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Analysis 2 MATH255

Course Outline:

Based on Lectures from Winter, 2024 by Prof. Dmitry Jackobson.

Contents

1 Introduction

1.1 Metric Spaces

↪ **Definition 1.1: Metric Space**

A set X is a *metric space* with distance d if

1. (symmetric) $d(x,y) = d(y,x) \geq 0$
2. $d(x,y) = 0 \iff x = y$
3. (triangle inequality) $d(x,y) + d(y,z) \geq d(x,z)$

Remark 1.1. *If 1., 3. are satisfied but not 2., d can be called a "pseudo-distance".*

↪ **Definition 1.2: Open Metric Space**

Let (X,d) be a metric space. A subset $A \subseteq X$ is open $\iff \forall x \in A, \exists r = r(x) > 0$ s.t. $B(x, r(x)) \subseteq A$.

↪ **Definition 1.3: Normed Space**

Let X be a vector space over $\mathbb{R}$. The norm on X, denoted $||x|| \in \mathbb{R}$, is a function that satisfies

1. $||x|| \geq 0$
2. $||x|| = 0 \iff x = 0$
3. $||c \cdot x|| = |c| \cdot ||x||$
4. $||x + y|| \leq ||x|| + ||y||$

If X is a normed vector space over $\mathbb{R}$, we can define a distance d on X by $d(x,y) = ||x - y||$.

↪ **Proposition 1.1**

If X is a normed vector space over $\mathbb{R}$, a distance d on X by $d(x,y) = ||x - y||$ makes (X,d) a metric space.

Proof.

1. $d(x,y) = ||x - y|| \geq 0$
2. $d(x,y) = 0 \iff ||x - y|| = 0 \iff x - y = 0 \iff x = y$
3. $d(x,y) + d(y,z) = ||x - y|| + ||y - z|| \geq ||(x - y) + (y - z)|| = ||x - z|| := d(x,z)$

■

⊛ **Example 1.1: L^p distance in $\mathbb{R}^n$**

Let $\overline{x} \in \mathbb{R}^n, x = (x_1, x_2, \dots, x_n)$. The L^p norm is defined

$$||x||_p := (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

In the case $p = 2, n = 2$, we simply have the standard Euclidean distance over $\mathbb{R}^2$.

Unit Balls: consider when $||x||_p \leq 1$, over $\mathbb{R}^2$.

- $p = 1 : |x_1| + |x_2| \leq 1$; this forms a "diamond ball" in the plane.
- $p = 2 : \sqrt{|x_1|^2 + |x_2|^2} \leq 1$; this forms a circle of radius 1. Clearly, this surrounds a larger area than in $p = 2$.

A natural question that follows is what happens as $p \to \infty$? Assuming $|x_1| \geq |x_2|$:

$$\begin{aligned} ||x||_p &= (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \\ &= \left[|x_1|^p \left(1 + \left|\frac{x_2}{x_1}\right|^p\right)\right]^{\frac{1}{p}} \\ &= |x_1| \left(1 + \left|\frac{x_2}{x_1}\right|^p\right)^{\frac{1}{p}} \end{aligned}$$

If $|x_1| > |x_2|$, this goes to $|x_1|$. If they are instead equal, then $||x||_p = |x_1| \cdot 2^{\frac{1}{p}} \to |x_1| \cdot 1$ as well. Hence, $\lim_{p\to\infty} ||x||_p = \max\{|x_1|, |x_2|\}$. Thus, the unit ball will approach $\max\{|x_1|, |x_2|\} \leq 1$, that is, the unit square.

↪ **Proposition 1.2**

Let $x \in \mathbb{R}^n$. Then, $||x||_p \to \max\{|x_1|, \dots, |x_n|\}$ as $p \to \infty$.

Remark 1.2. *This is an extension of the previous example to arbitrary real space; the proof follows nearly identically.*

↪ **Definition 1.4: Convex Set**

Let X be a normed space, and take $x, y \in X$. The line segment from x to y is the set

$$\{t \cdot x + (1 - t) \cdot y : 0 \leq t \leq 1\}.$$

Let $A \subseteq X$. A is *convex* $\iff$ $\forall\, x, y \in A$, we have that

$$(t \cdot x + (1 - t) \cdot y) \in A \,\forall\, 0 \leq t \leq 1.$$

Remark 1.3. *Think of this as saying "a set is convex iff every point on a line segment connected any two points is in the set".*

