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Nonlinear Dynamics and Chaos   MATH376

Contents

1 Introduction, Motivations, Overview

> **Definition 1.1** (Dynamical Systems). *Systems which evolve over time. We can categorize them as*
>
> - ***continuous,*** *which define ODEs, eg.* $\dot{u}(t) = f(u)$, *where* $u(t)$ *is define over some interval* t;
>
> - ***discrete,*** *which are defined by a* map, $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n, u \longmapsto S_u,$ *where* $u_1 = S_{u_0},$ $u_2 = S_{u_1},$ *etc., where* $n \in \mathbb{Z}.$

1.1 Examples of continuous dynamical systems

1.1.1 Exponential growth/decay

Consider $\dot{u} = \lambda u$, where $u \in \mathbb{R}$, $\dot{u} = \frac{du}{dt}$ $(u = f(t))$, and λ is a constant parameter. This is a linear, separable ODE;

$$\frac{du}{dt} = \lambda u$$
$$\int \frac{du}{u} = \int \lambda \, dt$$
$$\implies u(t) = u_0 e^{\lambda t}, \text{ where } u(0) = u_0$$

Assuming $\lambda \neq 0$ (otherwise $u(t) = u_0$), we can analyze the behavior of u as $t \longrightarrow \infty$ and $t \longrightarrow -\infty$.

- Clearly, if $u_0 = 0$, then $u(t) = 0 \forall t$. This is called a **steady state**.

- Else, (under the assumption $u_0 > 0$), we can consider the cases $\lambda > 0$ and $\lambda < 0$.
    - $\lambda > 0 \implies \lim_{t\to\infty} u(t) = \infty$
    - $\lambda < 0 \implies \lim_{t\to-\infty} u(t) = 0$

Using this fairly simple analysis, we can draw **phase diagrams** describing how the system changes based on initial conditions, for instance, given $\lambda > 0$:
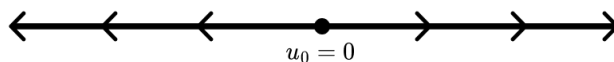

Figure 1: Exponential growth/decay, $\lambda > 0$

Note that the phase diagram is *independent* of the value of λ; naturally, a larger λ will result in a "faster" (so to speak) growth/decay, but the "asymptotic" behavior is identical. We can say that the **dynamics** of the system are independent of the constant λ.

In this case, at $u_0 = 0$, all other u_0 greater than or less than 0 diverge away from u_0; this would be called a *unstable equilibrium*. If $\lambda < 0$, we would see all u_0 converging to 0, which would be an *asymptotically stable equilibrium*.

1.1.2 Logistic ODE

Consider the *logistic* ode $\dot{x} = \lambda x(1-x), x(t) \in \mathbb{R}$. Normally, we would solve this ode (using separation of variables, resulting in a messy fraction decomposition, and lots of algebraic manipulation [1]) This will give the final explicit solution

$$x(t) = \frac{1}{(\frac{1}{x_0} - 1)e^{-\lambda t} + 1},$$

where $x(0) = x_0$. We can then analyze $x(t)$ similar to the previous example. Due to the complexity (the "embedded" exponential, etc), however, this is quite difficult.

Alternatively, we can consider the original ode

$$\dot{x} = f(x) = \lambda x(1-x),$$

without the exact solutions. Assuming $\lambda > 0$ (similar "methodology" for $\lambda < 0$), we can analyze the behavior:

- Steady states will occur when $\dot{x} = 0 \implies \lambda x(1-x) = 0 \implies x = 0$ or $x = 1$.

- Next, we can consider the behavior of x in the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$.

