

MATH255 - Analysis 2

Basic point-set topology; metric spaces; Hölder-Minkowski Inequalities; compactness.

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1 INTRODUCTION

1.1 Metric Spaces

↪ Definition 1.1: Metric Space

A set X is a *metric space* with distance d if

1. (symmetric) $d(x, y) = d(y, x) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. (triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$

Remark 1.1. If 1., 3. are satisfied but not 2., d can be called a “pseudo-distance”.

↪ Definition 1.2: Open Metric Space

Let (X, d) be a metric space. A subset $A \subseteq X$ is open $\iff \forall x \in A, \exists r = r(x) > 0$ s.t. $B(x, r(x)) \subseteq A$.

↪ Definition 1.3: Normed Space

Let X be a vector space over \mathbb{R} . The norm on X , denoted $\|x\| \in \mathbb{R}$, is a function that satisfies

1. $\|x\| \geq 0$
2. $\|x\| = 0 \iff x = 0$
3. $\|c \cdot x\| = |c| \cdot \|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

If X is a normed vector space over \mathbb{R} , we can define a distance d on X by $d(x, y) = \|x - y\|$.

↪ Proposition 1.1

If X is a normed vector space over \mathbb{R} , a distance d on X by $d(x, y) = \|x - y\|$ makes (X, d) a metric space.

Proof. 1. $d(x, y) = \|x - y\| \geq 0$

$$2. d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$$

$$3. d(x, y) + d(y, z) = \|x - y\| + \|y - z\| \geq \|(x - y) + (y - z)\| = \|x - z\| := d(x, z)$$

■

⊗ **Example 1.1: L^p distance in \mathbb{R}^n**

Let $\bar{x} \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$. The L^p norm is defined

$$||x||_p := (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{\frac{1}{p}}.$$

In the case $p = 2, n = 2$, we simply have the standard Euclidean distance over \mathbb{R}^2 .

Unit Balls: consider when $||x||_p \leq 1$, over \mathbb{R}^2 .

- $p = 1 : |x_1| + |x_2| \leq 1$; this forms a “diamond ball” in the plane.
- $p = 2 : \sqrt{|x_1|^2 + |x_2|^2} \leq 1$; this forms a circle of radius 1. Clearly, this surrounds a larger area than in $p = 2$.

A natural question that follows is what happens as $p \rightarrow \infty$? Assuming $|x_1| \geq |x_2|$:

$$\begin{aligned} ||x||_p &= (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \\ &= \left[|x_1|^p \left(1 + \left| \frac{x_2}{x_1} \right|^p \right) \right]^{\frac{1}{p}} \\ &= |x_1| \left(1 + \left| \frac{x_2}{x_1} \right|^p \right)^{\frac{1}{p}} \end{aligned}$$

If $|x_1| > |x_2|$, this goes to $|x_1|$. If they are instead equal, then $||x||_p = |x_1| \cdot 2^{\frac{1}{p}} \rightarrow |x_1| \cdot 1$ as well. Hence, $\lim_{p \rightarrow \infty} ||x||_p = \max\{|x_1|, |x_2|\}$. Thus, the unit ball will approach $\max\{|x_1|, |x_2|\} \leq 1$, that is, the unit square.

↪ **Proposition 1.2**

Let $x \in \mathbb{R}^n$. Then, $||x||_p \rightarrow \max\{|x_1|, \dots, |x_n|\}$ as $p \rightarrow \infty$.

Remark 1.2. This is an extension of the previous example to arbitrary real space; the proof follows nearly identically.

↪ **Definition 1.4: Convex Set**

Let X be a normed space, and take $x, y \in X$. The line segment from x to y is the set

$$\{t \cdot x + (1 - t) \cdot y : 0 \leq t \leq 1\}.$$

Let $A \subseteq X$. A is *convex* $\iff \forall x, y \in A$, we have that

$$(t \cdot x + (1 - t) \cdot y) \in A \forall 0 \leq t \leq 1.$$

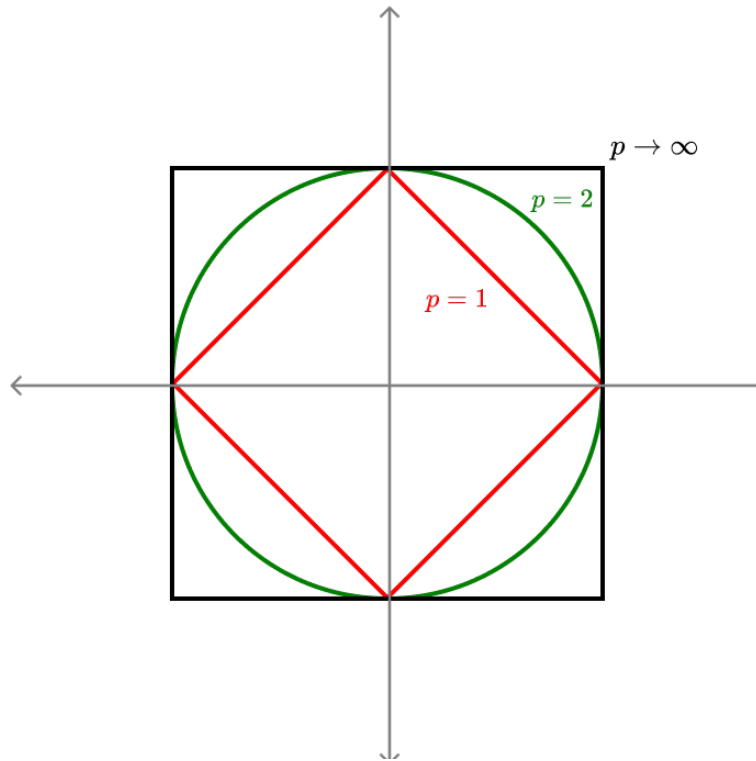


Figure 1: Regions of \mathbb{R}^2 where $\|x\|_p \leq 1$ for various values of p .

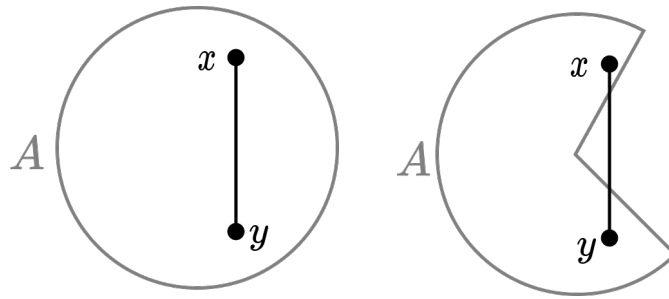


Figure 2: Convex (left) versus not convex (right) sets.

Remark 1.3. Think of this as saying “a set is convex iff every point on a line segment connected any two points is in the set”.

↪ Definition 1.5: ℓ_p

The space ℓ_p of sequences is defined as

$$\{x = (x_1, x_2, \dots, x_n, \dots) : \sum_{n=1}^{\infty} |x_n|^p < +\infty\} \quad *.$$

Then, $*$ defines the ℓ^p norm on the space of sequences; that is, $\|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$.

⊗ **Example 1.2:** $\ell_p, x_n = \frac{1}{n}$

. Let $x_n = \frac{1}{n}$. For which p is $x \in \ell_p$? We have, raising the norm to the power of p for ease:

$$\begin{aligned} \|x\|_p^p &= |x_1|^p + |x_2|^p + \cdots + |x_n|^p + \cdots \\ &= 1^p + \left(\frac{1}{2}\right)^p + \cdots < \infty \iff p > 1. \end{aligned}$$

In the case that $p = 1$, this becomes a harmonic sum, which diverges.

⊗ **Example 1.3:** L^p space of functions

Let $f(x)$ be a continuous function. We define the norm of f over an interval $[a, b]$

$$\|f\|_p = \left[\int_a^b |f(x)|^p dx \right]^{\frac{1}{p}}.$$

Remark 1.4. Triangle inequality for $\|x\|_p$ or $\|f\|_p$ is called Minkowski inequality; $\|x\|_p + \|y\|_p \geq \|x + y\|_p$. This will be discussed further.

⊗ **Example 1.4:** Distances between sets in \mathbb{R}^2

Let A, B be bounded, closed, “nice” sets in \mathbb{R}^2 . We define

$$d(A, B) := \text{Area}(A \Delta B),$$

where

$$A \Delta B : (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

It can be shown that this is a “valid” distance.

Remark 1.5. Δ denotes the “symmetric difference” of two sets.

⊗ **Example 1.5:** p -adic distance

Let p be a prime number. Let $x = \frac{a}{b} \in \mathbb{Q}$, and write $x = p^k \cdot \left(\frac{c}{d}\right)$, where c, d are not divisible by p . Then, the p -adic norm is defined $\|x\|_p := p^{-k}$. It can be shown that this is a norm.

Suppose $p = 2, x = 28 = 4 \cdot 7 = 2^2 \cdot 7$. Then, $\|28\|_2 = 2^{-2} = \frac{1}{4}$; similarly, $\|1024\|_2 = \|2^{10}\|_2 = 2^{-10}$.

More generally, we have that $\|2^k\|_2 = 2^{-k}$; conversely, $\|2^{-k}\| = 2^k$. That is, the closer to 0, the larger the distance, and vice versa, contrary to our notion of Euclidean distance.

↪ **Proposition 1.3**

$\|x\|_p$ as defined above is a well-defined norm over \mathbb{Q} .

Proof. Left as a (homework) exercise. ■

2 POINT-SET TOPOLOGY

2.1 Definitions

↪ [Definition 2.1: Topological space](#)

A set X is a topological space if we have a collection of subsets τ of X called *open sets* s.t.

1. $\emptyset \in \tau, X \in \tau$
2. Consider $\{A_\alpha\}_{\alpha \in I}$ where A_α an open set for any α ; then, $\bigcup_{\alpha \in I} A_\alpha \in \tau$, that is, it is also an open set.
3. If J is a finite set, and A_β open for all $\beta \in J$, then $\bigcap_{\beta \in J} A_\beta \in \tau$ is also open.

In other words, 2.: arbitrary unions of open sets are open, and 3.: finite intersections of open sets are open.

↪ [Definition 2.2: Closed sets](#)

Closed sets are complements of open sets; hence, axioms for closed sets follow appropriately;

- 1.* X, \emptyset closed;
- 2.* B_α closed $\forall \alpha \in I \implies \bigcap_{\alpha \in I} B_\alpha$ closed.
- 3.* B_β closed $\forall \beta \in J, J$ finite, then $\bigcup_{\beta \in J} B_\beta$ also closed.

↪ Lecture 01; Last Updated: Tue Feb 13 09:24:32 EST 2024

↪ Definition 2.3: Equivalence of Metrics

Suppose we have a metric space X with two distances d_1, d_2 ; will these necessarily admit the same topology?

A sufficient condition is that, if $\forall x \neq y \in X, \exists 1 < C < +\infty$ s.t.

$$\frac{1}{C} < \frac{d_1(x, y)}{d_2(x, y)} < C.$$

That is, the distances are equivalent, up to multiplication by a constant.

Indeed, this condition gives that $d_2 < C d_1$ and $d_2 > \frac{d_1}{C}$; this gives

$$B_{d_1}(x, \frac{r}{C}) \subseteq B_{d_2}(x, r) \subseteq B_{d_1}(x, C \cdot r).$$

Hence, d_1, d_2 define the same open/closed sets on X thus admitting the same topologies. We write $d_1 \asymp d_2$.

Remark 2.1. If $d_1 \asymp d_2$ and $d_2 \asymp d_3$, then also $d_1 \asymp d_3$. Moreover, clearly, $d_1 \asymp d_1$ and $d_1 \asymp d_2 \implies d_2 \asymp d_1$, hence this is a well-defined equivalence relation.

Hence, it's enough to show that $\forall 1 < p < +\infty$, we have $\|x\|_p \asymp \|x\|_\infty$ to show that any $\|x\|_q$ norm are equivalent for all q on \mathbb{R}^n .

↪ Definition 2.4: Interior, Boundary of a Topological Set

Let X be a topological space, $A \subseteq X$ and let $x \in X$. We have the following possibilities

1. $\exists U$ -open : $x \in U \subseteq A$. In this case, we say $x \in$ the *interior* of A , denoted

$$x \in \text{Int}(A).$$

2. $\exists V$ -open : $x \in V \subseteq X \setminus A = A^C$. In this case, we write

$$x \in \text{Int}(X^C).$$

3. $\forall U$ -open : $x \in U, U \cap A \neq \emptyset$ AND $U \cap A^C \neq \emptyset$. In this case, we say x is in the *boundary* of A , and denote

$$x \in \partial A.$$

↪ Definition 2.5: Closure

$x \in \text{Int}(A)$ or $x \in \partial A$ (that is, $x \in \text{Int}(A) \cup \partial A$) \iff every open set U that contains x intersects A .¹ Such points are called *limit points* of A . The set of all limit points of A is called the *closure* of A , denoted \overline{A} .

Remark 2.2. We have that

$$\text{Int}(A) \subseteq A \subseteq \overline{A} = \text{Int}(A) \cup \partial A.$$

↪ **Proposition 2.1: Properties of $\text{Int}(A)$**

$\text{Int}(A)$ is *open*, and it is the largest open set contained in A . It is the union of all U -open s.t. $U \subseteq A$. Moreover, we have that

$$\text{Int}(\text{Int}(A)) = \text{Int}(A).$$

↪ **Proposition 2.2: Properties of \overline{A}**

\overline{A} is *closed*; \overline{A} is the smallest closed set that contains A , that is, $\overline{A} = \bigcap B$ where B closed and $A \subseteq B$. We have too that

$$\overline{(\overline{A})} = \overline{A}.$$

↪ **Proposition 2.3**

1. A is open $\iff A = \text{Int}(A)$
2. A is closed $\iff A = \overline{A}$

2.2 Basis

↪ **Definition 2.6: Basis for a Topology**

Let τ be a topology on X . Let $\mathcal{B} \subseteq \tau$ be a collection of open sets in X such that every open set is a union of open sets in \mathcal{B} .

⊗ **Example 2.1: Example Basis**

$X = \mathbb{R}$, and $\mathcal{B} = \{\text{all open intervals } (a, b) : -\infty < a < b < +\infty\}$.

↪ **Proposition 2.4**

Let \mathcal{B} be a collection of open sets in X . Then, \mathcal{B} is a basis \iff

1. $\forall x \in X, \exists U\text{-open} \in \mathcal{B} \text{ s.t. } x \in U$.
2. If $U_1 \in \mathcal{B}$ and $U_2 \in \mathcal{B}$, and $x \in U_1 \cap U_2$, then $\exists U_3 \in \mathcal{B} \text{ s.t. } x \in U_3 \subseteq U_1 \cap U_2$.

¹"Requires" proof.

⊗ Example 2.2

Consider $X = \mathbb{R}$. Requirement 1. follows from taking $U = (x - \varepsilon, x + \varepsilon)$ for any $\varepsilon > 0$. For 2., suppose $x \in (a, b) \cap (c, d) =: U_1 \cap U_2$. Let $U_3 = (\max\{a, c\}, \min\{b, d\})$; then, we have that $U_3 \subseteq U_1 \cap U_2$, while clearly $x \in U_3$.

↪ Proposition 2.5

In a metric space, a basis for a topology is a collection of open balls,

$$\{B(x, r) : x \in X, r > 0\} = \{\{y \in X : d(x, y) < r\} : x \in X, r > 0\}.$$

Proof. We prove via proposition 2.4. Property 1. holds clearly; $x \in B(x, \varepsilon)$ -open $\subseteq \mathcal{B}$.

For property 2., let $x \in B(y_1, r_1) \cap B(y_2, r_2)$, that is, $d(x, y_1) < r_1$ and $d(x, y_2) < r_2$. Let

$$\delta := \min\{r_1 - d(x, y_1), r_2 - d(x, y_2)\}.$$

We claim that $B(x, \delta) \subseteq U_1 \cap U_2$.