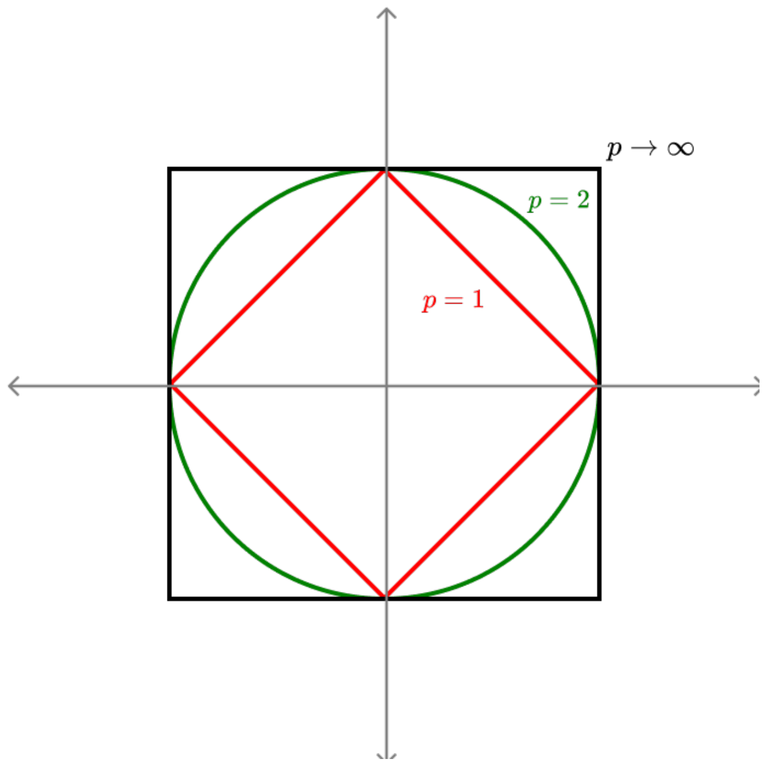

Figure 1: Regions of $\mathbb{R}^2$ where $||x||_p \leq 1$ for various values of p.

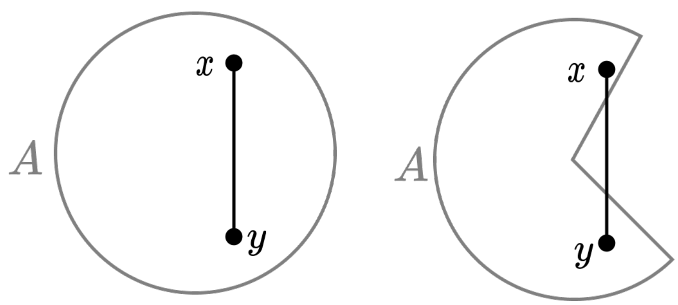

Figure 2: Convex (left) versus not convex (right) sets.

↪ **Definition 1.5:** ℓ_p

The space ℓ_p of sequences is defined as

$$\{x = (x_1, x_2, \ldots, x_n, \ldots) : \sum_{n=1}^{\infty} |x_n|^p < +\infty\} \quad * .$$

Then, $*$ defines the ℓ^p norm on the space of sequences; that is, $||x||_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}}$.

⊛ **Example 1.2:** ℓ_p, $x_n = \frac{1}{n}$

. Let $x_n = \frac{1}{n}$. For which p is $x \in \ell_p$? We have, raising the norm to the power of p for ease:

$$\begin{aligned} ||x||_p^p &= |x_1|^p + |x_2|^p + \cdots + |x_n|^p + \cdots \\ &= 1^p + \left(\frac{1}{2}\right)^p + \cdots < \infty \iff p > 1. \end{aligned}$$

In the case that $p = 1$, this becomes a harmonic sum, which diverges.

⊛ **Example 1.3: L^p space of functions**

Let $f(x)$ be a continuous function. We define the norm of f over an interval $[a, b]$

$$||f||_p = \left[\int_a^b |f(x)|^p \, dx\right]^{\frac{1}{p}}.$$

Remark 1.4. *Triangle inequality for $||x||_p$ or $||f||_p$ is called Minkowski inequality; $||x||_p + ||y||_p \geq ||x + y||_p$. This will be discussed further.*

⊛ **Example 1.4: Distances between sets in $\mathbb{R}^2$**

Let A, B be bounded, closed, "nice" sets in $\mathbb{R}^2$. We define

$$d(A, B) := \text{Area}(A \triangle B),$$

where

$$A \triangle B : (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

It can be shown that this is a "valid" distance.

Remark 1.5. *$\triangle$ denotes the "symmetric difference" of two sets.*

⊛ **Example 1.5: p-adic distance**

Let p be a prime number. Let $x = \frac{a}{b} \in \mathbb{Q}$, and write $x = p^k \cdot \left(\frac{c}{d}\right)$, where c, d are not divisible by p. Then, the p-adic norm is defined $||x||_p := p^{-k}$. It can be shown that this is a norm.

Suppose $p = 2$, $x = 28 = 4 \cdot 7 = 2^2 \cdot 7$. Then, $||28||_2 = 2^{-2} = \frac{1}{4}$; similarly, $||1024||_2 = ||2^{10}||_2 = 2^{-10}$.

More generally, we have that $||2^k||_2 = 2^{-k}$; coversely, $||2^{-k}|| = 2^k$. That is, the closer to 0, the larger the distance, and vice versa, contrary to our notion of Euclidean distance.

↪ **Proposition 1.3**

$||x||_p$ as defined above is a well-defined norm over $\mathbb{Q}$.

Proof. ■

2 Point-Set Topology

2.1 Definitions

↪ **Definition 2.1: Topological space**

A set X is a topological space if we have a collection of subsets τ of X called *open sets* s.t.

1. $\varnothing \in \tau, X \in \tau$
2. Consider $\{A_\alpha\}_{\alpha \in I}$ where A_α an open set for any α; then, $\bigcup_{\alpha \in I} A_\alpha \in \tau$, that is, it is also an open set.
3. If J is a finite set, and A_β open for all $\beta \in J$, then $\bigcap_{\beta \in J} A_\beta \in \tau$ is also open.

In other words, 2.: arbitrary unions of open sets are open, and 3.: finite intersections of open sets are open.

↪ **Definition 2.2: Closed sets**

Closed sets are complements of open sets; hence, axioms for closed sets follow appropriately;

1.* $X, \varnothing$ closed;

2.* B_α closed $\forall\, \alpha \in I \implies \bigcap_{\alpha \in I} B_\alpha$ closed.

