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Analysis 2 MATH255

Course Outline:

Based on Lectures from Winter, 2024 by Prof. Dmitry Jackobson.

Contents

1 Introduction

1.1 Metric Spaces

↪ **Definition 1.1: Metric Space**

A set X is a *metric space* with distance d if

1. (symmetric) $d(x, y) = d(y, x) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. (triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$

Remark 1.1. *If 1., 3. are satisfied but not 2., d can be called a "pseudo-distance".*

↪ **Definition 1.2: Normed Space**

Let X be a vector space over $\mathbb{R}$. The norm on X, denoted $||x|| \in \mathbb{R}$, is a function that satisfies

1. $||x|| \geq 0$
2. $||x|| = 0 \iff x = 0$
3. $||c \cdot x|| = |c| \cdot ||x||$
4. $||x + y|| \leq ||x|| + ||y||$

If X is a normed vector space over $\mathbb{R}$, we can define a distance d on X by $d(x, y) = ||x - y||$.

↪ **Proposition 1.1**

If X is a normed vector space over $\mathbb{R}$, a distance d on X by $d(x, y) = ||x - y||$ makes (X, d) a metric space.

Proof.

1. $d(x, y) = ||x - y|| \geq 0$
2. $d(x, y) = 0 \iff ||x - y|| = 0 \iff x - y = 0 \iff x = y$
3. $d(x, y) + d(y, z) = ||x - y|| + ||y - z|| \geq ||(x - y) + (y - z)|| = ||x - z|| := d(x, z)$

■

⊛ **Example 1.1:** L^p **distance in** $\mathbb{R}^n$

Let $\overline{x} \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n)$. The L^p norm is defined

$$||x||_p := (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

In the case $p = 2, n = 2$, we simply have the standard Euclidean distance over $\mathbb{R}^2$.

Unit Balls: consider when $||x||_p \leq 1$, over $\mathbb{R}^2$.

- $p = 1 : |x_1| + |x_2| \leq 1$; this forms a "diamond ball" in the plane.
- $p = 2 : \sqrt{|x_1|^2 + |x_2|^2} \leq 1$; this forms a circle of radius 1. Clearly, this surrounds a larger area than in $p = 2$.

A natural question that follows is what happens as $p \to \infty$? Assuming $|x_1| \geq |x_2|$:

$$\begin{aligned} ||x||_p &= (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \\ &= \left[|x_1|^p \left(1 + \left|\frac{x_2}{x_1}\right|^p\right)\right]^{\frac{1}{p}} \\ &= |x_1| \left(1 + \left|\frac{x_2}{x_1}\right|^p\right)^{\frac{1}{p}} \end{aligned}$$

If $|x_1| > |x_2|$, this goes to $|x_1|$. If they are instead equal, then $||x||_p = |x_1| \cdot 2^{\frac{1}{p}} \to |x_1| \cdot 1$ as well. Hence, $\lim_{p\to\infty} ||x||_p = \max\{|x_1|, |x_2|\}$. Thus, the unit ball will approach $\max\{|x_1|, |x_2|\} \leq 1$, that is, the unit square.

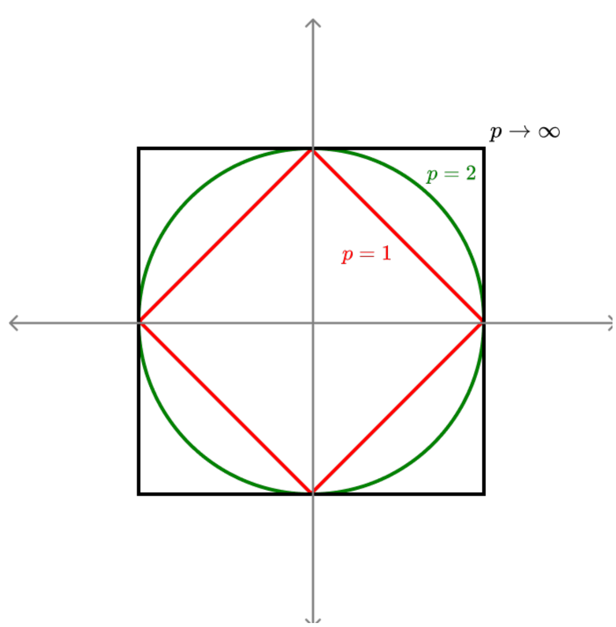

Figure 1: Regions of $\mathbb{R}^2$ where $||x||_p \leq 1$ for various values of p.