    - $(-\infty, 0)$: as $x \to -\infty$, $\dot{x} \to -\infty$ $((+) \times (-) \times (+) \sim (-))$
    - $(0, 1)$: as $x \to 1$, $\dot{x}$ increases.
    - $(1, \infty)$: as $x \to \infty$, $\dot{x} \to -\infty$ $((+) \times (+) \times (-) \sim (-))$

This means that $x_0 = 0$ and $x_0 = 1$ are unstable and stable, respectively. We can then draw the phase diagram:
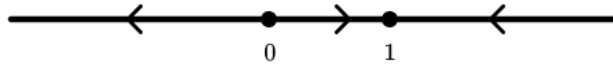
We can compare the logistic ODE to the seemingly unrelated [2] $\dot{x} = x - x^3$. Factoring, we write $\dot{x} = x(1-x)(1+x)$, indicating steady states at $x = -1, 0, 1$. This results in a very similar phase diagram:
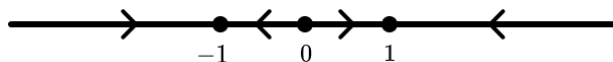
As is clear from the diagram, the two equations have very similar dynamics; however, at no initial condition does the second ODE tend to positive/negative infinity.

[1] Note that this is also a *Bernoulli ODE*, ie one of the form $y' + p(x)y = q(x)y^n$; you can solve it by dividing by y^n and making the substitution $u = y^{1-n}$ to get a far nicer (though obviously equivalent) linear equation which you can solve using an integrating factor.

[2] Also a Bernoulli

1.2 Analyzing the Lorenz Equations

The Lorenz equation is defined by

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases},$$

where solutions $u(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \in \mathbb{R}^3$. A trivial steady state exists at $(x, y, z) = (0, 0, 0), \forall r >$

0, and two more exist[3] $(x, y, z) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1), \forall r > 1$. Notice that when $r = 1$, the two non-trivial steady states collapse into the trivial steady state. This is what we call a **bifurcation**, or in this case specifically, a **pitchfork bifurcation**. This can make sense if we plot [4] (x, y, z) of the steady points as a function of r:
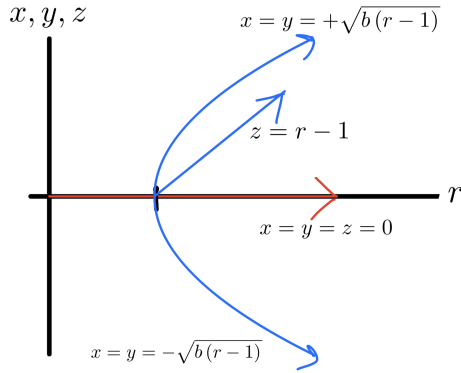
Further analyzing the dynamics of the system is a little trickier - we can't exactly use the same approaches as before in the $\mathbb{R}^2$ space. The system is clearly not linear because of the xz and xy terms. However, if we assume that x, y, z are small and remain small, then we can approximate the system as linear by dropping these terms [5]. Thus, we can approximate the system as $\dot{x} = \sigma(y - x), \dot{y} = rx - y, \dot{z} = -bz$, or, equivalently,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Now, if $x(0) = y(0) = 0$, then $\dot{x} = \dot{y} = 0 \implies x(t) = y(t) = 0 \forall t$, and thus the solution evolves solely on the z-axis; ie $\dot{z} = -bz \implies z = z(0)e^{-bt}$, which $\to 0$ as $t \to \infty$, supporting our assumption that x, y, z remain small.

On the other hand, let's assume[6] $|x(t), y(t), z(t)| \ll 1$; again, this results in x, y, z remaining small, and thus "allows" us to study the dynamics approximately near $(x, y, z) = (0, 0, 0)$.

Solving the matrix formulation of the system, we get

$$u(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{\lambda_1 t} \underline{\mathbf{v_1}} + c_2 e^{\lambda_2 t} \underline{\mathbf{v_2}} + c_3 e^{\lambda_3 t} \underline{\mathbf{v_3}},$$

where λ_i are eigenvalues and $\underline{\mathbf{v_i}}$ are corresponding eigenvectors.

