

Course Outline:

*Fundamentals of set theory. Properties of the reals. Limits, limsup, liminf. Continuity. Functions. Differentiation.*

References:

Understanding Analysis, *Abbott*; Introduction to Real Analysis, *Bartle*; Analysis I, *Tao*

Contents

1	Logic, Sets, and Functions	2
1.1	Mathematical Induction & The Naturals . . . . .	2
1.2	Extensions: Integers, Rationals, Reals . . . . .	5
1.2.1	The Insufficiency of the Rationals . . . . .	5
1.3	Sets & Set Operations . . . . .	6
1.4	Functions . . . . .	7
1.4.1	Properties of Functions . . . . .	7
1.5	Reals . . . . .	10
1.6	Density of Rationals in Reals . . . . .	13
1.7	Cardinality . . . . .	16
1.7.1	Power Sets . . . . .	23
2	Sequences	24
2.1	Definitions . . . . .	24
2.2	Properties of Limits . . . . .	26
2.3	Limit Superior, Inferior . . . . .	35
2.4	Subsequences and Bolzano-Weirestrass Theorem . . . . .	37

# 1 Logic, Sets, and Functions

## 1.1 Mathematical Induction & The Naturals

The **natural numbers**,  $\mathbb{N} = \{1, 2, 3, \dots\}$ , are specified by the 5 **Peano Axioms**:

- (1)  $1 \in \mathbb{N}$ <sup>1</sup>
- (2) every natural number has a successor in  $\mathbb{N}$
- (3) 1 is not the successor of any natural number
- (4) if the successor of  $x$  is equal to the successor of  $y$ , then  $x$  is equal to  $y$ <sup>2</sup>
- (5) **the axiom of induction**

The **Axiom of Induction** (AI), can be stated in a number of ways.

**Axiom 1.1** (AI.i). *Let  $S \subseteq \mathbb{N}$  with the properties:*

- (a)  $1 \in S$
- (b) *if  $n \in S$ , then  $n + 1 \in S$* <sup>3</sup>

*then  $S = \mathbb{N}$ .*

<sup>1</sup>using 0 instead of 1 is also valid, but we will use 1 here.

<sup>2</sup>axioms (2)-(4) can be equivalently stated in terms of a successor function  $s(n)$  more rigorously, but won't here

<sup>3</sup>(a) is called the **inductive base**; (b) the **inductive step**. All AI restatements are equivalent in having both of these, and only differentiate on their specific values.

**Example 1.1.** Prove that, for every  $n \in \mathbb{N}$ ,  $1 + 2 + \cdots + n = \frac{n(n+1)}{2} (\equiv (1))$

*Proof (via AI.i).* Let  $S$  be the subset of  $\mathbb{N}$  for which (1) holds; thus, our goal is to show  $S = \mathbb{N}$ , and we must prove (a) and (b) of AI.i.

- by inspection,  $1 \in S$  since  $1 = \frac{1(1+1)}{2} = 1$ , proving (a)
- assume  $n \in S$ ; then,  $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$  by definition of  $S$ . Adding  $n + 1$  to both sides yields:

$$1 + 2 + \cdots + n + (n + 1) = \frac{n(n + 1)}{2} + (n + 1) \quad (1)$$

$$= (n + 1)\left(\frac{n}{2} + 1\right) \quad (2)$$

$$= \frac{(n + 1)(n + 2)}{2} \quad (3)$$

$$= \frac{(n + 1)((n + 1) + 1)}{2} \quad (4)$$

Line (4) is equivalent to statement (1) (substituting  $n$  for  $n + 1$ ), and thus if  $n \in S$ , then  $n + 1 \in S$  and (b) holds. Thus, by AI.i,  $S = \mathbb{N}$  and  $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$  holds  $\forall n \in \mathbb{N}$ . ■

**Example 1.2.** Prove (by induction), that for every  $n \in \mathbb{N}$ ,  $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ .

*Proof.* Follows a similar structure to the previous example. Let  $S$  be the subset of  $\mathbb{N}$  for which the statement holds.  $1 \in S$  by inspection ((a) holds), and we prove (b) by assuming  $n \in S$  and showing  $n + 1 \in S$  (algebraically). Thus, by AI.i,  $S = \mathbb{N}$  and the statement holds  $\forall n \in \mathbb{N}$ . ■

*This can also be proven directly (Gauss' method).*

*Proof (Gauss' method).* Let  $A(n) = 1 + 2 + 3 + \cdots + n$ . We can write  $2 \cdot A(n) = 1 + 2 + 3 + \cdots + n + 1 + 2 + 3 + \cdots + n$ . Rearranging terms (1 with  $n$ , 2 with  $n - 1$ , etc.), we can say  $2 \cdot A(n) = (n + 1) + (n + 1) + \cdots$ , where  $(n + 1)$  is repeated  $n$  times; thus,  $2 \cdot A(n) = n(n + 1)$ , and  $A(n) = \frac{n(n+1)}{2}$ . ■

**Axiom 1.2 (AI.ii).** Let  $S \subseteq \mathbb{N}$  s.t.

(a)  $m \in S$

(b)  $n \in S \implies n + 1 \in S$

then  $\{m, m + 1, m + 2, \dots\} \subseteq S$ .

**Example 1.3.** Using AI.ii, prove that for  $n \geq 2$ ,  $n^2 > n + 1$

*Proof.* Let  $S \subseteq \mathbb{N}$  be the set of  $n$  for which the statement holds.  $n = 2 \implies 4 > 3$ , so the base case holds. Consider  $n^2 > n + 1$  for some  $n \geq 2$ . Then,  $(n + 1)^2 = n^2 + 2n + 1 > n + 1 + 2n + 1 = 3n + 2 > 2n + 2 > n + 2$ , hence  $S = \{2, 3, 4, \dots\}$  (all  $n \geq 2$ ). ■

**Axiom 1.3** (Principle of Complete Induction, AI.iii). *Let  $S \subseteq \mathbb{N}$  s.t.*

(a)  $1 \in S$

(b) *if  $1, 2, \dots, n - 1 \in S$ , then  $n \in S$*

*then  $S = \mathbb{N}$ .*

Finally, combining AI.ii and AI.iii;

**Axiom 1.4** (AI.iv). *Let  $S \subseteq \mathbb{N}$  s.t.:*

(a)  $m \in S$

(b) *if  $m, m + 1, \dots, m + n \in S$ , then  $m + n + 1 \in S$*

*then  $\{m, m + 1, m + 2, \dots\} \subseteq S$ .*

**Theorem 1.1** (Fundamental Theorem of Arithmetic). *Every natural number  $n$  can be written as a product of one or more primes.*<sup>4</sup>

<sup>4</sup>1 is not a prime number

*Proof of Theorem 1.1.* Let  $S$  be the set of all natural numbers that can be written as a product of one or more primes. We will use AI.iv to show  $S = \{2, 3, \dots\}$ .

- (a) holds; 2 is prime and thus  $2 \in S$
- suppose that  $2, 3, \dots, 2 + n \in S$ . Consider  $2 + (n + 1)$ :
  - if  $2 + (n + 1)$  is *prime*, then  $2 + (n + 1) \in S$ , as all primes are products of 1 and themselves and are thus in  $S$  by definition.
  - if  $2 + (n + 1)$  is *not prime*, then it can be written as  $2 + (n + 1) = a \cdot b$  where  $a, b \in \mathbb{N}$ , and  $1 < a < 2 + (n + 1)$  and  $1 < b < 2 + (n + 1)$ . By the definition of  $S$ ,  $a, b \in S$ , and can thus be written as the product of primes. Let  $a = p_1 \cdot \dots \cdot p_l$  and  $b = q_1 \cdot \dots \cdot q_j$ , where the  $p$ 's and  $q$ 's are prime and  $l, j \geq 1$ . Then,  $a \cdot b$  is a product of primes, and thus so is  $2 + (n + 1)$ . Thus,  $2 + (n + 1) \in S$ , and by AI.iv,  $S = \{2, 3, 4, \dots\}$

■

## 1.2 Extensions: Integers, Rationals, Reals

Consider the set of naturals  $\mathbb{N} = \{1, 2, 3, \dots\}$ . Adding 0 to  $\mathbb{N}$  defines  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ . We define the **integers** as the set  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ , or the set of all positive and negative whole numbers.

Within  $\mathbb{Z}$ , we can define multiplication, addition and subtraction, with the neutrals of 1 and 0, respectively. However, we cannot define division, as we are not guaranteed a quotient in  $\mathbb{Z}$ . This necessitates the **rational**s,  $\mathbb{Q}$ . We define

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \right\}.$$

On  $\mathbb{Q}$ , we have the familiar operations of multiplication, addition, subtraction and properties of associativity, distributivity, etc. We can also define division, as  $\frac{\frac{p}{q}}{\frac{p'}{q'}} = \frac{pq'}{qp'}$ .

We can also define a relation  $<$  between fractions, such that

- $x < y$  and  $y < z \implies x < z$
- $x < y \implies x + z < y + z$

$\mathbb{Q}$ , together with its operations and relations above, is called an **ordered field**.

### 1.2.1 The Insufficiency of the Rationals

We can consider historical reasoning for the extension of  $\mathbb{Q}$  to  $\mathbb{R}$ . Consider a right triangle of legs  $a, b$  and hypotenuse  $c$ . By the Pythagorean Theorem,  $a^2 + b^2 = c^2$ . Consider further the case there  $a = b = 1$ , and thus  $c^2 = 2$ . Does  $c$  exist in  $\mathbb{Q}$ ?

**Proposition 1.1.**  $c^2 = 2, c \notin \mathbb{Q}$ .

*Proof of Proposition 1.1.* Suppose  $c \in \mathbb{Q}$ . We can thus write  $c = \frac{p}{q}$ , where<sup>5</sup>  $p, q \in \mathbb{N}$ , and  $p, q$  share no common divisors, ie they are in “simplest form”. Notably,  $p$  and  $q$  cannot *both* be even (under our initial assumption), as they would then share a divisor of 2. We write

$$\begin{aligned} c &= \frac{p}{q} \\ c^2 = 2 &= \frac{p^2}{q^2} \\ 2q^2 &= p^2 \end{aligned}$$

$p \in \mathbb{N} \implies p^2 \in \mathbb{N}$ , and thus  $p^2$ , and therefore<sup>6</sup> $p$ , must be divisible by 2 ( $\implies p$  even). Therefore, we can write  $p = 2p_1, p_1 \in \mathbb{N}$ , and thus  $2q^2 = (2p_1^2)^2 \implies q^2 = 2p_1^2$ . By the same reasoning,  $q$  must now be even as well, contradicting our initial assumption that  $p$  and  $q$  share no common divisors. Thus,  $c \notin \mathbb{Q}$ . ■

<sup>5</sup>Note that in the definition of  $\mathbb{Q}$ ,  $p, q$  are defined to be in  $\mathbb{Z}$ ; however, as we are using a geometric argument, we can assume  $c > 0 \implies \text{Sign}(p) = \text{Sign}(q)$ , and we can just take  $p, q \in \mathbb{N}$  for convenience and wlog.

<sup>6</sup> $\sqrt{\text{even}} = \text{even}$

<sup>7</sup> $X$  is often omitted if it is clear from context.

### 1.3 Sets & Set Operations

- $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- $\bigcup_{i=1}^{\infty} A_i = \bigcup_{n \in \mathbb{N}} A_n = \{x : x \in A_n \text{ for some } n \in \mathbb{N}\}$
- $\bigcap_{i=1}^{\infty} A_i = \bigcap_{n \in \mathbb{N}} A_n = \{x : x \in A_n \forall n \in \mathbb{N}\}$
- $A^C = \{x : x \in X \text{ and } x \notin A\}$ <sup>7</sup>

**Theorem 1.2** (De Morgan's Theorem(s)). *Let  $A, B$  be sets. Then,*

$$(a) \quad (A \cap B)^C = A^C \cup B^C$$

and

$$(b) \quad (A \cup B)^C = A^C \cap B^C.$$

*Proof of Theorem 1.2.* (b) (A similar argument follows...) ■

**Proposition 1.2.**

$$(a) \quad \left( \bigcap_{n=1}^{\infty} A_n \right)^C = \bigcup_{n=1}^{\infty} A_n^C$$

$$(b) \quad \left( \bigcup_{n=1}^{\infty} A_n \right)^C = \bigcap_{n=1}^{\infty} A_n^C$$

*Proof of Proposition 1.2.* Consider Proposition (b). Working from the left-hand side, we have

$$\begin{aligned} \left( \bigcup_{n=1}^{\infty} A_n \right)^C &= \{x : x \notin \bigcup A_n\} \\ &= \{x : x \notin A_n \forall n \in \mathbb{N}\} \\ &= \bigcap \{x : x \notin A_n\} \\ &= \bigcap A_n^C \end{aligned}$$

(a) can be logically deduced from this result. Consider the RHS,  $\bigcup A_n^C$ . Taking the complement:

$$\begin{aligned} \left( \bigcup A_n^C \right)^C &\stackrel{\text{via (b)}}{=} \bigcap A_n^{CC} \\ &= \bigcap A_n \end{aligned}$$

Taking the complement of both sides, we have  $\bigcup A_n^C = (\bigcap A_n)^C$ , proving (a). ■

## 1.4 Functions

**Definition 1.1.** Let  $A, B$  be sets. A function  $f$  is a rule assigned to each  $x \in A$  a corresponding unique element  $f(x) \in B$ . We denote

$$f : A \rightarrow B.$$

**Definition 1.2.** The domain of a function  $f : A \rightarrow B$ , denoted  $\text{Dom}(f) = A$ . The range of  $f$ , denoted  $\text{Ran}(f) = \{f(x) : x \in A\}$ . Clearly,  $\text{Ran}(f) \subseteq B$ , though equality is not necessary.

