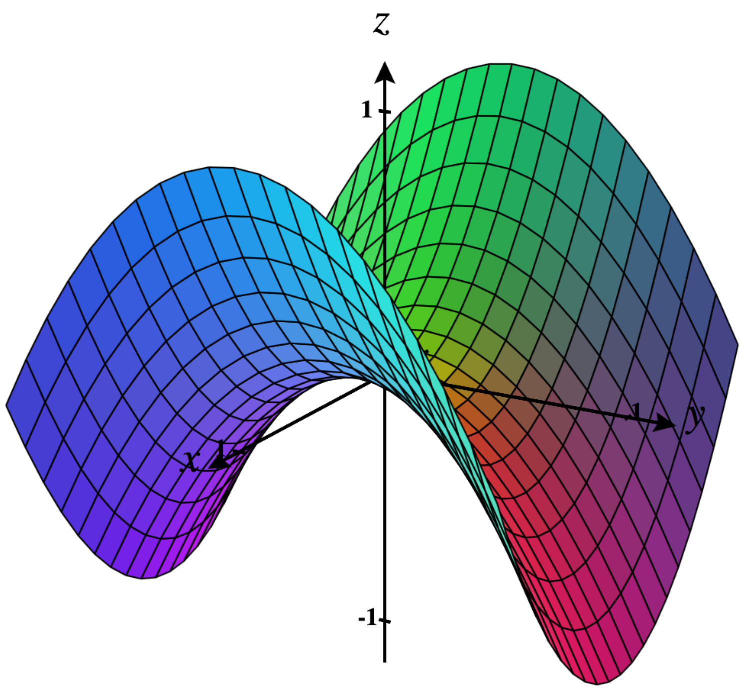
Calculus I, II, III



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Contents

1 Limits

1.1 Tricks for solving limits

1.1.1 $\frac{x}{x}$

In some situations, you can multiply a limit by some factor of $\frac{x}{x}$ to simplify an otherwise "impossible" limit

eg:

$$\begin{aligned}\lim_{x\to\infty}\frac{x+3}{\sqrt{9x^2-5x}} &= \lim_{x\to\infty}\frac{x+3}{\sqrt{9x^2-5x}}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} \\ &= \lim_{x\to\infty}\frac{\frac{x}{x}+\frac{3}{x}}{\sqrt{(\frac{1}{x^2})(9x^2-5x)}} \\ &= \lim_{x\to\infty}\frac{1+\frac{3}{x}}{\sqrt{9-\frac{5}{x}}}\end{aligned} \tag{1}$$

The terms $\frac{3}{x}$ and $\frac{5}{x}$ both tend to 0 as $x\to\infty$, so the final limit simplifies:

$$\begin{aligned}\lim_{x\to\infty}\frac{1}{\sqrt{9}} &= \frac{1}{\sqrt{9}} \\ &= \frac{1}{3}\end{aligned} \tag{2}$$

1.1.2 Intuition with ∞

In limits involving fractions where both numerator and denominator are polynomials, you can intuitively calculate the limit using the coefficients/powers.

$$\lim_{x\to\infty}\frac{ax^m}{bx^n}$$

1. $\mathbf{m} > \mathbf{n} : \infty$
2. $\mathbf{m} < \mathbf{n} : 0$
3. $\mathbf{m} = \mathbf{n} : \frac{m}{n}$ *(since both of the powers are the same "strength" ie ∞, you can think of this as the infinities "cancelling out"). To prove this idea actually works, use the method above.*

1.1.3 L'Hôpital's Rule

In short, if:

$$\frac{f(c)}{g(c)} = \frac{0}{0}, \frac{\infty}{\infty}, 0^0, \infty-\infty, \ldots(\textbf{indeterminate form})$$

then:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

This can also be repeated for multiple derivatives of f, g, as long as the resulting form is still indeterminate. Often times, you will need to manipulate the limit in order to express in an indeterminate form such that L'Hôpital's is usable.

While solving limits this way is very powerful (and, often, very easy), one should be careful to not overuse it when it is not applicable.

1.1.4 Solving for a given limit

When given a function with variables in place of constants and a final limit to solve for, there are some additional considerations to take into account.

eg:

$$\lim_{x\to 0}\frac{\sqrt{ax+b}-2}{x} = 1$$

In order for this limit to even exist, $\lim_{x\to 0}\sqrt{ax+b}-2$ MUST equal 0, following the quotient rule of limits, which states:

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$$

So:

$$\begin{aligned}\lim_{x\to 0}\sqrt{ax+b}-2 &= 0\\ \sqrt{a*0+b}-2 &= 0\\ \sqrt{(b)} &= 2\\ b &= 4\end{aligned} \tag{3}$$

To finish solving this limit, see:

1.1.5 "Radicalizing" fraction

Similar to the first point, but a little more complex/situation-specific.

When a limit is in the form:

$$\lim_{x\to 0}\frac{\sqrt[n]{(ax+b)}+c}{x}$$

The issue here is that we have an x alone on the bottom, which results in something being divided by 0. To solve this, we can radicalize the fraction; this results in an x ”alone”, cancelling with that on the bottom, and make the equation solvable.

For the example above:

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{ax+4}-2}{x} &= 1\\
\lim_{x\to 0}\frac{\sqrt{ax+4}-2}{x}\cdot\frac{\sqrt{ax+4}+2}{\sqrt{ax+4}+2} &= 1\\
\lim_{x\to 0}\frac{ax}{x(\sqrt{ax+4}+2)} &= 1\\
\lim_{x\to 0}\frac{a}{\sqrt{ax+4}+2} &= 1\\
&\dots\\
a &= 4
\end{aligned} \tag{4}$$

1.1.6 Squeeze Theorem

If:

- $f(x) \leq g(x) \leq h(x)$
- $\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$

then $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \lim_{x\to c} h(x)$

This makes a lot more sense in practice:

$$\begin{aligned}
\lim_{x\to 0} x^4 sin(\frac{x}{2}) &=?\\
-1 \leq sin(\frac{x}{2}) &\leq 1\\
-x^4 \leq x^4 sin(\frac{x}{2}) &\leq x^4\\
\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^4 = 0 \therefore \lim_{x\to 0} x^4 sin(\frac{x}{2}) &= 0
\end{aligned} \tag{5}$$

See Figure 1 to see this visually.

1.1.7 Using the definition of a derivative

A derivative, $f'(x)$, is defined as followed:

$$f'(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

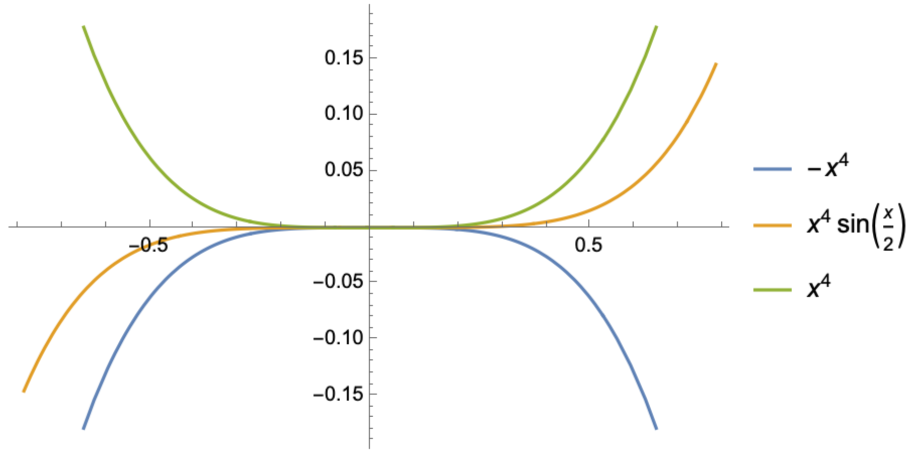

Figure 1: The squeeze theorem visualized

Or, alternatively, given $h = x - a$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$$

If you are given a limit in the form (or near) the above forms, you can use these definitions to solve the limit.

eg:

$$\begin{aligned}
\lim_{x \to \pi} \frac{e^{sin(x)} - 1}{x - \pi} &= \lim_{x \to \pi} \frac{e^{sin(x)} - e^{sin(\pi)}}{x - \pi} \\
\textit{Let } f(x) = e^{sin(x)} & \\
&= \lim_{x \to \pi} \frac{f(x) - f(\pi)}{x - \pi} \\
\textit{This is simply the definition of } f'(x) \textit{ at } a = \pi & \\
&= f'(\pi) = cos(\pi) e^{sin(\pi)} = -1
\end{aligned} \tag{6}$$

1.2 Intermediate Value Theorem

If

- f is continuous on $[a, b]$
- $f(a) \neq f(b)$
- $f(a) <= N <= f(b)$

there exists some c in (a, b) such that $N = f(c)$.

See Figure 2.

