: We will say $f : \mathcal{U} \to \mathbb{R}^m$ is $C^k$ on $\mathcal{U}$ if all the $k$th order partial derivatives of all of the component functions of $f$ are continuous. We say $f$ in $C^\infty$ if it is in $C^k$ for every $k \geq 1$. We write $C^0$ for the space of continuous functions.

**Remark 1.1**: $C^{k+1} \Rightarrow C^k$

## §2 Curves

↪**Definition 2.1** (Parametrized curve/path): A *parametrized curve/path* in $\mathbb{E}^n$ is a continuous function

$$\gamma : I \to \mathbb{E}^n,$$

where $I \subset \mathbb{R}$ an interval. We say $\gamma$ *compact* if $I$ is compact.

↪**Definition 2.2** ((Regular) $C^k$ parametrized curve): Fix coordinates in $\mathbb{E}^n$. Then, a (regular) $C^k$ parametrized curve is a parametrized curve in which $\gamma \in C^k(I)$ (and for which $\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) \neq 0 \forall t \in I$).

**Exercise 2.1**: Regularity and differentiability is preserved under rigid motion, i.e. if $\gamma$ a (regular) $C^k$ parametrized curve and $M$ a rigid motion on $\mathbb{R}^n$, then $\tilde{\gamma} := M \circ \gamma$ also (regular) $C^k$.

↪**Definition 2.3**: Given a curve $\gamma$, we define

- the *velocity*, $v = \frac{\mathrm{d}\gamma}{\mathrm{d}t} : I \to \mathbb{R}^n$
- the *acceleration*, $\alpha = \frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2} : I \to \mathbb{R}^n$
- the *speed*, $\sigma = \|v\| = \left\|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\| : I \to \mathbb{R}$,

whenever each of these quantities all exist.

**Exercise 2.2**: Speed is preserved by rigid motions.

↪**Definition 2.4**: Let $\gamma$ be a $C^1$ curve. The *arclength* of $\gamma$ is defined by

$$\ell(\gamma) := \int_I \sigma(t)\,\mathrm{d}t.$$

⊛ **Example 2.1**: Let $p, q \in \mathbb{E}^2$ with $d_{\mathbb{E}}(p, q) = 3$. Suppose $\gamma : [a, b] \to \mathbb{E}^2$ is a $C^1$-path with $\gamma(a) = p, \gamma(b) = q$. Prove that $\ell(\gamma) \geq 3$, with equality holding iff $\gamma(I)$ is a line segment, with no change of direction.

*(Hint: pick coordinates so that $p = 0$ and the x-axis passes through $q$ to simplify computations.)*

↪**Definition 2.5** (Curve): A set $\mathcal{C} \subset \mathbb{E}^n$ is a *curve* if it is connected, and for every $p \in \mathcal{C}$, there exists a compact neighborhood $N_p$ of $p$ and a one-to-one, compact, parametrized curve $\gamma : I \to \mathbb{E}^n$ such that $\gamma(I) = \mathcal{C} \cap N_p$.

A curve is called $C^k$ if there exists $\gamma$ as in the definition which are now required to be $C^k$.

I.e., a general curve is everywhere locally a compact parametrized curve.

**Remark 2.1**: One can relax the one-to-one/compact conditions to obtain either a global compact parametrization (which may not be one-to-one) or a parametrized curve with $I = \mathbb{R}$ with $\gamma(I) = C$ and $\gamma$ is periodic.

## §2.1 Classification Theorem for Curves

↪**Theorem 2.1** (Classification Theorem for Curves): Let $C \subset \mathbb{E}^n$ a connected subset. Then, $C$ is a (regular) $[C^k]$ curve iff it is the image of a (regular) $[C^k]$ path $\gamma : I \to \mathbb{E}^n$ satisfying either

1. $\gamma$ is one-to-one with $[C^k]$ continuous inverse
2. $I = \mathbb{R}$ and $\gamma$ is periodic, and the restriction of $\gamma$ to any interval $I'$ shorter than the period is one-to-one.

If $\gamma$ satisfies 1. or 2., we'll call it a *global parametrization* of $C$.