3.* B_β closed $\forall\, \beta \in J$, J finite, then $\bigcup_{\beta \in J} B_\beta$ also closed.

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↪ **Definition 2.3: Equivalence of Metrics**

Suppose we have a metric space X with two distances d_1, d_2; will these necessarily admit the same topology?

A sufficient condition is that, if $\forall\, x \neq y \in X, \exists 1 < C < +\infty$ s.t.

$$\frac{1}{C} < \frac{d_1(x,y)}{d_2(x,y)} < C.$$

That is, the distances are equivalent, up to multiplication by a constant.

Indeed, this condition gives that $d_2 < Cd_1$ and $d_2 > \frac{d_1}{C}$; this gives

$$B_{d_1}(x, \frac{r}{c}) \subseteq B_{d_2}(x, r) \subseteq B_{d_1}(x, C \cdot r).$$

Hence, d_1, d_2 define the same open/closed sets on X thus admitting the same topologies. We write $d_1 \asymp d_2$.

Remark 2.1. *If $d_1 \asymp d_2$ and $d_2 \asymp d_3$, then also $d_1 \asymp d_3$. Moreover, clearly, $d_1 \asymp d_1$ and $d_1 \asymp d_2 \implies d_2 \asymp d_1$, hence this is a well-defined equivalence relation.*

Hence, its enough to show that $\forall\, 1 < p < +\infty$, we have $||x||_p \asymp ||x||_\infty$ to show that any $||x||_q$ norm are equivalent for all q on $\mathbb{R}^n$.

↪ **Definition 2.4: Interior, Boundary of a Topological Set**

Let X be a topological space, $A \subseteq X$ and let $x \in X$. We have the following possibilities

1. $\exists U$-open : $x \in U \subseteq A$. In this case, we say $x \in$ the *interior* of A, denoted
$$x \in \text{Int}(A).$$
2. $\exists V$-open : $x \in V \subseteq X \setminus A = A^C$. In this case, we write
$$x \in \text{Int}(X^C).$$
3. $\forall\, U$-open : $x \in U$, $U \cap A \neq \varnothing$ AND $U \cap A^C \neq \varnothing$. In this case, we say x is in the *boundary* of A, and denote
$$x \in \partial A.$$

↪ **Definition 2.5: Closure**

$x \in \text{Int}(A)$ or $x \in \partial A$ (that is, $x \in \text{Int}(A) \cup \partial A$) $\iff$ every open set U that contains x intersects A.[1] Such points are called *limit points* of A. The set of all limits points of A is called the *closure* of A, denoted $\overline{A}$.

Remark 2.2. *We have that*
$$\text{Int}(A) \subseteq A \subseteq \overline{A} = \text{Int}(A) \cup \partial A.$$

↪ **Proposition 2.1: Properties of $\text{Int}(A)$**

$\text{Int}(A)$ is *open*, and it is the largest open set contained in A. It is the union of all U-open s.t. $U \subseteq A$. Moreover, we have that
$$\text{Int}(\text{Int}(A)) = \text{Int}(A).$$

↪ **Proposition 2.2: Properties of $\overline{A}$**

$\overline{A}$ is *closed*; $\overline{A}$ is the smallest closed set that contains A, that is, $\overline{A} = \bigcap B$ where B closed and $A \subseteq B$. We have too that
$$\overline{(\overline{A})} = \overline{A}.$$

[1] "Requires" proof.

↪ **Proposition 2.3**

1. A is open $\iff A = \mathrm{Int}(A)$
2. A is closed $\iff A = \overline{A}$

2.2 Basis

↪ **Definition 2.6: Basis for a Toplogy**

Let τ be a topology on X. Let $\mathcal{B} \subseteq \tau$ be a collection of open sets in X such that every open set is a union of open sets in $\mathcal{B}$.

⊛ **Example 2.1: Example Basis**

$X = \mathbb{R}$, and $\mathcal{B} = \{\text{all open intervals } (a, b) : -\infty < a < b < +\infty\}$.

↪ **Proposition 2.4**

Let $\mathcal{B}$ be a collection of open sets in X. Then, $\mathcal{B}$ is a basis $\iff$

1. $\forall x \in X, \exists U\text{-open} \in \mathcal{B}$ s.t. $x \in U$.
2. If $U_1 \in \mathcal{B}$ and $U_2 \in \mathcal{B}$, and $x \in U_1 \cap U_2$, then $\exists U_3 \in \mathcal{B}$ s.t. $x \in U_3 \subseteq U_1 \cap U_2$.

⊛ **Example 2.2**

Consider $X = \mathbb{R}$. Requirement 1. follows from taking $U = (x - \varepsilon, x + \varepsilon)$ for any $\varepsilon > 0$. For 2., suppose $x \in (a, b) \cap (c, d) =: U_1 \cap U_2$. Let $U_3 = (\max\{a, c\}, \min\{b, d\})$; then, we have that $U_3 \subseteq U_1 \cap U_2$, while clearly $x \in U_3$.

↪ **Proposition 2.5**

In a metric space, a basis for a topology is a collection of open balls,

$$\{B(x, r) : x \in X, r > 0\} = \{\{y \in X : d(x, y) < r\} : x \in X, r > 0\}.$$

Proof. We prove via proposition 2.4. Property 1. holds clearly; $x \in B(x, \varepsilon)\text{-open} \subseteq \mathcal{B}$.