Let $z \in B(x, \delta)$. Then,

$$d(z, y_1) \stackrel{\Delta\#}{\leq} d(z, x) + d(x, y_1) < \delta + d(x, y_1) \leq r_1 - d(x, y_1) + d(x, y_1) = r_1,$$

hence, as $d(z, y_1) < r_1 \implies z \in B(y_1, r_1) = U_1$. Replacing each occurrence of y_1, r_1 with y_2, r_2 respectively gives identically that $z \in B(y_2, r_2) = U_2$. Hence, we have that $B(x, \delta) \subseteq U_1 \cap U_2$ and 2. holds. ■

2.3 Subspaces

↪ Definition 2.7

Let X be a topological space and let $Y \subseteq X$. We define the subspace topology on Y :

1. Open sets in $Y = \{Y \cap \text{open sets in } X\}$

↪ Proposition 2.6: Consequences of Subspace Topologies

Suppose \mathcal{B} is a basis for a topology in X . Then, $\{U \cap Y : U \in \mathcal{B}\}$ forms a basis for the subspace $Y \subseteq X$.

Suppose X a metric space. Then, Y is also a metric space, with the same distance.

↪ Proposition 2.7

Let $Y \subseteq X$ - a metric space. Then, the metric space topology for (Y, d) is the same as the subspace topology.

Proof. (Sketch) A basis for the open sets in X can be written $\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)$; hence

$$Y \cap \left(\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha) \right) = \bigcup_{\alpha \in I} (Y \cap B(x_\alpha, r_\alpha))$$

is an open set topology for Y . ■

↪ **Lemma 2.1**

Let $A \subseteq X$ -open, $B \subseteq A$; B -open in subspace topology for $A \iff B$ -open in X .

↪ **Lemma 2.2**

Let $Y \subseteq X$, $A \subseteq Y$. Then, \overline{A} in $Y = Y \cap \overline{A}$ in X . We can denote this

$$\overline{A}_Y = \overline{A}_X \cap Y.$$

2.4 Continuous Functions

↪ **Definition 2.8: Continuous Function**

Let X, Y be topological spaces. Let $f : X \rightarrow Y$. f is *continuous* $\iff \forall$ open $V \in Y$, $f^{-1}(V)$ -open in X .

↪ **Proposition 2.8**

This definition is consistent with the normal ε - δ definition on the real line.

Proof. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, continuous; that is, $\forall \varepsilon > 0, \forall x \in \mathbb{R} \exists \delta > 0$ s.t. $|x_1 - x| < \delta$, then $|f(x_1) - f(x)| < \varepsilon$.

Let $V \subseteq \mathbb{R}$ open. Let $y \in V$. Then, $\exists \varepsilon : (y - \varepsilon, y + \varepsilon) \subseteq V$. Let $y = f(x)$, hence $y \in f^{-1}(V)$. Now, if $d(x, x_1) < \delta$, we have that $d(f(x_1), f(x)) < \varepsilon$ (by continuity of f), hence $f(x_1) \in (y - \varepsilon, y + \varepsilon) \subseteq V$; moreover, $(x - \delta, x + \delta) \subseteq f^{-1}(V)$, thus $f^{-1}(V)$ is open as required.

The inverse of this proof follows identically. ■

↪ Lecture 02; Last Updated: Thu Jan 11 08:52:09 EST 2024

↪ **Proposition 2.9**

Suppose \mathcal{B} forms a basis of topology for Y . Then, $f : X \rightarrow Y$ is continuous if $f^{-1}(U)$ open $\forall U \in \mathcal{B}$.

Proof. If U -open set in Y , then $\exists I$ -index set and a collection of open sets $\{A_\alpha\}_{\alpha \in I}, A_\alpha \in \mathcal{B}$, s.t. $U = \bigcup_{\alpha \in I} A_\alpha$. Then, we have

$$f^{-1}(U) = f^{-1}\left(\bigcup_{\alpha \in I} (A_\alpha)\right) = \bigcup_{\alpha \in I} \underbrace{f^{-1}(A_\alpha)}$$

Hence, if each $f^{-1}(A_\alpha)$ open, then $\cup_{\alpha \in I} f^{-1}(A_\alpha)$ open; hence it suffices to check if $f^{-1}(U) \forall U$ -open in V is open to see if f continuous. ■

↪ **Theorem 2.1: Continuity of Composition**

If $f : X \rightarrow Y$ continuous and $g : Y \rightarrow Z$ continuous, then $g \circ f$ continuous as well.

Proof. Let U -open in Z . Then

$$(g \circ f)^{-1}(U) = f^{-1}(\underbrace{g^{-1}(U)}_{\text{open in } Y})$$

$$\underbrace{\hspace{10em}}_{\text{open in } X}$$

↪ **Proposition 2.10**

If $f : X \rightarrow Y$ continuous and $A \subseteq X$, A has subspace topology, then $f|_A : A \rightarrow Y$ is also continuous.²

Proof. Let U -open in Y . Then

$$(f|_A)^{-1}(U) = \underbrace{f^{-1}(U)}_{\text{open}} \cap \underbrace{A}_{\text{open}}$$

By the definition of subspace topology, this is an open set and hence $f|_A$ is continuous. ■

2.5 Product Spaces

↪ **Definition 2.9: Finite Product Spaces**

Let X_1, \dots, X_n be topological spaces. We define

$$(X_1 \times X_2 \times \cdots \times X_n),$$

and aim to define a *product topology*; a basis of which consists of cylinder sets.

²We denote $f|_A$ as the restriction of the domain of f to A .

↪ **Definition 2.10: Cylinder Set**

A *cylinder set* has the form

$$A_1 \times A_2 \times \cdots \times A_n$$

where each A_j -open in X_j .

⊗ **Example 2.3**

Given an open interval $(a_1, b_1), (a_2, b_2) \subset \mathbb{R}$, the set $(a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$ is a basis for the topology on \mathbb{R}^2 .

↪ **Definition 2.11: Projection**

Let $X_1 \times X_2 \times \cdots \times X_n =: X$. The *projection* $\pi_j : X \rightarrow X_j$ maps $(x_1, \dots, x_n) \rightarrow x_j \in X_j$.

Remark 2.3. One can show π_j continuous.

↪ **Definition 2.12: Coordinate Function**

Given a function $f : Y \rightarrow X_1 \times \cdots \times X_n = (x_1(y), x_2(y), \dots, x_n(y))$. The *coordinate function* is

$$f_j = \pi_j \circ f; \quad f_j = x_j(y).$$

↪ **Proposition 2.11**

$f : Y \rightarrow X = X_1 \times \cdots \times X_n$ continuous $\iff f_j : Y \rightarrow X_j$ continuous.

Proof. Its enough to show that $\forall U \in \mathcal{B}$ -basis for X -product space, $f^{-1}(U)$ -open in Y . Take $U = A_1 \times \cdots \times A_n$ -open. Then, we claim that

$$f^{-1}(U) = f^{-1}(A_1 \times \cdots \times A_n) = f_1^{-1}(A_1) \cap f_2^{-1}(A_2) \cap \cdots \cap f_n^{-1}(A_n). \quad \star$$

If this holds, then as each f_i continuous (being a composition of continuous functions) and each A_i open in X_i , then each $f_i^{-1}(A_i)$ open in Y and hence \star , being the finite intersection of open sets in Y , is itself open in Y . ■

⊗ **Example 2.4: Fourier Transform: Motivation for Infinite Product Topologies**

Let $f \in C([0, 2\pi])$ is real-valued. We write the n th Fourier coefficients

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos(nx) dx - i \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin(nx) dx.\end{aligned}$$

And the Fourier transform of f as the infinite product

$$f(x) \mapsto (\dots, \hat{f}(-n), \hat{f}(-n+1), \dots, \hat{f}(-1), \hat{f}(0), \hat{f}(1), \dots, \hat{f}(n), \dots) \in \prod_{n \in \mathbb{Z}} (\mathbb{C})_n.$$

Hence, this is an (countably, as indexed by integers) infinite product space.

Now, let $f : \mathbb{R} \rightarrow \mathbb{R}$. Suppose $f(x) \rightarrow 0$ “fast enough” as $|x| \rightarrow \infty$ and f continuous. Then, we can define the Fourier coefficients

$$\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-itx} dx,$$

where $t \in \mathbb{R}$. We then have the transform

$$f \mapsto \{\hat{f}(t)\}_{t \in \mathbb{R}}.$$

In this case, our index set is \mathbb{R} is (uncountably) infinite.

↪ **Definition 2.13: Product Topology/Cylinder Sets for ∞ Products**

Let $X = \prod_{\alpha \in I} X_\alpha$. Then, a basis for X is given by cylinder sets of the form $A = \prod_{\alpha \in I} A_\alpha$ where A_α -open in X_α , AND $A_\alpha = X_\alpha$ except for finitely many indices α .

That is, there exists a finite set $J = (\alpha_1, \dots, \alpha_k) \subseteq I$, such that we can write $A = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$ (where A_α open in X_α).

↪ **Proposition 2.12**

Given $f : Y \rightarrow \prod_{\alpha \in I} X_\alpha = X$, then (taking $f_\alpha = \pi_\alpha \circ f$ as before) we have that f is continuous in $X \iff f_\alpha : Y \rightarrow X_\alpha$ continuous in $X_\alpha \forall \alpha \in I$.

Remark 2.4. Extension of proposition 2.11 to infinite product space.

Proof. Write $U = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$. Then,

$$f^{-1}(U) = \bigcap_{\alpha \in J} f_\alpha^{-1}(A_\alpha)$$

which is open in Y , hence f continuous. ■

Remark 2.5. The intersection of the entire spaces give no restriction.

↪ Lecture 03; Last Updated: Fri Jan 19 11:49:27 EST 2024

2.6 Metrizable

↪ Proposition 2.13

Different metrics can define the same topology.

⊗ Example 2.5

1. Different ℓ_p metrics in \mathbb{R}^n (PSET 1)
2. Let (X, d) be a metric space. Then,

$$\tilde{d}(x, y) := \frac{d(x, y)}{d(x, y) + 1}$$

is also a metric (the first two axioms are trivial), and defines the same topology. Note, moreover, that $\tilde{d}(x, y) \leq 1 \forall x, y$; this distance is bounded, and can often be more convenient to work with in particular contexts.

↪ Question 2.1

Suppose (X_k, d_k) are metric spaces $\forall k \geq 1$. Then, we can define the product topology τ on

$$X := \prod_{k=1}^{\infty} X_k.$$

Does the product topology τ come from a metric? That is, is τ metrizable?

Remark 2.6. There do indeed exist examples of non-metrizable topological spaces; this question is indeed well-founded.

Answer. Let $\underline{x} = (x_1, x_2, \dots, x_n, \dots)$, $\underline{y} = (y_1, y_2, \dots, y_n, \dots) \in \prod_{k=1}^{\infty} X_k$ (where $x_i, y_i \in X_i$) be infinite sequences of elements. Then, for each metric space X_k take the metric

$$\tilde{d}_k(x_k, y_k) = \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)}$$

(as in the example above). Then, we define

$$D(\underline{x}, \underline{y}) = \sum_{k=1}^{\infty} \frac{\tilde{d}_k(x_k, y_k)}{2^k},$$

noting that $D(\underline{x}, \underline{y}) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ (by our construction, “normalizing” each metric), hence this is a valid, *converging* metric (which wouldn’t otherwise be guaranteed if we didn’t normalize the metrics). It remains to show whether this metric omits the same topology as τ . ■

2.7 Compactness, Connectedness

↪ Definition 2.14: Compact

A set A in a topological space is said to be *compact* if every cover has a finite subcover. That is, if

$$A \subseteq \bigcup_{\alpha \in I} U_{\alpha} - \text{open},$$

then $\exists \{\alpha_1, \dots, \alpha_n \in I\}$ such that $A \subseteq \bigcup_{i=1}^n U_{\alpha_i}$.

↪ Proposition 2.14

A closed interval $[a, b]$ is compact.

Proof. If³ $a = b$, this is clear. Suppose $a < b$, and let $[a, b] \subseteq \bigcup_{i \in I} U_i =: \mathcal{U}$ be an arbitrary cover. Then, we proceed in the following steps:

1. **Claim:** Given $x \in [a, b]$, $x \neq b$, $\exists y \in [a, b]$ s.t. $[x, y]$ has a finite subcover.

Let $x \in [a, b]$, $x \neq b$. Then, $\exists U_{\alpha} \in \mathcal{U} : x \in U_{\alpha}$. Since U_{α} open, and $x \neq b$, we further have that $\exists c \in [a, b]$ s.t. $[x, c] \subseteq U_{\alpha}$.

Now, let $y \in (x, c)$; then, the interval $[x, y] \subseteq [x, c] \subseteq U_{\alpha}$, that is, $[x, y]$ has a finite subcover.

2. Define $C := \{y \in [a, b] : y > a, [a, y] \text{ has a finite subcover}\}$. We note that

- $C \neq \emptyset$; taking $x = a$ in Step 1. above, we have that $\exists y \in [a, b]$ such that $[a, y]$ has a finite step cover, so this $y \in C$.
- C bounded; by construction, $\forall y \in C, a < y \leq c$.

Thus, we can validly define $c := \sup C$, noting that $a < c \leq b$. Ultimately, we wish to prove that $c = b$, completing the proof that $[a, b]$ has a finite subcover.

3. **Claim:** $c \in C$.

Let $U_{\beta} \in \mathcal{U} : c \in U_{\beta}$. Then, by the openness of U_{β} , $\exists d \in [a, b]$ s.t. $(d, c] \subseteq U_{\beta}$.

³This proof is adapted from that of Theorem 27.1 in Munkre’s Topology, an identical theorem but applied to more general ordered topologies.

Supposing $c \notin C$, then $\exists z \in C$ such that $z \in (d, c)$; if one did not exist, then this would imply that d was a smaller upper bound than c , a contradiction. Thus, $[z, c] \subseteq (d, c] \subseteq U_\beta$.

Moreover, we have that, given $z \in C$, $[a, z]$ has a finite subcover; call it $U_z \subseteq \mathcal{U}$. This gives, then:

$$[a, c] = [a, z] \cup [z, c] \subseteq U_z \cup U_\beta.$$

But this is a finite subcover of $[a, c]$, contradicting the fact that $c \notin C$. We conclude, then, that $c \in C$ after all.

4. **Claim:** $c = b$.

Suppose not; then, since we have $c \leq b$, then assume $c < b$. Then, applying Step 1. with $x = c$ (which we can do, by our assumption of $c \neq b$!), then we have that $\exists y > c$ s.t. $[c, y]$ has a finite subcover, call this $U_y \subseteq \mathcal{U}$.

Moreover, we had $c \in C$, hence $[a, c]$ has a finite subcover, call this $U_c \subseteq \mathcal{U}$.

Then, this gives us that

$$[a, y] = [a, c] \cup [c, y] \subseteq U_c \cup U_y,$$

that is, $[a, y]$ has a finite subcover, and so $y \in C$. But recall that $y > c$; hence, this a contradiction to c being the least upper bound of C . We conclude that $c = b$, and thus $[a, b]$ has a finite subcover, and is thus compact. ■

Remark 2.7. A similar proof shows that $[a, b]$ is connected; we cannot cover it by two disjoint open sets.

↪ **Theorem 2.2: On Compactness**

Let $A \subseteq \mathbb{R}^n$. Then, A compact $\iff A$ closed and bounded.

↪ **Proposition 2.15**

If X, Y are compact topological spaces, then $X \times Y$ is compact.

Remark 2.8. By induction, if X_1, \dots, X_n compact, so is $\prod_{i=1}^n X_i$.

↪ **Proposition 2.16**

A closed subset of a compact topological space is compact in the subspace topology.

Proof. (Of theorem 2.2)

(\Leftarrow) If $A \subseteq \mathbb{R}^n$ closed and bounded, then $A \subseteq [-R, +R]^n$ for some $R > 0$ (it is contained in some “ n -cube”). Then, we have that $[-R, R]$ is compact, by proposition 2.14, proposition 2.15, and proposition 2.16, A itself compact.

(\Rightarrow) Suppose $A \subseteq \mathbb{R}^n$ is compact. Then, $\bigcup_{x \in A} B(x, \varepsilon)$ for some $\varepsilon > 0$ is an open cover of A . As A compact, there must exist a finite subcover of this cover, $A \subseteq \bigcup_{i=1}^N B(x_i, r_i)$. Let $R := \max_{i=1}^N (||x_i|| + r_i)$. Then, $A \subseteq \overline{B(0, R)}$, that is, A is bounded.

