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Analysis I, II MATH254

<u>Course Outline:</u>

Fundamentals of set theory. Properties of the reals. Limits, limsup, liminf. Continuity. Functions. Differentiation.

<u>*References:*</u>

Understanding Analysis, *Abbott;* Introduction to Real Analysis, *Bartle;* Analysis I, *Tao*

Contents

1 Logic, Sets, and Functions

1.1 Mathematical Induction & The Naturals

The **natural numbers**, $\mathbb{N} = \{1, 2, 3, \dots\}$, are specified by the 5 **Peano Axioms**:

(1) $1 \in \mathbb{N}$ [1]

(2) every natural number has a successor in $\mathbb{N}$

(3) 1 is not the successor of any natural number

(4) if the successor of x is equal to the successor of y, then x is equal to y [2]

(5) **the axiom of induction**

[1]using 0 instead of 1 is also valid, but we will use 1 here.

[2]axioms (2)-(4) can be equivalently stated in terms of a successor function $s(n)$ more rigorously, but won't here

The **Axiom of Induction** (AI), can be stated in a number of ways.

Axiom 1.1 (AI.i). *Let $S \subseteq \mathbb{N}$ with the properties:*

(a) $1 \in S$

(b) *if $n \in S$, then $n + 1 \in S$*[3]

then $S = \mathbb{N}$.

[3]*(a)* is called the **inductive base**; *(b)* the **inductive step**. All AI restatements are equivalent in having both of these, and only differentiate on their specific values.

Example 1.1. *Prove that, for every* $n \in \mathbb{N}$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2} (\equiv (1))$

Proof (via AI.i). Let S be the subset of $\mathbb{N}$ for which (1) holds; thus, our goal is to show $S = \mathbb{N}$, and we must prove *(a)* and (b) of AI.i.

- by inspection, $1 \in S$ since $1 = \frac{1(1+1)}{2} = 1$, proving *(a)*
- assume $n \in S$; then, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ by definition of S. Adding $n + 1$ to both sides yields:

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) && (1) \\ &= (n+1)(\frac{n}{2} + 1) && (2) \\ &= \frac{(n+1)(n+2)}{2} && (3) \\ &= \frac{(n+1)((n+1)+1)}{2} && (4) \end{aligned}$$

Line (4) is equivalent to statement (1) (substituting n for $n+1$), and thus if $n \in S$, then $n+1 \in S$ and *(b)* holds. Thus, by AI.i, $S = \mathbb{N}$ and $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ holds $\forall n \in \mathbb{N}$. ∎

Example 1.2. *Prove (by induction), that for every* $n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Proof. Follows a similar structure to the previous example. Let S be the subset of $\mathbb{N}$ for which the statement holds. $1 \in S$ by inspection (*(a)* holds), and we prove *(b)* by assuming $n \in S$ and showing $n + 1 \in S$ (algebraically). Thus, by AI.i, $S = \mathbb{N}$ and the statement holds $\forall n \in \mathbb{N}$. ∎

This can also be proven directly (Gauss' method).

Proof (Gauss' method). Let $A(n) = 1 + 2 + 3 + \cdots + n$. We can write $2 \cdot A(n) = 1 + 2 + 3 + \cdots + n + 1 + 2 + 3 + \cdots + n$. Rearranging terms (1 with n, 2 with $n - 1$, etc.), we can say $2 \cdot A(n) = (n+1) + (n+1) + \cdots$, where $(n+1)$ is repeated n times; thus, $2 \cdot A(n) = n(n+1)$, and $A(n) = \frac{n(n+1)}{2}$. ∎

Axiom 1.2 (AI.ii). *Let* $S \subseteq \mathbb{N}$ *s.t.*

(a) $m \in S$

(b) $n \in S \implies n + 1 \in S$

then $\{m, m+1, m+2, \dots\} \subseteq S$.

Example 1.3. *Using AI.ii, prove that for* $n \geq 2$, $n^2 > n + 1$

Proof. Let $S \subseteq \mathbb{N}$ be the set of n for which the statement holds. $n = 2 \implies 4 > 3$, so the base case holds. Consider $n^2 > n + 1$ for some $n \geq 2$. Then, $(n+1)^2 = n^2 + 2n + 1 > n + 1 + 2n + 1 = 3n + 2 > 2n + 2 > n + 2$, hence $S = \{2, 3, 4, \cdots\}$ (all $n \geq 2$). ∎

Axiom 1.3 (Principle of Complete Induction, AI.iii). *Let $S \subseteq \mathbb{N}$ s.t.*

(a) $1 \in S$

(b) if $1, 2, \ldots, n-1 \in S$, *then* $n \in S$

then $S = \mathbb{N}$.

Finally, combining AI.ii and AI.iii;

Axiom 1.4 (AI.iv). *Let $S \subseteq \mathbb{N}$ s.t.:*

(a) $m \in S$

(b) if $m, m+1, \ldots, m+n \in S$, *then* $m + n + 1 \in S$

then $\{m, m+1, m+2, \ldots\} \subseteq S$.

Theorem 1.1 (Fundamental Theorem of Arithmetic). *Every natural number n can be written as a product of one or more primes.* [4]

[4] 1 is not a prime number

Proof of Theorem 1.1. Let S be the set of all natural numbers that can be written as a product of one or more primes. We will use AI.iv to show $S = \{2, 3, \ldots\}$.

- *(a)* holds; 2 is prime and thus $2 \in S$
- suppose that $2, 3, \ldots, 2 + n \in S$. Consider $2 + (n+1)$:
  - if $2 + (n+1)$ is *prime*, then $2 + (n+1) \in S$, as all primes are products of 1 and themselves and are thus in S by definition.
  - if $2 + (n+1)$ is *not prime*, then it can be written as $2 + (n+1) = a \cdot b$ where $a, b \in \mathbb{N}$, and $1 < a < 2 + (n+1)$ and $1 < b < 2 + (n+1)$. By the definition of S, $a, b \in S$, and can thus be written as the product of primes. Let $a = p_1 \cdot \cdots \cdot p_l$ and $b = q_1 \cdot \cdots \cdot q_j$, where the p's and q's are prime and $l, j \geq 1$. Then, $a \cdot b$ is a product of primes, and thus so is $2 + (n+1)$. Thus, $2 + (n+1) \in S$, and by AI.iv, $S = \{2, 3, 4, \ldots\}$

∎

1.2 Extensions: Integers, Rationals, Reals

Consider the set of naturals $\mathbb{N} = \{1, 2, 3, \dots\}$. Adding 0 to $\mathbb{N}$ defines $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We define the **integers** as the set $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, or the set of all positive and negative whole numbers.

Within $\mathbb{Z}$, we can define multiplication, addition and subtraction, with the neutrals of 1 and 0, respectively. However, we cannot define division, as we are not guaranteed a quotient in $\mathbb{Z}$. This necessitates the **rationals**, $\mathbb{Q}$. We define

$$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}.$$

On $\mathbb{Q}$, we have the familiar operations of multiplication, addition, subtraction and properties of associativity, distributivity, etc. We can also define division, as $\frac{\frac{p}{q}}{\frac{p'}{q'}} = \frac{pq'}{qp'}$.

We can also define a relation $<$ between fractions, such that

- $x < y$ and $y < z \implies x < z$
- $x < y \implies x + z < y + z$

$\mathbb{Q}$, together with its operations and relations above, is called an **ordered field**.

1.2.1 The Insufficiency of the Rationals

We can consider historical reasoning for the extension of $\mathbb{Q}$ to $\mathbb{R}$. Consider a right triangle of legs a, b and hypotenuse c. By the Pythagorean Theorem, $a^2 + b^2 = c^2$. Consider further the case there $a = b = 1$, and thus $c^2 = 2$. Does c exist in $\mathbb{Q}$?

Proposition 1.1. $c^2 = 2$, $c \notin \mathbb{Q}$.

Proof of Proposition 1.1. Suppose $c \in \mathbb{Q}$. We can thus write $c = \frac{p}{q}$, where[5] $p, q \in \mathbb{N}$, and p, q share no common divisors, ie they are in "simplest form". Notably, p and q cannot *both* be even (under our initial assumption), as they would then share a divisor of 2. We write

$$\begin{aligned} c &= \frac{p}{q} \\ c^2 = 2 &= \frac{p^2}{q^2} \\ 2q^2 &= p^2 \end{aligned}$$

$p \in \mathbb{N} \implies p^2 \in \mathbb{N}$, and thus p^2, and therefore[6] p, must be divisible by 2 ($\implies p$ even). Therefore, we can write $p = 2p_1, p_1 \in \mathbb{N}$, and thus $2q^2 = (2p_1^2)^2 \implies q^2 = 2p_1^2$. By the same reasoning, q must now be even as well, contradicting our initial assumption that p and q share no common divisors. Thus, $c \notin \mathbb{Q}$. ■

[5] Note that in the definition of $\mathbb{Q}$, p, q are defined to be in $\mathbb{Z}$; however, as we are using a geometric argument, we can assume $c > 0 \implies \text{Sign}(p) = \text{Sign}(q)$, and we can just take $p, q \in \mathbb{N}$ for convenience and wlog.

[6] $\sqrt{\text{even}} = \text{even}$

1.3 Sets & Set Operations

- $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- $\bigcup_{i=1}^{\infty} A_n = \bigcup_{n \in \mathbb{N}} A_n = \{x : x \in A_n \text{ for some } n \in \mathbb{N}\}$
- $\bigcap_{i=1}^{\infty} A_n = \bigcap_{n \in \mathbb{N}} A_n = \{x : x \in A_n \, \forall \, n \in \mathbb{N}\}$
- $A^C = \{x : x \in X \text{ and } x \notin A\}$[7]

[7] X is often omitted if it is clear from context.

Theorem 1.2 (De Morgan's Theorem(s)). *Let A, B be sets. Then,*

$$(a) \qquad (A \cap B)^C = A^C \cup B^C$$

and

$$(b) \qquad (A \cup B)^C = A^C \cap B^C.$$

Proof of Theorem 1.2. (b) *(A similar argument follows…)*

■

Proposition 1.2.

$$(a) \left(\bigcap_{n=1}^{\infty} A_n\right)^C = \bigcup_{n=1}^{\infty} A_n^C$$

$$(b) \left(\bigcup_{n=1}^{\infty} A_n\right)^C = \bigcap_{n=1}^{\infty} A_n^C$$

Proof of Proposition 1.2. Consider Proposition (b). Working from the left-hand side, we have

$$\begin{aligned}\left(\bigcup_{n=1}^{\infty} A_n\right)^C &= \{x : x \notin \bigcup A_n\}\\ &= \{x : x \notin A_n \,\forall\, n \in \mathbb{N}\}\\ &= \bigcap \{x : x \notin A_n\}\\ &= \bigcap A_n^C\end{aligned}$$

(a) can be logically deduced from this result. Consider the RHS, $\bigcup A_n^C$. Taking the complement:

$$\begin{aligned}\left(\bigcup A_n^C\right)^C &\overset{\text{via (b)}}{=} \bigcap A_n^{C^C}\\ &= \bigcap A_n\end{aligned}$$

Taking the complement of both sides, we have $\bigcup A_n^C = \left(\bigcap A_n\right)^C$, proving (a). ■

1.4 Functions

Definition 1.1. *Let A, B be sets. A* function f *is a rule assigned to each $x \in A$ a corresponding unique element $f(x) \in B$. We denote*

$$f : A \to B.$$

Definition 1.2. *The* domain *of a function $f : A \to B$, denoted Dom$(f) = A$. The* range *of f, denoted Ran$(f) = \{f(x) : x \in A\}$. Clearly, Ran$(f) \subseteq B$, though equality is not necessary.*

Example 1.4. *The function $f(x) = \sin x$, $f : \mathbb{R} \to [-1, 1]$. Here, Dom$(f) = \mathbb{R}$, and Ran$(f) = [-1, 1]$.*

Example 1.5 (Dirichlet Function). $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \notin \mathbb{Q} \end{cases}$. *Despite not having a true "explicit" formula, so to speak, this is still a valid function (under modern definitions).*

1.4.1 Properties of Functions

Proposition 1.3. *Let $f : A \to B$, $C \subseteq A$, $f(C) = \{f(x) : x \in C\}$. We claim $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.*

Proof. We will prove this by showing (1) $\subseteq$ and (2) $\supseteq$.

