MATH454 - Analysis 3

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Contents

1 Sigma Algebras and Measures

1.1 A Review of Riemann Integration

Let $f : \mathbb{R} \to \mathbb{R}$ and $[a, b] \subset \mathbb{R}$. Define a **partition** of $[a, b]$ as the set

$$\operatorname{part}([a, b]) := \{a =: x_0 < x_1 < \cdots < x_N := b\}.$$

We can then define the upper and lower Riemann integrals of f over the region $[a, b]$ as

$$\text{upper:} \quad \overline{\int_a^b} f(x)\,\mathrm{d}x := \inf_{\operatorname{part}([a,b])} \left\{ \sum_{\{i=1\}}^{N} \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \right\}$$

$$\text{lower:} \quad \underline{\int_a^b} f(x)\,\mathrm{d}x := \sup_{\operatorname{part}([a,b])} \left\{ \sum_{\{i=1\}}^{N} \inf_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \right\}.$$

We then say f **Riemann integrable** if these two quantities are equal, and denote this value by $\int_a^b f(x)\,\mathrm{d}x$.

Many "nice-enough" (continuous, monotonic, etc.) functions are Riemann integrable, but many that we would like to be able to "integrate" are simply not, for instance Dirichlet's function $x \mapsto \begin{cases} 1 x \in \mathbb{Q} \setminus [a,b] \\ 0 x \in \mathbb{Q}^c \setminus [a,b] \end{cases}$. Hence, we need a more general notion of integration.

1.2 Sigma Algebras

↪**Definition 1.1** (Sigma algebra): Let X be a *space* (a nonempty set) and $\mathcal{F}$ a collection of subsets of X. $\mathcal{F}$ a *sigma algebra* or simply σ-algebra of X if the following hold:

1. $X \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ (closed under complement)
3. $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ (closed under countable unions)

↪**Proposition 1.1**:

4. $\emptyset \in \mathcal{F}$
5. $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
6. $A_1, \ldots, A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
7. $A, B \in \mathcal{F} \Rightarrow A \setminus B, B \setminus A \in \mathcal{F}$

⊛ **Example 1.1**: The "largest" sigma algebra of a set X is the power set 2^X, the smallest the trivial $\{\emptyset, X\}$.

Given a set $A \subset X$, the set $\mathcal{F}_A := \{\emptyset, X, A, A^c\}$ is a sigma algebra; given two disjoint sets $A, B \subset X$, then $\mathcal{F}_{A,B} := \{\emptyset, X, A, A^c, B, B^c, A \cup B, A^c \cap B^c\}$ a sigma algebra.

↪**Definition 1.2** (Generating a sigma algebra): Let X be a nonempty set, and $\mathcal{C}$ a collection of subsets of X. Then, the σ-algebra *generated* by $\mathcal{C}$, denoted $\sigma(\mathcal{C})$, is such that

1. $\sigma(\mathcal{C})$ a sigma algebra with $\mathcal{C} \subseteq \sigma(\mathcal{C})$
2. if $\mathcal{F}'$ a sigma algebra with $\mathcal{C} \subseteq \mathcal{F}'$, then $\mathcal{F}' \supseteq \sigma(\mathcal{C})$

Namely, $\sigma(\mathcal{C})$ is the smallest sigma algebra "containing" (as a subset) $\mathcal{C}$.

↪**Proposition 1.2**:

1. $\sigma(\mathcal{C}) = \bigcap\{\mathcal{F} : \mathcal{F} \quad \text{a sigma algebra containing } \mathcal{C}\}$
2. if $\mathcal{C}$ itself a sigma algebra, then $\sigma(\mathcal{C}) = \mathcal{C}$
3. if $\mathcal{C}_1, \mathcal{C}_2$ are two collections of subsets of X such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_2)$

↪**Definition 1.3** (The Borel sigma-algebra): The *Borel σ-algebra*, denoted $\mathfrak{B}_{\mathbb{R}}$, on the real line is given by

$$\mathfrak{B}_{\mathbb{R}} := \sigma(\{\text{open subsets of } \mathbb{R}\}).$$

We call sets in $\mathfrak{B}_{\mathbb{R}}$ *Borel sets.*

↪**Proposition 1.3**: $\mathfrak{B}_{\mathbb{R}}$ is also generated by the sets

- $\{(a, b) : a < b \in \mathbb{R}\}$
- $\{(a, b] : a < b \in \mathbb{R}\}$
- $\{[a, b] : a < b \in \mathbb{R}\}$
- $\{[a, b) : a < b \in \mathbb{R}\}$ ⊛
- $\{(-\infty, c) : c \in \mathbb{R}\}$
- $\{(-\infty, c] : c \in \mathbb{R}\}$
- etc.

Proof. We prove just ⊛. It suffices to show that the generating sets of each σ-algebra is contained in the other σ-algebra. Let $a < b \in \mathbb{R}$. Then,

$$(a, b) = \bigcup_{n=1}^{\infty} \underbrace{\left[a + \frac{1}{n}, b\right)}_{\in ⊛} \in \sigma(\{[a, b)\}) \Rightarrow \mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[a, b)\}).$$

Conversely,

$$[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right) \in \mathfrak{B}_{\mathbb{R}}.$$

■

↪**Proposition 1.4**: All intervals (open, closed, half open, half closed, finite, etc) are Borel sets; any set obtained from countable set operations of intervals are Borel; all singletons are Borel; any finite and countable sets are Borel.

1.3 Measures

↪**Definition 1.4** (Measurable Space): Let X be a space and $\mathcal{F}$ a σ-algebra. We call the tuple $(X, \mathcal{F})$ a *measurable space*.

↪**Definition 1.5** (Measure): Let $(X, \mathcal{F})$ be a measurable space. A *measure* is a function $\mu : \mathcal{F} \to [0, \infty]$ satisfying

(i) $\mu(\emptyset) = 0$;

(ii) if $\{A_n\} \subseteq \mathcal{F}$ a sequence of (pairwise) disjoint sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

i.e. μ is *countably additive*. We further call μ

- *finite* if $\mu(X) < \infty$,
- a *probability measure* if $\mu(X) = 1$,
- σ-finite if $\exists \{A_n\} \subseteq \mathcal{F}$ such that $X = \bigcup_{n=1}^{\infty} A_n$ with $\mu(A_n) < \infty \forall n \geq 1$,

and call the triple $(X, \mathcal{F}, \mu)$ a *measure space*.