**Example 1.4.** The function  $f(x) = \sin x$ ,  $f : \mathbb{R} \rightarrow [-1, 1]$ . Here,  $\text{Dom}(f) = \mathbb{R}$ , and  $\text{Ran}(f) = [-1, 1]$ .

**Example 1.5** (Dirichlet Function).  ${}^8 f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ . Despite not having a true “explicit” formula, so to speak, this is still a valid function (under modern definitions).

<sup>8</sup>Look up a **graph** of this function. Its beautiful. It’s also interesting to note that its integral is simply 0.

### 1.4.1 Properties of Functions

**Proposition 1.3.** Let  $f : A \rightarrow B$ ,  $C \subseteq A$ ,  $f(C) = \{f(x) : x \in C\}$ . We claim  $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$ .

*Proof.* We will prove this by showing  $(1) \subseteq$  and  $(2) \supseteq$ .

(1)  $y \in f(C_1 \cup C_2) \implies$  for some  $x \in C_1 \cup C_2, y = f(x)$ . This means that either for some  $x \in C_1, y = f(x)$ , or for some  $x \in C_2, y = f(x)$ . This implies that either  $y \in f(C_1)$ , or  $y \in f(C_2)$ , and thus  $y$  *must* be in their union, ie  $y \in C_1 \cup C_2$ .

(2)  $y \in f(C_1) \cup f(C_2) \implies y \in f(C_1)$  or  $y \in f(C_2)$ . This means that for some  $x \in C_1, y = f(x)$ , or for some  $x \in C_2, y = f(x)$ . Thus,  $x$  *must* be in  $C_1 \cup C_2$ , and for some  $x \in C_1 \cup C_2, y = f(x) \implies y \in f(C_1 \cup C_2)$ .

(1) and (2) together imply that  $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$ . ■

**Example 1.6.** Let  $A_n = 1, 2, \dots$  be a sequence of sets. Prove that  $f(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f(A_n)$ .

*Proof.* Let  $y \in f(\bigcup_{n=1}^{\infty} A_n)$ . This implies that  $\exists x \in \bigcup_{n=1}^{\infty} A_n$  s.t.  $f(x) = y$ . This implies that  $x \in A_n$  for some  $n$ , and  $y \in f(A_n)$  for that same “some”  $n$ , and thus  $y$  must be in the union of all possible  $f(A_n)$ , ie  $y \in \bigcup f(A_n)$ . This shows  $\subseteq$ , use similar logic for the reverse. ■

**Proposition 1.4.**  $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$  <sup>9</sup>

*Proof.*  $y \in f(C_1 \cap C_2) \implies$  for some  $x \in C_1 \cap C_2, y = f(x)$ . This implies that for some  $x \in C_1, y = f(x)$  **and** for some  $x \in C_2, y = f(x)$ . Note that this does *not* imply that these  $x$ 's are the same, ie this reasoning is not reversible as in the previous union case. This implies that  $y \in f(C_1)$  and  $y \in f(C_2) \implies y \in f(C_1) \cap f(C_2)$ . ■

**Example 1.7.** Prove that if  $A_n, n = 1, 2, \dots, f(\bigcap_{n=1}^{\infty} A_n) \subseteq \bigcap_{n=1}^{\infty} f(A_n)$ .

*Proof (Sketch).* Use the same idea as in Example 1.6, but, naturally, with intersections. ■

**Example 1.8.** Take  $f(x) = \sin x, A = \mathbb{R}, B = \mathbb{R}$ , and take  $C_1 = [0, 2\pi], C_2 = [2\pi, 4\pi]$ . Then,  $f(C_1) = [-1, 1]$ , and  $f(C_2) = [-1, 1]$ . But  $C_1 \cap C_2 = \{2\pi\}; f(\{2\pi\}) = \{\sin 2\pi\} = \{0\}$ , and thus  $f(C_1 \cap C_2) = \{0\}$ , while  $f(C_1) \cap f(C_2) = [-1, 1]$ , as shown in Proposition 1.4.

**Definition 1.3** (Inverse Image of a Set). Let  $f : A \rightarrow B$  and  $D \subseteq B$ . The inverse image of  $D$  by  $F$  is denoted  $f^{-1}(D)$ <sup>10</sup> and is defined as

$$f^{-1}(D) = \{x \in A : f(x) \in D\}.$$

<sup>9</sup>NB: the reverse is not always true, ie these sets are not always equal; “lack” of equality is more “common” than not.

<sup>10</sup>Note that this is **not** equivalent to the typical definition of an inverse function:  $f^{-1}$  may not



**Example 1.9.**  $A = [0, 2\pi]$ ,  $B = \mathbb{R}$ ,  $f(x) = \sin x$ ,  $D = [0, 1]$ .

$$f^{-1}(D) = \{x \in A : f(x) \in D\} = \{x \in [0, 2\pi] : \sin(x) \in [0, 1]\} = [0, \pi].$$

**Proposition 1.5.** Given function  $f$  and sets  $D_1, D_2$ ,

$$(a) f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$$

$$(b) f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)^{11}$$

**Proposition 1.6.** Let  $A_n, n = 1, 2, 3 \dots$ . Then,

$$(a) f^{-1}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$$

$$(b) f^{-1}(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} f^{-1}(A_n)$$

*Proof.*<sup>12</sup>

(a)

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) &\iff f(x) \in \bigcup_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for some } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for some } n \in \mathbb{N} \\ &\iff x \in \bigcup_{n=1}^{\infty} f^{-1}(A_n) \end{aligned}$$

(b)

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) &\iff f(x) \in \bigcap_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for all } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for all } n \in \mathbb{N} \\ &\iff x \in \bigcap_{n=1}^{\infty} f^{-1}(A_n)^{13} \end{aligned}$$

■

**Remark 1.1.**  $f : A \rightarrow B$ ,  $A_1 \subseteq A$ . Given  $f(A_1^C)$  and  $f(A_1)^C$ , there is **no general relation** between the two.

<sup>11</sup>Just see next proposition; if you really need convincing, just use 2 rather than  $\infty$  as the upper limit of the union-/intersections and use the same proof.

<sup>13</sup>This is a “proof by definitions” as I like to call it.

<sup>13</sup>Similar proof can be used to prove Proposition 1.5, less generally.

For instance, take  $A = [0, 6\pi]$ ,  $B = [-1, 2]$ ,  $C = [0, 2\pi]$ , and  $f(x) = \sin x$ . Then,  $f(C) = [-1, 1]$ , and  $f(C^C) = f([-1, 0)) = [-1, 1]$ , but  $f(C)^C = [-1, 1]^C = (1, 2]$ , and  $f(C^C) \neq f(C)^C$ ; in fact, these sets are disjoint.

**Proposition 1.7.** Let  $f : A \rightarrow B$  and let  $D \subseteq B$ . Then  $f^{-1}(D^C) = [f^{-1}(D)]^C$ .

*Proof.*

$$\begin{aligned} f^{-1}(D^C) &= \{x : f(x) \in D^C\} = \{x : f(x) \notin D\} \\ [f^{-1}(D)]^C &= [\{x : f(x) \in D\}]^C = \{x : x \notin f^{-1}(D)\} = \{x : f(x) \notin D\} \end{aligned}$$

■

## 1.5 Reals

**Axiom 1.5** (Of Completeness). Any non-empty subset of  $\mathbb{R}$  that is bound from above has at least one upper bound (also called the supremum).

In other words; let  $A \subseteq \mathbb{R}$  and suppose  $A$  is bounded from above ( $A$  has at a least upper bound). Then  $\sup(A)$  exists.

Real numbers, algebraically have the same properties as the rationals; we have addition, multiplication, inverse of non-zero real numbers, and we have the relation  $<$ . All together,  $\mathbb{R}$  is an ordered field.

**Definition 1.4.** Let  $A \subseteq \mathbb{R}$ . A number  $b \in \mathbb{R}$  is called an **upper bound** for  $A$  if for any  $x \in A$ ,  $x \leq b$ .

A number  $l \in \mathbb{R}$  is called a **lower bound** for  $A$  if for any  $x \in A$ ,  $x \geq l$ .

**Definition 1.5** (The Least Upper Bound). Let  $A \subseteq \mathbb{R}$ . A real number  $s$  is called the **least upper bound** for  $A$  if the following holds:

- (a)  $s$  is an upper bound for  $A$
- (b) if  $b$  is any other upper bound for  $A$ , then  $s \leq b$ .

The least upper bound of a set  $A$  is unique, if it exists; if  $s$  and  $s'$  are two least upper bounds, then by (a),  $s$  and  $s'$  are upper bound for  $A$ , and by (b),  $s \leq s'$  and  $s' \leq s$ , and thus  $s = s'$ .

This least upper bound is called the supremum of  $A$ , denoted  $\sup(A)$ .

**Definition 1.6** (The Greatest Lower Bound). Let  $A \subset \mathbb{R}$ . A number  $i \in \mathbb{R}$  is called the **greatest lower bound** for  $A$  if the following holds:

- (a)  $i$  is a lower bound for  $A$
- (b) if  $l$  is any other lower bound for  $A$ , then  $i \geq l$ .

If  $i$  exists, it is called the infimum of  $A$  and is denoted  $i = \inf(A)$ , and is unique by the same argument used for  $\sup(A)$ .

**Proposition 1.8.** Let<sup>14</sup>  $A \subseteq \mathbb{R}$  and let  $s$  be an upper bound for  $A$ . Then  $s = \sup(A)$  iff for any  $\varepsilon > 0$ , there exists  $x \in A$  s.t.  $s - \varepsilon < x$ .

*Proof.* We have two statements:

- I.  $s = \sup(A)$ ;
- II. For any  $\varepsilon > 0$ ,  $\exists x \in A$  s.t.  $s - \varepsilon < x$ ;

and we desire to show that  $I \iff II$ .

- $I \implies II$ : Let  $\varepsilon > 0$ . Then, since  $s = \sup(A)$ ,  $s - \varepsilon$  *cannot* be an upper bound for  $A$  (as  $s$  is the least upper bound, and thus  $s - \varepsilon < s$  cannot be an upper bound at all). Thus, there exists  $x \in A$  such that  $s - \varepsilon < x$ , and thus if I holds, II must hold.
- $II \implies I$ : suppose that this does not hold, ie II holds for an upper bound  $s$  for  $A$ , but  $s \neq \sup(A)$ . Then, there exists some upper bound  $b$  of  $A$  s.t.  $b < s$ . Take  $\varepsilon = s - b$ .  $\varepsilon > 0$ , and since II holds, there exists  $x \in A$  such that  $s - \varepsilon < x$ . But since  $s - \varepsilon = b$  and thus  $b < x$ , then  $b$  cannot be an upper bound for  $A$ , contradicting our initial condition. So, if  $II \implies I$  does *not* hold, we have a “impossibility”, ie a value  $b$  which is an upper bound for  $A$  which cannot be an upper bound, and thus  $II \implies I$ .

■

**Proposition 1.9.** Let  $A \subseteq \mathbb{R}$  and let  $i$  be a lower bound for  $A$ . Then  $i = \inf(A) \iff$  for every  $\varepsilon > 0$  there exists  $x \in A$  s.t.  $x < i + \varepsilon$ .<sup>15</sup>

**Remark 1.2.** Axiom 1.5 can also be expressed in terms of infimum. Define  $-A = \{-x : x \in A\}$ . Then, if  $b$  is an upper bound for  $A$ , then  $b \geq x \forall x \in A$ , then  $-b \leq -x \forall x \in A$ , ie  $-b$  is a lower bound of  $-A$ . Similarly, if  $l$  is a lower bound for  $A$ ,  $-l$  is an upper bound for  $-A$ .

<sup>14</sup>Note that this, and Proposition 1.9 that follows, are *not* definitions: they are restatements, and do technically require proof.

<sup>15</sup>Use similar argument to proof of previous proposition.

Thus, if  $A$  is bounded from above, then

$$-\sup(A) = \inf(-A),$$

and if  $A$  is bounded from below,

$$-\inf(A) = \sup(-A).$$

**Axiom 1.6** (AC (infimum)). Let  $A \subseteq \mathbb{R}$ ; if  $A$  bounded from below,  $\inf(A)$  exists.

**Definition 1.7** (max, min). Let  $A \subseteq \mathbb{R}$ . An  $M \in A$  is called a maximum of  $A$  if for any  $x \in A$ ,  $x \leq M$ .  $M$  is an upper bound for  $A$ , **but also**  $M \in A$ .

