

# MATH458 - Differential Geometry

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## §1 SOME REVIEW

We will work in  $\mathbb{R}^n$ , usually with  $n = 2, 3$ . For vectors  $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in \mathbb{R}^n$ , we denote the dot product

$$v \cdot w = \sum_{i=1}^n v_i w_i.$$

More generally, an *inner product* on  $\mathbb{R}^n$  is any function  $b : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  that is symmetric, bilinear and positive definite. For instance, if  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is linear and invertible  $b_T(v, w) := T(v) \cdot T(w)$  a new inner product. In fact, it turns out every inner product on  $\mathbb{R}^n$  is of this form; this implies that every inner product is just a coordinate-change away from the dot product.

We will say a linear transformation  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is *orthogonal* if it is inner product preserving, i.e.  $T(v) \cdot T(w) = v \cdot w$  for every  $v, w \in \mathbb{R}^n$ .

**Exercise 1.1:** Show that  $T$  is inner product preserving iff it is norm preserving ( $\|Tv\| = \|v\|$ ) iff it is distance preserving ( $\|T(v - w)\| = \|v - w\|$ ).

**Exercise 1.2:** Show that if  $T$  orthogonal, it is a bijection with determinant  $\pm 1$ .

We say  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , linear, is *orientation preserving* if  $\det(T) > 0$ .

→ **Definition 1.1** (Rigid Motion): A function  $M : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called a *rigid motion* if there exists an  $a \in \mathbb{R}^n$  and  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  orthogonal and orientation preserving such that

$$M(v) = a + Tv, \quad \forall v \in \mathbb{R}^n.$$

We view the space  $\mathbb{E}^n$  as  $\mathbb{R}^n$  equipped with the Euclidean distance, which we'll denote  $d_{\mathbb{E}}$  or  $d$  if no confusion arises, *up to rigid motions*. In practice, this means working in  $\mathbb{E}^n$  has no distinguished origin point or coordinate axes. However, also in practice, we will make the identification  $\mathbb{E}^n \simeq \mathbb{R}^n$  by picking an origin and axes, as we will see.

However, working in  $\mathbb{E}^n$ , abstractly, still preserves orientation and distance, since these are both preserved under rigid motions.

For  $r > 0$  and  $\rho \in \mathbb{E}^n$ , we write  $\mathbb{D}_r(\rho)$  for the open unit disk, and  $\mathbb{D}^n := \mathbb{D}_1(0) \subset \mathbb{R}^n$ .

→ **Theorem 1.1** (Heine-Borel):  $C \subset \mathbb{E}^n$  compact iff closed and bounded.

**Exercise 1.3:** Let  $r' > r > 0$  and  $\rho \in \mathbb{E}^n$ . Let  $f : \mathbb{D}_{r'}(\rho) \rightarrow \mathbb{E}^n$  be continuous. Show that  $f|_{\mathbb{D}_r(\rho)}$  uniformly continuous.

We'll denote the derivative of a function  $f : \mathcal{U} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  at a point  $a$  by  $D_a f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , which is represented by the Jacobian  $m \times n$  matrix  $J(f)_a = \left( \frac{\partial f}{\partial x_1}|_a, \dots, \frac{\partial f}{\partial x_n}|_a \right)$ .

→ **Definition 1.2:** We will say  $f : \mathcal{U} \rightarrow \mathbb{R}^m$  is  $C^k$  on  $\mathcal{U}$  if all the  $k$ th order partial derivatives of all of the component functions of  $f$  are continuous. We say  $f$  in  $C^\infty$  if it is in  $C^k$  for every  $k \geq 1$ . We write  $C^0$  for the space of continuous functions.

**Remark 1.1:**  $C^{k+1} \Rightarrow C^k$

## §2 CURVES

→**Definition 2.1** (Parametrized curve/path): A *parametrized curve/path* in  $\mathbb{E}^n$  is a continuous function

$$\gamma : I \rightarrow \mathbb{E}^n,$$

where  $I \subset \mathbb{R}$  an interval. We say  $\gamma$  *compact* if  $I$  is compact.

→**Definition 2.2** ((Regular)  $C^k$  parametrized curve): Fix coordinates in  $\mathbb{E}^n$ . Then, a (regular)  $C^k$  parametrized curve is a parametrized curve in which  $\gamma \in C^k(I)$  (and for which  $\frac{d\gamma}{dt}(t) \neq 0 \forall t \in I$ ).

**Exercise 2.1:** Regularity and differentiability is preserved under rigid motion, i.e. if  $\gamma$  a (regular)  $C^k$  parametrized curve and  $M$  a rigid motion on  $\mathbb{R}^n$ , then  $\tilde{\gamma} := M \circ \gamma$  also (regular)  $C^k$ .

→**Definition 2.3:** Given a curve  $\gamma$ , we define

- the *velocity*,  $\nu = \frac{d\gamma}{dt} : I \rightarrow \mathbb{R}^n$
- the *acceleration*,  $\alpha = \frac{d^2\gamma}{dt^2} : I \rightarrow \mathbb{R}^n$
- the *speed*,  $\sigma = \|\nu\| = \left\| \frac{d\gamma}{dt} \right\| : I \rightarrow \mathbb{R}$ ,

whenever each of these quantities all exist.

