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Analysis I, II **MATH254**

Course Outline:

Fundamentals of set theory. Properties of the reals. Limits, limsup, liminf. Continuity. Functions. Differentiation.

References:

Understanding Analysis, *Abbott;* Introduction to Real Analysis, *Bartle;* Analysis I, *Tao*

Contents

1 Logic, Sets, and Functions

1.1 Mathematical Induction & The Naturals

The **natural numbers**, $\mathbb{N} = \{1, 2, 3, \dots\}$, are specified by the 5 **Peano Axioms**:

(1) $1 \in \mathbb{N}$ [1]

(2) every natural number has a successor in $\mathbb{N}$

(3) 1 is not the successor of any natural number

(4) if the successor of x is equal to the successor of y, then x is equal to y [2]

(5) **the axiom of induction**

[1] using 0 instead of 1 is also valid, but we will use 1 here.

[2] axioms (2)-(4) can be equivalently stated in terms of a successor function $s(n)$ more rigorously, but won't here

The **Axiom of Induction** (AI), can be stated in a number of ways.

Axiom 1.1 (AI.i). *Let $S \subseteq \mathbb{N}$ with the properties:*

(a) $1 \in S$

(b) if $n \in S$, then $n + 1 \in S$ [3]

then $S = \mathbb{N}$.

[3] *(a)* is called the **inductive base**; *(b)* the **inductive step**. All AI restatements are equivalent in having both of these, and only differentiate on their specific values.

Example 1.1. *Prove that, for every $n \in \mathbb{N}$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2} (\equiv (1))$*

Proof (via AI.i). Let S be the subset of $\mathbb{N}$ for which (1) holds; thus, our goal is to show $S = \mathbb{N}$, and we must prove *(a)* and (b) of AI.i.

- by inspection, $1 \in S$ since $1 = \frac{1(1+1)}{2} = 1$, proving *(a)*
- assume $n \in S$; then, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ by definition of S. Adding $n+1$ to both sides yields:

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) && (1) \\ &= (n+1)(\frac{n}{2} + 1) && (2) \\ &= \frac{(n+1)(n+2)}{2} && (3) \\ &= \frac{(n+1)((n+1)+1)}{2} && (4) \end{aligned}$$

Line (4) is equivalent to statement (1) (substituting n for $n+1$), and thus if $n \in S$, then $n+1 \in S$ and *(b)* holds. Thus, by AI.i, $S = \mathbb{N}$ and $1+2+\cdots+n = \frac{n(n+1)}{2}$ holds $\forall\, n \in \mathbb{N}$. ■

Example 1.2. *Prove (by induction), that for every $n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.*

Proof. Follows a similar structure to the previous example. Let S be the subset of $\mathbb{N}$ for which the statement holds. $1 \in S$ by inspection (*(a)* holds), and we prove *(b)* by assuming $n \in S$ and showing $n+1 \in S$ (algebraically). Thus, by AI.i, $S = \mathbb{N}$ and the statement holds $\forall\, n \in \mathbb{N}$. ■

This can also be proven directly (Gauss' method).

Proof (Gauss' method). Let $A(n) = 1 + 2 + 3 + \cdots + n$. We can write $2 \cdot A(n) = 1 + 2 + 3 + \cdots + n + 1 + 2 + 3 + \cdots + n$. Rearranging terms (1 with n, 2 with $n-1$, etc.), we can say $2 \cdot A(n) = (n+1) + (n+1) + \cdots$, where $(n+1)$ is repeated n times; thus, $2 \cdot A(n) = n(n+1)$, and $A(n) = \frac{n(n+1)}{2}$. ■

Axiom 1.2 (AI.ii). *Let $S \subseteq \mathbb{N}$ s.t.*

(a) $m \in S$

(b) $n \in S \implies n+1 \in S$

then $\{m, m+1, m+2, \dots\} \subseteq S$.

Example 1.3. *Using AI.ii, prove that for $n \geq 2$, $n^2 > n+1$*

Proof. Let $S \subseteq \mathbb{N}$ be the set of n for which the statement holds. $n = 2 \implies 4 > 3$, so the base case holds. Consider $n^2 > n+1$ for some $n \geq 2$. Then, $(n+1)^2 = n^2 + 2n + 1 > n + 1 + 2n + 1 = 3n + 2 > 2n + 2 > n + 2$, hence $S = \{2, 3, 4, \cdots\}$ (all $n \geq 2$). ∎

Axiom 1.3 (Principle of Complete Induction, AI.iii)**.** *Let $S \subseteq \mathbb{N}$ s.t.*

(a) $1 \in S$

(b) if $1, 2, \ldots, n-1 \in S$, *then* $n \in S$

then $S = \mathbb{N}$.

Finally, combining AI.ii and AI.iii;

Axiom 1.4 (AI.iv)**.** *Let $S \subseteq \mathbb{N}$ s.t.:*

(a) $m \in S$

(b) if $m, m+1, \ldots, m+n \in S$, *then* $m+n+1 \in S$

then $\{m, m+1, m+2, \ldots\} \subseteq S$.

Theorem 1.1 (Fundamental Theorem of Arithmetic)**.** *Every natural number n can be written as a product of one or more primes.* [4]

[4] 1 is not a prime number

Proof of Theorem 1.1. Let S be the set of all natural numbers that can be written as a product of one or more primes. We will use AI.iv to show $S = \{2, 3, \ldots\}$.

- *(a)* holds; 2 is prime and thus $2 \in S$
- suppose that $2, 3, \ldots, 2+n \in S$. Consider $2 + (n+1)$:
  - if $2 + (n+1)$ is *prime*, then $2 + (n+1) \in S$, as all primes are products of 1 and themselves and are thus in S by definition.
  - if $2 + (n+1)$ is *not prime*, then it can be written as $2 + (n+1) = a \cdot b$ where $a, b \in \mathbb{N}$, and $1 < a < 2 + (n+1)$ and $1 < b < 2 + (n+1)$. By the definition of S, $a, b \in S$, and can thus be written as the product of primes. Let $a = p_1 \cdot \cdots \cdot p_l$ and $b = q_1 \cdot \cdots \cdot q_j$, where the p's and q's are prime and $l, j \geq 1$. Then, $a \cdot b$ is a product of primes, and thus so is $2 + (n+1)$. Thus, $2 + (n+1) \in S$, and by AI.iv, $S = \{2, 3, 4, \ldots\}$

∎

1.2 Extensions: Integers, Rationals, Reals

Consider the set of naturals $\mathbb{N} = \{1, 2, 3, \dots\}$. Adding 0 to $\mathbb{N}$ defines $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We define the **integers** as the set $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, or the set of all positive and negative whole numbers.

Within $\mathbb{Z}$, we can define multiplication, addition and subtraction, with the neutrals of 1 and 0, respectively. However, we cannot define division, as we are not guaranteed a quotient in $\mathbb{Z}$. This necessitates the **rationals**, $\mathbb{Q}$. We define

$$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}.$$

On $\mathbb{Q}$, we have the familiar operations of multiplication, addition, subtraction and properties of associativity, distributivity, etc. We can also define division, as $\frac{\frac{p}{q}}{\frac{p'}{q'}} = \frac{pq'}{qp'}$.

We can also define a relation $<$ between fractions, such that

- $x < y$ and $y < z \implies x < z$
- $x < y \implies x + z < y + z$

$\mathbb{Q}$, together with its operations and relations above, is called an **ordered field**.

1.2.1 The Insufficiency of the Rationals

We can consider historical reasoning for the extension of $\mathbb{Q}$ to $\mathbb{R}$. Consider a right triangle of legs a, b and hypotenuse c. By the Pythagorean Theorem, $a^2 + b^2 = c^2$. Consider further the case there $a = b = 1$, and thus $c^2 = 2$. Does c exist in $\mathbb{Q}$?

Proposition 1.1. $c^2 = 2, c \notin \mathbb{Q}$.

Proof of Proposition 1.1. Suppose $c \in \mathbb{Q}$. We can thus write $c = \frac{p}{q}$, where[5] $p, q \in \mathbb{N}$, and p, q share no common divisors, ie they are in "simplest form". Notably, p and q cannot *both* be even (under our initial assumption), as they would then share a divisor of 2. We write

$$\begin{aligned} c &= \frac{p}{q} \\ c^2 = 2 &= \frac{p^2}{q^2} \\ 2q^2 &= p^2 \end{aligned}$$

$p \in \mathbb{N} \implies p^2 \in \mathbb{N}$, and thus p^2, and therefore[6] p, must be divisible by 2 ($\implies$ p even). Therefore, we can write $p = 2p_1, p_1 \in \mathbb{N}$, and thus $2q^2 = (2p_1^2)^2 \implies q^2 = 2p_1^2$. By the same reasoning, q must now be even as well, contradicting our initial assumption that p and q share no common divisors. Thus, $c \notin \mathbb{Q}$. ■

[5]Note that in the definition of $\mathbb{Q}$, p, q are defined to be in $\mathbb{Z}$; however, as we are using a geometric argument, we can assume $c > 0 \implies \text{Sign}(p) = \text{Sign}(q)$, and we can just take $p, q \in \mathbb{N}$ for convenience and wlog.

[6]$\sqrt{\text{even}} = \text{even}$

1.3 Sets & Set Operations

- $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- $\bigcup_{i=1}^{\infty} A_n = \bigcup_{n \in \mathbb{N}} A_n = \{x : x \in A_n \text{ for some } n \in \mathbb{N}\}$
- $\bigcap_{i=1}^{\infty} A_n = \bigcap_{n \in \mathbb{N}} A_n = \{x : x \in A_n \, \forall \, n \in \mathbb{N}\}$
- $A^C = \{x : x \in X \text{ and } x \notin A\}$[7]

[7]X is often omitted if it is clear from context.

Theorem 1.2 (De Morgan's Theorem(s))**.** *Let A, B be sets. Then,*

$$(a) \qquad (A \cap B)^C = A^C \cup B^C$$

and

$$(b) \qquad (A \cup B)^C = A^C \cap B^C.$$

Proof of Theorem 1.2. (b) *(A similar argument follows...)*

■

Proposition 1.2.

$$(a) \left(\bigcap_{n=1}^{\infty} A_n \right)^C = \bigcup_{n=1}^{\infty} A_n^C$$

$$(b) \left(\bigcup_{n=1}^{\infty} A_n \right)^C = \bigcap_{n=1}^{\infty} A_n^C$$

Proof of Proposition 1.2. Consider Proposition (b). Working from the left-hand side, we have

$$\begin{aligned} \left(\bigcup_{n=1}^{\infty} A_n \right)^C &= \{x : x \notin \bigcup A_n\} \\ &= \{x : x \notin A_n \, \forall \, n \in \mathbb{N}\} \\ &= \bigcap \{x : x \notin A_n\} \\ &= \bigcap A_n^C \end{aligned}$$

(a) can be logically deduced from this result. Consider the RHS, $\bigcup A_n^C$. Taking the complement:

$$\left(\bigcup A_n^C\right)^C \overset{\text{via (b)}}{=} \bigcap A_n^{C^C}$$
$$= \bigcap A_n$$

Taking the complement of both sides, we have $\bigcup A_n^C = \left(\bigcap A_n\right)^C$, proving (a). ∎

1.4 Functions

Definition 1.1. *Let A, B be sets. A* function f *is a rule assigned to each $x \in A$ a corresponding unique element $f(x) \in B$. We denote*

$$f : A \to B.$$

Definition 1.2. *The* domain *of a function $f : A \to B$, denoted Dom$(f) = A$. The* range *of f, denoted Ran$(f) = \{f(x) : x \in A\}$. Clearly, Ran$(f) \subseteq B$, though equality is not necessary.*

Example 1.4. *The function $f(x) = \sin x$, $f : \mathbb{R} \to [-1, 1]$. Here, Dom$(f) = \mathbb{R}$, and Ran$(f) = [-1, 1]$.*

Example 1.5 (Dirichlet Function)**.** $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \notin \mathbb{Q} \end{cases}$. *Despite not having a true "explicit" formula, so to speak, this is still a valid function (under modern definitions).*

1.4.1 Properties of Functions

Proposition 1.3. *Let $f : A \to B, C \subseteq A, f(C) = \{f(x) : x \in C\}$. We claim $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.*

Proof. We will prove this by showing (1) $\subseteq$ and (2) $\supseteq$.

(1) $y \in f(C_1 \cup C_2) \implies$ for some $x \in C_1 \cup C_2, y = f(x)$. This means that either for some $x \in C_1, y = f(x)$, or for some $x \in C_2, y = f(x)$. This implies that either $y \in f(C_1)$, or $y \in f(C_2)$, and thus y *must* be in their union, ie $y \in C_1 \cup C_2$.