Now, suppose x is a limit point of A . Then, any neighborhood of x contains a point in A , so $\forall r > 0, B(x, r) \cap A \neq \emptyset$, and so $\overline{B}(x, r)$ also contains a point of A for any $r > 0$.

Now, suppose $x \notin A$ (looking for a contradiction). Then,

$$U := \bigcup_{r>0} U_r := \bigcup_{r>0} (\mathbb{R}^n \setminus \overline{B}(x, r)) = \mathbb{R}^n \setminus \{x\}$$

is an open cover for the set A . A being compact implies that U has a finite subcover such that $A \subset U_{r_1} \cup U_{r_2} \cup \dots \cup U_{r_N}$. Let $r_0 = \min_{i=1}^N r_i$. Then, $A \subseteq U_{r_0}$, and $A \cap B(x, r_0) = \emptyset$; but this is a contradiction to the definition of a limit point, hence any limit point x is contained in A and A is thus closed by definition. ■

↪ **Proposition 2.17**

Compact \implies sequentially compact; that is, every sequence in a compact set has a convergent subsequence.

↪ Lecture 04; Last Updated: Wed Jan 24 21:27:59 EST 2024

↪ **Definition 2.15: Connected**

A topological space X is *not connected* if $X = U \cup V$ for two open, nonempty, disjoint sets U, V .

If this does not hold, X is said to be *connected*.

A set $A \subseteq X$ is not connected if A is not connected in the subspace topology $\iff A = U \cup V$, for U, V -open in X , $(U \cap A) \neq \emptyset$, $(V \cap A) \neq \emptyset$ and $U \cap V = \emptyset$.

↪ **Theorem 2.3**

Let X be a connected topological space. Let $f : X \rightarrow Y$ be a continuous function. Then, $f(X)$ is also connected.

Proof. Suppose, seeking a contradiction, that X is connected, but $f(X)$ is not. Then, we can write $f(X) \subseteq Y$ as $f(X) \subseteq U \cup V$, such that U, V open in Y and $U \cap V = \emptyset$. Then,

$$(U \cap f(X)) \cap (V \cap f(X)) = \emptyset.$$

We also have that

$$X \subseteq \underbrace{f^{-1}(U)}_{\text{open in } X, \neq \emptyset} \cup \underbrace{f^{-1}(V)}_{\text{open in } X, \neq \emptyset}.$$

$f^{-1}(U) \cap f^{-1}(V) = \emptyset$ (that is, they are disjoint) by our assumption; this is a contradiction to the connectedness of X , as we are able to write it as a subset of two disjoint open sets. Hence, $f(X)$ is indeed connected. ■

↪ **Lemma 2.3**

Any interval $(a, b), [a, b], [a, b), \dots, \subseteq \mathbb{R}$ is connected.

Proof.

↪ **Theorem 2.4: “Intermediate Value Theorem”**

Suppose X is connected and $f : X \rightarrow \mathbb{R}$ is a continuous function. Then, f takes intermediate values.

More precisely, let $a = f(x), b = f(y)$ for $x, y \in X$. Assume $a < b$. Then, $\forall a < c < b, \exists z \in X$ s.t. $f(z) = c$.

Proof. Suppose, seeking a contradiction, that $\exists c : a < c < b$ s.t. $c \notin f(X)$ (that is, there exists an intermediate value that is “not reached” by the function).

Let $U = (-\infty, c)$ and $V = (c, +\infty)$; note that these are disjoint open sets. Then, we have that

$$X = f^{-1}(U) \cup f^{-1}(V),$$

by our assumption of $c \notin f(X)$. But this gives that X is not connected, as the union of two open (by continuity), disjoint, nonempty ($f(x) = a \in U \implies x \in f^{-1}(U)$, and $f(y) = b \in V \implies y \in f^{-1}(V)$) sets, a contradiction. ■

↪ **Theorem 2.5**

Suppose X is compact, Y -topological space, $f : X \rightarrow Y$ is a continuous function. Then, $f(X)$ is also compact.

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(X) \subseteq Y$, that is,

$$f(X) \subseteq \bigcup_{\alpha \in I} U_\alpha \implies X \subseteq f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(U_\alpha) =: \bigcup_{\alpha \in I} V_\alpha - \text{open}.$$

Then, this is an open cover of X ; X is compact, thus there exists a finite subcover, that is, indices $\{\alpha_1, \dots, \alpha_n\} \subseteq I$ such that $X = \bigcup_{i=1}^n V_{\alpha_i}$. Thus,

$$f(X) \subseteq \bigcup_{i=1}^n U_{\alpha_i},$$

which is a finite subcover of $f(X)$. Thus, $f(X)$ is compact. ■

Remark 2.9. Recall the “extreme value theorem”: let $f : [a, b] \rightarrow \mathbb{R}$ a continuous function; then, a minimum and maximum is obtained for $f(x)$ on this interval for values in this interval.

↪ **Theorem 2.6**

Let X compact, and $f : X \rightarrow \mathbb{R}$ a continuous function. Then,

$$\max_{x \in X} f(x) \text{ and } \min_{x \in X} f(x)$$

are both attained.

Proof. $f(X) \subseteq \mathbb{R}$ is compact by theorem 2.5, and so by theorem 2.2, $f(X)$ is closed and bounded. Let, then, $m = \inf f(X)$ and $M = \sup f(X)$; these necessarily exist, since $f(X)$ is bounded. Both m and M are limit points of $f(X)$. But $f(X)$ is closed, and hence contains all of its limit points, and thus $m \in f(X)$ and $M \in f(X)$, and thus $\exists y_m : f(y_m) = m$ and $y_M : f(y_M) = M$. ■

↪ **Definition 2.16: Path Connected**

A set $A \subseteq X$ is called *path connected* if $\forall x, y \in A, \exists f : [a, b] \rightarrow X$, continuous, s.t. $f(a) = x, f(b) = y$ and $f([a, b]) \subseteq A$.

The set $\{f(t) : a \leq t \leq b\}$ is called a *path* from x to y .

↪ **Theorem 2.7: Path connected \implies connected**

If $A \subseteq X$ is path connected, then A is connected.

Proof. Suppose, seeking a contradiction, that A is path connected, but not connected. Then, we can write $\overline{A} \subseteq U \cup V$, for open, disjoint, nonempty subsets $U, V \subseteq X$.

Let $x \in U \cap A$ and $y \in V \cap A$. Then, $\exists f : [a, b] \rightarrow A$ s.t. $f(a) = x, f(b) = y$, and $f([a, b]) \subseteq A$, by the path connectedness of A . Then,

$$[a, b] \subseteq f^{-1}(A) \subseteq \underbrace{f^{-1}(U \cap A)}_{\text{open}} \cup \underbrace{f^{-1}(V \cap A)}_{\text{open}} =: \underbrace{U_1}_{a \in} \cup \underbrace{U_2}_{b \in},$$

that is, $[a, b]$ is contained in a union of open, nonempty, disjoint sets, contradicting $[a, b]$ the connectedness of $[a, b]$ by lemma 2.3. Thus, A is connected. ■

Remark 2.10. A counterexample to the opposite side of the implication is the Topologist's sine curve, the set

$$\{(x, \sin\left(\frac{1}{x}\right)) : x \in (0, 1]\} \cup \{0\} \times [-1, 1].$$

This set is connected in \mathbb{R}^2 , but is not path connected.

↪ **Proposition 2.18**

For open sets in \mathbb{R}^n , path connected \iff connected.

2.8 Path Components, Connected Components

Remark 2.11. Remark that if a metric space X is not connected, then we can write $X = U \cup V$ where U, V are open, nonempty and disjoint. It follows, then, that $U = V^c$ (and vice versa) and hence U, V are both open and closed.

↪ Definition 2.17: Connected Component

A connected component of $x \in X$ is the largest connected subset of X that contains x .

⊗ Example 2.6

Let $X = (0, 1) \cup (1, 2)$. Here, we have two connected components, $(0, 1)$ and $(1, 2)$

⊗ Example 2.7: Middle Thirds Cantor Set

Let $C_0 := [0, 1]$, and given C_n , define $C_{n+1} := \frac{1}{3} (C_n \cup (2 + C_n))$ for $n \geq 0$. C_∞ is totally disconnected.

↪ Definition 2.18: Path Component

A path component $P(x)$ of $x \in X$ is the largest path connected subset of X that contains x .

↪ Proposition 2.19

$P(x) = \{x \in X : \exists \text{ continuous path } \gamma : [0, 1] \rightarrow X : \gamma(0) = x, \gamma(1) = y\}.$

Remark 2.12. Where we “start” a path does not matter. We write $x \sim y$ if $\exists \gamma$ from x to y ; this is an equivalence relation on the elements of X .

Remark 2.13. The choice of $[0, 1]$ here is arbitrary; any closed interval is homeomorphic.

↪ Lemma 2.4

If $P(x) \cap P(y) \neq \emptyset$, then $P(x) = P(y)$.

Proof. $P(x) \cap P(y) \neq \emptyset \implies \exists z : x \sim z \wedge y \sim z \implies x \sim y.$ ■

↪ Lemma 2.5

If $A \subseteq X$ is connected, then \overline{A} is also connected.

↪ Lemma 2.6

Suppose $A \subseteq X$ is both open and closed. Then, if $C \subseteq X$ is connected and $C \cap A \neq \emptyset$, then $C \subseteq A$.

Proof. If A is both open and closed, then $C \cap A$ is both open and closed in C . If $C \cap A^C \neq \emptyset$, then this is also open and closed in C . Hence, we can write $C = (C \cap A) \cup (C \cap A^C)$, that is, a disjoint union of two nonempty open sets, contradicting the connectedness of C . Hence, $C \cap A^C = \emptyset$, and so $C \subseteq A$. ■

↪ Proposition 2.20

Let $\{C_\alpha\}_{\alpha \in I}$ be a collection of nonempty connected subspaces of X s.t. $\forall \alpha, \beta \in I, C_\alpha \cap C_\beta \neq \emptyset$. Then, $\bigcup_{\alpha \in I} C_\alpha$ is connected.

↪ Proposition 2.21

Suppose each $x \in X$ has a path-connected neighborhood. Then, the path components in X are the same as the connected components in X .

2.8.1 Cantor Staircase Function

↪ Definition 2.19: An Explicit Definition

Let $x \in C : x = 0.a_1a_2a_3 \dots$ (base 3), ie $a_j = \begin{cases} 0 \\ 2 \end{cases}$. Define

$$f(x) = \begin{cases} \sum \frac{a_j/2}{2^j} & x \in C \\ \text{extend by continuity} & x \notin C. \end{cases}$$

That is, if $x \notin C$, set $f(y) = \sup_{x \in C, x < y} f(x) = \inf_{x \in C, x > y} f(x)$.

↪ Definition 2.20: Complement Definition

To construct the complement of the Cantor set, begin with $[0, 1]$ and at a step n , we remove 2^n open intervals from this interval. $f(x)$ will be constant on each of these intervals with values $\frac{k}{2^n}$ where k odd and $0 < k < 2^n$. Extend by continuity to all $x \in C$.

Remark 2.14. *Wikipedia's explanation of this is far better than whatever this definition is trying to say.*

3 L^p SPACES

3.1 Review of ℓ^p Norms

Remark 3.1. Recall that for $1 \leq p \leq +\infty$, we define for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the norm

$$\|x\|_p = (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}, \quad \|x\|_\infty = \max_{i=1}^n |x_i|.$$

Similarly, for infinite vector spaces, we had, for $x = (x_1, \dots, x_n, \dots)$, the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}, \quad \|x\|_\infty = \sup_{i \geq 1} |x_i|.$$

Here, we define

$$\ell_p := \{x = (x_1, \dots, x_n) : \|x\|_p < +\infty\}.$$

3.2 ℓ^p Norms, Hölder-Minkowski Inequalities

↪ **Definition 3.1:** Hölder Conjugates

For $1 \leq p, q \leq +\infty$, we say that p, q are said to be *Hölder conjugates* if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Remark 3.2. We refer to these simply as “conjugates” throughout as no other conception of conjugate numbers will be discussed.

Further, we take by convention $\frac{1}{\infty} = 0$.

↪ **Proposition 3.1:** Hölder’s Inequality

Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$. Suppose $p, q : 1 \leq p, q \leq +\infty$ are conjugate. Then,

$$\langle x, y \rangle_{\mathbb{R}^n} := \left| \sum_{i=1}^n x_i y_i \right| \leq \|x\|_p \cdot \|y\|_q$$

⊗ **Example 3.1**

For the case $p = 1$ or ∞ (functionally, the same case):

↪ **Lemma 3.1**

Let p, q be conjugates, and $x, y \geq 0$. Then,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Remark 3.3. If the inequality holds, then, for some $t > 0$, let $\tilde{x} = t^{\frac{1}{p}} \cdot x$, $\tilde{y} = t^{\frac{1}{q}} y$. Substituting x for \tilde{x} and y for \tilde{y} , we have

$$\text{LHS: } \tilde{x}\tilde{y} = t^{\frac{1}{p}}x \cdot t^{\frac{1}{q}}y = t^{\frac{1}{p}+\frac{1}{q}} \cdot xy = xy$$

$$\text{RHS: } \dots = t\left(\frac{x^p}{p} + \frac{y^q}{q}\right).$$

That is, we have

$$t \cdot xy \leq t\left(\frac{x^p}{p} + \frac{y^q}{q}\right),$$

hence, the inequality is preserved under multiplication by a positive scalar; moreover, the original inequality holds iff this “scaled” version holds. Hence, choosing t such that $\tilde{y} = 1$ (let $t = \left(\frac{1}{y}\right)^q$), it suffices to prove the lemma for $y = 1$.

Proof. If $x = 0$ or $y = 0$, then the entire LHS becomes 0 and we are done; assume $x, y > 0$; by the previous remark, assume wlog $y = 1$. Then, we have

$$\begin{aligned} x \cdot y \leq \frac{x^p}{p} + \frac{y^q}{q} &\iff x \cdot 1 \leq \frac{x^p}{p} + \frac{1}{q} \\ &\iff \frac{x^p}{p} - x + \frac{1}{q} =: f(x) \geq 0. \end{aligned}$$

Taking the derivative, we have

$$\begin{aligned} f'(x) &= \frac{px^{p-1}}{p} - 1 = x^{p-1} - 1 \\ p > 1 &\implies p - 1 > 0 \implies \begin{cases} f'(x) > 0 & \forall x > 1 \\ f'(x) = 0 & x = 0 \\ f'(x) < 0 & \forall 0 < x < 1 \end{cases} \end{aligned}$$

Hence, $x = 1$ is a local minimum of the function, and thus $f(x) \geq f(1) \forall 0 < x \leq 1$. But $f(1) = \frac{1^p}{p} - 1 + \frac{1}{q} = 1 - 1 = 0$, hence $f(x) \geq 0 \forall x \geq 0$, as desired, and the inequality holds. ■

Proof. Assume $\|x\|_p = \|y\|_q = 1$. Then,

$$\begin{aligned}
\left| \sum_{i=1}^n x_i y_i \right| &\leq \sum_{i=1}^n |x_i y_i| && \text{(by triangle inequality)} \\
&\leq \sum_{i=1}^n \left| \frac{x_i^p}{p} + \frac{y_i^q}{q} \right| && \text{(by lemma 3.1)} \\
&= \frac{1}{p} \left(\sum_{i=1}^n |x_i|^p \right) + \frac{1}{q} \left(\sum_{i=1}^n |y_i|^q \right) \\
&= \frac{1}{p} \|x\|_p^p + \frac{1}{q} \|y\|_q^q && \text{(by staring)} \\
&= \frac{1}{p} \cdot 1^p + \frac{1}{q} \cdot 1^q = \frac{1}{p} + \frac{1}{q} = 1 && \text{(by assumption)} \\
&= \|x\|_p \cdot \|y\|_q,
\end{aligned}$$

and the proposition holds, in the special case $\|x\|_p = \|y\|_q = 1$.