**Exercise 2.2:** Speed is preserved by rigid motions.

→**Definition 2.4:** Let  $\gamma$  be a  $C^1$  curve. The *arclength* of  $\gamma$  is defined by

$$\ell(\gamma) := \int_I \sigma(t) dt.$$

⊕ **Example 2.1:** Let  $p, q \in \mathbb{E}^2$  with  $d_{\mathbb{E}}(p, q) = 3$ . Suppose  $\gamma : [a, b] \rightarrow \mathbb{E}^2$  is a  $C^1$ -path with  $\gamma(a) = p, \gamma(b) = q$ . Prove that  $\ell(\gamma) \geq 3$ , with equality holding iff  $\gamma(I)$  is a line segment, with no change of direction.

(Hint: pick coordinates so that  $p = 0$  and the  $x$ -axis passes through  $q$  to simplify computations.)

→**Definition 2.5** (Curve): A set  $\mathcal{C} \subset \mathbb{E}^n$  is a *curve* if it is connected, and for every  $p \in \mathcal{C}$ , there exists a compact neighborhood  $N_p$  of  $p$  and a one-to-one, compact, parametrized curve  $\gamma : I \rightarrow \mathbb{E}^n$  such that  $\gamma(I) = \mathcal{C} \cap N_p$ .

A curve is called  $C^k$  if there exists  $\gamma$  as in the definition which are now required to be  $C^k$ .

I.e., a general curve is everywhere locally a compact parametrized curve.

**Remark 2.1:** One can relax the one-to-one/compact conditions to obtain either a global compact parametrization (which may not be one-to-one) or a parametrized curve with  $I = \mathbb{R}$  with  $\gamma(I) = \mathcal{C}$  and  $\gamma$  is periodic.

## §2.1 Classification Theorem for Curves

↪**Theorem 2.1** (Classification Theorem for Curves): Let  $\mathcal{C} \subset \mathbb{E}^n$  a connected subset. Then,  $\mathcal{C}$  is a (regular)  $[C^k]$  curve iff it is the image of a (regular)  $[C^k]$  path  $\gamma : I \rightarrow \mathbb{E}^n$  satisfying either

1.  $\gamma$  is one-to-one with  $[C^k]$  continuous inverse
2.  $I = \mathbb{R}$  and  $\gamma$  is periodic, and the restriction of  $\gamma$  to any interval  $I'$  shorter than the period is one-to-one.

If  $\gamma$  satisfies 1. or 2., we'll call it a *global parametrization* of  $\mathcal{C}$ .

**Remark 2.2:** This means we just need *one* path to describe a curve; but it may, in 2., loop back onto itself.

## §2.2 Reparametrizations of Curves

↪**Definition 2.6** (Reparametrization): Let  $I, \tilde{I} \subset \mathbb{R}$  be intervals and  $t : \tilde{I} \rightarrow I$  a continuous bijection (we'll call it a *change of parameters*). Then, the *reparametrization* of  $\gamma : I \rightarrow \mathbb{E}^n$  using  $t$  is the composition  $\tilde{\gamma} := \gamma \circ t : \tilde{I} \rightarrow \mathbb{E}^n$ .

Suppose  $\gamma$  a regular  $C^k$  path and  $t : \tilde{I} \rightarrow I$  a  $C^k$  bijection with a  $C^k$  inverse. Then  $\tilde{\gamma}$  is a  $C^k$ -reparametrization of  $\gamma$ .

We say  $t$  is *orientation-preserving* (-reversing) if it is monotone increasing (decreasing).

**Remark 2.3:**  $\gamma$  also a reparametrization of  $\tilde{\gamma}$  using the inverse  $s := t^{-1}$ .

↪**Theorem 2.2:** Suppose  $\gamma : I \rightarrow \mathbb{R}^n$  is  $C^1$  and  $\tilde{\gamma} : \tilde{I} \rightarrow \mathbb{R}^n$  a  $C^1$  reparametrization of  $\gamma$ . Then  $\ell(\gamma) = \ell(\tilde{\gamma})$ , that is, arclength is invariant under change of parameters.

↪**Theorem 2.3** (Arc-Length Parametrization): Let  $\gamma : I \rightarrow \mathbb{E}^n$  be a regular  $C^k$  path. Then, there exists an orientation-preserving  $C^k$  reparametrization of  $\gamma$ ,  $\tilde{\gamma} : \tilde{I} \rightarrow \mathbb{E}^n$ , with unit speed, i.e.  $\|\dot{\tilde{\gamma}}\| \equiv 1$ .