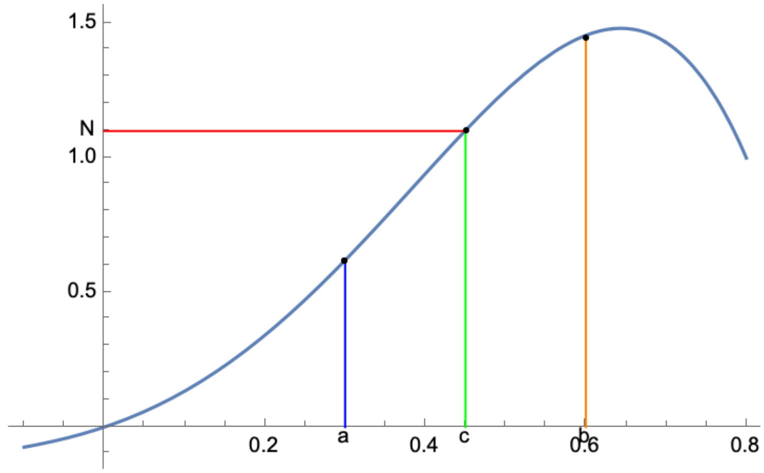

Figure 2: The intermediate value theorem visualized

2 Exponential and Logarithmic Functions

2.1 Exponential Functions

2.1.1 Defining e

An exponential function is any $f(x)$ of the form $f(x) = a^x$. To find the derivative, $f'(x)$:

$$\begin{aligned} f'(x) &= \lim_{h \to 0}[\frac{a^{x+h} - a^x}{h}] \\ &= a^x \lim_{h \to 0}[\frac{a^h - 1}{h}] \end{aligned} \tag{7}$$

We can rewrite this for the derivative of f at $x = 0$:

$$\begin{aligned} f'(0) &= a^0 \lim_{h \to 0}[\frac{a^h - 1}{h}] \\ &= \lim_{h \to 0}[\frac{a^h - 1}{h}] \end{aligned} \tag{8}$$

We can use this to find for what value of a that $f'(0) = f(0)$. $f(0) = a^0 = 1$, therefore $f'(0) = f(0) = 1 = \lim_{h \to 0}[\frac{a^h - 1}{h}]$.

$$
\begin{aligned}
1 &= \lim_{h\to 0}[\frac{a^h-1}{h}] \\
\mathbf{a=2}: \lim_{h\to 0}[\frac{2^h-1}{h}] &= 0.693... \\
\mathbf{a=3}: \lim_{h\to 0}[\frac{2^h-1}{h}] &= 1.099...
\end{aligned}
\tag{9}
$$

By the Intermediate Value Theorem, we know that $2 < a < 3$, which can be defined by e.

$$
1 = e^0 = \lim_{h\to 0}[\frac{e^h-1}{h}] \tag{10}
$$

Extending this further for e;

$$
\begin{aligned}
\frac{d}{dx}e^x &= \lim_{h\to 0}[\frac{e^{x+h}-e^x}{h}] \\
&= e^x \lim_{h\to 0}[\frac{e^h-1}{h}] \\
\textit{Substituting from eqn. 10} \\
&= \mathbf{e^x}
\end{aligned}
\tag{11}
$$

2.1.2 Generalizing $\frac{d}{dx}a^x$

From eqn. 8:

$$
\frac{d}{dx}a^x = a^x \lim_{h\to 0}\frac{a^h-1}{h} = f'(0)a^x
$$

We can rewrite using rules of logs:

$$
\frac{d}{dx}a^x = \frac{d}{dx}e^{\ln a^x} = \frac{d}{dx}e^{x\ln a}
$$

We can let $u = x\ln a$, and thus $a^x = e^u$. We can then use implicit differentiation as follows:

$$
\begin{aligned}
\frac{d}{dx}a^x &= \frac{d}{dx}e^u \\
&= \frac{du}{dx}\frac{d}{du}[e^u] \\
\textit{From eqn. 11} & \\
&= e^u\frac{du}{dx} \\
&= e^u\frac{d(x\ln a)}{dx} \textit{Differentiating in terms of } x \\
&= e^u \ln a \\
&= e^{\ln a^x} \ln a \\
&= a^x \ln a
\end{aligned}
\tag{12}
$$

Thus, $\frac{d}{dx}a^x = a^x \ln a$.

2.2 Logarithmic Functions

Let $f(x) = y = \log_a x$, so $x = a^y$. To find the derivative of $f(x)$:

$$
\begin{aligned}
\frac{d}{dx}x &= \frac{d}{dx}a^y \\
1 &= [\frac{d}{dy}a^y]\frac{dy}{dx} \\
\textit{Using eqn. 12}: & \\
1 &= a^y \ln a\frac{dy}{dx} \\
\frac{1}{a^y \ln a} &= \frac{dy}{dx} \\
\textit{Subbing in for } y: & \\
\frac{1}{a^{log_a x} \ln a} &= \frac{d}{dx}log_a x \\
\frac{1}{x \ln a} &= \frac{d}{dx}\log_a x
\end{aligned}
\tag{13}
$$

Therefore, $f'(x) = \frac{d}{dx}log_a x = \frac{1}{x\ln a}$.

When $a = e$:

$$\begin{aligned}\frac{d}{dx}log_e x &= \frac{d}{dx}\ln x \\ &= \frac{1}{x\ln e} \\ &= \frac{1}{x}\end{aligned} \tag{14}$$

2.3 Obtaining e as a limit

Let $f(x) = \ln x, \therefore\ f(1) = \ln 1 = 0$. In addition, $f'(x) = \frac{1}{x}, \therefore\ f'(1) = 1/1 = 1$. We can use the definition of a limit to say the following:

$$\begin{aligned}1 &= \lim_{h\to 0}[\frac{f(1+h)-f(1)}{h}] \\ &= \lim_{h\to 0}\frac{1}{h}[f(1+h)-f(1)] \\ &= \lim_{h\to 0}\frac{1}{h}[\ln(1+h)-\ln 1] \\ &= \lim_{h\to 0}\frac{1}{h}\ln(1+h) \\ &= \lim_{h\to 0}\ln[(1+h)^{\frac{1}{h}}] \\ &= \ln[\lim_{h\to 0}[(1+h)^{\frac{1}{h}}]] \\ &\therefore \lim_{h\to 0}[(1+h)^{\frac{1}{h}}] = e\end{aligned} \tag{15}$$

Note that the penultimate step of this work relies on the following (very helpful) theorem:

If f is continuous at $\lim_{x\to a} g(x)$, then,

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x))$$

A proof can be found for this on page A39 of the Stewart's book.

Alternatively, we can write e as a limit going to ∞; Let $y = \frac{1}{h}$, and substituting into eqn. 15:

$$\lim_{y\to\infty}[1+\frac{1}{y}]^y = e \tag{16}$$

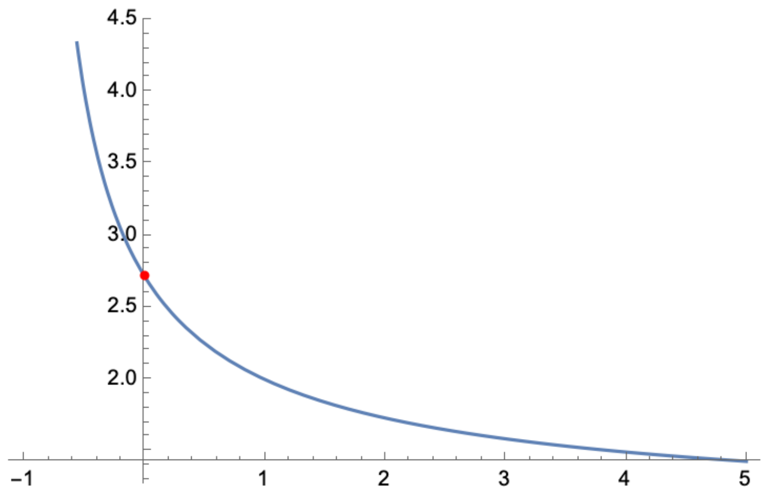

Figure 3: Eqn. 15 visualized

3 Linearization

3.1 Linear Approximation

We can approximate a function by using the equation for its tangent at a particular point.

For example; For a function $y = f(x)$, there exists a point $P = (a, f(a))$. Let $L(x)$ be the line tangent to $f(x)$ at P. The slope of L would be defined as:

$$m_L = \frac{\Delta y}{\Delta x} = \frac{L(x) - f(a)}{x - a} = f'(a)$$

Solving for $L(x)$:

$$\begin{aligned} L(x) - f(a) &= (x-a)f'(a) \\ L(x) &= f(a) + (x-a)f'(a) \end{aligned} \tag{17}$$

This function is a fairly good approximation of $f(x)$ near $x = a$, at least for most continuous functions. It's also very important to note that, by definition $f(a) = L(a)$, AND $f'(a) = L'(a)$. This will be expanded later.

For example; take $f(x) = sin(x)$, @ $a = 0$:

$$\begin{aligned} f(x) \approx L(x) &= f(0) + (x-0)f'(0) \\ &= f(0) + x * cos(0) \\ &= 0 + x * 1 \\ &= x = L(x) \end{aligned} \tag{18}$$

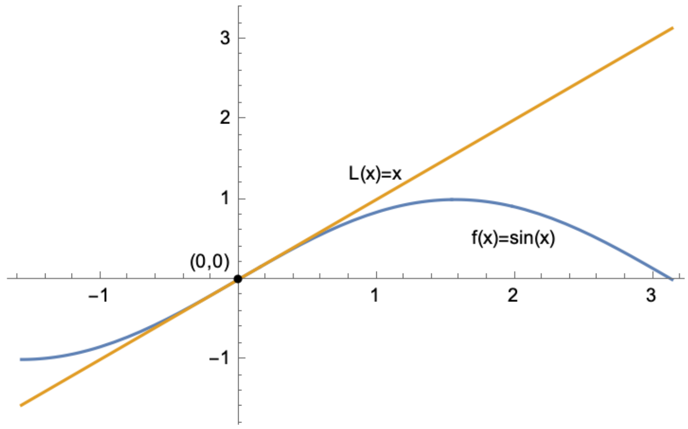

Figure 4: Linearization of $sin(x)$ at $x = 0$

Clearly, in the vicinity of a, this is a pretty good approximation, but quickly becomes inaccurate.

3.2 Quadratic Approximation

From the section earlier, we defined $L(x)$, a linear approximation of $f(x)$, where $f(a) = L(a)$ AND $f'(a) = L'(a)$. Now, what if we wanted to continue this trend, and define a new function, $P_2(x)$, where $f(a) = P_2(a)$, $f'(a) = P_2'(a)$, and $f''(a) = P_2''(a)$? Rather than a "linear" approximation, this would, logically, become a "quadratic" approximation. We can say:

$$\begin{aligned}
P_2(x) &= f(a) + (x-a)f'(a) + {\color{red}k(x-a)^2} \\
P_2(x) &= L(a) + k(x-a)^2 \\
P_2'(x) &= L'(a) + 2k(x-a) \\
P_2''(a) &= 2k \\
k &= \frac{P_2''(a)}{2} = \frac{f''(a)}{2}
\end{aligned} \tag{19}$$

Note that $L''(a) = 0$; since $L(x)$ is a line, its first derivative is a constant, and its second is 0.