If  $M$  exists, then  $M = \sup(A)$ ;  $M$  is an upper bound, and if  $b$  any other upper bound, then  $b \geq M$ , because  $M \in A$ , and thus  $M = \sup(A)$ .

NB:  $M = \max(A)$  **need not** exist, while  $\sup(A)$  must exist. Consider  $A = [0, 1)$ ;  $\sup(A) = 1$ , but there exists no  $\max(A)$ .

The same logic exists for the existence of minimum vs infimum (consider  $(0, 1)$ , with no maximum nor minimum).

**Theorem 1.3** (Nested interval property of  $\mathbb{R}$ ). Let  $I_n = [a_n, b_n] = \{x : a_n \leq x \leq b_n\}$ ,  $n = 1, 2, 3 \dots$  be an infinite sequence of bounded, closed intervals s.t.

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots I_n \supseteq I_{n+1} \supseteq \dots$$

Then,  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$  (note that this does not hold in  $\mathbb{Q}$ ).

*Proof.* <sup>16</sup> We have  $I_n = [a_n, b_n]$ ,  $I_{n+1} = [a_{n+1}, b_{n+1}]$ ,  $\dots$ . And the inclusion  $I_n \supseteq I_{n+1}$ .  $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ ,  $\forall n \geq 1$ . So, the sequence  $a_n$  (left-end) is increasing, and the sequence  $b_n$  (right-end) is decreasing.

We also have that for any  $n, k \geq 1$ ,  $a_n \leq b_k$ . We see this by considering two cases:

- Case 1:  $n \leq k$ , then  $a_n \leq a_k$  (as  $a_n$  is increasing), and thus  $a_n \leq a_k \leq b_k$ .
- Case 2:  $n > k$ , then  $a_n \leq b_n \leq b_k$  (again, as  $b_n$  is decreasing).

Let  $A = \{a_n : n \in \mathbb{N}\}$ . Then,  $A$  is bounded from above by **any**  $b_k$  (as in our inequality we showed above). Let  $x = \sup(A)$ , which must exist by Axiom 1.5.

<sup>16</sup>Sketch: show that the left-end points are increasing and the right-end points are decreasing. Show either that all the left-end points are bounded from above or that all the right-end points are bounded from below. As a result, there exists a sup/inf (depending on which end you choose) of the set of all the right/left points. For the sup case, all upper bounds must be  $\geq \sup$ , and thus the sup is in all  $I_n$ , and thus in their intersect, and thus the intersect is not empty.

Note that as a result,  $x \geq a_n$  for all  $n$ , and for all  $k$ ,  $x \leq b_k$ , as  $x$  is the lowest upper bound and must be  $\leq$  all other upper bounds, and so for all  $n \geq 1$ ,  $a_n \leq x \leq b_n$ , ie  $x \in I_n \forall n \geq 1$ , and thus  $x \in \bigcap_{n=1}^{\infty} I_n$  and so  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ . ■

**Remark 1.3.** *The proof above emphasized the left-end points; it can equivalently be proven via the right-end points, and using  $y = \inf(\{b_n : n \in \mathbb{N}\}) = \inf(B)$ , rather than  $\sup(A)$ , and showing that  $y \in \bigcap I_n$ .*

**Remark 1.4.** *Note too that, if  $x = \sup(A)$  and  $y = \inf(B)$ , then  $x, y \in \bigcap_{n=1}^{\infty} I_n$ ; in fact,  $\bigcap_{n=1}^{\infty} I_n = [x, y]$ .*

**Remark 1.5.** *The intervals  $I_n$  must be closed; if not, eg  $I_n = (0, \frac{1}{n})$ , then  $\bigcap_{n=1}^{\infty} I_n = \emptyset$ .*

## 1.6 Density of Rationals in Reals

**Proposition 1.10** (Archimedean Property). (a) *For any  $x \in \mathbb{R}$ , there exists a natural number  $n$  s.t.  $n > x$ .*

(b) *For any  $y \in \mathbb{R}$  satisfying  $y > 0$ ,  $\exists n \in \mathbb{N}$  such that  $\frac{1}{n} < y$ .*

**Remark 1.6.** (a) states that  $\mathbb{N}$  is not a bounded subset of  $\mathbb{R}$ .

**Remark 1.7.** (b) follows from (a) by taking  $x = \frac{1}{y}$  in (a), then  $\exists n \in \mathbb{N}$  s.t.  $n > \frac{1}{y} \implies \frac{1}{n} < y$ , and thus we need only prove (a).

**Remark 1.8.** Recall that  $\mathbb{Q}$  is an ordered field (operations  $+, \cdot$  and a relation  $<$ ).  $\mathbb{Q}$  can be extended to a larger ordered field with extended definitions of these operations/relations, such that it contains elements that are larger than any natural numbers (ie, not bounded above). This is impossible in  $\mathbb{R}$  due to AC.

*Proof.* Suppose (a) not true in  $\mathbb{R}$ , ie  $\mathbb{N}$  is bounded from above in  $\mathbb{R}$ . Let  $\alpha = \sup \mathbb{N}$ , which exists by AC.

Consider  $\alpha - 1$ ; since  $\alpha - 1 < \alpha$ ,  $\alpha - 1$  is not an upper bound of  $\mathbb{N}$ . So, there exists some  $n \in \mathbb{N}$  s.t.  $\alpha - 1 < n$ ; then,  $\alpha < n + 1$  where  $n + 1 \in \mathbb{N}$ , and thus  $\alpha$  is also not an upper bound, as there exists a natural number that is greater than  $\alpha$ . This contradicts the assumption that  $\alpha = \sup \mathbb{N}$ , so (a) must be true. ■

**Theorem 1.4** (Density). *Let  $a, b \in \mathbb{R}$  s.t.  $a < b$ . Then,  $\exists x \in \mathbb{Q}$  s.t.  $a < x < b$ .*

**Remark 1.9.** *If you take  $a \in \mathbb{R}$  and  $\varepsilon > 0$ , then by the theorem,  $\exists x \in \mathbb{Q}$  where  $x \in (a - \varepsilon, a + \varepsilon)$ . So any real number can be approximated arbitrarily closely (via choose of  $\varepsilon$ ) by a rational number.*

*Proof.* Since  $b - a > 0$ , by (b) of Proposition 1.10,  $\exists n \in \mathbb{N}$  s.t.  $\frac{1}{n} < b - a$ , ie  $na + 1 < nb$ .

Let  $m \in \mathbb{Z}$  s.t.  $m - 1 \leq na < m$ . Such an integer must exist since  $\bigcup_{m \in \mathbb{Z}} [m - 1, m) = \mathbb{R}$ , the family  $[m - 1, m), m \in \mathbb{Z}$  makes partitions of  $\mathbb{R}$ . Then,  $na < m$  gives that  $a < \frac{m}{n}$ . On the other hand,  $m - 1 \leq na$  gives  $m \leq na + 1 < nb$ . So  $\frac{m}{n} < b$  and it follows that  $\frac{m}{n}$  satisfies  $a < \frac{m}{n} < b$ . ■

In the proof, we used the claim:

**Proposition 1.11.** *If  $z \in \mathbb{R}$ , then there exists  $m \in \mathbb{Z}$  s.t.  $m - 1 \leq z < m$ .*

*Proof.* Let  $S$  be a non-empty subset of  $\mathbb{N}$ . Then  $S$  has the least element;  $\exists m \in S$  s.t.  $m \leq n, \forall n \in S$ .

We can assume  $z \geq 0$ ; if  $0 \leq z < 1$ , then we are done (take  $m = 1$ ), and assume that  $z \geq 1$ . Let now  $S = \{n \in \mathbb{N} : z < n\}$ ,  $\neq \emptyset$  by Proposition 1.10, (a). Let  $m$  be the least element of  $S$ . It exists by Well-Ordering Property; then, since  $m \in S$ ,  $z < m$ . But, we also have  $m - 1 \leq z$ , otherwise, if  $z < m - 1$  then  $m - 1 \in S$  and then  $m$  is not the least element of  $S$ . Thus, we have  $m - 1 \leq z < m$ , as required. ■

**Theorem 1.5.** *The set  $J$  of irrationals is also dense in  $\mathbb{R}$ . That is, if  $a, b \in \mathbb{R}, a < b$ ,  $\exists$  irrational  $y$  s.t.  $a < y < b$  (noting that  $J = \mathbb{R} \setminus \mathbb{Q}$ ).*

*Proof.* Fix  $y_0 \in \mathbb{J}$ . Consider  $a - y_0, b - y_0$ .  $a - y_0 < b - y_0$ , and by density of rationals,  $\exists x \in \mathbb{Q}$  s.t.  $a - y_0 < x < b - y_0$ . Then,  $a < y_0 + x < b$ ; let  $y = x + y_0$ , and we have  $a < y < b$ .

Note that  $y$  cannot be rational; if  $y \in \mathbb{Q}, y = x + y_0 \implies y - x = y_0$ , and since  $x \in \mathbb{Q}, y - x \in \mathbb{Q} \implies y_0 \in \mathbb{Q}$ , contradicting the original choice of  $y_0 \notin \mathbb{Q}$ . Thus,  $y \in J$ . ■

**Theorem 1.6.**  *$\exists$  a unique positive real number  $\alpha$  s.t.  $\alpha^2 = 2$ .*

*Proof.* We show both uniqueness, existence:<sup>17</sup>

Uniqueness: if  $\alpha^2 = 2$  and  $\beta^2 = 2, \alpha \geq 0, \beta \geq 0$ , then  $0 = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta) > 0$ , and so  $\alpha - \beta = 0 \implies \alpha = \beta$ .

- Existence: consider the set  $A = \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 < 2\}$ .  $A$  is not empty as  $1 \in A$ . The set of  $A$  is bounded above by 2, since if  $x \geq 2$ , then  $x^2 \geq 4 > 2$ , so  $x \notin A$ . So, by AC,  $\sup A$  exists; let  $\alpha = \sup A$ . We will show that  $\alpha^2 = 2$ , by showing that both  $\alpha^2 < 2$  and  $\alpha^2 > 2$  are contradictions.

$$\alpha^2 < 2$$

For any  $n \in \mathbb{N}$  we expand

$$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \leq \alpha^2 + \frac{2\alpha + 1}{n},$$

noting that  $\frac{1}{n^2} \leq \frac{1}{n}$  for  $n \geq 1$ .

Let  $y = \frac{2-\alpha^2}{2\alpha+1}$ , which is strictly positive. By Proposition 1.10,  $\exists n_0 \in \mathbb{N}$  s.t.

$$\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1} \text{ or } \frac{2\alpha+1}{n_0} < 2-\alpha^2.$$

Substituting this  $n_0$  into our inequality, we have

$$\left(\alpha + \frac{1}{n_0}\right)^2 \leq \alpha^2 + \frac{2\alpha+1}{n_0} < \alpha^2 + 2 - \alpha^2 = 2.$$

Since  $\alpha + \frac{1}{n_0}$  is positive,  $\alpha + \frac{1}{n_0} \in A$ . But, since  $\alpha = \sup A$ ,  $\alpha + \frac{1}{n_0} \leq \alpha$ , which is impossible, so  $\alpha^2 < 2$  cannot be true.

$$\alpha^2 > 2$$

Take  $n \in \mathbb{N}$ ;

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}.$$

Now, let  $y = \frac{\alpha^2-2}{2\alpha}$ ;  $y > 0$ , and by Proposition 1.10,  $\exists n_0 \in \mathbb{N}$  s.t.

$$\frac{1}{n_0} < \frac{\alpha^2-2}{2\alpha}, \text{ or } \frac{2\alpha}{n_0} < \alpha^2 - 2.$$

Substituting this  $n_0$ , we have

$$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 - \frac{2\alpha}{n_0} > \alpha^2 + 2 - \alpha^2 = 2.$$

So for any  $x \in A$ , we have  $\left(\alpha - \frac{1}{n_0}\right)^2 > 2 > x^2$ .  $\alpha - \frac{1}{n_0} > 0$ , and  $x > 0$ , since  $x \in A$ . Then,  $\left(\alpha - \frac{1}{n_0}\right)^2 > x^2$  gives that  $\alpha - \frac{1}{n_0} > x$ .

So,  $\alpha - \frac{1}{n_0} > x$  for all  $x \in A$ . So  $\alpha - \frac{1}{n_0}$  is an upper bound for  $A$ , but since  $\alpha = \sup A$ ,  $\alpha - \frac{1}{n_0} \geq \alpha$  ie  $\alpha \geq \alpha + \frac{1}{n_0}$ , which is impossible. So  $\alpha^2 > 2$  cannot be true.

Thus,  $\alpha^2 = 2$ .

■

<sup>17</sup>Proof sketch:  
uniqueness is clear.  
Existence follows from  
showing that  $\alpha^2$  cannot  
be either  $<$  or  $> 2$ . This

**Remark 1.10.** A similar argument gives that for any  $x \in \mathbb{R}$ ,  $x \geq 0$ ,  $\exists! \alpha \in \mathbb{R}$ ,  $\alpha \geq 0$  such that  $\alpha^2 = x$ . This  $\alpha$  is called the square root of  $x$ , denoted  $\alpha = \sqrt{x}$ .