PROOF. Pick  $t_0 \in I$  and definte

$$s : I \rightarrow \mathbb{R}, \quad s(t) := \int_{t_0}^t \|\dot{\gamma}(r)\| dr.$$

This integral exists and is bounded, and moreover,

$$\frac{ds}{dt} = \|\dot{\gamma}(t)\| > 0,$$

since  $\gamma$  regular. In particular, we see that  $s$  is invertible on its image  $\tilde{I} := s(I)$ , and increasing. Then,  $s : I \rightarrow \tilde{I}$  an orientation-preserving,  $C^1$  bijection with  $s' > 0$ . By the

inverse function theorem,  $t := s^{-1} : \tilde{I} \rightarrow I$  exists and has the same desired properties.

Moreover,

$$t'(s) = \frac{1}{s'(t(s))} = \frac{1}{\|\dot{\gamma}(t(s))\|}.$$

Letting  $\tilde{\gamma} := \gamma \circ t$ , then we see that

$$\|\dot{\tilde{\gamma}}(s)\| = \|\dot{\gamma} \circ t(s) \cdot t'(s)\| = \frac{1}{\|\dot{\gamma}(t(s))\|} \|\dot{\gamma}(t(s))\| \equiv 1.$$

■

**Exercise 2.3:** Any two arc-length parametrizations differ by some shifting in the domain, i.e. if  $\gamma_i : I_i \rightarrow \mathbb{R}^n$  are two arc-length reparametrizations of a regular path  $\gamma : I \rightarrow \mathbb{R}^n$  using a change of parameters  $t_i : I_i \rightarrow I$  for  $i = 1, 2$ , then  $h := t_2^{-1} \circ t_1 : I_1 \rightarrow I_2$  is a restriction of a rigid motion of  $\mathbb{R}$ ; specifically  $h' \equiv 1$ .

With this, we can try to define the length of a general curve  $\mathcal{C}$ . Suppose  $\mathcal{C} \subset \mathbb{E}^n$  a compact curve with boundary  $\{p, q\}$  (so satisfies the first point of the classification theorem).

1. If  $\mathcal{C}$  a line segment, then we just define

$$\mathcal{L}_1(\mathcal{C}) := d_{\mathbb{E}}(p, q).$$

2. If  $\mathcal{C}$  regular, then we define

$$\mathcal{L}_2(\mathcal{C}) := \ell(\gamma),$$

where  $\gamma$  is any parametrization of  $\mathcal{C}$ .

**Exercise 2.4:** This definition of  $\mathcal{L}_2$  is well-defined, i.e. independent of choice of parametrization.

→ **Definition 2.7** (Rectifiable): Let  $\mathcal{C}$  be a compact curve with boundary  $\{p, q\}$ . An *inscribed polygon* in  $\mathcal{C}$  is a finite increasing sequence of points  $\mathcal{P} = \{p_i\}_{i=0}^N$  of points in  $\mathcal{C}$  with endpoints  $p_0 = p, p_N = q$ . We write

$$L(\mathcal{P}) := \sum_{i=0}^{N-1} d_{\mathbb{E}}(p_i, p_{i+1})$$

for the length of  $\mathcal{P}$ , and

$$|\mathcal{P}| := \max_{i=0}^{N-1} d_{\mathbb{E}}(p_i, p_{i+1})$$

for the size of  $\mathcal{P}$ .

A curve  $\mathcal{C}$  is said to be *rectifiable* if there exists a real number  $\mathcal{L}_3(\mathcal{C}) \geq 0$  such that for all sequence  $\{\mathcal{P}_m\}$  of inscribed polygons in  $\mathcal{C}$  with  $|\mathcal{P}_m| \xrightarrow{m \rightarrow \infty} 0$ , we have

$$\lim_{m \rightarrow \infty} L(\mathcal{P}_m) = \mathcal{L}_3(\mathcal{C}).$$

→ **Proposition 2.1:** A unit-speed reparametrization is essentially unique, up to a shift in the domain  $I$ .

**Exercise 2.5:** Compute the arc-length parametrization of  $\gamma(t) := (t, t^2)$ .

→ **Lemma 2.1:** Let  $\tilde{\gamma} : \tilde{I} \rightarrow \mathbb{R}^n$  be a regular  $C^2$  path with constant speed. Then,  $\ddot{\tilde{\gamma}}$  will always be orthogonal to  $\dot{\tilde{\gamma}}$ .

PROOF. Suppose  $\|\dot{\tilde{\gamma}}\| \equiv c$ . We apply the product rule for dot products, to obtain

$$\begin{aligned} 0 &= \frac{d}{dt}(c^2) = \frac{d}{dt}\|\dot{\tilde{\gamma}}\|^2 \\ &= \frac{d}{dt}\dot{\tilde{\gamma}} \cdot \dot{\tilde{\gamma}} \\ &= 2\ddot{\tilde{\gamma}} \cdot (\dot{\tilde{\gamma}}), \end{aligned}$$

which gives the proof. ■

### §2.3 Curvature

Let  $\gamma$  be a regular  $C^2$ -path  $\gamma : I \rightarrow \mathbb{R}^n$ , there exists an orientation-preserving change of parameters  $t : \tilde{I} \rightarrow I$  such that  $\tilde{\gamma} := \gamma \circ t : \tilde{I} \rightarrow \mathbb{R}^n$  has unit speed. Let  $s := t^{-1} : I \rightarrow \tilde{I}$ .