⊛ **Example 1.2**: The measure on $\mathfrak{B}_{\mathbb{R}}$ given by

$$A \mapsto \begin{cases} |A| & \text{if } A \text{ finite} \\ \infty & \text{else} \end{cases}$$

is called the *counting measure*.

Fix $x_0 \in \mathbb{R}$, then the measure on $\mathfrak{B}_{\mathbb{R}}$ given by

$$A \mapsto \begin{cases} 1 \text{ if } x_0 \in A \\ 0 \text{ else} \end{cases}$$

is called the *point mass at* x_0.

↪**Theorem 1.1** (Properties of Measures): Fix a measure space $(X, \mathcal{F}, \mu)$. The following properties hold:

1. (finite additivity) For any sequence $\{A_n\}_{n=1}^N \subseteq \mathcal{F}$ of disjoint sets,
$$\mu\left(\bigcup_{n=1}^N A_n\right) = \sum_{n=1}^N \mu(A_n).$$
2. (monotonicity) For any $A \subseteq B \in \mathcal{F}$, then $\mu(A) \leq \mu(B)$.
3. (countable/finite subadditivity) For any sequence $\{A_n\} \subseteq \mathcal{F}$ (**not** necessarily disjoint),
$$\mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n),$$
an analogous statement holding for a finite collection of sets $A_1, ..., A_N$.
4. (continuity from below) For $\{A_n\} \subseteq \mathcal{F}$ such that $A_n \subseteq A_{n+1} \forall n \geq 1$ (in which case we say $\{A_n\}$ "increasing" and write $A_n \uparrow$) we have
$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n\to\infty} \mu(A_n).$$
5. (continuity from above) For $\{A_n\} \subseteq \mathcal{F}$, $A_n \supseteq A_{n+1} \forall n \geq 1$ (we write $A_n \downarrow$) we have that **if** $\mu(A_1) < \infty$,
$$\mu\left(\bigcap_{n=1}^\infty A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Remark 1.3.1: In 4., note that since A_n increasing, that the union $\bigcup_{n=1}^\infty A_n \supseteq A_m$ for any arbitrarily large m; indeed, one could logically right $\lim_{n\to\infty} A_n = \bigcup_{n=1}^\infty A_n$. This this notation, then, 4. simply states that we may interchange limit and measure. A similar argument can be viewed for 5. (how?).

Remark 1.3.2: The finiteness condition in 5. may be slightly modified such as to state that $\mu(A_n) < \infty$ for some n; remark why this would suffice to ensure the entire rest of the sequence has finite measure.

PROOF.

1. Extend $A_1, ..., A_N$ to an infinite sequence by $A_n := \emptyset$ for $n > N$. Then this simply follows from countable additivity and $\mu(\emptyset) = 0$.
2. We may write $B = A \cup (B \setminus A)$; this is a disjoint union of sets. By finite additivity, then,
$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A),$$
since the measure is positive.
3. We prove only for a countable union; use the technique from 1. to extend to finite. We first "disjointify" the sequence such that we can use the countable additivity axiom. Let $B_1 =$

$A_1, B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$ for $n \geq 2$. Remark then that $\{B_n\} \subseteq \mathcal{F}$ is a disjoint sequence of sets, and that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$. By countable additivity and subadditivity,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

4. We again "disjointify" the sequence $\{A_n\}$. Put $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$ for all $n \geq 2$ (remark that this is equivalent to the construction from the previous proof because the sets are increasing). Then, again, $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$, and in particular, for all $N \geq 1$, $\bigcup_{n=1}^{N} B_n = A_N$. Then

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \sum_{n=1}^{\infty} \mu(B_n) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n) \\ &= \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} B_n\right) = \lim_{N\to\infty} \mu(A_N).\end{aligned}$$

5. We yet again disjointify, backwards (in a way) from the previous case. Put $B_n = A_1 \setminus A_n$ for all $n \geq 1$. Then, $\{B_n\} \subseteq \mathcal{F}$, B_n increasing, and $\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \bigcap_{n=1}^{\infty} A_n$. Then, by continuity from below,

$$\mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(A_1 \setminus A_n)$$

and also

$$\begin{aligned}\mu(A_1) &= \mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) + \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \\ &= \mu(A_1 \setminus A_n) + \mu(A_n),\end{aligned}$$

and combining these two equalities yields the desired result.

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1.4 Constructing the Lebesgue Measure on $\mathbb{R}$

↪**Definition 1.6** (Lebesgue outer measure): For all $A \subseteq \mathbb{R}$, define

$$m^*(A) := \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n, I_n \text{ open intervals}\right\},$$

called the *Lebesgue outer measure* of A (where $\ell(I)$ is the length of interval I, i.e. the absolute value of the difference of its endpoints, if finite, or ∞ if not).

↪**Proposition 1.5**: The following properties of m^* hold:

1. $m^*(A) \geq 0$ for all $A \subseteq \mathbb{R}$, and $m^*(\emptyset) = 0$.
2. (monotonicity) For $A \subseteq B$, $m^*(A) \leq m^*(B)$.
3. (countable subadditivity) For $\{A_n\}$, $A_n \subseteq \mathbb{R}$, $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$.[1]
4. If $I \subseteq \mathbb{R}$ an interval, then $m^*(I) = \ell(I)$.
5. m^* is translation invariant; for any $A \subseteq R, x \in \mathbb{R}$, $m^*(A) = m^*(A + x)$ where $A + x := \{a + x : a \in A\}$.
6. For all $A \subseteq \mathbb{R}$, $m^*(A) = \inf\{m^*(B) : A \subseteq B \subseteq R, B - \text{open}\}$.
7. If $A = A_1 \cup A_2 \subseteq \mathbb{R}$ with $d(A_1, A_2) > 0$,[2] then $m^*(A_1) + m^*(A_2) = m^*(A)$.
8. If $A = \bigcup_{k=1}^{\infty} J_k$ where J_k's are "almost disjoint intervals" (i.e. share at most endpoints), then $m^*(A) = \sum_{k=1}^{\infty} m^*(J_k) = \sum_{k=1}^{\infty} \ell(J_k)$.

Proof.

3. If $m^*(A_n) = \infty$, for any n, we are done, so assume wlog $m^*(A_n) < \infty$ for all n. Then, for each n and $\varepsilon > 0$, one can choose open intervals $\{I_{n,i}\}_{i \geq 1}$ such that $A_n \subseteq \bigcup_{i=1}^{\infty} I_{n,i}$ and $\sum_{i=1}^{\infty} \ell(I_{n,i}) \leq m^*(A_n) + \frac{\varepsilon}{2^n}$. Hence

$$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1,i=1}^{\infty} I_{n,i}$$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n,i=1}^{\infty} \ell(I_{n,i}) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} \ell(I_{n,i}) \leq \sum_{n=1}^{\infty}\left(m^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon,$$

and as ε arbitrary, the statement follows.