**Remark 1.11.** For any natural number  $m \geq 2$  and  $x \geq 0$ ,  $\exists! \alpha \in \mathbb{R}$ ,  $\alpha \geq 0$  s.t.  $\alpha^m = x$ . The proof is similar, and we call  $\alpha$  the  $m$ -th root of  $x$ .

**Remark 1.12.** Our last proof also gives that  $\mathbb{Q}$  cannot satisfy AC. Suppose it does, ie any set in  $\mathbb{Q}$  bounded from above has a supremum  $\in \mathbb{Q}$ . Then, consider  $B = \{x \in \mathbb{Q} : x \geq 0 \text{ and } x^2 < 2\}$ ; set  $\alpha = \sup B$ . The exact same proof can be used, but we will not be able to find an upper bound in  $\mathbb{Q}$ .

## 1.7 Cardinality

**Definition 1.8.** Let  $f : A \rightarrow B$ .

1.  $f$  injective (one-to-one) if  $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$
2.  $f$  surjective (onto) if for any  $b \in B \exists a \in A$  s.t.  $f(a) = b$ .
3.  $f$  bijective if both.

**Definition 1.9** (Composition). If  $f : A \rightarrow B$ ,  $g : B \rightarrow C$ , the composite map  $h = g \circ f$  is define by  $h(x) = g(f(x))$ . Note that  $h : A \rightarrow C$ .

**Example 1.10.** Consider functions  $f, g$ .

1. If  $f, g$  injective, so is  $h = g \circ f$
2. If  $f, g$  bijective, then so is  $h$
3. If  $\exists E \subseteq C$ , then  $h^{-1}(E) = f^{-1}(g^{-1}(E))$

**Definition 1.10.** The inverse function<sup>18</sup> is defined only for bijective map  $f : A \rightarrow B$ .  $y \in B$ ,  $f^{-1}(y) = x$  where  $x \in A$  s.t.  $f(x) = y$ .

**Example 1.11.** 1.  $A = \mathbb{R}$ ,  $B = (0, \infty)$ ,  $f(x) = e^x$ .  $f$  is a bijection, and  $f^{-1}(y) = \ln y$ ,  $y \in (0, \infty)$ .

2.  $A = (-\frac{\pi}{2}, \frac{\pi}{2})$ ,  $B = \mathbb{R}$ .  $f(x) = \tan x$ ,  $f^{-1}(y) = \arctan y$

<sup>18</sup>Not the same as the inverse *image* of a set by a function, which is defined for any function.



**Definition 1.11** (Equal Cardinalities). Let  $A, B$  be two sets. We say  $A, B$  have the same cardinality, denote  $A \sim B$  if there exists a bijective function  $f : A \rightarrow B$ .

**Example 1.12.** Let  $E = \{2, 4, 6, \dots\}$  (even natural numbers). Define  $f : \mathbb{N} \rightarrow E$  by  $f(n) = 2n$ . Thus,  $f$  is a bijection, and  $\mathbb{N} \sim E$ .<sup>19</sup>

<sup>19</sup>See [these independent notes](#) for more.

**Theorem 1.7.** The relation  $\sim$  is a relation of equivalence.

1.  $A \sim A$
2. if  $A \sim B$ , then  $B \sim A$
3. if  $A \sim B$  and  $B \sim C$ , then  $A \sim C$

**Definition 1.12** (Countable). A set  $A$  is countable if  $\mathbb{N} \sim A$ .

**Remark 1.13.** According to this, finite sets are not countable; this is just a convention. Sometimes, we say a set is countable if it is finite or to above definition holds, where we say that a set is countably infinite if it is infinite and countable.

Other times, finite sets are treated separately than countable sets.

**Theorem 1.8.** Suppose that  $A \subseteq B$ .

1. If  $B$  is finite or countable, then so is  $A$
2. If  $A$  is infinite and uncountable, then so is  $B$

**Definition 1.13** (Cartesian Product). If  $A, B$  sets,  $A \times B = \{(a, b) : a, b \in A, B\}$ .

**Proposition 1.12.**  $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$ ; there exists a bijection  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ .

**Proposition 1.13.** Let  $A$  be a set. The following are equivalent statements:

- (a)  $A$  is finite or a countable set;
- (b) there exists a surjection from  $\mathbb{N}$  onto  $A$ ;
- (c) there exists an injection from  $A$  into  $\mathbb{N}$ .

*Proof.* We proceed by proving that each statement implies the next (and thus are equivalent).

- (a)  $\implies$  (b): Suppose  $A$  is finite and has  $\mathbb{N}$  elements. Then there exists a bijection  $h : \{1, 2, \dots, n\} \rightarrow A$ . We now define a map  $f : \mathbb{N} \rightarrow A$ , by setting

$$f(m) = \begin{cases} h(m) & \text{if } m \leq n \\ h(n) & \text{if } m > n \end{cases}.$$

$f$  is surjective, and thus (b) holds. If (a) countable,  $\exists$  bijection  $f : \mathbb{N} \rightarrow A$ , and any bijection is a surjection, so (b) also holds.

- (b)  $\implies$  (c): Let  $h : \mathbb{N} \rightarrow A$  be a surjection, whose existence is guaranteed by (b). Then, for any  $a \in A$ , the set

$$h^{-1}(\{a\}) = \{m \in \mathbb{N} : h(m) = a\} \neq \emptyset,$$

since  $h$  is a surjection. Then, by the well-ordering property of  $\mathbb{N}$ , the set  $h^{-1}(\{a\})$  has a least element.

If  $n$  is the least element of  $h^{-1}(\{a\})$ , we set  $f(a) = n$ . This defines a function

$$f : A \rightarrow \mathbb{N},$$

and we aim to show that  $f$  is injective, ie that  $f(a_1) = f(a_2) \implies a_1 = a_2$ .

Suppose  $f(a_1) = f(a_2) = n$ . Then,  $n$  is the least element of  $h^{-1}(\{a_1\})$  and of  $h^{-1}(\{a_2\})$ , and in particular,  $h(n) = a_1$  and  $h(n) = a_2$ , and thus  $a_1 = a_2$  and so  $f$  is indeed injective.

- (c)  $\implies$  (a): Let  $f : A \rightarrow \mathbb{N}$  be an injection, whose existence is guaranteed by (c). Consider the range of  $f$ , ie

$$f(A) = \{f(a) : a \in A\}.$$

Since  $f$  an injection,  $f$  is a bijection between  $A$  and  $f(A)$ .

Otoh,  $f(A) \subseteq \mathbb{N}$ , and so by Theorem 1.8,  $f(A)$  is either finite or countable, and there exists a bijection between  $A$  and some set that is either finite or countable. Thus,  $A$  must also be finite or countable, and so (a) holds.

■

**Theorem 1.9.** Let  $A_n, n = 1, 2, \dots$  be a sequence of sets such that each  $A_n$  is either finite or countable. Then, their union

$$A = \bigcup_{n=1}^{\infty} A_n$$

is also either finite or countable.

*Proof.* We will use (a)  $\iff$  (b) from Proposition 1.13 to prove this.

Since each  $A_n$  finite or countable, by (a)  $\implies$  (b), there exists a surjection

$$\varphi_n : \mathbb{N} \rightarrow A_n.$$

Now, let  $h : \mathbb{N} \times \mathbb{N} \rightarrow A$ , (the union) by setting

$$h(n, m) = \varphi_n(m).$$

We aim to show that  $h$  is also surjective.

If  $a \in \bigcup_{n=1}^{\infty} A_n$ , then  $a \in A_n$  for some  $n \in \mathbb{N}$ . Since  $\varphi_n : \mathbb{N} \rightarrow A_n$  is a surjection, there exists an  $m \in \mathbb{N}$  s.t.  $\varphi_n(m) = a$ . By definition of  $h$ , we have

$$h(n, m) = a,$$

and thus  $h$  is a surjection.

By Proposition 1.12, there exists a bijection  $f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ , and we can define the composite map

$$h \circ f : \mathbb{N} \rightarrow A (= \bigcup_{n=1}^{\infty} A_n),$$

which is a surjection as both  $h, f$  are surjections. So, there exists a surjection from  $\mathbb{N} \rightarrow A$ , and by Proposition 1.13, (b)  $\implies$  (a), and thus  $A = \bigcup_{n=1}^{\infty} A_n$  is also finite or countable. ■

**Remark 1.14.** If  $A = \bigcup_{n=1}^{\infty} A_n$ , where each  $A_n$  is either finite or countable, and at least one  $A_n$  is countable, then  $A$  is countable.

**Remark 1.15.** If  $A_1, \dots, A_n$  are finitely many finite or countable sets then their union  $A_1 \cup \dots \cup A_n$  is also finite or countable (essentially just previous proof where we use  $n$  instead of  $\infty$  for the upper limit of the union...).

**Theorem 1.10.** The set  $\mathbb{Q}$  of rational numbers is countable.

*Proof.* We write

$$\mathbb{Q} = A_0 \cup A_1 \cup A_2,$$

where  $A_0 = \{0\}$ ,  $A_1 = \{\frac{m}{n} : m, n \in \mathbb{N}\}$ , and  $A_2 = \{-\frac{m}{n} : m, n \in \mathbb{N}\}$ .

Let us show that  $A_1$  is countable; define

$$h : \mathbb{N} \times \mathbb{N} \rightarrow A_1, f(m, n) = \frac{m}{n}.$$

$h$  is clearly a surjection; if  $f : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$  is a bijection, then by Proposition 1.12,  $h \circ f : \mathbb{N} \rightarrow A_1$  is a surjection. By Proposition 1.13,  $A_1$  is countable.

We prove that  $A_2$  countable in essentially the same way.

Then,  $A_0 \cup A_1 \cup A_2$  is also countable, as it is the union of countable sets, and thus  $\mathbb{Q}$  is also countable. ■

**Theorem 1.11.** *The set  $\mathbb{R}$  of real numbers is uncountable.*<sup>20</sup>

*Proof.* We will argue by contradiction; suppose  $\mathbb{R}$  is countable, then show that the nested interval property (Theorem 1.3) of the real line fails.

Let  $f : \mathbb{N} \rightarrow \mathbb{R}$  be a bijection, setting  $f(1) = x_1, f(2) = x_2, \dots, f(n) = x_n, \dots$ ; we can then list the elements of  $\mathbb{R}$  as  $\mathbb{R} = \{x_1, x_2, x_3, \dots, x_n, \dots\}$ .

We can now construct a sequence  $I_n, n \in \mathbb{N}$  of bounded, closed intervals, such that  $I_1$  does not contain  $x_1$ .

If  $x_2 \notin I_1$ , then  $I_2 = I_1$ . If  $x_2 \in I_1$ , then divide  $I_1$  into four equal closed intervals.

Call the leftmost/rightmost of these intervals  $I'_1$  and  $I''_1$  respectively. We know that  $x_2 \in I_1$ , so we must have that either  $x_2 \notin I'_1$  or  $x_2 \notin I''_1$ . If  $x_2 \notin I'_1$ , then  $I_2 = I'_1$ . If  $x_2 \notin I''_1$ , then  $I_2 = I''_1$ .

Thus, we have constructed  $I_1, I_2$  s.t.

$$I_1 \supseteq I_2 \text{ and } x_1 \notin I_1, x_2 \notin I_2.$$

Consider  $x_3$ ; if  $x_3 \notin I_2$ , then  $I_3 = I_2$ . If  $x_3 \in I_2$ , we repeat the “dividing” process as before.

Since  $x_3 \in I_2$ , either  $x_3 \notin I'_2$  or  $x_3 \notin I''_2$ . If  $x_3 \notin I'_2$ ,  $I_3 = I'_2$ . Else, if  $x_3 \notin I''_2$ ,  $I_3 = I''_2$ .

We have now that

$$I_1 \supseteq I_2 \supseteq I_3 \text{ and } x_1 \notin I_1, x_2 \notin I_2, x_3 \notin I_3,$$

and we can continue this construction to obtain an infinite sequence of bounded, closed intervals  $I_n$  s.t.

$$I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq I_{n+1} \supseteq \dots,$$

and for each  $n, x_n \notin I_n$ .

Consider the intersection of all these  $I_n$ 's,

$$\bigcap_{n=1}^{\infty} I_n.$$

For every  $m, x_m \notin I_m$ , so for every  $m \in \mathbb{N}, x_m \notin \bigcap_{n=1}^{\infty} I_n$ , and so  $\mathbb{R} = \{x_1, x_2, \dots, x_m, \dots\}$  has an empty intersection with this intersection, ie

$$\mathbb{R} \cap \left( \bigcap_{n=1}^{\infty} I_n \right) = \emptyset.$$

Otoh,  $\bigcap_{n=1}^{\infty} I_n \subseteq \mathbb{R}$ , so we must have that  $\bigcap_{n=1}^{\infty} I_n = \emptyset$  contradicting the nested interval property of the real line which states that this intersection must not be empty. We thus have a contradiction, and our assumption that  $\mathbb{R}$  countable fails.<sup>21</sup> ■

<sup>20</sup>Proof sketch: by contradiction. Assume that a bijection exists, and show that it cannot be a surjection by the previous props/thms. Specifically, carefully construct nested intervals  $I_n$ , for which  $x_i \notin I_i$ , and then show that the intersection of all these intervals is empty, contradicting the nested interval property of the real line. See pg. 25 of Abbott's Analysis for a more concise proof in the same language.