(1) $y \in f(C_1 \cup C_2) \implies$ for some $x \in C_1 \cup C_2, y = f(x)$. This means that either for some $x \in C_1, y = f(x)$, or for some $x \in C_2, y = f(x)$. This implies that either $y \in f(C_1)$, or $y \in f(C_2)$, and thus y *must* be in their union, ie $y \in C_1 \cup C_2$.

(2) $y \in f(C_1) \cup f(C_2) \implies y \in f(C_1)$ or $y \in f(C_2)$. This means that for some $x \in C_1, y = f(x)$, or for some $x \in C_2, y = f(x)$. Thus, x *must* be in $C_1 \cup C_2$, and for some $x \in C_1 \cup C_2, y = f(x) \implies y \in f(C_1 \cup C_2)$.

(1) and (2) together imply that $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$. ■

Example 1.6. *Let* $A_n = 1, 2, \ldots$ *be a sequence of sets. Prove that* $f(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f(A_n)$.

Proof. Let $y \in f(\bigcup_{n=1}^{\infty} A_n)$. This implies that $\exists x \in \bigcup_{n=1}^{\infty} A_n$ s.t. $f(x) = y$. This implies that $x \in A_n$ for some n, and $y \in f(A_n)$ for that same "some" n, and thus y must be in the union of all possible $f(A_n)$, ie $y \in \bigcup f(A_n)$. This shows $\subseteq$, use similar logic for the reverse. ■

Proposition 1.4. $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$ [8]

Proof. $y \in f(C_1 \cap C_2) \implies$ for some $x \in C_1 \cap C_2, y = f(x)$. This implies that for some $x \in C_1, y = f(x)$ **and** for some $x \in C_2, y = f(x)$. Note that this does *not* imply that these x's are the same, ie this reasoning is not reversible as in the previous union case. This implies that $y \in f(C_1)$ and $y \in f(C_2) \implies y \in f(C_1) \cap f(C_2)$. ■

[8]NB: the reverse is not always true, ie these sets are not always equal; "lack" of equality is more "common" than not.

Example 1.7. *Prove that if* $A_n, n = 1, 2, \ldots, f(\bigcap_{n=1}^{\infty} A_n) \subseteq \bigcap_{n=1}^{\infty} f(A_n)$.

Proof (Sketch). Use the same idea as in Example 1.6, but, naturally, with intersections. ■

Example 1.8. *Take* $f(x) = \sin x$, $A = \mathbb{R}, B = \mathbb{R}$, *and take* $C_1 = [0, 2\pi], C_2 = [2\pi, 4\pi]$. *Then,* $f(C_1) = [-1, 1]$, *and* $f(C_2) = [-1, 1]$. *But* $C_1 \cap C_2 = \{2\pi\}$; $f(\{2\pi\}) = \{\sin 2\pi\} = \{0\}$, *and thus* $f(C_1 \cap C_2) = \{0\}$, *while* $f(C_1) \cap f(C_2) = [-1, 1]$, *as shown in Proposition 1.4.*

Definition 1.3 (Inverse Image of a Set)**.** *Let* $f : A \to B$ *and* $D \subseteq B$. *The* inverse image *of* D *by* F *is denoted* $f^{-1}(D)$[9] *and is defined as*

$$f^{-1}(D) = \{x \in A : f(x) \in D\}.$$

[9]Note that this is **not** equivalent to the typical definition of an inverse *function*: f^{-1} may not

Example 1.9. $A = [0, 2\pi], B = \mathbb{R}, f(x) = \sin x, D = [0, 1]$.

$f^{-1}(D) = \{x \in A : f(x) \in D\} = \{x \in [0, 2\pi] : \sin(x) \in [0, 1]\} = [0, \pi]$.

Proposition 1.5. *Given function f and sets D_1, D_2,*

(a) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$

(b) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$[10]

Proposition 1.6. *Let $A_n, n = 1, 2, 3 \ldots$. Then,*

(a) $f^{-1}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f(A_n)$

(b) $f^{-1}(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} f(A_n)$

Proof. [11]

(a)

$$\begin{aligned} x \in f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) &\iff f(x) \in \bigcup_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for some } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for some } n \in \mathbb{N} \\ &\iff x \in \bigcup_{n=1}^{\infty} f^{-1}(A_n) \end{aligned}$$

(b)

$$\begin{aligned} x \in f^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) &\iff f(x) \in \bigcap_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for all } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for all } n \in \mathbb{N} \\ &\iff x \in \bigcap_{n=1}^{\infty} f^{-1}(A_n)^{12} \end{aligned}$$

■

Remark 1.1. $f : A \to B$, $A_1 \subseteq A$. *Given $f(A_1^C)$ and $f(A_1)^C$, there is* ***no general relation*** *between the two.*

[10] Just see next proposition; if you really need convincing, just use 2 rather than ∞ as the upper limit of the unions/intersections and use the same proof.

[12] This is a "proof by definitions" as I like to call it.

[12] Similar proof can be used to prove Proposition 1.5, less generally.

For instance, take $A = [0, 6\pi], B = [-1, 2], C = [0, 2\pi]$, and $f(x) = \sin x$. Then, $f(C) = [-1, 1]$, and $f(C^C) = f([-1, 0)) = [-1, 1]$, but $f(C)^C = [-1, 1]^C = (1, 2]$, and $f(C^C) \neq f(C)^C$; in fact, these sets are disjoint.

Proposition 1.7. *Let $f : A \to B$ and let $D \subseteq B$. Then $f^{-1}(D^C) = [f^{-1}(D)]^C$.*

Proof.

$$f^{-1}(D^C) = \{x : f(x) \in D^C\} = \{x : f(x) \notin D\}$$
$$[f^{-1}(D)]^C = [\{x : f(x) \in D\}]^C = \{x : x \notin f^{-1}(D)\} = \{x : f(x) \notin D\}$$

■

1.5 Reals

Axiom 1.5 (Of Completeness)**.** *Any non-empty subset of $\mathbb{R}$ that is bound from above has at least one upper bound (also called the supremum).*

In other words; let $A \subseteq \mathbb{R}$ and suppose A is bounded from above (A has at a least upper bound). Then $\sup(A)$ exists.

Real numbers, algebraically, have the same properties as the rationals; we have addition, multiplication, inverse of non-zero real numbers, and we have the relation $<$. All together, $\mathbb{R}$ is an ordered field.

Definition 1.4. *Let $A \subseteq \mathbb{R}$. A number $b \in \mathbb{R}$ is called an **upper bound** for A if for any $x \in A$, $x \leq B$.*

*A number $l \in \mathbb{R}$ is called a **lower bound** for A if for any $x \in A$, $x \geq l$.*

Definition 1.5 (The Least Upper Bound)**.** *Let $A \subseteq \mathbb{R}$. A real number s is called the **least upper bound** for A if the following holds:*

(a) s is an upper bound for A

(b) if b is any other upper bound for A, then $s \leq b$.

The least upper bound of a set A is unique, *if it exists; if s and s' are two least upper bounds, then by (a), s and s' are upper bound for A, and by (b), $s \leq s'$ and $s' \leq s$, and thus $s = s'$.*

This least upper bound is called the supremum *of A, denoted $\sup(A)$.*

Definition 1.6 (The Greatest Lower Bound). *Let $A \subset \mathbb{R}$. A number $i \in \mathbb{R}$ is called the* ***greatest lower bound*** *for A if the following holds:*

(a) i is a lower bound for A

(b) if l is any other lower bound for A, then $i \geq l$.

If i exists, it is called the infimum *of A and is denoted $i = \inf(A)$, and is unique by the same argument used for* $\sup(A)$.

Proposition 1.8. *Let*[13] *$A \subseteq \mathbb{R}$ and let s be an upper bound for A. Then $s = \sup(A)$ iff for any $\varepsilon > 0$, there exists $x \in A$ s.t. $s - \varepsilon < x$.*

[13]Note that this, and Proposition 1.9 that follows, are *not* definitions: they are restatements, and do technically require proof.

Proof. We have two statements:

I. $s = \sup(A)$;

II. For any $\varepsilon > 0$, $\exists x \in A$ s.t. $s - \varepsilon < x$;

and we desire to show that I $\iff$ II.

- I $\implies$ II: Let $\varepsilon > 0$. Then, since $s = \sup(A)$, $s - \varepsilon$ *cannot* be an upper bound for A (as s is the least upper bound, and thus $s - \varepsilon < s$ cannot be an upper bound at all). Thus, there exists $x \in A$ such that $s - \varepsilon < x$, and thus if I holds, II must hold.
- II $\implies$ I: suppose that this does not hold, ie II holds for an upper bound s for A, but $s \neq \sup(A)$. Then, there exists some upper bound b of A s.t. $b < s$. Take $\varepsilon = s - b$. $\varepsilon > 0$, and since II holds, there exists $x \in A$ such that $s - \varepsilon < x$. But since $s - \varepsilon = b$ and thus $b < x$, then b cannot be an upper bound for A, contradicting our initial condition. So, if II $\implies$ I does *not* hold, we have a "impossibility", ie a value b which is an upper bound for A which cannot be an upper bound, and thus II $\implies$ I.

■

Proposition 1.9. *Let $A \subseteq \mathbb{R}$ and let i be a lower bound for A. Then $i = \inf(A) \iff$ for every $\varepsilon > 0$ there exists $x \in A$ s.t. $x < i + \varepsilon$.*[14]

[14]Use similar argument to proof of previous proposition.

Remark 1.2. *Axiom 1.5 can also be expressed in terms of infimum. Define $-A = \{-x : x \in A\}$. Then, if b is an upper bound for A, then $b \geq x \,\forall\, x \in A$, then $-b \leq -x \,\forall\, x \in A$, ie -b is a lower bound of $-A$. Similarly, if l is a lower bound for A, $-l$ is an upper bound for $-A$.*

Thus, if A is bounded from above, then

$$-\sup(A) = \inf(-A),$$

and if A is bounded from below,

$$-\inf(A) = \sup(-A).$$

Axiom 1.6 (AC (infimum)). *Let $A \subseteq \mathbb{R}$; if A bounded from below, $\inf(A)$ exists.*

Definition 1.7 (max, min). *Let $A \subseteq \mathbb{R}$. An $M \in A$ is called a* maximum *of A if for any $x \in A$, $x \leq M$. M is an upper bound for A,* ***but also*** *$M \in A$.*

If M exists, then $M = \sup(A)$; M is an upper bound, and if b any other upper bound, then $b \geq M$, because $M \in A$, and thus $M = \sup(A)$.

NB: $M = \max(A)$ ***need not*** *exist, while $\sup(A)$ must exist. Consider $A = [0, 1)$; $\sup(A) = 1$, but there exists no $\max(A)$.*

The same logic exists for the existence of minimum vs infimum (consider $(0, 1)$, with no maximum nor minimum).