→ **Definition 2.8** (Curvature of a parametrized curve): Define the curvature of  $\gamma$  as above at some time  $t \in I$  to be

$$\kappa_\gamma : I \rightarrow \mathbb{R}_+, \quad \kappa_\gamma(t) := \|(\ddot{\tilde{\gamma}} \circ s)(t)\|.$$

**Exercise 2.6:** Show that this definition is well-defined, i.e. independent of choice of unit-speed parametrization.

→ **Definition 2.9** (Curvature of a curve): Given a regular  $C^2$  curve  $\mathcal{C} \subset \mathbb{R}^n$ , there exists (by the classification theorem) a global, regular,  $C^2$  parametrization of  $\mathcal{C}$ ,  $\gamma : I \rightarrow \mathbb{R}^n$ . For a point  $p \in \mathcal{C}$ , then, there exists some  $t \in I$  such that  $\gamma(t) = p$ . Define, then, the curvature of  $\mathcal{C}$  at  $p$ , then, to be the curvature of  $\gamma$  at time  $t$ .

**Exercise 2.7:** Show that this definition is well-defined, i.e., independent of choice of regular global parametrization. One will need to appeal to the inverse function theorem, to show that any two such parametrizations differ by an orientation-preserving change of parameters.

**Exercise 2.8:** Show that curvature is preserved by rigid motions of  $\mathbb{R}^n$ , i.e. given  $M$  a rigid motion of  $\mathbb{R}^n$  and a regular  $C^2$  curve  $\gamma$ , then

$$\kappa_{M \circ \gamma} = \kappa_\gamma.$$

**Remark 2.4:** In particular, this exercise gives the curvature is an *inherit property* of curves in  $\mathbb{E}^n$ , not just in  $\mathbb{R}^n$ .

**Remark 2.5:** The definition of  $\kappa_\gamma$  is a little bothersome in the sense that it requires computing an arc-length parametrization. The follow result shows how we can compute it regardless.

↪**Proposition 2.2:**

$$\kappa_\gamma = \frac{1}{\|\dot{\gamma}\|^2} \left\| \ddot{\gamma} - \frac{\ddot{\gamma} \cdot \dot{\gamma}}{\dot{\gamma} \cdot \dot{\gamma}} \dot{\gamma} \right\| = \frac{\|\ddot{\gamma}^\perp\|}{\|\dot{\gamma}\|^2},$$

where we use the “ $\perp$ ” notation to indicate the orthogonal complement of  $\ddot{\gamma}$  with respect to  $\dot{\gamma}$ .

PROOF. I'll add it later. It's just repeated application of the chain rule and product rule. ■

**Exercise 2.9:** Compute the curvature of parabola  $\mathcal{C} := \{(x, y) \mid y = x^2\} \subset \mathbb{R}^2$  at any point.

↪**Theorem 2.4:** The quantity  $\frac{\|\ddot{\gamma}^\perp\|}{\|\dot{\gamma}\|^2}$  is preserved under reparametrization.

**Remark 2.6:** This is really more of a corollary of the previous proposition. Moreover, this implies that our definition of curvature is “correct” as a property of curves in  $\mathbb{E}^n$  rather than just  $\mathbb{R}^n$ .

↪**Definition 2.10:** Let  $\gamma : I \rightarrow \mathbb{R}^n$  a regular path. We define

- $T(t) := \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$ , then *unit tangent* at time  $t$

If  $\gamma \in C^2$ ,

- $N(t) := \frac{\ddot{\gamma}(t)^\perp}{\|\ddot{\gamma}(t)^\perp\|}$ , the *unit normal* at time  $t$
- the *osculating plane* at time  $t$  is the plane in  $\mathbb{R}^n$  that contains the point  $\gamma(t)$  and is spanned by  $\{\dot{\gamma}(t), \ddot{\gamma}(t)\}$  (supposing  $\kappa_\gamma \neq 0$ )
- the *osculating circle* at time  $t$  as the circle laying in the osculating plane of radius  $\frac{1}{\kappa(t)}$  and centered at  $\gamma(t) + \frac{N(t)}{\kappa(t)}$
- the *evolute* of  $\gamma$  is the map

$$t \in I \mapsto \gamma(t) + \frac{N(t)}{\kappa(t)} = \text{center of oscualting circle at } t$$

**Remark 2.7:**  $\ddot{\gamma}^\perp \neq 0 \Leftrightarrow \kappa_\gamma \neq 0 \Leftrightarrow \{\dot{\gamma}, \ddot{\gamma}\}$  a linearly independent set.

**Exercise 2.10:** A circle of radius  $r$ , i.e. the curve defined implicitly by  $\{x^2 + y^2 = r^2\}$ , has curvature  $\frac{1}{r}$ .

This exercise shows that the osculating circle at a point on a curve has the same curvature as the curve at that point.