For property 2., let $x \in B(y_1, r_1) \cap B(y_2, r_2)$, that is, $d(x, y_1) < r_1$ and $d(x, y_2) < r_2$. Let

$$\delta := \min\{r_1 - d(x, y_1), r_2 - d(x, y_2)\}.$$

We claim that $B(x, \delta) \subseteq U_1 \cap U_2$.

Let $z \in B(x, \delta)$. Then,

$$d(z, y_1) \overset{\triangle\neq}{\leq} d(z, x) + d(x, y_1) < \delta + d(x, y_1) \leq r_1 - d(x, y_1) + d(x, y_1) = r_1,$$

hence, as $d(z, y_1) < r_1 \implies z \in B(y_1, r_1) = U_1$. Replacing each occurrence of y_1, r_1 with y_2, r_2 respectively gives identically that $z \in B(y_2, r_2) = U_2$. Hence, we have that $B(x, \delta) \subseteq U_1 \cap U_2$ and 2. holds. ■

2.3 Subspaces

↪ **Definition 2.7**

Let X be a topological space and let $Y \subseteq X$. We define the subspace topology on Y:

1. Open sets in $Y = \{Y \cap \text{ open sets in } X\}$

↪ **Proposition 2.6: Consequences of Subspace Topologies**

Suppose $\mathcal{B}$ is a basis for a topology in X. Then, $\{U \cap Y : U \in \mathcal{B}\}$ forms a basis for the subspace $Y \subseteq X$.

Suppose X a metric space. Then, Y is also a metric space, with the same distance.

↪ **Proposition 2.7**

Let $Y \subseteq X$- a metric space. Then, the metric space topology for (Y, d) is the same as the subspace topology.

Proof. (Sketch) A basis for the open sets in X can be written $\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)$; hence

$$Y \cap (\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)) = \bigcup_{\alpha \in I} (Y \cap B(x_\alpha, r_\alpha))$$

is an open set topology for Y. ■

↪ **Lemma 2.1**

Let $A \subseteq X$-open, $B \subseteq A$; B-open in subspace topology for $A \iff B$-open in X.

↪ **Lemma 2.2**

Let $Y \subseteq X$, $A \subseteq Y$. Then, $\overline{A}$ in $Y = Y \cap \overline{A}$ in X. We can denote this

$$\overline{A}_Y = \overline{A}_X \cap Y.$$

2.4 Continuous Functions

↪ Definition 2.8: Continuous Function

Let X, Y be topological spaces. Let $f : X \to Y$. f is *continuous* $\iff$ $\forall$ open $V \in Y$, $f^{-1}(V)$-open in X.

↪ Proposition 2.8

This definition is consistent with the normal ε-δ definition on the real line.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$, continuous; that is, $\forall \varepsilon > 0$, $\forall x \in \mathbb{R} \exists \delta > 0$ s.t. $|x_1 - x| < \delta$, then $|f(x_1) - f(x)| < \varepsilon$.

Let $V \subseteq \mathbb{R}$ open. Let $y \in V$. Then, $\exists \varepsilon : (y - \varepsilon, y + \varepsilon) \subseteq V$. Let $y = f(x)$, hence $y \in f^{-1}(V)$. Now, if $d(x, x_1) < \delta$, we have that $d(f(x_1), f(x)) < \varepsilon$ (by continuity of f), hence $f(x_1) \in (y - \varepsilon, y + \varepsilon) \subseteq V$; moreover, $(x - \delta, x + \delta) \subseteq f{-}1(V)$, thus $f^{-1}(V)$ is open as required.

The inverse of this proof follows identically. ■

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↪ Proposition 2.9

Suppose $\mathcal{B}$ forms a basis of topology for Y. Then, $f : X \to Y$ is continuous if $f^{-1}(U)$ open $\forall U \in \mathcal{B}$.

Proof. If U-open set in Y, then $\exists I$-index set and a collection of open sets $\{A_\alpha\}_{\alpha \in I}$, $A_\alpha \in \mathcal{B}$, s.t. $U = \cup_{\alpha \in I} A_\alpha$. Then, we have

$$f^{-1}(U) = f^{-1}(\cup_{\alpha \in I}(A_\alpha)) = \cup_{\alpha \in I} \underbrace{f^{-1}(A_\alpha)}$$

Hence, if each $f^{-1}(A_\alpha)$ open, then $\cup_{\alpha \in I} f^{-1}(A_\alpha)$ open; hence it suffices to check if $f^{-1}(U) \, \forall U$-open in V is open to see if f continuous. ■

↪ Theorem 2.1: Continuity of Composition

If $f : X \to Y$ continuous and $g : Y \to Z$ continuous, then $g \circ f$ continuous as well.

Proof. Let U-open in Z. Then

$$(g \circ f)^{-1}(U) = \underbrace{f^{-1}(\underbrace{g^{-1}(U)}_{\text{open in } Y})}_{\text{open in } X}$$

■

↪ **Proposition 2.10**

If $f : X \to Y$ continuous and $A \subseteq X$, A has subspace topology, then $f|_A : A \to Y$ is also continuous.[2]

Proof. Let U-open in Y. Then

$$(f|_A)^{-1}(U) = \underbrace{f^{-1}(U)}_{\text{open}} \cap \underbrace{A}_{\text{open}}$$

By the definition of subspace topology, this is an open set and hence $f|_A$ is continuous. ■

2.5 Product Spaces

↪ **Definition 2.9: Finite Product Spaces**

Let $X_1, \ldots, X_n$ be topological spaces. We define

$$(X_1 \times X_2 \times \cdots \times X_n),$$

and aim to define a *product topology*; a basis of which consists of cylinder sets.