4. We prove first for $I = [a, b]$. For any $\varepsilon > 0$, set $I_1 = (a - \varepsilon, b + \varepsilon)$; then $I \subseteq I_1$ so $m^*(I) \leq \ell(I_1) = (b - 1) + 2\varepsilon$ hence $m^*(I) \leq b - a = \ell(I)$. Conversely, let $\{I_n\}$ be any open-interval convering of I (wlog, each of finite length; else the statement holds trivially). Since I compact, it can be covered by finitely many of the I_n's, say $\{I_n\}_{n=1}^{N}$, denoting $I_n = (a_n, b_n)$ (with relabelling, etc). Moreover, we can pick the a_n, b_n's such that $a_1 < a, b_N > b$, and generally $a_n < b_{n-1} \forall 2 \leq n \leq N$. Then,

$$\begin{aligned}\sum_{n=1}^{\infty} \ell(I_n) \geq \sum_{n=1}^{N} \ell(I_n) &= b_1 - a_1 + \sum_{n=2}^{N}(b_n - a_n) \\ &\geq b_1 - a_1 + \sum_{n=2}^{N}(b_n - b_{n-1}) \\ &= b_N - a_1 \geq b - 1 = \ell(I),\end{aligned}$$

hence since the cover was arbitrary, $m^*(A) \geq \ell(I)$, and equality holds.

Now, suppose I finite, with endpoints $a < b$. Then for any $\frac{b-a}{2} > \varepsilon > 0$, then

$$[a + \varepsilon, b - \varepsilon] \subseteq I \subseteq [a - \varepsilon, b + \varepsilon],$$

[1]More generally, any set function on $2^{\mathbb{R}}$ that satisfies 1., 2., and 3. is called an *outer measure*.

[2]Remark: this is a stronger requirement than disjointness!

hence by monotonicity and the previous part of this proof

$$m^*([a+\varepsilon, b-\varepsilon]) = b - a - 2\varepsilon \leq m^*(I) \leq b - a + 2\varepsilon = m^*([a-\varepsilon, b+\varepsilon]),$$

from which it follows that $m^*(I) = b - a = \ell(I)$.

Finally, suppose I infinite. Then, $\forall M \geq 0, \exists$ closed, finite interval I_M with $I_M \subseteq I$ and $\ell(I_M) \geq M$. Hence, $m^*(I) \geq m^*(I_M) \geq M$ and thus as M arbitrary it must be that $m^*(I) = \infty = \ell(I)$.

6. Denote $\tilde{m}(A) := \inf\{m^*(B) : A \subseteq B \subseteq R, B - \text{open}\}$. For any $A \subseteq B \subseteq \mathbb{R}$ with B open, monotonicity gives that $m^*(A) \leq m^*(B)$, hence $m^*(A) \leq \tilde{m}(A)$. Conversely, assuming wlog $m^*(A) < \infty$ (else holds trivially), then for all $\varepsilon > 0$, there exists $\{I_n\}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ with $\sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon$. Setting $B := \bigcup_{n=1}^{\infty} I_n$, we have that $A \subseteq B$ and $m^*(B) = m^*(\bigcup I_n) \leq$ (by finite subadditivity) $\sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon$ hence $m^*(B) \leq m^*(A)$ for all B. Thus $m^*(A) \geq \tilde{m}(A)$ and equality holds.

7. Put $\delta := d(A_1, A_2) > 0$. Clearly $m^*(A) \leq m^*(A_1) + m^*(A_2)$ by finite subadditivity. wlog, $m^*(A) < \infty$ (and hence $m^*(A_i) < \infty, i = 1, 2$) (else holds trivially). Then $\forall \varepsilon > 0, \exists \{I_n\} : A \subseteq \bigcup I_n$ and $\sum \ell(I_n) \leq m^*(A) + \varepsilon$. Then, for all n, we consider a "refinement" of I_n; namely, let $\left\{I_{n,i}\right\}_{i \geq 1}$ such that $I_n \subseteq \bigcup_i I_{n,i}$ and $\ell\left(I_{n,i}\right) < \delta$ and $\sum_i \ell\left(I_{n,i}\right) \leq \ell(I_n) + \frac{\varepsilon}{2^n}$. Relabel $\left\{I_{n,i} : n, i \geq 1\right\} \rightsquigarrow \{J_m : m \geq 1\}$ (both are countable). Then, $\{J_m\}$ defines an open-interval cover of A, and since $\ell(J_m) < \delta$ for each m, J_m intersects at most one A_i. For each m and $p = 1, 2$, put

$$M_p := \left\{m : J_m \cap A_p \neq \emptyset\right\},$$

noting that $M_1 \cap M_2 = \emptyset$. Then $\left\{J_m : m \in M_p\right\}$ is an open covereing of A_p, and so

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\leq \sum_{m \in M_1} \ell(J_m) + \sum_{m \in M_2} \ell(J_m) \\ &\leq \sum_{m=1}^{\infty} \ell(J_m) = \sum_{n,i=1}^{\infty} \ell(I_n, i) \\ &\leq \sum_n \left(\ell(I_n) + \frac{\varepsilon}{2^n}\right) \\ &= \sum_n \ell(I_n) + \varepsilon \\ &\leq m^*(A) + 2\varepsilon, \end{aligned}$$

and hence equality follows.

8. If $\ell(J_k) = \infty$ for some k, then since $J_k \subseteq A$, subadditivity gives us that $m^*(J_k) \leq m^*(A)$ and so $m^*(A) = \infty = \sum_{k=1}^{\infty} \ell(J_k)$ (since if any J_k infinite, the sum of the lengths of all of them will also be infinite).