**Exercise 2.11:** Suppose  $n = 2$  and a curve is given *explicitly* by  $y = f(x)$  where  $f : \mathbb{R} \rightarrow \mathbb{R}$  sufficiently differentiable. Compute the curvature in terms of  $f$  and its derivatives. Do the same if the curve is given *implicitly* as the set of  $(x, y) \in \mathbb{R}^2$  such that  $g(x, y) = 0$  where  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  sufficiently differentiable.

Fix now  $n = 2$ . Let  $\gamma : I \rightarrow \mathbb{R}^2$  be a regular  $C^2$  curve and fix  $t \in I$ . Let us assume (by changing coordinates if necessary) that  $\gamma(t) = 0$  and the  $x$ -axis is parallel to  $T(t)$ , i.e.  $T(t) = (1, 0)$ . Then, we see that we may write

$$\frac{\ddot{\gamma}(t)^\perp}{\|\dot{\gamma}(t)\|^2} = \text{constant} \times (0, 1).$$

Specifically, the “constant” here is what we call the *signed curvature* of  $\gamma$  at time  $t$ , and is computed as :

↪ **Definition 2.11** (Signed curvature): Let  $\gamma$  as in the above, then the *signed curvature* is the quantity

$$\kappa_\gamma^\pm(t) = \frac{1}{\|\dot{\gamma}(t)\|^2} \frac{\ddot{\gamma}(t) \cdot \dot{\gamma}(t)^*}{\|\dot{\gamma}(t)\|},$$

where we use the notation  $v^*$  as a rotation of  $v = (v_1, v_2)$  by an angle of  $\frac{\pi}{2}$ , counter-clockwise, i.e.  $v^* = (-v_2, v_1)$ .

**Exercise 2.12:**  $\kappa_\gamma^\pm(-t) = -\kappa_\gamma^\pm(t)$ .

**Exercise 2.13:** Suppose  $\gamma(t) = (x(t), y(t))$ , then show

$$\kappa_\gamma = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

↪ **Definition 2.12** (Angle function): Let  $\gamma : I \rightarrow \mathbb{R}^2$  be a regular  $C^2$  curve parametrized by arc length with basepoint  $s_0 \in I$ . We assume wlog  $s_0 = 0$  (by translating if necessary) and that  $\dot{\gamma}(0) = (1, 0)$  (by changing coordinates). We define the *angle function* of  $\gamma$  by

$$\theta : I \rightarrow \mathbb{R}, \quad \theta(0) = 0, \quad \theta(s) := \int_0^s \kappa_\gamma^\pm(u) \, du.$$

In particular,  $\frac{d\theta}{ds} = \kappa_\gamma^\pm(s)$ .

**Remark 2.8:** We can view  $s \mapsto \dot{\gamma}(s)$  as a new  $C^1$ -parametrized curve, in which case its arc length is given by

$$\int_0^s \|\ddot{\gamma}(u)\| du = \int_0^s \kappa_\gamma(u) du.$$

So, in a sense, the  $\theta$  angle function is the “signed arc-length” of  $\dot{\gamma}$ , i.e. it accounts for backtracking.

Moreover, since we have an arc length parametrization, we know  $\dot{\gamma}$  a unit vector, hence we can view the map  $t \mapsto \dot{\gamma}(t)$  as a map from  $I$  to the unit circle in  $\mathbb{R}^2$ . Hence,  $\theta$  is meant to capture the angle of this unit vector for any  $t$ , i.e.  $\dot{\gamma} = (\cos, \sin) \circ \theta$ .

↪ **Theorem 2.5** (Fundamental Theorem of Plane Paths): Let  $s_0 \in I$  be a given base point and let  $\kappa : I \rightarrow \mathbb{R}$  be a  $C^{k-2}$  function ( $2 \leq k \leq \infty$ ). Then, for each  $p \in \mathbb{R}^2$  and  $\theta_0 \in \mathbb{R}$ , there is a unique regular  $C^k$  path  $\gamma : I \rightarrow \mathbb{R}^2$ , parametrized by arc-length, such that

1.  $\kappa_\gamma^\pm = \kappa$ ,
2.  $\dot{\gamma}(s_0) = (\cos(\theta_0), \sin(\theta_0))$ ,
3.  $\gamma(s_0) = p$ .

**Remark 2.9:** The choice of  $p, \theta_0$  just correspond to a translation, rotation (resp.) of  $\mathbb{R}^2$  of our curve, i.e. this means our curve is uniquely determined up to rigid motion.

