Louis Meunier

# Analysis 2 MATH255

Course Outline:

*Based on Lectures from Winter, 2024 by Prof. Dmitry Jackobson.*

## Contents

# 1 Introduction

## 1.1 Metric Spaces

↪ **Definition 1.1: Metric Space**

A set $X$ is a *metric space* with distance $d$ if

1. (symmetric) $d(x, y) = d(y, x) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. (triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$

**Remark 1.1.** *If 1., 3. are satisfied but not 2., $d$ can be called a "pseudo-distance".*

↪ **Definition 1.2: Normed Space**

Let $X$ be a vector space over $\mathbb{R}$. The norm on $X$, denoted $||x|| \in \mathbb{R}$, is a function that satisfies

1. $||x|| \geq 0$
2. $||x|| = 0 \iff x = 0$
3. $||c \cdot x|| = |c| \cdot ||x||$
4. $||x + y|| \leq ||x|| + ||y||$

If $X$ is a normed vector space over $\mathbb{R}$, we can define a distance $d$ on $X$ by $d(x, y) = ||x - y||$.

↪ **Proposition 1.1**

If $X$ is a normed vector space over $\mathbb{R}$, a distance $d$ on $X$ by $d(x, y) = ||x - y||$ makes $(X, d)$ a metric space.

*Proof.*

1. $d(x, y) = ||x - y|| \geq 0$
2. $d(x, y) = 0 \iff ||x - y|| = 0 \iff x - y = 0 \iff x = y$
3. $d(x, y) + d(y, z) = ||x - y|| + ||y - z|| \geq ||(x - y) + (y - z)|| = ||x - z|| := d(x, z)$

■

⊛ **Example 1.1: $L^p$ distance in $\mathbb{R}^n$**

Let $\overline{x} \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n)$. The $L^p$ norm is defined

$$||x||_p := (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

In the case $p = 2, n = 2$, we simply have the standard Euclidean distance over $\mathbb{R}^2$.

Unit Balls: consider when $||x||_p \leq 1$, over $\mathbb{R}^2$.

- $p = 1 : |x_1| + |x_2| \leq 1$; this forms a "diamond ball" in the plane.
- $p = 2 : \sqrt{|x_1|^2 + |x_2|^2} \leq 1$; this forms a circle of radius 1. Clearly, this surrounds a larger area than in $p = 2$.

A natural question that follows is what happens as $p \to \infty$? Assuming $|x_1| \geq |x_2|$:

$$\begin{aligned} ||x||_p &= (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \\ &= \left[|x_1|^p \left(1 + \left|\frac{x_2}{x_1}\right|^p\right)\right]^{\frac{1}{p}} \\ &= |x_1| \left(1 + \left|\frac{x_2}{x_1}\right|^p\right)^{\frac{1}{p}} \end{aligned}$$

If $|x_1| > |x_2|$, this goes to $|x_1|$. If they are instead equal, then $||x||_p = |x_1| \cdot 2^{\frac{1}{p}} \to |x_1| \cdot 1$ as well. Hence, $\lim_{p\to\infty} ||x||_p = \max\{|x_1|, |x_2|\}$. Thus, the unit ball will approach $\max\{|x_1|, |x_2|\} \leq 1$, that is, the unit square.

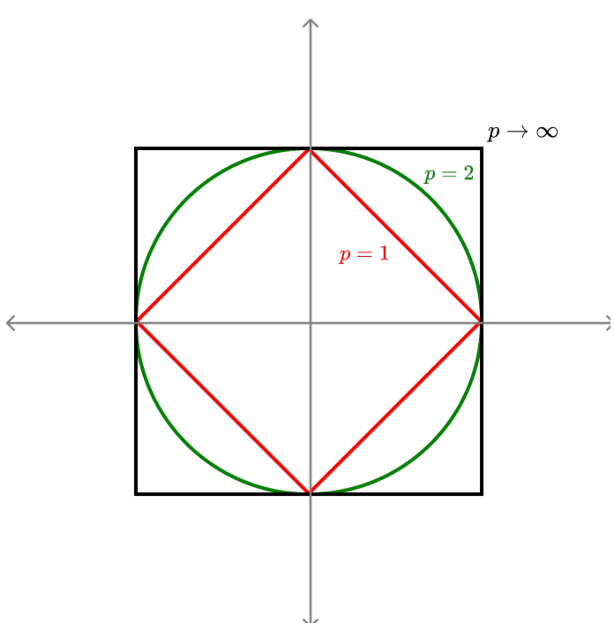


Figure 1: Regions of $\mathbb{R}^2$ where $||x||_p \leq 1$ for various values of $p$.

↪ **Proposition 1.2**

Let $x \in \mathbb{R}^n$. Then, $||x||_p \to \max\{|x_1|, \dots, |x_n|\}$ as $p \to \infty$.