Theorem 1.3 (Nested interval property of $\mathbb{R}$). *Let $I_n = [a_n, b_n] = \{x : a_n \leq x \leq b_n\}, n = 1, 2, 3 \ldots$ be an infinite sequence of bounded, closed intervals s.t.*

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots I_n \supseteq I_{n+1} \supseteq \ldots$$

Then, $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$ (note that this does not *hold in $\mathbb{Q}$).*

Proof. [15] We have $I_n = [a_n, b_n], I_{n+1} = [a_{n+1}, b_{n+1}], \ldots$. And the inclusion $I_n \supseteq I_{n+1}$. $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$, $\forall n \geq 1$. So, the sequence a_n (left-end) is increasing, and the sequence b_n (right-end) is decreasing.

We also have that for any $n, k \geq 1$, $a_n \leq b_k$. We see this by considering two cases:

- Case 1: $n \leq k$, then $a_n \leq a_k$ (as a_n is increasing), and thus $a_n \leq a_k \leq b_k$.
- Case 2: $n > k$, then $a_n \leq b_n \leq b_k$ (again, as b_n is decreasing).

Let $A = \{a_n : n \in \mathbb{N}\}$. Then, A is bounded from above by *any* b_k (as in our inequality we showed above). Let $x = \sup(A)$, which must exist by Axiom 1.5.

[15] Sketch: show that the left-end points are increasing and the right-end points are decreasing. Show either that all the left-end points are bounded from above or that all the right-end points are bounded from below. As a result, there exists a sup/inf (depending on which end you choose) of the set of all the right/left points. For the sup case, all upper bounds must be $\geq$ sup, and thus the sup is in all I_n, and thus in their intersect, and thus the intersect is not empty.

Note that as a result, $x \geq a_n$ for all n, and for all k, $x \leq b_k$, as x is the lowest upper bound and must be $\leq$ all other upper bounds, and so for all $n \geq 1$, $a_n \leq x \leq b_n$, ie $x \in I_n \,\forall\, n \geq 1$, and thus $x \in \bigcap_{n=1}^{\infty} I_n$ and so $\bigcap_{n=1}^{\infty} \neq \varnothing$. ■

Remark 1.3. *The proof above emphasized the left-end points; it can equivalently be proven via the right-end points, and using* $y = \inf(\{b_n : n \in \mathbb{N}\}) = \inf(B)$, *rather than* $\sup(A)$, *and showing that* $y \in \bigcap I_n$.

Remark 1.4. *Note too that, if* $x = \sup(A)$ *and* $y = \inf(B)$, *then* $x, y \in \bigcap_{n=1}^{\infty} I_n$; *in fact,* $\bigcap_{n=1}^{\infty} I_n = [x, y]$. *This can be done by*

- *Use the main proof to show* $x \in \bigcap I_n$
- *Use the previous remark to show* $y \in \bigcap I_n$
- *Show* $x \leq y \implies [x, y] \subseteq \bigcap I_n$
- *Show* $\bigcap I_n \subseteq [x, y] \implies$ *equality.*

Remark 1.5. *The intervals* I_n must *be closed; if not, eg* $I_n = (0, \frac{1}{n})$, *then* $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

1.6 Density of Rationals in Reals

Proposition 1.10 (Archimedian Property). *(a) For any* $x \in \mathbb{R}$, *there exists a natural number* n *s.t.* $n > x$.

(b) For any $y \in \mathbb{R}$ *satisfying* $y > 0$, $\exists n \in \mathbb{N}$ *such that* $\frac{1}{n} < y$.

Remark 1.6. *(a) states that* $\mathbb{N}$ *is not a bounded subset of* $\mathbb{R}$.

Remark 1.7. *(b) follows from (a) by taking* $x = \frac{1}{y}$ *in (a), then* $\exists n \in \mathbb{N}$ *s.t.* $n > \frac{1}{y} \implies \frac{1}{n} < y$, *and thus we need only prove (a).*

Remark 1.8. *Recall that* $\mathbb{Q}$ *is an ordered field (operations* $+, \cdot$ *and a relation* $<$*).* $\mathbb{Q}$ *can be extended to a larger ordered field with extended definitions of these operations/relations, such that it contains elements that are larger than any natural numbers (ie, not bounded above). This is impossible in* $\mathbb{R}$ *due to AC.*

Proof. Suppose (a) not true in $\mathbb{R}$, ie $\mathbb{N}$ is bounded from above in $\mathbb{R}$. Let $\alpha = \sup \mathbb{N}$, which exists by AC.

Consider $\alpha - 1$; since $\alpha - 1 < \alpha$, $\alpha - 1$ is not an upper bound of $\mathbb{N}$. So, there exists some $n \in \mathbb{N}$ s.t. $\alpha - 1 < n$; then, $\alpha < n + 1$ where $n + 1 \in \mathbb{N}$, and thus α is also not an upper bound, as there exists a natural number that is greater than α. This contradicts the assumption that $\alpha = \sup \mathbb{N}$, so (a) must be true. ∎

Theorem 1.4 (Density). *Let $a, b \in \mathbb{R}$ s.t. $a < b$. Then, $\exists x \in \mathbb{Q}$ s.t. $a < x < b$.*

Remark 1.9. *If you take $a \in \mathbb{R}$ and $\varepsilon > 0$, then by the theorem, $\exists x \in \mathbb{Q}$ where $x \in (a-\varepsilon, a+\varepsilon)$. So any real number can be approximated arbitrarily closely (via choose of ε) by a rational number.*

Proof. Since $b - a > 0$, by (b) of Proposition 1.10, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a$, ie $na + 1 < nb$.

Let $m \in \mathbb{Z}$ s.t. $m - 1 \leq na < m$. Such an integer must exists since $\bigcup_{m\in\mathbb{Z}}[m-1, m) = \mathbb{R}$, the family $[m-1, m), m \in \mathbb{Z}$ makes partitions of $\mathbb{R}$. Then, $na < m$ gives that $a < \frac{m}{n}$. On the other hand, $m - 1 \leq na$ gives $m \leq na + 1 < nb$. So $\frac{m}{n} < b$ and it follows that $\frac{m}{n}$ satisfies $a < \frac{m}{n} < b$. ∎

In the proof, we used the claim:

Proposition 1.11. *If $z \in \mathbb{R}$, then there exists $m \in \mathbb{Z}$ s.t. $m - 1 \leq z < m$.*

Proof. Let S be a non-empty subset of $\mathbb{N}$. Then S has the least element; $\exists m \in S$ s.t. $m \leq n, \forall n \in S$.

We can assume $z \geq 0$; if $0 \leq z < 1$, then we are done (take $m = 1$), and assume that $z \geq 1$. Let now $S = \{n \in \mathbb{N} : z < n\}, \neq \varnothing$ by Proposition 1.10, (a). Let m be the least element of S. It exists by Well-Ordering Property; then, since $m \in S$, $z < m$. But, we also have $m - 1 \leq z$, otherwise, if $z < m - 1$ then $m - 1 \in S$ and then m is not the least element of S. Thus, we have $m - 1 \leq z < m$, as required. ∎

Theorem 1.5. *The set J of irrationals is also dense in $\mathbb{R}$. That is, if $a, b \in \mathbb{R}, a < b$, $\exists$ irrational y s.t. $a < y < b$ (noting that $J = \mathbb{R} \setminus \mathbb{Q}$).*

Proof. Fix $y_0 \in \mathbb{J}$. Consider $a - y_0, b - y_0$. $a - y_0 < b - y_0$, and by density of rationals, $\exists x \in \mathbb{Q}$ s.t. $a - y_0 < x < b - y_0$. Then, $a < y_0 + x < b$; let $y = x + y_0$, and we have $a < y < b$.

Note that y cannot be rational; if $y \in \mathbb{Q}$, $y = x + y_0 \implies y - x = y_0$, and since $x \in \mathbb{Q}$, $y - x \in \mathbb{Q} \implies y_0 \in \mathbb{Q}$, contradicting the original choice of $y_0 \notin \mathbb{Q}$. Thus, $y \in J$. ∎

Theorem 1.6. *$\exists$ a unique positive real number α s.t. $\alpha^2 = 2$.*

Proof. We show both uniqueness, existence:[16]

Uniqueness: if $\alpha^2 = 2$ and $\beta^2 = 2$, $\alpha \geq 0, \beta \geq 0$, then $0 = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta) > 0$, and so $\alpha - \beta = 0 \implies \alpha = \beta$.

- Existence: consider the set $A = \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 < 2\}$. A is not empty as $1 \in A$. The set of A is bounded above by 2, since if $x \geq 2$, then $x^2 \geq 4 > 2$, so $x \notin A$. So, by AC, $\sup A$ exists; let $\alpha = \sup A$. We will show that $\alpha^2 = 2$, by showing that both $\alpha^2 < 2$ and $\alpha^2 > 2$ are contradictions.

  $\alpha^2 < 2$

  For any $n \in \mathbb{N}$ we expand

$$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \leq \alpha^2 + \frac{2\alpha + 1}{n},$$

  noting that $\frac{1}{n^2} \leq \frac{1}{n}$ for $n \geq 1$.

  Let $y = \frac{2-\alpha^2}{2\alpha+1}$, which is strictly positive. By Proposition 1.10, $\exists n_0 \in \mathbb{N}$ s.t.

$$\frac{1}{n_0} < \frac{2 - \alpha^2}{2\alpha + 1} \text{ or } \frac{2\alpha + 1}{n_0} < 2 - \alpha^2.$$

  Substituting this n_0 into our inequality, we have

$$\left(\alpha + \frac{1}{n_0}\right)^2 \leq \alpha^2 + \frac{2\alpha + 1}{n_0} < \alpha^2 + 2 - \alpha^2 = 2.$$

  Since $\alpha + \frac{1}{n_0}$ is positive, $\alpha + \frac{1}{n_0} \in A$. But, since $\alpha = \sup A$, $\alpha + \frac{1}{n_0} \leq \alpha$, which is impossible, so $\alpha^2 < 2$ cannot be true.

  $\alpha^2 > 2$

  Take $n \in \mathbb{N}$;

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}.$$

  Now, let $y = \frac{\alpha^2-2}{2\alpha}$; $y > 0$, and by Proposition 1.10, $\exists n_0 \in \mathbb{N}$ s.t.

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}, \text{ or } \frac{2\alpha}{n_0} < \alpha^2 - 2.$$

  Substituting this n_0, we have

$$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 - \frac{2\alpha}{n_0} > \alpha^2 + 2 - \alpha^2 = 2.$$

So for any $x \in A$, we have $\left(\alpha - \frac{1}{n_0}\right)^2 > 2 > x^2$. $\alpha - \frac{1}{n_0} > 0$, and $x > 0$, since $x \in A$. Then, $\left(\alpha - \frac{1}{n_0}\right)^2 > x^2$ gives that $\alpha - \frac{1}{n_0} > x$.

So, $\alpha - \frac{1}{n_0} > x$ for all $x \in A$. So $\alpha - \frac{1}{n_0}$ is an upper bound for A, but since $\alpha = \sup A$, $\alpha - \frac{1}{n_0} \geq \alpha$ ie $\alpha \geq \alpha + \frac{1}{n_0}$, which is impossible. So $\alpha^2 > 2$ cannot be true.

Thus, $\alpha^2 = 2$.

■

[16]Proof sketch: uniqueness is clear. Existence follows from showing that α^2 cannot be either $<$ or $>$ 2. This is done by contradiction, taking some number slightly larger/smaller than α for the $<$ / $>$ resp., then showing that this number cannot be greater/less than α. In the $<$ case, we show that $\alpha + \frac{1}{n_0}$ for a particular n_0 must be in A, and so α cannot be $\sup A$ and thus a contradiction is reached. For the $>$ case, we need slightly different logic (really, more algebra), and get to another contradiction, this time by showing that $\alpha - \frac{1}{n_0}$ is an upper bound for A by our assumption, contradicting.