Suppose then $\ell(J_k) < \infty$ for all k. Fix $\varepsilon > 0$. Then for all $k \geq 1$, choose $I_k \subseteq J_k$ such that $\ell(J_k) \leq \ell(I_k) + \frac{\varepsilon}{2^k}$. For any $N \geq 1$, we can choose a subset $\{I_1, \ldots, I_N\}$ of intervals such that all are disjoint, with strictly positive distance between them, and so

$$\bigcup_{k=1}^{N} I_k \subseteq \bigcup_{k=1}^{N} I_k \subseteq A$$

$$\begin{aligned} \Rightarrow m^*(A) \geq m^*\left(\bigcup_{k=1}^{N} I_k\right) &\geq \sum_{k=1}^{N} \ell(I_k) \\ &\geq \sum_{k=1}^{N} \left(\ell(J_k) - \frac{\varepsilon}{2^k}\right) \\ &\geq \sum_{k=1}^{N} \ell(J_k) - \varepsilon \end{aligned}$$

$$\Rightarrow m^*(A) \geq \sum_{k=1}^{\infty} \ell(J_k),$$

the second inequality following from finite subadditivity. The converse of the final inequality holds trivially. ■

1.5 Lebesgue-Measurable Sets

↪**Definition 1.7**: $A \subseteq \mathbb{R}$ is m^**-measurable* if $\forall B \subseteq \mathbb{R}$,

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c).$$

Remark 1.5.1: By subadditivity, $\leq$ always holds in the definition above.

↪**Theorem 1.2** (Carathéodary's Theorem): Let

$$\mathcal{M} := \{A \subseteq \mathbb{R} : A \;\; m^* - \text{measurable}\}.$$

Then, $\mathcal{M}$ is a σ-algebra of subsets of $\mathbb{R}$.

Define $m : \mathcal{M} \to [0, \infty]$, $m(A) = m^*(A)$. Then, m is a measure on $\mathcal{M}$, called the *Lebesgue measure* on $\mathbb{R}$. We call sets in $\mathcal{M}$ *Lebesgue-measurable* or simply *measurable* (if clear from context) accordingly. We call $(\mathbb{R}, \mathcal{M}, m)$ the *Lebesgue measure space*.

Proof. The first two σ-algebra axioms are easy. We have for any $B \subseteq \mathbb{R}$ that

$$m^*(B \cap \mathbb{R}) + m^*(B \cap \mathbb{R}^c) = m^*(B) + m^*(B \cap \emptyset) = m^*(B)$$

so $\mathbb{R} \in \mathcal{M}$. Further, $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$ by the symmetry of the requirement for sets to be in $\mathcal{M}$.

The final axiom takes more work. We show first $\mathcal{M}$ closed under finite unions; by induction it suffices to show for 2 sets. Let $A_1, A_2 \in \mathcal{M}$. Then, for all $B \subseteq \mathbb{R}$,

$$
\begin{aligned}
m^*(B) &= m^*(B \cap A_1) + m^*(B \cap A_1^c) \\
&= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*(B \cap A_1^c \cap A_2^c) \\
&= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*(B \cap (A_1 \cup A_2)^c)
\end{aligned}
$$

Note that $(B \cap A_1) \cup (B \cap A_1^c \cap A_2^c) = B \cap (A_1 \cup A_2)$, hence by subadditivity,

$$m^*(B) \geq m^*(B \cap (A_1 \cup A_2)) + m^*(B \cap (A_1 \cup A_2)^c),$$

and since the other direction of the inequality comes for free, we conclude $A_1 \cup A_2 \in \mathcal{M}$.

Let now $\{A_n\} \subseteq \mathcal{M}$. We "disjointify" $\{A_n\}$; put $B_1 := A_1$, $B_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i, n \geq 2$, noting $\bigcup_n A_n = \bigcup_n B_n$, and each $B_n \in \mathcal{M}$, as each is but a finite number of set operations applied to the A_n's, and thus in $\mathcal{M}$ as demonstrated above. Put $E_n := \bigcup_{i=1}^n B_i$, noting again $E_n \in \mathcal{M}$. Then, for all $B \subseteq \mathbb{R}$,

$$
\begin{aligned}
m^*(B) &= m^*\left(\underbrace{B \cap E_n}_{\text{chop up } B_n}\right) + m^*\left(\underbrace{B \cap E_n^c}_{E_n \subseteq \cup B_n \Rightarrow E_n^c \supseteq (\cup B_n)^c}\right) \\
&\geq m^*\left(B \cap \underbrace{E_n \cap B_n}_{=B_n}\right) + m^*\left(B \cap \underbrace{E_n \cap B_n^c}_{=E_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\
&\geq m^*(B \cap B_n) + m^*\left(\underbrace{B \cap E_{n-1}}_{\text{chop up } B_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\
&\geq m^*(B \cap B_n) + m^*(B \cap E_{n-1} \cap B_{n-1}) \\
&\quad + m^*(B \cap E_{n-1} \cap B_{n-1}^c) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right).
\end{aligned}
$$

Notice that the last line is essentially the second applied to B_{n-1}; hence, we have a repeating (essentially, "descending") pattern in this manner, which we repeat until $n \to 1$. We have, thus, that

$$m^*(B) \geq \sum_{i=1}^{n} [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right),$$

so taking $n \to \infty$,

$$
\begin{aligned}
m^*(B) &\geq \sum_{i=1}^{\infty} [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\
&\geq m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right).
\end{aligned}
$$

As usual, the inverse inequality comes for free, and thus we can conclude $\bigcup_{n=1}^{\infty} B_n$ also m^*-measurable, and thus so is $\bigcup_{n=1}^{\infty} A_n$. This proves $\mathcal{M}$ a σ-algebra.

We show now m a measure. By previous propositions, we have that $m \geq 0$ and $m(\emptyset) = 0$ (since $m = m^* \mid_{\mathcal{M}}$), so it remains to prove countable subadditivity.

Let $\{A_n\} \subseteq \mathcal{M}$-disjoint. Following precisely the same argument as above, used to prove that $\mathcal{M}$ closed under countable unions, shows that for any $n \geq 1$

$$m\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(A_i),$$

that is, finite additivity holds, and thus by subadditivity

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq m\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(A_i),$$

and so taking the limit of $n \to \infty$, we have

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} m(A_i),$$

with the converse inequality coming for free. Thus, m indeed a measure on $\mathcal{M}$. ■

↪**Proposition 1.6**: $\mathcal{M}, m$ translation invariant; for all $A \in \mathcal{M}$, $x \in \mathbb{R}$, $x + A = \{x + a : a \in A\} \in \mathcal{M}$ and $m(A) = m(A + x)$.

Remark 1.5.2: We would like this to hold, heuristically, since if we shift sets on the real line, we should expect their length to remain constant.

Proof. For all $B \subseteq \mathbb{R}$, we have (since m^* translation invariant)

$$\begin{aligned} m^*(B) = m^*(B - x) &= m^*\left(\underbrace{(B - x) \cap A}_{= B \cap (A + x)}\right) + m^*\left(\underbrace{(B - x) \cap A^c}_{= B \cap (A^c + x) = B \cap (A + x)^c}\right) \\ &= m^*(B \cap (A + x)) + m^*(B \cap (A + x)^c), \end{aligned}$$

thus $A + x \in \mathcal{M}$, and since m^* translation invariant, it follows that m is. ■

↪**Theorem 1.3**: $\forall a, b \in \mathbb{R}$ with $a < b$, $(a, b) \in \mathcal{M}$, and $m((a, b)) = b - a$.