**Remark 2.2**: This means we just need *one* path to describe a curve; but it may, in 2., loop back onto itself.

## §2.2 Reparametrizations of Curves

↪**Definition 2.6** (Reparametrization): Let $I, \tilde{I} \subset \mathbb{R}$ be intervals and $t : \tilde{I} \to I$ a continuous bijection (we'll call it a *change of parameters*). Then, the *reparametrization* of $\gamma : I \to \mathbb{E}^n$ using $t$ is the composition $\tilde{\gamma} := \gamma \circ t : \tilde{I} \to \mathbb{E}^n$.

Suppose $\gamma$ a regular $C^k$ path and $t : \tilde{I} \to I$ a $C^k$ bijection with a $C^k$ inverse. Then $\tilde{\gamma}$ is a $C^k$-reparametrization of $\gamma$.

We say $t$ is *orientation-preserving (-reversing)* if it is monotone increasing (decreasing).

**Remark 2.3**: $\gamma$ also a reparametrization of $\tilde{\gamma}$ using the inverse $s := t^{-1}$.

↪**Theorem 2.2**: Suppose $\gamma : I \to \mathbb{R}^n$ is $C^1$ and $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^n$ a $C^1$ reparametrization of $\gamma$. Then $\ell(\gamma) = \ell(\tilde{\gamma})$, that is, arclength is invariant under change of parameters.

↪**Theorem 2.3** (Arc-Length Parametrization): Let $\gamma : I \to \mathbb{E}^n$ be a regular $C^k$ path. Then, there exists an orientation-preserving $C^k$ reparametrization of $\gamma$, $\tilde{\gamma} : \tilde{I} \to \mathbb{E}^n$, with unit speed, i.e. $\|\dot{\tilde{\gamma}}\| \equiv 1$.

Proof. Pick $t_0 \in I$ and definte

$$s : I \to \mathbb{R}, \quad s(t) := \int_{t_0}^{t} \|\dot{\gamma}(r)\| \, \mathrm{d}r.$$

This integral exists and is bounded, and moreover,

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \|\dot{\gamma}(t)\| > 0,$$

since $\gamma$ regular. In particular, we see that $s$ is invertible on its image $\tilde{I} := s(I)$, and increasing. Then, $s : I \to \tilde{I}$ an orientation-preserving, $C^1$ bijection with $s' > 0$. By the inverse function theorem, $t := s^{-1} : \tilde{I} \to I$ exists and has the same desired properties. Moreover,

$$t'(s) = \frac{1}{s'(t(s))} = \frac{1}{\|\dot{\gamma}(t(s))\|}.$$

Letting $\tilde{\gamma} := \gamma \circ t$, then we see that

$$\|\dot{\tilde{\gamma}}(s)\| = \|\dot{\gamma} \circ t(s) \cdot t'(s)\| = \frac{1}{\|\dot{\gamma}(t(s))\|}\|\dot{\gamma}(t(s))\| \equiv 1.$$

■

With this, we can try to define the length of a general curve $C$. Suppose $C \subset \mathbb{E}^n$ a compact curve with boundary $\{p, q\}$ (so satisfies the first point of the classification theorem).

1. If $C$ a line segment, then we just define
$$\mathcal{L}_1(C) := d_{\mathbb{E}}(p, q).$$
2. If $C$ regular, then we define
$$\mathcal{L}_2(C) := \ell(\gamma),$$
where $\gamma$ is any parametrization of $C$.

**Exercise 2.3**: This definition of $\mathcal{L}_2$ is well-defined, i.e. independent of choice of parametrization.

↪**Definition 2.7** (Rectifiable): Let $C$ be a compact curve with boundary $\{p, q\}$. An *inscribed polygon* in $C$ is a finite increasing sequence of points $\mathcal{P} = \{p_i\}_{i=0}^N$ of points in $C$ with endpoints $p_0 = p, p_N = q$. We write

$$L(\mathcal{P}) := \sum_{i=0}^{N-1} d_{\mathbb{E}}(p_i, p_{i+1})$$

for the length of $\mathcal{P}$, and

$$|\mathcal{P}| := \max_{i=0}^{N-1} d_{\mathbb{E}}(p_i, p_{i+1})$$

for the size of $\mathcal{P}$.