**Remark 1.2.** *This is an extension of the previous example to arbitrary real space; the proof follows nearly identically.*

↪ **Definition 1.3: Convex Set**

Let $X$ be a normed space, and take $x, y \in X$. The line segment from $x$ to $y$ is the set

$$\{t \cdot x + (1-t) \cdot y : 0 \leq t \leq 1\}.$$

Let $A \subseteq X$. $A$ is *convex* $\iff$ $\forall\, x, y \in A$, we have that

$$(t \cdot x + (1-t) \cdot y) \in A \, \forall\, 0 \leq t \leq 1.$$

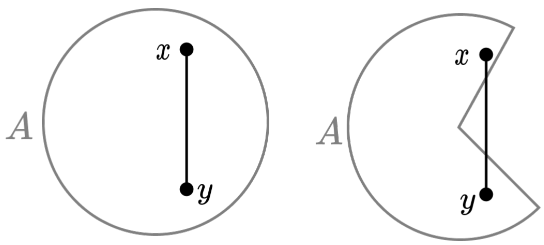


Figure 2: Convex (left) versus not convex (right) sets.

**Remark 1.3.** *Think of this as saying "a set is convex iff every point on a line segment connected any two points is in the set".*

↪ **Definition 1.4:** $\ell_p$

The space $\ell_p$ of sequences is defined as

$$\left\{x = (x_1, x_2, \dots, x_n, \dots) : \sum_{n=1}^{\infty} |x_n|^p < +\infty\right\} \quad *\,.$$

Then, $*$ defines the $\ell^p$ norm on the space of sequences; that is, $||x||_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$.

⊛ **Example 1.2:** $\ell_p$, $x_n = \frac{1}{n}$

. Let $x_n = \frac{1}{n}$. For which $p$ is $x \in \ell_p$? We have, raising the norm to the power of $p$ for ease:

$$\begin{aligned} ||x||_p^p &= |x_1|^p + |x_2|^p + \cdots + |x_n|^p + \cdots \\ &= 1^p + \left(\frac{1}{2}\right)^p + \cdots < \infty \iff p > 1. \end{aligned}$$

In the case that $p = 1$, this becomes a harmonic sum, which diverges.

⊛ **Example 1.3: $L^p$ space of functions**

Let $f(x)$ be a continuous function. We define the norm of $f$ over an interval $[a, b]$

$$||f||_p = \left[\int_a^b |f(x)|^p \, dx\right]^{\frac{1}{p}}.$$

**Remark 1.4.** *Triangle inequality for $||x||_p$ or $||f||_p$ is called Minkowski inequality; $||x||_p + ||y||_p \geq ||x + y||_p$. This will be discussed further.*

⊛ **Example 1.4: Distances between sets in $\mathbb{R}^2$**

Let $A, B$ be bounded, closed, "nice" sets in $\mathbb{R}^2$. We define

$$d(A, B) := \text{Area}(A \triangle B),$$

where

$$A \triangle B : (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

It can be shown that this is a "valid" distance.

**Remark 1.5.** *$\triangle$ denotes the "symmetric difference" of two sets.*

⊛ **Example 1.5: $p$-adic distance**

Let $p$ be a prime number. Let $x = \frac{a}{b} \in \mathbb{Q}$, and write $x = p^k \cdot \left(\frac{c}{d}\right)$, where $c, d$ are not divisible by $p$. Then, the $p$-adic norm is defined $||x||_p := p^{-k}$. It can be shown that this is a norm.

Suppose $p = 2$, $x = 28 = 4 \cdot 7 = 2^2 \cdot 7$. Then, $||28||_2 = 2^{-2} = \frac{1}{4}$; similarly, $||1024||_2 = ||2^{10}||_2 = 2^{-10}$.

More generally, we have that $||2^k||_2 = 2^{-k}$; coversely, $||2^{-k}|| = 2^k$. That is, the closer to $0$, the larger the distance, and vice versa, contrary to our notion of Euclidean distance.

↪ **Proposition 1.3**

$||x||_p$ as defined above is a well-defined norm over $\mathbb{Q}$.

*Proof.* ■

## 1.2 Topological Spaces

↪ **Definition 1.5: Topological space**

A set $X$ is a topological space if we have a collection of subsets $\tau$ of $X$ called *open sets* s.t.

1. $\varnothing \in \tau, X \in \tau$
2. Consider $\{A_\alpha\}_{\alpha \in I}$ where $A_\alpha$ an open set for any $\alpha$; then, $\bigcup_{\alpha \in I} A_\alpha \in \tau$, that is, it is also an open set.
3. If $J$ is a finite set, and $A_\beta$ open for all $\beta \in J$, then $\bigcap_{\beta \in J} A_\beta \in \tau$ is also open.

In other words, 2.: arbitrary unions of open sets are open, and 3.: finite intersections of open sets are open.

↪ **Definition 1.6: Closed sets**

Closed sets are complements of open sets; hence, axioms for closed sets follow appropriately;

1.* $X, \varnothing$ closed;

2.* $B_\alpha$ closed $\forall\, \alpha \in I \implies \bigcap_{\alpha \in I} B_\alpha$ closed.

3.* $B_\beta$ closed $\forall\, \beta \in J$, $J$ finite, then $\bigcup_{\beta \in J} B_\beta$ also closed.

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↪ *Sat Jan 6 10:31:52 EST 2024*