If $\|x\|_p = 0$ or $\|y\|_q = 0$, then $x_1 = \dots = x_n = 0$ or $y_1 = \dots = y_n = 0$, resp, then we'd have ($\|x\|_p = 0$ case)

$$0 \cdot y_1 + \dots + 0 \cdot y_n \leq 0,$$

which clearly holds.

Assume, then, $\|x\|_p > 0, \|y\|_q > 0$. Let $\tilde{x} := \frac{x}{\|x\|_p}, \tilde{y} := \frac{y}{\|y\|_q}$. Then,

$$\|\tilde{x}\|_p^p = \frac{(\sum_{i=1}^n |x_i|^p)}{\|x\|_p^p} = \frac{\|x\|_p^p}{\|x\|_p^p} = 1 \implies \|\tilde{x}\|_p = 1.$$

The same case holds for \tilde{y} , hence $\|\tilde{y}\|_q = 1$; that is, we have “rescaled” both vectors. Hence, we can use the case we proved above for when the norms were identically 1 on \tilde{x}, \tilde{y} . We have:

$$\left| \sum_{i=1}^n \tilde{x}_i \tilde{y}_i \right| \leq 1$$

But by definition of \tilde{x}, \tilde{y} , we have

$$\left| \sum_{i=1}^n \tilde{x}_i \tilde{y}_i \right| = \left| \frac{1}{\|x\|_p \|y\|_q} \sum_{i=1}^n x_i y_i \right| \leq 1 \implies \left| \sum_{i=1}^n x_i y_i \right| \leq \|x\|_p \cdot \|y\|_q,$$

and the proof is complete. ■

↪ **Proposition 3.2: Minkowski Inequality**

Let $1 \leq p \leq \infty$, $x, y \in \mathbb{R}^n$. Then,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Remark 3.4. This is just the triangle inequality for ℓ_p norms.

Proof. The cases $p = 1, \infty$ are left as an exercise.

Assume $1 < p < \infty$. Then,

$$\begin{aligned} \|x + y\|_p^p &= \sum_{j=1}^n |x_j + y_j|^p = \sum_{j=1}^n |x_j + y_j| |x_j + y_j|^{p-1} \\ &\leq \sum_{j=1}^n (|x_j| + |y_j|) \cdot |x_j + y_j|^{p-1} \\ &= \underbrace{\sum_{j=1}^n |x_j| \cdot |x_j + y_j|^{p-1}}_{:=A} + \overbrace{\sum_{j=1}^n |y_j| \cdot |x_j + y_j|^{p-1}}^{:=B} \quad \circledast \end{aligned}$$

Let $\vec{u} = (|x_1|, \dots, |x_n|)$ and $\vec{v} = (|x_1 + y_1|^{p-1}, \dots, |x_n + y_n|^{p-1})$, then, $A = \vec{u} \cdot \vec{v} = \langle \vec{u}, \vec{v} \rangle_{\mathbb{R}^n}$. We have

$$\begin{aligned} \|\vec{u}\|_p &= \left(\sum_{i=1}^n (|x_i|^p) \right)^{\frac{1}{p}} = \|x\|_p \\ \|\vec{v}\|_q &= \left(\sum_{i=1}^n (|x_i + y_i|^{p-1})^q \right)^{\frac{1}{q}} \\ &= \left[\sum_{i=1}^n (|x_i + y_i|^{p-1})^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}} \\ &= \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\ &= \|x + y\|_p^{p-1} \end{aligned}$$

where the second-to-last line follows from p, q being conjugate, hence $q = \frac{p}{p-1}$. Thus, by Hölder's Inequality, we have that

$$A = \langle \vec{u}, \vec{v} \rangle \leq \|\vec{u}\|_p \cdot \|\vec{v}\|_q = \|x\|_p \cdot \|x + y\|_p^{p-1}.$$

By a similar construction, we can show that

$$B \leq \|y\|_p \cdot \|x + y\|_p^{p-1}.$$

Thus, returning to our original inequality in \circledast , we have

$$\begin{aligned} \|x + y\|_p^p &\leq A + B \\ &\leq \|x\|_p \cdot \|x + y\|_p^{p-1} + \|y\|_p \cdot \|x + y\|_p^{p-1} \\ &\implies \|x + y\|_p \leq \|x\|_p + \|y\|_p, \end{aligned}$$

and the proof is complete. ■

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3.3 Complete Metric Spaces, Completeness of ℓ_p

↪ **Theorem 3.1**

The sequence of centers of balls with monotonically decreasing radii is a Cauchy sequence in X .

Proof. Let $\varepsilon > 0$ and let $N : \forall j > N, r_j < \varepsilon$. Then,

$$d(x_j, x_k) < r_{\min(j,k)} = r_j$$
■

↪ **Definition 3.2: Complete Metric Space**

A metric space is complete if every Cauchy sequence converges to a limit in that space.

\circledast **Example 3.2: Examples of Complete Metric Spaces**

1. \mathbb{R} , p -adic integers $(\mathbb{Z}_p)/\text{rationals}(\mathbb{Q}_p)$.
2. $\ell_p = \{x = (x_i)_{i=1}^\infty : \sum_{i=1}^\infty |x_i|^p < +\infty\}, 1 \leq p \leq +\infty$
3. $\ell_\infty = \{x = (x_i) : \sup_{i=1}^\infty |x_i| < +\infty\}$.

↪ **Proposition 3.3**

Hölder's Inequality and Minkowski Inequality inequalities hold for infinite sequences. that is,

1. if $x = (x_i) \in \ell_p$ and $y = (y_i) \in \ell_q$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left| \sum_{i=1}^\infty x_i y_i \right| \leq \|x\|_{\ell_p} \|y\|_{\ell_q}.$$

2. if $x, y \in \ell_p$, then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Remark 3.5. 2. gives the triangle inequality for the $\|x\|_p$ norm on ℓ_p .

Moreover,

$$\begin{aligned}\|c \cdot x\|_p &= \|(c_1 x_1, \dots, c_n x_n, \dots)\|_p \\ &= \left(\sum_{i=1}^{\infty} |c x_i|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty} |c|^p |x_i|^p \right)^{\frac{1}{p}} \\ &= (|c|^p)^{\frac{1}{p}} \|x\|_p = c \cdot \|x\|_p\end{aligned}$$

Proof. (of 2.) If $x, y \in \ell_p$, we have that $\sum_{i=1}^{\infty} |x_i|^p < +\infty$, $\sum_{i=1}^{\infty} |y_i|^p < +\infty$, so $\exists N > 0 : \sum_{i=N+1}^{\infty} |x_i|^p < \varepsilon$, $\sum_{i=N+1}^{\infty} |y_i|^p < \varepsilon$. Let $x_i^{(n)} = (x_1, \dots, x_n, 0, 0, \dots)$ be (x) truncated after n (finite) coordinates. This gives

$$\|(x_i + y_i)^{(n)}\|_p \leq \|x_i^{(n)}\|_p + \|y_i^{(n)}\|_p \leq \|x\|_p + \|y\|_p$$

by Minkowski on finite spaces. Taking $n \rightarrow \infty$ (ie, “detruncating”), we have $(x + y) \in \ell_p$, and thus $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

1. left as an exercise. ■

↪ Proposition 3.4

Let $1 \leq p \leq +\infty$, and $\|x\|_{\infty} = \sup_{i=1}^{\infty} |x_i| = A < +\infty$, $\|y\|_{\infty} = \sup_{i=1}^{\infty} |y_i| = B < +\infty$. Then, the triangle inequality $\|x + y\|_{\infty} \leq \|x\|_{\infty} + \|y\|_{\infty}$ holds.

Proof. We have

$$\sup_{i=1}^{\infty} |x_i + y_i| \leq \sup_{i=1}^{\infty} (|x_i| + |y_i|) \leq \sup_{i=1}^{\infty} |x_i| + \sup_{i=1}^{\infty} |y_i| = \|x\|_{\infty} + \|y\|_{\infty}.$$

↪ Proposition 3.5

$\|x\|_{\infty} := \sup_{i=1}^{\infty} |x_i|$ is a well-defined norm on ℓ_{∞} .

Proof. The triangle inequality is prove in proposition 3.4. The remainder of the requirements are left as an exercise. ■

↪ Proposition 3.6

$\ell_p \subseteq \ell_q$ if $p < q$.

Proof. Let $x \in \ell_p$. If $\sum_{i=1}^{\infty} |x_i|^p < +\infty$, then $\exists N : \forall i \geq N, |x_i| \leq 1$. Then,

$$\begin{aligned} \sum_{i \geq N} |x_i|^q &\leq \sum_{i \geq N} |x_i|^p < \infty \\ \Rightarrow \sum_{i=1}^{\infty} |x_i|^q < +\infty &\Rightarrow x \in \ell_q \\ &\Rightarrow \ell_p \subseteq \ell_q \end{aligned}$$

■

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3.4 Contraction Mapping Theorem

↪ Definition 3.3: Contraction Mapping

Let (X, d) be a metric space. A *contraction mapping* on X is a function $f : X \rightarrow X$ for which \exists a constant $0 < c < 1$ such that

$$d(f(x), f(y)) \leq c \cdot d(x, y) \quad \forall x, y \in X.$$

↪ Theorem 3.2: Contraction Mapping Theorem

Let (X, d) be a complete metric space, and let $f : X \rightarrow X$ be a contraction. Then, there exists a unique fixed point z of f such that $f(z) = z$.

Moreover, $f^{[n]}(x) := f \circ f \circ \dots \circ f(x) \rightarrow z$ as $n \rightarrow \infty$ for any $x \in X$.

Remark 3.6. The “functional construction” of the Cantor set is an example of a contraction mapping, with $f_1(x) = \frac{x}{3}$, $f_2(x) = \frac{x+2}{3}$. The first has a fixed point of 0, and the second a fixed point of 1.

Remark 3.7. This is a generalization of [this proof](#) done in Analysis I, an equivalent claim over the reals.

Proof. Fix $x \in X$. Consider the sequence $\{x_0, x_1, x_2, \dots, x_n, \dots\} := \{x, f(x), f \circ f(x), \dots, f^{[n]}(x), \dots\}$ (we call $f^{[n]}$ the *orbit* of x under iterations of f). We claim that this is a Cauchy sequence. Let $n \in \mathbb{N}$ arbitrary, then we have, by the property of the contraction mapping,

$$d(f^{[n+1]}(x) - f^{[n]}(x)) \leq c \cdot d(f^{[n]}(x) - f^{[n-1]}(x)) \leq c^2 d(f^{[n-1]}(x) - f^{[n-2]}(x)).$$

Arguing inductively, it follows that

$$d(f^{[n+1]}(x) - f^{[n]}(x)) \leq c^n d(f(x), x). \quad \star$$

Let now $m, k \in \mathbb{N}, m, k > 0$. It follows that

$$\begin{aligned} d(f^{[m]}, f^{[m+k]})(x) &\leq d(f^{[m]}(x), f^{[m+1]}(x)) + d(f^{[m+1]}(x), f^{[m]}(x)) + \dots + d(f^{[m+k-1]}(x), f^{[m+k]}(x)) \\ &\stackrel{\star}{\leq} d(x, f(x)) [c^m + c^{m+1} + \dots + c^{m+k-1}] \\ &\leq c^m d(x, f(x)) [1 + c + \dots + c^k + c^{k+1} + \dots] = \frac{c^m d(x, f(x))}{1 - c} \end{aligned}$$

Now, given $\varepsilon > 0$, choose N such that $\frac{c^N d(x, f(x))}{1 - c} < \varepsilon$. It follows, then, that $\{f^{[n]}(x)\}_{n \in \mathbb{N}}$ a Cauchy sequence, and thus converges, $f^{[n]}(x) \rightarrow z$ as $n \rightarrow \infty$ for some z .

We further have to show that $f(z) = z$. It is easy to show that f continuous due to the contraction mapping (it is clearly Lipschitz with constant c), and it thus follows that

$$\lim_{n \rightarrow \infty} f(f^{[n]}(x)) = \lim_{n \rightarrow \infty} f^{[n]}(x) \implies f(z) = z,$$

by sequential characterization of continuous functions.

Finally, we need to show that this limit is unique. Suppose $\exists y_1 \neq y_2$, ie two fixed points with $f(y_1) = y_1$ and $f(y_2) = y_2$. Then, by the property of the contraction mapping,

$$d(f(y_1), f(y_2)) \leq c \cdot d(y_1, y_2),$$

but by assumption of being fixed points,

$$d(f(y_1), f(y_2)) = d(y_1, y_2),$$

implying $d(y_1, y_2) \leq c \cdot d(y_1, y_2)$. This is only possible if $d(y_1, y_2) = 0$, and thus $y_1 = y_2$ and the fixed point is indeed unique. ■

↪ **Theorem 3.3:** ℓ_p complete

The space ℓ_p is complete for all $1 \leq p \leq +\infty$.

Equivalently, if $(x^1), (x^2), \dots, (x^n)$ is a Cauchy sequence in ℓ^p , $\exists y \in \ell^p$ s.t. $x^n \rightarrow y$ as $n \rightarrow \infty$.

Proof. (Sketch) We suppose first $p < +\infty$. Consider an arbitrary number of Cauchy sequences in ℓ_p :

$$\begin{aligned} x^{(1)} &= (x_1^{(1)}, \dots, x_n^{(1)}, \dots) \\ x^{(2)} &= (x_1^{(2)}, \dots, x_n^{(2)}, \dots) \\ &\vdots \quad \vdots \quad \vdots \\ x^{(k)} &= (x_1^{(k)}, \dots, x_n^{(k)}, \dots) \in \ell_p \end{aligned}$$

We claim that, for any $k \in \mathbb{N}$, the $(x_k^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence; note that in this definition we are taking a *fixed-index* (namely, the k th) element from different sequences (namely, the n th sequence).

Since $x^{(1)}, x^{(2)}, \dots, x^{(n)}, \dots$ are Cauchy sequences in ℓ^p , we have for a fixed $\varepsilon > 0$, $\exists N \in \mathbb{N} : \forall m, n > N$, $d_p(x^{(m)}, x^{(n)}) < \varepsilon$:

$$\begin{aligned} d_p(x^{(m)}, x^{(n)})^p &= \|x^{(m)} - x^{(n)}\|_p^p = \sum_{i=1}^{\infty} |x_i^{(m)} - x_i^{(n)}|^p < \varepsilon^p \\ |x_k^{(m)} - x_k^{(n)}|^p &\leq \sum_{i=1}^{\infty} |x_i^{(m)} - x_i^{(n)}|^p \implies |x_k^{(m)} - x_k^{(n)}|^p < \varepsilon^p \\ &\implies |x_k^{(m)} - x_k^{(n)}| < \varepsilon, \end{aligned}$$

since we are taking “less of the summands in the second line”. It follows, then, that for each k , $\exists z_k : x_k^{(n)} \rightarrow z_k$ as $n \rightarrow \infty$. Let $z = (z_1, \dots, z_n, \dots)$. We claim that $x^{(n)} \rightarrow z \in \ell_p$ as $n \rightarrow \infty$.

First, we show that $d_p(x^{(n)}, z) \rightarrow 0$ as $n \rightarrow \infty$ (that is, $x^{(n)} \rightarrow z$ as $n \rightarrow \infty$). Fix $\varepsilon > 0$, and choose $N \in \mathbb{N}$ for which $d_p(x^{(m)}, x^{(n)}) < \varepsilon \forall m, n \geq N$ (by Cauchy). Fix $K \in \mathbb{N}, K > 0$.

$$\begin{aligned} d_p^p(x^{(n)}, z) &= \|x^{(n)} - z\|_p^p = \sum_{i=1}^{\infty} |x_i^{(n)} - z_i|^p \\ \|x^{(m)} - x^{(n)}\|_p^p < \varepsilon^p &\implies \sum_{i=1}^K |x_i^{(m)} - x_i^{(n)}|^p \leq \varepsilon^p \end{aligned}$$

Let $m \rightarrow \infty$; then $x_i^{(m)} \rightarrow z_i$ (note that i fixed!), and we have

$$\sum_{i=1}^K |z_i - x_i^{(n)}|^p \leq \varepsilon^p.$$

Let $K \rightarrow \infty$; then,

$$\sum_{i=1}^{\infty} |z_i - x_i^{(n)}|^p \leq \varepsilon^p \implies \|z - x^{(n)}\|_p \leq \varepsilon \implies d_p(z, x^{(n)}) \leq \varepsilon,$$

and thus $x^{(n)} \rightarrow z$ as $n \rightarrow \infty$.