(2) $y \in f(C_1) \cup f(C_2) \implies y \in f(C_1)$ or $y \in f(C_2)$. This means that for some $x \in C_1, y = f(x)$, or for some $x \in C_2, y = f(x)$. Thus, x *must* be in $C_1 \cup C_2$, and for some $x \in C_1 \cup C_2, y = f(x) \implies y \in f(C_1 \cup C_2)$.

(1) and (2) together imply that $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$. ■

Example 1.6. *Let* $A_n = 1, 2, \ldots$ *be a sequence of sets. Prove that* $f(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f(A_n)$.

Proof. Let $y \in f(\bigcup_{n=1}^{\infty} A_n)$. This implies that $\exists x \in \bigcup_{n=1}^{\infty} A_n$ s.t. $f(x) = y$. This implies that $x \in A_n$ for some n, and $y \in f(A_n)$ for that same "some" n, and thus y must be in the union of all possible $f(A_n)$, ie $y \in \bigcup f(A_n)$. This shows $\subseteq$, use similar logic for the reverse. ■

Proposition 1.4. $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$ [8]

[8] NB: the reverse is not always true, ie these sets are not always equal; "lack" of equality is more "common" than not.

Proof. $y \in f(C_1 \cap C_2) \implies$ for some $x \in C_1 \cap C_2, y = f(x)$. This implies that for some $x \in C_1, y = f(x)$ **and** for some $x \in C_2, y = f(x)$. Note that this does *not* imply that these x's are the same, ie this reasoning is not reversible as in the previous union case. This implies that $y \in f(C_1)$ and $y \in f(C_2) \implies y \in f(C_1) \cap f(C_2)$. ■

Example 1.7. *Prove that if* $A_n, n = 1, 2, \ldots, f(\bigcap_{n=1}^{\infty} A_n) \subseteq \bigcap_{n=1}^{\infty} f(A_n)$.

Proof (Sketch). Use the same idea as in Example 1.6, but, naturally, with intersections. ■

Example 1.8. *Take* $f(x) = \sin x$, $A = \mathbb{R}, B = \mathbb{R}$, *and take* $C_1 = [0, 2\pi], C_2 = [2\pi, 4\pi]$. *Then,* $f(C_1) = [-1, 1]$, *and* $f(C_2) = [-1, 1]$. *But* $C_1 \cap C_2 = \{2\pi\}$; $f(\{2\pi\}) = \{\sin 2\pi\} = \{0\}$, *and thus* $f(C_1 \cap C_2) = \{0\}$, *while* $f(C_1) \cap f(C_2) = [-1, 1]$, *as shown in Proposition 1.4.*

Definition 1.3 (Inverse Image of a Set). *Let* $f : A \to B$ *and* $D \subseteq B$. *The* inverse image *of* D *by* F *is denoted* $f^{-1}(D)$[9] *and is defined as*

$$f^{-1}(D) = \{x \in A : f(x) \in D\}.$$

[9] Note that this is **not** equivalent to the typical definition of an inverse *function*; f^{-1} may not exist

Example 1.9. $A = [0, 2\pi], B = \mathbb{R}, f(x) = \sin x, D = [0, 1]$.

$$f^{-1}(D) = \{x \in A : f(x) \in D\} = \{x \in [0, 2\pi] : \sin(x) \in [0, 1]\} = [0, \pi].$$

Proposition 1.5. *Given function* f *and sets* D_1, D_2,

(a) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$

(b) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$[10]

[10] Just see next proposition; if you really need convincing, just use 2 rather than ∞ as the upper limit of the

Proposition 1.6 ($\star$). *Let $A_n, n = 1, 2, 3 \dots$. Then,*

(a) $f^{-1}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$

(b) $f^{-1}(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} f^{-1}(A_n)$

Proof. [11]

(a)

$$\begin{aligned} x \in f^{-1}(\bigcup_{n=1}^{\infty} A_n) &\iff f(x) \in \bigcup_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for some } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for some } n \in \mathbb{N} \\ &\iff x \in \bigcup_{n=1}^{\infty} f^{-1}(A_n) \end{aligned}$$

(b)

$$\begin{aligned} x \in f^{-1}(\bigcap_{n=1}^{\infty} A_n) &\iff f(x) \in \bigcap_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for all } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for all } n \in \mathbb{N} \\ &\iff x \in \bigcap_{n=1}^{\infty} f^{-1}(A_n)^{12} \end{aligned}$$

■

[12]This is a "proof by definitions" as I like to call it.

[12]Similar proof can be used to prove Proposition 1.5, less generally.

Remark 1.1. *$f : A \to B$, $A_1 \subseteq A$. Given $f(A_1^C)$ and $f(A_1)^C$, there is* ***no general relation*** *between the two.*

For instance, take $A = [0, 6\pi], B = [-1, 2], C = [0, 2\pi]$, and $f(x) = \sin x$. Then, $f(C) = [-1, 1]$, and $f(C^C) = f([-1, 0)) = [-1, 1]$, but $f(C)^C = [-1, 1]^C = (1, 2]$, and $f(C^C) \neq f(C)^C$; in fact, these sets are disjoint.

Proposition 1.7. *Let $f : A \to B$ and let $D \subseteq B$. Then $f^{-1}(D^C) = [f^{-1}(D)]^C$.*

Proof.

$$f^{-1}(D^C) = \{x : f(x) \in D^C\} = \{x : f(x) \notin D\}$$
$$[f^{-1}(D)]^C = [\{x : f(x) \in D\}]^C = \{x : x \notin f^{-1}(D)\} = \{x : f(x) \notin D\}$$

■

1.5 Reals

Axiom 1.5 (Of Completeness)**.** *Any non-empty subset of* $\mathbb{R}$ *that is bound from above has at least one upper bound (also called the supremum).*

In other words; let $A \subseteq \mathbb{R}$ *and suppose* A *is bounded from above (*A *has at a least upper bound). Then* $\sup(A)$ *exists.*

Real numbers, algebraically, have the same properties as the rationals; we have addition, multiplication, inverse of non-zero real numbers, and we have the relation $<$. All together, $\mathbb{R}$ is an ordered field.

Definition 1.4. *Let* $A \subseteq \mathbb{R}$*. A number* $b \in \mathbb{R}$ *is called an* ***upper bound*** *for* A *if for any* $x \in A$, $x \leq B$*.*

A number $l \in \mathbb{R}$ *is called a* ***lower bound*** *for* A *if for any* $x \in A$, $x \geq l$*.*

Definition 1.5 (The Least Upper Bound)**.** *Let* $A \subseteq \mathbb{R}$*. A real number* s *is called the* ***least upper bound*** *for* A *if the following holds:*

(a) s *is an upper bound for* A

(b) if b *is any other upper bound for* A*, then* $s \leq b$*.*

The least upper bound of a set A *is* unique*, if it exists; if* s *and* s' *are two least upper bounds, then by (a),* s *and* s' *are upper bound for* A*, and by (b),* $s \leq s'$ *and* $s' \leq s$*, and thus* $s = s'$*.*

This least upper bound is called the supremum *of* A*, denoted* $\sup(A)$*.*

Definition 1.6 (The Greatest Lower Bound)**.** *Let* $A \subset \mathbb{R}$*. A number* $i \in \mathbb{R}$ *is called the* ***greatest lower bound*** *for* A *if the following holds:*

(a) i *is a lower bound for* A

(b) if l *is any other lower bound for* A*, then* $i \geq l$*.*

If i *exists, it is called the* infimum *of* A *and is denoted* $i = \inf(A)$*, and is unique by the same argument used for* $\sup(A)$*.*

Proposition 1.8. *Let*[13] $A \subseteq \mathbb{R}$ *and let* s *be an upper bound for* A*. Then* $s = \sup(A)$ *iff for any* $\varepsilon > 0$*, there exists* $x \in A$ *s.t.* $s - \varepsilon < x$*.*

[13]Note that this, and Proposition 1.9 that follows, are *not* definitions: they are restatements, and do

Proof. We have two statements:

I. $s = \sup(A)$;

II. For any $\varepsilon > 0$, $\exists x \in A$ s.t. $s - \varepsilon < x$;

and we desire to show that I $\iff$ II.

- I $\implies$ II: Let $\varepsilon > 0$. Then, since $s = \sup(A)$, $s - \varepsilon$ *cannot* be an upper bound for A (as s is the least upper bound, and thus $s - \varepsilon < s$ cannot be an upper bound at all). Thus, there exists $x \in A$ such that $s - \varepsilon < x$, and thus if I holds, II must hold.
- II $\implies$ I: suppose that this does not hold, ie II holds for an upper bound s for A, but $s \neq \sup(A)$. Then, there exists some upper bound b of A s.t. $b < s$. Take $\varepsilon = s - b$. $\varepsilon > 0$, and since II holds, there exists $x \in A$ such that $s - \varepsilon < x$. But since $s - \varepsilon = b$ and thus $b < x$, then b cannot be an upper bound for A, contradicting our initial condition. So, if II $\implies$ I does *not* hold, we have a "impossibility", ie a value b which is an upper bound for A which cannot be an upper bound, and thus II $\implies$ I.

■

Proposition 1.9 (⋆). *Let $A \subseteq \mathbb{R}$ and let i be a lower bound for A. Then $i = \inf(A) \iff$ for every $\varepsilon > 0$ there exists $x \in A$ s.t. $x < i + \varepsilon$.*[14]

[14]Use similar argument to proof of previous proposition.

Remark 1.2. *Axiom 1.5 can also be expressed in terms of infimum. Define $-A = \{-x : x \in A\}$. Then, if b is an upper bound for A, then $b \geq x \, \forall \, x \in A$, then $-b \leq -x \, \forall \, x \in A$, ie -b is a lower bound of $-A$. Similarly, if l is a lower bound for A, $-l$ is an upper bound for $-A$.*

Thus, if A is bounded from above, then

$$-\sup(A) = \inf(-A),$$

and if A is bounded from below,

$$-\inf(A) = \sup(-A).$$

Axiom 1.6 (AC (infimum)). *Let $A \subseteq \mathbb{R}$; if A bounded from below, $\inf(A)$ exists.*

Definition 1.7 (max, min)**.** *Let $A \subseteq \mathbb{R}$. An $M \in A$ is called a* maximum *of A if for any $x \in A$, $x \leq M$. M is an upper bound for A,* ***but also*** *$M \in A$.*

If M exists, then $M = \sup(A)$; M is an upper bound, and if b any other upper bound, then $b \geq M$, because $M \in A$, and thus $M = \sup(A)$.

NB: $M = \max(A)$ ***need not*** *exist, while $\sup(A)$ must exist. Consider $A = [0, 1)$; $\sup(A) = 1$, but there exists no $\max(A)$.*

The same logic exists for the existence of minimum vs infimum (consider $(0, 1)$, with no maximum nor minimum).

Theorem 1.3 (Nested interval property of $\mathbb{R}$)**.** *Let $I_n = [a_n, b_n] = \{x : a_n \leq x \leq b_n\}$, $n = 1, 2, 3 \ldots$ be an infinite sequence of bounded, closed intervals s.t.*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots I_n \supseteq I_{n+1} \supseteq \ldots$$

Then, $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$ (note that this does not *hold in $\mathbb{Q}$).*

Proof. [15] We have $I_n = [a_n, b_n], I_{n+1} = [a_{n+1}, b_{n+1}], \ldots$. And the inclusion $I_n \supseteq I_{n+1}$. $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n, \forall\, n \geq 1$. So, the sequence a_n (left-end) is increasing, and the sequence b_n (right-end) is decreasing.

We also have that for any $n, k \geq 1$, $a_n \leq b_k$. We see this by considering two cases:

- Case 1: $n \leq k$, then $a_n \leq a_k$ (as a_n is increasing), and thus $a_n \leq a_k \leq b_k$.
- Case 2: $n > k$, then $a_n \leq b_n \leq b_k$ (again, as b_n is decreasing).

Let $A = \{a_n : n \in \mathbb{N}\}$. Then, A is bounded from above by *any* b_k (as in our inequality we showed above). Let $x = \sup(A)$, which must exist by Axiom 1.5.

Note that as a result, $x \geq a_n$ for all n, and for all k, $x \leq b_k$, as x is the lowest upper bound and must be $\leq$ all other upper bounds, and so for all $n \geq 1$, $a_n \leq x \leq b_n$, ie $x \in I_n \,\forall\, n \geq 1$, and thus $x \in \bigcap_{n=1}^{\infty} I_n$ and so $\bigcap_{n=1}^{\infty} \neq \varnothing$. ■

[15] Sketch: show that the left-end points are increasing and the right-end points are decreasing. Show either that all the left-end points are bounded from above or that all the right-end points are bounded from below. As a result, there exists a sup/inf (depending on which end you choose) of the set of all the right/left points. For the sup case, all upper bounds must be $\geq$ sup, and thus the sup is in all I_n, and thus in their intersect, and thus the intersect is not empty.