**Remark 2.10:** This essentially says that, given the curvature of a curve in the plane, we can reconstruct the curve.

PROOF. We seek to find  $\gamma : I \rightarrow \mathbb{R}^2$  and  $\theta : I \rightarrow \mathbb{R}$  such that

$$\frac{d\gamma}{ds} = (\cos \theta, \sin \theta), \gamma(s_0) = p$$

and

$$\frac{d\theta}{ds} = \kappa, \theta(s_0) = \theta_0.$$

By the fundamental theorem of calculus, we know

$$\theta(s) = \int_{s_0}^s \kappa(u) du + \theta_0$$

is the unique solution for  $\theta(s)$  with the given properties, which in turn implies

$$\gamma(s) = \left( \int_{s_0}^s \cos(\theta(u)) du, \int_{s_0}^s \sin(\theta(u)) du \right) + p,$$

which is again unique by FTC. ■

**Remark 2.11:** This theorem essentially says that a curve is uniquely determined by its signed curvature. However, the same is not true if we just take the curvature. For instance, the curves given explicitly by  $y = x^3$ ,  $y = |x|^3$  have the same curvature everywhere but clearly do not described the same curves.

A more abstract manner of characterizing the angle function for a more general curve is as follows. If  $\gamma : I \rightarrow \mathbb{R}^2$  a regular,  $C^2$  curve, then the angle function  $\theta : I \rightarrow I'$  where  $I'$  some other interval of  $\mathbb{R}$ , is such that

$$T = \rho \circ \theta,$$

where  $\rho : I' \rightarrow \mathbb{R}^2$  is the standard parametrization of the circle given by  $\rho(\theta) := (\cos(\theta), \sin(\theta))$  and  $T$  the unit tangent vector viewed as a map  $I \rightarrow \mathbb{R}^2$ .

**Exercise 2.14:** Show that the signed curvature of  $\gamma$  is preserved under rigid motion, hence is well-defined as a property of a curve in  $\mathbb{E}^n$ . (Note that the signed curvature is the derivative of the  $\theta$  function, hence it suffices to prove this property for  $\theta$ )

## §2.4 3-Dimensional Space Paths

We wish to derive an analogous “fundamental” result for curves in  $\mathbb{R}^3$ . However, we have no notion of “signed curvature” in this case. Moreover, as we’ll see, we actually need a second “intrinsic” (called *torsion*) of the curve to uniquely identify it.

Fix  $\gamma : I \rightarrow \mathbb{R}^3$  regular and  $C^2$  and with strictly positive curvature (turns out, there’s not much we can say when the curvature is 0).

Define as before

$$T := \frac{\dot{\gamma}}{\|\dot{\gamma}\|}, \quad N := \frac{\ddot{\gamma}^\perp}{\|\ddot{\gamma}^\perp\|}$$

the unit tangent and normal vectors. Remark that  $T \cdot N = 0$ . Since we are in  $\mathbb{R}^3$ , there exists a unique third vector, which we denote  $B$  and call it the *binormal* such that  $\{T, N, B\}$  is an orthonormal, positively oriented basis (in the sense that the matrix consisting of columns  $T, N, B$  in that order is orthogonal with determinant 1) of  $\mathbb{R}^3$ , i.e.

$$B := T \times N.$$

The basis  $\{T, N, B\} \subset \mathbb{R}^3$  is called the *Frenet frame* associated to  $\gamma$ .

We’ll be interested in the dynamics of this frame, i.e. how  $T, N, B$  resp. change in time. We need to additionally assume  $\gamma \in C^3$  for this, so that we may take derivatives of  $N$ . We’ll also assume  $\gamma$  is parametrized by arc-length for convenience. We find that with these assumptions,

$$\begin{aligned} T &= \dot{\gamma} \\ \Rightarrow \dot{T} &= \ddot{\gamma} = \|\ddot{\gamma}\|N = \kappa N. \end{aligned}$$

In addition,

$$\|B\| = 1 \Rightarrow \dot{B} \cdot B = 0$$

and

$$B = T \times N \Rightarrow \dot{B} = \dot{T} \times N + T \times \dot{N} = \kappa \underbrace{N \times N}_{=0} + T \times \dot{N} \Rightarrow \dot{B} \cdot T = 0,$$

hence  $\dot{B}$  is simultaneously orthogonal to  $B$  and  $T$ , hence

$$\dot{B} = \text{const}(-N).$$

We call this constant the *torsion*  $\tau$  of  $\gamma$  at time  $s$ , which is given by

$$\tau := -\dot{B} \cdot N.$$

Finally, to compute  $\dot{N}$ , we have that

$$\begin{aligned}\|N\| = 1 &\Rightarrow \dot{N} \cdot N = 0 \\ T \cdot N = 0 &\Rightarrow 0 = \dot{T} \cdot N + T \cdot \dot{N} = \kappa \underbrace{\|N\|^2}_{=1} + T \cdot \dot{N} \Rightarrow T \cdot \dot{N} = -\kappa \\ B \cdot N = 0 &\Rightarrow 0 = \dot{B} \cdot N + BN = -\tau + B \cdot \dot{N} \Rightarrow B \cdot \dot{N} = \tau.\end{aligned}$$

This implies

$$\dot{N} = -\kappa T + \tau B.$$

In summary, we can succinctly write

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \quad (\text{The Frenet equations}).$$