↪ **Definition 2.10: Cylinder Set**

A *cylinder set* has the form

$$A_1 \times A_2 \times \cdots \times A_n$$

where each A_j-open in X_j.

⊛ **Example 2.3**

Given an open interval $(a_1, b_1), (a_2, b_2) \subset \mathbb{R}$, the set $(a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$ is a basis for the topology on $\mathbb{R}^2$.

↪ **Definition 2.11: Projection**

Let $X_1 \times X_2 \times \cdots X_n =: X$. The *projection* $\pi_j : X \to X_j$ maps $(x_1, \ldots, x_n) \to x_j \in X_j$.

Remark 2.3. *One can show π_j continuous.*

↪ **Definition 2.12: Coordinate Function**

[2] We denote $f|_A$ as the restriction of the domain of f to A.

Given a function $f : Y \to X_1 \times \cdots X_n = (x_1(y), x_2(y), \ldots, x_n(y))$. The *coordinate function* is

$$f_j = \pi_j \circ f; \quad f_j = x_j(y).$$

↪ **Proposition 2.11**

$f : Y \to X = X_1 \times \cdots \times X_n$ continuous $\iff$ $f_j : Y \to X_j$ continuous.

Proof. Its enough to show that $\forall U \in \mathcal{B}$-basis for X-product space, $f^{-1}(U)$-open in Y. Take $U = A_1 \times \cdots A_n$-open. Then, we claim that

$$f^{-1}(U) = f^{-1}(A_1 \times \cdots \times A_n) = f_1^{-1}(A_1) \cap f_2^{-1}(A_2) \cap \cdots \cap f_n^{-1}(A_n). \quad \star$$

If this holds, then as each f_i continuous (being a composition of continuous functions) and each A_i open in X_i, then each $f_i^{-1}(A_i)$ open in Y and hence $\star$, being the finite intersection of open sets in Y, is itself open in Y.

■

⊛ **Example 2.4: Fourier Transform: Motivation for Infinite Product Toplogies**

Let $f \in C([0, 2\pi])$ is real-valued. We write the nth Fourier coefficients

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos(nx)\,\mathrm{d}x - i\frac{1}{2\pi}\int_0^{2\pi} f(x)\sin(nx)\,\mathrm{d}x\,.\end{aligned}$$

And the Fourier transform of f as the infinite product

$$f(x) \mapsto (\ldots, \hat{f}(-n), \hat{f}(-n+1), \ldots, \hat{f}(-1), \hat{f}(0), \hat{f}(1), \cdots, \hat{f}(n), \ldots) \in \prod_{n\in\mathbb{Z}} (\mathbb{C})_n.$$

Hence, this is an (countably, as indexed by integers) infinite product space.

Now, let $f : \mathbb{R} \to \mathbb{R}$. Suppose $f(x) \to 0$ "fast enough" as $|x| \to \infty$ and f continuous. Then, we can define the Fourier coefficients

$$\hat{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-itx}\,\mathrm{d}x\,,$$

where $t \in \mathbb{R}$. We then have the transform

$$f \mapsto \{\hat{f}(t)\}_{t\in\mathbb{R}}.$$

In this case, our index set is $\mathbb{R}$ is (uncountably) infinite.

↪ **Definition 2.13: Product Topology/Cylinder Sets for ∞ Products**

Let $X = \prod_{\alpha \in I} X_\alpha$. Then, a basis for X is given by cylinder sets of the form $A = \prod_{\alpha \in I} A_\alpha$ where A_α-open in X_α, AND $A_\alpha = X_\alpha$ except for finitely many indices α.

That is, there exists a finite set $J = (\alpha_1, \ldots, \alpha_k) \subseteq I$, such that we can write $A = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$ (where A_α open in X_α).

↪ **Proposition 2.12**

Given $f : Y \to \prod_{\alpha \in I} X_\alpha = X$, then (taking $f_\alpha = \pi_\alpha \circ f$ as before) we have that f is continuous in $X \iff f_\alpha : Y \to X_\alpha$ continuous in $X_\alpha \, \forall \, \alpha \in I$.

Remark 2.4. *Extension of proposition 2.11 to infinite product space.*

Proof. Write $U = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$. Then,

$$f^{-1}(U) = \bigcap_{\alpha \in J} f_\alpha^{-1}(A_\alpha)$$

which is open in Y, hence f continuous. ■

Remark 2.5. *The intersection of the entire spaces give no restriction.*

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2.6 Metrizability

↪ **Proposition 2.13**

Different metrics can define the same topology.

⊛ **Example 2.5**

1. Different ℓ_p metrics in $\mathbb{R}^n$ (PSET 1)

2. Let (X, d) be a metric space. Then,

$$\tilde{d}(x, y) := \frac{d(x, y)}{d(x, y) + 1}$$

   is also a metric (the first two axioms are trivial), and defines the same topology. Note, moreover, that $\tilde{d}(x, y) \leq 1 \, \forall \, x, y$; this distance is bounded, and can often be more convenient to work with in particular contexts.

↪ **Question 2.1**

Suppose (X_k, d_k) are metric spaces $\forall\, k \geq 1$. Then, we can define the product topology τ on

$$X := \prod_{k=1}^{\infty} X_k.$$

Does the product topology τ come from a metric? That is, is τ *metrizable*?