Remark 1.3. *The proof above emphasized the left-end points; it can equivalently be proven via the right-end points, and using $y = \inf(\{b_n : n \in \mathbb{N}\}) = \inf(B)$, rather than $\sup(A)$, and showing that $y \in \bigcap I_n$.*

Remark 1.4 ($\star$)**.** *Note too that, if $x = \sup(A)$ and $y = \inf(B)$, then $x, y \in \bigcap_{n=1}^{\infty} I_n$; in fact, $\bigcap_{n=1}^{\infty} I_n = [x, y]$. This can be done by*

- *Use the main proof to show $x \in \bigcap I_n$*

- *Use the previous remark to show $y \in \bigcap I_n$*
- *Show $x \leq y \implies [x, y] \subseteq \bigcap I_n$*
- *Show $\bigcap I_n \subseteq [x, y] \implies$ equality.*

Remark 1.5. *The intervals I_n* must *be closed; if not, eg $I_n = (0, \frac{1}{n})$, then $\bigcap_{n=1}^{\infty} I_n = \varnothing$. Say $\bigcap I_n \neq \varnothing$; take then some $x \in \bigcap I_n$. Then, $x \in (0, \frac{1}{n})\, \forall n \in \mathbb{N}$. But by Proposition 1.10, $\forall x \in \mathbb{R}, \exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < x$. Clearly, x must be greater than 0 to exist in the intersection; hence, there will always exist some sufficiently large N such that $\frac{1}{N} < x \implies x \notin (1, \frac{1}{N}) \implies x \notin \bigcap I_n \implies \bigcap I_n = \varnothing$.*

1.6 Density of Rationals in Reals

Proposition 1.10 (Archimedian Property). *(a) For any $x \in \mathbb{R}$, there exists a natural number n s.t. $n > x$.*

(b) For any $y \in \mathbb{R}$ satisfying $y > 0$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < y$.

Remark 1.6. *(a) states that $\mathbb{N}$ is not a bounded subset of $\mathbb{R}$.*

Remark 1.7. *(b) follows from (a) by taking $x = \frac{1}{y}$ in (a), then $\exists n \in \mathbb{N}$ s.t. $n > \frac{1}{y} \implies \frac{1}{n} < y$, and thus we need only prove (a).*

Remark 1.8. *Recall that $\mathbb{Q}$ is an ordered field (operations $+, \cdot$ and a relation $<$). $\mathbb{Q}$ can be extended to a larger ordered field with extended definitions of these operations/relations, such that it contains elements that are larger than any natural numbers (ie, not bounded above). This is impossible in $\mathbb{R}$ due to AC.*

Proof. Suppose (a) not true in $\mathbb{R}$, ie $\mathbb{N}$ is bounded from above in $\mathbb{R}$. Let $\alpha = \sup \mathbb{N}$, which exists by AC.

Consider $\alpha - 1$; since $\alpha - 1 < \alpha$, $\alpha - 1$ is not an upper bound of $\mathbb{N}$. So, there exists some $n \in \mathbb{N}$ s.t. $\alpha - 1 < n$; then, $\alpha < n + 1$ where $n + 1 \in \mathbb{N}$, and thus α is also not an upper bound, as there exists a natural number that is greater than α. This contradicts the assumption that $\alpha = \sup \mathbb{N}$, so (a) must be true. ■

Theorem 1.4 (Density). *Let $a, b \in \mathbb{R}$ s.t. $a < b$. Then, $\exists x \in \mathbb{Q}$ s.t. $a < x < b$.*

Remark 1.9. *If you take $a \in \mathbb{R}$ and $\varepsilon > 0$, then by the theorem, $\exists x \in \mathbb{Q}$ where $x \in (a - \varepsilon, a + \varepsilon)$. So any real number can be approximated arbitrarily closely (via choose of ε) by a rational number.*

Proof. Since $b - a > 0$, by (b) of Proposition 1.10, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a$, ie $na + 1 < nb$.

Let $m \in \mathbb{Z}$ s.t. $m-1 \leq na < m$. Such an integer must exists since $\bigcup_{m\in\mathbb{Z}}[m-1, m) = \mathbb{R}$, the family $[m-1, m), m \in \mathbb{Z}$ makes partitions of $\mathbb{R}$. Then, $na < m$ gives that $a < \frac{m}{n}$. On the other hand, $m - 1 \leq na$ gives $m \leq na + 1 < nb$. So $\frac{m}{n} < b$ and it follows that $\frac{m}{n}$ satisfies $a < \frac{m}{n} < b$. ∎

In the proof, we used the claim:

Proposition 1.11. *If $z \in \mathbb{R}$, then there exists $m \in \mathbb{Z}$ s.t. $m - 1 \leq z < m$.*

Proof. Let S be a non-empty subset of $\mathbb{N}$. Then S has the least element; $\exists m \in S$ s.t. $m \leq n, \forall n \in S$.

We can assume $z \geq 0$; if $0 \leq z < 1$, then we are done (take $m = 1$), and assume that $z \geq 1$. Let now $S = \{n \in \mathbb{N} : z < n\}, \neq \varnothing$ by Proposition 1.10, (a). Let m be the least element of S. It exists by Well-Ordering Property; then, since $m \in S$, $z < m$. But, we also have $m - 1 \leq z$, otherwise, if $z < m - 1$ then $m - 1 \in S$ and then m is not the least element of S. Thus, we have $m - 1 \leq z < m$, as required. ∎

Theorem 1.5. *The set J of irrationals is also dense in $\mathbb{R}$. That is, if $a, b \in \mathbb{R}, a < b$, $\exists$ irrational y s.t. $a < y < b$ (noting that $J = \mathbb{R} \setminus \mathbb{Q}$).*

Proof. Fix $y_0 \in \mathbb{J}$. Consider $a - y_0, b - y_0$. $a - y_0 < b - y_0$, and by density of rationals, $\exists x \in \mathbb{Q}$ s.t. $a - y_0 < x < b - y_0$. Then, $a < y_0 + x < b$; let $y = x + y_0$, and we have $a < y < b$.

Note that y cannot be rational; if $y \in \mathbb{Q}$, $y = x + y_0 \implies y - x = y_0$, and since $x \in \mathbb{Q}$, $y - x \in \mathbb{Q} \implies y_0 \in \mathbb{Q}$, contradicting the original choice of $y_0 \notin \mathbb{Q}$. Thus, $y \in J$. ∎

Theorem 1.6. *$\exists$ a unique positive real number α s.t. $\alpha^2 = 2$.*

Proof. We show both uniqueness, existence:[16]

Uniqueness: if $\alpha^2 = 2$ and $\beta^2 = 2$, $\alpha \geq 0, \beta \geq 0$, then $0 = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta) > 0$, and so $\alpha - \beta = 0 \implies \alpha = \beta$.

- Existence: consider the set $A = \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 < 2\}$. A is not empty as $1 \in A$. The set of A is bounded above by 2, since if $x \geq 2$, then $x^2 \geq 4 > 2$, so $x \notin A$. So, by AC, $\sup A$ exists; let $\alpha = \sup A$. We will show that $\alpha^2 = 2$, by showing that both $\alpha^2 < 2$ and $\alpha^2 > 2$ are contradictions.

$\alpha^2 < 2$

For any $n \in \mathbb{N}$ we expand

$$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \leq \alpha^2 + \frac{2\alpha + 1}{n},$$

noting that $\frac{1}{n^2} \leq \frac{1}{n}$ for $n \geq 1$.

Let $y = \frac{2-\alpha^2}{2\alpha+1}$, which is strictly positive. By Proposition 1.10, $\exists n_0 \in \mathbb{N}$ s.t.

$$\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1} \text{ or } \frac{2\alpha+1}{n_0} < 2 - \alpha^2.$$

Substituting this n_0 into our inequality, we have

$$\left(\alpha + \frac{1}{n_0}\right)^2 \leq \alpha^2 + \frac{2\alpha+1}{n_0} < \alpha^2 + 2 - \alpha^2 = 2.$$

Since $\alpha + \frac{1}{n_0}$ is positive, $\alpha + \frac{1}{n_0} \in A$. But, since $\alpha = \sup A$, $\alpha + \frac{1}{n_0} \leq \alpha$, which is impossible, so $\alpha^2 < 2$ cannot be true.

$\alpha^2 > 2$

Take $n \in \mathbb{N}$;

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}.$$

Now, let $y = \frac{\alpha^2-2}{2\alpha}$; $y > 0$, and by Proposition 1.10, $\exists n_0 \in \mathbb{N}$ s.t.

$$\frac{1}{n_0} < \frac{\alpha^2-2}{2\alpha}, \text{ or } \frac{2\alpha}{n_0} < \alpha^2 - 2.$$

Substituting this n_0, we have

$$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 - \frac{2\alpha}{n_0} > \alpha^2 + 2 - \alpha^2 = 2.$$

So for any $x \in A$, we have $\left(\alpha - \frac{1}{n_0}\right)^2 > 2 > x^2$. $\alpha - \frac{1}{n_0} > 0$, and $x > 0$, since $x \in A$. Then, $\left(\alpha - \frac{1}{n_0}\right)^2 > x^2$ gives that $\alpha - \frac{1}{n_0} > x$.

So, $\alpha - \frac{1}{n_0} > x$ for all $x \in A$. So $\alpha - \frac{1}{n_0}$ is an upper bound for A, but since $\alpha = \sup A$, $\alpha - \frac{1}{n_0} \geq \alpha$ ie $\alpha \geq \alpha + \frac{1}{n_0}$, which is impossible. So $\alpha^2 > 2$ cannot be true.

Thus, $\alpha^2 = 2$.

■

[16]Proof sketch: uniqueness is clear. Existence follows from showing that α^2 cannot be either $<$ or $>$ 2. This is done

Remark 1.10. *A similar argument gives that for any $x \in \mathbb{R}$, $x \geq 0$, $\exists! \alpha \in \mathbb{R}$, $\alpha \geq 0$ such that $\alpha^2 = x$. This x is called the* square root *of x, denoted $\alpha = \sqrt{x}$.*

Remark 1.11. *For any natural number $m \geq 2$ and $x \geq 0$, $\exists! \alpha \in \mathbb{R}, \alpha \geq 0$ s.t. $\alpha^m = x$. The proof is similar, and we call α the m-th root of x.*

Remark 1.12. *Our last proof also gives that $\mathbb{Q}$ cannot satisfy AC. Suppose it does, ie any set in $\mathbb{Q}$ bounded from above has a supremum $\in \mathbb{Q}$. Then, consider $B = \{x \in \mathbb{Q} : x \geq 0$ and $x^2 < 2\}$; set $\alpha = \sup B$. The exact same proof can be used, but we will not be able to find an upper bound in $\mathbb{Q}$.*

1.7 Cardinality

Definition 1.8. *Let $f : A \to B$.*

1. *f injective (one-to-one) if $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$*
2. *f surjective (onto) if for any $b \in B \exists a \in A$ s.t. $f(a) = b$.*
3. *f bijective if both.*

Definition 1.9 (Composition)**.** *If $f : A \to B, g : B \to C$, the* composite map *$h = g \circ f$ is define by $h(x) = g(f(x))$. Note that $h : A \to C$.*

Example 1.10. *Consider functions f, g.*

1. *If f, g injective, so is $h = g \circ f$*
2. *If f, g bijective, then so is h*
3. *If $\exists E \subseteq C$, then $h^{-1}(E) = f^{-1}(g^{-1}(E))$*

Definition 1.10. *The inverse function*[17] *is defined only for bijective map $f : A \to B$. $y \in B$, $f^{-1}(y) = x$ where $x \in A$ s.t. $f(x) = y$.*

[17] Not the same as the inverse *image* of a set by a function, which is defined for any function.

Example 1.11.

1. *$A = \mathbb{R}, B = (0, \infty), f(x) = e^x$. f is a bijection, and $f^{-1}(y) = \ln y, y \in (0, \infty)$.*
2. *$A = (-\frac{\pi}{2}, \frac{\pi}{2}, B = \mathbb{R})$. $f(x) = \tan x$, $f^{-1}(y) = \arctan y$*

Definition 1.11 (Equal Cardinalities)**.** *Let A, B be two sets. We say A, B have the same cardinality, denote $A \sim B$ if there exists a bijective function $f : A \to B$.*

Example 1.12. *Let $E = \{2, 4, 6, \dots\}$ (even natural numbers). Define $f : \mathbb{N} \to E$ by $f(n) = 2n$. Thus, f is a bijection, and $\mathbb{N} \sim E$.*[18]

[18] See these independent notes for more.

Theorem 1.7. *The relation $\sim$ is a relation of equivalence.*

1. $A \sim A$
2. *if* $A \sim B$*, then* $B \sim A$
3. *if* $A \sim B$ *and* $B \sim C$*, then* $A \sim C$

Definition 1.12 (Countable). *A set A is* countable *if* $\mathbb{N} \sim A$.