The purpose of this work was to find some k that we can multiple by a quadratic factor $(x-a)^2$ to maintain the previously stated desired properties of P_2. As such, we can rewrite:

$$P_2(x) = f(a) + (x-a)f'(a) + {\color{red}\frac{(x-a)^2}{2}f''(a)}$$

3.2.1 Taylor Polynomials

We can, logically, continue this trend for cubic, quartic, etc. approximations. In general, you can approximate any function $f(x)$ to the Nth power;

$$f(x) \approx P_N(x) = \sum_{n=0}^{N} \frac{(x-a)^n f^{(n)}(a)}{n!}$$

This approximation:

- passes through the point $(a, f(a))$
- has the same slope, concavity, ..., ie $P_N^{N-1}(a) = f_N^{N-1}(a)$

This Nth degree polynomial is called a **Taylor Polynomial**, and, intuitively, better approximates $f(x)$ as $N \to \infty$.

When $a = 0$, this polynomial becomes called a **Maclaurin Polynomial**:

$$P_n(x) = \sum_{n=0}^{N} \frac{x^n f^n(0)}{n!}$$

3.2.2 Relationship between $f(x)$ and $P_N(x)$

While, intuitively, it makes sense that $P_N(x)$ becomes a better approximation of $f(x)$ as $N \to \infty$, this isn't necessarily true.

Taylor's Theorem states the following:

$$f(x) = P_N(x) + R_N(x);$$

where $P_N(x)$ is an approximation (via Taylor Polynomial) of $f(x)$, and $R_N(x)$ is the error in said approximation, defined as:

$$R_N(x) = \frac{(x-a)^{N+1}}{(N+1)!} f^{(N+1)}(c), c \in \mathbb{R} \textit{ and between } x, a$$

.

We can then rewrite Taylor's Theorem, as N trends to ∞:

$$f(x) = \lim_{N\to\infty} [\sum_{n=0}^{N} \frac{(x-a)^n f^N(a)}{n!} + \frac{(x-a)^{N+1}}{(N+1)!} f^{(N+1)}(c)]$$

Therefore, $f(x) = P_N(x)$ if, and only if, $\lim_{N\to\infty} R_N(x) = 0$. Logically, this should make sense; if there is no remainder, then the approximation is equal to the original function.

3.3 Taylor Polynomials, Applied Example

Using Taylor Polynomials, we can explore a number of interesting properties of functions, and even derive some unique expressions. For instance (assuming $R_N(x) = 0$):

$$
\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + ... \\
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + ... \\
e^{ix} &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} ... + i[x - \frac{x^3}{3!} + \frac{x^5}{5!} ...]
\end{aligned}
\tag{20}
$$

This may look like it just made things more complicated. However, take a look at the Taylor Polynomial representations of $sin(x)$:

$$
\begin{aligned}
f(x) = \sin(x) &= \sum_{n=0}^{\infty} \frac{(x-a)^n f^n(a)}{n!} \\
f'(x) &= \cos(x) \\
f''(x) &= -\sin(x) \\
f'''(x) &= -\cos(x) \\
f^{IV}(x) &= \sin(x) \\
\therefore\ f(x) &= 0 + x + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + ... \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\end{aligned}
\tag{21}
$$

and, very similarly, for $\cos x$:

$$
\begin{aligned}
f(x) &= \cos(x) \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\end{aligned}
\tag{22}
$$

Thus, we can rewrite equation 22 by substituting in sin and cos, for the odd and even factors respectively:

$$
e^{ix} = \cos(x) + i\sin(x) \tag{23}
$$

This is a very helpful equation, known as **Euler's Formula**. Some applications are present in Section 5.

From here, we can solve for $x = \pi$, to obtain:

$$\begin{aligned} e^{i\pi} &= \cos(\pi) + i\sin(\pi) \\ &= -1 + i * 0 \\ e^{i\pi} + 1 &= 0 \end{aligned} \tag{24}$$

This (”the most beautiful formula in mathematics”) is known as **Euler’s Identity**.

To take this formula further we can solve for $\sin(x)$ and $\cos(x)$ in the imaginary plane.

$$\begin{aligned} e^{ix} = \cos(x) + i\sin(x) \implies \cos(x) &= e^{ix} - i\sin(x) \\ e^{-ix} = \cos(x) - i\sin(x) \implies \cos(x) &= e^{-ix} + i\sin(x) \\ e^{ix} - i\sin(x) &= e^{-ix} + i\sin(x) \\ e^{ix} - e^{-ix} &= 2i\sin(x) \\ \sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} \\ \cos(x) &= e^{ix} - i\sin(x) \\ \cos(x) &= e^{ix} - i(\frac{e^{ix} - e^{-ix}}{2i}) \\ \cos(x) &= \frac{e^{ix} + e^{-ix}}{2} \end{aligned} \tag{25}$$

From here, we can solve for $\cos(ix)$ and $\sin(ix)$:

$$\begin{aligned} \cos({\color{red}i}x) &= \frac{e^{i^2x} + e^{-i(ix)}}{2} \\ &= \frac{e^{x} + e^{-x}}{2} = \cosh(x) \\ {\color{red}i}\sin(x) &= i(\frac{e^{i^2x} - e^{-i(ix)}}{2}) \\ &= i(\frac{e^{x} - e^{-x}}{2}) = i\sinh(x) \end{aligned} \tag{26}$$

And thus, we can define the hyperbolic trigonometric functions using the complex plane, Taylor Series, and Euler’s Identity.

4 Trigonometry

4.1 Trigonometric Identities

There are many, many trigonometric identities that are often (some more than others) useful. See here for a list of quite a few, accompanied by proofs.

4.2 Inverse Trig Functions

As with all functions, we can define the inverse of the basic trig functions, as the functions that undo their respective trig functions. The following shows each of the three basic trig functions, their inverses, and how to derive their respective derivatives.

- $y = \arcsin x = sin^{-1} x$

$$\begin{aligned} x &= \sin y \\ \frac{d}{dx} x &= \frac{d}{dx} \sin y \\ 1 &= \frac{d}{dy}[\sin y] \frac{dy}{dx} \\ 1 &= \cos y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\cos y} \\ &= \frac{1}{\sqrt{1 - \sin^2(y)}} = \frac{1}{\sqrt{1 - x^2}} \end{aligned} \tag{27}$$

- $y = \arccos x = cos^{-1} x$

$$\begin{aligned} x &= \cos y \\ \frac{d}{dx} x &= \frac{d}{dx} \cos y \\ 1 &= \frac{d}{dy}[\cos y] \frac{dy}{dx} \\ 1 &= -\sin y \frac{dy}{dx} \\ \frac{dy}{dx} &= -\frac{1}{\sin y} \\ &= -\frac{1}{\sqrt{1 - \cos^2(y)}} = -\frac{1}{\sqrt{1 - x^2}} \end{aligned} \tag{28}$$

- $y = \arctan x = \tan^{-1} x$

$$\begin{aligned} x &= \tan y \\ \frac{d}{dx}x &= \frac{d}{dx}\tan y \\ 1 &= \frac{d}{dy}\tan y \frac{dy}{dx} \\ 1 &= \sec^2 y \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sec^2 y} \\ &= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \end{aligned} \tag{29}$$

4.3 Hyperbolic Functions

4.3.1 Definitions, Identities, Derivatives

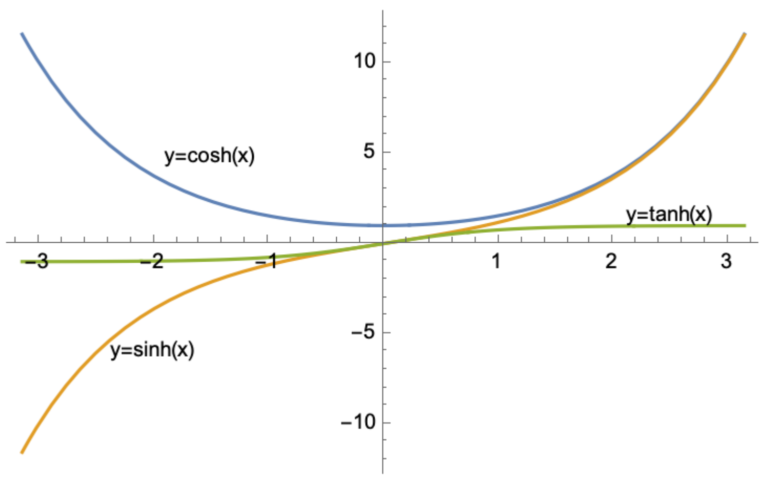

Figure 5: $\cosh(x), \sinh(x), \tanh(x)$

Def:

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} \\ \cosh x &= \frac{e^x + e^{-x}}{2} \end{aligned} \tag{30}$$

Hyperbolic trigonometric functions are the ”hyperbolic equivalent” of the basic trig functions. This is an admittedly abstract definition, but there is a more concrete definition in section 3.3, using Taylor Polynomials and complex numbers.

The derivatives are as follows; for $\sinh x$:

$$\frac{d}{dx}\sinh x = \frac{d}{dx}[\frac{e^x - e^{-x}}{2}] = \frac{e^x + e^{-x}}{2} = \cosh x;$$

and for $\cosh x$:

$$\frac{d}{dx}\cosh x = \frac{d}{dx}[\frac{e^x + e^{-x}}{2}] = \frac{e^x - e^{-x}}{2} = \sinh x.$$

Note that this is not quite the same relationship between the derivatives of $\sin x$ and $\cos x$.

Similarly, there exist several properties of hyperbolic functions that, while similar in appearance to their equivalent trig functions, have some key differences.