Clearly[7], $\underline{\mathbf{v_3}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ with a corresponding $\lambda_3 = -b$. Thus, $\underline{\mathbf{v_1}}$ and $\underline{\mathbf{v_2}}$ must lie in the xy-plane, and similarly, are eigenvectors of the top left, $\begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$ (with 0 z-component, of course). Solving for these by standard methods, we write

$$0 = \det \left(\begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} - \lambda I \right)$$

$$= \begin{vmatrix} -\sigma - \lambda & \sigma \\ r & -1 - \lambda \end{vmatrix} = (\sigma + \lambda)(1 + \lambda) - r\sigma$$

$$= \lambda^2 + (1 + \sigma)\lambda - (r - 1)$$

$$\implies \lambda_{1,2} = \frac{-(1 + \sigma) \pm \sqrt{(1 + \sigma)^2 + 4(r - 1)\sigma}}{2}$$

Notice that if $r \in (0, 1)$, then[8] $(1 + \sigma)^2 + 4(r - 1)\sigma < (1 + \sigma)^2$. Assuming[9] the lhs of this inequality is greater than 0, we can further say that $\left| \sqrt{(1 + \sigma)^2 + 4(r - 1)\sigma} \right| < |1 + \sigma|$. Thus, both λ_1 and λ_2 are < 0, as taking either the positive or negative sign in the quadratic necessarily yields a negative[10]. We could work out a full solution, but this is unncessary; clearly, as all $\lambda_i < 0$ when $r < 1$, $(x, y, z) \to 0$ as $t \to \infty$, which supports our original assumption in simpifying the system that (x, y, z) remain "small".[11] Thus, in all, $\underline{\mathbf{u}}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \sum_{j=1}^{3} \left[c_j e^{\lambda_j t} \underline{\mathbf{v_j}} \right] \to 0$

as $t \to \infty$, $\implies (|x(t)|, |y(t)|, |z(t)|)$ "small" $\forall t > 0$.[12]

Again, this is all under $r > 1$; when $r > 1$, $4(r - 1)\sigma > 0 \implies \sqrt{(1 + \sigma)^2 + 4(r - 1)\sigma} > 0 \implies -(1 + \sigma) + \sqrt{(1 + \sigma)^2 + 4(r - 1)\sigma} > 0$, ie the root of the characteristic polynomial when we take the positive is now greater than one. In practical terms, this indicates a positive eigenvalue (we will take this to be λ_1 and $\underline{v_1}$), and thus one of our terms will grow as $t \to \infty$; however, the other two eigenvalues remain < 1 and will continue to shrink with time.

Remember that, this whole time, we are working with a linearized version of the origi-

[7]Note the $-b$ in the bottom right of the matrix, surround by 0's; the only component of the "position" vector that will multiply to that $-b$ is z, thus, any vector with only a z component will remain unchanged, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ being the "unit" of these.

[8]As the (r-1) term is thus negative.

[9]Allowing us to operate in $\mathbb{R}^n$, as this is the part under the radical.

[10]Based on the reasoning "above", we are essentially saying (in "pseudomath") $-\alpha + (\alpha - \epsilon) < 0$, as does $-\alpha - (\alpha - \epsilon)$, taking α to represent the "terms" of the quadratic and ϵ the undetermined-but-clearly-there difference.