Remark 1.13. *According to this, finite sets are not countable; this is just a convention. Sometimes, we say a set is countable if it is finite or to above definition holds, where we say that a set is* countably infinite *if it is infinite and countable.*

Other times, finite sets are treated separately than countable sets.

Theorem 1.8. *Suppose that $A \subseteq B$.*

1. *If B is finite or countable, then so is A*
2. *If A is infinite and uncountable, then so is B*

Definition 1.13 (Cartesian Product). *If A, B sets, $A \times B = \{(a, b) : a, b \in A, B\}$.*

Proposition 1.12. $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$*; there exists a bijection* $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Proposition 1.13. *Let A be a set. The following are equivalent statements:*

(a) *A is finite or a countable set;*

(b) *there exists a surjection from $\mathbb{N}$ onto A;*

(c) *there exists a injection from A into $\mathbb{N}$.*

Proof. We proceed by proving that each statement implies the next (and thus are equivalent).

- (a) $\implies$ (b): Suppose A is finite and has $\mathbb{N}$ elements. Then there exists a bijection $h : \{1, 2, \dots n\} \to A$. We now define a map $f : \mathbb{N} \to A$, by setting

$$f(m) = \begin{cases} h(m) & \text{if } m \leq n \\ h(n) & \text{if } m > n \end{cases}.$$

f is surjective, and thus (b) holds. If (a) countable, $\exists$ bijection $f : \mathbb{N} \to A$, and any bijection is a surjection, so (b) also holds.

- (b) $\implies$ (c): Let $h : \mathbb{N} \to A$ be a surjection, whose existence is guaranteed by (b). Then, for any $a \in A$, the set

$$h^{-1}(\{a\}) = \{m \in \mathbb{N} : h(m) = n\} \neq \varnothing,$$

since h is a surjection. Then, by the well-ordering property of $\mathbb{N}$, the set $h^{-1}(\{a\})$ has a least element.

If n is the least element of $h^{-1}(\{a\})$, we set $f(a) =$. This defines a function

$$f : A \to \mathbb{N},$$

and we aim to show that f is injective, ie that $f(a_1) = f(a_2) \implies a_1 = a_2$.

Suppose $f(a_1) = f(a_2) = n$. Then, n is the least element of $h^{-1}(\{a_1\})$ and of $h^{-1}(\{a_2\})$, and in particular, $h(n) = a_1$ and $h(n) = a_2$, and thus $a_1 = a_2$ and so f is indeed injective.

- (c) $\implies$ (a): Let $f : A \to \mathbb{N}$ be an injection, whose existence is guaranteed by (c). Consider the range of f, ie

$$f(A) = \{f(a) : a \in A\}.$$

Since f an injection, f is a bijection between A and $f(A)$.

Otoh, $f(A) \subseteq \mathbb{N}$, and so by Theorem 1.8, $f(A)$ is either finite or countable, and there exists a bijection between A and some set that is either fininte or countable. Thus, A must also be finite or countable, and so (a) holds.

■

Theorem 1.9. *Let $A_n, n = 1, 2, \ldots$ be a sequence of sets such that each A_n is either finite or countable. Then, their union*

$$A = \bigcup_{n=1}^{\infty} A_n$$

is also either finite or countable.

Proof. We will use (a) $\iff$ (b) from Proposition 1.13 to prove this.

Since each A_n finite or countable, by (a) $\implies$ (b), there exists a surjection

$$\varphi_n : \mathbb{N} \to A_n.$$

Now, let $h : \mathbb{N} \times \mathbb{N} \to A$, (the union) by setting

$$h(n, m) = \varphi_n(m).$$

We aim to show that h is also surjective.
If $a \in \bigcup_{n=1}^{\infty} A_n$, then $a \in A_n$ for some $n \in \mathbb{N}$. Since $\varphi_n : \mathbb{N} \to A_n$ is a surjection, there exists an $m \in \mathbb{N}$ s.t. $\varphi_n(m) = a$. By definition of h, we have

$$h(n, m) = a,$$

and thus h is a surjection.

By Proposition 1.12, there exists a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, and we can define the composite map

$$h \circ f : \mathbb{N} \to A \, (= \cup_{n=1}^{\infty} A_n),$$

which is a surjection as both h, f are surjections. So, there exists a surjection from $\mathbb{N} \to A$, and by Proposition 1.13, (b) $\implies$ (a), and thus $A = \bigcup_{n=1}^{\infty} A_n$ is also finite or countable.

■

Remark 1.14. *If $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is either finite or countable, and at least one A_n is countable, then A is countable.*

Remark 1.15. *If $A_1, \ldots, A_n$ are finitely many finite or countable sets then their union $A_1 \cup \cdots \cup A_n$ is also finite or countable (essentially just previous proof where we use n instead of ∞ for the upper limit of the union...).*

Theorem 1.10. *The set $\mathbb{Q}$ of rational numbers is countable.*

Proof. We write

$$\mathbb{Q} = A_0 \cup A_1 \cup A_2,$$

where $A_0 = \{0\}, A_1 = \{\frac{m}{n} : m, n \in \mathbb{N}\}$, and $A_2 = \{-\frac{m}{n} : m, n \in \mathbb{N}\}$.
Let us show that A_1 is countable; define

$$h : \mathbb{N} \times \mathbb{N} \to A_1, f(m, n) = \frac{m}{n}.$$

h is clearly a surjection; if $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection, then by Proposition 1.12, $h \circ f : \mathbb{N} \to A_1$ is a surjection. By Proposition 1.13, A_1 is countable.
We prove that A_2 countable in essentially the same way.

Then, $A_0 \cup A_1 \cup A_2$ is also countable, as it is the union of countable sets, and thus $\mathbb{Q}$ is also countable. ■

Theorem 1.11. *The set $\mathbb{R}$ of real numbers is uncountable.*[19]

Proof. We will argue by contradiction; suppose $\mathbb{R}$ is countable, then show that the nested interval property (Theorem 1.3) of the real line fails.

Let $f : \mathbb{N} \to \mathbb{R}$ be a bijection, setting $f(1) = x_1, f(2) = x_2, \ldots, f(n) = x_n, \ldots$; we can then list the elements of $\mathbb{R}$ as $\mathbb{R} = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$.

We can now construct a sequence $I_n, n \in \mathbb{N}$ of bounded, closed intervals, such that I_1 does not contain x_1.

If $x_2 \notin I_1$, then $I_2 = I_1$. If $x_2 \in I_1$, then divide I_1 into four equal closed intervals.

Call the leftmost/rightmost of these intervals I_1' and I_1'' respectively. We know that $x_2 \in I_1$, so we must have that either $x_2 \notin I_1'$ or $x_2 \notin I_1''$ If $x_2 \notin I_1'$, then $I_2 = I_1'$. If $x_2 \notin I_1''$, then $I_2 = I_1''$.

Thus, we have constructed I_1, I_2 s.t.

$$I_1 \supseteq I_2 \text{ and } x_1 \notin I_1, x_2 \notin I_2.$$

Consider x_3; if $x_3 \notin I_2$, then $I_3 = I_2$. If $x_3 \in I_2$, we repeat the "dividing" process as before. Since $x_3 \in I_2$, either $x_3 \notin I_2'$ or $x_3 \notin I_2''$. If $x_3 \notin I_2'$, $I_3 = I_2'$. Else, if $x_3 \notin I_2''$, $I_3 = I_2''$.

We have now that

$$I_1 \supseteq I_2 \supseteq I_3 \text{ and } x_1 \notin I_1, x_2 \notin I_2, x_3 \notin I_3,$$

and we can continue this construction to obtain an infinite sequence of bounded, closed intervals I_n s.t.

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots,$$

and for each n, $x_n \notin I_n$.

Consider the intersection of all these I_n's,

$$\bigcap_{n=1}^{\infty} I_n.$$

For every m, $x_m \notin I_m$, so for every $m \in \mathbb{N}$, $x_m \notin \bigcap_{n=1}^{\infty} I_n$, and so $\mathbb{R} = \{x_1, x_2, \ldots x_m, \ldots\}$ has an empty intersection with this intersection, ie

$$\mathbb{R} \cap \left(\bigcap_{n=1}^{\infty} I_n \right) = \varnothing.$$

Otoh, $\bigcap_{n=1}^{\infty} I_n \subseteq \mathbb{R}$, so we must have that $\bigcap_{n=1}^{\infty} I_n = \varnothing$ contradicting the nested interval property of the real line which states that this intersection must not be empty. We thus have a contradiction, and our assumption that $\mathbb{R}$ countable fails. [20] ■

[19] Proof sketch: by contradiction. Assume that a bijection exists, and show that it cannot be a surjection by the previous props/thms. Specifically, carefully construct nested intervals I_n, for which $x_i \notin I_i$, and then show that the intersection of all these intervals is empty, contradicting the nested interval property of the real line.
See pg. 25 of Abbott's Analysis for a more concise proof in the same language.

[20] Note that Theorem 1.3 is built upon the Axiom of Completeness, a "fact" of $\mathbb{R}$ (what makes it "distinct" from $\mathbb{Q}, \mathbb{N}$, etc). Thus, we are really just using AC, with some abstractions sts.

Proposition 1.14. *The set J of all irrational numbers in $\mathbb{R}$ is uncountable.*

Proof. We have that $\mathbb{R} = \mathbb{Q} \cup J$. If J countable, then $\mathbb{R}$ would also be countable as the union of two countable sets (as we showed $\mathbb{Q}$ countable in Theorem 1.10). $\mathbb{R}$ uncountable, so J is also uncountable. ■

Proposition 1.15. *The set $(-1, 1) \subseteq \mathbb{R}$ is uncountable.*

Proof. We can write $\mathbb{R} = \bigcup_{n=1}^{\infty}(-n, n)$. If each $(-n, n)$ is countable, then $\mathbb{R}$ would also be countable, as a countable union of countable sets. Thus, there must exist some $n_0 \in \mathbb{N}$ s.t. $(-n_0, n_0)$ is not countable. The map

$$f : (-n_0, n_0) \to (-1, 1), f(x) = \frac{x}{n_0}$$

is a bijection, and so $(-1, 1)$ is uncountable. ■

Example 1.13. *Show that the map*

$$f(x) = \frac{x}{1 - x^2}$$

is a bijection between $(-1, 1)$ and $\mathbb{R}$ ie $(-1, 1) \sim \mathbb{R}$.

Proof. Surjection is fairly trivial (if stuck, consider the graph of the function).
Injection; given $f(x) = f(y)$ where $x, y \in (-1, 1)$,

$$\begin{aligned}
\frac{x}{1-x^2} &= \frac{y}{1-y^2} \\
x - xy^2 &= y - yx^2 \\
x - y = xy^2 - yx^2 &= xy(y - x) \\
x - y &= -xy(x - y) \\
\implies -xy = 1 \implies xy &= -1, \text{ or } x - y = 0
\end{aligned}$$

$xy = -1$ is impossible given the domain of the function, hence $x - y = 0 \implies x = y$, as desired. ■

Proposition 1.16. *Any bounded non-empty open interval $(a, b) \in \mathbb{R}$ is uncountable.*

Proof. We will construct a bijection $f : (a, b) \to \mathbb{R}$ so that $(a, b) \sim \mathbb{R}$. Since $\mathbb{R}$ is uncountable, so must (a, b).

The map
$$f(x) = \frac{2(x-a)}{b-a} - 1$$
is a bijection between (a, b) and $(-1, 1)$, and we have shown that $(-1, 1) \sim \mathbb{R}$, so $(a, b) \sim \mathbb{R}$, and thus any open interval has the same cardinality as $\mathbb{R}$. ∎

Example 1.14. *Prove that $\exists$ bijection between $[0, 1)$ and $(0, 1)$, and conclude that $[0, 1) \sim (0, 1) \sim \mathbb{R}$. Then conclude for any $a < b$, $[a, b) \sim \mathbb{R}$.*

1.7.1 Power Sets

Definition 1.14 (Power Set)**.** *Let A be a set. The* power set *of Am denoted $\mathcal{P}(A)$ is the collection of all subsets of A.*

Generally, if A finite of size n, $\mathcal{P}(A)$ has 2^n elements.

Theorem 1.12 (Cantor Power Set Theorem)**.** *Let A be any set. Then there exists no surjection from A onto $\mathcal{P}(A)$.* [21]

[21]Certified Classic

Proof. Suppose that there exists a surjection,
$$f : A \to \mathcal{P}(A).$$
Let $D \subseteq A$ defined as
$$D = \{a \in A : a \notin f(a)\}.$$
Since $D \subseteq \mathcal{P}(A)$, and f is surjective, there must exist some $a_0 \in A$ s.t. $f(a_0) = D$.
We have two cases:

1. $a_0 \in D$. But then, by definition of D, $a_0 \notin f(a_0) = D$, so $a_0 \in D$ is not possible as it implies $a_0 \notin D$.