Remark 1.10. *A similar argument gives that for any $x \in \mathbb{R}$, $x \geq 0$, $\exists! \alpha \in \mathbb{R}$, $\alpha \geq 0$ such that $\alpha^2 = x$. This x is called the* square root *of x, denoted $\alpha = \sqrt{x}$.*

Remark 1.11. *For any natural number $m \geq 2$ and $x \geq 0$, $\exists! \alpha \in \mathbb{R}, \alpha \geq 0$ s.t. $\alpha^m = x$. The proof is similar, and we call α the m-th root of x.*

Remark 1.12. *Our last proof also gives that $\mathbb{Q}$ cannot satisfy AC. Suppose it does, ie any set in $\mathbb{Q}$ bounded from above has a supremum $\in \mathbb{Q}$. Then, consider $B = \{x \in \mathbb{Q} : x \geq 0 \text{ and } x^2 < 2\}$; set $\alpha = \sup B$. The exact same proof can be used, but we will not be able to find an upper bound in $\mathbb{Q}$.*

1.7 Cardinality

Definition 1.8. *Let $f : A \to B$.*

1. *f injective (one-to-one) if $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$*
2. *f surjective (onto) if for any $b \in B \exists a \in A$ s.t. $f(a) = b$.*
3. *f bijective if both.*

Definition 1.9 (Composition). *If $f : A \to B, g : B \to C$, the* composite map *$h = g \circ f$ is define by $h(x) = g(f(x))$. Note that $h : A \to C$.*

Example 1.10. *Consider functions f, g.*

1. *If f, g injective, so is $h = g \circ f$*
2. *If f, g bijective, then so is h*
3. *If $\exists E \subseteq C$, then $h^{-1}(E) = f^{-1}(g^{-1}(E))$*

Definition 1.10. *The inverse function*[17] *is defined only for bijective map $f : A \to B$. $y \in B$, $f^{-1}(y) = x$ where $x \in A$ s.t. $f(x) = y$.*

[17]Not the same as the inverse *image* of a set by a function, which is defined for any function.

Example 1.11. *1. $A = \mathbb{R}, B = (0, \infty), f(x) = e^x$. f is a bijection, and $f^{-1}(y) = \ln y, y \in (0, \infty)$.*

2. $A = (-\frac{\pi}{2}, \frac{\pi}{2}, B = \mathbb{R})$. $f(x) = \tan x$, $f^{-1}(y) = \arctan y$

Definition 1.11 (Equal Cardinalities). *Let A, B be two sets. We say A, B have the same cardinality, denote $A \sim B$ if there exists a bijective function $f : A \to B$.*

Example 1.12. *Let $E = \{2, 4, 6, \dots\}$ (even natural numbers). Define $f : \mathbb{N} \to E$ by $f(n) = 2n$. Thus, f is a bijection, and $\mathbb{N} \sim E$.*[18]

[18]See these independent notes for more.

Theorem 1.7. *The relation $\sim$ is a relation of equivalence.*

1. *$A \sim A$*
2. *if $A \sim B$, then $B \sim A$*
3. *if $A \sim B$ and $B \sim C$, then $A \sim C$*

Definition 1.12 (Countable). *A set A is* countable *if $\mathbb{N} \sim A$.*

Remark 1.13. *According to this, finite sets are not countable; this is just a convention. Sometimes, we say a set is countable if it is finite or to above definition holds, where we say that a set is* countably infinite *if it is infinite and countable.*

Other times, finite sets are treated separately than countable sets.

Theorem 1.8. *Suppose that $A \subseteq B$.*

1. *If B is finite or countable, then so is A*
2. *If A is infinite and uncountable, then so is B*

Definition 1.13 (Cartesian Product). *If A, B sets, $A \times B = \{(a, b) : a, b \in A, B\}$.*

Proposition 1.12. *$\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$; there exists a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.*

Proposition 1.13. *Let A be a set. The following are equivalent statements:*

(a) A is finite or a countable set;

(b) there exists a surjection from $\mathbb{N}$ onto A;

(c) there exists a injection from A into $\mathbb{N}$.

Proof. We proceed by proving that each statement implies the next (and thus are equivalent).

- (a) $\implies$ (b): Suppose A is finite and has $\mathbb{N}$ elements. Then there exists a bijection $h : \{1, 2, \ldots n\} \to A$. We now define a map $f : \mathbb{N} \to A$, by setting

$$f(m) = \begin{cases} h(m) & \text{if } m \leq n \\ h(n) & \text{if } m > n \end{cases}.$$

  f is surjective, and thus (b) holds. If (a) countable, $\exists$ bijection $f : \mathbb{N} \to A$, and any bijection is a surjection, so (b) also holds.

- (b) $\implies$ (c): Let $h : \mathbb{N} \to A$ be a surjection, whose existence is guaranteed by (b). Then, for any $a \in A$, the set

$$h^{-1}(\{a\}) = \{m \in \mathbb{N} : h(m) = n\} \neq \varnothing,$$

  since h is a surjection. Then, by the well-ordering property of $\mathbb{N}$, the set $h^{-1}(\{a\})$ has a least element.
  If n is the least element of $h^{-1}(\{a\})$, we set $f(a) =$. This defines a function

$$f : A \to \mathbb{N},$$

  and we aim to show that f is injective, ie that $f(a_1) = f(a_2) \implies a_1 = a_2$.
  Suppose $f(a_1) = f(a_2) = n$. Then, n is the least element of $h^{-1}(\{a_1\})$ and of $h^{-1}(\{a_2\})$, and in particular, $h(n) = a_1$ and $h(n) = a_2$, and thus $a_1 = a_2$ and so f is indeed injective.

- (c) $\implies$ (a): Let $f : A \to \mathbb{N}$ be an injection, whose existence is guaranteed by (c). Consider the range of f, ie

$$f(A) = \{f(a) : a \in A\}.$$

  Since f an injection, f is a bijection between A and $f(A)$.
  Otoh, $f(A) \subseteq \mathbb{N}$, and so by Theorem 1.8, $f(A)$ is either finite or countable, and there

exists a bijection between A and some set that is either fininte or countable. Thus, A must also be finite or countable, and so (a) holds.

■

Theorem 1.9. *Let $A_n, n = 1, 2, \ldots$ be a sequence of sets such that each A_n is either finite or countable. Then, their union*

$$A = \bigcup_{n=1}^{\infty} A_n$$

is also either finite or countable.

Proof. We will use (a) $\iff$ (b) from Proposition 1.13 to prove this.

Since each A_n finite or countable, by (a) $\implies$ (b), there exists a surjection

$$\varphi_n : \mathbb{N} \to A_n.$$

Now, let $h : \mathbb{N} \times \mathbb{N} \to A$, (the union) by setting

$$h(n, m) = \varphi_n(m).$$

We aim to show that h is also surjective.
If $a \in \bigcup_{n=1}^{\infty} A_n$, then $a \in A_n$ for some $n \in \mathbb{N}$. Since $\varphi_n : \mathbb{N} \to A_n$ is a surjection, there exists an $m \in \mathbb{N}$ s.t. $\varphi_n(m) = a$. By definition of h, we have

$$h(n, m) = a,$$

and thus h is a surjection.

By Proposition 1.12, there exists a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, and we can define the composite map

$$h \circ f : \mathbb{N} \to A \, (= \cup_{n=1}^{\infty} A_n),$$

which is a surjection as both h, f are surjections. So, there exists a surjection from $\mathbb{N} \to A$, and by Proposition 1.13, (b) $\implies$ (a), and thus $A = \bigcup_{n=1}^{\infty} A_n$ is also finite our countable.

■

Remark 1.14. *If $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is either finite or countable, and at least one A_n is countable, then A is countable.*

Remark 1.15. *If $A_1, \ldots, A_n$ are finitely many finite or countable sets then their union $A_1 \cup \cdots \cup A_n$ is also finite or countable (essentially just previous proof where we use n instead of ∞ for the upper limit of the union...).*

Theorem 1.10. *The set $\mathbb{Q}$ of rational numbers is countable.*

Proof. We write

$$\mathbb{Q} = A_0 \cup A_1 \cup A_2,$$

where $A_0 = \{0\}$, $A_1 = \{\frac{m}{n} : m, n \in \mathbb{N}\}$, and $A_2 = \{-\frac{m}{n} : m, n \in \mathbb{N}\}$.
Let us show that A_1 is countable; define

$$h : \mathbb{N} \times \mathbb{N} \to A_1, f(m, n) = \frac{m}{n}.$$

h is clearly a surjection; if $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection, then by Proposition 1.12, $h \circ f : \mathbb{N} \to A_1$ is a surjection. By Proposition 1.13, A_1 is countable.
We prove that A_2 countable in essentially the same way.
Then, $A_0 \cup A_1 \cup A_2$ is also countable, as it is the union of countable sets, and thus $\mathbb{Q}$ is also countable. ■

Theorem 1.11. *The set $\mathbb{R}$ of real numbers is uncountable.*[19]

[19]Proof sketch: by contradiction. Assume that a bijection exists, and show that it cannot be a surjection by the previous props/thms. Specifically, carefully construct nested intervals I_n, for which $x_i \notin I_i$, and then show that the intersection of all these intervals is empty, contradicting the nested interval property of the real line.
See pg. 25 of Abbott's Analysis for a more concise proof in the same language.

Proof. We will argue by contradiction; suppose $\mathbb{R}$ is countable, then show that the nested interval property (Theorem 1.3) of the real line fails.

Let $f : \mathbb{N} \to \mathbb{R}$ be a bijection, setting $f(1) = x_1, f(2) = x_2, \dots, f(n) = x_n, \dots$; we can then list the elements of $\mathbb{R}$ as $\mathbb{R} = \{x_1, x_2, x_3, \dots, x_n, \dots\}$.

We can now construct a sequence $I_n, n \in \mathbb{N}$ of bounded, closed intervals, such that I_1 does not contain x_1.

If $x_2 \notin I_1$, then $I_2 = I_1$. If $x_2 \in I_1$, then divide I_1 into four equal closed intervals.

Call the leftmost/rightmost of these intervals I_1' and I_1'' respectively. We know that $x_2 \in I_1$, so we must have that either $x_2 \notin I_1'$ or $x_2 \notin I_1''$ If $x_2 \notin I_1'$, then $I_2 = I_1'$. If $x_2 \notin I_1''$, then $I_2 = I_1''$.

Thus, we have constructed I_1, I_2 s.t.

$$I_1 \supseteq I_2 \text{ and } x_1 \notin I_1, x_2 \notin I_2.$$

Consider x_3; if $x_3 \notin I_2$, then $I_3 = I_2$. If $x_3 \in I_2$, we repeat the "dividing" process as before. Since $x_3 \in I_2$, either $x_3 \notin I_2'$ or $x_3 \notin I_2''$. If $x_3 \notin I_2'$, $I_3 = I_2'$. Else, if $x_3 \notin I_2''$, $I_3 = I_2''$.

We have now that

$$I_1 \supseteq I_2 \supseteq I_3 \text{ and } x_1 \notin I_1, x_2 \notin I_2, x_3 \notin I_3,$$

and we can continue this construction to obtain an infinite sequence of bounded, closed intervals I_n s.t.

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots,$$

and for each n, $x_n \notin I_n$.