[11]NB: just because the assumption "held" sts does not mean that it is *always true*; it simply validates the approximation we made in the particular scenario
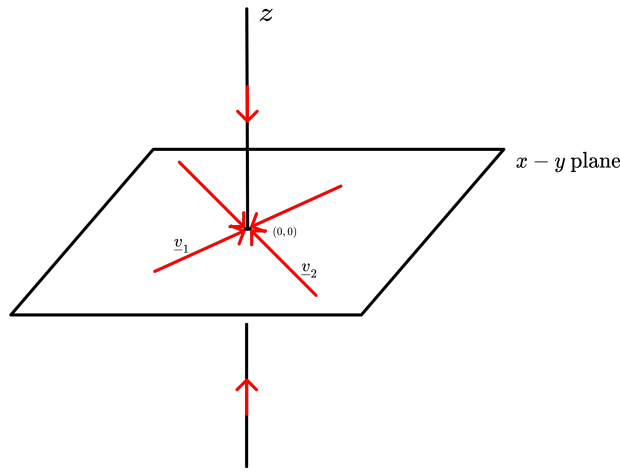
nal Lorenz equations, and thus these diagrams are not fully reflective of reality. As shown, for instance, the Lorenz equations have two other steady states (unless bifurcation...), which influence trajectories in the original system. However, this linearization is still useful in analyzing the dynamics of the system near the steady state $(0, 0, 0)$.

The other two steady states influence the dynamics of the system such that, at certain initial conditions, the trajectories will tend to spiral towards one of the two steady states, as well as (in chaotic systems) jump "randomly" from one steady state to the other, hence the "attractor" name.

1.3   Motivations of Maps

While aforementioned dynamical systems were defined via ordinary differential equations, we can also define them via **maps**, which are discrete dynamical systems. These can be defined:

1. Taking the maxima of a function in a system plotted against the previous maxima (for instance, in the Lorenz map, taking the maxima of $z(t)$ as z_n and plotting z_n against z_{n+1} for natural n).

2. "Redefining" ODE's as maps. For instance, the **logistic map** defined

$$x_{n+1} = f(x_n) = \lambda x_n(1 - x_n)$$

2 Chapter 2

2.1 Flows on the Line

Consider $\dot{x} = f(x)$ or $\dot{x} = f(x, \mu)$ where μ some parameter. Solutions will be $x(t) \in \mathbb{R}$, and will either have $f : \mathbb{R} \to \mathbb{R}(f(x))$ or $f : \mathbb{R}^2 \to \mathbb{R}(f(x, \mu))$.

We may consider initial conditions $x(t = 0) = x_0 \in \mathbb{R}$; different x_0 lead to different solutions. Typically, we do not plot $x(t)$ against t, rather, we show the dynamics in $\mathbb{R}$ on a phase plot.

We could, in principle solve ODEs exactly; eg $\dot{x} = f(x), t \geq 0, x(0) = x_0$. We have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

$$\int \frac{dx}{f(x)} = \int dt$$

$$\int_{x(0)}^{x(t)} \frac{dx}{f(x)} = \int_0^t dt$$

$$\int_{x(0)}^{x(t)} \frac{dx}{f(x)} = t$$

From here, we would have to solve the integral on the left and solve for $x(t)$.

However, we will approach this by determining the dynamics graphically. We do the following:

1. Graph $f(x)$

2. Draw steady states when $\dot{x} = f(x) = 0$

3. For $f(x) \neq 0$ we have either $\dot{x} = f(x) > 0, x(t)$ increasing, or $\dot{x} = f(x) < 0, x(t)$ decreasing

Remark 2.1. *If $f(x) = 0$ and $f'(x) < 0$, then x is a **stable steady state**. If $f(x) = 0$ and $f'(x) > 0$, then x is an **unstable steady state**.*

If $f(x) = f'(x) = 0$, we have a steady state which is "half-stable", eg $\dot{x} = f(x) = x^2$.

Example 2.1. *$\dot{x} = f(x, \mu) = x^2 + \mu$. If $\mu = 0$, we have a "half-stable" point.*

If $\mu > 0$, we have $f(x, \mu) \geq \mu > 0 \forall x$, and we have no steady state.

If $\mu < 0$, we have two stable steady states (on stable, one unstable).

*There is a **bifurcation** at $x = 0$ when $\mu = 0$; the number of steady states changes.*

In short; $\text{sign}(f'(x))$ determines the stability of the steady state, given $f'(x) \neq 0$, in which case we need to study further.