2. $a_0 \notin D$. But then, since $D = f(a_0)$, $a_0 \notin f(a_0)$, and so by definition of D, $a_0 \in D$, which is again not possible.

So, the assumption of a surjection existing has led to $a_0 \in A$ such that neither $a_0 \in D$ nor $a_0 \notin D$, which is impossible. Thus there can be no surjective f.
Notice, though, that there exists an injection $A \to \mathcal{P}(A), a \mapsto \{a\}$, and thus there is an

injection but no bijection.

Thus, we can say that $\mathcal{P}(A)$ is strictly bigger than A.

■

2 Sequences

2.1 Definitions

Definition 2.1. *Let A be a set. An A-valued sequence indexed by $\mathbb{R}$ is a map*

$$x : \mathbb{N} \to A.$$

The value $x(n)$ is called the n-th element of the sequence. One writes $x(n) = x_n$, or lists its elements

$$\{x_1, x_2, x_3, \dots\} \equiv \{x_n\}_{n\in\mathbb{N}} \equiv (x_n)_{n\in\mathbb{N}} \equiv \{x_n\}.$$

Definition 2.2 (Convergence). *We say that a sequence (x_n) converges to a real number x if for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for all $n \geq N$ we have*

$$|x_n - x| < \varepsilon.$$

If sequence (x_n) converges to x, we write $\lim_{n\to\infty} x_n = x$.

Example 2.1. *Let (x_n) be a sequence defined by $x_n = \frac{1}{n}$, $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n = 0$.*

Proof. Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$. Then for $n \geq N$, we have that

$$0 < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

So, for $n \geq N$, $|x_n - 0| < \varepsilon$, and so the limit is 0. ■

Definition 2.3 (Quantifier of Limit ⋆). *The limit can be written in terms of quantifiers.*

$$\lim_{n\to\infty} x_n = x$$

means that

$$(\forall\, \varepsilon > 0)(\exists N \in \mathbb{N})(\forall\, n \geq N)(|x_n - x| < \varepsilon).$$

Example 2.2. *Prove that*

$$\lim_{n\to\infty} \frac{n^2+1}{n^2} = 1.$$

Proof. Let $\varepsilon > 0$. Let N be a natural number such that $N > \frac{1}{\sqrt{\varepsilon}}$. Then, for $n \geq N$,

$$|\frac{n^2+1}{n^2} - 1| = |\frac{n^2+1-n^2}{n^2}| = \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

■

Definition 2.4 (Divergent Sequences)**.** *If a sequence (x_n) does not converge to any real number x, we say that the sequence is divergent. For instance, consider*

$$x_n = (-1)^n, n \geq 1.$$

The sequence alternates between 1 and -1 and so intuitively does not converge. How do we prove it?

Proof. By contradiction; suppose that $x_n = (-1)^n$ be a converging sequence. Let $x = \lim_{n\to\infty} x_n$. Take $\varepsilon = 1$, then $\exists N \in \mathbb{N}$ s.t. for all $n \geq N$ we have that $|x - x_n| < \varepsilon = 1$. Consider indices $n = N, n = N+1$. We have

$$|x_{N+1} - x_N| = |x_{n+1} - x + x - x_N| \leq \underbrace{|x_{N+1} - x| + |x - x_N|}_{\text{triangle inequality}} < 1 + 1 = 2.$$

But we also have that

$$|(-1)^{N+1} - (-1)^N| = |(-1)^{N+1} + (-1)^{N+1}| = 2,$$

We thus have that $2 < 2$, which is a contradiction. Thus, x_n is not convergent. ■

Example 2.3. *Evaluate the following examples using the ε definition:*

1. $\lim_{n\to\infty} \frac{\sin n}{\sqrt[3]{n}} = 0$
2. $\lim_{n\to\infty} \frac{n!}{n^n} = 0$
3. $\lim_{n\to\infty} \frac{(1+2+\cdots+n)^2}{n^4} = \frac{1}{4}$

Proof. 1. For all $\varepsilon > 0$; take $\frac{1}{N} < \varepsilon^3 \implies \frac{1}{\sqrt[3]{N}} < \varepsilon$. Then, $\forall\, n \geq N$,

$$n \geq N \implies \sqrt[3]{n} \geq \sqrt[3]{N} \implies \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}}$$
$$-1 \leq \sin n \leq 1 \implies |\sin n| \leq 1 \implies \left|\frac{\sin n}{\sqrt[3]{n}}\right| \leq \left|\frac{1}{\sqrt[3]{N}}\right| \leq \frac{1}{\sqrt[3]{N}} < \varepsilon$$
$$\implies \lim_{n\to\infty} \frac{\sin n}{\sqrt[3]{n}} = 0$$

2. Take $\frac{1}{N} \leq \varepsilon$. Then, $\forall\, \varepsilon > 0$, $\forall\, n \geq N \implies \frac{1}{n} \leq \frac{1}{N}$,

$$\frac{n!}{n^n} > 0 \implies \left|\frac{n!}{n^n}\right| = \frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots 1}{n \cdot n \cdots n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{1}{n}$$
$$\leq 1 \cdot 1 \cdots 1 \cdot \frac{1}{n}$$
$$\leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$
$$\implies \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

3. Note first that $(1 + 2 + \cdots + n)^2 = (\frac{n(n+1)}{2})^2$ (see Example 1.1). Take $\frac{1}{N} < \frac{\varepsilon}{2}$; then, $\forall\, \varepsilon > 0$, we have that $\forall\, n \geq N$,

$$\left|\frac{(1 + 2 + \cdots + n)^2}{n^4} - \frac{1}{4}\right| = \frac{\frac{n^2(n+1)^2}{4}}{n^4} - \frac{n^4}{n^4} = \frac{n^4 + 2n^3 + n^2 - n^4}{n^4}$$
$$= \frac{2n^3 + n^2}{n^4} = \frac{2n+1}{n^2} \leq \frac{2n}{n^2} \leq \frac{2}{n} \leq \frac{2}{N} < \varepsilon$$
$$\implies \lim_{n\to\infty} \frac{(1 + 2 + \cdots + n)^2}{n^4} = \frac{1}{4}$$

■

2.2 Properties of Limits

Lemma 2.1 (Triangle Inequality). *For $x, y, z \in \mathbb{R}$,*

$$(i) \quad |x + y| \leq |x| + |y|; \qquad (ii) \quad |x - y| \leq |x - z| + |z - y|$$[22]

Sketch proof. (i): $|x + y| = \begin{cases} x + y & x + y \geq 0 \\ -(x + y) & x + y \leq 0 \end{cases}$. So if $x + y \geq 0$, $|x + y| = x + y \leq |x| + |y|$.

If $x + y > 0$, $|x + y| = -(x + y) = (-x) + (-y) \leq |x| + |y|$.

[22] Generally, proofs involving limits will consist of 1) picking/defining an ε based on given limit/series definitions, and then 2) using triangle inequality/related techniques to reach the desired conclusion.

(ii): $|x-y| = |x-z+z-y| \le |x-z| + |z-y|$ (using (i)). ■

Theorem 2.1 ($\star$). *A limit of a sequence is unique. In other words, if the sequence is converging, then its limit is unique. The sequence cannot converge to two distinct numbers x and y.*[23]

[23] Proof sketch: contradiction, assume two distinct limits, and take ε as their midpoint. Arrive at a contradiction by using triangle inequalities to show that $|x-y| < |x-y|$, and thus the limits cannot be distinct.

Proof. By contradiction; suppose $\exists (x_n)$ s.t. $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$, and that $x \neq 0$.

Take $\varepsilon = \frac{|x-y|}{2}$. Since $x \neq y$, we have that $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, $\exists N_1 \in \mathbb{N}$ s.t. for $n \geq N_1$, $|x_n - x| < \varepsilon$.

Similarly, since $\lim x_n = y$, $\exists N_2 \in \mathbb{N}$ s.t for $g \geq N_2$, $|x_n - y| < \varepsilon$.

Take some $n \geq \max(N_1, N_2)$; then

$$\begin{aligned} |x-y| = |x - x_n + x_n - y| &\le |x - x_n| + |x_n - y| \\ &< \varepsilon + \varepsilon = |x-y| \\ &\implies |x-y| < |x-y|, \perp \end{aligned}$$

■

Theorem 2.2. *Any converging sequence is bounded.*[24]

In other words, if (x_n) is a converging sequence,

$$\exists M > 0 \text{ s.t. } |x_n| \le M \, \forall n \geq 1.$$

[24] Take $\varepsilon = 1$, which is greater than $|x_n - x|$ by limit definition for $n \geq N$ for some N. We then use this to show that $|x_n| < 1 + |x|$, then construct a summation M such that it bounds $|x_n|$; it is equal to $|x_1| + |x_2| + \cdots$ up to $|x_{N-1}|$, then plus $1 + |x|$. We have finished.

Proof. Let (x_n) be a converging sequence, and $x = \lim_{n\to\infty} x_n$. Take $\varepsilon = 1$ in the definition of the limit; then, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|x_n - x| < 1$.

This gives that for $n \geq N$, $|x_n| = |x_n - x + x| \le |x_n - x| + |x| < 1 + |x|$.

Let now $M = |x_1| + |x_2| + \cdots + |x_{N-1}| + (1 + |x|)$. Then, for any $n \geq 1$, $|x_n| \le M$;

If $n \le N - 1$, then $|x_n|$ is a summand in M, and thus $|x_n| \le M$.

If $n \geq N$, then we have by the choice of N that $|x_n| < 1 + |x| \le M$.

Thus, for all $n \geq 1$, $|x_n| \le M$, and is thus bounded given (x_n) converges. ■

Proposition 2.1 (Algebraic Properties of Limits). *Let $(x_n), (y_n)$ be sequences such that*[25]

$$\lim x_n = x, \quad \lim y_n = y.$$

Then:

1. *For any constant c, $\lim c \cdot x_n = c \cdot \lim x_n = c \cdot x$*
2. $\lim(x_n + y_n) = \lim x_n + \lim y_n = x + y$
3. $\lim x_n \cdot y_n = (\lim x_n)(\lim y_n) = x \cdot y$
4. *Suppose $y \neq 0$, $y_n \neq 0 \, \forall \, n \geq 1$. Then, $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n} = \frac{x}{y}$*

Remark 2.1. *Let X be the collection of all sequences of real numbers, $X = \{(x_n) : x_n$ is a sequence$\}$. If $(x_n) \in X$ and $c \in \mathbb{R}$, we can define $c \cdot (x_n) = (c \cdot x_n)$*[26]*; this defines* scalar multiplication *on X.*
We can also define addition*; if (x_n) and (y_n) are two sequences in X, then $(x_n) + (y_n) = (x_n + y_n)$. Then, with these two operations X is a* vector space.

Example 2.4. *Take $x_n = (-1)^n, y_n = (-1)^{n+1}, n \geq 1$.*
$(x_n) + (y_n) = 0, x_n \cdot y_n = -1$, and so $\lim x_n + y_n = 0, \lim x_n \cdot y_n = -1$, while neither $\lim x_n$ nor $\lim y_n$ exist.

Proof (part 3. of Proposition 2.1). Take[27] $\lim x_n = x, \lim y_n = y$. Since (x_n) is converging, it is bound by Theorem 2.2, and there exists $M > 0$ s.t. $\forall \, n \geq 1, |x_n| \leq M$.
Now,

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n y_n - x_n y| + |x_n y - xy| \\ &= |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \\ &\leq M \cdot |y_n - y| + |y| \cdot |x_n - x| \quad (i) \end{aligned}$$

Let $\varepsilon > 0$; since $\lim y_n = y$, there exists $N_1 \in \mathbb{N}$ s.t. $n \geq N_1, |y_n - y| < \frac{\varepsilon}{2M}$. Sim, since $\lim x_n = x, \exists N_2 \in \mathbb{N}$ s.t. $|x_n - x| < \frac{\varepsilon}{2(|y|+1)}$
Let $N = \max(N_1, N_2)$, $n \geq N$. Then, we have, with (i),

$$\begin{aligned} (i) \quad |x_n y_n - xy| &\leq M \cdot |y_n - y| + |y| \cdot |x_n| - x \\ &< M \cdot \frac{\varepsilon}{2M} + |y| \cdot \frac{\varepsilon}{2(|y|+1)} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \end{aligned}$$

[25] Note that the contrary of these statements need not hold; ie, if $\lim(x_n \cdot y_n)$ exists, this does not imply the existence of $\lim x_n$ and $\lim y_n$. Consider Example 2.4

[26] NB: this denotes c multiplying to each nth element in x_n, ie $c \cdot x_1, c \cdot x_2$, etc

Thus, for $n \geq N$, $|x_n y_n - xy| < \varepsilon$, and by definition of the limit, $\lim x_n y_n = xy$. ■

[27]Proof sketch: take an upper bound of x_n. Then, show that $|x_n y_n - xy| < \varepsilon$, by using triangle inequalities to show inequality to a combination of M, arbitrarily small values (by def of limits of x_n, y_n resp,), and $|y|$.