<sup>21</sup>Note that Theorem 1.3 is built upon the Axiom of Completeness, a “fact” of  $\mathbb{R}$  (what makes it “distinct” from  $\mathbb{Q}, \mathbb{N}$ , etc). Thus, we are really just using AC, with some abstractions sts.

**Proposition 1.14.** *The set  $J$  of all irrational numbers in  $\mathbb{R}$  is uncountable.*

*Proof.* We have that  $\mathbb{R} = \mathbb{Q} \cup J$ . If  $J$  countable, then  $\mathbb{R}$  would also be countable as the union of two countable sets (as we showed  $\mathbb{Q}$  countable in Theorem 1.10).  $\mathbb{R}$  uncountable, so  $J$  is also uncountable. ■

**Proposition 1.15.** *The set  $(-1, 1) \subseteq \mathbb{R}$  is uncountable.*

*Proof.* We can write  $\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n)$ . If each  $(-n, n)$  is countable, then  $\mathbb{R}$  would also be countable, as a countable union of countable sets. Thus, there must exist some  $n_0 \in \mathbb{N}$  s.t.  $(-n_0, n_0)$  is not countable. The map

$$f : (-n_0, n_0) \rightarrow (-1, 1), f(x) = \frac{x}{n_0}$$

is a bijection, and so  $(-1, 1)$  is uncountable. ■

**Example 1.13.** *Show that the map*

$$f(x) = \frac{x}{1 - x^2}$$

*is a bijection between  $(-1, 1)$  and  $\mathbb{R}$  ie  $(-1, 1) \sim \mathbb{R}$ .*

*Proof.* Surjection is fairly trivial (if stuck, consider the graph of the function).

Injection; given  $f(x) = f(y)$  where  $x, y \in (-1, 1)$ ,

$$\begin{aligned} \frac{x}{1 - x^2} &= \frac{y}{1 - y^2} \\ x - xy^2 &= y - yx^2 \\ x - y &= xy^2 - yx^2 = xy(y - x) \\ x - y &= -xy(x - y) \\ \implies -xy &= 1 \implies xy = -1, \text{ or } x - y = 0 \end{aligned}$$

$xy = -1$  is impossible given the domain of the function, hence  $x - y = 0 \implies x = y$ , as desired. ■

**Proposition 1.16.** *Any bounded non-empty open interval  $(a, b) \in \mathbb{R}$  is uncountable.*

*Proof.* We will construct a bijection  $f : (a, b) \rightarrow \mathbb{R}$  so that  $(a, b) \sim \mathbb{R}$ . Since  $\mathbb{R}$  is uncountable, so must  $(a, b)$ .

The map

$$f(x) = \frac{2(x - a)}{b - a} - 1$$

is a bijection between  $(a, b)$  and  $(-1, 1)$ , and we have shown that  $(-1, 1) \sim \mathbb{R}$ , so  $(a, b) \sim \mathbb{R}$ , and thus any open interval has the same cardinality as  $\mathbb{R}$ . ■

**Example 1.14.** Prove that  $\exists$  bijection between  $[0, 1)$  and  $(0, 1)$ , and conclude that  $[0, 1) \sim (0, 1) \sim \mathbb{R}$ . Then conclude for any  $a < b$ ,  $[a, b) \sim \mathbb{R}$ .

### 1.7.1 Power Sets

**Definition 1.14** (Power Set). Let  $A$  be a set. The power set of  $A$  denoted  $\mathcal{P}(A)$  is the collection of all subsets of  $A$ .

Generally, if  $A$  finite of size  $n$ ,  $\mathcal{P}(A)$  has  $2^n$  elements.

**Theorem 1.12** (Cantor Power Set Theorem). Let  $A$  be any set. Then there exists no surjection from  $A$  onto  $\mathcal{P}(A)$ .<sup>22</sup>

<sup>22</sup>Certified Classic

*Proof.* Suppose that there exists a surjection,

$$f : A \rightarrow \mathcal{P}(A).$$

Let  $D \subseteq A$  defined as

$$D = \{a \in A : a \notin f(a)\}.$$

Since  $D \subseteq \mathcal{P}(A)$ , and  $f$  is surjective, there must exist some  $a_0 \in A$  s.t.  $f(a_0) = D$ .

We have two cases:

1.  $a_0 \in D$ . But then, by definition of  $D$ ,  $a_0 \notin f(a_0) = D$ , so  $a_0 \in D$  is not possible as it implies  $a_0 \notin D$ .
2.  $a_0 \notin D$ . But then, since  $D = f(a_0)$ ,  $a_0 \notin f(a_0)$ , and so by definition of  $D$ ,  $a_0 \in D$ , which is again not possible.

So, the assumption of a surjection existing has led to  $a_0 \in A$  such that neither  $a_0 \in D$  nor  $a_0 \notin D$ , which is impossible. Thus there can be no surjective  $f$ .

Notice, though, that there exists an injection  $A \rightarrow \mathcal{P}(A)$ ,  $a \mapsto \{a\}$ , and thus there is an injection but no bijection.

Thus, we can say that  $\mathcal{P}(A)$  is strictly bigger than  $A$ . ■

## 2 Sequences

### 2.1 Definitions

**Definition 2.1.** Let  $A$  be a set. An  $A$ -valued sequence indexed by  $\mathbb{N}$  is a map

$$x : \mathbb{N} \rightarrow A.$$

The value  $x(n)$  is called the  $n$ -th element of the sequence. One writes  $x(n) = x_n$ , or lists its elements

$$\{x_1, x_2, x_3, \dots\} \equiv \{x_n\}_{n \in \mathbb{N}} \equiv (x_n)_{n \in \mathbb{N}} \equiv \{x_n\}.$$

**Definition 2.2 (Convergence).** We say that a sequence  $(x_n)$  converges to a real number  $x$  if for every  $\varepsilon > 0$ ,  $\exists N \in \mathbb{N}$  s.t. for all  $n \geq N$  we have

$$|x_n - x| < \varepsilon.$$

If sequence  $(x_n)$  converges to  $x$ , we write  $\lim_{n \rightarrow \infty} x_n = x$ .

**Example 2.1.** Let  $(x_n)$  be a sequence defined by  $x_n = \frac{1}{n}$ ,  $n \in \mathbb{N}$ , then  $\lim_{n \rightarrow \infty} x_n = 0$ .

*Proof.* Let  $\varepsilon > 0$ . Let  $N \in \mathbb{N}$  s.t.  $N > \frac{1}{\varepsilon}$ . Then for  $n \geq N$ , we have that

$$0 < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

So, for  $n \geq N$ ,  $|x_n - 0| < \varepsilon$ , and so the limit is 0. ■

**Definition 2.3 (Quantifier of Limit).** The limit can be written in terms of quantifiers.

$$\lim_{n \rightarrow \infty} x_n = x$$

means that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(|x_n - x| < \varepsilon).$$



**Example 2.2.** Prove that

$$\lim_{n \rightarrow \infty} \frac{n^2 + 1}{n^2} = 1.$$

*Proof.* Let  $\varepsilon > 0$ . Let  $N$  be a natural number such that  $N > \frac{1}{\sqrt{\varepsilon}}$ . Then, for  $n \geq N$ ,

$$\left| \frac{n^2 + 1}{n^2} - 1 \right| = \left| \frac{n^2 + 1 - n^2}{n^2} \right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

■

**Definition 2.4** (Divergent Sequences). If a sequence  $(x_n)$  does not converge to any real number  $x$ , we say that the sequence is divergent. For instance, consider

$$x_n = (-1)^n, n \geq 1.$$

The sequence alternates between 1 and  $-1$  and so intuitively does not converge. How do we prove it?

*Proof.* By contradiction; suppose that  $x_n = (-1)^n$  be a converging sequence. Let  $x = \lim_{n \rightarrow \infty} x_n$ .

Take  $\varepsilon = 1$ , then  $\exists N \in \mathbb{N}$  s.t. for all  $n \geq N$  we have that  $|x - x_n| < \varepsilon = 1$ .

Consider indices  $n = N, n = N + 1$ . We have

$$|x_{N+1} - x_N| = |x_{N+1} - x + x - x_N| \leq \underbrace{|x_{N+1} - x| + |x - x_N|}_{\text{triangle inequality}} < 1 + 1 = 2.$$

But we also have that

$$|(-1)^{N+1} - (-1)^N| = |(-1)^{N+1} + (-1)^{N+1}| = 2,$$

We thus have that  $2 < 2$ , which is a contradiction. Thus,  $x_n$  is not convergent. ■

**Example 2.3.** Evaluate the following examples using the  $\varepsilon$  definition:

1.  $\lim_{n \rightarrow \infty} \frac{\sin n}{\sqrt[3]{n}} = 0$

2.  $\lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$

3.  $\lim_{n \rightarrow \infty} \frac{(1+2+\dots+n)^2}{n^4} = \frac{1}{4}$

*Proof.* 1. For all  $\varepsilon > 0$ ; take  $\frac{1}{N} < \varepsilon^3 \implies \frac{1}{\sqrt[3]{N}} < \varepsilon$ . Then,  $\forall n \geq N$ ,

$$\begin{aligned} n \geq N &\implies \sqrt[3]{n} \geq \sqrt[3]{N} \implies \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} \\ -1 \leq \sin n \leq 1 &\implies |\sin n| \leq 1 \implies \left| \frac{\sin n}{\sqrt[3]{n}} \right| \leq \left| \frac{1}{\sqrt[3]{N}} \right| \leq \frac{1}{\sqrt[3]{N}} < \varepsilon \\ &\implies \lim_{n \rightarrow \infty} \frac{\sin n}{\sqrt[3]{n}} = 0 \end{aligned}$$

2. Take  $\frac{1}{N} \leq \varepsilon$ . Then,  $\forall \varepsilon > 0, \forall n \geq N \implies \frac{1}{n} \leq \frac{1}{N}$ ,

$$\begin{aligned} \frac{n!}{n^n} > 0 &\implies \left| \frac{n!}{n^n} \right| = \frac{n!}{n^n} = \frac{n(n-1)(n-2) \cdots 1}{n \cdot n \cdots n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{1}{n} \\ &\leq 1 \cdot 1 \cdots 1 \cdot \frac{1}{n} \\ &\leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon \\ &\implies \lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0 \end{aligned}$$

3. Note first that  $(1 + 2 + \cdots + n)^2 = \left(\frac{n(n+1)}{2}\right)^2$  (see Example 1.1). Take  $\frac{1}{N} < \frac{\varepsilon}{2}$ ; then,  $\forall \varepsilon > 0$ , we have that  $\forall n \geq N$ ,

$$\begin{aligned} \left| \frac{(1 + 2 + \cdots + n)^2}{n^4} - \frac{1}{4} \right| &= \frac{\frac{n^2(n+1)^2}{4}}{n^4} - \frac{n^4}{n^4} = \frac{n^4 + 2n^3 + n^2 - n^4}{n^4} \\ &= \frac{2n^3 + n^2}{n^4} = \frac{2n + 1}{n^2} \leq \frac{2n}{n^2} \leq \frac{2}{n} \leq \frac{2}{N} < \varepsilon \\ &\implies \lim_{n \rightarrow \infty} \frac{(1 + 2 + \cdots + n)^2}{n^4} = \frac{1}{4} \end{aligned}$$

■

## 2.2 Properties of Limits

**Lemma 2.1** (Triangle Inequality). For  $x, y, z \in \mathbb{R}$ ,

$$(i) \quad |x + y| \leq |x| + |y|; \quad (ii) \quad |x - y| \leq |x - z| + |z - y|^{23}$$

*Sketch proof.* (i):  $|x + y| = \begin{cases} x + y & x + y \geq 0 \\ -(x + y) & x + y \leq 0 \end{cases}$ . So if  $x + y \geq 0$ ,  $|x + y| = x + y \leq |x| + |y|$ .

If  $x + y < 0$ ,  $|x + y| = -(x + y) = (-x) + (-y) \leq |x| + |y|$ .

<sup>23</sup>Generally, proofs involving limits will consist of 1) picking/defining an  $\varepsilon$  based on given limit/series definitions, and then 2) using triangle inequality/related techniques to reach the desired conclusion.

(ii):  $|x - y| = |x - z + z - y| \leq |x - z| + |z - y|$  (using (i)). ■

**Theorem 2.1.** *A limit of a sequence is unique. In other words, if the sequence is converging, then its limit is unique. The sequence cannot converge to two distinct numbers  $x$  and  $y$ .<sup>24</sup>*

*Proof.* By contradiction; suppose  $\exists(x_n)$  s.t.  $\lim_{n \rightarrow \infty} x_n = x$  and  $\lim_{n \rightarrow \infty} x_n = y$ , and that  $x \neq y$ .