↪ **Proposition 1.2**

Let $x \in \mathbb{R}^n$. Then, $||x||_p \to \max\{|x_1|, \ldots, |x_n|\}$ as $p \to \infty$.

Remark 1.2. *This is an extension of the previous example to arbitrary real space; the proof follows nearly identically.*

↪ **Definition 1.3: Convex Set**

Let X be a normed space, and take $x, y \in X$. The line segment from x to y is the set

$$\{t \cdot x + (1-t) \cdot y : 0 \leq t \leq 1\}.$$

Let $A \subseteq X$. A is *convex* $\iff$ $\forall\, x, y \in A$, we have that

$$(t \cdot x + (1-t) \cdot y) \in A \,\forall\, 0 \leq t \leq 1.$$

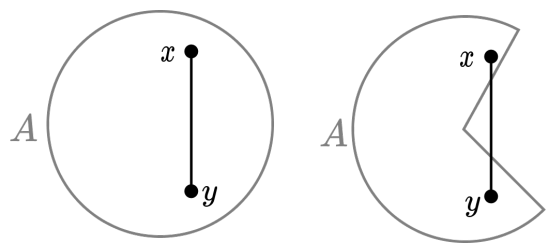

Figure 2: Convex (left) versus not convex (right) sets.

Remark 1.3. *Think of this as saying "a set is convex iff every point on a line segment connected any two points is in the set".*

↪ **Definition 1.4: ℓ_p**

The space ℓ_p of sequences is defined as

$$\left\{x = (x_1, x_2, \ldots, x_n, \ldots) : \sum_{n=1}^{\infty} |x_n|^p < +\infty\right\} \quad * .$$

Then, $*$ defines the ℓ^p norm on the space of sequences; that is, $||x||_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$.

⊛ **Example 1.2: ℓ_p, $x_n = \frac{1}{n}$**

. Let $x_n = \frac{1}{n}$. For which p is $x \in \ell_p$? We have, raising the norm to the power of p for ease:

$$\begin{aligned} ||x||_p^p &= |x_1|^p + |x_2|^p + \cdots + |x_n|^p + \cdots \\ &= 1^p + \left(\frac{1}{2}\right)^p + \cdots < \infty \iff p > 1. \end{aligned}$$

In the case that $p = 1$, this becomes a harmonic sum, which diverges.

⊛ **Example 1.3: L^p space of functions**

Let $f(x)$ be a continuous function. We define the norm of f over an interval $[a, b]$

$$||f||_p = \left[\int_a^b |f(x)|^p \, dx\right]^{\frac{1}{p}}.$$

Remark 1.4. *Triangle inequality for $||x||_p$ or $||f||_p$ is called Minkowski inequality; $||x||_p + ||y||_p \geq ||x + y||_p$. This will be discussed further.*

⊛ **Example 1.4: Distances between sets in $\mathbb{R}^2$**

Let A, B be bounded, closed, "nice" sets in $\mathbb{R}^2$. We define

$$d(A, B) := \text{Area}(A \triangle B),$$

where

$$A \triangle B : (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

It can be shown that this is a "valid" distance.

Remark 1.5. *$\triangle$ denotes the "symmetric difference" of two sets.*

⊛ **Example 1.5: p-adic distance**

Let p be a prime number. Let $x = \frac{a}{b} \in \mathbb{Q}$, and write $x = p^k \cdot \left(\frac{c}{d}\right)$, where c, d are not divisible by p. Then, the p-adic norm is defined $||x||_p := p^{-k}$. It can be shown that this is a norm.

Suppose $p = 2$, $x = 28 = 4 \cdot 7 = 2^2 \cdot 7$. Then, $||28||_2 = 2^{-2} = \frac{1}{4}$; similarly, $||1024||_2 = ||2^{10}||_2 = 2^{-10}$.

More generally, we have that $||2^k||_2 = 2^{-k}$; coversely, $||2^{-k}|| = 2^k$. That is, the closer to 0, the larger the distance, and vice versa, contrary to our notion of Euclidean distance.

↪ **Proposition 1.3**

$||x||_p$ as defined above is a well-defined norm over $\mathbb{Q}$.