Remark 2.6. *There do indeed exist examples of non-metrizable topological spaces; this question is indeed well-founded.*

Answer. Let $\underline{x} = (x_1, x_2, \dots, x_n, \dots), \underline{y} = (y_1, y_2, \dots, y_n, \dots) \in \prod_{k=1}^{\infty}$ (where $x_i, y_i \in X_i$) be infinite sequences of elements. Then, for each metric space X_k take the metric

$$\tilde{d}_k(x_k, y_k) = \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)}$$

(as in the example above). Then, we define

$$D(\underline{x}, \underline{y}) = \sum_{k=1}^{\infty} \frac{\tilde{d}_k(x_k, y_k)}{2^k},$$

noting that $D(\underline{x}, \underline{y}) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ (by our construction, "normalizing" each metric), hence this is a valid, *converging* metric (which wouldn't otherwise be guaranteed if we didn't normalize the metrics). It remains to show whether this metric omits the same topology as τ. ■

2.7 Compactness, Connectedness

↪ Definition 2.14: Compact

A set A in a topological space is said to be *compact* if every cover has a finite subcover. That is, if

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha - \text{open},$$

then $\exists \{\alpha_1, \dots, \alpha_n \in I\}$ such that $A \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$.

↪ Proposition 2.14

A closed interval $[a, b]$ is compact.

Proof. If $a = b$, this is clear. Suppose $a < b$. Then, suppose

$$[a, b] \subseteq \bigcup_{\alpha \in I} U_\alpha - \text{open}.$$

Then, it must be that $a \in U_{\alpha_1}$ for some α_1; hence, $\exists \varepsilon > 0$ $(a - \varepsilon, a + \varepsilon) \subseteq U_{\alpha_1}$. Let $c = a + \frac{\varepsilon}{2}$, then $[a, c] \subseteq U_{\alpha_1}$. Then, $[a, c]$ has a finite subcover; it is covered by the single open set U_{α_1}.

Let, then, $l =$ least upper bound for all $c \leq b$ s.t. $[a, c]$ has a finite subcover. Then, $l \in [a, b]$ and so $\exists \beta \in I$ such that $l \in U_\beta$. It follows that $\exists \varepsilon > 0$ s.t. $[l - \varepsilon, l] \subseteq U_\beta$.

By our definition of l, $\exists c_j$ s.t. $d(l, c_j) < \frac{1}{j}$ s.t. $[a, c_j]$ is contained in a finite union of U_α's. Let $c_j \in [l - \varepsilon, l]$. Then, $[a, c_j] \subseteq \bigcup_{i=1}^k U_{\alpha_i}$, and then, the interval $[a, l] \subseteq \bigcup_{i=1}^k U_{\alpha_1} \cup U_\beta$.

But we can "choose" U_β such that $[l, l + \varepsilon] \subseteq U_\beta$, and so we'd further have that $[a, l + \varepsilon] \subseteq \bigcup_{i=1}^k U_{\alpha_1} \cup U_\beta$; but we have that $L + \varepsilon > l$, and we have a contradiction to l being the least upper bound for which this holds, unless $l = b$ itself. Then, we cannot "increase" further, and hence $l = b$ and so $[a, l] = [a, b] \subseteq$ a finite union of open sets. ■

Remark 2.7. *A similar proof shows that $[a, b]$ is* connected*; we cannot cover it by two disjoint open sets.*

↪ Theorem 2.2: On Compactness

Let $A \subseteq \mathbb{R}^n$. Then, A compact $\iff$ A closed and bounded.

↪ Proposition 2.15

If X, Y are compact topological spaces, then $X \times Y$ is compact.

Remark 2.8. *By induction, if $X_1, \ldots, X_n$ compact, so is $\prod_{i=1}^n X_i$.*

↪ Proposition 2.16

A closed subset of a compact topological space is compact in the subspace topology.

Proof. (Of theorem 2.2)

($\Longleftarrow$) If $A \subseteq \mathbb{R}^n$ closed and bounded, then $A \subseteq [-R, +R]^n$ for some $R > 0$ (it is contained in some "n-cube"). Then, we have that $[-R, R]$ is compact, by proposition 2.14, proposition 2.15, and proposition 2.16, A itself compact.

($\Longrightarrow$) Suppose $A \subseteq \mathbb{R}^n$ is compact. Then, $\bigcup_{x \in A} B(x, \varepsilon)$ for some $\varepsilon > 0$ is an open cover of A. As A compact, there must exist a finite subcover of this cover, $A \subseteq \bigcup_{i=1}^N B(x_i, r_i)$. Let $R := \max_{i=1}^N (||x_i|| + r_i)$. Then, $A \subseteq \overline{B(0, R)}$, that is, A is bounded.

Now, suppose x is a limit point of A. Then, any neighborhood of x contains a point in A, so $\forall\, r > 0$, $B(x, r) \cap A \neq \varnothing$, and so $\overline{B}(x, r)$ also contains a point of A for any $r > 0$.