Take  $\varepsilon = \frac{|x-y|}{2}$ . Since  $x \neq y$ , we have that  $\varepsilon > 0$ . Since  $\lim_{n \rightarrow \infty} x_n = x$ ,  $\exists N_1 \in \mathbb{N}$  s.t. for  $n \geq N_1$ ,  $|x_n - x| < \varepsilon$ .

Similarly, since  $\lim x_n = y$ ,  $\exists N_2 \in \mathbb{N}$  s.t for  $n \geq N_2$ ,  $|x_n - y| < \varepsilon$ .

Take some  $n \geq \max(N_1, N_2)$ ; then

$$\begin{aligned} |x - y| &= |x - x_n + x_n - y| \leq |x - x_n| + |x_n - y| \\ &< \varepsilon + \varepsilon = |x - y| \\ \implies |x - y| &< |x - y|, \perp \end{aligned}$$

■

**Theorem 2.2.** *Any converging sequence is bounded.<sup>25</sup>*

*In other words, if  $(x_n)$  is a converging sequence,*

$$\exists M > 0 \text{ s.t. } |x_n| \leq M \forall n \geq 1.$$

*Proof.* Let  $(x_n)$  be a converging sequence, and  $x = \lim_{n \rightarrow \infty} x_n$ . Take  $\varepsilon = 1$  in the definition of the limit; then,  $\exists N \in \mathbb{N}$  s.t.  $\forall n \geq N$ ,  $|x_n - x| < 1$ .

This gives that for  $n \geq N$ ,  $|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$ .

Let now  $M = |x_1| + |x_2| + \cdots + |x_{N-1}| + (1 + |x|)$ . Then, for any  $n \geq 1$ ,  $|x_n| \leq M$ ;

If  $n \leq N - 1$ , then  $|x_n|$  is a summand in  $M$ , and thus  $|x_n| \leq M$ .

If  $n \geq N$ , then we have by the choice of  $N$  that  $|x_n| < 1 + |x| \leq M$ .

Thus, for all  $n \geq 1$ ,  $|x_n| \leq M$ , and is thus bounded given  $(x_n)$  converges. ■

<sup>24</sup>Proof sketch: contradiction, assume two distinct limits, and take  $\varepsilon$  as their midpoint. Arrive at a contradiction by using triangle inequalities to show that  $|x - y| < |x - y|$ , and thus the limits cannot be distinct.

<sup>25</sup>Take  $\varepsilon = 1$ , which is greater than  $|x_n - x|$  by limit definition for  $n \geq N$  for some  $N$ . We then use this to show that  $|x_n| < 1 + |x|$ , then construct a summation  $M$  such that it bounds  $|x_n|$ ; it is equal to  $|x_1| + |x_2| + \cdots$  up to  $|x_{N-1}|$ , then plus  $1 + |x|$ . We have finished.

**Proposition 2.1** (Algebraic Properties of Limits). *Let  $(x_n), (y_n)$  be sequences such that*<sup>26</sup>

$$\lim x_n = x, \quad \lim y_n = y.$$

*Then:*

1. *For any constant  $c$ ,  $\lim c \cdot x_n = c \cdot \lim x_n = c \cdot x$*
2.  $\lim(x_n + y_n) = \lim x_n + \lim y_n = x + y$
3.  $\lim x_n \cdot y_n = (\lim x_n)(\lim y_n) = x \cdot y$
4. *Suppose  $y \neq 0$ ,  $y_n \neq 0 \forall n \geq 1$ . Then,  $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n} = \frac{x}{y}$*

<sup>26</sup>Note that the contrary of these statements need not hold; ie, if  $\lim(x_n \cdot y_n)$  exists, this does not imply the existence of  $\lim x_n$  and  $\lim y_n$ . Consider Example 2.4

**Remark 2.1.** Let  $X$  be the collection of all sequences of real numbers,  $X = \{(x_n) : x_n \text{ is a sequence}\}$ .

If  $(x_n) \in X$  and  $c \in \mathbb{R}$ , we can define  $c \cdot (x_n) = (c \cdot x_n)$ <sup>27</sup>; this defines scalar multiplication on  $X$ .

We can also define addition; if  $(x_n)$  and  $(y_n)$  are two sequences in  $X$ , then  $(x_n) + (y_n) = (x_n + y_n)$ .

Then, with these two operations  $X$  is a vector space.

<sup>27</sup>NB: this denotes  $c$  multiplying to each  $n$ th element in  $x_n$ , ie  $c \cdot x_1$ ,  $c \cdot x_2$ , etc

**Example 2.4.** Take  $x_n = (-1)^n$ ,  $y_n = (-1)^{n+1}$ ,  $n \geq 1$ .

$(x_n) + (y_n) = 0$ ,  $x_n \cdot y_n = -1$ , and so  $\lim x_n + y_n = 0$ ,  $\lim x_n \cdot y_n = -1$ , while neither  $\lim x_n$  nor  $\lim y_n$  exist.

*Proof (part 3. of Proposition 2.1).* Take<sup>28</sup> $\lim x_n = x$ ,  $\lim y_n = y$ . Since  $(x_n)$  is converging, it is bound by Theorem 2.2, and there exists  $M > 0$  s.t.  $\forall n \geq 1, |x_n| \leq M$ .

Now,

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n y_n - x_n y| + |x_n y - xy| \\ &= |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \\ &\leq M \cdot |y_n - y| + |y| \cdot |x_n - x| \quad (i) \end{aligned}$$

Let  $\varepsilon > 0$ ; since  $\lim y_n = y$ , there exists  $N_1 \in \mathbb{N}$  s.t.  $n \geq N_1, |y_n - y| < \frac{\varepsilon}{2M}$ . Sim, since  $\lim x_n = x$ ,  $\exists N_2 \in \mathbb{N}$  s.t.  $|x_n - x| < \frac{\varepsilon}{2(|y|+1)}$

Let  $N = \max(N_1, N_2)$ ,  $n \geq N$ . Then, we have, with (i),

$$\begin{aligned} (i) \quad |x_n y_n - xy| &\leq M \cdot |y_n - y| + |y| \cdot |x_n - x| \\ &< M \cdot \frac{\varepsilon}{2M} + |y| \cdot \frac{\varepsilon}{2(|y|+1)} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \end{aligned}$$

Thus, for  $n \geq N$ ,  $|x_n y_n - xy| < \varepsilon$ , and by definition of the limit,  $\lim x_n y_n = xy$ . ■

<sup>28</sup>Proof sketch: take an upper bound of  $x_n$ . Then, show that  $|x_n y_n - xy| < \varepsilon$ , by using triangle inequality to show inequality to a combination of  $M$ , arbitrarily small values (by def of limits of  $x_n, y_n$  resp.), and  $|y|$ .

**Theorem 2.3 (Order Properties of Limits).** Let  $(x_n), (y_n)$  be two sequences such that

$$\lim x_n = x, \quad \lim y_n = y.$$

$$1. \quad x_n \geq 0 \forall n \implies x \geq 0.$$

$$2. \quad x_n \geq y_n \forall n \implies x \geq y.$$

$$3. \quad c \text{ is constant since } c \leq x_n \forall n \geq 1 \implies c \leq x. \quad x_n \leq c \forall n \geq 1 \implies x_n \leq c.$$

**Remark 2.2.** 2., 3. follow from 1. Set  $z_n = x_n - y_n \forall n \geq 1$ . Then,  $z_n \geq 0 \forall n \geq 1$ ,  $\lim z_n = \lim(x_n - y_n) = \lim x_n - \lim y_n$  (as these limits exist)  $= x - y$ . By 1.,  $\lim z_n \geq 0$ , and so either  $x - y \geq 0$  or  $x \geq y$ .

*Proof of 1.* Suppose 1. does not hold; suppose  $\exists(x_n)$  s.t.  $\lim x_n = x$ ,  $x_n \geq 0 \forall n \geq 1$ , but  $x < 0$ . Let  $\varepsilon > 0$  s.t.  $x < -2\varepsilon < 0$ . With this  $\varepsilon$ ,  $\lim x_n = x$  gives that  $\exists N \in \mathbb{N}$  s.t.  $\forall n \geq N$ ,  $|x_n - x| < \varepsilon$ , or particularly,  $x_n - x < \varepsilon$ . Then,  $x_n < \varepsilon + x$ , and since  $x < -2\varepsilon$ , we have  $\forall n \geq N$ ,  $x_n < -\varepsilon$ , and thus  $\forall n \geq N$ ,  $x_n < 0$ , a contradiction. ■

**Theorem 2.4 (The Squeeze Theorem).** Let  $(x_n), (y_n), (z_n)$  be sequences such that  $x_n \leq y_n \leq z_n$ ,  $\forall n \geq 1$ , and  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = \ell$ , then  $\lim_{n \rightarrow \infty} y_n = \ell$ .<sup>29</sup>

*Proof.* Let  $\varepsilon > 0$ . Since  $\lim x_n = \ell$ , there  $\exists N_1 \in \mathbb{N}$  s.t.  $\forall n \geq N_1$ ,  $|x_n - \ell| < \varepsilon$ . Since  $\lim z_n = \ell$ , there  $\exists N_2 \in \mathbb{N}$  s.t.  $\forall n \geq N_2$ ,  $|z_n - \ell| < \varepsilon$ . Take  $N = \max\{N_1, N_2\}$  and take  $n \geq N$ . Then,

$$y_n \leq z_n \implies y_n - \ell \leq z_n - \ell \leq |z_n - \ell| < \varepsilon,$$

since  $n \geq \max\{N_1, N_2\} \implies n \geq N_2$ .

Now, we have that

$$y_n \geq x_n \implies y_n - \ell \geq x_n - \ell > -\varepsilon,$$

since  $|x_n - \ell| < \varepsilon$  for  $n \geq N_1$ , and our  $n$  is  $\geq \max\{N_1, N_2\}$ . Thus, for  $n \geq N$ ,

$$-\varepsilon < y_n - \ell < \varepsilon \implies |y_n - \ell| < \varepsilon,$$

and thus  $\lim y_n = \ell$ , by definition. ■

**Definition 2.5 (Increasing/Decreasing).** A sequence  $(x_n)$  is called increasing if  $x_{n+1} \geq x_n \forall n \in \mathbb{N}$ , and is decreasing if  $x_{n+1} \leq x_n \forall n \in \mathbb{N}$ .

**Definition 2.6 (Bounded from above/below).** A sequence  $(x_n)$  is called bounded from above if there exists some  $M \in \mathbb{R}$  s.t.  $x_n \leq M \forall n \geq 1$ .

Sequence  $(x_n)$  is bounded from below if there exists some  $M \in \mathbb{R}$  s.t.  $x_n \geq M \forall n \geq 1$ .

<sup>29</sup>Sketch: This follows a similar technique to many that follow. Use the definitions of the limits of  $x_n, z_n$  to take an arbitrary  $\varepsilon$ , and an  $N$  for each respective limit. Take the max of these  $N$ 's, and show that for all  $n \geq \max N_i$ , you can show that  $y_n - \ell$  is less than  $\varepsilon$  and greater than  $-\varepsilon$ . Really, this is just a proof of applying definitions correctly.

**Theorem 2.5** (Monotone Convergence Theorem). *The following relate to bounded above/below and increasing/decreasing sequences:*<sup>30</sup>

1. Let  $(x_n)$  be an increasing sequence that is bounded from above. Then  $(x_n)$  is converging.
2. Let  $(x_n)$  be a decreasing sequence that is bounded from below. then  $(x_n)$  is converging.

*Proof (of 1).* Let  $A = \{x_n : n \geq 1\}$ . Since  $(x_n)$  is bounded above by  $M$ , the set  $A$  is bounded from above. Let  $\alpha = \sup A$ , which exists by AC.

Let  $\varepsilon > 0$ . Since  $\alpha$  is the least upper bound for  $A$ ,  $\alpha - \varepsilon$  is *not* an upper bound of  $A$  ( $\alpha - \varepsilon < \alpha$ ). Hence, there must exist some  $N \in \mathbb{N}$  such that  $\alpha - \varepsilon < x_N$  (if it didn't exist, then  $\alpha$  wouldn't be the supremum ...). Then, for  $n \geq N$ , and since  $(x_n)$  increasing,

$$\alpha - \varepsilon < x_N \leq x_n \leq \alpha.$$

Then, for all  $n \geq N$ ,

$$\alpha - \varepsilon < x_n \leq \alpha \implies -\varepsilon < x_n - \alpha \leq 0,$$

and so  $|x_n - \alpha| < \varepsilon$  for  $n \geq N$ . By definition,  $\alpha = \lim x_n$ . ■

**Example 2.5.** A sequence  $(x_n)$  is called eventually increasing if there exists some  $N_0 \in \mathbb{N}$  s.t.  $\forall n \geq N_0, x_{n+1} \geq x_n$ . If  $(x_n)$  is eventually increasing and bounded from above,  $\lim x_n = \alpha$  exists.

<sup>30</sup>Sketch: 1,2 are proven very similarly. For 1., take the set of all  $x_n$  in the given sequence. Since the sequence is bounded, then so is the set, and so we can take its supremum. Use the  $\varepsilon$  definition of sup to show that this supremum is also the limit of the sequence (basically, a bunch of inequalities, and being careful with definitions). 2. follows identically but using the infimum.