A curve $C$ is said to be *rectifiable* if there exists a real number $\mathcal{L}_3(C) \geq 0$ such that for all sequence $\{\mathcal{P}_m\}$ of inscribed polygons in $C$ with $|\mathcal{P}_m| \underset{m\to\infty}{\to} 0$, we have

$$\lim_{m\to\infty} L(\mathcal{P}_m) = \mathcal{L}_3(C).$$

↪**Proposition 2.1**: A unit-speed reparametrization is essentially unique, up to a shift in the domain $I$.

**Exercise 2.4**: Compute the arc-length parametrization of $\gamma(t) := (t, t^2)$.

↪**Lemma 2.1**: Let $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^n$ be a regular $C^2$ path with constant speed. Then, $\ddot{\tilde{\gamma}}$ will always be orthogonal to $\dot{\tilde{\gamma}}$.

Proof. Suppose $\|\dot{\tilde{\gamma}}\| \equiv c$. We apply the product rule for dot products, to obtain

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}(c^2) &= \frac{\mathrm{d}}{\mathrm{d}t}\|\dot{\tilde{\gamma}}\|^2 \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\dot{\tilde{\gamma}} \cdot \dot{\tilde{\gamma}} \\ &= 2\ddot{\tilde{\gamma}} \cdot (\dot{\tilde{\gamma}}), \end{aligned}$$

which gives the proof. ■

## §2.3 Curvature

Let $\gamma$ be a regular $C^2$-path $\gamma : I \to \mathbb{R}^n$, there exists an orientation-preserving change of parameters $t : \tilde{I} \to I$ such that $\tilde{\gamma} := \gamma \circ t : \tilde{I} \to \mathbb{R}$ has unit speed. Let $s := t^{-1} : I \to \tilde{I}$.

↪**Definition 2.8** (Curvature of a parametrized curve): Define the curvature of $\gamma$ as above at some time $t \in I$ to be

$$\kappa_\gamma : I \to \mathbb{R}_+, \quad \kappa_\gamma(t) := \|(\ddot{\tilde{\gamma}} \circ s)(t)\|.$$

**Exercise 2.5**: Show that this definition is well-defined, i.e. independent of choice of unit-speed parametrization.

↪**Definition 2.9** (Curvature of a curve): Given a regular $C^2$ curve $\mathcal{C} \subset \mathbb{R}^n$, there exists (by the classification theorem) a global, regular, $C^2$ parametrization of $\mathcal{C}$, $\gamma : I \to \mathbb{R}^n$. For a point $p \in \mathcal{C}$, then, there exists some $t \in I$ such that $\gamma(t) = p$. Define, then, the curvature of $\mathcal{C}$ at $p$, then, to be the curvature of $\gamma$ at time $t$.

**Exercise 2.6**: Show that this definition is well-defined, i.e., independent of choice of regular global parametrization. One will need to appeal to the inverse function theorem, to show that any two such parametrizations differ by an orientation-preserving change of parameters.

**Exercise 2.7**: Show that curvature of preserved by rigid motions of $\mathbb{R}^n$, i.e. given $M$ a rigid motion of $\mathbb{R}^n$ and a regular $C^2$ curve $\gamma$, then

$$\kappa_{M \circ \gamma} = \kappa_\gamma.$$

**Remark 2.4**: In particular, this exercise gives the curvature is an *inherit property* of curves in $\mathbb{E}^n$, not just in $\mathbb{R}^n$.

**Remark 2.5**: The definition of $\kappa_\gamma$ is a little bothersome in the sense that it requires computing an arc-length parametrization. The follow result shows how we can compute it regardless.

↪**Proposition 2.2**:

$$\kappa_\gamma = \frac{1}{\|\dot{\gamma}\|^2}\left\|\ddot{\gamma} - \frac{\ddot{\gamma}\cdot\dot{\gamma}}{\dot{\gamma}\cdot\dot{\gamma}}\dot{\gamma}\right\| = \frac{\|\ddot{\gamma}^\perp\|}{\|\dot{\gamma}\|^2},$$

where we use the "$\perp$" notation to indicate the orthogonal complement of $\ddot{\gamma}$ with respect to $\dot{\gamma}$.

Proof. I'll add it later. It's just repeated application of the chain rule and product rule. ■

**Exercise 2.8**: Compute the curvature of parabola $C := \{(x, y) \mid y = x^2\} \subset \mathbb{R}^2$ at any point.