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= 1 \\ \sinh(x+y) &= \sinh x \cosh y + \cosh x \sinh y \\ \cosh(x+y) &= \sinh x \sinh y + \cosh x \cosh y\end{aligned} \tag{31}$$

We can also define $\tanh x$:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} = \frac{e^x - e^{-x}}{e^x + e^{-x}};$$

and its derivative:

$$\frac{d}{dx}\tanh x = \frac{d}{dx}[\frac{\sinh x}{\cosh x}] = \frac{\sinh x \sinh x - \cosh x \cosh x}{\cosh^2} = \frac{\sinh^2 x - \cosh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = sech^2 x.$$

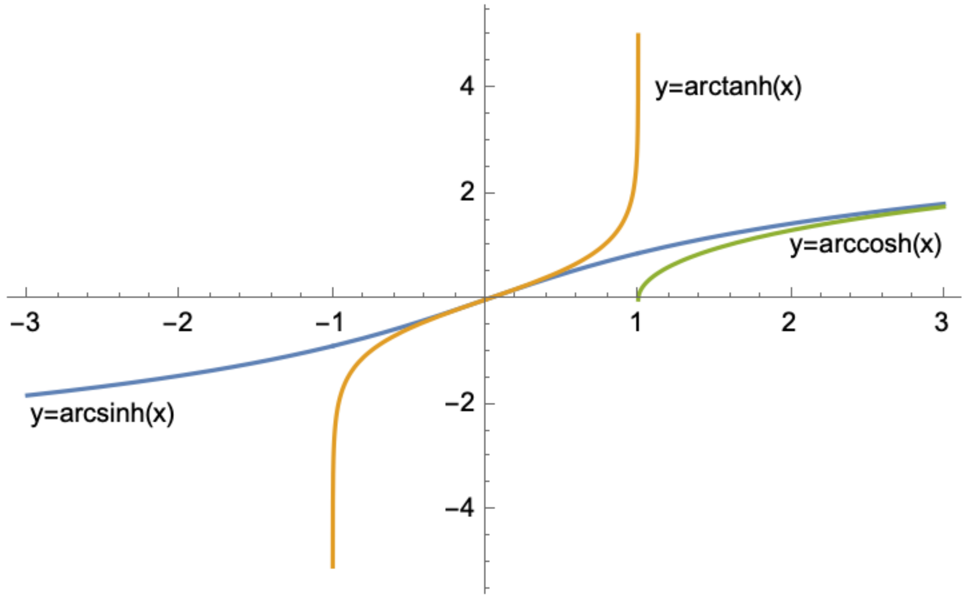

Figure 6: Inverse Hyperbolic Functions

4.3.2 Inverse Hyperbolic Functions and their Derivatives

$$
\begin{aligned}
y = \sinh^{-1}(x) \Rightarrow x &= \sinh(y) \\
\frac{d}{dx}x &= \frac{d}{dx}\sinh(y) \\
1 &= \frac{dy}{dx}\cosh(y) \\
\frac{dy}{dx} &= \frac{1}{\cosh(y)} = \frac{1}{\sqrt{1+\sinh^2(x)}} = \frac{1}{\sqrt{1+x^2}} \\
y = \cosh^{-1}(x) \Rightarrow x &= \cosh(y) \\
\frac{d}{dx}x &= \frac{d}{dx}\cosh(y) \\
1 &= \frac{dy}{dx}\sinh y \\
\frac{dy}{dx} &= \frac{1}{\sinh y} = \frac{1}{\sqrt{cosh^2(y)-1}} = \frac{1}{\sqrt{x^2-1}} \\
y = \tanh^{-1}(x) \Rightarrow x &= \tanh(y) \\
\frac{d}{dx}x &= \frac{d}{dx}\tanh(y) \\
1 &= \frac{d}{dy}\tanh(y)\frac{dy}{dx} \\
1 &= sech^2(y)\frac{dy}{dx} \\
\frac{dy}{dx} &= \frac{1}{sech^2(y)} = \frac{1}{1-\tanh^2(y)} = \frac{1}{1-x^2}
\end{aligned}
\tag{32}
$$

Note that for $\cosh^{-1}(x)$, the domain of $\cosh(x)$ must be restricted to $x >= 0$ so that $\cosh^{-1}(x)$ is a valid one-to-one function. You can see this graphically in figure 6.

4.3.3 Explicitly Expressing Inverse Hyperbolic Functions

Using the rules of natural logarithms, we can re-express each inverse hyperbolic function explicitly in terms of x.

$$
\begin{aligned}
y = \sinh^{-1}(x) \Rightarrow x = \sinh y &= \frac{e^y - e^{-y}}{2} \\
2x &= e^y - e^{-y} \\
2xe^y &= (e^y - e^{-y})e^y \\
2xe^y &= e^{2y} - 1 \\
0 &= e^{2y} - 2xe^y - 1
\end{aligned}
\tag{33}
$$

This is just a quadratic formula in terms of e^y, and can be solved accordingly:

$$
\begin{aligned}
e^y &= \frac{2x \pm \sqrt{4x^2 + 4}}{2} \\
e^y &= x + \sqrt{x^2 + 1}
\end{aligned}
\tag{34}
$$

We don't take the negative, as e^y is always > 0.

$$
\begin{aligned}
\ln(e^y) &= \ln(x + \sqrt{x^2 + 1}) \\
y &= \sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})
\end{aligned}
\tag{35}
$$

A very similar process follows for cosh and tanh, so it won't be shown, but for reference:

$$
\begin{aligned}
y &= \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \\
y &= \tanh^{-1} x = \frac{1}{2} \ln(\frac{1 + x}{1 - x})
\end{aligned}
\tag{36}
$$

5 Complex Numbers

5.1 Complex Quadratic Equations

To introduce the concept of complex numbers, take an arbitrary quadratic equation of the form

$$\alpha x^2 + \beta x + \gamma = 0, \alpha, \beta, \gamma \in \mathbb{R}$$

.

We can solve this for x as follows:

$$x = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

.

If the **discriminant**, $\beta^2 - 4\alpha\gamma$, is < 0, then the equation has no real roots. However, it does have two imaginary roots, of the form

$$\begin{aligned} x &= \frac{-\beta \pm \sqrt{(-4\alpha)(-\beta^2)}}{2\alpha} \\ &= \frac{-\beta}{2\alpha} \pm i \frac{\sqrt{4\alpha\gamma - \beta^2}}{2\alpha} \\ &= a \pm ib; a, b \in \mathbb{R}. \end{aligned} \tag{37}$$

This final, simplified form, $a \pm ib$, is also known as "complex conjugates", which has one part (a) that is real, and one part (ib) that is imaginary. Generally, this is written as $z = a + ib$

5.2 De Moivre's Formula and Related

Take the equation from the previous section, $z = a + ib$. Figure 7 shows this formula in the complex plane, with r representing the distance between z and the origin $(0, 0)$.

Note that the complex plane is also often referred to as the ***Argand plane****.*

Using this visualization, we can rewrite the formula in the polar form

$$z = r \cos\theta + ir \sin\theta$$

.

We can use **Euler's formula** to simplify this;

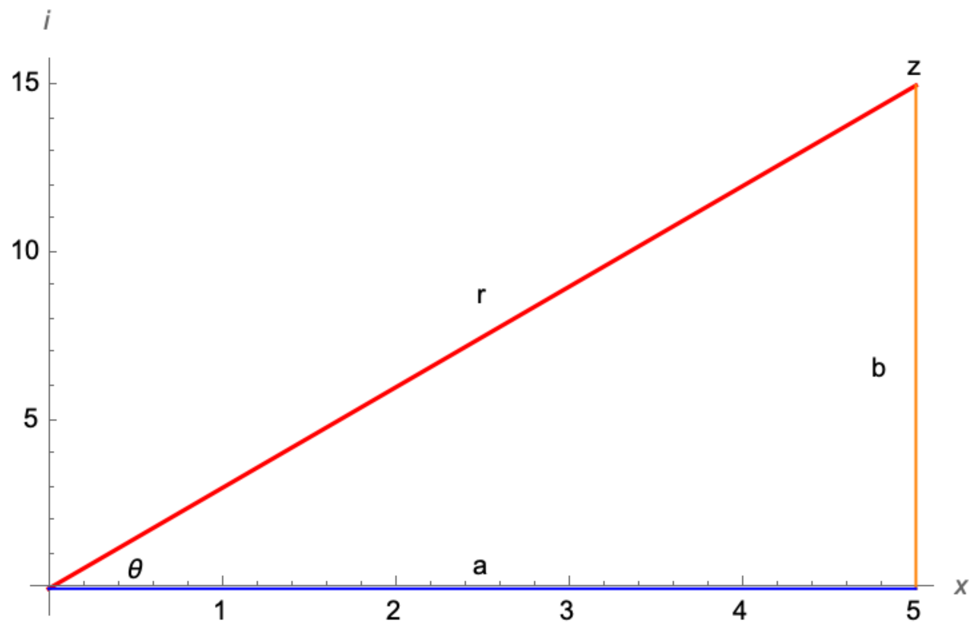

Figure 7: The complex plane

$$
\begin{aligned}
z &= r(\cos\theta + i\sin\theta) \\
z &= re^{i\theta}
\end{aligned}
\tag{38}
$$

From this definition, we can define several other properties to aid in solving of equations involving complex numbers.

- **Products:** $z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$
- **Quotients:** $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$
- **Powers:** $z^n = r^n e^{in\theta}, n \in \mathbb{Z}$

Example; **calculate** $(-1-i)^{20}$. See figure 8 for a visual representation.

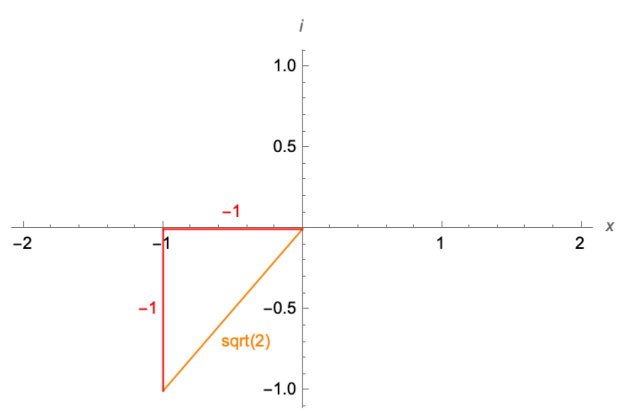

Figure 8: $(-1-i)$ represented on the complex plane

$$\begin{aligned}(-1-i)^{20} = z^n &= r^n e^{in\theta} \\ &= \sqrt(2)^{20} e^{i20*\frac{5\pi}{4}} \\ &= 2^{10} e^{25\pi i} \\ &= 1024 e^{24\pi i} e^{\pi i} \\ &= 1024 * (1)(-1) \\ &= -1024\end{aligned} \tag{39}$$

Note that the second to last step is done using Euler's Identity; specifically:

$$\begin{aligned}e^{i\pi} &= -1 \\ e^{ni\pi} &= e^{(i\pi)^n} = -1^n, n \in \mathbb{Z} \\ n &= \{0, 2, 4, ...\}, e^{ni\pi} = 1 \\ n &= \{1, 3, 5, ...\}, e^{ni\pi} = -1\end{aligned} \tag{40}$$