**Example 2.6.** Let  $(x_n)$  be a sequence defined recursively by  $x_1 = \sqrt{2}$  and  $x_{n+1} = \sqrt{2 + x_n}$ ,  $n \geq 1$ . So  $x_2 = \sqrt{2 + \sqrt{2}}$ ,  $x_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$ ,  $x_n = 2 \cos \frac{\pi}{2^{n+1}}$ ,  $n \geq 1$ . Show that  $\lim x_n = 2$ .

*Proof.* We will prove this using the Monotone Convergence Thm by showing that the  $x_n$  is bounded from above and increasing, which implies that the limit exists. We will then find the actual limit.

Recall that  $n \geq 1, x_n \leq 2$ . We will prove this by induction. Let  $S \subseteq \mathbb{N}$  be the set of indices such that  $x_n \leq 2$ . Since  $x_1 = \sqrt{2} < 2$ ,  $1 \in S$ . Now suppose some  $n \in S$ , ie  $x_n \leq 2$ . Then, we have that  $x_{n+1} = \sqrt{2 + x_n} \leq \sqrt{2 + 2} = 2 \implies x_{n+1} \leq 2$ . Thus, by induction,  $n \in S \implies n + 1 \in S \implies S = \mathbb{N}$ , ie  $x_n \leq 2 \forall n \in \mathbb{N}$ . Thus, our sequence is bounded from above.

We now prove that  $(x_n)$  is increasing. Let  $S \subseteq \mathbb{N}$  s.t.  $n \in S \iff x_{n+1} \leq x_n$ .  $x_2 = \sqrt{2 + \sqrt{2}} \geq \sqrt{2} = x_1 \implies x_1 \leq x_2 \implies 1 \in S$ . Suppose  $n \in S \implies x_{n+1} \geq x_n$ . Then,  $x_{n+2} = \sqrt{2 + x_{n+1}} \geq \sqrt{2 + x_n} = x_{n+1} \implies n + 1 \in S$ . Thus,  $S = \mathbb{N}$ , so  $x_{n+1} \geq x_n \forall n \in \mathbb{N}$ . So the sequence  $(x_n)$  is increasing and bounded from above, and thus  $\exists \lim x_n = \alpha$ . To find the value of  $\alpha$ , consider  $x_{n+1} = \sqrt{2 + x_n}$ , or  $x_{n+1}^2 = 2 + x_n$ . We can also write that  $\alpha = \lim x_n = \lim x_{n+1}$ .<sup>31</sup> We then have that  $\lim x_{n+1} = \alpha \implies \lim x_{n+1}^2 = \alpha^2$ , and thus  $x_{n+1}^2 = 2 + x_n \implies \lim x_{n+1}^2 = \lim(2 + x_n) \implies \alpha^2 = 2 + \alpha \implies \alpha = 2, -1$ .  $x_n \geq 0 \forall n$ , by Order Limit Theorem, and so  $\alpha \geq 0$  and thus  $\alpha = 2$ . ■

<sup>31</sup>Add proof

**Corollary 2.1.** For  $a, b > 0$ , then  $\frac{1}{2}(a + b) \geq \sqrt{ab}$

*Proof.*  $\left[\frac{1}{2}(a + b)\right]^2 = \frac{1}{4}(a^2 + 2ab + b^2) \geq ab \implies \frac{1}{2}(a + b) \geq \sqrt{ab}$  ■

**Example 2.7.** Let  $(x_n)$  be defined recursively by  $x_1 = 2$  and  $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$  for  $n \geq 1$ . Then,  $(x_n)$  is converging and  $\lim x_n = \sqrt{2}$ .

*Proof.* We<sup>32</sup>will show that  $(x_n)$  bounded from below and decreasing, implying the limit exists. We will show that for  $n$ ,  $x_n \geq \sqrt{2}$ . For  $n = 1$ ,  $2 \geq \sqrt{2}$ . For  $n > 1$ , we will Corollary 2.1. We then have that  $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \geq \dots \geq \sqrt{2} \implies x_n \geq \sqrt{2} \forall n \geq 1$ , ie, it is bounded from below.

We will now show that the sequence is decreasing.

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) = \frac{1}{2}x_n - \frac{1}{x_n} = \frac{1}{2x_n}(x_n^2 - 2).$$

<sup>32</sup>This example, as well as the more general one after it, rely on applying 1) the monotone convergence theorem



**Example 2.8.** Let  $a > 0$  and let  $(x_n)$  be a sequence defined recursively by  $x_1$  is arbitrary (positive), and

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n \geq 1.$$

Show that  $\lim_{n \rightarrow \infty} x_n = \sqrt{a}$ .

*Proof.* By Corollary 2.1,  $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \geq \sqrt{x_n \cdot \frac{a}{x_n}} = \sqrt{a}$ , hence,  $x_n$  is bounded from below by  $\sqrt{a}$ .

We also have that  $x_n - x_{n+1} = x_n - \frac{1}{2}x_n - \frac{a}{2x_n} = \frac{x_n}{2} - \frac{a}{2x_n} = \frac{1}{x_n}(x_n^2 - a)$ . We have that  $x_n \geq \sqrt{a} \implies x_n^2 \geq a \implies x_n^2 - a \geq 0$ . Further, since the sequence is bounded from below by  $\sqrt{a} > 0$  ( $\iff a > 0$ ), then  $\frac{1}{x_n} > 0$  as well. Hence,  $\frac{1}{x_n}(x_n^2 - a) \geq 0$ , and thus  $x_n - x_{n+1} \geq 0 \implies x_n \geq x_{n+1}$  and thus  $x_n$  is decreasing.

Thus, by the Monotone Convergence Theorem,  $x_n$  is convergent. Let  $X := \lim_{n \rightarrow \infty} x_n$ . We have from the recursive definition,  $\lim x_n = \lim \left(\frac{1}{2}\left(x_n + \frac{a}{x_n}\right)\right)$ . Since we know  $x_n$  convergent, we can “split up” this limit using algebraic properties, hence

$$\begin{aligned} \lim x_n &= \lim \frac{1}{2}x_n + \lim \frac{a}{2x_n} = \frac{1}{2} \lim x_n + \frac{a}{2} \lim \frac{1}{x_n} \\ &\implies X = \frac{1}{2}X + \frac{a}{2X} \\ &\implies \frac{X}{2} = \frac{a}{2X} \implies X^2 = a \implies X = \sqrt{a}, \end{aligned}$$

which completes the proof. ■

**Example 2.9.** Evaluate<sup>33</sup> the limit of  $x_n$  given the recursive relation  $x_{n+1} = \frac{1}{4-x_n}$ ,  $x_1 = 3$ .

<sup>33</sup>Abbott, pg 54 exercise 2.4.2

*Proof.* We aim to show that  $(x_n)$  is bounded from below and decreasing.

**Bounded from below:** we claim  $x_n > 0$ ; we proceed by induction.  $x_1 = 3 > 0$  holds; say  $x_n > 0$  for some  $n \geq 1$ . Then, we have

$$x_n > 0 \implies -x_n < 0 \implies 4 - x_n < 4 \implies \frac{1}{4 - x_n} > \frac{1}{4} > 0 \implies x_{n+1} = \frac{1}{4 - x_n} > 0,$$

so the sequence is bounded from below by 0.

**Decreasing:**  $(x_n)$  decreasing iff  $x_{n+1} \leq x_n \forall n$ . We have  $x_2 = \frac{1}{4-3} = 1 \implies x_1 = 3 \geq 1$  holds. Say  $x_{n-1} \geq x_n$  for some  $n \geq 1$ . Then, we have

$$x_{n-1} \geq x_n \implies 4 - x_{n-1} \leq 4 - x_n \implies \frac{1}{4 - x_{n-1}} \geq \frac{1}{4 - x_n} = x_{n+1} \implies x_n \geq x_{n+1}$$

and thus the sequence decreases, and by Theorem 2.5 the limit exists. Let  $X = \lim_{n \rightarrow \infty} x_n =$

$\lim_{n \rightarrow \infty} \frac{1}{4-x_{n-1}} \implies X = \frac{1}{4-X} \implies 4X - X^2 = 1 \implies 0 = X^2 - 4X + 1 \implies X =$   
 $\dots = 2 \pm \sqrt{3}$ . We must take the negative root, since  $X$  is decreasing and thus must be less  
than 3. ■

## 2.3 Limit Superior, Inferior

**Definition 2.7** (limsup, liminf). Recall Theorem 2.2, stating that a convergence sequence is bounded. Let  $(x_n)$  be a convergent sequence bounded by  $m$  and  $M$  from below/above resp, ie

$$m \leq x_n \leq M, \forall n$$

and let  $A_n = \{x_k : k \geq n\}$  (the set of elements in the sequence “after” a particular index). Let  $y_n = \sup A_n$ ; by definition,  $y_n \leq M$ , and  $y_n \geq m$ , since  $y_n \geq x_n \geq m$ . Thus, we have

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots ,$$

and further,

$$y_1 \geq y_2 \geq \cdots \geq y_n \geq y_{n+1} \geq \cdots ;$$

since  $A_2 \subseteq A_1$ ,  $y_1$  also an upper bound for  $A_2$ , and thus  $y_2 \leq y_1$  by definition of a supremum.

So, the sequence  $(y_n)$  is decreasing, and bounded from below; by MCT,  $\lim_{n \rightarrow \infty} y_n = y$  exists. Note too that since  $m \leq y_n \leq M$ , we have that  $m \leq y \leq M$ .

This  $y$  is called the limit superior of  $(x_n)$  denoted by

$$\overline{\lim}_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n.$$

Now, similarly, note that  $A_n$  is bounded below by  $m$  and thus  $z_n = \inf A_n$  exists. We further have that  $z_n \leq x_n \leq M$ , and that  $z_n \geq m \forall n$ , and we have

$$z_1 \leq z_2 \leq \cdots \leq z_n \leq z_{n+1} \leq \cdots ,$$

by a similar argument as before. So, as before, the sequence  $(z_n)$  is increasing, and bounded from above by  $M$ . Again, by MCT,  $\lim_{n \rightarrow \infty} z_n = z$  exists. We call  $z$  the limit inferior of  $(x_n)$ , and denote

$$\underline{\lim}_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n.$$

We note that  $y_n \geq z_n$ , so  $\overline{\lim}_{n \rightarrow \infty} x_n \geq \underline{\lim}_{n \rightarrow \infty} x_n$  ( $y \geq z$ ).

Further,  $\liminf$  and  $\limsup$  exist for any bounded sequence, regardless if whether or not the limit itself exists.

**Example 2.10.** Let  $(x_n) = (-1)^n, n \in \mathbb{N}$ . We showed previously that this is a divergent sequence, so the limit does not exist. However, the sequence is bounded, since  $-1 \leq x_n \leq 1 \forall n$ . We have  $A_n = \{(-1)^k : k \geq n\} = \{-1, 1\}$ . So,  $y_n = \sup A_n = 1$ , and  $z_n = \inf A_n = -1, \forall n$ . Thus,  $\limsup x_n = \lim y_n = 1$ , and  $\liminf x_n = \lim z_n = -1$ , despite  $\lim x_n$  not existing. More specifically, we have a divergent sequence, and  $\liminf \neq \limsup$ .

**Theorem 2.6** (lim inf, lim sup and convergence). Let  $(x_n)$  be a bounded sequence. The following are equivalent;

1. The sequence  $(x_n)$  is convergent, and  $\lim_{n \rightarrow \infty} x_n = x$ .
2.  $\overline{\lim}_{n \rightarrow \infty} x_n = \underline{\lim}_{n \rightarrow \infty} x_n = x$ .

*Proof.* Let  $A_n, y_n, z_n$  be as in the definition of  $\limsup, \liminf$ .

**(1)  $\implies$  (2):** Suppose  $(x_n)$  is converging, and  $\lim_{n \rightarrow \infty} x_n = x$ . Let  $\varepsilon > 0$ . Then, there exists some  $N \in \mathbb{N}$  s.t.  $\forall n \geq N$ ,

$$|x_n - x| < \frac{\varepsilon}{2},$$

or equivalently,

$$x - \frac{\varepsilon}{2} < x_n < x + \frac{\varepsilon}{2}, \forall n \geq N.$$

Since  $A_n = \{x_k : k \geq n\}$ , if  $n \geq N$ , then  $x + \frac{\varepsilon}{2}$  is an upper bound for  $A_n$ , and  $x - \frac{\varepsilon}{2}$  is a lower bound for  $A_n$ . This gives that

$$y_n = \sup A_n \leq x + \frac{\varepsilon}{2}; \quad z_n = \inf A_n \geq x - \frac{\varepsilon}{2}.$$

This gives that for  $n \geq N$ ,

$$x - \frac{\varepsilon}{2} \leq z_n \leq x_n \leq y_n \leq x + \frac{\varepsilon}{2},$$

ie  $z_n, y_n \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$ . So, for all  $n \geq N$ ,  $|z_n - x| \leq \frac{\varepsilon}{2} < \varepsilon$ , and  $|y_n - x| \leq \frac{\varepsilon}{2} < \varepsilon$ , so by definition of the limit, this gives

$$\lim_{n \rightarrow \infty} y_n = x \text{ and } \lim_{n \rightarrow \infty} z_n = x,$$

ie,  $\overline{\lim}_{n \rightarrow \infty} x_n = \underline{\lim}_{n \rightarrow \infty} x_n = x$ .