Consider the intersection of all these I_n's,

$$\bigcap_{n=1}^{\infty} I_n.$$

For every m, $x_m \notin I_m$, so for every $m \in \mathbb{N}, x_m \notin \bigcap_{n=1}^{\infty} I_n$, and so $\mathbb{R} = \{x_1, x_2, \dots x_m, \dots\}$ has an empty intersection with this intersection, ie

$$\mathbb{R} \cap \left(\bigcap_{n=1}^{\infty} I_n \right) = \varnothing.$$

Otoh, $\bigcap_{n=1}^{\infty} I_n \subseteq \mathbb{R}$, so we must have that $\bigcap_{n=1}^{\infty} I_n = \varnothing$ contradicting the nested interval property of the real line which states that this intersection must not be empty. We thus have a contradiction, and our assumption that $\mathbb{R}$ countable fails. [20] ∎

[20]Note that Theorem 1.3 is built upon the Axiom of Completeness, a "fact" of $\mathbb{R}$ (what makes it "distinct" from $\mathbb{Q}, \mathbb{N}$, etc). Thus, we are really just using AC, with some abstractions sts.

Proposition 1.14. *The set J of all irrational numbers in $\mathbb{R}$ is uncountable.*

Proof. We have that $\mathbb{R} = \mathbb{Q} \cup J$. If J countable, then $\mathbb{R}$ would also be countable as the union of two countable sets (as we showed $\mathbb{Q}$ countable in Theorem 1.10). $\mathbb{R}$ uncountable, so J is also uncountable. ∎

Proposition 1.15. *The set $(-1, 1) \subseteq \mathbb{R}$ is uncountable.*

Proof. We can write $\mathbb{R} = \bigcup_{n=1}^{\infty}(-n, n)$. If each $(-n, n)$ is countable, then $\mathbb{R}$ would also be countable, as a countable union of countable sets. Thus, there must exist some $n_0 \in \mathbb{N}$ s.t. $(-n_0, n_0)$ is not countable. The map

$$f : (-n_0, n_0) \to (-1, 1), f(x) = \frac{x}{n_0}$$

is a bijection, and so $(-1, 1)$ is uncountable. ∎

Example 1.13. *Show that the map*

$$f(x) = \frac{x}{1 - x^2}$$

is a bijection between $(-1, 1)$ and $\mathbb{R}$ ie $(-1, 1) \sim \mathbb{R}$.

Proof. Surjection is fairly trivial (if stuck, consider the graph of the function).
Injection; given $f(x) = f(y)$ where $x, y \in (-1, 1)$,

$$\begin{aligned} \frac{x}{1-x^2} &= \frac{y}{1-y^2} \\ x - xy^2 &= y - yx^2 \\ x - y = xy^2 - yx^2 &= xy(y - x) \\ x - y &= -xy(x - y) \\ \implies -xy = 1 \implies xy = -1, &\text{ or } x - y = 0 \end{aligned}$$

$xy = -1$ is impossible given the domain of the function, hence $x - y = 0 \implies x = y$, as desired. ∎

Proposition 1.16. *Any bounded non-empty open interval $(a, b) \in \mathbb{R}$ is uncountable.*

Proof. We will construct a bijection $f : (a, b) \to \mathbb{R}$ so that $(a, b) \sim \mathbb{R}$. Since $\mathbb{R}$ is uncountable, so must (a, b).

The map

$$f(x) = \frac{2(x-a)}{b-a} - 1$$

is a bijection between (a, b) and $(-1, 1)$, and we have shown that $(-1, 1) \sim \mathbb{R}$, so $(a, b) \sim \mathbb{R}$, and thus any open interval has the same cardinality as $\mathbb{R}$. ■

Example 1.14. *Prove that $\exists$ bijection between $[0, 1)$ and $(0, 1)$, and conclude that $[0, 1) \sim (0, 1) \sim \mathbb{R}$. Then conclude for any $a < b$, $[a, b) \sim \mathbb{R}$.*

1.7.1 Power Sets

Definition 1.14 (Power Set)**.** *Let A be a set. The* power set *of Am denoted $\mathcal{P}(A)$ is the collection of all subsets of A.*

Generally, if A finite of size n, $\mathcal{P}(A)$ has 2^n elements.

Theorem 1.12 (Cantor Power Set Theorem)**.** *Let A be any set. Then there exists no surjection from A onto $\mathcal{P}(A)$.* [21]

[21]Certified Classic

Proof. Suppose that there exists a surjection,

$$f : A \to \mathcal{P}(A).$$

Let $D \subseteq A$ defined as

$$D = \{a \in A : a \notin f(a)\}.$$

Since $D \subseteq \mathcal{P}(A)$, and f is surjective, there must exist some $a_0 \in A$ s.t. $f(a_0) = D$.
We have two cases:

1. $a_0 \in D$. But then, by definition of D, $a_0 \notin f(a_0) = D$, so $a_0 \in D$ is not possible as it implies $a_0 \notin D$.

2. $a_0 \notin D$. But then, since $D = f(a_0)$, $a_0 \notin f(a_0)$, and so by definition of D, $a_0 \in D$, which is again not possible.

So, the assumption of a surjection existing has led to $a_0 \in A$ such that neither $a_0 \in D$ nor $a_0 \notin D$, which is impossible. Thus there can be no surjective f.

Notice, though, that there exists an injection $A \to \mathcal{P}(A), a \mapsto \{a\}$, and thus there is an injection but no bijection.
Thus, we can say that $\mathcal{P}(A)$ is strictly bigger than A.

■

2 Sequences

2.1 Definitions

Definition 2.1. *Let A be a set. An A-valued sequence indexed by $\mathbb{R}$ is a map*

$$x : \mathbb{N} \to A.$$

The value $x(n)$ is called the n-th element of the sequence. One writes $x(n) = x_n$, or lists its elements

$$\{x_1, x_2, x_3, \dots\} \equiv \{x_n\}_{n\in\mathbb{N}} \equiv (x_n)_{n\in\mathbb{N}} \equiv \{x_n\}.$$

Definition 2.2 (Convergence)**.** *We say that a sequence (x_n) converges to a real number x if for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for all $n \geq N$ we have*

$$|x_n - x| < \varepsilon.$$

If sequence (x_n) converges to x, we write $\lim_{n\to\infty} x_n = x$.

Example 2.1. *Let (x_n) be a sequence defined by $x_n = \frac{1}{n}, n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n = 0$.*

Proof. Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$. Then for $n \geq N$, we have that

$$0 < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

So, for $n \geq N$, $|x_n - 0| < \varepsilon$, and so the limit is 0. ■

Definition 2.3 (Quantifier of Limit)**.** *The limit can be written in terms of quantifiers.*

$$\lim_{n\to\infty} x_n = x$$

means that

$$(\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(|x_n - x| < \varepsilon).$$

Example 2.2. *Prove that*

$$\lim_{n\to\infty} \frac{n^2+1}{n^2} = 1.$$

Proof. Let $\varepsilon > 0$. Let N be a natural number such that $N > \frac{1}{\sqrt{\varepsilon}}$. Then, for $n \geq N$,

$$|\frac{n^2+1}{n^2} - 1| = |\frac{n^2+1-n^2}{n^2}| = \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

■

Definition 2.4 (Divergent Sequences)**.** *If a sequence (x_n) does not converge to any real number x, we say that the sequence is divergent. For instance, consider*

$$x_n = (-1)^n, n \geq 1.$$

The sequence alternates between 1 and -1 and so intuitively does not converge. How do we prove it?

Proof. By contradiction; suppose that $x_n = (-1)^n$ be a converging sequence. Let $x = \lim_{n\to\infty} x_n$.
Take $\varepsilon = 1$, then $\exists N \in \mathbb{N}$ s.t. for all $n \geq N$ we have that $|x - x_n| < \varepsilon = 1$.
Consider indices $n = N, n = N+1$. We have

$$|x_{N+1} - x_N| = |x_{n+1} - x + x - x_N| \leq \underbrace{|x_{N+1} - x| + |x - x_N|}_{\text{triangle inequality}} < 1 + 1 = 2.$$

But we also have that

$$|(-1)^{N+1} - (-1)^N| = |(-1)^{N+1} + (-1)^{N+1}| = 2,$$

We thus have that $2 < 2$, which is a contradiction. Thus, x_n is not convergent. ■

Example 2.3. *Evaluate the following examples using the ε definition:*

1. $\lim_{n\to\infty} \frac{\sin n}{\sqrt[3]{n}} = 0$
2. $\lim_{n\to\infty} \frac{n!}{n^n} = 0$
3. $\lim_{n\to\infty} \frac{(1+2+\cdots+n)^2}{n^4} = \frac{1}{4}$

Proof. 1. For all $\varepsilon > 0$; take $\frac{1}{N} < \varepsilon^3 \implies \frac{1}{\sqrt[3]{N}} < \varepsilon$. Then, $\forall\, n \geq N$,

$$\begin{aligned} n \geq N \implies \sqrt[3]{n} \geq \sqrt[3]{N} \implies \frac{1}{\sqrt[3]{n}} &\leq \frac{1}{\sqrt[3]{N}} \\ -1 \leq \sin n \leq 1 \implies |\sin n| \leq 1 \implies \left|\frac{\sin n}{\sqrt[3]{n}}\right| \leq \left|\frac{1}{\sqrt[3]{N}}\right| &\leq \frac{1}{\sqrt[3]{N}} < \varepsilon \\ \implies \lim_{n\to\infty} \frac{\sin n}{\sqrt[3]{n}} &= 0 \end{aligned}$$

2. Take $\frac{1}{N} \leq \varepsilon$. Then, $\forall\, \varepsilon > 0$, $\forall\, n \geq N \implies \frac{1}{n} \leq \frac{1}{N}$,

$$\begin{aligned} \frac{n!}{n^n} > 0 \implies \left|\frac{n!}{n^n}\right| = \frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots 1}{n \cdot n \cdots n} &= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{1}{n} \\ &\leq 1 \cdot 1 \cdots 1 \cdot \frac{1}{n} \\ &\leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon \\ &\implies \lim_{n\to\infty} \frac{n!}{n^n} = 0 \end{aligned}$$

3. Note first that $(1 + 2 + \cdots + n)^2 = (\frac{n(n+1)}{2})^2$ (see Example 1.1). Take $\frac{1}{N} < \frac{\varepsilon}{2}$; then, $\forall\, \varepsilon > 0$, we have that $\forall\, n \geq N$,

$$\begin{aligned} \left|\frac{(1+2+\cdots+n)^2}{n^4} - \frac{1}{4}\right| = \frac{\frac{n^2(n+1)^2}{4}}{n^4} - \frac{n^4}{n^4} &= \frac{n^4 + 2n^3 + n^2 - n^4}{n^4} \\ = \frac{2n^3+n^2}{n^4} = \frac{2n+1}{n^2} \leq \frac{2n}{n^2} \leq \frac{2}{n} &\leq \frac{2}{N} < \varepsilon \\ \implies \lim_{n\to\infty} \frac{(1+2+\cdots+n)^2}{n^4} &= \frac{1}{4} \end{aligned}$$

■

2.2 Properties of Limits

Lemma 2.1 (Triangle Inequality). *For $x, y, z \in \mathbb{R}$,*

$$(i) \quad |x+y| \leq |x| + |y|; \qquad (ii) \quad |x-y| \leq |x-z| + |z-y|$$ [22]

Sketch proof. (i): $|x+y| = \begin{cases} x+y & x+y \geq 0 \\ -(x+y) & x+y \leq 0 \end{cases}$. So if $x+y \geq 0$, $|x+y| = x+y \leq |x|+|y|$. If $x+y > 0$, $|x+y| = -(x+y) = (-x)+(-y) \leq |x|+|y|$.

[22] Generally, proofs involving limits will consist of 1) picking/defining an ε based on given limit/series definitions, and then 2) using triangle inequality/related techniques to reach the desired conclusion.