Now, suppose $x \notin A$ (looking for a contradiction). Then,

$$U := \bigcup_{r > 0} U_r := \bigcup_{r > 0} (\mathbb{R}^n \setminus \overline{B(x, r)}) = \mathbb{R}^n \setminus \{x\}$$

is an open cover for the set A. A being compact implies that U has an finite subcover such that $A \subset U_{r_1} \cup U_{r_2} \cup \cdots \cup U_{r_N}$. Let $r_0 = \min_{i=1}^N r_i$. Then, $A \subseteq U_{r_0}$, and $A \cap B(x, r_0) = \varnothing$; but this is a contradiction to the definition of a limit point, hence any limit point x is contained in A and A is thus closed by definition. ■

↪ Proposition 2.17

Compact $\Longrightarrow$ sequentially compact; that is, every sequence in a compact set has a convergent subsequence.

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↪ **Definition 2.15: Connected**

A topological space X is *not connected* if $X = U \cup V$ for two open, nonempty, disjoint sets U, V.

If this does not hold, X is said to be *connected.*

A set $A \subseteq X$ is not connected if A is not connected in the subspace topology $\iff$ $A =\subseteq U \cup V$, for U, V-open in X, $(U \cap A) \neq \varnothing$, $(V \cap A) \neq \varnothing$ and $U \cap V = \varnothing$.

↪ **Theorem 2.3**

Let X be a connected topological space. Let $f : X \to Y$ be a continuous. Then, $f(X)$ is also connected.

Proof. Suppose, seeking a contradiction, that X is connected, but $f(X)$ is not. Then, we can write $f(X) \subseteq Y$ as $f(X) \subseteq U \cup V$, such that U, V open in Y and $U \cap V = \varnothing$. Then,

$$(U \cap f(X)) \cap (V \cap f(X)) = \varnothing.$$

We also have that

$$X \subseteq \underbrace{f^{-1}(U)}_{\text{open in } X, \neq \varnothing} \cup \underbrace{f^{-1}(V)}_{\text{open in } X, \neq \varnothing}.$$

$f^{-1}(U) \cap f^{-1}(V) = \varnothing$ (that is, they are disjoint) by our assumption; this is a contradiction to the connectedness of X, as we are able to write it as a subset of two disjoint open sets. Hence, $f(X)$ is indeed connected. ■

↪ **Lemma 2.3**

Any interval $(a, b), [a, b], [a, b), \ldots, \subseteq \mathbb{R}$ is connected.

Proof. ■

↪ **Theorem 2.4: "Intermediate Value Theorem"**

Suppose X is connected and $f : X \to \mathbb{R}$ is a continuous function. Then, f takes intermediate values.

More precisely, let $a = f(x), b = f(y)$ for $x, y \in X$. Assume $a < b$. Then, $\forall\, a < c < b, \exists z \in X$ s.t. $f(z) = c$.

Proof. Suppose, seeking a contradiction, that $\exists c : a < c < b$ s.t. $c \notin f(X)$ (that is, there exists an intermediate value that is "not reached" by the function).

Let $U = (-\infty, c)$ and $V = (c, +\infty)$; note that these are disjoint open sets. Then, we have that

$$X = f^{-1}(U) \cup f^{-1}(V),$$

by our assumption of $c \notin f(X)$. But this gives that X is not connected, as the union of two open (by continuity), disjoint, nonempty ($f(x) = a \in U \implies x \in f^{-1}(U)$, and $f(y) = b \in V \implies y \in f^{-1}(V)$) sets, a contradiction. ■

↪ **Theorem 2.5**

Suppose X is compact, Y-topological space, $f : X \to Y$ is a continuous function. Then, $f(X)$ is also compact.

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(X) \subseteq Y$, that is,

$$f(X) \subseteq \bigcup_{\alpha \in I} U_\alpha \implies X \subseteq f^{-1}(\bigcup_{\alpha \in I} U_\alpha) = \bigcup_{\alpha \in I} f^{-1}(U_\alpha) =: \bigcup_{\alpha \in I} V_\alpha - \text{open}.$$

Then, this is an open cover of X; X is compact, thus there exists a finite subcover, that is, indices $\{\alpha_1, \dots, \alpha_n\} \subseteq I$ such that $X = \bigcup_{i=1}^n V_{\alpha_i}$. Thus,

$$f(X) \subseteq \bigcup_{i=1}^n U_{\alpha_i},$$

which is a finite subcover of $f(X)$. Thus, $f(X)$ is compact. ∎

Remark 2.9. *Recall the "extreme value theorem": let $f : [a, b] \to \mathbb{R}$ a continuous function; then, a minimum and maximum is obtained for $f(x)$ on this interval for values in this interval.*

↪ **Theorem 2.6**

Let X compact, and $f : X \to \mathbb{R}$ a continuous function. Then,

$$\max_{x \in X} f(x) \text{ and } \min_{x \in X} f(x)$$

are both attained.

Proof. $f(X) \subseteq \mathbb{R}$ is compact by theorem 2.5, and so by theorem 2.2, $f(X)$ is closed and bounded. Let, then, $m = \inf f(X)$ and $M = \sup f(X)$; these necessarily exist, since $f(X)$ is bounded. Both m and M are limit points of $f(X)$. But $f(X)$ is closed, and hence contains all of its limit points, and thus $m \in f(X)$ and $M \in f(X)$, and thus $\exists y_m : f(y_m) = m$ and $y_M : f(y_M) = M$. ∎

↪ **Definition 2.16: Path Connected**

A set $A \subseteq X$ is called *path connected* if $\forall\, x, y \in A$, $\exists f : [a, b] \to X$, continuous, s.t. $f(a) = x, f(b) = y$ and $f([a, b]) \subseteq A$.

The set $\{f(t) : a \leq t \leq b\}$ is called a *path* from x to y.