5.2.1 Roots

We can use some of the identities above to find a formula to find the real AND complex roots of number:

$$\begin{aligned}z &= re^{i\theta} = re^{i(\theta+2k\pi)}, k \in \mathbb{Z} \geq 0 \\ z^{\frac{1}{n}} &= z^{\frac{1}{n}} e^{\frac{i(\theta+2k\pi}{n}}, k = 0, 1, ..(n-1)\end{aligned} \tag{41}$$

For example, to find all of the cube roots of 8:

$$\begin{aligned}z &= 8 = 8e^{2k\pi i} \\ z^{\frac{1}{3}} &= 8^{\frac{1}{3}} = 8^{\frac{1}{3}} e^{\frac{2k\pi i}{3}}, k = 0, 1, 2 \\ z_{k=0} &= 8^{\frac{1}{3}} e^0 = 2 \\ z_{k=1} &= 8^{\frac{1}{3}} e^{\frac{2\pi i}{3}} = 2e^{\frac{2\pi}{3}} = 2[cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}] = 2[-\frac{1}{2} + i\frac{\sqrt(3)}{2}] = -1 + i\sqrt{3} \\ z_{k=2} &= 8^{\frac{1}{3}} e^{\frac{4\pi i}{3}} = ... = -1 - i\sqrt{3}\end{aligned} \tag{42}$$

Thus, the cube roots of 8 are $2, -1 + i\sqrt{3}, -1 - i\sqrt{3}$.

Another method of solving for the roots of a complex number (which can also be extended to other problems involving complex numbers, but is most obvious when calculating roots) involves a little more intuition rather than just using the polar form immediately.

Say you are solving for the square roots of $-8-15i$, ie $(-8-15i)^{\frac{1}{2}}$. We can then say:

$$\begin{aligned}(-8-15i)^{\frac{1}{2}} &= a+bi, a, b \in \mathbb{R} \\ -8-15i &= (a+bi)^2 = a^2+2abi-b^2 = a^2-b^2+2abi \\ &\Rightarrow -8 = a^2-b^2 \\ &\Rightarrow -15 = 2ab\end{aligned} \tag{43}$$

Logically, the roots of a complex number involve some real part (a) and some imaginary part (bi). Using this knowledge, we can use the above steps to find a and b using more basic algebra. Doing the work out fully will result in a quadratic formula for a and b, which can finally be solved for the values of the roots (the ± in the quadratic formula allows for the creation of two roots).

5.2.2 Logarithms

We can use some of the identities above to also find a logs involving complex numbers:

$$\begin{aligned}z &= re^{i\theta} \\ \ln(z) &= \ln(re^{i\theta}) \\ &= \ln(r)+\ln(e^{i\theta}) \\ &= \ln(r)+i\theta\end{aligned} \tag{44}$$

For example, to find $\ln(-1)$:

$$\begin{aligned}\ln(-1) &= \ln(1)+i(\pi+2n\pi) \\ &= i(\pi+2n\pi), n = 0, \pm 1, \pm 2, \ldots\end{aligned} \tag{45}$$

6 Applications of Differentiation

6.1 Related Rates

"Related rates" problems are very common, and often follow very similar patterns that can be exploited to make individual problems very straightforward. Generally:

1. Draw a picture to represent the scenario
2. Write a formula involving the variable who's change is given, in relation to the variable who's change you're solving for
   (a) Many times, said formula will have to be simplified/rewritten in terms of a single variable. This often involves similar triangles in geometric questions.
3. Differentiate, then solve accordingly
   (a) This pretty much always involves implicit differentiation

The best way to approach these questions is to simply practice. The patterns between questions will quickly become obvious.

6.2 Rolle's Theorem

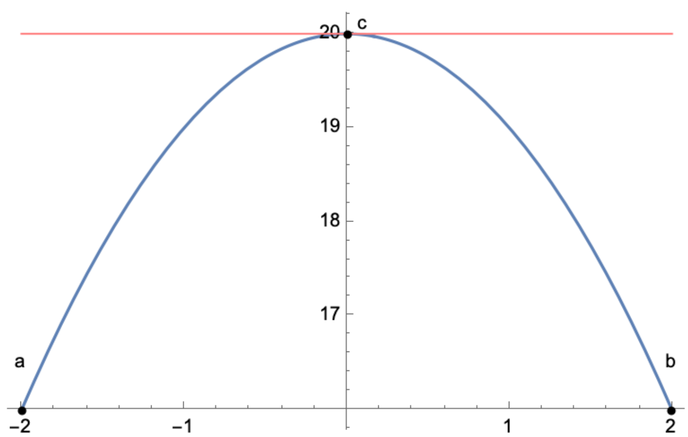

Figure 9: Rolle's Theorem visualized

If a function f

- is *continuous* on $[a, b]$
- is *differentiable* on (a, b)
- $f(a) = f(b)$

then there is a number c in (a, b) such that $f'(c) = 0$.

The proof for this theorem is fairly straightforward (and intuitive), and also makes a lot of sense graphically (see figure 9).

6.3 Mean Value Theorem

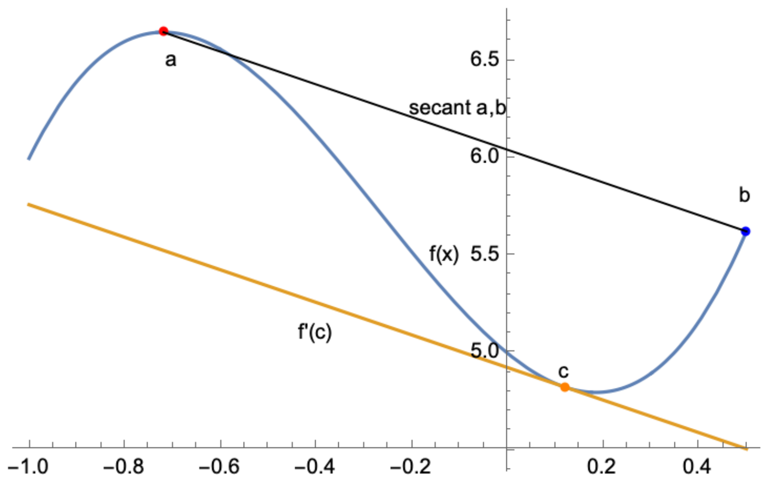

Figure 10: The Mean Value Theorem visualized

If a function f is

- *continuous* on $[a, b]$
- *differentiable* on (a, b)

then there exists a c in (a, b) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$. See figure 10.

This theorem is essentially an application of Rolle's Theorem, but for any a, b that are not necessarily equal. The "official" proof for this theorem is fairly straightforward as well, but again, is fairly intuitive graphically.

This, and Rolle's, are useful in questions involving proving the number of roots in an equation, as well as proving other theorems.

6.4 Optimization

Optimization questions are (*at least to me*) very similar to related rates questions. In both, you have to find some sort of formula representing a situation, then differentiating. However, for optimization questions, the intention is to solve for a min/max value, by finding when the derivative ("rate of change") is 0. Logically this should make a lot of sense: when something is not changing ($\frac{dx}{dy} = 0$), then it has to be either increasing until that point then being to decrease, or vice versa. From here, checking the second derivative will reveal whether it was a min or max.

In many situations, you will have to ensure that your min/max fits within some endpoints, which depend on the situation at hand; eg, if you have a geometric question, then your value logically can't be negative.

A summary of general steps:

- Draw a diagram, assign relevant variables, create a formula
- Differentiate the formula, set it equal to 0, and solve for the unknown(s) that satisfy the question

- Check the sign of the second derivative of the formula to ensure that your value really was a min/max
- Check the "endpoints" of the problem (are either logical, or imposed by the wording in the question), and ensure that your min/max falls within that range
  - If it doesn't, this usually means your ACTUAL min/max is one of the endpoints, so be careful for this

6.5 Error and Uncertainty

6.5.1 Uncertainty

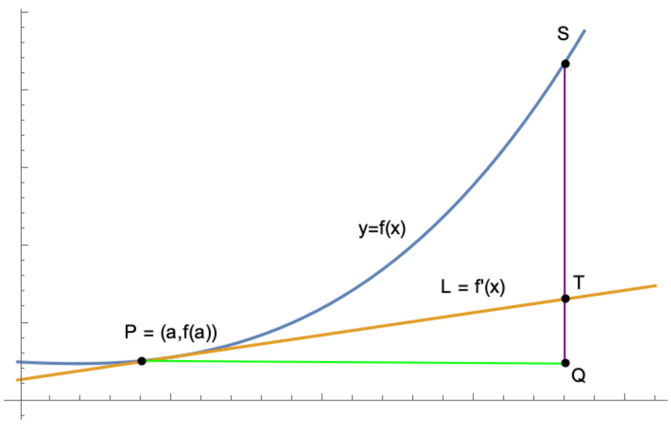

Figure 11: Visual representation of using a differential to approximate error

One very helpful use of derivatives is in estimating how some variable changes when given the relative change in another. This same idea is very similar to the rationale behind using differentials to estimate an error in a measurement, but this will be further explained a little later.

To show how a derivative of a function at a point can be used to estimate its change over a particular range, consider a function $y = f(x)$, as shown in figure 11.

As discussed in 3, we can define L as the linear approximation of $f(x)$ at P as $L = f(a) + (x - a)f'(a)$

As shown in the graph, $PQ = \Delta x = dx$; note that this is the change in x of both $f(x)$ AND L. Conversely, $QS = \Delta y$, the change in $f(x)$, and $QT = dy$, the change in L. In summary:

$$\begin{aligned} L &= f(a) + (x - a)f'(a) \\ PQ &= \Delta x = dx \\ QS &= \Delta(f(x)) = \Delta y \\ QT &= \Delta(L(x)) = dy \end{aligned} \tag{46}$$

From here, we can define θ as the angle between PQ and PT. As such:

$$\begin{aligned}
\frac{QT}{PQ} &= \tan\theta = f'(a) \\
\therefore QT &= f'(a)PQ \\
dy &= f'(a)\Delta x = f'(a)dx \\
\therefore QS &\approx QT \therefore \Delta y \approx dy = f'(a)\Delta x = f'(a)dx
\end{aligned} \tag{47}$$

As shown, dy is a good approximation for Δy when δx is relatively small. This reasoning can also be applied to the logic behind why a linear approximation $L(x)$ of a function $f(x)$ at a is most accurate when $x \approx a$.