Theorem 2.3 (Order Properties of Limits). *Let $(x_n), (y_n)$ be two sequences such that*

$$\lim x_n = x, \quad \lim y_n = y.$$

1. $x_n \geq 0 \, \forall \, n \implies x \geq 0.$
2. $x_n \geq y_n \, \forall \, n \implies x \geq y.$
3. *c is constant since $c \leq x_n \, \forall \, n \geq 1 \implies c \leq x$. $x_n \leq c \, \forall \, n \geq 1 \implies x_n \leq c$.*

Remark 2.2. *2., 3. follow from 1. Set $z_n = x_n - y_n \, \forall \, n \geq 1$. Then, $z_n \geq 0 \, \forall \, b \geq 1$, $\lim z_n = \lim(x_n - y_n) = \lim x_n - \lim y_n$ (as these limits exist) $= x - y$. By 1., $\lim z_n \geq 0$, and so either $x - y \geq 0$ or $x \geq y$.*

Proof of 1. Suppose 1. does not hold; suppose $\exists (x_n)$ s.t. $\lim x_n = x$, $x_n \geq 0 \, \forall \, \geq$, but $x < 0$.

Let $\varepsilon > 0$ s.t. $x < -2\varepsilon < 0$. With this ε, $\lim x_n = x$ gives that $\exists N \in \mathbb{N}$ s.t. $\forall \, n \geq N$, $|x_n - x| < \varepsilon$, or particularly, $x_n - x < \varepsilon$.

Then, $x_n < \varepsilon + x$, and since $x < -2\varepsilon$, we have $\forall \, n \geq N$, $x_n < -\varepsilon$, and thus $\forall \, n \geq N$, $x_n < 0$, a contradiction. ■

Theorem 2.4 (The Squeeze Theorem). *Let $(x_n), (y_n), (z_n)$ be sequences such that $x_n \leq y_n \leq z_n$, $\forall \, n \geq 1$, and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell$, then $\lim_{n\to\infty} y_n = \ell$.*[28]

[28]Sketch: This follows a similar technique to many that follow. Use the definitions of the limits of x_n, z_n to take an arbitrary ε, and an N for each respective limit. Take the max of these N's, and show that for all $n \geq \max N_i$, you can show that f $y_n - l$ is less than ε and greater than $-\varepsilon$. Really, this is just a proof of applying definitions correctly.

Proof. Let $\varepsilon > 0$. Since $\lim x_n = \ell$, there $\exists N_1 \in \mathbb{N}$ s.t. $\forall \, n \geq N_1, |x_n - \ell| < \varepsilon$.
Since $\lim z_n = \ell$, there $\exists N_2 \in \mathbb{N}$ s.t. $\forall \, n \geq N_2, |z_n - \ell| < \varepsilon$.
Take $N = \max\{N_1, N_2\}$ and take $n \geq N$. Then,

$$y_n \leq z_n \implies y_n - \ell \leq z_n - \ell \leq |z_n - \ell| < \varepsilon,$$

since $n \geq \max\{N_1, N_2\} \implies n \geq N_2$.
Now, we have that

$$y_n \geq x_n \implies y_n - \ell \geq x_n - \ell > -\varepsilon,$$

since $|x_n - \ell| < \varepsilon$ for $n \geq N_1$, and our n is $\geq \max\{N_1, N_2\}$. Thus, for $n \geq N$,

$$-\varepsilon < y_n - \ell < \varepsilon \implies |y_n - \ell| < \varepsilon,$$

and thus $\lim y_n = \ell$, by definition. ■

Definition 2.5 (Increasing/Decreasing). *A sequence (x_n) is called* increasing *if $x_{n+1} \geq x_n \,\forall\, n \in \mathbb{N}$, and is* decreasing *if $x_{n_1} \leq x_n \,\forall\, n \in \mathbb{N}$.*

Definition 2.6 (Bounded from above/below). *A sequence (x_n) is called* bounded *from above if there exists some $M \in \mathbb{R}$ s.t. $x_n \leq M \,\forall\, n \geq 1$.*
Sequence (x_n) is bounded from below if there exists some $M \in \mathbb{R}$ s.t. $x_n \geq M \,\forall\, n \geq 1$.

Theorem 2.5 (Monotone Convergence Theorem). *The following relate to bounded above/-below and increasing/decreasing sequences:*[29]

1. *Let (x_n) be an increasing sequence that is bounded from above. Then (x_n) is converging.*
2. *Let (x_n) be a decreasing sequence that is bounded from below. then (x_n) is converging.*

[29] Sketch: 1,2 are proven very similarly. For 1., take the set of all x_n in the given sequence. Since the sequence is bounded, then so is the set, and so we can take its supremum. Use the ε definition of sup to show that this supremum is also the limit of the sequence (basically, a bunch of inequalities, and being careful with definitions). 2. follows identically but using the infimum.

Proof (of 1). Let $A = \{x_n : n \geq 1\}$. Since (x_n) is bounded above by M, the set A is bounded from above. Let $\alpha = \sup A$, which exists by AC.
Let $\varepsilon > 0$. Since α is the least upper bound for A, $\alpha - \varepsilon$ is *not* an upper bound of A ($\alpha - \varepsilon < \alpha$). Hence, there must exist some $N \in \mathbb{N}$ such that $\alpha - \varepsilon < x_N$ (if it didn't exist, then α wouldn't be the supremum …). Then, for $n \geq N$, and since (x_n) increasing,

$$\alpha - \varepsilon < x_N \leq x_n \leq \alpha.$$

Then, for all $n \geq N$,

$$\alpha - \varepsilon < x_n \leq \alpha \implies -\varepsilon < x_n - \alpha \leq 0,$$

and so $|x_n - \alpha| < \varepsilon$ for $n \geq N$. By definition, $\alpha = \lim x_n$. ∎

Example 2.5. *A sequence (x_n) is called* eventually increasing *if there exists some $N_0 \in \mathbb{N}$ s.t. $\forall\, n \geq N_0, x_{n+1} \geq x_n$. If (x_n) is eventually increasing and bounded from above, $\lim x_n = \alpha$ exists.*

Example 2.6. *Let* (x_n) *be a sequence defined recursively by* $x_1 = \sqrt{2}$ *and* $x_{n+1} = \sqrt{2+x_n}, n \geq 1$. *So* $x_2 = \sqrt{2+\sqrt{2}}, x_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} \cdots, x_n = 2\cos\frac{\pi}{2^{n+1}}, n \geq 1$. *Show that* $\lim x_n = 2$.

Proof. We will prove this using the Monotone Convergence Thm by showing that the x_n is bounded from above and increasing, which implies that the limit exists. We will then find the actual limit.

Recall that $n \geq 1, x_n \leq 2$. We will prove this by induction. Let $S \subseteq \mathbb{N}$ be the set of indices such that $x_n \leq 2$. Since $x_1 = \sqrt{2} < 2$, $1 \in S$. Now suppose some $n \in S$, ie $x_n \leq 2$. Then, we have that $x_{n+1} = \sqrt{2+x_n} \leq \sqrt{2+2} = 2 \implies x_{n+1} \leq 2$. Thus, by induction, $n \in S \implies n+1 \in S \implies S = \mathbb{N}$, ie $x_n \leq 2 \,\forall\, n \in \mathbb{N}$. Thus, our sequence is bounded from above.

We now prove that (x_n) is increasing. Let $S \subseteq \mathbb{N}$ s.t. $n \in S \iff x_{n+1} \leq x_n$. $x_2 = \sqrt{2+\sqrt{2}} \geq \sqrt{2} = x_1 \implies x_1 \leq x_2 \implies 1 \in S$. Suppose $n \in S \implies x_{n+1} \geq x_n$. Then, $x_{n+2} = \sqrt{2+x_{n+1}} \geq \sqrt{2+x_n} = x_{n+1} \implies n+1 \in S$. Thus, $S = \mathbb{N}$, so $x_{n+1} \geq x_n \,\forall\, n \in \mathbb{N}$.

So the sequence (x_n) is increasing and bounded from above, and thus $\exists \lim x_n = \alpha$. To find the value of α, consider $x_{n+1} = \sqrt{2+x_n}$, or $x_{n+1}^2 = 2 + x_n$. We can also write that $\alpha = \lim x_n = \lim x_{n+1}$.[30] We then have that $\lim x_{n+1} = \alpha \implies \lim x_{n+1}^2 = \alpha^2$, and thus $x_{n+1}^2 = 2+x_n \implies \lim x_{n+1}^2 = \lim(2+x_n) \implies \alpha^2 = 2+\alpha \implies \alpha = 2, -1$. $x_n \geq 0 \,\forall\, n$, by Order Limit Theorem, and so $\alpha \geq 0$ and thus $\alpha = 2$. ■

[30] Add proof

Corollary 2.1. *For* $a, b > 0$, *then* $\frac{1}{2}(a+b) \geq \sqrt{ab}$

Proof. $\left[\frac{1}{2}(a+b)\right]^2 = \frac{1}{4}(a^2+2ab+b^2) \geq ab \implies \frac{1}{2}(a+b) \geq \sqrt{ab}$ ■

Example 2.7. *Let* (x_n) *be defined recursively by* $x_1 = 2$ *and* $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ *for* $n \geq 1$. *Then,* (x_n) *is converging and* $\lim x_n = \sqrt{2}$.

Proof. We[31] will show that (x_n) bounded from below and decreasing, implying the limit exists. We will show that for n, $x_n \geq \sqrt{2}$. For $n = 1$, $2 \geq \sqrt{2}$. For $n > 1$, we will Corollary 2.1. We then have that $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n}) \geq \cdots \geq \sqrt{2} \implies x_n \geq \sqrt{2} \,\forall\, n \geq 1$, ie, it is bounded from below.

We will now show that the sequence is decreasing.

$$x_n - x_{n+1} = x_n - \frac{1}{2}(x_n + \frac{2}{x_n}) = \frac{1}{2}x_n - \frac{1}{x_n} = \frac{1}{2x_n}(x_n^2 - 2).$$

■

Example 2.8 ($\star$). *Let $a > 0$ and let (x_n) be a sequence defined recursively by x_1 is arbitrary (positive), and*

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}), \quad n \geq 1.$$

Show that $\lim_{n\to\infty} x_n = \sqrt{a}$.

[31] This example, as well as the more general one after it, rely on applying 1) the monotone convergence theorem, then 2) using Algebraic Limit Properties to turn the problem into an algebraic problem, using the given recursive relation.

Proof. By Corollary 2.1, $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}) \geq \sqrt{x_n \cdot \frac{a}{x_n}} = \sqrt{a}$, hence, x_n is bounded from below by $\sqrt{a}$.

We also have that $x_n - x_{n+1} = x_n - \frac{1}{2}x_n - \frac{a}{2x_n} = \frac{x_n}{2} - \frac{a}{2x_n} = \frac{1}{x_n}(x_n^2 - a)$. We have that $x_n \geq \sqrt{a} \implies x_n^2 \geq a \implies x_n^2 - a \geq 0$. Further, since the sequence is bounded from below by $\sqrt{a} > 0 (\impliedby a > 0)$, then $\frac{1}{x_n} > 0$ as well. Hence, $\frac{1}{x_n}(x_n^2 - a) \geq 0$, and thus $x_n - x_{n+1} \geq 0 \implies x_n \geq x_{n+1}$ and thus x_n is decreasing.

Thus, by the Monotone Convergence Theorem, x_n is convergent. Let $X := \lim_{n\to\infty} x_n$. We have from the recursive definition, $\lim x_n = \lim \left(\frac{1}{2}(x_n + \frac{a}{x_n})\right)$. Since we know x_n convergent, we can "split up" this limit using algebraic properties, hence

$$\begin{aligned} \lim x_n = \lim \frac{1}{2}x_n + \lim \frac{a}{2x_n} &= \frac{1}{2}\lim x_n + \frac{a}{2}\lim \frac{1}{x_n} \\ &\implies X = \frac{1}{2}X + \frac{a}{2X} \\ \implies \frac{X}{2} = \frac{a}{2X} \implies X^2 = a &\implies X = \sqrt{a}, \end{aligned}$$

which completes the proof. ∎

Example 2.9. *Evaluate[32] the limit of x_n given the recursive relation $x_{n+1} = \frac{1}{4-x_n}$, $x_1 = 3$.*

[32] Abbott, pg 54 exercise 2.4.2

Proof. We aim to show that (x_n) is bounded from below and decreasing.