•

**(2)  $\implies$  (1):** Let  $\varepsilon > 0$ . Since  $\lim_{n \rightarrow \infty} y_n = x$ ,  $\exists N_1$  s.t.  $\forall n \geq N_1, |y_n - x| < \varepsilon$ . Similarly, since  $\lim z_n = x$ ,  $\exists N_2$  s.t.  $\forall n \geq N_2, |z_n - x| < \varepsilon$ .

Take  $N = \max\{N_1, N_2\}$ . Then, for  $n \geq N$ , we have

$$x - \varepsilon < z_n \leq x_n \leq y_n < x + \varepsilon.$$

So, for  $n \geq N$ ,  $|x_n - x| < \varepsilon$ , thus  $\lim x_n = x$  as desired. ■

**Example 2.11.** Let<sup>34</sup> $(x_n)$  be a bounded sequence. Then

$$\limsup_{n \rightarrow \infty} (-x_n) = -\liminf_{n \rightarrow \infty} x_n.$$

*Proof.* Recall Remark 1.2; Let  $A_n := \{x_k : k \geq n\}$  as in the definition of  $\limsup$ ,  $\liminf$ . Let  $y_n := \sup A_n$ ,  $z_n := \inf A_n$ . By Theorem 2.6,  $\lim y_n = \lim z_n$ . Further,  $\sup(-A_n) = -\inf(A_n)$ , where  $-A_n = \{-x_k : k \geq n\}$ ; hence,  $\limsup(-x_n) = -\liminf x_n$ , as desired. ■

<sup>34</sup>Midterm material ends here. There will be 5 questions. Memorize **everything**; homeworks, exercises, class material. Study the solutions until you can recite it upwards, backwards, sideways.

## 2.4 Subsequences and Bolzano-Weirestrass Theorem

**Definition 2.8** (Subsequence). Let  $(x_n)$  be a sequence of real numbers, and let  $n_1 < n_2 < n_3 < \dots < n_k < n_{k+1} < \dots$  be a strictly increasing sequence of natural numbers. Then, the sequence

$$(x_{n_1}, x_{n_2}, \dots, x_{n_k}, x_{n_{k+1}}, \dots)$$

is called a subsequence of  $(x_n)$  and is denoted  $(x_{n_k})_{k \in \mathbb{N}}$ .

**Remark 2.3.**  $k$  is the index of the subsequence,  $(x_{n_k})_{k \in \mathbb{N}}$ , **not**  $n$ ;  $x_{n_1}$  is the 1st element,  $\dots$ ,  $x_{n_k}$  is the  $k$ -th element.

**Example 2.12.** Let  $x_n = \frac{1}{n}$ ,  $(\frac{1}{n})_{n \in \mathbb{N}}$ , and let  $n_k = 2k + 1$ ,  $k \in \mathbb{N}$ .  $n_1 = 3, n_2 = 5, n_3 = 7, \dots, n_k = 2k + 1$ . Our subsequence is then

$$(x_{n_1}, x_{n_2}, \dots, x_{n_k}, \dots) = \left( \frac{1}{3}, \frac{1}{5}, \dots, \frac{1}{2k+1}, \dots \right) = \left( \frac{1}{2k+1} \right)_{k \in \mathbb{N}}$$

is our subsequence of  $(x_n)$ .

**Remark 2.4.** Note that for any  $k$ ,  $n_k \geq k$ .

Let  $S = \{k \in \mathbb{N} : n_k \geq k\}$ . Then,  $1 \in S$ , since  $n_1 \in \mathbb{N}$ ,  $n_1 \geq 1$ . If  $k \in S$ , then  $n_k \geq k$ , and so, since  $n_{k+1} > n_k$  (increasing), we have that  $n_{k+1} > k \implies n_{k+1} \geq k + 1$ . So,  $k + 1 \in S$ ,  $S = \mathbb{N}$ .

**Remark 2.5.**  $\lim_{k \rightarrow \infty} x_{n_k} = x$  if  $\forall \varepsilon > 0, \exists K \in \mathbb{N}$  s.t.  $\forall k \geq K, |x_{n_k} - x| < \varepsilon$ .

**Theorem 2.7.** Let  $(x_n)$  be a sequence such that  $\lim_{n \rightarrow \infty} x_n = x$ . Then, for any subsequence  $(x_{n_k})_{k \in \mathbb{N}}$ , we have that  $\lim_{k \rightarrow \infty} x_{n_k} = x$

*Proof.* Let  $\varepsilon > 0$ . Since  $\lim_{n \rightarrow \infty} x_n = x$ ,  $\exists N \in \mathbb{N}$  s.t.  $\forall n \geq N$ ,  $|x_n - x| < \varepsilon$ . Take  $K = N$  (from Remark 2.5). Then, for  $k \geq K$ , we have from Remark 2.4 that

$$n_k \geq k \geq K = N,$$

and hence  $|x_{n_k} - x| < \varepsilon \implies \lim_{k \rightarrow \infty} x_{n_k} = x$ . ■

**Theorem 2.8** (Bolzano-Weirestrass Theorem). <sup>35</sup>Any bounded sequence  $(x_n)$  has a convergent subsequence.

<sup>35</sup>Fundamental property of the real line; equivalent to AC.

**Example 2.13.** Take  $x_n = (-1)^n$ ,  $n \in \mathbb{N}$ . This sequence does not converge. However, if we take a subsequence with  $n_k = 2k$ ,  $k \in \mathbb{N}$ .  $x_{n_k} = (-1)^{2k} = 1$ , so  $(x_{n_k})$  is a constant sequence 1 and converges to 1.

Similarly, if  $n_k = 2k + 1$ ,  $k \in \mathbb{N}$ , then  $x_{n_k} = (-1)^{2k+1} = -1$ , and the subsequence converges to  $-1$ .

**Proposition 2.2.** If  $0 < b < 1$ , then  $\lim_{n \rightarrow \infty} b^n = 0$ .

*Proof.* Let  $x_n = b^n$ . Then  $x_n > 0$ , and  $x_{n+1} = b^{n+1} = bx_n > x_n$ , and since  $0 < b < 1$ ,  $(x_n)$  is decreasing and bounded from below,  $(x_n)$  converges by the Monotone Convergence Theorem. Let  $x = \lim_{n \rightarrow \infty} x_n$ . Again,  $x_{n+1} = bx_n$ , so  $\lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} bx_n = b \lim_{n \rightarrow \infty} x_n$ , so  $x = bx \implies (1 - b)x = 0$ .  $0 < b < 1 \implies x = 0$ . ■

*BW Proof (1): using Nested Interval Property.* <sup>36</sup>Since  $(x_n)$  bounded,  $\exists M > 0$  s.t.  $|x_n| \leq M \forall n \in \mathbb{N}$ . Let  $I_1 = [-M, M]$  and  $n_1 = 1$ . We now construct  $I_2, n_2$  as follows.

Divide  $I_1$  into two intervals of the same size,  $I'_1 = [-M, 0]$ ,  $I''_1 = [0, M]$ . Now, consider the sets

$$A_1 = \{n \in \mathbb{N} : n > n_1 (= 1), x_n \in I'_1\}, \quad A_2 = \{n \in \mathbb{N} : n > n_1, x_n \in I''_1\}$$

(ie, all the indices of all the elements in  $I'_1, I''_1$  resp.).

Hence,  $A_1 \cup A_2 = \{n : n > n_1\}$ , an infinite set, and hence, one of  $A_1, A_2$  must be infinite (by Theorem 1.9). If  $A_1$  infinite, set  $I_2 = I'_1$ ,  $n_2 = \min A_1$ . If  $A_1$  finite, then  $A_2$  infinite, and set  $I_2 = I''_1$ ,  $n_2 = \min A_2$ .

Suppose now that  $I_k, n_k$  are chosen, and that  $I_k$  contains infinitely many elements of the sequence  $(x_n)$ . Divide  $I_k$  into two equal sub-intervals,  $I'_k, I''_k$ . We now introduce

$$A_1^{(k)} = \{n \in \mathbb{N} : n > n_k \text{ and } x_n \in I'_k\}, \quad A_2^{(k)} = \{n \in \mathbb{N} : n > n_k \text{ and } x_n \in I''_k\},$$

(similar to our construction of  $A_1, A_2$ ).  $A_1^{(k)} \cup A_2^{(k)}$  must be infinite, so one of the two must be infinite. If  $A_1$  infinite, set  $I_{k+1} = I'_k, n_{k+1} = \min A_1^{(k)}$ . If  $A_2$  infinite, set  $I_{k+1} = I''_k, n_{k+1} = \min A_2^{(k)}$ .

This gives now that  $I_{k+1}$  and  $n_{k+1}$ , where  $I_{k+1} \subseteq I_k, I_{k+1}$  contains infinitely many elements of the sequence. Further, by construction,  $n_{k+1} > n_k$ . This gives us a sequence of closed intervals  $I_k = [a_k, b_k], k \in \mathbb{N}$  such that  $I_1 \supseteq I_2 \supseteq \dots \supseteq I_k \supseteq I_{k+1} \supseteq \dots$ , such that  $x_{n_k} \in I_k$ , and that  $n_k$  is a strictly increasing sequence of natural numbers, defining subsequence  $(x_{n_k})$ .

Now, by construction, the length of  $I_{k+1}$  is  $\frac{1}{2}$  of the length of  $I_k$ . Since  $I_k = [a_k, b_k]$ , then

$$b_k - a_k = \frac{b_{k-1} - a_{k-1}}{2} = \dots = \frac{b_1 - a_1}{2^{k-1}} = \frac{2M}{2^{k-1}} = \frac{M}{2^{k-2}}.$$

Since  $I_k, k \in \mathbb{N}$ , is a nested sequence of closed intervals and by the nested interval property of the real line (AC),  $\exists x \in \bigcap_{k=1}^{\infty} I_k$ .

We claim now that our subsequence  $(x_{n_k})$  satisfies  $\lim_{k \rightarrow \infty} x_{n_k} = x$ . To see this, let  $\varepsilon > 0$ . Since  $\lim_{k \rightarrow \infty} \frac{M}{2^{k-2}} = \lim_{k \rightarrow \infty} \frac{4M}{2^k} = 0$ , by Proposition 2.2, with  $b = \frac{1}{2}$ . There exists  $K \in \mathbb{N}$  such that  $\forall k \geq K$ , we have  $\frac{M}{2^{k-2}} = b_k - a_k < \varepsilon$ . So, since  $I_k$  is a nested sequence of intervals,  $\forall k \geq K, x_{n_k} \in I_K (x_{n_k} \in I_k \subseteq I_K)$ . We also have that  $x \in I_K$ , since  $x \in \bigcap I_k$ . So,  $x, x_{n_k} \in [a_K, b_K] \forall k \geq K$ . So, for  $k \geq K, |x_{n_k} - x| \leq |b_k - a_k| < \varepsilon$ . So for  $\varepsilon > 0, \exists K \in \mathbb{N}$  s.t.  $\forall k \geq K, |x_{n_k} - x| < \varepsilon$ , and so  $\lim_{k \rightarrow \infty} x_{n_k} = x$ , as desired. ■

<sup>36</sup>Sketch:  
See Abbott, pg 57, for good diagram.

**Definition 2.9 (Peak).** Let  $(x_n)$  be a sequence of real numbers. An element  $x_m$  is called a peak of this sequence if  $x_m \geq x_n \forall n \geq m$ .  $x_m$  is bigger or equal then to any element of the sequence that follows it.

If a sequence is decreasing, then any element of the sequence is a peak.

If a sequence is increasing, then there is no peak.

*BW Proof (2): using Peaks.* Take sequence  $(x_n)$ . Then,

- **Case 1:**  $(x_n)$  has *infinitely* many peaks; enumerate the indices of those peaks as  $n_1 < n_2 < n_3 < \dots$ , then  $x_{n_k} < x_{n_{k+1}} \forall k$ , since  $x_{n_k}$  is a peak,  $n_{k+1} > n_k$ . This gives a decreasing subsequence  $(x_{n_k})$ .

- **Case 2:**  $(x_n)$  has *finitely* many peaks, with indices  $m_1 < m_2 < \cdots < m_r$ . Set  $n_1 = m_r + 1$ . Then  $x_{n_1}$  is not a peak, and so  $\exists n_2 > n_1$  s.t.  $x_{n_2} > x_{n_1}$ . Now,  $x_{n_2}$  is also not a peak, ( $n_2 > n_1 > m_r$ ), and so there exists  $n_3 > n_2$  such that  $x_{n_3} > x_{n_2}$ , and so on. In this way, we construct a subsequence  $(x_{n_k})$  that is strictly increasing, that is,  $x_{n_{k+1}} > x_{n_k}$ .

■