Louis Meunier

# ODEs                                                    MATH325

---

# Contents

# 1   Introduction

## 1.1   Definitions

> ↪ **Definition** 1.1: **Diffferential equation**
>
> A *diffferential equation* (DE) is an equation with derivatives. *Ordinary* DE's (ODE) will be covered in this course; other types (PDE's, SDE's, DDE's, FDE's, etc.) exist as well but won't be discussed. ODE's only have one independent variable (typically, $y = f(x)$ or $y = f(t)$).

> ⊛ **Example 1.1: A Trivial Example**
>
> $\frac{dy}{dx} = 6x$. Integrating both sides:
>
> $$\int \frac{dy}{dx} \, dx = \int 6x \, dx \implies y(x) = 3x^2 + C.$$

> ⊛ **Example 1.2: Another One**
>
> $$\frac{d^2 u}{dt^2} = 0 \implies y = at + b.$$

> ↪ **Definition** 1.2: **Order**
>
> The order of a differential equation is defined as the order of the highest derivative in the equation.

## 1.2   Initival Values

**Remark 1.1.** *Note the existence of arbitrary constants in the previous examples, indicating infinite solutions. We often desire unique solutions by fixing these coefficients. For first order ODEs, we simply specify a single initial condition (say, some $y(x_0) = \alpha_0$). For higher order ODEs of degree $n$, we can either specify $n-1$ initial conditions for $n-1$ derivatives (say, $y(x_0) = \alpha_0$, $y'(x_0) = \beta_0$), or boundary conditions (say, $y(x_0) = \alpha_0, y(x_1) = \alpha_1$) where values for the solution itself are specified.*

> ⊛ **Example 1.3: A Less Trivial Example**
>
> $\frac{dy}{dx} = y$. We cannot simply integrate both sides as before, as we have no way to know what $\int y \, dx$ (the RHS) is equal to. We can fairly easily guess that $y = e^x$ is a solution; its derivative is equal to itself, hence it does indeed solve the equation. This is not the

only solution; indeed, given $y = ce^x$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ce^x = y = ce^x.$$

Luckily, we were rather limited in how many places constants could appear; this doesn't always hold.

## 1.3   Physical Applications

⊛ **Example 1.4: Simple Pendulum**

Let $\theta$ be the angle of a pendulum of mass $m$ from vertical and length $l$. Then, we have the equation of motion

$$ml\ddot{\theta} = -mg\sin\theta \implies \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \implies \ddot{\theta} + \omega^2\sin\theta = 0.$$

Take $\theta$ small, then, $\sin\theta \approx \theta$. Then, $\ddot{\theta} + \omega^2\theta = 0$. This is linear simple harmonic motion, and has periodic solutions; how do we know this is a valid solution to the non-linear model?

---

↪ *Thu Jan 4 15:16:18 EST 2024*

⊛ **Example 1.5: Lorenz Equations**

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = rx - y - xz$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz$$

These are a famous set of equations originally derived from atmospheric modeling, known for its chaotic behavior for particular parameters. This is a nonlinear system of de's, and beyond the scope of this class (indeed, it is not solvable exactly).

## 1.4   Uniqueness

Given an ODE of the general form $y^{(n)} = f(t, y, y', \ldots, y^{n-1})$, if we wish to determine $y^{(n)}(t_0)$ uniquely, we need to specify the initial conditions

$$y(t_0), y'(t_0), \ldots, y^{(n-1)}(t_0).$$

Moreover, this not only determines uniqueness of $y^{(n)}(t_0)$, byt the uniqueness of solution $y$ for $t \in I$ for some "interval of validity" $I$.

---

↪ **Definition** 1.3: **Autonomous/Nonautonomous**

An ODE of the form

$$y^{(n)} = f(y, y', \ldots, y^{(n-1)})$$

is called *autonomous*; that is, if it has no explicit dependence on the independent variable. Otherwise, the system is called *nonautonomous*.

↪ **Definition** 1.4: **Linear/Nonlinear**

Linear ODEs of dimension $n$ have a solution space which is a vector space of dimension $n$. As a result, solutions can be written as a linear combination of $n$ basis solutions (or "fundamental set of solutions"). Solutions to nonlinear ODEs cannot be written this way (except locally).

Alternatively (but equivalently), if we can write an $n$th order ODE in the form

$$a_n(t)y^n(t) + \cdots a_1(t)y'(t) + a_0(t)y(t) = g(t),$$

or equivalently,

$$\sum_{i=0}^{n} a_i(t)y^i(t) = g(t), \quad \circledast$$

where each $a_i(t)$ and $g(t)$ are known functions of $t$, then we say that the ODE is linear. Otherwise, it is nonlinear.

⊛ **Example 1.6**

The pendulum

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

is autonomous and linear;

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

is autonomous and nonlinear, due to the $\sin\theta$ term (indeed, this is a nonlinear oscillator equation); a damped-forced oscillator

$$\ddot{\theta} + k^2\dot{\theta} + \omega^2\theta = A\sin(\mu t)$$

is nonautonomous and linear.

**Remark 1.2.** *Note that the following definitions apply only to linear ODEs.*

↪ **Definition** 1.5: **Homogeneous/Nonhomogeneous**

A linear ODE of the form ⊛ is *homogeneous* if $g(t) = 0$; otherwise it is *nonhomogeneous.*

↪ **Definition** 1.6: **Constant/Variable**

A linear ODE of the form $*$ is *constant coefficient* if $a_j(t) = $ constant $\forall\, j$; if at least one $a_j$ not constant, it is *non-constant* or *variable coefficient.*

**Remark 1.3.** *Note that while we define linearity of ODEs in terms of the form of* $y^{(n)} = f(t, y, \dots)$, *this more "helpfully" relates to the form of the solution of such an ODE, which is indeed linear.*

## 1.5 Solutions

Given an $n$ order ODE $y^{(n)} = f(t, y, \dots)$, and assuming $f$ continuous, then for $y(t)$ to be a solution, we need $y$ to be $n$-times differentiable; hence, $y, \dots, y^{(n-1)}$ must all exist and be continuous. Then, $y^{(n)}$, being a continuous function of continuous functions, is, itself, continuous.

↪ **Definition** 1.7: **Solution**

The function $y(t) : I \to \mathbb{R}$ is a solution to an ODE on an interval $I \subseteq \mathbb{R}$ if it is $n$-times differentiable on $I$, and satisfies the ODE on this interval.

Given an well-defined IVP with $n-1$ initial values defined at $t_0$, then $y(t)$ is a solution if $t_0 \in I$, $y$ satisfies the initial values, and $y(t)$ is a solution on the interval.

↪ **Definition** 1.8: **Interval of Validity**

The largest $I$ on which $y(t) : I \to \mathbb{R}$ solves an ODE is called the *interval of validity* of the problem.

↪ *Tue Jan 9 11:21:25 EST 2024*