Bounded from below: we claim $x_n > 0$; we proceed by induction. $x_1 = 3 > 0$ holds; say $x_n > 0$ for some $n \geq 1$. Then, we have

$$x_n > 0 \implies -x_n < 0 \implies 4 - x_n < 4 \implies \frac{1}{4 - x_n} > \frac{1}{4} > 0 \implies x_{n+1} = \frac{1}{4 - x_n} > 0,$$

so the sequence is bounded from below by 0.

Decreasing: (x_n) decreasing iff $x_{n+1} \leq x_n \, \forall n$. We have $x_2 = \frac{1}{4-3} = 1 \implies x_1 = 3 \geq 1$ holds. Say $x_{n-1} \geq x_n$ for some $n \geq 1$. Then, we have

$$x_{n-1} \geq x_n \implies 4 - x_{n-1} \leq 4 - x_n \implies \frac{1}{4 - x_{n-1}} \geq \frac{1}{4 - x_n} = x_{n+1} \implies x_n \geq x_{n+1}$$

and thus the sequence decreases, and by Theorem 2.5 the limit exists. Let $X = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{4-x_{n-1}} \implies X = \frac{1}{4-X} \implies 4X - X^2 = 1 \implies 0 = X^2 - 4X + 1 \implies X = \cdots = 2 \pm \sqrt{3}$. We must take the negative root, since X is decreasing and thus must be less than 3. ■

2.3 Limit Superior, Inferior

Definition 2.7 (limsup, liminf). *Recall Theorem 2.2, stating that a convergence sequence is bounded. Let (x_n) be a convergent sequence bounded by m and M from below/above resp, ie*

$$m \leq x_n \leq M, \forall n$$

and let $A_n = \{x_k : k \geq n\}$ (the set of elements in the sequence "after" a particular index). Let $y_n = \sup A_n$; by definition, $y_n \leq M$, and $y_n \geq m$, since $y_n \geq x_n \geq m$. Thus, we have

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots,$$

and further,

$$y_1 \geq y_2 \geq \cdots \geq y_n \geq y_{n+1} \geq \cdots;$$

since $A_2 \subseteq A_1$, y_1 also an upper bound for A_2, and thus $y_2 \leq y_1$ by definition of a supremum. So, the sequence (y_n) is decreasing, and bounded from below; by MCT, $\lim_{n\to\infty} y_n = y$ exists. Note too that since $m \leq y_n \leq M$, we have that $m \leq y \leq M$.
This y is called the limit superior *of (x_n) denoted by*

$$\overline{\lim}_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Now, similarly, note that A_n is bounded below by m and thus $z_n = \inf A_n$ exists. We further have that $z_n \leq x_n \leq M$, and that $z_n \geq m \, \forall n$, and we have

$$z_1 \leq z_2 \leq \cdots \leq z_n \leq z_{n+1} \leq \cdots,$$

by a similar argument as before. So, as before, the sequence (z_n) is increasing, and bounded from above by M. Again, by MCT, $\lim_{n\to\infty} z_n = z$ exists. We call z the limit inferior *of (x_n), and denote*

$$\underline{\lim}_{n\to\infty} x_n = \liminf_{n\to\infty} x_n.$$

We note that $y_n \geq z_n$, so $\overline{\lim}_{n\to\infty} x_n \geq \underline{\lim}_{n\to\infty} x_n \quad (y \geq z)$.
Further, $\liminf$ and $\limsup$ exist for any bounded sequence, regardless if whether or not the limit itself exists.

Example 2.10. *Let $(x_n) = (-1)^n, n \in \mathbb{N}$. We showed previously that this is a divergent sequence, so the limit does not exist. However, the sequence is bounded, since $-1 \leq x_n \leq 1 \, \forall \, n$. We have $A_n = \{(-1)^k : k \geq n\} = \{-1, 1\}$. So, $y_n = \sup A_n = 1$, and $z_n = \inf A_n = -1$, $\forall \, n$. Thus, $\limsup x_n = \lim y_n = 1$, and $\liminf x_n = \lim z_n = -1$, despite $\lim x_n$ not existing.*
More specifically, we have a divergent sequence, and $\liminf \neq \limsup$.

Theorem 2.6 ($\liminf$, $\limsup$ and convergence). *Let (x_n) be a* bounded *sequence. The following are equivalent;*

1. *The sequence (x_n) is convergent, and $\lim_{n\to\infty} x_n = x$.*
2. $\overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n = x$.

Proof. Let A_n, y_n, z_n be as in the definition of $\limsup, \liminf$.

(1) $\implies$ (2): Suppose (x_n) is converging, and $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Then, there exists some $N \in \mathbb{N}$ s.t. $\forall \, n \geq N$,

$$|x_n - x| < \frac{\varepsilon}{2},$$

or equivalently,

$$x - \frac{\varepsilon}{2} < x_n < x + \frac{\varepsilon}{2}, \forall \, n \geq N.$$

Since $A_n = \{x_k : k \geq n\}$, if $n \geq N$, then $x + \frac{\varepsilon}{2}$ is an upper bound for A_n, and $x - \frac{\varepsilon}{2}$ is a lower bound for A_n. This gives that

$$y_n = \sup A_n \leq x + \frac{\varepsilon}{2}; \quad z_n = \inf A_n \geq x - \frac{\varepsilon}{2}.$$

This gives that for $n \geq N$,

$$x - \frac{\varepsilon}{2} \leq z_n \leq x_n \leq y_n \leq x + \frac{\varepsilon}{2},$$

ie $z_n, y_n \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$. So, for all $n \geq N$, $|z_n - x| \leq \frac{\varepsilon}{2} < \varepsilon$, and $|y_n - x| \leq \frac{\varepsilon}{2} < \varepsilon$, so by definition of the limit, this gives

$$\lim_{n\to\infty} y_n = x \text{ and } \lim_{n\to\infty} z_n = x,$$

ie, $\overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n = x$.

•

(2) $\implies$ (1): Let $\varepsilon > 0$. Since $\lim_{n\to\infty} y_n = x$, $\exists N_1$ s.t. $\forall \, n \geq N_1, |y_n - x| < \varepsilon$. Similarly, since $\lim z_n = x$, $\exists N_2$ s.t. $\forall \, n \geq N_2, |z_n - x| < \varepsilon$.

Take $N = \max\{N_1, N_2\}$. Then, for $n \geq N$, we have

$$x - \varepsilon < z_n \leq x_n \leq y_n < x + \varepsilon.$$

So, for $n \geq N$, $|x_n - x| < \varepsilon$, thus $\lim x_n = x$ as desired. ■

Example 2.11. *Let*[33] (x_n) *be a bounded sequence. Then*

$$\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n.$$

[33] Midterm material ends here. There will be 5 questions. Memorize **everything**; homeworks, exercises, class material. Study the solutions until you can recite it upwards, backwards, sideways.

Proof. Recall Remark 1.2; Let $A_n := \{x_k : k \geq n\}$ as in the definition of $\limsup, \liminf$. Let $y_n := \sup A_n, z_n := \inf A_n$. By Theorem 2.6, $\lim y_n = \lim z_n$. Further, $\sup(-A_n) = -\inf(A_n)$, where $-A_n = \{-x_k : k \geq n\}$; hence, $\limsup(-x_n) = -\liminf x_n$, as desired. ■

2.4 Subsequences and Bolzano-Weirestrass Theorem

Definition 2.8 (Subsequence)**.** *Let* (x_n) *be a sequence of real numbers, and let* $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$ *be a strictly increasing sequence of natural numbers. Then, the sequence*

$$(x_{n_1}, x_{n_2}, \cdots, x_{n_k}, x_{n_{k+1}}, \cdots)$$

is called a subsequence *of* (x_n) *and is denoted* $(x_{n_k})_{k\in\mathbb{N}}$.

Remark 2.3. k *is the index of the subsequence,* $(x_{n_k})_{k\in\mathbb{N}}$, ***not*** n; x_{n_1} *is the 1st element, …,* x_{n_k} *is the* k*-th element.*

Example 2.12. *Let* $x_n = \frac{1}{n}, (\frac{1}{n})_{n\in\mathbb{N}}$, *and let* $n_k = 2k+1, k \in \mathbb{N}$. $n_1 = 3, n_2 = 5, n_3 = 7, \ldots, n_k = 2k+1$. *Our subsequence is then*

$$(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots) = \left(\frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2k+1}, \ldots\right) = \left(\frac{1}{2k+1}\right)_{k\in\mathbb{N}}$$

is our subsequence of (x_n).

Remark 2.4. *Note that for any* $k, n_k \geq k$.
Let $S = \{k \in \mathbb{N} : n_k \geq k\}$. *Then,* $1 \in S$, *since* $n_1 \in \mathbb{N}, n_1 \geq 1$. *If* $k \in S$, *then* $n_k \geq k$, *and so, since* $n_{k+1} > n_k$ *(increasing), we have that* $n_{k+1} > k \implies n_{k+1} \geq k+1$. *So,* $k+1 \in S, S = \mathbb{N}$.

Remark 2.5. $\lim_{k\to\infty} x_{n_k} = x$ *if* $\forall\, \varepsilon > 0, \exists K \in \mathbb{N}$ *s.t.* $\forall\, k \geq K, |x_{n_k} - x| < \varepsilon$.

Theorem 2.7. *Let (x_n) be a sequence such that $\lim_{n\to\infty} x_n = x$. Then, for any subsequence $(x_{n_k})_{k\in\mathbb{N}}$, we have that $\lim_{k\to\infty} x_{n_k} = x$*

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, $\exists N \in \mathbb{N}$ s.t. $\forall\, n \geq N$, $|x_n - x| < \varepsilon$. Take $K = N$ (from Remark 2.5). Then, for $k \geq K$, we have from Remark 2.4 that

$$n_k \geq k \geq K = N,$$

and hence $|x_{n_k} - x| < \varepsilon \implies \lim_{k\to\infty} x_{n_k} = x$. ∎

Theorem 2.8 (Bolzano-Weirestrass Theorem). [34]*Any bounded sequence (x_n) has a convergent subsequence.*

[34]Fundamental property of the real line; equivalent to AC.

Example 2.13. *Take $x_n = (-1)^n, n \in \mathbb{N}$. This sequence does not converge. However, if we take a subsequence with $n_k = 2k, k \in \mathbb{N}$. $x_{n_k} = (-1)^{2k} = 1$, so (x_{n_k}) is a constant sequence 1 and converges to 1.*
Similarly, if $n_k = 2k+1, k \in \mathbb{N}$, then $x_{n_k} = (-1)^{2k+1} = -1$, and the subsequence converges to -1.

Proposition 2.2. *If $0 < b < 1$, then $\lim_{n\to\infty} b^n = 0$.*

Proof. Let $x_n = b^n$. Then $x_n > 0$, and $x_{n+1} = b^{n+1} = bx_n > x_n$, and since $0 < b < 1$, (x_n) is decreasing and bounded from below, (x_n) converges by the Monotone Convergence Theorem. Let $x = \lim_{n\to\infty} x_n$. Again, $x_{n+1} = bx_n$, so $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} bx_n = b\lim_{n\to\infty} x_n$, so $x = bx \implies (1-b)x = 0$. $0 < b < 1 \implies x = 0$. ∎

BW Proof (1): using Nested Interval Property. [35]Since (x_n) bounded, $\exists M > 0$ s.t. $|x_n| \leq M\, \forall\, n \in \mathbb{N}$. Let $I_1 = [-M, M]$ and $n_1 = 1$. We now construct I_2, n_2 as follows.

Divide I_1 into two intervals of the same size, $I_1' = [-M, 0], I_1'' = [0, M]$. Now, consider the sets

$$A_1 = \{n \in \mathbb{N} : n > n_1(=1), x_n \in I_1'\}, \quad A_2 = \{n \in \mathbb{N} : n > n_1, x_n \in I_1''\}$$

(ie, all the indices of all the elements in I_1', I_1'' resp.).
Hence, $A_1 \cup A_2 = \{n : n > n_1\}$, an infinite set, and hence, one of A_1, A_2 must be infinite (by Theorem 1.9). If A_1 infinite, set $I_2 = I_1'$, $n_2 = \min A_1$. If A_1 finite, then A_2 infinite, and set $I_2 = I_1''$, $n_2 = \min A_2$.

Suppose now that I_k, n_k are chosen, and that I_k contains infinitely many elements of the sequence (x_n). Divide I_k into two equal sub-intervals, I_k', I_k''. We now introduce

$$A_1^{(k)} = \{n \in \mathbb{N} : n > n_k \text{ and } x_n \in I_k'\}, \quad A_2^{(k)} = \{n \in \mathbb{N} : n > n_k \text{ and } x_n \in I_k''\},$$

(similar to our construction of A_1, A_2). $A_1^{(k)} \cup A_2^{(k)}$ must be infinite, so one of the two must be infinite. If A_1 infinite, set $I_{k+1} = I_k'$, $n_{k+1} = \min A_1^{(k)}$. If A_2 infinite, set $I_{k+1} = I_k''$, $n_{k+1} = \min A_2^{(k)}$.