In particular, cases where $f(x) = f'(x) = 0$ are "delicate", and small parameter changes can cause large changes in the dynamics.

2.2 Linear Stability Analysis

$\dot{x} = f(x)$, let $x^* \in \mathbb{R}$ be a s.s., ie $f(x^*) = 0$; what does the dynamics look like near x^*?

First, change variables such that the s.s. is at the origin; let $v(t) = x(t) - x^*$, then

$$\dot{v} = \frac{\mathrm{d}}{\mathrm{d}t}(x(t) - x^*) = \frac{\mathrm{d}x}{\mathrm{d}t} - 0$$
$$= f(x(t)) = f(x^* + v(t)) = g(v(t))$$

Note that this "new" system has a steady state at $v = 0$; $g(0) = f(x^*) = 0$. Here, we can Taylor expand $\dot{v}$:

$$\dot{v} = \sum_{j=0}^{\infty} \frac{v^j f^j(x^*)}{j!}$$
$$= f(x^*) + v f'(x^*) + \frac{v^2}{2} f''(x^*) + \dots$$
$$f(x^*) = 0 \implies \dot{v} = f'(x^*)v + \frac{1}{2}f''(x^*)v^2 + O(v^3)$$

For $x(t) \approx x^*$, we have $|v(t)| = |x(t) - x^*| \ll 1$, then $1 \gg |v(t)| \gg |v(t)|^2 \gg |v(t)|^3 \gg \dots > 0$. Provided $f'(x^*) \neq 0$ for $|v|$ sufficiently small the $f'(x^*)v$ term will dominate others and we can write

$$\dot{v} \approx f(x^*)v.$$

Let $\lambda = f(x^*)$ (just a constant), then we say

$$\dot{v} = \lambda v$$
$$\implies \frac{\mathrm{d}v}{\mathrm{d}t} = \lambda v$$
$$\implies \int_{v_0}^{v(t)} \frac{\mathrm{d}v}{v} = \int_0^t \lambda dt$$
$$\implies [\ln|v|]_{v_0}^{v(t)} = \lambda t$$
$$\implies \ln \frac{v(t)}{v_0} = \lambda t$$

We can drop the absolute value bars as $v(t)$ and v_0 must have the same sign.

$$v(t) = v_0 e^{\lambda t}$$

Thus, if $\lambda < 0$, $v(t) \to 0$ as $t \to \infty$, then $v(t) = x - x^* \implies x(t) \to x^*$ as $t \to \infty$. Else, if

$\lambda > 0$, then $|v_0 e^{\lambda t}|$ grows with t and $|v(t)| \to \infty$ as $t \to \infty$. Importantly, $|v(t)|$ becomes large before it becomes unbounded, meaning that our initial assumption doesn't work as the $O(v^2)$ term becomes significant. However, we can still make conclusions about the (in)stability local to x^*, and we can't draw conclusions for dynanmics $t \to \infty$.

Example 2.2. $\dot{x} = f(x) = x - x^3 = x(1 - x^2) = x(1 - x)(1 + x)$. *We have steady states at* $x = -1, 0, 1$.

$f(x) = x - x^3 \implies f'(x) = 1 - 3x^2$, *so* $f(0) = 1 \implies 0$ *unstable,* $f'(\pm 1) = 1 - 3 = -2 \implies \pm 1$ *stable.*

Alternatively, graph $f(x)$ *and the steady states are visually obvious.*

This is called **linear stability analysis**, as are reducing the nonlinear differential equation $\dot{x} = f(x)$ to a linear ODE $\dot{v} = \lambda v$ (we'll see in higher dimensions that we replace λ with some Jacobian).