Remark 1.5.3: Again, we'd like this to hold, heuristically, since we would like the measure of an interval to simply be its length; we'd moreover like to be able to measure intervals, i.e. have intervals be contained in $\mathcal{M}$.

↪**Corollary 1.1**: $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M}$

PROOF. $\mathfrak{B}_{\mathbb{R}}$ is generated by open intervals of the form (a, b). All such intervals are in $\mathcal{M}$ by the previous theorem, and hence the proof. ■

1.6 Properties of the Lebesgue Measure

↪**Proposition 1.7** (Regularity Assumptions on m): For all $A \in \mathcal{M}$, the following hold.

- For all $\varepsilon > 0, \exists G$ open such that $A \subseteq G$ and $m(G \setminus A) < \varepsilon$.
- For all $\varepsilon > 0$, $\exists F$-closed such that $F \subseteq A$ and $m(A \setminus F) \leq \varepsilon$.
- $m(A) = \inf\{m(G) : G \text{ open}, G \supseteq A\}$.
- $m(A) = \sup\{m(K) : K \text{ compact}, K \subseteq A\}$.
- If $m(A) < \infty$, then for all $\varepsilon > 0$, $\exists K \subseteq A$ compact, such that $m(A \setminus K) < \varepsilon$.
- If $m(A) < \infty$, then for all $\varepsilon \geq 0$, $\exists$ finite collection of open intervals $I_1, ..., I_N$ such that $m\left(A \triangle \left(\bigcup_{n=1}^{N} I_n\right)\right) \leq \varepsilon$.

↪**Proposition 1.8** (Completeness of m): $(\mathbb{R}, \mathcal{M}, m)$ is *complete*, in the sense that for all $A \subseteq \mathbb{R}$, if $\exists B \in \mathcal{M}$ such that $A \subseteq B$ and $m(B) = 0$, then $A \in \mathcal{M}$ and $m(A) = 0$.

Equivalently, any subset of a null set is again a null set.

Remark 1.6.1: In general, $A \in \mathcal{F}, B \subseteq A \not\Rightarrow B \in \mathcal{F}$.

↪**Proposition 1.9**: Up to rescaling, m is the unique, nontrivial measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ that is finite on compact sets and is translation invariant, i.e. if μ another such measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ with $\mu = c \cdot m$ for $c > 0$, then $\mu = m$.

Remark 1.6.2: Such a c is simply $c = \mu((0, 1))$.

To prove this proposition, we first introduce some helpful tooling:

↪**Theorem 1.4** (Dynkin's π-d): Given a space X, let $\mathcal{C}$ be a collection of subsets of X. $\mathcal{C}$ is called a π*-system* if $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$ (that is, it is closed under finite intersections).

Let $\mathcal{F} = \sigma(\mathcal{C})$, and suppose μ_1, μ_2 are two finite measures on $(X, \mathcal{F})$ such that $\mu_1(X) = \mu_2(X)$ and $\mu_1 = \mu_2$ when restricted to $\mathcal{C}$. Then, $\mu_1 = \mu_2$ on all of $\mathcal{F}$.

↪**Proposition 1.10**: $\{\emptyset\} \cup \{(a, b) : a < b \in \mathbb{R}\}$ a π-system.

↪**Proposition 1.11**: If μ a measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ such that for all intervals I, $\mu(I) = \ell(I)$, then $\mu = m$.

Proof. Consider for all $n \geq 1$ $\mu|_{\mathfrak{B}_{[-n,n]}}$. Clearly, $\mu([-n, n]) = m([-n, n]) = 2n$, and for all $a, b \in \mathbb{R}$, $\mu((a, b) \cap [-n, n]) = \ell((a, b) \cap [-n, n]) = m((a, b) \cap [-n, n])$. Thus, by the previous theorem, μ must match m on all of $\mathfrak{B}_{[-n,n]}$.

Let now $A \in \mathfrak{B}_{\mathbb{R}}$. Let $A_n := A \cap [-n, n] \in \mathfrak{B}_{[-n,n]}$. By continuity of m from below,

$$\begin{aligned}\mu(A) &= \lim_{n\to\infty} \mu(A_n)\\ &= \lim_{n\to\infty} m(A_n)\\ &= m(A),\end{aligned}$$

hence $\mu = m$. ■

↪**Proposition 1.12**: If μ a measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ assigning finite values to compact sets and is translation invariant, then $\mu = cm$ for some $c > 0$.

Remark 1.6.3: This proposition is also tacitly stating that $\mathfrak{B}_{\mathbb{R}}$ translation invariant; this needs to be shown.

↪**Lemma 1.1**: $\mathfrak{B}_{\mathbb{R}}$ translation invariant; for any $A \in \mathfrak{B}_{\mathbb{R}}, x \in \mathbb{R}$, $A + x \in \mathfrak{B}_{\mathbb{R}}$.

Proof. We employ the "good set strategy"; fix some $x \in \mathbb{R}$ and let

$$\Sigma := \{B \in \mathfrak{B}_{\mathbb{R}} : B + x \in \mathfrak{B}_{\mathbb{R}}\}.$$

One can check that Σ a σ-algebra, and so $\Sigma \subseteq \mathfrak{B}_{\mathbb{R}}$. But in addition, its easy to see that $\{(a, b) : a < b \in \mathbb{R}\} \subseteq \Sigma$, since a translated interval is just another interval, and since these sets generate $\mathfrak{B}_{\mathbb{R}}$, it must be further that $\mathfrak{B}_{\mathbb{R}} \subseteq \Sigma$, completing the proof. ■

Proof. (of the proposition) Let $c = \mu((0, 1])$, noting that $c > 0$ (why? Consider what would happen if $c = 0$).