It remains to show that $z \in \ell_p$, ie $\|z\|_p < +\infty$. We have:

$$\|z\|_p \leq \underbrace{\|z - x^{(n)}\|_p}_{\rightarrow 0} + \|x^{(n)}\|_p.$$

For sufficiently large n , $\|z - x^{(n)}\| \leq 1$ (for instance); $x^{(n)} \in \ell_p$, hence $\|x^{(n)}\|_p < +\infty$ (say, $\|x^{(n)}\|_p \leq M$). Thus:

$$\|z\|_p \leq 1 + M < +\infty \implies z \in \ell_p,$$

and the proof is complete. ■

3.5 Equivalent Notions of Compactness in Metric Spaces

↪ Definition 3.4: Totally Bounded

Let (X, d) be a metric space. If for every $\varepsilon > 0$, $\exists x_1, \dots, x_n \in X, n = n(\varepsilon) : \bigcup_{i=1}^n B(x_i, \varepsilon) = X$, we say X is *totally bounded*.

↪ Lecture 09; Last Updated: Thu Mar 28 09:13:48 EDT 2024

↪ Theorem 3.4

Let (X, d) be a metric space. TFAE:

1. X is complete and totally bounded;
2. X is compact;
3. X is sequentially compact (every sequence has a convergent subsequence).

Proof. (1. \implies 2.) Suppose X complete and totally bounded. Assume towards a contradiction that X not compact, ie there exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of X with no finite subcover.

X being totally bounded gives that it can be covered by finitely many open balls of radius $\frac{1}{2}$. It must be that at least one of these open balls cannot be finitely covered, otherwise we would have a finite subcover. Let F_1 be the closure of this ball. F_1 closed, with diameter $\text{diam}(F_1) \leq 1$. X .

We also have that X can be covered by finitely many balls of radius $\frac{1}{4}$; again, there must be at least one ball B_1 such that $B_1 \cap F_1$ cannot be covered by finitely many open sets from the cover. Let $F_2 = \overline{B_1} \cap F_1$ -closed, with $\text{diam}(F_2) \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.⁴

Arguing inductively, at some step n , X can be covered by finitely many balls of radius $\frac{1}{2^n}$; at least one of these balls B cannot be covered by a finite subcover hence $B \cap F_{n-1}$ cannot be covered by finitely many U_α 's. Let $F_n = \overline{B} \cap F_{n-1}$ -closed, with $\text{diam}(F_n) \leq \frac{1}{2^{n-1}}$.

As such, we have a nested sequence $F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$ of closed sets, where $\text{diam}(F_k) \leq \frac{1}{2^{k-1}} \rightarrow 0$ as $k \rightarrow \infty$.

↪ Lemma 3.1 (Cantor Intersection Theorem). $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$.

Proof. (Of Lemma) Let $x_k \in F_k$. Then, $\{x_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence, since

$$d(x_n, x_{n+k}) \leq \text{diam}(F_n) + \dots + \text{diam}(F_{n+k}) \leq \frac{1}{2^{n-1}},$$

by the nested property, which can be made arbitrarily small for sufficiently large n, k . Hence, $x_n \rightarrow y \in X$ for some y , as X complete. The tail of x_n lies in F_n for all sufficiently large n , and as each F_n closed, the limit must lie in F_n for all sufficiently large n . We conclude the intersection nonempty. ■

⁴ B_1 has radius $\frac{1}{4}$ and hence diameter $\frac{1}{2}$. The intersection of B_1 with a set with a larger diameter must have diameter $\leq \frac{1}{2}$

This y from the lemma is covered by some U_{α_0} -open for some $\alpha_0 \in I$. Being open, $\exists \varepsilon > 0 : B(y, \varepsilon) \subseteq U_{\alpha_0}$. Let $n : \frac{1}{2^{n-1}} < \varepsilon$. Then, $y \in F_n$, and as $\text{diam}(F_n) \leq \frac{1}{2^{n-1}}$, we have that $F_n \subseteq B(y, \frac{1}{2^{n-1}}) \subseteq B(y, \varepsilon) \subseteq U_{\alpha_0}$. But then, we have that F_n covered by a single open set U_{α_0} , a contradiction to our inductive construction of F_n . We conclude X compact.

(2. \implies 3.) Suppose X compact. Let $\{x_n\}_{n \in \mathbb{N}} \in X$. Let $F_n = \overline{\bigcup_{k \geq n} \{x_k\}}$ -closed; we have too that $F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$.

↪ Definition 3.5: Finite Intersection Property

\mathcal{F} has finite intersection property provided any finite subcollection of sets in \mathcal{F} has a non-empty intersection.

↪ Lemma 3.2 (Finite Intersection Formulation of Compactness). X -compact \iff every collection \mathcal{F} of closed subsets of X with finite intersection property has non-empty intersection.

Proof. ■

This lemma directly gives that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, $\{F_n\}_{n \in \mathbb{N}}$ being a collection of closed subsets with any subset having nonempty intersection (by the nestedness). Let $y \in \bigcap_{n=1}^{\infty} F_n$. Take $B(y, \frac{1}{k})$, which thus has nonempty intersection with $\{x_k\}_{k \geq n} \forall n$, ie $\exists n_1 : d(y, x_{n_1}) < 1$ and $\exists n_2 > n_1 : d(y, x_{n_2}) < \frac{1}{2}$. Arguing inductively, $\exists n_j > n_{j-1} : d(y, x_{n_j}) < \frac{1}{j}$ for any given n_{j-1} . It follows that $\lim_{j \rightarrow \infty} x_{n_j} = y$, and thus $\{x_{n_j}\}$ is a convergent subsequence of $\{x_n\}$ that converges within X , and thus X is sequentially compact.

(3. \implies 1.) Suppose X sequentially compact. Let $\{x_n\} \in X$ be a Cauchy sequence in X , which thus have a convergent subsequence $\{x_{n_k}\} \rightarrow y$.

↪ Lemma 3.3. Let $\{x_n\}$ be a Cauchy sequence in X where X sequentially compact. Then, if $\{x_{n_k}\} \rightarrow y$, so does $\{x_n\} \rightarrow y$

Proof. ■

Then, $\{x_n\}_n \rightarrow y$ and so X complete.

Suppose X not totally bounded, ie $\exists \varepsilon > 0 : X$ cannot be covered by a finite union of balls of $B(x_j, \varepsilon)$. Let $x_1 \in X$ s.t. $B(x_1, \varepsilon) \not\supseteq X$; $\exists x_2 \in X \setminus B(x_1, \varepsilon)$, and so $X \not\subseteq B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$ by assumption. Then, choose $x_3 \in X \setminus (B(x_1, \varepsilon) \cup B(x_2, \varepsilon))$. Arguing inductively, we have that $\exists x_n \in X \setminus (\bigcup_{i=1}^n B(x_i, \varepsilon))$, noting that $d(x_n, x_j) \geq \varepsilon \forall 1 \leq j \leq n$.

Consider the sequence $\{x_j\}_{j \in \mathbb{N}}$:

↪ Lemma 3.4. $\{x_j\}$ cannot have a convergent subsequence.

Proof. Follows by $d(x_m, x_n) \geq \varepsilon \forall m, n$. ■

This contradicts our assumption that X sequentially compact, and we conclude X must be totally bounded. ■

⊗ **Example 3.3: Complete Metric Space Example: L^p norm**

Let $f \in C([a, b])$. We define the norm

$$\|f\|_p := \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

As desired, $\|f\|_p \geq 0$; $\|f\|_p = 0 \iff f \equiv 0$; $\|c \cdot f\|_p = c \cdot \|f\|_p$.

Hölder's and Minkowski's inequalities for functions also hold; for $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$,

$$\int |fg| \leq \|f\|_p \cdot \|g\|_q; \quad \|f + g\|_p \leq \|f\|_p + \|g\|_p,$$

respectively.

We similarly have the L^∞ norm, namely, for a function $f : [a, b] \rightarrow \mathbb{R}$,

$$\|f\|_\infty = \sup_{x \in [a, b]} |f(x)|,$$

which obeys all the necessary properties as well.

Let $f_n \rightarrow f$ in $C([a, b])$, wrt $\|\cdots\|_\infty$, where $\{f_n\}_{n \in \mathbb{N}}$ a sequence of functions. Namely, we say that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \sup_{x \in [a, b]} |f_n(x) - f(x)| < \varepsilon.$$

If this holds, we say that f_n *uniformly converges*.

We say that $f_n(x) \rightarrow f(x)$ *pointwise* on $[a, b]$ if $\forall x \in [a, b], f_n(x) \rightarrow f(x)$. Note that uniform convergence implies pointwise convergence, but not the converse.

↪ **Theorem 3.5**

Suppose $f_n(x)$ continuous, and $f_n(x) \rightarrow f(x)$ uniformly on $[a, b]$. Then, $f(x)$ also continuous on $[a, b]$.

Proof. Fix $\varepsilon > 0, x_0 \in [a, b]$. We have that $\exists N : n \geq N, |f_n(x) - f(x)| < \frac{\varepsilon}{3}, \forall x \in [a, b]$.

Let $n \geq N$. $f_n(x)$ continuous at x_0 , hence $\exists \delta(x_0) > 0 : |y - x_0| \implies |f_n(y) - f_n(x_0)| < \frac{\varepsilon}{3}$. We have

$$\begin{aligned} |f(x_0) - f(y)| &\leq |f(x_0) - f_n(x_0)| + |f_n(x_0) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

completing the proof. ■

Remark 3.8. This does not hold with pointwise convergence.

Remark 3.9. We will prove later that $C([a, b])$ is complete for $\|f\|_\infty$, but not for arbitrary $\|f\|_p$, $1 \leq p < +\infty$. To “complete” $C([a, b])$ for $p \neq \infty$, we will need to consider measurable functions and redefine our notion of integration.

↪ Lecture 11; Last Updated: Thu Feb 8 09:51:13 EST 2024

4 DERIVATIVES

4.1 Introduction

↪ Definition 4.1: Differentiable

We say $f(x)$ differentiable at c if $\exists \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$. If so, we denote the limit $f'(c)$.

Remark 4.1. For x close to c , then $f(x) \approx f(c) + f'(c)(x - c)$; this is a linear approximation of f at c .

⊗ Example 4.1: Weierstrass

$f(x) = \sum_{n=1}^{\infty} \frac{\cos(3^n)x}{2^n}$ is continuous in \mathbb{R} , but nowhere differentiable.

↪ Definition 4.2

The derivative, dx , is a linear map $C([a, b]) \rightarrow C^0([a, b])$.

4.2 Chain Rule

Remark 4.2. See [Analysis I notes](#) as well.

↪ Theorem 4.1: Caratheodory's Theorem

Let $f : I \rightarrow \mathbb{R}$, $c \in I$. f is differentiable at $x = c$ iff $\exists \varphi(x) : I \rightarrow \mathbb{R}$ s.t. φ continuous at c and $f(x) - f(c) = \varphi(x)(x - c)$.⁵

Proof. If $f'(c)$ exists, let

$$\varphi(x) = \begin{cases} \frac{f(x) - f(c)}{x - c} & x \neq c \\ f'(c) & x = c. \end{cases},$$

which is well defined. Moreover, for $x \neq c$, $\varphi(x)(x - c) = \frac{f(x) - f(c)}{x - c}(x - c) = f(x) - f(c)$ as desired; the case for $x = c$ is clear. Continuity at c :

$$\lim_{x \rightarrow c} \varphi(x) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = f'(c) = \varphi(c).$$

⁵If not stated otherwise, sets named I or J are intervals.

Conversely, suppose such a φ exists. Then, by continuity,

$$\exists \varphi(c) = \lim_{x \rightarrow c} \varphi(x) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

which gives directly that f differentiable at c . ■

↪ **Theorem 4.2: Chain Rule**

Let $f : J \rightarrow \mathbb{R}, g : I \rightarrow \mathbb{R}, f(J) \subseteq I$. If $f(x)$ differentiable at c and $g(y)$ is differentiable at $y = f(c)$, then $g \circ f(x)$ is also differentiable at c , and

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

Proof. Using Caratheodory's Theorem, $\exists \varphi : f(x) - f(c) = \varphi(x)(x - c)$ with $\varphi(c) = f'(c)$. Let $d = f(c)$, then similarly $\exists \psi : g(y) - g(d) = \psi(y)(y - d)$ with $\psi(d) = g'(d)$, with φ, ψ continuous at c, d resp. Then,

$$g(f(x)) - g(f(c)) = \psi(f(x))(f(x) - f(c)) = (\psi \circ f)(x) \cdot (\varphi(x)(x - c))$$

$\psi \circ f$ is continuous at c , as a composition of continuous functions (ψ, ϕ continuous by construction, f differentiable and thus continuous). It follows, then, that

$$\lim_{x \rightarrow c} \frac{g(f(x)) - g(f(c))}{x - c} = \lim_{x \rightarrow c} (\psi \circ f)(x) \cdot \varphi(x) = \psi(f(c))\varphi(c) = g'(f(c)) \cdot f'(c),$$

by construction. ■

4.3 Critical Points

↪ **Definition 4.3**

$f : I \rightarrow \mathbb{R}$ has a max/min c if $\exists J \subseteq I : x \in J$ s.t. $\max_{x \in J} f(x) / \min_{x \in J} f(x) = f(c)$.

↪ **Theorem 4.3: Rolle's**

Let $f : [a, b] \rightarrow \mathbb{R}$ continuous. Suppose $f'(x)$ exists for all $x \in (a, b)$ and $f(a) = f(b) = 0$. Then, $\exists c \in (a, b) : f'(c) = 0$.

Remark 4.3. A “complex-version” of Rolle's:

↪ **Theorem 4.4: Gauss-Lucas**

Let $P(z)$ be a complex-valued polynomial. Then, the roots of $P'(z)$ lie inside the convex hull of roots of $P(z)$, where a convex hull is the smallest polygon with vertices at the roots of $P(z)$.

↪ Definition 4.4

Consider $P(z) = z^n - 1$ for some $n \in \mathbb{N}$. If z a root, we can show that $(|z|)^n = 1$, hence all roots lie on the unit circle in the complex plane at multiples of the same angle. This gives us a regular n -gon in the complex plane. We then have that $P'(z) = nz^{n-1}$, with has root $z = 0$, which clearly lies within the n -gon hull.

↪ Theorem 4.5: Mean Value

Let f be continuous on $[a, b]$ and differentiable on (a, b) . Then, $\exists c \in (a, b)$ s.t. $f(b) - f(a) = f'(c)(b - a)$.

Proof. Let $\varphi(x) = f(x) - f(a) = \frac{f(b)-f(a)}{(b-a)}(x - a)$, where $\varphi(a) = 0 = \varphi(b)$. By Rolle's theorem, $\exists c \in (a, b) : \varphi'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{(b-a)}$, as desired. ■

↪ Lecture 12; Last Updated: Thu Feb 15 09:47:18 EST 2024

4.4 Aside: Continued Fractions

We have that, for any $x \in \mathbb{R}$, $x = \lfloor x \rfloor + \{x\}$, with $\{x\} \in (0, 1)$; $\lfloor x \rfloor$ and $\{x\}$ are the integral and fractional parts of x respectively.

Fix $x \in \mathbb{R}$, assuming $x \neq 0$.

Let $x_1 := \frac{1}{\{x\}}$. We can write

$$x = \lfloor x \rfloor + \frac{1}{x_1}.$$

If $\{x_1\} \neq 0$, let $x_2 := \frac{1}{\{x_1\}}$ and write

$$x = \lfloor x \rfloor + \frac{1}{\lfloor x_1 \rfloor + \frac{1}{x_2}}.$$

Continuing in this manner, this process stops if $\{x_i\} = 0$ for some i ; if $x \in \mathbb{Q}$, this process will stop, else, it will continue infinitely. For instance, the Golden Ratio $x = \frac{\sqrt{5}+1}{2}$ has continued fraction expansion

$$x = \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}$$

More succinctly, we can denote $a_0 := \lfloor x \rfloor$ and $a_i = \lfloor x_i \rfloor, i \geq 1$, and write

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}}$$

We notate, accordingly, $x := (a_1, a_2, a_3, \dots)$; in this case, the Golden Ratio can be notated $(1, 1, 1, \dots)$.