↪ **Theorem 2.7: Path connected $\implies$ connected**

If $A \subseteq X$ is path connected, then A is connected.

Proof. Suppose, seeking a contradiction, that A is path connected, but not connected. Then, we can write $A \subseteq U \cup V$, for open, disjoint, nonempty subsets $U, V \subseteq X$.

Let $x \in U \cap A$ and $y \in V \cap A$. Then, $\exists f : [a, b] \to A$ s.t. $f(a) = x, f(b) = y$, and $f([a, b]) \subseteq A$, by the path connectedness of A. Then,

$$[a, b] \subseteq f^{-1}(A) \subseteq \underbrace{f^{-1}(U \cap A)}_{\text{open}} \cup \underbrace{f^{-1}(V \cap A)}_{\text{open}} =: \underbrace{U_1}_{a \in} \cup \underbrace{U_2}_{b \in},$$

that is, $[a, b]$ is contained in a union of open, nonempty, disjoint sets, contradicting $[a, b]$ the connectedness of $[a, b]$ by lemma 2.3. Thus, A is connected. ■

Remark 2.10. *A counterexample to the opposite side of the implication is the Topologist's sine curve, the set*

$$\{(x, \sin\left(\frac{1}{x}\right)) : x \in (0, 1]\} \cup \{0\} \times [-1, 1].$$

This set is connected in $\mathbb{R}^2$, but is not path connected.

↪ **Proposition 2.18**

For open sets in $\mathbb{R}^n$, path connected $\iff$ connected.

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2.8 Path Components, Connected Components

Remark 2.11. *Remark that if a metric space X is not connected, then we can write $X = U \cup V$ where U, V are open, nonempty and disjoint. It follows, then, that $U = V^C$ (and vice versa) and hence U, V are both open and closed.*

↪ **Definition 2.17: Connected Component**

A connected component of $x \in X$ is the largest connected subset of X that contains x.

⊛ **Example 2.6**

Let $X = (0, 1) \cup (1, 2)$. Here, we have two connected components, $(0, 1)$ and $(1, 2)$

⊛ **Example 2.7: Middle Thirds Cantor Set**

Let $C_0 := [0, 1]$, and given C_n, define $C_{n+1} := \frac{1}{3}(C_n \cup (2 + C_n))$ for $n \geq 0$. C_∞ is totally disconnected.

↪ **Definition 2.18: Path Component**

A *path component* $P(x)$ of $x \in X$ is the largest path connected subset of X that contains x.

↪ **Proposition 2.19**

$P(x) = \{x \in X : \exists$ conintuous path $\gamma : [0,1] \to X : \gamma(0) = x, \gamma(1) = y\}$.

Remark 2.12. *Where we "start" a path does not matter. We write $x \sim y$ if $\exists\gamma$ from x to y; this is an equivalence relation on the elements of X.*

Remark 2.13. *The choice of $[0,1]$ here is arbitrary; any closed interval is homeomorphic.*

↪ **Lemma 2.4**

If $P(x) \cap P(y) \neq \varnothing$, then $P(x) = P(y)$.

Proof. $P(x) \cap P(y) \neq \varnothing \implies \exists z : x \sim z \land y \sim z \implies x \sim y$. ■

↪ **Lemma 2.5**

If $A \subseteq X$ is connected, then $\overline{A}$ is also connected.

↪ **Lemma 2.6**

Suppose $A \subseteq X$ is both open and closed. Then, if $C \subseteq X$ is connected and $C \cap A \neq \varnothing$, then $C \subseteq A$.

Proof. If A is both open and closed, then $C \cap A$ is both open and closed in C. If $C \cap A^C \neq \varnothing$, then this is also open and closed in C. Hence, we can write $C = (C \cap A) \cup (C \cap A^C)$, that is, a disjoint union of two nonempty open sets, contradicting the connectedness of C. Hence, $C \cap A^C = \varnothing$, and so $C \subseteq A$. ■

↪ **Proposition 2.20**

Let $\{C_\alpha\}_{\alpha \in I}$ be a collection of nonempty connected subspaces of X s.t. $\forall \alpha, \beta \in I, C_\alpha \cap C_\beta \neq \varnothing$. Then, $\cup_{\alpha \in I} C_\alpha$ is connected.

↪ **Proposition 2.21**

Suppose each $x \in X$ has a path-connected neighborhood. Then, the path components in X are the same as the connected components in X.

2.8.1 Cantor Staircase Function

↪ **Definition 2.19: An Explicit Definition**

Let $x \in C : x = 0.a_1a_2a_3\dots$ (base 3), ie $a_j = \begin{cases} 0 \\ 2 \end{cases}$. Define

$$f(x) = \begin{cases} \sum \frac{a_j/2}{2^j} & x \in C \\ \text{extend by continuity} & x \notin C. \end{cases}$$

That is, if $x \notin C$, set $f(y) = \sup_{x \in C, x < y} f(x) = \inf_{x \in C, x > y} f(x)$.

↪ **Definition 2.20: Complement Definition**

To construct the complement of the Cantor set, begin with $[0, 1]$ and at a step n, we remove 2^n open intervals from this interval. $f(x)$ will be constant on each of these intervals with values $\frac{k}{2^n}$ where k odd and $0 < k < 2^n$. Extend by continuity to all $x \in C$.

Remark 2.14. *Wikipedia's explanation of this is far better than whatever this definition is trying to say.*

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