This implies that $\forall n \geq 1$, $\mu\left(\left(0, \frac{1}{n}\right]\right) = \frac{c}{n}$ (obtained by "chopping up" $(0, 1]$ into n disjoint intervals); from here we can draw many further conclusions:

$$\forall m = 1, ..., n-1, \mu\left(\left(0, \frac{m}{n}\right]\right) = \frac{m}{n} c$$
$$\Rightarrow \forall q \in \mathbb{Q} \cap (0,1], \mu((0,q]) = qc$$
$$\Rightarrow \forall q \in \mathbb{Q}^+, \mu((0,q]) = q \cdot c \text{ (translate)}$$
$$\Rightarrow \forall a \in \mathbb{R}, \mu((a, a+q]) = q \cdot c$$
$$\Rightarrow \forall \text{ intervals } I, \mu(I) = c \cdot \ell(I) \text{ (continuity)}$$
$$\Rightarrow \forall n \geq 1, a, b \in \mathbb{R}, \mu((a,b) \cap [-n,n]) = c \cdot \ell((a,b) \cap [-n,n]) = c \cdot m((a,b) \cap [-n,n]),$$

but then, $\mu = c \cdot m$ on $\mathfrak{B}_{\mathbb{R}_{[-n,n]}}$, and by appealing again the Dynkin's, $\mu = c \cdot m$ on all of $\mathfrak{B}_{\mathbb{R}}$. ■

↪**Proposition 1.13** (Scaling): m has the *scaling property* that $\forall A \in \mathcal{M}, c \in \mathbb{R}, c \cdot A = \{cx : x \in A\} \in \mathcal{M}$, and $m(c \cdot A) = |c|\ m(A)$.

Proof. Assume $c \neq 0$. Given $A \subseteq \mathbb{R}$, remark that $\{I_n\}$ an open interval cover of A iff $\{cI_n\}$ and open interval cover of cA, and $\ell(cI_n) = |c|\ \ell(I_n)$, and thus $m^*(cA) = |c|\ m^*(A)$.

Now, suppose $A \in \mathcal{M}$. Then, we have for any $B \subseteq \mathbb{R}$,

$$\begin{aligned} m^*(B) = |c|\ m^*\left(\frac{1}{c}B\right) &= |c|\ m^*\left(\frac{1}{c}B \cap A\right) + |c|\ m^*\left(\frac{1}{c}B \cap A^c\right) \\ &= m^*(B \cap cA) + m^*(B \cap (cA)^c), \end{aligned}$$

so $cA \in \mathcal{M}$. ■

1.7 Relationship between $\mathfrak{B}_{\mathbb{R}}$ and $\mathcal{M}$

↪**Definition 1.8**: Given $(X, \mathcal{F}, \mu)$, consider the following collection of subsets of X,

$$\mathcal{N} := \{B \subseteq X : \exists A \in \mathcal{F} \text{ s.t. } \mu(A) = 0, B \subseteq A\}.$$

Put $\overline{\mathcal{F}} := \sigma(\mathcal{F} \cup \mathcal{N})$; this is called the *completion* of $\mathcal{F}$ with respect to μ.

↪**Proposition 1.14**: $\overline{\mathcal{F}} = \{F \subseteq X : \exists E, G \in \mathcal{F} \text{ s.t. } \exists E \subseteq F \subseteq G \text{ and } m(G \setminus E) = 0\}$.

Proof. Put $\mathcal{G}$ the set on the right; one can check $\mathcal{G}$ a σ-algebra. Since $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{N} \subseteq \mathcal{G}$, we have $\overline{\mathcal{F}} \subseteq \mathcal{G}$.

Conversely, for any $F \in \mathcal{G}$, we have $E, G \in \mathcal{F}$ such that $E \subseteq F \subseteq G$ with $m(G \setminus E) = 0$. We can rewrite

$$F = \underbrace{E}_{\in \mathcal{F}} \cup \underbrace{(F \setminus E)}_{\substack{\subseteq G \setminus E \\ \Rightarrow \mu(F \setminus E) = 0 \\ \Rightarrow G \setminus E \in \mathcal{N}}},$$

hence $F \in \mathcal{F} \cup \mathcal{N}$ and thus in $\mathcal{F}$, and equality holds. ■

↪**Definition 1.9**: Given $(X, \mathcal{F}, \mu)$, μ can be *extended* to $\overline{\mathcal{F}}$ by, for each $F \in \overline{\mathcal{F}}$ with $E \subseteq F \subseteq G$ s.t. $\mu(G \setminus E) = 0$, put

$$\mu(F) = \mu(E) = \mu(G).$$

We call then $(X, \overline{\mathcal{F}}, \mu)$ a *complete measure space.*

Remark 1.7.1: It isn't obvious that this is well defined a priori; in particular, the E, G sets are certainly not guaranteed to be unique in general, so one must check that this definition is valid regardless of choice of "sandwich sets".

↪**Theorem 1.5**: $(\mathbb{R}, \mathcal{M}, m)$ is the completion of $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, m)$.

PROOF. Given $A \in \mathcal{M}$, then $\forall n \geq 1, \exists G_n$-open with $A \subseteq G_n$ s.t. $m^*(G_n \setminus A) \leq \frac{1}{n}$ and $\exists F_n$-closed with $F_n \subseteq A$ s.t. $m^*(A \setminus F_n) \leq \frac{1}{n}$.

Put $C := \bigcap_{n=1}^{\infty} G_n$, $B := \bigcap_{n=1}^{\infty} F_n$, remarking that $C, B \in \mathfrak{B}_{\mathbb{R}}$, $B \subseteq A \subseteq C$, and moreover

$$m(C \setminus A) \leq \frac{1}{n}, m(A \setminus B) \leq \frac{1}{n}$$

$$\Rightarrow m(C \setminus B) = m(C \setminus A) + m(A \setminus B) \leq \frac{2}{n},$$

but n can be arbitrarily large, hence $m(C \setminus B) = 0$; in short, given a measurable set, we can "sandwich it" arbitrarily closely with Borel sets. Thus, $A \in \overline{\mathfrak{B}_{\mathbb{R}}} \Rightarrow \mathcal{M} \subseteq \overline{\mathfrak{B}_{\mathbb{R}}}$. But recall that $\mathcal{M}$ complete, so $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M} \Rightarrow \overline{\mathfrak{B}_{\mathbb{R}}} \subseteq \overline{\mathcal{M}} = \mathcal{M}$, and thus $\overline{\mathfrak{B}_{\mathbb{R}}} = \mathcal{M}$ indeed.

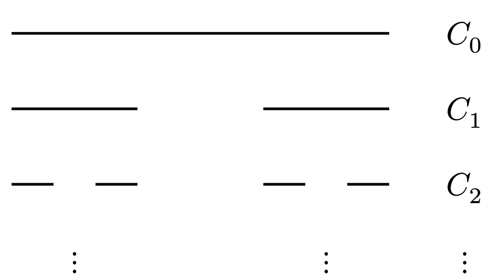
Heuristically, this means that any measurable set is "different" from a Borel set by at most a null set. ■

1.8 Some Special Sets

1.8.1 Uncountable Null Set?

Remark that for any countable set $A \in \mathcal{M}$, $m(A) = 0$. One naturally asks the opposite question, does there exist a measurable, uncountable set with measure 0? We construct a particular one here, the Cantor set, C.