Proof. ■

2 Point-Set Topology

2.1 Definitions

↪ **Definition 2.1: Topological space**

A set X is a topological space if we have a collection of subsets τ of X called *open sets* s.t.

1. $\varnothing \in \tau, X \in \tau$
2. Consider $\{A_\alpha\}_{\alpha \in I}$ where A_α an open set for any α; then, $\bigcup_{\alpha \in I} A_\alpha \in \tau$, that is, it is also an open set.
3. If J is a finite set, and A_β open for all $\beta \in J$, then $\bigcap_{\beta \in J} A_\beta \in \tau$ is also open.

In other words, 2.: arbitrary unions of open sets are open, and 3.: finite intersections of open sets are open.

↪ **Definition 2.2: Closed sets**

Closed sets are complements of open sets; hence, axioms for closed sets follow appropriately;

1.* $X, \varnothing$ closed;

2.* B_α closed $\forall\, \alpha \in I \implies \bigcap_{\alpha \in I} B_\alpha$ closed.

3.* B_β closed $\forall\, \beta \in J$, J finite, then $\bigcup_{\beta \in J} B_\beta$ also closed.

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↪ **Definition 2.3: Interior, Boundary of a Topological Set**

Let X be a topological space, $A \subseteq X$ and let $x \in X$. We have the following possibilities

1. $\exists U$-open : $x \in U \subseteq A$. In this case, we say $x \in$ the *interior* of A, denoted
$$x \in \text{Int}(A).$$
2. $\exists V$-open : $x \in V \subseteq X \setminus A = A^C$. In this case, we write
$$x \in \text{Int}(X^C).$$

3. $\forall U$-open : $x \in U, U \cap A \neq \varnothing$ AND $U \cap A^C \neq \varnothing$. In this case, we say x is in the *boundary* of A, and denote
$$x \in \partial A.$$

↪ **Definition 2.4: Closure**

$x \in \text{Int}(A)$ or $x \in \partial A$ (that is, $x \in \text{Int}(A) \cup \partial A$) $\iff$ every open set U that contains x intersects A.[1] Such points are called *limit points* of A. The set of all limits points of A is called the *closure* of A, denoted $\overline{A}$.

[1] "Requires" proof.

Remark 2.1. *We have that*

$$\text{Int}(A) \subseteq A \subseteq \overline{A} = \text{Int}(A) \cup \partial A.$$

↪ **Proposition 2.1: Properties of $\text{Int}(A)$**

$\text{Int}(A)$ is *open*, and it is the largest open set contained in A. It is the union of all U-open s.t. $U \subseteq A$. Moreover, we have that

$$\text{Int}(\text{Int}(A)) = \text{Int}(A).$$

↪ **Proposition 2.2: Properties of $\overline{A}$**

$\overline{A}$ is *closed*; $\overline{A}$ is the smallest closed set that contains A, that is, $\overline{A} = \bigcap B$ where B closed and $A \subseteq B$. We have too that

$$\overline{(\overline{A})} = \overline{A}.$$

↪ **Proposition 2.3**

1. A is open $\iff$ $A = \text{Int}(A)$
2. A is closed $\iff$ $A = \overline{A}$

2.2 Basis

↪ **Definition 2.5: Basis for a Toplogy**

Let τ be a topology on X. Let $\mathcal{B} \subseteq \tau$ be a collection of open sets in X such that every open set is a union of open sets in $\mathcal{B}$.

⊛ **Example 2.1: Example Basis**

$X = \mathbb{R}$, and $\mathcal{B} = \{\text{all open intervals } (a,b) : -\infty < a < b < +\infty\}$.

↪ **Proposition 2.4**

Let $\mathcal{B}$ be a collection of open sets in X. Then, $\mathcal{B}$ is a basis $\iff$

1. $\forall x \in X, \exists U$-open $\in \mathcal{B}$ s.t. $x \in U$.
2. If $U_1 \in \mathcal{B}$ and $U_2 \in \mathcal{B}$, and $x \in U_1 \cap U_2$, then $\exists U_3 \in \mathcal{B}$ s.t. $x \in U_3 \subseteq U_1 \cap U_2$.