2.3 Existence & Uniqueness

We are studying the *qualitative* behavior of solutions, which only makes sense if these solutions exist. Usually we require that the IVP $\dot{x} = f(x), x(0) = x_0$ to have a unique solution $x(t) \forall t \geq 0$, for every $x_0 \in \mathbb{R}$. If they were not unique, they have multiple solutions starting at some point or worse, at points x_0 where multiple solutions are possible orbits will cross each other. The very aspect that this doesn't happen is what makes phase plots useful.

This is also why only autonomous ODEs are considered; in nonautonomous ODES $\dot{x} = f(t, x)$, the value of $\dot{x}(t)$ at a particular value of x will depend of t. For instance, consider $\dot{x} = -x + t$; without a t, this has very straightforward dynamics, but with the t parameter, there are no fixed points and can't be analyzed as easily without solving exactly. Time-dependent problems will not be considered in this course, but they can be dealt with; esp, consider periodic forcing, eg $\dot{x} = x - x^3 + \cos t$. To study system such as this, we define a *map*.

Let T be the period of the forcing, $T = 2\pi$ here. Define a map

$$S_T : x_0 \mapsto x(2\pi),$$

where $x(2\pi) = x(T)$ solves the ODE with initial $x(0) = x_0$. Note that $\cos$ takes the same values in $[0, 2\pi]$ as in $[2\pi, 4\pi]$, so we can use the same map S_T to map from $x(2\pi)$ to $x(4\pi)$.

A fixed point of the map S_T is a periodic orbit of the ODE of period T. In forced systems, period orbits must have period nT, an integer multiple of the forcing period; we'll stick with $\dot{x} = f(x)$ with $f : \mathbb{R} \to \mathbb{R}$.

Lets consider $x(0) = x_0$; when is a solution $x(t)$ unique? When does it exist? What can "go wrong"/when?

Example 2.3. $\dot{u} = u^{\frac{1}{3}}$, $u(0) = 0$.

Clearly, if $u(t) = 0 \forall t$ satisfies the ODE, and the initial condition (stays at 0). But we can also solve it exactly:

$$\frac{du}{dt} = u^{\frac{1}{3}} \implies \int_{u(0)}^{u(t)} \frac{du}{u^{\frac{1}{3}}} = \int_0^t dt$$

$$\left[\frac{3}{2} u^{\frac{2}{3}}\right]_0^{u(t)} = t$$

$$u(t) = \left(\frac{2t}{3}\right)^{\frac{3}{2}}$$

This is different to previously; is it still a valid solution? Clearly, satisfies $u(0) = 0$. And for $t \geq 0$, $u(t) = (\frac{2t}{3})^{3/2} \implies \frac{du}{dt} = \frac{3}{2}\left(\frac{2t}{3}\right)^{1/2} \cdot \frac{2}{3} = u^{1/3}$, and thus also satisfies ODE.

Notice that this fails when $t < 0$, as we would have to take the square root of a negative number. Thus, the solution only valid for $t \geq 0$ and at $t = 0$ it has $u(0) = \dot{u}(0) = 0$. "Appending" our first solution, we write:

$$u(t) = \begin{cases} 0 & t < 0 \\ \left(\frac{2t}{3}\right)^{\frac{3}{2}} & t \geq 0 \end{cases},$$

which is defined for all $t \in \mathbb{R}$, continuous for all $t \in \mathbb{R}$, differentiable for all $t \in \mathbb{R}$, $\dot{u}(t)$ is continuous, and satisfies $u(0) = 0$.

However; we actually have two "new" solutions;

$$u(t) = \begin{cases} 0 & t < 0 \\ +\left(\frac{2t}{3}\right)^{\frac{3}{2}} & t \geq 0 \end{cases}$$

or

$$u(t) = \begin{cases} 0 & t < 0 \\ -\left(\frac{2t}{3}\right)^{\frac{3}{2}} & t \geq 0 \end{cases}.$$

For any $t_0 \geq 0$, $u(t) = \begin{cases} 0 & t \leq 0 \\ \left(\frac{2}{3}(t - t_0)\right)^{\frac{3}{2}} & t \geq t_0 \end{cases}$, ie staying at 0 for arbitrarily long time, then "spitting" away at $t = t_0$. Thus, this IVP has infinitely many solutions, and thus is not unique.