(ii): $|x-y| = |x-z+z-y| \le |x-z| + |z-y|$ (using (i)). ■

Theorem 2.1. *A limit of a sequence is unique. In other words, if the sequence is converging, then its limit is unique. The sequence cannot converge to two distinct numbers x and y.*[23]

[23] Proof sketch: contradiction, assume two distinct limits, and take ε as their midpoint. Arrive at a contradiction by using triangle inequalities to show that $|x-y| < |x-y|$, and thus the limits cannot be distinct.

Proof. By contradiction; suppose $\exists (x_n)$ s.t. $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$, and that $x \neq 0$.
Take $\varepsilon = \frac{|x-y|}{2}$. Since $x \neq y$, we have that $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, $\exists N_1 \in \mathbb{N}$ s.t. for $n \ge N_1$, $|x_n - x| < \varepsilon$.
Similarly, since $\lim x_n = y$, $\exists N_2 \in \mathbb{N}$ s.t for $g \ge N_2$, $|x_n - y| < \varepsilon$.
Take some $n \ge \max(N_1, N_2)$; then

$$
\begin{aligned}
|x-y| = |x - x_n + x_n - y| &\le |x - x_n| + |x_n - y| \\
&< \varepsilon + \varepsilon = |x-y| \\
&\implies |x-y| < |x-y|, \perp
\end{aligned}
$$

■

Theorem 2.2. *Any converging sequence is bounded.*[24]
In other words, if (x_n) is a converging sequence,

$$\exists M > 0 \text{ s.t. } |x_n| \le M \,\forall\, n \ge 1.$$

[24] Take $\varepsilon = 1$, which is greater than $|x_n - x|$ by limit definition for $n \ge N$ for some N. We then use this to show that $|x_n| < 1 + |x|$, then construct a summation M such that it bounds $|x_n|$; it is equal to $|x_1| + |x_2| + \cdots$ up to $|x_{N-1}|$, then plus $1 + |x|$. We have finished.

Proof. Let (x_n) be a converging sequence, and $x = \lim_{n\to\infty} x_n$. Take $\varepsilon = 1$ in the definition of the limit; then, $\exists N \in \mathbb{N}$ s.t. $\forall\, n \ge N$, $|x_n - x| < 1$.
This gives that for $n \ge N$, $|x_n| = |x_n - x + x| \le |x_n - x| + |x| < 1 + |x|$.
Let now $M = |x_1| + |x_2| + \cdots + |x_{N-1}| + (1 + |x|)$. Then, for any $n \ge 1$, $|x_n| \le M$;
If $n \le N - 1$, then $|x_n|$ is a summand in M, and thus $|x_n| \le M$.
If $n \ge N$, then we have by the choice of N that $|x_n| < 1 + |x| \le M$.
Thus, for all $n \ge 1$, $|x_n| \le M$, and is thus bounded given (x_n) converges. ■

Proposition 2.1 (Algebraic Properties of Limits). *Let $(x_n), (y_n)$ be sequences such that*[25]

$$\lim x_n = x, \quad \lim y_n = y.$$

Then:

1. *For any constant* c, $\lim c \cdot x_n = c \cdot \lim x_n = c \cdot x$
2. $\lim(x_n + y_n) = \lim x_n + \lim y_n = x + y$
3. $\lim x_n \cdot y_n = (\lim x_n)(\lim y_n) = x \cdot y$
4. *Suppose* $y \neq 0$, $y_n \neq 0 \, \forall \, n \geq 1$. *Then,* $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n} = \frac{x}{y}$

[25]Note that the contrary of these statements need not hold; ie, if $\lim(x_n \cdot y_n)$ exists, this does not imply the existence of $\lim x_n$ and $\lim y_n$. Consider Example 2.4

Remark 2.1. *Let X be the collection of all sequences of real numbers, $X = \{(x_n) : x_n$ is a sequence$\}$. If $(x_n) \in X$ and $c \in \mathbb{R}$, we can define $c \cdot (x_n) = (c \cdot x_n)$[26]; this defines* scalar multiplication *on X.*
We can also define addition*; if (x_n) and (y_n) are two sequences in X, then $(x_n)+(y_n) = (x_n+y_n)$. Then, with these two operations X is a* vector space.

[26] NB: this denotes c multiplying to each nth element in x_n, ie $c \cdot x_1$, $c \cdot x_2$, etc

Example 2.4. *Take $x_n = (-1)^n, y_n = (-1)^{n+1}, n \geq 1$.*
$(x_n) + (y_n) = 0, x_n \cdot y_n = -1$, and so $\lim x_n + y_n = 0, \lim x_n \cdot y_n = -1$, while neither $\lim x_n$ nor $\lim y_n$ exist.

Proof (part 3. of Proposition 2.1). Take[27] $\lim x_n = x, \lim y_n = y$. Since (x_n) is converging, it is bound by Theorem 2.2, and there exists $M > 0$ s.t. $\forall n \geq 1, |x_n| \leq M$.
Now,

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&\leq |x_n y_n - x_n y| + |x_n y - xy| \\
&= |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \\
&\leq M \cdot |y_n - y| + |y| \cdot |x_n - x| \quad (i)
\end{aligned}$$

Let $\varepsilon > 0$; since $\lim y_n = y$, there exists $N_1 \in \mathbb{N}$ s.t. $n \geq N_1, |y_n - y| < \frac{\varepsilon}{2M}$. Sim, since $\lim x_n = x, \exists N_2 \in \mathbb{N}$ s.t. $|x_n - x| < \frac{\varepsilon}{2(|y|+1)}$
Let $N = \max(N_1, N_2)$, $n \geq N$. Then, we have, with (i),

$$\begin{aligned}
(i) \quad |x_n y_n - xy| &\leq M \cdot |y_n - y| + |y| \cdot |x_n| - x \\
&< M \cdot \frac{\varepsilon}{2M} + |y| \cdot \frac{\varepsilon}{2(|y|+1)} \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\end{aligned}$$

Thus, for $n \geq N$, $|x_n y_n - xy| < \varepsilon$, and by definition of the limit, $\lim x_n y_n = xy$. ∎

[27] Proof sketch: take an upper bound of x_n. Then, show that $|x_n y_n - xy| < \varepsilon$, by using triangle inequalities to show inequality to a combination of M, arbitrarily small values (by def of limits of x_n, y_n resp,), and $|y|$.

Theorem 2.3 (Order Properties of Limits). *Let $(x_n), (y_n)$ be two sequences such that*

$$\lim x_n = x, \quad \lim y_n = y.$$

1. $x_n \geq 0 \,\forall\, n \implies x \geq 0$.
2. $x_n \geq y_n \,\forall\, n \implies x \geq y$.
3. *c is constant since $c \leq x_n \,\forall\, n \geq 1 \implies c \leq x$. $x_n \leq c \,\forall\, n \geq 1 \implies x_n \leq c$.*

Remark 2.2. *2., 3. follow from 1. Set* $z_n = x_n - y_n \forall n \geq 1$*. Then,* $z_n \geq 0 \forall b \geq 1$, $\lim z_n = \lim(x_n - y_n) = \lim x_n - \lim y_n$ *(as these limits exist)* $= x - y$*. By 1.,* $\lim z_n \geq 0$*, and so either* $x - y \geq 0$ *or* $x \geq y$*.*

Proof of 1. Suppose 1. does not hold; suppose $\exists (x_n)$ s.t. $\lim x_n = x$, $x_n \geq 0 \forall \geq$, but $x < 0$. Let $\varepsilon > 0$ s.t. $x < -2\varepsilon < 0$. With this ε, $\lim x_n = x$ gives that $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N, |x_n - x| < \varepsilon$, or particularly, $x_n - x < \varepsilon$.

Then, $x_n < \varepsilon + x$, and since $x < -2\varepsilon$, we have $\forall n \geq N$, $x_n < -\varepsilon$, and thus $\forall n \geq N$, $x_n < 0$, a contradiction. ■

Theorem 2.4 (The Squeeze Theorem). *Let* $(x_n), (y_n), (z_n)$ *be sequences such that* $x_n \leq y_n \leq z_n,\ \forall n \geq 1$*, and* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell$*, then* $\lim_{n\to\infty} y_n = \ell$*.*[28]

[28]Sketch: This follows a similar technique to many that follow. Use the definitions of the limits of x_n, z_n to take an arbitrary ε, and an N for each respective limit. Take the max of these N's, and show that for all $n \geq \max N_i$, you can show that f $y_n - l$ is less than ε and greater than $-\varepsilon$. Really, this is just a proof of applying definitions correctly.

Proof. Let $\varepsilon > 0$. Since $\lim x_n = \ell$, there $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1, |x_n - \ell| < \varepsilon$.
Since $\lim z_n = \ell$, there $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2, |z_n - \ell| < \varepsilon$.
Take $N = \max\{N_1, N_2\}$ and take $n \geq N$. Then,

$$y_n \leq z_n \implies y_n - \ell \leq z_n - \ell \leq |z_n - \ell| < \varepsilon,$$

since $n \geq \max\{N_1, N_2\} \implies n \geq N_2$.
Now, we have that

$$y_n \geq x_n \implies y_n - \ell \geq x_n - \ell > -\varepsilon,$$

since $|x_n - \ell| < \varepsilon$ for $n \geq N_1$, and our n is $\geq \max\{N_1, N_2\}$. Thus, for $n \geq N$,

$$-\varepsilon < y_n - \ell < \varepsilon \implies |y_n - \ell| < \varepsilon,$$

and thus $\lim y_n = \ell$, by definition. ■

Definition 2.5 (Increasing/Decreasing). *A sequence* (x_n) *is called* increasing *if* $x_{n+1} \geq x_n \forall n \in \mathbb{N}$*, and is* decreasing *if* $x_{n_1} \leq x_n \forall n \in \mathbb{N}$*.*

Definition 2.6 (Bounded from above/below). *A sequence* (x_n) *is called* bounded *from above if there exists some* $M \in \mathbb{R}$ *s.t.* $x_n \leq M \forall n \geq 1$*.*
Sequence (x_n) *is bounded from below if there exists some* $M \in \mathbb{R}$ *s.t.* $x_n \geq M \forall n \geq 1$*.*

Theorem 2.5 (Monotone Convergence Theorem)**.** *The following relate to bounded above/below and increasing/decreasing sequences:*[29]

1. *Let (x_n) be an increasing sequence that is bounded from above. Then (x_n) is converging.*
2. *Let (x_n) be a decreasing sequence that is bounded from below. then (x_n) is converging.*

Proof (of 1). Let $A = \{x_n : n \geq 1\}$. Since (x_n) is bounded above by M, the set A is bounded from above. Let $\alpha = \sup A$, which exists by AC.

Let $\varepsilon > 0$. Since α is the least upper bound for A, $\alpha - \varepsilon$ is *not* an upper bound of A ($\alpha - \varepsilon < \alpha$). Hence, there must exist some $N \in \mathbb{N}$ such that $\alpha - \varepsilon < x_N$ (if it didn't exist, then α wouldn't be the supremum …). Then, for $n \geq N$, and since (x_n) increasing,

$$\alpha - \varepsilon < x_N \leq x_n \leq \alpha.$$

Then, for all $n \geq N$,

$$\alpha - \varepsilon < x_n \leq \alpha \implies -\varepsilon < x_n - \alpha \leq 0,$$

and so $|x_n - \alpha| < \varepsilon$ for $n \geq N$. By definition, $\alpha = \lim x_n$. ■

[29]Sketch: 1,2 are proven very similarly. For 1., take the set of all x_n in the given sequence. Since the sequence is bounded, then so is the set, and so we can take its supremum. Use the ε definition of sup to show that this supremum is also the limit of the sequence (basically, a bunch of inequalities, and being careful with definitions). 2. follows identically but using the infimum.