We denote $\frac{p_n}{q_n}$ as the n th continued fraction of a given x . It turns out that this is the best possible rational approximation for $x \notin \mathbb{Q}$.

4.5 Back To Derivatives

↪ Theorem 4.6

$f : I \rightarrow \mathbb{R}$, differentiable. f is increasing (resp decreasing) iff $f'(x) \geq 0 \forall x \in I$ (resp $f'(x) \leq 0 \forall x \in I$).

↪ Proposition 4.1

Let f continuous on $I = [a, b]$. Let $a < c < b$ and suppose f differentiable on (a, c) and (c, b) . Suppose $f'(x) \geq 0$ on $(c - \delta, c)$ and $f'(x) \leq 0$ on $(c, c + \delta)$ for some $\delta > 0$. Then, f has local max at $x = c$.

↪ Lemma 4.1

Let $I \subseteq \mathbb{R}$, and assume $f : I \rightarrow \mathbb{R}$ is differentiable at $x = c \in I$.

1. If $f'(c) > 0$, then $\exists \delta > 0 : f(x) > f(c) \forall x \in I, x \in (c, c + \delta)$.
2. (Reverse statement for $f'(c) < 0$)

↪ Theorem 4.7: Darboux

Suppose f differentiable on $I := [a, b]$ and $f'(a) < k < f'(b)$. Then, $\exists c \in (a, b)$ such that $f'(c) = k$.

↪ Lecture 13; Last Updated: Thu Feb 15 09:49:55 EST 2024

4.6 L'Hopital's Rules

↪ Proposition 4.2

Suppose $f(x), g(x) : [a, b] \rightarrow \mathbb{R}$ with $f(a) = g(a) = 0$, and $g(x) \neq 0 \forall a < x < b$. Suppose f, g are differentiable at $x = a$ and $g'(a) \neq 0$. Then, $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)}$ exists, and moreover, it is equal to $f'(a)g'(a)$.

Proof.

$$\lim_{x \rightarrow a^+} \left(\frac{f(x) - f(a)}{x - a} \right) / \left(\frac{g(x) - g(a)}{x - a} \right) = \lim_{x \rightarrow a^+} \frac{f(x)}{x - a} \frac{x - a}{g(x)} = \lim_{x \rightarrow a^+} \frac{f(x)}{g(x)},$$

but the original line is simply $\frac{f'(a)}{g'(a)}$. ■

⊗ Example 4.2

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{\cos(0)}{1} = 1.$$

↪ **Theorem 4.8: Cauchy Mean Value**

Let $f(x), g(x) : [a, b] \rightarrow \mathbb{R}$ where f, g continuous on $[a, b]$ and differentiable on (a, b) . Assuming $g'(x) \neq 0, \forall x \in (a, b)$, then $\exists c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

↪ **Proposition 4.3: More General L'Hopital**

let $-\infty \leq a < b \leq +\infty$ and f, g differentiable on (a, b) . Suppose $\lim_{x \rightarrow a^+} f(x) = 0 = \lim_{x \rightarrow a^+} g(x)$.

1. If $\exists L := \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}$ where L some real number, then $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$ as well.
2. If $\exists L := \lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)}$ where $L = +\infty$ or $-\infty$, then $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$ as well.

↪ **Proposition 4.4**

Let $-\infty \leq a < b \leq +\infty, f, g$ differentiable on (a, b) and $g'(x) \neq 0 \forall x \in (a, b)$. Suppose $\lim_{x \rightarrow a^+} g(x) = \pm\infty$.

1. If $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} =: L$ exists and is some finite real number, then $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$ as well.
2. If $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} =: L$ exists and is $\pm\infty$, then $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$ as well.

4.7 Taylor's Theorem

↪ **Theorem 4.9: Taylor's Theorem**

Let $I = [a, b] \subseteq \mathbb{R}, f : I \rightarrow \mathbb{R}, f \in C^n(I)$ and suppose $f^{(n+1)}(x)$ exists on (a, b) . Let $x_0 \in [a, b]$. Then, for any $x \in [a, b], \exists c$ between x, x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

↪ **Theorem 4.10: Relative Extrema**

⁶Let $I \subseteq \mathbb{R}$ be an open interval, $x_0 \in I$, and $n \geq 2$. Suppose $f', f'', \dots, f^{(n)}$ are continuous in a neighborhood of x_0 , and $f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) \neq 0$. Then:

1. if n is even and $f^{(n)}(x_0) > 0$, then f has a local minimum at x_0 ;
2. if n is even and $f^{(n)}(x_0) < 0$, then f has a local maximum at x_0 ;
3. if n is odd, then f has neither a local minimum nor maximum at x_0 .

Proof. If $n := 2m$ -even and $f^{(2m)}(x_0) > 0$, then $f^{(n)}(c) > 0$ so $f(x) - f(x_0) = f^{(2m)}(c)(x - x_0) > 0$. ■

4.8 Convex Sets

↪ **Definition 4.5: Convex Set**

$A \subseteq V$ -vector space over \mathbb{R} is *convex* if for any $x, y \in A$ and any $0 \leq t \leq 1$, $t \cdot x + (1 - t) \cdot y \in A$.

↪ **Definition 4.6: Convex Function**

Let $f : I \rightarrow \mathbb{R}$. f is *convex* if $\forall x_1, x_2 \in I$ and $0 \leq t \leq 1$,

$$f((1 - t)x_1 + tx_2) \leq (1 - t)f(x_1) + tf(x_2).$$

↪ Lecture 15; Last Updated: Thu Feb 22 21:53:23 EST 2024

5 RIEMANN INTEGRAL

5.1 Introduction

↪ **Definition 5.1: Partitions**

A *partition* is a division of an interval (a, b) , denoted

$$\mathcal{P} = \{a = x_0, x_1, \dots, x_{n-1}, x_n = b\}.$$

We define $\text{diam}(\mathcal{P}) := \max_n |x_i - x_{i-1}|$.

A *marked partition*, denoted $\dot{\mathcal{P}}$, is one in which, for each interval we choose some $t_i \in (x_i, x_{i+1}]$.

⁶Bartle-Sherbert, Theorem 6.4.4

↪ **Definition 5.2: Riemann Sum**

We denote

$$S(f, \dot{\mathcal{P}}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1}).$$

↪ **Definition 5.3: Riemann Integrable**

A function f is *Riemann Integrable* if $[a, b]$ if $S(f, \dot{\mathcal{P}}) \rightarrow L$ as $\text{diam}(\dot{\mathcal{P}}) \rightarrow 0$ for any choice of $t_i \in [x_i, x_{i+1}]$.

That is, $\forall \varepsilon > 0, \exists \delta : \text{if } \text{diam}(\mathcal{P}) < \delta, \text{ then for any choice of } t_i \in [x_i, x_{i+1}] \text{ we have } |L - S(f, \dot{\mathcal{P}})| < \varepsilon$.

↪ **Proposition 5.1**

1. If L exists, it is unique.
2. The integral is linear in $f(x)$; if $\int_a^b f(x) dx$ and $\int_a^b g(x) dx$ exist, then $\int_a^b (c_1 f + c_2 g) dx = c_1 \int_a^b f dx + c_2 \int_a^b g dx$.
3. If $f \leq g$ are Riemann integrable on $[a, b]$, then $\int_a^b f(x) dx \leq \int_a^b g(x) dx$.

↪ **Proposition 5.2**

If $f(x)$ integrable on $[a, b]$, the $f(x)$ is bounded on $[a, b]$.

Proof. Suppose $\int_a^b f$ exists. Let $\varepsilon > 0$, and δ such that if $\text{diam}(\dot{\mathcal{P}}) < \delta$ then $|L - S(f, \dot{\mathcal{P}})| < \varepsilon$. Let $\varepsilon = 1$. Then, $S(f, \dot{\mathcal{P}}) \leq |L| + 1$.

Let $Q = \{a = x_0, \dots, x_n = b\}$ be a partition of $[a, b]$ such that $\text{diam}(Q) < \delta$. Suppose towards a contradiction that f is not bounded on $[a, b]$. Then, f is unbounded on at least one interval $[x_i, x_{i+1}]$, say, on $[x_k, x_{k+1}]$. Let $t_i = x_i$ for $i \neq k$ and choose $t_k \in [x_k, x_{k+1}]$ such that $|f(t_k)|(x_{k+1} - x_k) > |L| + 1 + \left| \sum_{i \neq k} f(t_i)(x_{i+1} - x_i) \right|$ (which we can do by assumption of f being unbounded).

By assumption, $|S(f, \dot{Q})| \leq |L| + 1$, but we have that

$$S(f, \dot{Q}) = \underbrace{\sum_{i \neq k} f(t_i)(x_{i+1} - x_i)}_{:=N} + |f(t_k)|(x_{k+1} - x_k) > 2N + |L| + 1,$$

contradiction. ■

5.2 Cauchy Criterion

↪ **Proposition 5.3: Cauchy Criterion for Integrability**

$f \in \mathcal{R}[a, b] \iff \forall \varepsilon > 0, \exists \delta > 0$: if \dot{P} and \dot{Q} are tagged partitions of $[a, b]$ s.t. $\text{diam } \dot{P} < \delta$ and $\text{diam } \dot{Q} < \delta$, then $|S(f, \dot{P}) - S(f, \dot{Q})| < \varepsilon$.⁷

5.3 Squeeze Theorem

↪ **Theorem 5.1**

Let $f : [a, b] \rightarrow \mathbb{R}$. Then $\int_a^b f$ exists $\iff \forall \varepsilon > 0, \exists \alpha_\varepsilon(x), \omega_\varepsilon(x) \in \mathcal{R}[a, b], \alpha_\varepsilon \leq f \leq \omega_\varepsilon$, and

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon$$

Proof. If $f \in \mathcal{R}[a, b]$ then take $\alpha_\varepsilon = f = \omega_\varepsilon$.

Conversely, let $\varepsilon > 0$. Since $\alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}[a, b]$, then, $\exists \delta > 0$ such that for any tagged partition with $\text{diam } \dot{P} < \delta$, then $|S(\alpha_\varepsilon, \dot{P}) - \int_a^b \alpha_\varepsilon| < \varepsilon$ and $|S(\omega_\varepsilon, \dot{P}) - \int_a^b \omega_\varepsilon| < \varepsilon$, thus

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(\alpha_\varepsilon, \dot{P}) \leq S(f, \dot{P}) \leq S(\omega_\varepsilon, \dot{P}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

Let \dot{Q} be any other tagged partition with $\text{diam } \dot{Q} < \delta$; then, the same inequality holds ie $\int_a^b \alpha_\varepsilon - \varepsilon < S(f, \dot{Q}) < \int_a^b \omega_\varepsilon + \varepsilon$. Subtracting one from the other, we see that

$$|S(f, \dot{P}) - S(f, \dot{Q})| < \int_a^b \omega_\varepsilon - \int_a^b \alpha_\varepsilon + 2\varepsilon < 3\varepsilon,$$

and thus $f \in \mathcal{R}[a, b]$ by Cauchy Criterion. ■

↪ Lecture 16; Last Updated: Tue Feb 27 13:18:19 EST 2024

↪ **Lemma 5.1: BS-7.2.4**

Let $J \subseteq [a, b]$ an interval with endpoints $c < d$. If

$$\varphi_J(x) := \begin{cases} 1 & x \in J \\ 0 & x \notin J \end{cases}.$$

Then, $\varphi_J \in \mathcal{R}[a, b]$ and $\int_a^b \varphi_J = d - c$.

⁷Note that $\mathcal{R}[a, b]$ is the set of all real-valued functions integrable on the interval $[a, b]$.

↪ **Theorem 5.2**

Let $\varphi : [a, b] \rightarrow \mathbb{R}$, $\varphi \in \mathcal{R}[a, b]$; that is, step functions are integrable.

↪ **Theorem 5.3**

f continuous on $[a, b]$ implies $f \in \mathcal{R}[a, b]$.

Proof. (Sketch) f uniform continuous; use this to construct step functions that “bound” f from above and below. Apply the squeeze theorem. ■

↪ Lecture 17; Last Updated: Thu Mar 28 14:33:47 EDT 2024

↪ **Theorem 5.4: BS-7.2.7**

Monotone functions on $[a, b]$ are integrable.

Proof. We show only for increasing. Let $f : [a, b] \rightarrow \mathbb{R}$ be monotone increasing. If f constant, then it is a step function and we are done.

Otherwise, $f(b) - f(a) > 0$. Let $\varepsilon > 0$ and $q \in \mathbb{N}$ such that $h := \frac{f(b)-f(a)}{q} < \frac{\varepsilon}{b-a}$, effectively subdividing the y -axis into q equal-sized parts. Then, let

$$y_i := f(a) + ih, \quad 0 \leq i \leq q,$$

and take

$$A_k := f^{-1}([y_{k+1}, y_k)) = \begin{cases} \emptyset \\ \{x\} \\ I_i \end{cases}.$$

We disregard each $A_k : A_k = \emptyset$, and adjoin the isolated points $\{x\}$ to the I_i 's, and hence have a partition $\cup_k A_k = [a, b]$. Letting $\alpha(x) = y_{k-1}$ and $\omega(x) = y_k$ for $x \in A_k$, then $\alpha(x) \leq f(x) \leq \omega(x) \forall x \in [a, b]$ (effectively, we are created a “series of squeezes”). Then,

$$\int_a^b \omega(x) - \alpha(x) dx = \sum_{k=1}^q (y_k - y_{k-1})(x_k - x_{k-1}) = h(b - a) < \varepsilon,$$

and the proof is completed by applying the squeeze theorem. ■

↪ **Theorem 5.5: Additivity; BS-7.2.8**

Let $f : [a, b] \rightarrow \mathbb{R}$ and $a < c < b$. Then, $f \in \mathcal{R}[a, b] \iff f \in \mathcal{R}[a, c]$ and $f \in \mathcal{R}[c, b]$. Moreover, $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$.

Proof. See book. Remark that this holds for finite summations of integrals as such by induction. ■

5.4 Fundamental Theorem of Calculus

↪ Definition 5.4

Call $F(x)$ a *primitive* of $f(x)$ if F differential and $F'(x) = f(x)$.

↪ Theorem 5.6: Fundamental Theorem of Calculus

Let $F, f : [a, b] \rightarrow \mathbb{R}$ and $E \subseteq [a, b]$ a finite set s.t.

1. F continuous on $[a, b]$
2. $F'(x) = f(x) \forall x \in [a, b] \setminus E$; ie they agree for all but finitely many points
3. $f \in \mathcal{R}[a, b]$

Then, $\int_a^b f(x) = F(b) - F(a)$.

Proof. (Sketch) Remark first that it suffices to prove for $E := \{a, b\}$; using additivity, we can subdivide any other such E into such subsets of 1 or 2 elements.