In $\dot{u} = f(u) = u^{1/3}$, $f(u)$ is continuous on $\mathbb{R}$, and differential except at $u = 0$, causing the non-uniqueness here; any other $u(t_0)$ would yield a unique solution.

In phase space, $f(u)$ has an unsteady state at 0, and is increasing for $u > 0$ and decreasing for $u < 0$.

Example 2.4. $\dot{u} = f(u) = u^3$. Now, f is continuous and differentiable on $\mathbb{R}$. We have an unsteady ss at 0 again. Solving:

$$\dot{u} = u^3 \implies \int \frac{du}{u^3} = \int dt$$

$$-\frac{1}{2u^2} = t + C$$

$$\frac{1}{u^2} = -2t - 2c = k - 2t$$

$$u^2 = \frac{1}{k - 2t}$$

$$u = \pm \frac{1}{(k - 2t)^{\frac{1}{2}}}.$$

Let $u(0) = u_0 > 0$ (symmetrical over $u_0 < 0$). We need to take the positive square root for a positive u_0, so $u = \frac{+1}{(k - 2t)^{1/2}}$.

$$u(0) = u_0 \implies u_0 = \frac{1}{(k - 0)^{1/2}}$$

$$\cdots k = \frac{1}{u_0^2}$$

$$\implies u(t) = \frac{1}{\left(\frac{1}{u_0^2} - 2t\right)^{1/2}}$$

For $u_0 > 0$ this solution becomes unbounded in finite time. $u(0) = u_0 > 0$, then as $t \to \frac{1}{2u_0^2}$, $u(t) \to +\infty$.

"Purely" speaking, this is an issue, as the solution does not exist $\forall$ time. In "applications", we don't worry about this since if $u(t)$ getting "very large", it is already outside of the reasonable range of validity for the model.

Theorem 2.1. *Consider the IVP $\dot{x} = f(x)$, $x(0) = x_0 \in \mathbb{R}$. Suppose $f(x), f'(x)$ are both continuous on an open interval $I \subset \mathbb{R}$ and $x_0 \in I$. The, $\exists T > 0$ s.t. the problem has a solution $x(t)$ for $t \in (-T, T)$ with $x(t) \in I$ for all $t \in (-T, T)$, and this solution is unique.*

Remark 2.2. *Won't be proven.*

Remark 2.3. *For Example 2.3, $f'(x)$ not continuous at $x = 0$, and thus the theorem does not apply. Notice that if we consider the same system with $x(0) \neq 0$, we can choose some I which contains x_0 but not 0 and apply it there.*

Corollary 2.1. *If $f(x), f'(x)$ both continuous on all $\mathbb{R}$, then the solution exists and is unique $\forall (T_-, T_+)$, and either $T_- = -\infty$ or $|x(t)| \to \infty$ as $t \to T_-$ (and similar for T_+).*

In this course, we'll always consider f as differentiable as we need it to be. $f \in C^1$ ([13])then solutions are unique while bounded; this is often sufficient, but we'll need more differentiability for bifurcation theory, ie often take infinitely differentiable functions.

2.4   Impossibility of Oscillations

Suppose $\dot{x} = f(x)$ has a periodic solution so $x(t + T) = x(t)\forall t \in \mathbb{R}$ for some period $T > 0$.

For a given value of x, $\dot{x}$ changes sign each time solution crosses x, so $f(x)$ also changes sign. But $f(x)$ uniquely defined for any particular value of x, so this cannot happen, ie, $\dot{x} = f(x)$ *cannot* have periodic solutions.

The same argument shows that all solutions $x(t)$ of $\dot{x} = f(x)$ for $x \in \mathbb{R}$ are **monotonic**.