For example: say you were asked to find $\cos(\frac{\pi}{3} + 0.05)$. In this case, we can say $f(x) = \cos(x)$, $a = \frac{\pi}{3}$, and $\Delta x = dx = 0.05$. From here:

$$\begin{aligned}
f'(x) &= -\sin(x) \\
f'(a) &= -\sin(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2} \\
dy \approx f'(a)dx &= \frac{-\sqrt{3}}{2} * 0.05 \approx -0.043
\end{aligned} \tag{48}$$

In reality, $\cos(\frac{\pi}{3} + 0.05) = -0.044$, demonstrating how accurate (and easy-to-compute) this method is.

6.5.2 Error

Using very similar reasoning to above, we can use differentials to find the range of error in a calculation. If given the error (change) of one variable in a formula, you can then use implicit differentiation, solve for the error (change) in the desired variable.

Often times, the relative change is also asked for. This is simple $\frac{dx}{x}$.

For example: given a circle with radius $r = 24$, with a max error in measurement of 0.2. To find the max error in the circle's area:

$$\begin{aligned}
A &= \pi r^2 = f(r) \\
\Delta A \approx dA &= f'(r)dr \\
&= 2\pi r dr \\
&= 2\pi(24)(0.2) = 9.6\pi \approx 30.16 \\
\textit{rel. error:} \frac{\Delta A}{A} &\approx \frac{dA}{A} = \frac{1}{60}
\end{aligned} \tag{49}$$

6.6 Newton's Method

Newton's Method is an application of derivatives that can be used to estimate the solutions/roots of a function. The steps to the method are as follows, for a function $f(x)$:

1. Pick some x_1, then find the point $P = (x_1, f(x_1))$
2. Find the tangent line of $f(x)$ at the point P, defined by $y - f(x_1) = f'(x_1)(x - x_1)$
3. Find the point x_2 where the tangent line intersects the x-axis, by setting $y = 0$:

$$\begin{aligned} 0 &= f(x_1) + f'(x_1)(x_2 - x_1) \\ x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \end{aligned} \tag{50}$$

4. Repeat steps 1-3 for increasing x_n and x_{n+1}. As n increases, the x_n will approach a root of $f(x)$. The general formula follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

See figure 12 to see this process visually.

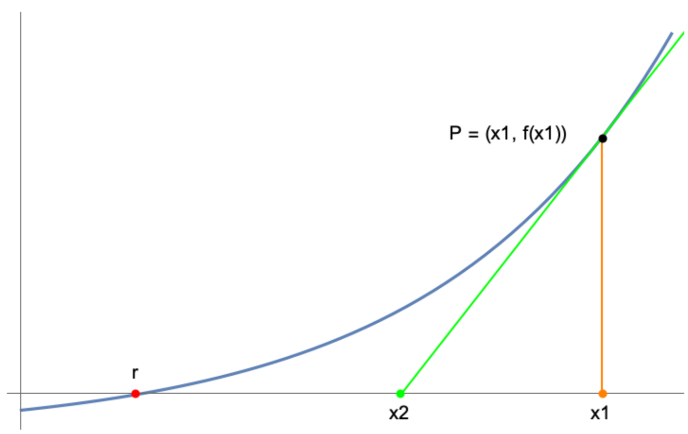

Figure 12: Newton's Method Visualized

6.7 Derivative-Related Theorems

Following are a number of theorems related to derivatives, and limits. While many of them are already described above, some are not.

These theorems have various applications in determining limits, derivatives, and can aid in graphing functions.

1. **Intermediate Value Theorem**

   If f is continuous on $[a, b]$, let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, there exists a c in (a, b) such that $f(c) = N$.

2. **Extreme Value Theorem**

   If f is continuous on $[a, b]$, then f attains a minimum and maximum value (extrema) on that interval. *Not that these extrema may occur at a or b; make sure to check this case when answering questions using this theorem.*

3. **Fermat's Theorem**

   If f has a local extrema at c, and $f'(c)$ exists, then $f'(c) = 0$.

   Note that the inverse of this theorem is not necessarily true: if $f'(b) = 0$, then b is not necessarily a local extrema.

4. **Rolle's Theorem**

   If f is continuous on $[a, b]$, f' is continuous on (a, b), and $f(a) = f(b)$, then there exists a c in (a, b) such that $f'(c) = 0$.

6.8 Graph Sketching

Using calculus (and other properties of functions), we can sketch curves with fairly high accuracy, without needing calculators. To do so, there is a "shopping list" to follow, to graph a function $f(x)$:

1. **Domain**

   Find all x for which $f(x)$ is defined. If $f(x)$ is undefined for some x, you should also find how f behaves as it approaches x (is it a hole? An asymptote? Or does the function simply only exist on a closed interval?)

2. **Intercepts**

   Find both the x and y intercepts, i.e. the ordered pair $(x, f(x))$ at $y = 0$ and $x = 0$ respectively.

3. **Symmetry (even/odd)**

   - $f(x) = f(-x)$: $f(x)$ is **even**
   - $f(-x) = -f(x)$: $f(x)$ is **odd**

   Graphically, an even function is symmetrical over the y axis, and an odd function is symmetrical over $y = x$.

4. **Asymptotes**

   Determine $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty}$, to determine whether the function has any **horizontal asymptotes**. In addition, find the limit of $f(x)$ to any holes in the domain, from both right and left, to determine if the function has any **vertical asymptotes**.

5. **Intervals of Increase/Decrease & Local Min/Max**

   Find $f'(x)$, and find when $f'(x)$ is greater/less than 0 to find when $f(x)$ is increasing/decreasing respectively. The points where $f'(x) = 0$ are also the place where the function has a local min/max, AS

LONG AS $f'(x)$ is not an inflection point...

6. **Concavity & Points of Inflection**

   Find $f''(x)$, and find when $f''(x)$ is greater/less than 0 to find when $f(x)$ is concave up/down respectively. The points where $f''(x) = 0$ are also the inflection points of $f(x)$.

7 Integration

7.1 Defining Antiderivatives

For a function $f(x)$, we can say that:

$$\int f(x)dx = F(x) + k$$

,

where $\frac{d}{dx}[F(x) + k] = f(x)$, and k is a constant.

For example: $\int x^2 dx = \frac{x^3}{3} + k$.

You can (and should) think of $\int dx$ and $\frac{d}{dx}$ as *inverse operators* of one another, hence the term "antiderivative". As such, numerous derivative rules have corresponding rules for integrals, which is discussed later. Also note that this type of integration is called *indefinite* because it defines the integral of a function over an *indefinite range*, ie, for all x. *Definite integrals* will be discussed later and have their own very particular applications, and define an integral $\int_m^n$ over a range $[m, n]$.

7.2 Techniques of Integration

7.2.1 Known Formulas of Derivatives

eg) $\int \frac{1}{1+x^2} dx = \arctan x + c$

"Simply" recognize that the integral to solve for is the derivative of a common function. Most other techniques of integration involve simplifying/modifying a given integral into this type of easy-to-solve form.

7.2.2 By Substitution

For many integrals, we can substitute part of a complex expression for a single variable (typically u; this technique is often shorthanded "u-sub"), and, following some manipulation, find an integral much more easily. The best way to understand this method is through an example:

$$\int \sqrt{x^3 + 2}x^2 dx$$

We can let $u = x^3 + 2$, and differentiating u in terms of x, we get

$$\begin{aligned} \frac{du}{dx} &= 3x^2 \\ du &= 3x^2 dx \\ x^2 dx &= \frac{du}{3} \end{aligned} \tag{51}$$

From here, we can substitute our expressions back into the original integral, and solve:

$$\begin{aligned}\int \sqrt{x^3+2}x^2 dx &= \int u^{\frac{1}{2}}\frac{du}{3} \\ &= \frac{1}{3}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + k \\ &= \frac{2(x^3+2)^{\frac{3}{2}}}{9} + k\end{aligned} \tag{52}$$

Note that solving this integral involved using the inverse of the ”power rule” of derivatives, which should hopefully be intuitive.

7.2.3 By Parts

Given two functions u and v, we can write:

$$\begin{aligned}\frac{d}{dx}[uv] &= \frac{du}{dx}v + u\frac{dv}{dx} \\ \int (\frac{d}{dx}[uv] &= \frac{du}{dx}v + u\frac{dv}{dx})dx \\ \int \frac{d}{dx}[uv]dx &= \int [v\frac{du}{dx}]dx + \int [u\frac{dv}{dx}]dx \\ uv &= \int vdu + \int udv \\ \int udv &= uv - \int vdu\end{aligned} \tag{53}$$

For example, to calculate the integral of xe^{ax}:

$$\begin{aligned}&\int xe^{ax}dx \\ &\qquad x = u, du = dx \\ &e^{ax}dx = dv, v = \frac{e^{ax}}{a} \\ &\qquad \int xe^{ax}dx = x\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}dx \\ &\qquad = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2} + C\end{aligned} \tag{54}$$

Generally, you want to find some u such that du becomes ”simpler”, and a dv such that v does not get ”too much more complicated”.

Another integral that is commonly computed using integration by parts is as follows:

$$\begin{aligned}\int \ln x dx \\ &= x\ln x - \int x\frac{dx}{x} \\ &= x\ln x - x + C\end{aligned} \tag{55}$$

7.2.4 Trigonometric Integrals

Finding the integral of a combination of trigonometric functions is largely a question of using trig. identities in such a way that the integral becomes easier to solve using more "basic" methods.

While a lot of this becomes intuitive over time, some general guidelines are as follows:

- $\int \sin^m(x)\cos^n(x)dx$

  If n **is odd** ($n = 2k+1$), rewrite all cos, except one, with $\cos^2(x) = 1 - \sin^2(x)$.

$$\begin{aligned}\int \sin^m(x)\cos^{2k+1}(x)dx &= \int \sin^m(x)(\cos^2 x)^k\cos(x)dx \\ &= \int \sin^m(x)(1-\sin^2(x))^k\cos(x)dx\end{aligned} \tag{56}$$

  From here, substitute $u = \sin(x)$ and solve accordingly.