This gives now that I_{k+1} and n_{k+1}, where $I_{k+1} \subseteq I_k, I_{k+1}$ contains infinitely many elements of the sequence. Further, by construction, $n_{k+1} > n_k$. This gives us a sequence of closed intervals $I_k = [a_k, b_k], k \in \mathbb{N}$ such that $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq I_{k+1} \supseteq \cdots$, such that $x_{n_k} \in I_k$, and that n_k is a strictly increasing sequence of natural numbers, defining subsequence (x_{n_k}).

Now, by construction, the length of I_{k+1} is $\frac{1}{2}$ of the length of I_k. Since $I_k = [a_k, b_k]$, then

$$b_k - a_k = \frac{b_{k-1} - a_{k-1}}{2} = \cdots \frac{b_1 - a_1}{2^{k-1}} = \frac{2M}{2^{k-1}} = \frac{M}{2k^{k-2}}.$$

Since $I_k, k \in \mathbb{N}$, is a nested sequence of closed intervals and by the nested interval property of the real line (AC), $\exists x \in \bigcap_{k=1}^{\infty} I_k$.

We claim now that our subsequence (x_{n_k}) satisfies $\lim_{k\to\infty} x_{n_k} = x$. To see this, let $\varepsilon > 0$. Since $\lim_{k\to\infty} \frac{M}{2^{k-2}} = \lim_{k\to\infty} \frac{4M}{2k} = 0$, by Proposition 2.2, with $b = \frac{1}{2}$. There exists $K \in \mathbb{N}$ such that $\forall\, k \geq K$, we have $\frac{M}{2^{k-2}} = b_k - a_k < \varepsilon$. So, since I_k is a nested sequence of intervals, $\forall\, k \geq K$, $x_{n_k} \in I_K$ ($x_{n_k} \in I_k \subseteq I_K$). We also have that $x \in I_K$, since $x \in \bigcap I_k$. So, $x, x_{n_k} \in [a_K, b_K] \forall\, k \geq K$. So, for $k \geq K$, $|x_{n_k} - x| \leq |b_k - a_k| < \varepsilon$. So for $\varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\forall\, k \geq K$, $|x_{n_k} - x| < \varepsilon$, and so $\lim_{k\to\infty} x_{n_k} = x$, as desired. ■

[35]Sketch:
See Abbott, pg 57, for good diagram.

Definition 2.9 (Peak). *Let (x_n) be a sequence of real numbers. An element x_m is called a* peak *of this sequence if $x_m \geq x_n \forall\, n \geq m$. x_m is bigger or equal then to any element of the sequence that follows it.*

If a sequence is decreasing, then any element of the sequence is a peak.

If a sequence is increasing, then there is no peak.

BW Proof (2): using Peaks. Take sequence (x_n). Then,

- **Case 1:** (x_n) has *infinitely* many peaks; enumerate the indices of those peaks as $n_1 < n_2 < n_3 < \cdots$, then $x_{n_k} < x_{n_{k+1}} \forall\, k$, since x_{n_k} is a peak, $n_{k+1} > n_k$. This gives a decreasing subsequence (x_{n_k}).

- **Case 2:** (x_n) has *finitely* many peaks, with indices $m_1 < m_2 < \cdots < m_r$. Set $n_1 = m_r + 1$. Then x_{n_1} is not a peak, and so $\exists\, n_2 > n_1$ s.t. $x_{n_2} > x_{n_1}$. Now, x_{n_2} is also not a peak, ($n_2 > n_1 > m_r$), and so there exists $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$, and so on. In this way, we construct a subsequence (x_{n_k}) that is strictly increasing, that is, $x_{n_{k+1}} > x_{n_k}$.

If in addition (x_n) is bounded, say $|x_n| \leq M\,\forall\, n$, then the monotone subsequence constructed in **Cases 1, 2** is also bounded; ie $|x_{n_k}| \leq M\,\forall\, k$. Thus, by Monotone Convergence Theorem, (x_{n_k}) is converging. ■

2.5 Cauchy Sequences

Definition 2.10 (Cauchy Criterion). *A sequence (x_n) is called* Cauchy *if for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall\, n, m \geq N, |x_n - x_m| < \varepsilon$.*

Theorem 2.9. *A sequence (x_n) is convergent iff it is Cauchy.*

Remark 2.6. *This is, again, an "equivalent" formulation of AC; at least, the direction (x_n) Cauchy $\implies$ convergent is. The other direction, convergent $\implies$ Cauchy, does not rely on AC.*

Remark 2.7. *AC $\iff$ BW, AC $\iff$ MCT, AC $\iff$ NIP; AC $\iff$ Cauchy Criterion + Archimedian Property*

Remark 2.8. *Beyond the real line, AC (in terms of* $\sup$*) cannot be formulated, because of the lack of ordering. In this case, the Cauchy criterion can be used to extend AC to other spaces.*

Proof. (Theorem 2.9; (x_n) **Convergent** $\implies$ **Cauchy**)
Suppose $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$, $N \in \mathbb{N}$ s.t. $\forall\, n \geq N$, $|x_n - x| < \frac{\varepsilon}{2}$. Then, for $n, m \geq N$,

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
$$\implies |x_n - x_m| < \varepsilon,$$

hence (x_n) is Cauchy. ■

Remark 2.9. *To prove $\impliedby$, we first introduce the following theorem(s); see section 2.5 for the remainder.*

Theorem 2.10. *Let (x_n) be a Cauchy sequence and suppose that (x_n) has a convergent subsequence (x_{n_k}). Then (x_n) is also convergent.*

Proof. Let $x = \lim_{n\to\infty} x_{n_k}$. Let $\varepsilon > 0$. Then, $\exists K \in \mathbb{N}$ such that $\forall k \geq K$, $|x_{n_k} - x| < \varepsilon$. We have too that (x_n) Cauchy, ie $\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$, $|x_n - x_m| < \frac{\varepsilon}{2}$.
Let now $K_0 \geq \max\{K, N\}$. Recall that $n_{K_0} \geq K_0 \geq N$. Take now $n \geq N$, and estimate

$$|x_n - x| = \left|x_n - x_{n_{K_0}} + x_{n_{K_0}} - x\right| \leq \left|x_n - x_{n_{K_0}}\right| + \left|x_{n_{K_0}} - x\right|$$

Since $K_0 \geq K$, $\left|x_{n_{K_0}} - x\right| < \frac{\varepsilon}{2}$. Since $n_{K_0} \geq N$, we also have $\left|x_n - x_{n_{K_0}}\right| < \frac{\varepsilon}{2}$. Thus, we have

$$|x_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $\lim_{n\to\infty} x_n = x$. ■

Remark 2.10. *This did not use AC.*

Theorem 2.11. *Any Cauchy sequence is bounded.*

Proof. Let (x_n) be Cauchy. We aim tos hwo that $\exists M > 0$ s.t. $\forall n \in \mathbb{N}, |x_n| \leq M$.
Take $\varepsilon = 1$ in the definition of Cauchy sequence. Let N be such that $\forall n, m \geq N$, $|x_n - x_m| < 1$. We can take $m = N$, and so for all $n \geq N$, $|x_n - x_N| < 1$, which gives that for $n \geq N$,

$$|x_n| = |x_n - x_N + x_N| \leq |x_n - x_N| + |x_N| < 1 + |x_N|$$

Let

$$M = |x_1| + |x_2| + \dots |x_{N-1}| + |x_N| + 1.$$

Then, if $n \leq N$, $|x_n| \leq M$; if $n \geq M, |x_n| \leq M$, so $\forall n \geq 1, |x_n| \leq M$, hence (x_n) is bounded. ■

Remark 2.11. *This did not use AC.*

Proof. (Theorem 2.9; (x_n) **Convergent** $\Longleftarrow$ **Cauchy**)
If (x_n) Cauchy, then (x_n) is bounded by Theorem 2.11, and thus by Bolzano-Weirestrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). Then, by Theorem 2.10, (x_n) must converge. ■

Example 2.14. *Let (x_n) be a sequence defined recursively by $x_1 = 1$, $x_2 = 2$, $x_{n+1} = \frac{1}{2}(x_n + x_{n-1}), n \geq 2$. Prove that (x_n) is a convergence sequence, and find its limit.*

Remark 2.12. *Before solving, we establish a number of properties about the sequence.*

Proposition 2.3 (Property I)**.** $1 \leq x_n \leq 2 \, \forall \, n \geq 1$

Proof. We proceed by induction. Let $S \subseteq \mathbb{N}$ be the set of all n such that $1 \leq x_n \leq 2$.
Base Case: $1 \in x$, since $x_1 = 1$.
Assumption: suppose $\{1, 2, \dots, n\} \in S$. We want to show that $n+1 \in S$.
If $n = 1$, then $x_2 = 2$, so $x_2 \in S$. If $n > 1$, then

$$x_{n+1} = \frac{1}{2}(x_n + x_{n+1}),$$

and by inductive assumption, $1 \leq x_n \leq 2$ and $1 \leq x_{n-1} \leq 2$, hence

$$1 \leq x_{n+1} \leq 2,$$

hence $n+1 \in S$, and thus $S = \mathbb{N}$. ■

Proposition 2.4 (Property II)**.** *For all* $n \geq 1$, $|x_{n+1} - x_n| = \frac{1}{2^{n-1}}$.

Proof. We proceed by induction. Let $S \subseteq \mathbb{N}$ be the set of all n such that the statement holds for x_n.
Base Case: $x_2 = 2, x_1 = 1$, hence $2 - 1 = 1 = \frac{1}{2^0} = 1$, holds.
Assumption: suppose $n \in S$, ie $|x_{n+1} - x_n| = \frac{1}{2^{n-1}}$ holds for n. Then,

$$\begin{aligned} |x_{n+2} - x_{n+1}| &= \left| \frac{1}{2}(x_{n+1} + x_n) - x_{n+1} \right| \\ &= \left| \frac{1}{2}x_n - \frac{1}{2}x_{n+1} \right| = \frac{1}{2} \left| x_{n+1} - x_n \right| \\ \textit{(assumption} \implies) \qquad &= \frac{1}{2} \cdot \frac{1}{2^{n-1}} = \frac{1}{2^n}, \end{aligned}$$

hence the statement holds for $n+1$, and $S = \mathbb{N}$. ■

Corollary 2.2. *For any* $r \neq 1$, *and any* $k \in \mathbb{N}$, $1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}$.

Proposition 2.5 (Property III)**.** (x_n) *a Cauchy sequence.*

Proof. Let $\varepsilon > 0$. We need to find $N \in \mathbb{N}$ such that $\forall \, n, m \geq N$, $|x_n - x_m| < \varepsilon$. Let N be such that[36] $\frac{1}{2^{N-2}} = \frac{4}{2^N} < \varepsilon$. Let, now, $n, m \geq N$, and suppose $n > m$ (when $n = m$, we are done; when $n < m$, simply switch the variables wlog). We can write

$$\begin{aligned}|x_n - x_m| &= |x_n - x_{n-1} + x_{n+1} - x_{n-2} + x_{n-2} + \cdots - x_{m+1} + x_{m+1} - x_m| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|^{37}\end{aligned}$$

Using Property II we can write

$$\begin{aligned}|x_n - x_m| &\leq \frac{1}{2^{m-1}} + \frac{1}{2^m} + \cdots \frac{1}{2^{n-2}} \\ &= \frac{1}{2^{m-1}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-1}}\right)\end{aligned}$$

By Corollary 2.2, with $r = \frac{1}{2}$ and $k = n - m - 1$, we have

$$\frac{1}{2^{m-1}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-1}}\right) = \frac{1}{2^{m-1}}\left(\frac{1 - \left(\frac{1}{2}\right)^{n-m}}{1 - \frac{1}{2}}\right) < \frac{1}{2^{m-2}} \leq \frac{1}{2^{N-2}}.$$

We have chosen N so that $\frac{1}{2^{N-2}} < \varepsilon$, hence for $n, m \geq N$, $|x_n - x_m| < \varepsilon$, and thus our sequence is Cauchy, so $\lim_{n\to\infty} x_n = X$ exists. ■

[36] $\lim \frac{1}{2^n} = 0$, so such an N exists.

[37] "Telescoping" the sequence; the inequality follows directly from the triangle inequality.

Proof. (Of Example 2.14)

By Property III, the limit $\lim x_n = X$ exists. From the recursive definition, we can write

$$\begin{aligned}X = \lim x_n &= \lim\left(\frac{1}{2}(x_{n-1} + x_{n-2})\right) \\ \implies X &= \frac{1}{2}(X + X) = X,\end{aligned}$$

which, while true, is useless. How do we find the limit?

■