**Theorem 2.6 (Fundamental Theorem of Space Paths):** Let  $I \subset \mathbb{R}$  be an interval with basepoint  $s_0 \in I$ . Suppose  $\tau : I \rightarrow \mathbb{R}$  is a  $C^{k-3}$  function and  $\kappa : I \rightarrow \mathbb{R}_{>0}$  is a  $C^{k-2}$  function (where  $3 \leq k \leq \infty$ ). Then, for any initial point  $p_0 \in \mathbb{R}^3$ , initial velocity  $v_0 \in \mathbb{R}^3$ , and initial normal direction  $n_0 \in \mathbb{R}^3$  such that  $\|v_0\| = \|n_0\| = 1$  and  $v_0 \cdot n_0 = 1$ , there is a *unique* regular  $C^k$  path  $\gamma : I \rightarrow \mathbb{R}^3$  parametrized by arc length and satisfying:

1.  $\kappa_\gamma = \kappa$ ,
2.  $\tau_\gamma = \tau$ ,
3.  $\gamma(s_0) = p_0$ ,
4.  $\dot{\gamma}(s_0) = v_0$ ,
5.  $\ddot{\gamma} \frac{s_0}{\|\ddot{\gamma}(s_0)\|} = n_0$ .

**Remark 2.12:** The last three requirements say that this curve is uniquely defined up to rigid motion, hence unique in  $\mathbb{E}^3$ ; translations will simply change the initial point  $p_0$ , and rotations will change the angles of  $v_0, n_0$ .

**PROOF.** Remark that the Frenet equations are a system of (9) first order ODEs with given initial condition. The Picard-Lindelhoff theorem from ODEs says that there exist unique function  $T, N, B : I \rightarrow \mathbb{R}^3$  satisfying the equations with  $T(s_0) = v_0, N(s_0) = n_0, B(s_0) = v_0 \times n_0$ . We need to show that these are the Frenet frame of some curve.

First, we show they are a positively oriented orthogonal basis. Indeed, remark that, using the Frenet equations,

$$\begin{aligned}
\frac{d}{ds}(T \cdot N) &= \kappa(N \cdot N) - \kappa(T \cdot T) + \tau(T \cdot B) \\
\frac{d}{ds}(T \cdot B) &= \kappa(N \cdot B) - \tau(T \cdot N) \\
\frac{d}{ds}(N \cdot B) &= -\kappa(T \cdot B) + \tau(B \cdot B) - \tau(N \cdot N) \\
\frac{d}{ds}(T \cdot T) &= 2\kappa(T \cdot N) \\
\frac{d}{ds}(N \cdot N) &= -2\kappa(T \cdot N) + 2\tau(N \cdot B) \\
\frac{d}{ds}(B \cdot B) &= -2\tau(N \cdot B).
\end{aligned}$$

These are a system of ODEs for the quantities  $T \cdot N, T \cdot B$ , etc with initial conditions  $0, 0, 0, 1, 1, 1$ . However, the system can also be solved by  $T \cdot N \equiv 0, T \cdot B \equiv 0$ , etc, and so by uniqueness of solutions to linear ODEs, it follows that  $T \cdot N = 0$ , etc, which proves the orthonormality. To show positive orientation, it suffices to show that  $(T \times N) \cdot B \equiv 1$ . This is true at the basepoint of time by choice of initial conditions, and if we take the derivative, we find

$$\frac{d}{ds}((T \times N) \cdot B) = \kappa(N \times N) \cdot B + [(T \times (-\kappa T)) + T \times (\tau B)] \cdot B + (T \times N) \cdot (-\tau N),$$

which we see to be equal to zero by our orthonormality proof from above. Thus,  $\{T, N, B\}$  is indeed a positively-oriented orthonormal basis.

Finally, we need to show that there exists a unique curve with  $T$  as its unit tangent (from which the remainder of the quantities  $N$ , etc will follow); indeed, we have

$$\gamma : I \rightarrow \mathbb{R}^3, \quad \gamma(s) = p_0 + \int_{s_0}^s T(u) du$$

is the unique curve with  $\dot{\gamma} = T$ ; the fact that  $\gamma \in C^k$  follows from  $T \in C^{k-1}$ . ■

**Exercise 2.15:** With the same assumptions as above, also assume  $\sigma : I \rightarrow \mathbb{R}_{>0} \in C^{k-1}$ . Then, there exists a unique  $C^k$  regular path  $\gamma \in \mathbb{E}^3$  such that

$$\|\dot{\gamma}\| = \sigma, \quad \kappa_\gamma = \kappa, \quad \tau_\gamma = \tau.$$

We're interested in defining the torsion for more general paths in a consistent way. Let  $\gamma$  a regular  $C^3$  curve in  $\mathbb{R}^3$  with  $\kappa > 0$ . Let  $\tilde{\gamma} : \tilde{I} \rightarrow \mathbb{R}^3$  be a arc-length reparametrization using  $t : \tilde{I} \rightarrow I$ , and let  $s = t^{-1}$ , and define

$$\tau := \tilde{\tau} \circ s,$$

where  $\tilde{\tau}$  the torsion of  $\tilde{\gamma}$ , as defined above.