This requires an "inductive" construction. Define $C_0 = [0, 1]$, and define C_k to be C_{k-1} after removing the middle third from each of its disjoint components. For instance $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, then $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, and so on. This may be clearest graphically:



Remark that the $C_n \downarrow$. Put finally

$$C := \bigcap_{n=1}^{\infty} C_n.$$

↪**Proposition 1.15**: The following hold for the Cantor set C:

1. C is closed (and thus $C \in \mathfrak{B}_{\mathbb{R}}$);
2. $m(C) = 0$;
3. C is uncountable.

Proof.

1. For each n, C_n is the countable (indeed, finite) union of 2^n-many disjoint, closed intervals, hence each C_n closed. C is thus a countable intersection of closed sets, and is thus itself closed.
2. For each n, each of the 2^n disjoint closed intervals in C_n has length $\frac{1}{3^n}$, hence

$$m(C_n) = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$

Since $\{C_n\} \downarrow$, by continuity of m we have

$$m(C) = \lim_{n \Rightarrow \infty} m(C_n) = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0.$$

3. This part is a little trickier. Notice that for any $x \in [0, 1]$, we can define a sequence (a_n) where each $a_n \in \{0, 1, 2\}$, and such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n};$$

in particular, this is just the base-3 representation of x, which we denote $(x)_3 = (a_1 a_2 \cdots)$.

I claim now that

$$C = \left\{x \in [0, 1] : (x)_3 \text{ has no 1's}\right\}.$$

Indeed, at each stage n of the construction of the Cantor set, we get rid of the segment of the real line that would correspond to the $a_n = 1$. One should note that $(x)_3$ not necessarily unique; for instance $\left(\frac{1}{3}\right)_3 = (1, 0, 0, ...) = (0, 2, 2, ...)$, but if we specifically consider all x such that there *exists* a base three representation with no 1's, i.e. like $\frac{1}{3}$, then C indeed captures all the desired numbers.

Thus, we have that

$$\text{card } (C) = \text{card } (\{\{a_n\} : a_n = 0, 2\}).$$

Define now the function

$$f : C \to [0, 1], \quad x \mapsto \sum_{n=1}^{\infty} \frac{a_n}{2} \frac{1}{2^n}, \text{where } (x)_3 = (a_n)$$

i.e., we "squish" the base-3 representation into a base-2 representation of a number. This is surjective; for any $y \in [0,1]$, $(b_n) := (y)_2$ contains only $0'$s and $1'$s, hence $(2b_n)$ contains only 0's and 1's, so let x be the number such that $(x)_3 = (2b_n)$. This necessarily exists, indeed, we simply take our definitions backwards:

$$x := \sum_{n=1}^{\infty} \frac{2b_n}{3^n},$$

which maps to y under f and is contained in C. Hence, $\mathrm{card}(C) \geq \mathrm{card}([0,1])$; but $[0,1]$ uncountable, and thus so must C. ■

We can naturally extend the function f to map the entire interval $[0,1] \to [0,1]$ as follows

$$f(x) := \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot \frac{1}{2^n} \text{ if } x \in C, (x)_3 = (a_n) \\ f(a) \text{ if } x \notin C \text{ then } x \in (a,b) \text{ s.t. } (a,b) \text{ removed from } [0,1] \end{cases}.$$

This function is often called the *Devil's Staircase* or *Cantor-Lebesgue function.*

↪**Proposition 1.16**:
1. $f(0) = 0, f(1) = 1, f \equiv \frac{1}{2}$ on $\left(\frac{1}{3}, \frac{2}{3}\right)$, $f \equiv \frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9}\right)$
2. $f : [0,1] \to [0,1]$ a surjection
3. f is nondecreasing
4. f is continuous

Proof. 1., 2., clear from construction.

For 3., let $x_1 < x_2 \in C$, and suppose $(x_1)_3 = (a_n), (x_2)_3 = (b_n)$. Then, since $x_1 < x_2$, it must be that a_n, b_n can only be equal up to some finite N; then the next $0 = a_{N+1} < b_{N+1} = 2$. Hence, it follows that the "modified binary expansion" that arises from f gives directly that $f(x_1) \leq f(x_2)$.

For 4., f is clearly continuous on $[0,1] - C$, since it is piecewise-constant here. Also, f is "one-sided continuous" at each of the "boundary points" $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \ldots$. If $x \in C$, for any $n \geq 1$, there must be $x_n, {x_n}'$ such that $x_n < x < {x_n}'$ (if $x = 0$, only need ${x_n}'$, if $x = 1$, only need x_n) and $f({x_n}') - f(x_n) \leq \frac{1}{2^n}$. Then, f is continuous at x by monotonicity of f. ■

1.8.2 Non-Measurable Sets?

We've shown then that there is indeed an uncountable set of measure 0. Another question we may ask ourselves is, is there a $A \subseteq \mathbb{R}$ that is non-measurable? The answer to this turns out to be yes, but the construction requires invoking the axiom of choice:

Axiom 1 (Of Choice): If Σ a collection of nonempty sets, then $\exists$ a function

$$S : \Sigma \to \bigcup_{A \in \sigma} A,$$

such that $A \in \sigma$, $S(A) \in A$. Such a function is called a *selection function*, and $S(A)$ a *representative* of A.

We construct now a non-measurable set, assuming the above. Consider $[0, 1]$, and define an equivalence relation $\sim$ on $[0, 1]$ by

$$a \sim b \Leftrightarrow a - b \in \mathbb{Q}.$$

Its easy to check that this is indeed an equivalence relation. Denote by E_a the equivalence class containing a, and set $\Sigma = \{E_a : a \in [0, 1]\}$. Note that for any $E_a \in \Sigma$, $E_a \neq \emptyset$.

Invoking the axiom of choice, we can select exactly one element S_a from E_a for each $E_a \in \Sigma$. Set

$$N := \{S_a : S_a \text{ is a representative of } E_a, E_a \in \Sigma\}.$$

↪**Proposition 1.17**: N, called a *Vitali set*, is non-measurable.