  Very similarly, **if** m **is odd** ($m = 2k+1$):

$$\begin{aligned}\int \sin^{2k+1}(x)\cos^n(x)dx &= \int (\sin^2(x))^k\cos^n(x)\sin(x)dx \\ &= \int (1-\cos^2 x)^k\cos^n(x)\sin^n(x)dx\end{aligned} \tag{57}$$

  Let $u = \cos(x)$, etc..

  If both m **and** n **are even**, you can use the half-angle theorems.

- $\int \tan^m(x)\sec^n(x)dx$

  Very similar to the strategy for sin and cos: if n is even, substitute $\sec^2(x) = 1+\tan^2(x)$ for all sec except one, and the opposite for $\tan^2(x)$ if m is even.

7.2.5 Trigonometric Substitution

Using a strategy similar to ”u-substitution”, we can carefully replace variables in an integral with trigonometric functions to make the integral easier to solve.

In general, we can say:

$$\int f(x)dx = \int f(g(t))g'(t)dt$$

There are three main forms to look out for in integrals that can be used to effectively substitute trigonometric functions. Notice that, in each case, a form similar to a common trigonometric identity is present.

- $\sqrt{a^2 - x^2} \Rightarrow x = a\sin\theta$
- $\sqrt{a^2 + x^2} \Rightarrow x = a\tan\theta$
- $\sqrt{x^2 - a^2} \Rightarrow x = a\sec\theta$

Using this strategy makes a lot more sense when used in an example; say $\int \frac{\sqrt{9-x^2}}{x^2}dx$. This is of the first form listed above, so we can substitute $x = 3\sin\theta$, meaning $dx = 3\cos\theta d\theta$. Assume $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\begin{aligned}
\int \frac{\sqrt{9-x^2}}{x^2}dx &= \int \frac{\sqrt{9-9\sin^2(\theta)}}{9\sin^2(\theta)}3\cos(\theta)d\theta \\
&= \int \frac{\sqrt{9\cos^2(\theta)}}{9\sin^2(\theta)}3\cos(\theta)d\theta \\
&= \int \frac{3\cos(\theta)}{9\sin^2(\theta)}3\cos(\theta)d\theta \\
&= \int \frac{\cos^2)(\theta)}{\sin^2(\theta)}d\theta = \int \cot^2(\theta)d\theta \\
&= \int (\csc^2(\theta) - 1)d\theta = -\cot(\theta) - \theta + C
\end{aligned} \tag{58}$$

From here, the integral can be rewritten in terms of x using some trig rules (drawing a diagram of a triangle can help here):

$$-cot(\theta) - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}(\frac{x}{3}) + C$$

7.3 Applications of Integrals

8 Differential Equations

8.1 Definitions

A **differential equation** is an equation that contains an unknown function and some of its derivatives. We can more specifically describe differential equations by their **order**, which is the number of the highest derivative in the equation, ie, a first order differential equation is of the form $F(x, y, y')$.

8.2 Variables Separable

While not all differential equations are solvable (or at least, easily solvable), one (of many) special cases is of the form:

$$\frac{dy}{dx} = f(x)g(y) = \frac{f(x)}{h(y)}, g(y) = \frac{1}{h(y)}, h(y) \neq 0$$

This is the form of a function of x times a function of y, indicating the parts are *separable*. We can rewrite this:

$$\begin{aligned} \frac{dy}{dx} &= f(x)g(y) \\ dy &= [\frac{dy}{dx}]dx = f(x)g(y)dx \\ \int \frac{dy}{g(y)} &= \int \frac{dx}{f(x)} \end{aligned} \tag{59}$$

From here, integrating both sides yields either y as a function of x, or x as function of y (which you solve for depends on the context of the question).

For example, take: $\frac{dy}{dx} = x^2y - y + x^2 - 1$. This can be rewritten as a separable equation:

$$\begin{aligned} \frac{dy}{dx} &= x^2y - y + x^2 - 1 \\ &= y(x^2 - 1) + 1(x^2 - 1) \\ &= (y + 1)(x^2 - 1) \\ \int \frac{dy}{y + 1} &= \int (x^2 - 1)dx \\ \ln(|y + 1|) &= \frac{x^3}{3} - x + k \\ y + 1 &= \pm e^{(\frac{x^3}{3} - x + k)} = \pm e^{(\frac{x^3}{3} - x)}e^k \\ y &= Ce^{(\frac{x^3}{3} - x)} - 1, C = \pm e^k \end{aligned} \tag{60}$$

From here, if given $y(0)$, you can solve for the value of C and rewrite the equation appropriately. This would then become what is called an **initial value problem** (IVP); you are given the initial value, after all.

In this example, say $y(0) = 3$. We can write:

$$\begin{aligned} 3 &= Ce^{\frac{(0)^3}{3}-0} - 1 \\ 3 &= Ce^{0} - 1 \\ C &= 4 \\ y &= 4e^{\frac{x^3}{3}-x} - 1 \end{aligned} \tag{61}$$

8.3 Mathematical Models

A number of real-world scenarios can be represented using differential equations, and the following are some of more common/useful forms, as well as their general solutions and applications.

I: $\alpha y = \frac{dy}{dt}, \alpha \in \mathbb{R}$

When solved for a function y of t, this differential equation becomes a (hopefully) familiar form;

$$\begin{aligned} \frac{dy}{dt} &= \alpha y \\ \int \frac{dy}{y} &= \int \alpha dt \\ \ln|y| &= \alpha t + C \\ y &= \pm e^{(\alpha t + C)} = \pm e^{C} e^{\alpha t} = k e^{\alpha t}, k = \pm e^{C} \end{aligned} \tag{62}$$

If we say $y(0) = y_0$, then we can rewrite $y(t) = y_0 e^{\alpha t}$. This is the typical form used to represent simple **exponential growth/decay**; which of these two the function represents depends on α:

- $\alpha > 0$;**growth:** used in scenarios such as simple population models.
- $\alpha < 0$; **decay:** used in scenarios such as radioactive decay.

See figure 13 for a visual representation of these two.

II: $\frac{dy}{dt} = k(N - y)$

This general form of equation is used in a number of scenarios, such as Newton's Law of cooling/heating and modeling deprecation. Solving this differential for y as a function of t goes as follows:

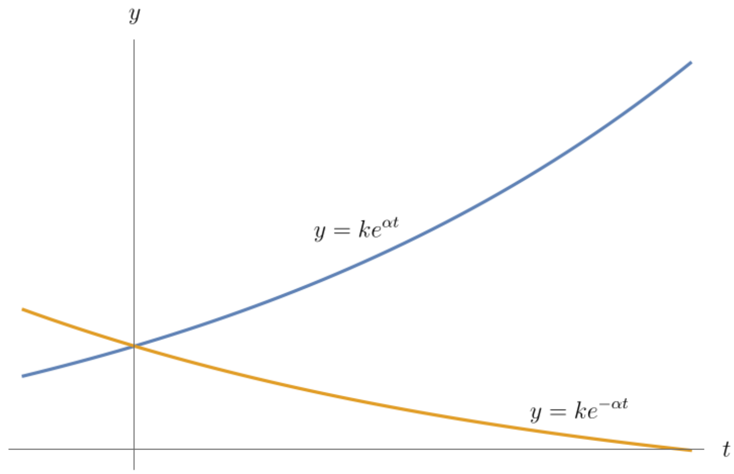

Figure 13: Exponential growth and decay

$$\begin{aligned}
\frac{dy}{dt} &= k(N-y) \\
\frac{dy}{N-y} &= k\,dt \\
\int \frac{dy}{N-y} &= \int k\,dt \\
-\ln|N-y| &= kt + C \\
N - y &= \pm e^{-C}e^{-kt} = Ke^{-kt} \\
y &= N - Ke^{-kt}
\end{aligned} \tag{63}$$

If we state that $y(0) = y_0$, we can rewrite this equation as $y = N + (y_0 - N)e^{-kt}$. This is the same form that Newton's Law of Cooling takes: $T(t) = T_{env} + (T(0) - T_{env})e^{-rt}$.

It's important to note that $y(t)$ approaches N as time approaches infinity, ie:

$$\lim_{t\to\infty} [N + (y_0 - N)e^{-kt}] = N$$

If the value of $y_0 - N$ is positive, then this limit is approached from above (ie, the function is *lower bounded* by N), and if $y_0 - N$ is negative, then the limit is approached from below (*upper bounded* by N). See figure 14 for a visual representation of the difference, assuming a constant y_0.

III: $\frac{dy}{dt} = \frac{k}{N}y(N-y)$

This differential equation represents a general form of the **logistic equation**, commonly used in population modeling in ecology and other fields. Solving this differential equation for y as a function of x goes

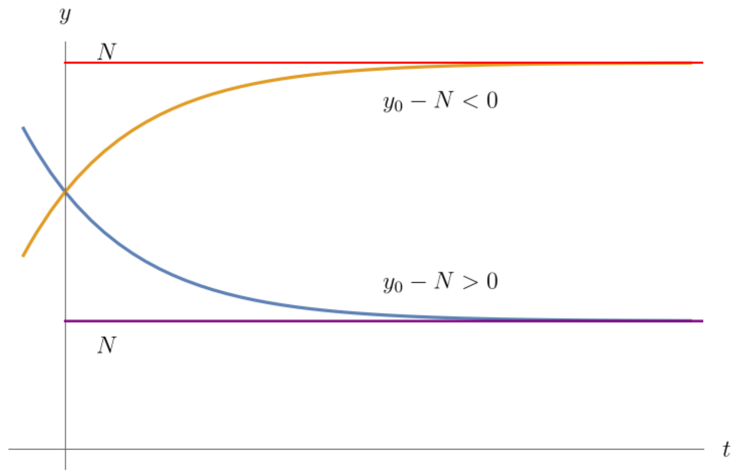

Figure 14: $y = N + (y_0 - N)e^{-kt}$, $y_0 - N > 0$ vs $y_0 - N < 0$

as follows:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{k}{N} y(N-y) \\
\frac{dy}{y(N-y)} &= \frac{k\,dt}{N} \\
\int \frac{dy}{y(N-y)} &= \int \frac{k\,dt}{N} \\
\int [\frac{\frac{1}{N}}{y} + \frac{\frac{1}{N}}{N-y}]dy &= \frac{k}{N}t + C \\
\frac{1}{N}(\ln|y| - ln|N-y|) &= \frac{k}{N}t + C \\
\ln|\frac{y}{N-y}| &= kt + D, D = CN \\
\frac{y}{N-y} &= Ke^{kt}, K = \pm e^{D} \\
y &= \frac{NKe^{kt}}{1+Ke^{tx}} = \frac{NKe^{kt}}{1+Ke^{kt}}\frac{\frac{e^{-kt}}{K}}{\frac{e^{-kt}}{K}} \\
&= \frac{N}{\frac{e^{-kt}}{K}+1} = \frac{N}{1+be^{-kt}}, b = \frac{1}{K}
\end{aligned} \tag{64}$$