Example 2.5. *A sequence (x_n) is called* eventually increasing *if there exists some $N_0 \in \mathbb{N}$ s.t. $\forall n \geq N_0, x_{n+1} \geq x_n$. If (x_n) is eventually increasing and bounded from above,* $\lim x_n = \alpha$ *exists.*

Example 2.6. *Let* (x_n) *be a sequence defined recursively by* $x_1 = \sqrt{2}$ *and* $x_{n+1} = \sqrt{2 + x_n}, n \geq 1$. *So* $x_2 = \sqrt{2+\sqrt{2}}, x_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} \cdots, x_n = 2\cos\frac{\pi}{2^{n+1}}, n \geq 1$. *Show that* $\lim x_n = 2$.

Proof. We will prove this using the Monotone Convergence Thm by showing that the x_n is bounded from above and increasing, which implies that the limit exists. We will then find the actual limit.

Recall that $n \geq 1, x_n \leq 2$. We will prove this by induction. Let $S \subseteq \mathbb{N}$ be the set of indices such that $x_n \leq 2$. Since $x_1 = \sqrt{2} < 2$, $1 \in S$. Now suppose some $n \in S$, ie $x_n \leq 2$. Then, we have that $x_{n+1} = \sqrt{2+x_n} \leq \sqrt{2+2} = 2 \implies x_{n+1} \leq 2$. Thus, by induction, $n \in S \implies n+1 \in S \implies S = \mathbb{N}$, ie $x_n \leq 2 \, \forall \, n \in \mathbb{N}$. Thus, our sequence is bounded from above.

We now prove that (x_n) is increasing. Let $S \subseteq \mathbb{N}$ s.t. $n \in S \iff x_{n+1} \leq x_n$. $x_2 = \sqrt{2+\sqrt{2}} \geq \sqrt{2} = x_1 \implies x_1 \leq x_2 \implies 1 \in S$. Suppose $n \in S \implies x_{n+1} \geq x_n$. Then, $x_{n+2} = \sqrt{2+x_{n+1}} \geq \sqrt{2+x_n} = x_{n+1} \implies n+1 \in S$. Thus, $S = \mathbb{N}$, so $x_{n+1} \geq x_n \, \forall \, n \in \mathbb{N}$.

So the sequence (x_n) is increasing and bounded from above, and thus $\exists \lim x_n = \alpha$. To find the value of α, consider $x_{n+1} = \sqrt{2+x_n}$, or $x_{n+1}^2 = 2 + x_n$. We can also write that $\alpha = \lim x_n = \lim x_{n+1}$.[30] We then have that $\lim x_{n+1} = \alpha \implies \lim x_{n+1}^2 = \alpha^2$, and thus $x_{n+1}^2 = 2 + x_n \implies \lim x_{n+1}^2 = \lim(2+x_n) \implies \alpha^2 = 2 + \alpha \implies \alpha = 2, -1$. $x_n \geq 0 \, \forall \, n$, by Order Limit Theorem, and so $\alpha \geq 0$ and thus $\alpha = 2$. ■

[30] Add proof

Corollary 2.1. *For* $a, b > 0$, *then* $\frac{1}{2}(a+b) \geq \sqrt{ab}$

Proof. $\left[\frac{1}{2}(a+b)\right]^2 = \frac{1}{4}(a^2 + 2ab + b^2) \geq ab \implies \frac{1}{2}(a+b) \geq \sqrt{ab}$ ■

Example 2.7. *Let* (x_n) *be defined recursively by* $x_1 = 2$ *and* $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ *for* $n \geq 1$. *Then,* (x_n) *is converging and* $\lim x_n = \sqrt{2}$.

Proof. We[31] will show that (x_n) bounded from below and decreasing, implying the limit exists. We will show that for n, $x_n \geq \sqrt{2}$. For $n = 1$, $2 \geq \sqrt{2}$. For $n > 1$, we will Corollary 2.1. We then have that $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n}) \geq \cdots \geq \sqrt{2} \implies x_n \geq \sqrt{2} \, \forall \, n \geq 1$, ie, it is bounded from below.

We will now show that the sequence is decreasing.

$$x_n - x_{n+1} = x_n - \frac{1}{2}(x_n + \frac{2}{x_n}) = \frac{1}{2}x_n - \frac{1}{x_n} = \frac{1}{2x_n}(x_n^2 - 2).$$

■

[31] This example, as well as the more general one after it, rely on applying 1) the monotone convergence theorem

Example 2.8. *Let $a > 0$ and let (x_n) be a sequence defined recursively by x_1 is arbitrary (positive), and*

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}), \quad n \geq 1.$$

Show that $\lim_{n\to\infty} x_n = \sqrt{a}$.

Proof. By Corollary 2.1, $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}) \geq \sqrt{x_n \cdot \frac{a}{x_n}} = \sqrt{a}$, hence, x_n is bounded from below by $\sqrt{a}$.

We also have that $x_n - x_{n+1} = x_n - \frac{1}{2}x_n - \frac{a}{2x_n} = \frac{x_n}{2} - \frac{a}{2x_n} = \frac{1}{x_n}(x_n^2 - a)$. We have that $x_n \geq \sqrt{a} \implies x_n^2 \geq a \implies x_n^2 - a \geq 0$. Further, since the sequence is bounded from below by $\sqrt{a} > 0(\impliedby a > 0)$, then $\frac{1}{x_n} > 0$ as well. Hence, $\frac{1}{x_n}(x_n^2 - a) \geq 0$, and thus $x_n - x_{n+1} \geq 0 \implies x_n \geq x_{n+1}$ and thus x_n is decreasing.

Thus, by the Monotone Convergence Theorem, x_n is convergent. Let $X := \lim_{n\to\infty} x_n$. We have from the recursive definition, $\lim x_n = \lim\left(\frac{1}{2}(x_n + \frac{a}{x_n})\right)$. Since we know x_n convergent, we can "split up" this limit using algebraic properties, hence

$$\begin{aligned}\lim x_n = \lim \frac{1}{2}x_n + \lim \frac{a}{2x_n} = \frac{1}{2}\lim x_n + \frac{a}{2}\lim \frac{1}{x_n} \\ \implies X = \frac{1}{2}X + \frac{a}{2X} \\ \implies \frac{X}{2} = \frac{a}{2X} \implies X^2 = a \implies X = \sqrt{a},\end{aligned}$$

which completes the proof. ■

Example 2.9. *Evaluate*[32] *the limit of x_n given the recursive relation $x_{n+1} = \frac{1}{4-x_n}$, $x_1 = 3$.*

[32]Abbott, pg 54 exercise 2.4.2

Proof. We aim to show that (x_n) is bounded from below and decreasing.

Bounded from below: we claim $x_n > 0$; we proceed by induction. $x_1 = 3 > 0$ holds; say $x_n > 0$ for some $n \geq 1$. Then, we have

$$x_n > 0 \implies -x_n < 0 \implies 4 - x_n < 4 \implies \frac{1}{4 - x_n} > \frac{1}{4} > 0 \implies x_{n+1} = \frac{1}{4 - x_n} > 0,$$

so the sequence is bounded from below by 0.

Decreasing: (x_n) decreasing iff $x_{n+1} \leq x_n \,\forall\, n$. We have $x_2 = \frac{1}{4-3} = 1 \implies x_1 = 3 \geq 1$ holds. Say $x_{n-1} \geq x_n$ for some $n \geq 1$. Then, we have

$$x_{n-1} \geq x_n \implies 4 - x_{n-1} \leq 4 - x_n \implies \frac{1}{4 - x_{n-1}} \geq \frac{1}{4 - x_n} = x_{n+1} \implies x_n \geq x_{n+1}$$

and thus the sequence decreases, and by Theorem 2.5 the limit exists. Let $X = \lim_{n\to\infty} x_n =$

$\lim_{n\to\infty} \frac{1}{4-x_{n-1}} \implies X = \frac{1}{4-X} \implies 4X - X^2 = 1 \implies 0 = X^2 - 4X + 1 \implies X = \cdots = 2 \pm \sqrt{3}$. We must take the negative root, since X is decreasing and thus must be less than 3. ■

2.3 Limit Superior, Inferior

Definition 2.7 (limsup, liminf). *Recall Theorem 2.2, stating that a convergence sequence is bounded. Let (x_n) be a convergent sequence bounded by m and M from below/above resp, ie*

$$m \leq x_n \leq M, \forall\, n$$

and let $A_n = \{x_k : k \geq n\}$ (the set of elements in the sequence "after" a particular index).
Let $y_n = \sup A_n$; by definition, $y_n \leq M$, and $y_n \geq m$, since $y_n \geq x_n \geq m$. Thus, we have

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots,$$

and further,

$$y_1 \geq y_2 \geq \cdots \geq y_n \geq y_{n+1} \geq \cdots;$$

since $A_2 \subseteq A_1$, y_1 also an upper bound for A_2, and thus $y_2 \leq y_1$ by definition of a supremum.
So, the sequence (y_n) is decreasing, and bounded from below; by MCT, $\lim_{n\to\infty} y_n = y$ exists. Note too that since $m \leq y_n \leq M$, we have that $m \leq y \leq M$.
This y is called the limit superior *of (x_n) denoted by*

$$\overline{\lim}_{\mathrm{n}\to\infty}\, x_n = \limsup_{n\to\infty} x_n.$$

Now, similarly, note that A_n is bounded below by m and thus $z_n = \inf A_n$ exists. We further have that $z_n \leq x_n \leq M$, and that $z_n \geq m\, \forall\, n$, and we have

$$z_1 \leq z_2 \leq \cdots \leq z_n \leq z_{n+1} \leq \cdots,$$

by a similar argument as before. So, as before, the sequence (z_n) is increasing, and bounded from above by M. Again, by MCT, $\lim_{n\to\infty} z_n = z$ exists. We call z the limit inferior *of (x_n), and denote*

$$\underline{\lim}_{\mathrm{n}\to\infty}\, x_n = \liminf_{n\to\infty} x_n.$$

We note that $y_n \geq z_n$, so $\overline{\lim}_{\mathrm{n}\to\infty}\, x_n \geq \underline{\lim}_{\mathrm{n}\to\infty}\, x_n \quad (y \geq z)$.
Further, $\liminf$ and $\limsup$ exist for any bounded sequence, regardless if whether or not the limit itself exists.

Example 2.10. *Let* $(x_n) = (-1)^n, n \in \mathbb{N}$. *We showed previously that this is a divergent sequence, so the limit does not exist. However, the sequence is bounded, since* $-1 \leq x_n \leq 1\, \forall\, n$. *We have* $A_n = \{(-1)^k : k \geq n\} = \{-1, 1\}$. *So,* $y_n = \sup A_n = 1$, *and* $z_n = \inf A_n = -1$, $\forall\, n$. *Thus,* $\limsup x_n = \lim y_n = 1$, *and* $\liminf x_n = \lim z_n = -1$, *despite* $\lim x_n$ *not existing.*
More specifically, we have a divergent sequence, and $\liminf \neq \limsup$.

Theorem 2.6 (lim inf, lim sup and convergence). *Let* (x_n) *be a* bounded *sequence. The following are equivalent;*

1. *The sequence* (x_n) *is convergent, and* $\lim_{n\to\infty} x_n = x$.
2. $\overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n = x$.

Proof. Let A_n, y_n, z_n be as in the definition of $\limsup, \liminf$.