Proof. Assume towards a contradiction that N indeed measurable, $N \in \mathcal{M}$. Consider $[-1, 1] \cap \mathbb{Q}$; this is countable, so we can enumerate it $\{q_k\}$, $k \geq 1$. For each k, put

$$N_k := N + q_k.$$

By the assumption of measurability and translation invariance of m, it must be that each N_k measurable and has the same measure as N.

We claim each N_k disjoint. Assume not, then $\exists k \neq \ell$ (i.e. $q_k \neq q_\ell$) and $S_a, S_b \in N$ such that $S_a + q_k = S_b + q_\ell$. But then $S_a - S_b = q_\ell - q_k \in \mathbb{Q}$, hence $S_a \sim S_b$. But we constructed N to have only one representative from each equivalence class, hence it must be that $S_a = S_b$, and so $S_a + q_k = S_a + q_\ell \Rightarrow q_k = q_\ell$, contradicting the assumed distinctness of the q's; hence, the N_k's indeed disjoint.

We claim next that $[0, 1] \subseteq \bigcup_{n-1}^{\infty} N_k$. Let $x \in [0, 1]$. Then, $x \sim S_a$ for some unique $S_a \in N$ and so $x - S_a \in \mathbb{Q}$. But also, $x, S_a \in [0, 1]$, hence $x - S_a \in [-1, 1]$ (moreover, $x - S_a \in [-1, 1] \cap \mathbb{Q}$) and there must exist a k such that $x - S_a = q_k$, since the q_k's enumerate the entire $[-1, 1] \cap \mathbb{Q}$. Thus, $x \in N_k$ by the construction of the N_k's. Thus, $[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k$ indeed.

On the other hand, $\bigcup_{k=1}^{\infty} N_k \subseteq [-1, 2]$ and so we have the "bound"

$$[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k \subseteq [-1, 2].$$

Taking the measure of all sides then, we have the bound

$$1 \leq \mu\left(\bigcup_{n=1}^{\infty} N_k\right) \leq 3.$$

Invoking the disjointness of the N_k's, we can also use countable additivity to write

$$\mu\left(\bigcup_{n=1}^{\infty} N_k\right) = \sum_{k=1}^{\infty} m(N_k) = \sum_{k=1}^{\infty} m(N),$$

but this final line is a sequence of positive, constant real numbers; hence, it is impossible for it to be within 1 and 3, and we have a contradiction. Hence, N indeed not measurable.

Remark that this proof also shows that $m^*(N_k) > 0$ so $m^*(N) > 0$ (given the interval bound on N we've found). ■

↪**Proposition 1.18**: For every $A \in \mathcal{M}$ such that $m(A) > 0$, there exists $B \subseteq A$ such that B is non-measurable.

Proof. Assume otherwise, that there is a $A \in \mathcal{M}$ with $m(A) > 0$ such that any subset B of A is also measurable.

Remark that $A \subseteq \bigcup_{n\in\mathbb{Z}} A \cap [n, n+1]$. Then, there exists an n such that $m(A \cap [n, n+1]) > 0$ and thus, translating $A' := A \cap [n, n+1] - n)$, $m(A') > 0$, noting that $A' \subseteq [0, 1]$. Now, for any $B' \subseteq A'$, $B' + n \subseteq A$. By assumption, then $B' + n$ must be measurable so B' measurable.

In summary, then, we have $A' \subseteq [0, 1]$ with $m(A') > 0$ such that (by assumption) B' measurable for all $B' \subseteq A'$.

Let $N, \{q_k\}, N_k$ be as in the previous proof. Set

$$A_k{}' := A' \cap N_k, k \geq 1.$$

Then, $A_k{}'$ disjoint, and

$$A' = [0, 1] \cap A' \subseteq \bigcup_{k=1}^{\infty} (N_k \cap A') = \bigcup_{k=1}^{\infty} A_{k'}.$$

Since $m(A') > 0$, there exists a k such that $m(A_k{}') > 0$. Set, for this k,

$$L := \{\ell \geq 1 : q_\ell + q_k \in [-1, 1]\}.$$

This set is again countably infinite. We translate, obtaining a disjoint sequence of sets $\{q_\ell + A_k{}' : \ell \in L\}$; since $q_\ell + q_k \in [-1, 1] \cap \mathbb{Q}$, then $q_\ell + q_k = q_m$ for some unique m, and so $q_\ell + A_k{}' = q_\ell + A' \cap (N + q_k) \subseteq N_m$. Hence, we have on the one hand that by countable additivity

$$\bigcup_{\ell \in L} (q_\ell + A_k{}') \subseteq [-1, 2] \Rightarrow \sum_{\ell \in L} m(q_\ell + A_k{}') \leq 3,$$

and so it must be that $m(q_\ell + A_k{}') = m(A_k{}') = 0$ (else the series couldn't be finite), contradicting the finiteness assumption on $m(A_k{}')$. ■

1.8.3 Non-Borel Measurable Set?

We may ask, is there $A \in \mathcal{M}$ such that $A \notin \mathfrak{B}_{\mathbb{R}}$?

Let $f : [0, 1] \to [0, 1]$ be the Cantor-Lebesgue function, and put $g(x) = f(x) + x$; note that g is continuous and strictly increasing, and is defined $g : [0, 1] \to [0, 2]$. Remark that g bijective; the strictly increasing gives injective, and moreover $g(0) = 0, g(1) = 2$ hence by intermediate value theorem it is surjective. Hence, $g^{-1} : [0, 2] \to [0, 1]$ exists, and is also continuous, so in short g is a homeomorphism; it maps open to open, closed to closed. In particular, if $A \in \mathfrak{B}_{\mathbb{R}}$, then $g(A) \in \mathfrak{B}_{\mathbb{R}}$.

Recall that if (a, b) an open interval that gets removed from the construction of C, then f is constant and so g will map (a, b) to another open interval of the same length $b - a$. Thus,

$$m(g([0, 1] \setminus C)) = m([0, 1] \setminus C) = 1.$$

Hence, $m(g(C)) = 2 - 1 = 1 > 0$, since $g(C \cup [0, 1] \setminus C) = [0, 2]$. Hence, there exists a $B \subseteq G(C)$ such that $B \notin \mathcal{M}$, as per the previous proposition.

Let $A := g^{-1}(B)$; then $A \subseteq g^{-1}(g(C)) = C$. Since $m(C) = 0$, $A \in \mathcal{M}$ and $m(A) = 0$. But, $A \notin \mathfrak{B}_{\mathbb{R}}$; if it were, then $g(A) = B \in \mathfrak{B}_{\mathbb{R}}$, since g “maintains” Borel sets, but B is not even Lebesgue measurable and so this is a contradiction).