⊛ **Example 2.2**

Consider $X = \mathbb{R}$. Requirement 1. follows from taking $U = (x - \varepsilon, x + \varepsilon)$ for any $\varepsilon > 0$. For 2., suppose $x \in (a,b) \cap (c,d) =: U_1 \cap U_2$. Let $U_3 = (\max\{a,c\}, \min\{b,d\})$; then, we have that $U_3 \subseteq U_1 \cap U_2$, while clearly $x \in U_3$.

↪ **Proposition 2.5**

In a metric space, a basis for a topology is a collection of open balls,

$$\{B(x,r) : x \in X, r > 0\} = \{\{y \in X : d(x,y) < r\} : x \in X, r > 0\}.$$

Proof. We prove via proposition 2.4. Property 1. holds clearly; $x \in B(x, \varepsilon)$-open $\subseteq \mathcal{B}$.

For property 2., let $x \in B(y_1, r_1) \cap B(y_2, r_2)$, that is, $d(x, y_1) < r_1$ and $d(x, y_2) < r_2$. Let

$$\delta := \min\{r_1 - d(x, y_1), r_2 - d(x, y_2)\}.$$

We claim that $B(x, \delta) \subseteq U_1 \cap U_2$.

Let $z \in B(x, \delta)$. Then,

$$d(z, y_1) \overset{\triangle\neq}{\leq} d(z,x) + d(x, y_1) < \delta + d(x, y_1) \leq r_1 - d(x, y_1) + d(x, y_1) = r_1,$$

hence, as $d(z, y_1) < r_1 \implies z \in B(y_1, r_1) = U_1$. Replacing each occurrence of y_1, r_1 with y_2, r_2 respectively gives identically that $z \in B(y_2, r_2) = U_2$. Hence, we have that $B(x, \delta) \subseteq U_1 \cap U_2$ and 2. holds. ■

2.3 Subspaces

↪ **Definition 2.6**

Let X be a topological space and let $Y \subseteq X$. We define the subspace topology on Y:

1. Open sets in $Y = \{Y \cap$ open sets in $X\}$

↪ **Proposition 2.6: Consequences of Subspace Topologies**

Suppose $\mathcal{B}$ is a basis for a topology in X. Then, $\{U \cap Y : U \in \mathcal{B}\}$ forms a basis for the subspace $Y \subseteq X$.

Suppose X a metric space. Then, Y is also a metric space, with the same distance.

↪ **Proposition 2.7**

Let $Y \subseteq X$- a metric space. Then, the metric space topology for (Y, d) is the same as the subspace topology.

Proof. (Sketch) A basis for the open sets in X can be written $\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)$; hence

$$Y \cap (\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)) = \bigcup_{\alpha \in I} (Y \cap B(x_\alpha, r_\alpha))$$

is an open set topology for Y. ■

↪ **Lemma 2.1**

Let $A \subseteq X$-open, $B \subseteq A$; B-open in subspace topology for $A \iff B$-open in X.

↪ **Lemma 2.2**

Let $Y \subseteq X$, $A \subseteq Y$. Then, $\overline{A}$ in $Y = Y \cap \overline{A}$ in X. We can denote this

$$\overline{A}_Y = \overline{A}_X \cap Y.$$

2.4 Continuous Functions

↪ **Definition 2.7: Continuous Function**

Let X, Y be topological spaces. Let $f : X \to Y$. f is *continuous* $\iff \forall$ open $V \in Y$, $f^{-1}(V)$-open in X.

↪ **Proposition 2.8**

This definition is consistent with the normal ε-δ definition on the real line.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$, continuous; that is, $\forall \varepsilon > 0$, $\forall x \in \mathbb{R} \exists \delta > 0$ s.t. $|x_1 - x| < \delta$, then $|f(x_1) - f(x)| < \varepsilon$.

Let $V \subseteq \mathbb{R}$ open. Let $y \in V$. Then, $\exists \varepsilon : (y - \varepsilon, y + \varepsilon) \subseteq V$. Let $y = f(x)$, hence $y \in f^{-1}(V)$. Now, if $d(x, x_1) < \delta$, we have that $d(f(x_1), f(x)) < \varepsilon$ (by continuity of f), hence $f(x_1) \in (y - \varepsilon, y + \varepsilon) \subseteq V$; moreover, $(x - \delta, x + \delta) \subseteq f{-}1(V)$, thus $f^{-1}(V)$ is open as required.

The inverse of this proof follows identically. ■

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