Fix $\varepsilon > 0$ and take $\delta > 0$ such that for any \dot{P} of $[a, b]$ s.t. $\text{diam } \dot{P} < \delta$, $\left| S(f, \dot{P}) - \int_a^b f(x) \right| < \varepsilon$. Applying the mean value theorem to F on each $[x_{i-1}, x_i]$ of \dot{P} :

$$\begin{aligned} F(x_i) - F(x_{i-1}) &= F'(u_i)(x_i - x_{i-1}), \quad u_i \in [x_{i-1}, x_i] \\ &= f(u_i)(x_i - x_{i-1}) \end{aligned}$$

Hence, summing over each of these,

$$\begin{aligned} F(x_1) - \cancel{F(x_0)} + \cancel{F(x_2)} - F(x_1) + \cdots + \cancel{F(x_n)} - F(x_{n-1}) &= f(u_1)(x_1 - a) + \cdots + f(u_n)(x_n - x_{n-1}) \\ \implies F(b) - F(a) &= \sum_{i=1}^n f(u_i)(x_i - x_{i+1}) =: S(f, \dot{P}_1) \end{aligned}$$

by construction, $\text{diam}(\dot{P}_1) < \delta$ since the only change we have made from \dot{P} is the tags, hence $\left| S(f, \dot{P}_1) - \int_a^b f(x) \right| < \varepsilon$. Thus,

$$\begin{aligned} \left| S(f, \dot{P}_1) - \int_a^b f(x) \right| &= \left| F(b) - F(a) - \int_a^b f(x) \right| < \varepsilon \\ \implies F(b) - F(a) &= \int_a^b f(x) \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

■

5.5 Upper and Lower Riemann Sums

↪ Definition 5.5: Upper/Lower Riemann Sums

For a partition P ,

- $\bar{S}(f, P) := \sum_{i=1}^n (\sup_{t \in [x_{i-1}, x_i]} f(t)) \cdot (x_i - x_{i-1})$
- $\underline{S}(f, P) := \sum_{i=1}^n (\inf_{t \in [x_{i-1}, x_i]} f(t)) \cdot (x_i - x_{i-1})$

↪ Proposition 5.4

For any tagged partition \dot{P} ,

$$\underline{S}(f, P) \leq S(f, \dot{P}) \leq \bar{S}(f, P).$$

Moreover, $f \in \mathcal{R}[a, b]$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\text{diam}(P) < \delta \implies \left| \bar{S}(f, P) - \underline{S}(f, P) \right| < \varepsilon$.

Proof. (Sketch) Remark that this is a similar idea to saying that $\inf = \sup \implies$ limit exists. ■

↪ Proposition 5.5

Let P_1, P_2 be partitions of $[a, b]$, and let P_3 be the *common refinement* of P_1, P_2 . Then

$$\underline{S}(f, P_i) \leq \underline{S}(f, P_3) \leq \bar{S}(f, P_3) \leq \bar{S}(f, P_i), \quad i = 1, 2,$$

that is, the finer refinement always gives a better approximation.

5.6 Indefinite Integral

↪ Definition 5.6

For $f \in \mathcal{R}[a, b]$ and any $z \in [a, b]$, define

$$F(z) := \int_a^z f(x) \, dx.$$

↪ Theorem 5.7

$F(z)$ continuous on $[a, b]$.

Proof. $f \in \mathcal{R}[a, b] \implies f$ bounded $\implies \exists M$ s.t. $|f(x)| \leq M \, \forall x \in [a, b]$, so (assuming $z < w$),

$$|F(z) - F(w)| = \left| \int_a^z f(x) \, dx - \int_a^w f(x) \, dx \right| = \left| \int_z^w f(x) \, dx \right| \leq M \cdot |z - w|,$$

so taking $w \rightarrow z$, $|F(z) - F(w)| \rightarrow 0$. ■

↪ **Theorem 5.8: Another Fundamental Theorem of Calculus**

Let $f \in \mathcal{R}[a, b]$, f -continuous at $c \in [a, b]$. Then $F(z)$ differentiable at c and $F'(c) = f(c)$.

↪ **Corollary 5.1**

If $f(x)$ continuous on $[a, b]$ $F'(x) = f(x) \forall x \in [a, b]$.

↪ **Theorem 5.9: Substitution/Change of Variables**

Let $J := [\alpha, \beta]$, $\varphi : J \rightarrow \mathbb{R}$, $\varphi \in C^1([a, b])$. Suppose $\varphi(J) \subseteq I \subseteq \mathbb{R}$, and let $f : I \rightarrow \mathbb{R}$ be continuous on I . Then,

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x) dx = \int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t) dt.$$

Proof. Left as a (homework) exercise; make use of the chain rule! ■

⊗ **Example 5.1**

Compute $\int_1^4 \frac{\sin(\sqrt{t})}{\sqrt{t}} dt$ using the previous theorem.

5.7 Lebesgue Integrability Criterion

↪ **Definition 5.7: Lebesgue Measure 0**

$A \subseteq \mathbb{R}$ has *Lebesgue measure 0* iff $\forall \varepsilon > 0$, A can be covered by a countable union of intervals $J_k := [a_k, b_k]$ such that $\sum_{k=1}^{\infty} |J_k| \leq \varepsilon$. We also call such an A a null set.

For some set $S \subseteq \mathbb{R}$ and statement P , we say “ P holds for almost every $x \in S$ ” if $\{x \in S : P \text{ false}\}$ has Lebesgue measure 0.

⊗ **Example 5.2**

1. Any countable set is a null set.
2. The Cantor set is a null set.

↪ **Theorem 5.10: Lebesgue Integrability Criterion**

Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Then

$$\begin{aligned} f \in \mathcal{R}[a, b] &\iff f \text{ - continuous for almost every } x \in [a, b] \\ &\iff \{z \in [a, b] : f \text{ discontinuous}\} \text{ has Lebesgue measure } 0. \end{aligned}$$

Remark 5.1. The proof is rather involved, but is in the appendix of Bartle. Its important to remark that this is a necessary and sufficient condition.

⊗ **Example 5.3**

1. Let $f : [0, 1] \rightarrow \mathbb{R}, f(x) := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. f discontinuous everywhere, so $f \notin \mathcal{R}[a, b]$.
2. Let $f(x) := \begin{cases} \frac{1}{b} & x = \frac{a}{b} \in \mathbb{Q} \text{ s.t. } (a, b) = 1 \\ 0 & x \notin \mathbb{Q} \end{cases}$. One can show that f continuous on $x \in \mathbb{R} \setminus \mathbb{Q}$ and only discontinuous on \mathbb{Q} . But this is a countable set so certainly has Lebesgue measure 0 and so $f \in \mathcal{R}[0, 1]$.

↪ Lecture 19; Last Updated: Thu Mar 28 09:11:46 EDT 2024

↪ **Theorem 5.11: Composition**

$f \in \mathcal{R}[a, b], f([a, b]) \subseteq [c, d], \varphi : [c, d] \rightarrow \mathbb{R}$ continuous, then $\varphi \circ f \in \mathcal{R}[a, b]$.

Proof.

$$\{x \text{ s.t. } \varphi \circ f \text{ discontinuous at } x\} \subseteq \{x : f \text{ discontinuous at } x\}$$

since φ continuous. The RHS has Lebesgue measure 0, and thus so does the LHS, hence the proof. ■

↪ **Theorem 5.12: Product Theorem**

$f, g \in \mathcal{R}[a, b] \implies f \cdot g \in \mathcal{R}[a, b]$.

Proof. $f \cdot g = \frac{1}{4} [(f + g)^2 - (f - g)^2]$. $f \pm g \in \mathcal{R}[a, b]$ and so so is $(f \pm g)^2$ by taking $\varphi(x) := x^2$ as in the previous theorem. It follows that $f \cdot g \in \mathcal{R}[a, b]$. ■

5.8 Integration by Parts

↪ **Theorem 5.13**

Let F, G be differentiable on $[a, b]$, with $f := F', g := G'$. Suppose $f, g \in \mathcal{R}[a, b]$, then

$$\int_a^b f(x)G(x) \, dx = F(x)G(x)|_a^b - \int_a^b F(x)g(x) \, dx.$$

Proof. Remark that $(FG)' = F'G + FG' = fG + Fg$, so on the one hand

$$\int_a^b (FG)' \, dx = \int_a^b (fG + Fg) \, dx = \int_a^b fG \, dx + \int_a^b Fg \, dx,$$

but on the other hand, by the fundamental theorem of calculus,

$$\int_a^b (FG)' \, dx = [F \cdot G](a) - [F \cdot G](b) = F(x)G(x)|_a^b,$$

and so

$$\begin{aligned} F(x)G(x)|_a^b &= \int_a^b fG \, dx + \int_a^b Fg \, dx \\ \Rightarrow \int_a^b f(x)G(x) \, dx &= F(x)G(x)|_a^b - \int_a^b F(x)g(x) \, dx, \end{aligned}$$

and hence the result. ■

↪ **Theorem 5.14: Taylor's Theorem, Remainder's Version**

Suppose $f', f'', \dots, f^{(n)}$ exist on $[a, b]$ and $f^{(n+1)} \in \mathcal{R}[a, b]$.⁸ Then,

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!} + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n,$$

with $R_n := \frac{1}{n!} \int_a^b f^{(n+1)}(t)(b-t)^n \, dt$.

Proof. See Bartle; makes use of integration by parts. ■

6 FUNCTION SEQUENCES, SERIES

6.1 Pointwise and Uniform Convergence

⁸Remark that this is a weaker condition than continuity as was used in our previous statement of Taylor's theorem.

↪ Definition 6.1: Pointwise vs Uniform Convergence

We say a sequence of functions $f_n \rightarrow f$ *pointwise* on a set E if $\forall x \in E, f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$.

On the other hand, $f_n \rightarrow f$ *uniformly* on E if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N$ and $x \in E$, $|f_n(x) - f(x)| < \varepsilon$.

Remark 6.1. Notice that uniformly implies pointwise convergence.

⊗ Example 6.1

Let $f_n := \begin{cases} 2nx & 0 \leq x \leq \frac{1}{2n} \\ 0 & x > \frac{1}{2n} \end{cases}$. Show that $f_n \rightarrow 0$ pointwise but not uniformly (hint: $f_n(\frac{1}{2n}) = 1 \forall n$).

↪ Theorem 6.1

Suppose $\lim_{n \rightarrow \infty} f_n$ continuous on $[a, b]$ where each f_n also continuous on $[a, b]$. Then, the space of function $C([a, b])$ equipped with the sup norm is complete.

Proof. Proven in tutorials. ■

↪ Theorem 6.2: Interchange of Limits

Let $J \subseteq \mathbb{R}$ be a bounded interval such that $\exists x_0 \in J : f_n(x_0) \rightarrow f(x_0)$. Suppose $f'_n(x) \rightarrow g(x)$ uniformly $\forall x \in J$. Then, $\exists f : f_n(x) \rightarrow f(x)$ uniformly on J , $f(x)$ differentiable on J , and $f'(x) = g(x) \forall x \in J$.

Proof. This is a rather painful proof; one needs to make use of the “multiple epsilons” from each given continuity/convergence/differentiability statement. ■

↪ Lecture 20; Last Updated: Thu Mar 28 11:59:48 EDT 2024

↪ Theorem 6.3

Let $f_n \in \mathcal{R}[a, b]$, $f_n \rightarrow f$ uniformly on $[a, b]$. Then, $f \in \mathcal{R}[a, b]$ and $\int_a^b f_n(x) dx \rightarrow \int_a^b f(x) dx$.

↪ Theorem 6.4: Bounded Convergence Theorem

$f_n \in \mathcal{R}[a, b]$, $f_n \rightarrow f \in \mathcal{R}[a, b]$, not necessarily uniformly. Suppose $\exists B > 0$ s.t. $|f_n(x)| \leq B \forall x \in [a, b]$. Then, $\int_a^b f_n \rightarrow \int_a^b f$ as $n \rightarrow \infty$.

↪ **Theorem 6.5: Dimi's Theorem/Monotone Convergence**

$f_n \in C([a, b])$, $f_n(x)$ monotone (as a sequence). Suppose $f_n \rightarrow f \in C([a, b])$. Then, $f_n \rightarrow f$ uniformly on $[a, b]$.

↪ Lecture 21; Last Updated: Thu Mar 28 11:58:26 EDT 2024

6.2 Series

↪ **Definition 6.2: Absolute Convergence**

Let $\{x_j\} \in X$ where X a normed vector space (say, \mathbb{R}). We say

$$\sum_{j=1}^{\infty} x_j \text{ converges absolutely} \iff \sum_{j=1}^{\infty} \|x_j\| < +\infty.$$

↪ **Theorem 6.6**

Any rearrangement of absolutely convergent series given the same sum.

↪ **Definition 6.3: Conditional Convergence**

$\sum_{j=1}^{\infty} \vec{x}^{(j)}$ conditionally convergent if $\sum_{j=1}^{\infty} x^{(j)}$ converges (ie each component converges) but $\sum_{j=1}^{\infty} \|\vec{x}^{(j)}\| = \infty$.

↪ **Theorem 6.7**

If $\sum_{i=1}^{\infty} a_i \in \mathbb{R}$ conditionally convergent, you can change the order of summation such that $\forall x \in \mathbb{R}$, $\exists \sigma$ -permutation such that $\sum_{i=1}^{\infty} a_{\pi(i)} = x$.

Proof. (Sketch) Separate a_i into positive, negative parts. Since conditionally convergent, $\sum_{a_j > 0} a_j = +\infty$ and $\sum_{a_j < 0} a_j = -\infty$. Add positive a_i 's until the partial sum $\geq x$, then add negative a_i 's until the partial sum $\leq x$, and repeat. The final rearrangement will converge as desired. ■

↪ Lecture 22; Last Updated: Thu Mar 28 12:14:46 EDT 2024

↪ **Theorem 6.8**

Suppose $\sum_{i=1}^{\infty} \vec{v}_i$, $\vec{v}_i \in \mathbb{R}^n$, converges, but $\sum_{i=1}^{\infty} \|\vec{v}_i\| = +\infty$. Then, the set of rearranged sums $\sum_{i=1}^{\infty} \vec{v}_{\sigma(i)}$ for each $\sigma : \mathbb{N} \leftrightarrow \mathbb{N}$ permutation form an *affine subspace* of \mathbb{R}^n .

6.3 Tests for Absolute Convergence

↪ Proposition 6.1

Let x_n, y_n be sequences and $r := \lim_{n \rightarrow \infty} \left| \frac{x_n}{y_n} \right|$.

1. If $r \neq 0$, $\sum_{n=1}^{\infty} x_n$ converges absolutely iff $\sum_{n=1}^{\infty} y_n$ converges absolutely. In addition, if $0 < r_1 := \liminf \left| \frac{x_n}{y_n} \right| \leq \limsup \left| \frac{x_n}{y_n} \right| =: r_2 < +\infty$, this still holds.
2. If $r = 0$, and if $\sum y_n$ converges absolutely, so does $\sum x_n$.

↪ Proposition 6.2: Root Test

If there $\exists r < 1$ such that $|x_n|^{1/n} \leq r$ for sufficiently large $n \geq K$, then $\sum_{n=K}^{\infty} |x_n| \leq \sum_{n=K}^{\infty} r^{-n}$ converges.

If $|x_n|^{1/n} \geq n$ for $n \geq K$, $\sum x_n$ does not converge absolutely.

↪ Proposition 6.3: Ratio Test

Let $x_n \neq 0$. If $\exists 0 < r < 1$, $\left| \frac{x_{n+1}}{x_n} \right| \leq r$ for sufficiently large n , $\sum x_n$ absolutely convergent.

If $\left| \frac{x_{n+1}}{x_n} \right| \geq 1$ for $n \geq K$, $\sum x_n$ diverges.

↪ Proposition 6.4: Integral Test

Let $f(x) \geq 0$ be non-increasing/non-decreasing function of $x \geq 1$. Then $\sum_{k=1}^{\infty} f(k)$ converges $\iff \lim_{k \rightarrow \infty} \int_1^k f(x) dx$ finite.

↪ Proposition 6.5: Raabe's Test

Let $x_n \neq 0$.

1. Suppose $\exists a > 1$ s.t. $\left| \frac{x_{n+1}}{x_n} \right| \leq 1 - \frac{1}{n}, n \geq K$. Then $\sum x_n$ converges absolutely.
2. If $\exists a \leq 1$ s.t. $\left| \frac{x_{n+1}}{x_n} \right| \geq 1 - \frac{1}{n}, n \geq K$. Then $\sum x_n$ does not converge absolutely.

↪ Corollary 6.1

Let $a := \lim_{n \rightarrow \infty} n(1 - \left| \frac{x_{n+1}}{x_n} \right|)$, if such a limit exists. Then, if $a > 1$, $\sum x_n$ converges absolutely, and if $a < 1$, $\sum x_n$ does not.

6.4 Tests for Non-Absolute Convergence

↪ **Proposition 6.6: Alternating Series**

If $x_n > 0$, $x_{n+1} \leq x_n$, $\lim_{n \rightarrow \infty} x_n = 0 \implies \sum (-1)^n x_n$ converges.