→**Proposition 2.3:** Let  $\gamma$  be as above. Then,

$$\kappa = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3}, \quad \tau = \left( \frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} \right) \cdot \ddot{\gamma}$$

PROOF. We know  $\kappa = \frac{\|\dot{\gamma}^\perp\|}{\|\dot{\gamma}\|^2}$ . In  $\mathbb{R}^3$ ,  $\|\dot{\gamma} \times \ddot{\gamma}\|$  is the area of the parallelogram with sides  $\dot{\gamma}, \ddot{\gamma}$ , or equivalently, twice the area of the triangle with base  $\dot{\gamma}$  and height  $\dot{\gamma}^\perp$  (the perpendicular to the base  $\dot{\gamma}$ ), i.e.

$$\|\dot{\gamma} \times \ddot{\gamma}\| = \|\dot{\gamma}\| \|\dot{\gamma}^\perp\|,$$

which proves the first claim. The second claim follows from lots of careful chain rules.

■

**Exercise 2.16:** Is torsion preserved by reversals? i.e., if  $\bar{\gamma} := \gamma \circ \bar{t}$  where  $\bar{t}(t) = -t$ , is  $\tau_{\bar{\gamma}}(\bar{t}) = \tau_\gamma(t)$ ?

**Exercise 2.17:** (Twisted Cubic) Let  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$  be given by  $\gamma(t) = (t, t^2, t^3)$ . Show that  $\kappa(0) = 2, \tau(0) = 3$ .

**Exercise 2.18:** (Helix) Let  $\gamma(t) = (\cos(t), \sin(t), t)$ . Show that  $\kappa \equiv \frac{1}{2}, \tau \equiv \frac{1}{2}$ .

**Exercise 2.19:** Find an example where  $\kappa \equiv \frac{1}{2}, \tau \equiv -\frac{1}{2}$ .

## §2.5 Global Theorems/Properties of Plane Curves

Let  $\mathbb{S}^1$  denote the unit circle in  $\mathbb{R}^2$  centered at the origin, with global periodic parametrization  $\rho(t) = (\cos(t), \sin(t))$ . Given a  $C^0$  curve in  $\mathbb{S}^1$  by  $g : I \rightarrow \mathbb{S}^1$ , a function  $\theta : I \rightarrow \mathbb{R}$  is called a *lift* of  $g$  via  $\rho$  if

1. it is  $C^0$
2.  $g = \rho \circ \theta$

→**Theorem 2.7:** Fix  $t_0 \in \mathbb{R}, \theta_0 \in \mathbb{R}$  such that  $g(t_0) = (\cos \theta_0, \sin \theta_0)$ . Then, there exists a unique lift  $\theta$  of  $g$  such that  $\theta(t_0) = \theta_0$ .

If  $g : \mathbb{R} \rightarrow \mathbb{S}^1$  a periodic path with period  $[a, b]$ , then for any lift  $\theta$  of  $g$ ,

$$|\theta(b) - \theta(a)| = 2\pi n, \quad n \in \mathbb{Z}_+,$$

where  $n$  the number of times the curve “goes around”  $\mathbb{S}^1$ .

→**Theorem 2.8 (Hopf's Umlaufsatz):** If  $\mathcal{C} \subset \mathbb{R}^2$  a regular closed curve periodic (with period  $[a, b]$ ) parametrization  $\gamma : R \rightarrow \mathbb{R}^2$ , then for any lift  $\theta$  of its tangent vector  $T$  (i.e.,  $\theta$  is an angle function),  $|\theta(b) - \theta(a)| = 2\pi$ .

We say  $\gamma$  is *positively/ccw oriented* if  $\theta(b) - \theta(a) = 2\pi$ , and *negatively/cw oriented* if  $\theta(b) - \theta(a) = -2\pi$ .

→ **Theorem 2.9** (Jordan Curve Theorem): Let  $\mathcal{C} \subset \mathbb{R}^2$  a regular closed curve. Then,  $\mathbb{R}^2 \setminus \mathcal{C}$  has two connected components; one bounded (“inside” of  $\mathcal{C}$ ) and one unbounded (“outside” of  $\mathcal{C}$ ).

We can then say  $\gamma$  is *positively oriented* if  $T^*$  points inside  $\mathcal{C}$ , and *negatively oriented* if  $T^*$  points outside  $\mathcal{C}$ . It turns out these different notions of orientation are equivalent.

→ **Theorem 2.10** (Isoperimetric Inequality): Let  $\mathcal{C} \subset \mathbb{R}^2$  a regular closed curve. Let  $\ell = \text{length of } \mathcal{C}$  and  $A = \text{area of inside of } \mathcal{C}$ . Then,

$$4\pi A \leq \ell^2,$$

with equality iff  $\mathcal{C}$  is a circle.

PROOF. (Sketch of Hopf's) ■

### §3 APPENDIX

→ **Proposition 3.1:** For  $u, v \in \mathbb{R}^3$ ,  $\|u \times v\|$  is the area of the parallelogram with side  $u, v$ .

→ **Proposition 3.2:**

→ **Proposition 3.3:**