(1) $\implies$ (2): Suppose (x_n) is converging, and $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Then, there exists some $N \in \mathbb{N}$ s.t. $\forall\, n \geq N$,

$$|x_n - x| < \frac{\varepsilon}{2},$$

or equivalently,

$$x - \frac{\varepsilon}{2} < x_n < x + \frac{\varepsilon}{2}, \forall\, n \geq N.$$

Since $A_n = \{x_k : k \geq n\}$, if $n \geq N$, then $x + \frac{\varepsilon}{2}$ is an upper bound for A_n, and $x - \frac{\varepsilon}{2}$ is a lower bound for A_n. This gives that

$$y_n = \sup A_n \leq x + \frac{\varepsilon}{2}; \quad z_n = \inf A_n \geq x - \frac{\varepsilon}{2}.$$

This gives that for $n \geq N$,

$$x - \frac{\varepsilon}{2} \leq z_n \leq x_n \leq y_n \leq x + \frac{\varepsilon}{2},$$

ie $z_n, y_n \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$. So, for all $n \geq N$, $|z_n - x| \leq \frac{\varepsilon}{2} < \varepsilon$, and $|y_n - x| \leq \frac{\varepsilon}{2} < \varepsilon$, so by definition of the limit, this gives

$$\lim_{n\to\infty} y_n = x \text{ and } \lim_{n\to\infty} z_n = x,$$

ie, $\overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n = x$.

•

(2) $\implies$ (1): Let $\varepsilon > 0$. Since $\lim_{n\to\infty} y_n = x$, $\exists N_1$ s.t. $\forall\, n \geq N_1, |y_n - x| < \varepsilon$. Similarly, since $\lim z_n = x$, $\exists N_2$ s.t. $\forall\, n \geq N_2, |z_n - x| < \varepsilon$.

Take $N = \max\{N_1, N_2\}$. Then, for $n \geq N$, we have

$$x - \varepsilon < z_n \leq x_n \leq y_n < x + \varepsilon.$$

So, for $n \geq N$, $|x_n - x| < \varepsilon$, thus $\lim x_n = x$ as desired. ■

Example 2.11. *Let*[33] (x_n) *be a bounded sequence. Then*

$$\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n.$$

[33] Midterm material ends here. There will be 5 questions. Memorize **everything**; homeworks, exercises, class material. Study the solutions until you can recite it upwards, backwards, sideways.

Proof. Recall Remark 1.2; Let $A_n := \{x_k : k \geq n\}$ as in the definition of $\limsup, \liminf$. Let $y_n := \sup A_n$, $z_n := \inf A_n$. By Theorem 2.6, $\lim y_n = \lim z_n$. Further, $\sup(-A_n) = -\inf(A_n)$, where $-A_n = \{-x_k : k \geq n\}$; hence, $\limsup(-x_n) = -\liminf x_n$, as desired. ■

2.4 Subsequences and Bolzano-Weirestrass Theorem

Definition 2.8 (Subsequence). *Let* (x_n) *be a sequence of real numbers, and let* $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$ *be a strictly increasing sequence of natural numbers. Then, the sequence*

$$(x_{n_1}, x_{n_2}, \cdots, x_{n_k}, x_{n_{k+1}}, \cdots)$$

is called a subsequence *of* (x_n) *and is denoted* $(x_{n_k})_{k\in\mathbb{N}}$.

Remark 2.3. k *is the index of the subsequence,* $(x_{n_k})_{k\in\mathbb{N}}$, ***not*** n*;* x_{n_1} *is the 1st element, …,* x_{n_k} *is the* k*-th element.*

Example 2.12. *Let* $x_n = \frac{1}{n}, (\frac{1}{n})_{n\in\mathbb{N}}$*, and let* $n_k = 2k+1, k \in \mathbb{N}$. $n_1 = 3, n_2 = 5, n_3 = 7, \ldots, n_k = 2k+1$*. Our subsequence is then*

$$(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots) = \left(\frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2k+1}, \ldots\right) = \left(\frac{1}{2k+1}\right)_{k\in\mathbb{N}}$$

is our subsequence of (x_n).

Remark 2.4. *Note that for any* $k, n_k \geq k$.

Let $S = \{k \in \mathbb{N} : n_k \geq k\}$*. Then,* $1 \in S$*, since* $n_1 \in \mathbb{N}, n_1 \geq 1$ *. If* $k \in S$*, then* $n_k \geq k$*, and so, since* $n_{k+1} > n_k$ *(increasing), we have that* $n_{k+1} > k \implies n_{k+1} \geq k+1$*. So,* $k+1 \in S, S = \mathbb{N}$.

Remark 2.5. $\lim_{k\to\infty} x_{n_k} = x$ *if* $\forall\, \varepsilon > 0, \exists K \in \mathbb{N}$ *s.t.* $\forall\, k \geq K, |x_{n_k} - x| < \varepsilon$.

Theorem 2.7. *Let (x_n) be a sequence such that $\lim_{n\to\infty} x_n = x$. Then, for any subsequence $(x_{n_k})_{k\in\mathbb{N}}$, we have that $\lim_{k\to\infty} x_{n_k} = x$*

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|x_n - x| < \varepsilon$. Take $K = N$ (from Remark 2.5). Then, for $k \geq K$, we have from Remark 2.4 that

$$n_k \geq k \geq K = N,$$

and hence $|x_{n_k} - x| < \varepsilon \implies \lim_{k\to\infty} x_{n_k} = x$. ∎

Theorem 2.8 (Bolzano-Weirestrass Theorem). [34] *Any bounded sequence (x_n) has a convergent subsequence.*

[34] Fundamental property of the real line; equivalent to AC.

Example 2.13. *Take $x_n = (-1)^n, n \in \mathbb{N}$. This sequence does not converge. However, if we take a subsequence with $n_k = 2k, k \in \mathbb{N}$. $x_{n_k} = (-1)^{2k} = 1$, so (x_{n_k}) is a constant sequence 1 and converges to 1.*
Similarly, if $n_k = 2k+1, k \in \mathbb{N}$, then $x_{n_k} = (-1)^{2k+1} = -1$, and the subsequence converges to -1.

Proposition 2.2. *If $0 < b < 1$, then $\lim_{n\to\infty} b^n = 0$.*

Proof. Let $x_n = b^n$. Then $x_n > 0$, and $x_{n+1} = b^{n+1} = bx_n > x_n$, and since $0 < b < 1$, (x_n) is decreasing and bounded from below, (x_n) converges by the Monotone Convergence Theorem. Let $x = \lim_{n\to\infty} x_n$. Again, $x_{n+1} = bx_n$, so $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} bx_n = b\lim_{n\to\infty} x_n$, so $x = bx \implies (1-b)x = 0$. $0 < b < 1 \implies x = 0$. ∎

BW Proof (1): using Nested Interval Property. [35] Since (x_n) bounded, $\exists M > 0$ s.t. $|x_n| \leq M \, \forall n \in \mathbb{N}$. Let $I_1 = [-M, M]$ and $n_1 = 1$. We now construct I_2, n_2 as follows.

Divide I_1 into two intervals of the same size, $I_1' = [-M, 0]$, $I_1'' = [0, M]$. Now, consider the sets

$$A_1 = \{n \in \mathbb{N} : n > n_1 (= 1), x_n \in I_1'\}, \quad A_2 = \{n \in \mathbb{N} : n > n_1, x_n \in I_1''\}$$

(ie, all the indices of all the elements in I_1', I_1'' resp.).
Hence, $A_1 \cup A_2 = \{n : n > n_1\}$, an infinite set, and hence, one of A_1, A_2 must be infinite (by Theorem 1.9). If A_1 infinite, set $I_2 = I_1'$, $n_2 = \min A_1$. If A_1 finite, then A_2 infinite, and set $I_2 = I_1''$, $n_2 = \min A_2$.

Suppose now that I_k, n_k are chosen, and that I_k contains infinitely many elements of the sequence (x_n). Divide I_k into two equal sub-intervals, I_k', I_k''. We now introduce

$$A_1^{(k)} = \{n \in \mathbb{N} : n > n_k \text{ and } x_n \in I_k'\}, \quad A_2^{(k)} = \{n \in \mathbb{N} : n > n_k \text{ and } x_n \in I_k''\},$$

(similar to our construction of A_1, A_2). $A_1^{(k)} \cup A_2^{(k)}$ must be infinite, so one of the two must be infinite. If A_1 infinite, set $I_{k+1} = I_k'$, $n_{k+1} = \min A_1^{(k)}$. If A_2 infinite, set $I_{k+1} = I_k''$, $n_{k+1} = \min A_2^{(k)}$.

This gives now that I_{k+1} and n_{k+1}, where $I_{k+1} \subseteq I_k$, I_{k+1} contains infinitely many elements of the sequence. Further, by construction, $n_{k+1} > n_k$. This gives us a sequence of closed intervals $I_k = [a_k, b_k]$, $k \in \mathbb{N}$ such that $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq I_{k+1} \supseteq \cdots$, such that $x_{n_k} \in I_k$, and that n_k is a strictly increasing sequence of natural numbers, defining subsequence (x_{n_k}).

Now, by construction, the length of I_{k+1} is $\frac{1}{2}$ of the length of I_k. Since $I_k = [a_k, b_k]$, then

$$b_k - a_k = \frac{b_{k-1} - a_{k-1}}{2} = \cdots \frac{b_1 - a_1}{2^{k-1}} = \frac{2M}{2^{k-1}} = \frac{M}{2k^{k-2}}.$$

Since I_k, $k \in \mathbb{N}$, is a nested sequence of closed intervals and by the nested interval property of the real line (AC), $\exists x \in \bigcap_{k=1}^{\infty} I_k$.

We claim now that our subsequence (x_{n_k}) satisfies $\lim_{k\to\infty} x_{n_k} = x$. To see this, let $\varepsilon > 0$. Since $\lim_{k\to\infty} \frac{M}{2^{k-2}} = \lim_{k\to\infty} \frac{4M}{2k} = 0$, by Proposition 2.2, with $b = \frac{1}{2}$. There exists $K \in \mathbb{N}$ such that $\forall\, k \geq K$, we have $\frac{M}{2^{k-2}} = b_k - a_k < \varepsilon$. So, since I_k is a nested sequence of intervals, $\forall\, k \geq K$, $x_{n_k} \in I_K$ ($x_{n_k} \in I_k \subseteq I_K$). We also have that $x \in I_K$, since $x \in \bigcap I_k$. So, $x, x_{n_k} \in [a_K, b_K] \,\forall\, k \geq K$. So, for $k \geq K$, $|x_{n_k} - x| \leq |b_k - a_k| < \varepsilon$. So for $\varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\forall\, k \geq K$, $|x_{n_k} - x| < \varepsilon$, and so $\lim_{k\to\infty} x_{n_k} = x$, as desired. ■

[35]Sketch: See Abbott, pg 57, for good diagram.

Definition 2.9 (Peak). *Let (x_n) be a sequence of real numbers. An element x_m is called a* peak *of this sequence if $x_m \geq x_n \,\forall\, n \geq m$. x_m is bigger or equal then to any element of the sequence that follows it.*

If a sequence is decreasing, then any element of the sequence is a peak.

If a sequence is increasing, then there is no peak.

BW Proof (2): using Peaks. Take sequence (x_n). Then,

- **Case 1:** (x_n) has *infinitely* many peaks; enumerate the indices of those peaks as $n_1 < n_2 < n_3 < \cdots$, then $x_{n_k} < x_{n_{k+1}} \,\forall\, k$, since x_{n_k} is a peak, $n_{k+1} > n_k$. This gives a decreasing subsequence (x_{n_k}).

- **Case 2:** (x_n) has *finitely* many peaks, with indices $m_1 < m_2 < \cdots < m_r$. Set $n_1 = m_r + 1$. Then x_{n_1} is not a peak, and so $\exists\, n_2 > n_1$ s.t. $x_{n_2} > x_{n_1}$. Now, x_{n_2} is also not a peak, ($n_2 > n_1 > m_r$), and so there exists $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$, and so on. In this way, we construct a subsequence (x_{n_k}) that is strictly increasing, that is, $x_{n_{k+1}} > x_{n_k}$.

$\blacksquare$