↪ **Lemma 6.1: Abel's Lemma**

Let $x_n, y_n \in \mathbb{R}$. Let $s_0 := 0$, $s_n := \sum_{k=1}^n y_k$. Then, for $m > n$,

$$\sum_{k=n+1}^m x_k y_k = x_m s_m - x_{n+1} s_{n+1} + \sum_{k=n+1}^m (x_k - x_{k+1}) s_k$$

↪ Lecture 23; Last Updated: Thu Mar 28 12:36:05 EDT 2024

7 APPENDIX

7.1 Notes from Tutorials

↪ **Theorem 7.1**

Let (X, d) be a compact metric space.⁹ Let $C(X) := \{f : X \rightarrow \mathbb{R} : f \text{ continuous}\}$ be a vector space. Take the uniform norm $\|f\| := \sup_{x \in X} |f(x)|$ on $C(X)$. Then, $(C(X), \|\cdot\|)$ is complete.¹⁰

Proof. Denote the “canonical norm” $\rho(f, g) := \|f - g\|$.

Let $(f_n) \in C(X)$ be a Cauchy sequence. Then, $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall m, n \geq N, \rho(f_n, f_m) < \varepsilon$.

Fix $x \in X$, noting that

$$|f_n(x) - f_m(x)| \leq \sup_{y \in X} |f_n(y) - f_m(y)| = \rho(f_n, f_m) < \varepsilon. \quad *^1$$

Define, for this fixed x , a sequence in $\mathbb{R} \{f_n(x)\}_{n \in \mathbb{N}}$. By $*^1$, we have that this sequence is Cauchy in \mathbb{R} , but as \mathbb{R} complete, $f_n(x)$ hence converges, to some limit we call $f(x) := \lim_{n \rightarrow \infty} f_n(x)$. Note that x is still fixed at this point; these are but real numbers we are working with here.

Now, as x was completely arbitrary, we can repeat this process for all of X , and define a function $f : X \rightarrow \mathbb{R}$ where $f(x) := \lim_{n \rightarrow \infty} f_n(x)$.

For a fixed x , we have that $f_m(x) \rightarrow f(x)$ as $m \rightarrow \infty$. This implies:

$$\begin{aligned} 0 &\leq \lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| \leq \lim_{m \rightarrow \infty} \varepsilon = \varepsilon \\ &\implies |f_n(x) - f(x)| \leq \varepsilon \quad \forall n \geq N \\ &\implies \rho(f_n, f) = \sup_{x \in X} |f_n(x) - f(x)| \leq \varepsilon \implies f_n \rightarrow f \end{aligned}$$

It remains to show that $f \in C(X)$. Let $c \in X$ and $\varepsilon > 0$, and the corresponding $N \in \mathbb{N} : \rho(f_n, f) < \frac{\varepsilon}{3} \quad \forall n \geq N$. By construction, $f_N \in C(X)$, and is thus continuous at c . This gives that $\exists \delta > 0 : |f_N(x) - f_N(c)| < \frac{\varepsilon}{3}$ whenever $d(x, c) < \delta$.¹¹

Hence, if $d(x, c) < \delta$, we have

$$\begin{aligned} |f(x) - f(c)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &\leq \rho(f, f_N) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

¹⁰In this proof, the compactness is necessary for the norm to be well-defined.

¹⁰In this way, this becomes a Banach Space: a complete, normed vector space.

¹¹Be careful here, there are three different metrics going on; ρ from the vector space, d from the underlying metric space, and $|\cdot|$ from \mathbb{R} .

hence f continuous at c , which was completely arbitrary, and thus $f \in C(X)$. ■

↪ **Theorem 7.2**

Let (X, d) -complete. Let $\{F_n\}$ be a decreasing family of non-empty closed sets with $\lim_{n \rightarrow \infty} \text{diam}(F_n) = 0$. Then, $\exists z : \bigcap_{n \in \mathbb{N}} F_n = \{z\}$.

↪ **Theorem 7.3**

Let (X, d) -complete, and $f : X \rightarrow X$ an “expanding map”, such that $d(x, y) \leq d(f(x), f(y)) \forall x, y \in X$. Then, f is a surjective isometry, ie, $f(X) = X$ and $d(f(x), f(y)) = d(x, y) \forall x, y \in X$.

↪ **Lemma 7.1**

Differentiable \implies Continuous.

Proof. Let $f : I \rightarrow \mathbb{R}$, and $c \in I$ arbitrary. Notice that $\forall x \neq c \in I, f(x) - f(c) = (x - c) \frac{f(x) - f(c)}{x - c}$. Hence,

$$\begin{aligned} \lim_{x \rightarrow c} (f(x) - f(c)) &= \lim_{x \rightarrow c} (x - c) \frac{f(x) - f(c)}{x - c} \\ &= \lim_{x \rightarrow c} (x - c) \cdot \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \\ &= 0 \cdot f'(c) = 0 \\ &\implies \lim_{x \rightarrow c} f(x) = f(c), \end{aligned}$$

hence f continuous, noting that the splitting of the limits is valid as both are defined. ■

⊗ **Example 7.1**

$$\text{Let } f : \mathbb{R} \rightarrow \mathbb{R}, f(x) := \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Claim: f discontinuous at all $x \neq 0$.

Proof. Let $x \neq 0 \in \mathbb{R}$. By density of $\mathbb{Q} \subseteq \mathbb{R}$, there exist sequences $(r_n) \in \mathbb{Q}$ s.t. $r_n \rightarrow x$ and $(z_n) \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $z_n \rightarrow x$. Then:

$$\begin{aligned} \lim_{n \rightarrow \infty} f(r_n) &= \lim_{n \rightarrow \infty} r_n^2 = x^2 \\ \lim_{n \rightarrow \infty} f(z_n) &= \lim_{n \rightarrow \infty} 0 = 0, \end{aligned}$$

hence f discontinuous by the sequential criterion at $x \neq 0$. ■

Claim: $f'(0) = 0$.

Proof. Let $\varepsilon > 0$ and $\delta = \varepsilon$. Notice that $f(x) \leq x^2 \forall x$. Then, we have that $\forall |x| < \delta$,

$$\begin{aligned} \left| \frac{f(x) - f(0)}{x - 0} - 0 \right| &= \left| \frac{f(x)}{x} \right| \\ &\leq \left| \frac{x^2}{x} \right| = |x| < \delta = \varepsilon \end{aligned}$$

■

↪ **Definition 7.1**

Let $f : I \rightarrow \mathbb{R}$. A point $c \in I$ is a local max (resp min) if $\exists \delta > 0$ s.t. $f(x) \leq f(c)$ (resp $f(x) \geq f(c)$) $\forall x \in (c - \delta, c + \delta) \cap I$.

↪ **Lemma 7.2**

Let $f : I \rightarrow \mathbb{R}$ be differentiable at $c \in I^\circ$. If c a local extrema of f , then $f'(c) = 0$.

Proof. Assume wlog that c a local max; if a local min, take $\tilde{f} := -f$ and continue.

Since I° open, $\exists \delta_1 > 0 : (c - \delta_1, c + \delta_1) \subseteq I^\circ \subseteq I$. We also have that $\exists \delta_2 > 0 : f(x) \leq f(c) \forall x \in (c - \delta_2, c + \delta_2) \cap I$, by c an extrema.

Let $\delta := \min\{\delta_1, \delta_2\}$. Then, we have both $(c - \delta, c + \delta) \subseteq I$ and $f(x) \leq f(c) \forall x \in (c - \delta, c + \delta)$.

Since $f'(c)$ exists, $\lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c}$. But we have from the property of being a maximum

that

$$\lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \geq 0, \quad \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \leq 0,$$

hence, as these two limits must agree, they must equal 0 and thus $f'(c) = 0$. ■

7.2 Miscellaneous

⊗ Example 7.2: Rudin, Chapter 7: Differentiability

1. Let f be defined $\forall x \in \mathbb{R}$, and suppose that $|f(x) - f(y)| \leq (x - y)^2$, $\forall x, y \in \mathbb{R}$. Prove that f is constant.¹²

Proof. Let $x > y \in \mathbb{R}$. Then, as $|x - y| = x - y$, we have

$$\begin{aligned} |f(x) - f(y)| \leq (x - y)^2 &\implies \left| \frac{f(x) - f(y)}{x - y} \right| \leq x - y = |x - y| \rightarrow 0 \text{ as } y \rightarrow x \\ &\implies \left| \frac{f(x) - f(y)}{x - y} \right| \rightarrow 0 \end{aligned}$$

This implies, then, that $f'(x)$ is defined $\forall x \in \mathbb{R}$, and moreover, that $f'(x) = 0 \forall x \in \mathbb{R}$. We conclude, then, that $f(x)$ constant $\forall x \in \mathbb{R}$. ■

2. Suppose $f'(x) > 0$ in (a, b) . Prove that f is strictly increasing in (a, b) , and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

Proof. Fix $x > y \in (a, b)$. Then, by the mean value theorem, $\exists z \in (x, y) : f'(z) = \frac{f(x) - f(y)}{x - y}$. Since $f'(z) > 0$, it follows that

$$\frac{f(x) - f(y)}{x - y} > 0 \implies f(x) - f(y) > x - y > 0 \implies f(x) > f(y),$$

hence, f increasing, as $x > y$ arbitrary.

Let now $g := f^{-1}$. ■

¹²Note that this means that f Hölder continuous with constant $\alpha = 2$. Indeed, Hölder continuous functions with $\alpha > 1$ are always constant by a similar proof. For $0 < \alpha \leq 1$, we have the inclusion continuously differentiable \implies Lipschitz $\implies \alpha$ -Hölder \implies uniformly continuous \implies continuous.

7.3 Class Midterm Solutions

↪ Question 7.1

Let X be a topological space, and let $f, g : X \rightarrow \mathbb{R}$ be two continuous functions. Show that the set $\{x \in X : f(x) > g(x)\}$ is an open subset of X .

Proof. Let $A := \{x \in X : f(x) > g(x)\}$. Letting $\varphi(x) := f(x) - g(x) = (f - g)(x)$, then remark that $A \equiv \{x \in X : \varphi(x) > 0\}$, and since differences of continuous functions are continuous, φ continuous. Letting $B := (0, \infty) \subseteq \mathbb{R}$, then, we have that $A = \varphi^{-1}(B)$. But B an open set, and the inverse images of open sets by continuous functions are open, hence A open. ■

↪ Question 7.2

- (a) List three equivalent properties (definitions) of compact sets in metric spaces; you don't need to prove anything.
- (b) Is the unit ball¹³ in the space ℓ^2 of infinite sequences compact? Prove or disprove. You may use any of the properties from (a).

Proof. (a) Every open cover admits a finite subcover \iff sequentially compact \iff complete and totally bounded.

- (b) Denote the closed unit ball centered at $(0, 0, \dots)$ in ℓ^2 , $B := \{x \in \ell^2 : d_2^2(0, x) = \sum_{i=1}^{\infty} |x_i|^2 \leq 1\}$. Consider the sequence of “unit sequences”

$$\{e^n\}_{n \in \mathbb{N}} \in B, \quad e_i^n := \delta_{in}.$$

Then, for any $i \neq j$, $d_2(e^n, e^m) = \sqrt{2} > 1$. It follows that, although $e^n \in B$ for any n , there cannot exist a subsequence of x^n that converges within B (verify why this is!). Thus, B cannot be sequentially compact and thus not compact. ■

¹³Jakobson said in class this is supposed to be a closed ball.

↪ **Question 7.3**

- (a) Define a complete metric space.
- (b) State (without proof) the contraction mapping theorem.
- (c) Let $f : (0, 1) \rightarrow (0, 1)$ be defined by $f(x) = x/2$. Is f a contraction?
- (d) Does f have a fixed point in the open interval $I = (0, 1)$? Does that contradict the contraction mapping theorem?

Proof. (a) A complete metric space is a metric space in which every Cauchy sequence converges within that space.

(b) Let (X, d) be a complete metric space, and let $f : X \rightarrow X$ be a contraction mapping, ie for any $x, y \in X$, $d(f(x), f(y)) \leq c \cdot d(x, y)$ for some $c \in (0, 1)$. Then, the contraction mapping states that f has a unique fixed point $z \in X$, ie $f(z) = z$ and $\lim_{n \rightarrow \infty} f^{(n)}(x) = z$ for any x .

(c) For any $x, y \in (0, 1)$, we have

$$d(f(x), f(y)) = |f(x) - f(y)| = \left| \frac{x - y}{2} \right| = \frac{1}{2} |x - y| = c \cdot d(x, y),$$

so f indeed a contraction mapping with $c := \frac{1}{2}$.

(d) We have that for any $x \in I$, $f^{(n)}(x) = \frac{x}{2^n}$ so x a fixed point iff $\frac{x}{2^n} = \frac{x}{2^{n-1}}$ for some n , which is only possible if $x = 0$, but $0 \notin I$, so indeed f has no fixed point in I . This is not a contradiction to the contraction mapping theorem since $I := (0, 1)$ not complete (indeed, $\frac{1}{n} \in I \forall n$ but $\frac{1}{n} \rightarrow 0 \notin I$).

■

↪ **Question 7.4**

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ be infinite real sequences satisfying $\|x\|_2 \leq 2$ and $\|y\|_2 \leq 3$, where $\|\cdots\|_2$ the ℓ^2 norm.

(a) State Holder's inequality and Minkowski inequality for sequences.

(b) Give an upper bound for $x \cdot y = \sum_i x_i y_i$, and for $\|x + y\|$.

Proof. (a) Holder's inequality: for p, q Holder conjugates and $x \in \ell^p, y \in \ell^q$ we have

$$\left| \sum_{i=1} x_i y_i \right| \leq \|x\|_p \|y\|_q.$$

Minkowski inequality: for $x, y \in \ell^p$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

(b) For x, y as given; by Holders, $x \cdot y \leq \|x\|_p \|y\|_q = 2 \cdot 3 = 6$, and by Minkowski's, $\|x + y\| \leq \|x\| + \|y\| = 2 + 3 = 5$, so 6, 5 are upper bounds for $x \cdot y, \|x + y\|$ respectively.

■

→ **Question 7.5**

(a) State (without proof) Taylor's theorem.

(b) Let $f \in C^4([0, 2])$, and let $f'(1) = f''(1) = f'''(1) = 0$ while $f^{(4)}(1) = 2$. Use (a) to show that $f(x)$ has a local extremum at $x = 1$, and determine its type.

Proof. (a) Let $I := [a, b] \subseteq \mathbb{R}$ and let $f : I \rightarrow \mathbb{R}$ such that $f \in C^n(I)$, and $f^{(n+1)}(x)$ exists on (a, b) . Then, for $x_0 \in [a, b]$, there exists some $c \in (\min(x, x_0), \max(x, x_0))$ such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

(b) By Taylor's, for any $x \in [0, 2]$, there exists some c between x and 1 such that

$$\begin{aligned} f(x) &= f(1) + \underbrace{f'(1)(\cdots) + f''(1)(\cdots) + f'''(1)(\cdots)}_{=0} + \frac{f^{(4)}(c)}{4!}(x - 1)^4 \\ &= f(1) + \frac{f^{(4)}(c)}{4!}(x - 1)^4 \\ \implies f(x) - f(1) &\geq \frac{f^{(4)}(c)}{4!}(x - 1)^4 \forall x \in [0, 2] \end{aligned}$$

By continuity of $f^{(4)}$, there exists some neighborhood V of $x_0 = 1$ such that $f^{(4)}(c)$ has the same sign of $f^{(4)}(1)$. So, for any $x \in V$, $\frac{f^{(4)}(c)}{4!} \geq 0$, since $\frac{f^{(4)}(1)}{4!} = \frac{2}{4!} \geq 0$. Thus, since $(x - 1)^4 \geq 0$, we have that for such x in V ,

$$f(x) - f(1) \geq 0 \implies f(x) \geq f(1).$$

Hence, we have a neighborhood of 1 such that for all x in the neighborhood $f(x) \geq f(1)$. It follows that 1 a local minimum of f .

■