If we say $y(0) = P_0$, or the initial population, then we can simplify further and say:

$$\begin{aligned}
y(0) = P_0 &= \frac{N}{1+b} \\
\implies b &= \frac{N}{P_0} - 1 = \frac{N - P_0}{P_0} \\
\therefore y(t) &= \frac{N}{1+be^{-kt}}, b = \frac{N-P_0}{P_0} \\
y(t) &= \frac{NP_0}{P_0 + (N-P_0)e^{-kt}}
\end{aligned} \tag{65}$$

Just as with Model II, the function approaches N as the time goes to infinity, and as such we can call N the "carrying capacity" or "max population", depending on the context of the problem.

Another interesting property of the logistic model comes about from exploring its derivatives. It's first derivative is clear, as it is given in the expression of the logistic model as a differential equation. This first derivative ($\frac{dy}{dt}$) is *always positive*, which should be clear from inspection of the original formula for $\frac{dy}{dt}$, as both N and y are always positive, and N must be greater than y.

Recall that the second derivative represents the derivative of the first derivative, and in the context of population growth, for example, represents the change in how fast the population is growing. By extension, the inflection point of the logistic model would represent when the change in the population growth rate is at a maximum (or minimum, but the work below will show that it is indeed a maximum).

We can find this inflection point (IP) by doing the follow:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{k}{N}y(N-y) = ky - \frac{ky^2}{N} \\
\frac{d^2y}{dt^2} &= k - \frac{2ky}{N} \\
k - \frac{2ky}{N} &= 0 \\
y &= \frac{N}{2} \\
f(t) &= \frac{N}{2} = \frac{N}{1+be^{-kt}} \\
1 + be^{-kt} &= 2 \\
e^{-kt} &= \frac{1}{b} \\
t &= -\frac{1}{k}\ln(\frac{1}{b}) = \ln((\frac{1}{b})^{-\frac{1}{k}}) \\
t &= \ln[(\frac{N-P_0}{P_0})^{\frac{1}{k}}]
\end{aligned} \tag{66}$$

As such, the point $(\ln[(\frac{N-P_0}{P_0})^{\frac{1}{k}}], \frac{N}{2})$ is an inflection point, and represents the maximum change in population growth (proving this is relatively straightforward, and is omitted here).

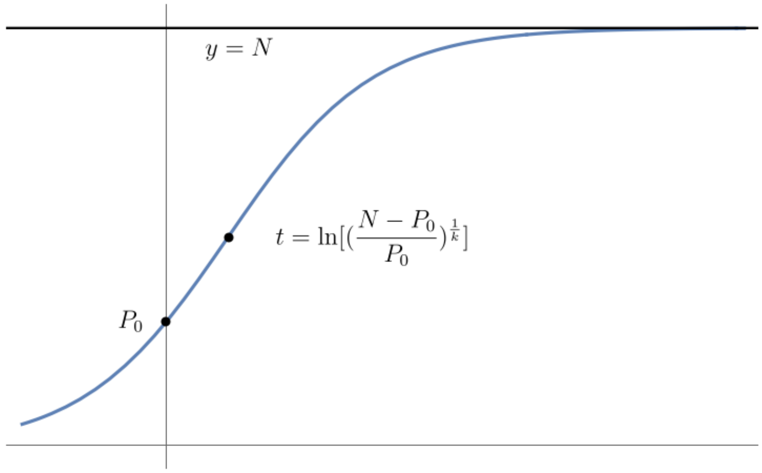

Figure 15: $y(t) = \frac{N}{1+be^{-kt}}, b = \frac{N-P_0}{P_0}$

8.4 Linear Equations

A **first-order linear differential equation** is a differential equation of the general form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

,

where P and Q are continuous over a particular interval. Notice that this form *does not* fit the form of a separable equation (see 8.2), and thus is not solvable using the same method.

Instead, we have to use a new function, called an **integrating factor**, called $I(x)$, which will make the equation solvable for y. To understand the motivation behind $I(x)$, recall the product rule for calculating derivatives:

$$(xy)' = xy' + y$$

We can define $I(x)$ as a factor such that the left-hand side of the general equation becomes the derivative of $I(x)y$, and simplifies as follows:

$$\begin{aligned} I(x)(\frac{dy}{dx} + P(x)y) = (I(x)y)' &= I(x)Q(x) \\ \int (I(x)y)' &= \int I(x)Q(x)dx \\ I(x)y &= \int I(x)Q(x)dx + C \\ y(x) &= \frac{1}{I(x)}[\int I(x)Q(x)dx + C] \end{aligned} \tag{67}$$

This provides a general solvable form for the equation, but we still need to find $I(x)$ to make this useful. We can do so by rearranging $I(x)(\frac{dy}{dx} + P(x)y) = (I(x)y)'$ as follows:

$$\begin{aligned} I(x)(\frac{dy}{dx} + P(x)y) = I(x)(y' + P(x)y) &= (I(x)y)' \\ I(x)y' + I(x)P(x)y &= I'(x)y + I(x)y' \\ I(x)P(x)y &= I'(x)y \\ I(x)P(x) &= I'(x) \end{aligned} \tag{68}$$

This is just a separable equation, which can be solved for I:

$$\begin{aligned} \int \frac{dI}{I} &= \int P(x)dx \\ \ln|I| &= \int P(x)dx \\ I &= \pm e^{\int P(x)dx + C} = Ae^{\int P(x)dx}, A = \pm e^C \end{aligned} \tag{69}$$

We can typically set $A = 1$ for a general I. Using these general steps, we can thus solve a linear first order differential equation by multiplying it by the appropriate integrating factor $I(x) = e^{\int P(x)dx}$. For instance, to solve $x^2y' + xy = 1$:

$$
\begin{aligned}
x^2y' + xy &= 1 \\
y' + \frac{1}{x}y &= \frac{1}{x^2} \\
I(x) &= e^{\int \frac{1}{x}dx} = e^{\ln x} = x \\
I(x)(y' + \frac{1}{x}y) &= I(x)\frac{1}{x^2} \\
xy' + y &= \frac{1}{x} \\
(xy)' &= \frac{1}{x} \\
\int (xy)' &= \int \frac{1}{x}dx \\
xy &= \ln x + C \\
y &= \frac{\ln x + C}{x}
\end{aligned} \tag{70}
$$

Note that the original equation was not in standard form, but we were able to simply divide each factor by x^2 to get it into standard form.

Orthogonal Families of Curves

Using differential equation, we can determine the general equation of curves orthogonal (ie ”perpendicular”) to a given curve (or, typically, family of curves). Consider the general family of curves described by;

$$ax^2 + by^2 = k,$$

where a, b, k are real constants. The derivative of any perpendicular curve must be the negative reciprocal of the derivative of the original curve (by the very definition of perpendicularity), so we can find the derivative of the original curve as follows, using implicit differentiation:

$$
\begin{aligned}
ax^2 + by^2 &= k \\
2ax + 2by\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= \frac{-2ax}{2by} = -\frac{ax}{by} \\
\therefore m_{orth.} &= -(\frac{dy}{dx})^{-1} = \frac{by}{ax}
\end{aligned} \tag{71}
$$

If we define P as the orthogonal curve’s y-value, we can then find a general equation for the orthogonal curve as a function of x by rewriting it as a (*separable*) differential equation:

$$\begin{aligned}\frac{dP}{dx} &= m_{orth.} = \frac{by}{ax} = \frac{bP}{ax}\\ \frac{dP}{bP} &= \frac{dx}{ax}\\ \int \frac{dP}{bP} &= \int \frac{dx}{ax}\\ \frac{1}{b}\ln|P| &= \frac{1}{a}\ln|x| + C\\ \ln|P| &= \frac{b}{a}\ln|x| + C\\ P(x) = Ke^{\ln|x^{\frac{b}{a}}|} &= Kx^{\frac{b}{a}}, K = \pm e^{C}\end{aligned} \tag{72}$$

This general method can be used to find the general equation of any curve orthogonal to a given curve (assuming the given curve is differentiable). See figure 16 for a visualization of this concept.

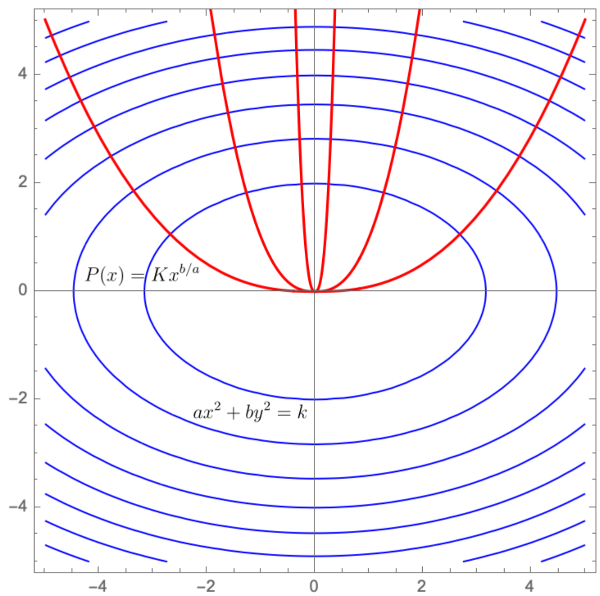

Figure 16: $ax^2 + by^2 = k$ and its orthogonal curves

9 Partial Derivatives