MATH455 - Analysis 4

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Contents

§1 Abstract Metric and Topological Spaces

§1.1 Review of Metric Spaces

Throughout fix X a nonempty set.

↪**Definition 1.1** (Metric): $\rho : X \times X \to \mathbb{R}$ is called a *metric*, and thus (X, ρ) a *metric space*, if for all $x, y, z \in X$,
- $\rho(x, y) \geq 0$,
- $\rho(x, y) = 0 \Leftrightarrow x = y$,
- $\rho(x, y) = \rho(y, x)$, and
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

↪**Definition 1.2** (Norm): Let X a linear space. A function $\|\cdot\| : X \to [0, \infty)$ is called a *norm* if for all $u, v \in X$ and $\alpha \in \mathbb{R}$,
- $\|u\| = 0 \Leftrightarrow u = 0$,
- $\|u + v\| \leq \|u\| + \|v\|$, and
- $\|\alpha u\| = |\alpha| \, \|u\|$.

Remark 1.1: A norm induces a metric by $\rho(x, y) := \|x - y\|$.

↪**Definition 1.3**: Given two metrics ρ, σ on X, we say they are *equivalent* if $\exists \, C > 0$ such that $\frac{1}{C}\sigma(x, y) \leq \rho(x, y) \leq C\sigma(x, y)$ for every $x, y \in X$. A similar definition follows for equivalence of norms.

Given a metric space (X, ρ), then, we have the notion of
- open balls $B(x, r) = \{y \in X : \rho(x, y) < r\}$,
- open sets (subsets of X with the property that for every $x \in X$, there is a constant $r > 0$ such that $B(x, r) \subseteq X$), closed sets, closures, and
- *convergence*.

↪**Definition 1.4** (Convergence): $\{x_n\} \subseteq X$ converges to $x \in X$ if $\lim_{n\to\infty} \rho(x_n, x) = 0$.

We have several (equivalent) notions, then, of continuity; via sequences, $\varepsilon - \delta$ definition, and by pullbacks (inverse images of open sets are open).

↪**Definition 1.5** (Uniform Continuity): $f : (X, \rho) \to (Y, \sigma)$ uniformly continuous if f has a "modulus of continuity", i.e. there is a continuous function $\omega : [0, \infty) \to [0, \infty)$ such that

$$\sigma(f(x_1), f(x_2)) \leq \omega(\rho(x_1, x_2))$$

for every $x_1, x_2 \in X$.

Remark 1.2: For instance, we say f Lipschitz continuous if there is a constant $C > 0$ such that $\omega(\cdot) = C(\cdot)$. Let $\alpha \in (0, 1)$. We say f α-Holder continuous if $\omega(\cdot) = C(\cdot)^\alpha$ for some constant C.

↪**Definition 1.6** (Completeness): We say (X, ρ) *complete* if every cauchy sequence in (X, ρ) converges to a point in X.

Remark 1.3: If (X, ρ) complete and $E \subseteq X$, then (E, ρ) is complete iff E closed in X.

§1.2 Compactness, Separability

↪**Definition 1.7** (Open Cover, Compactness): $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$, where X_λ open in X and Λ an arbitrary index set, an *open cover* of X if for every $x \in X$, $\exists \lambda \in \Lambda$ such that $x \in X_\lambda$.

X is *compact* if every open cover of X admits a compact subcover. We say $E \subseteq X$ compact if (E, ρ) compact.

↪**Definition 1.8** (Totally Bounded, ε-nets): (X, ρ) *totally bounded* if $\forall \varepsilon > 0$, there is a finite cover of X of balls of radius ε. If $E \subseteq X$, an ε-*net* of E is a collection $\{B(x_i, \varepsilon)\}_{i=1}^N$ such that $E \subseteq \bigcup_{i=1}^N B(x_i, \varepsilon)$ and $x_i \in X$ (note that x_i need not be in E).

↪**Definition 1.9** (Sequentially Compact): (X, ρ) *sequentially compact* if every sequence in X has a convergence subsequence whose limit is in X.

↪**Definition 1.10** (Relatively/Pre- Compact): $E \subseteq X$ *relatively compact* if $\overline{E}$ compact.

↪**Theorem 1.1**: TFAE:

- X complete and totally bounded;
- X compact;
- X sequentially compact.

Remark 1.4: $E \subseteq X$ relatively compact if every sequence in E has a convergent subsequence.

Let $f : (X, \rho) \to (Y, \sigma)$ continuous with (X, ρ) compact. Then,

- $f(X)$ compact in Y;
- if $Y = \mathbb{R}$, the max and min of f over X are achieved;
- f is uniformly continuous.

Let $C(X) := \{f : X \to \mathbb{R} \mid f \text{ continuous}\}$ and $\|f\|_\infty := \max_{x \in X} |f(x)|$ the sup (max, in this case) norm. Then,

↪**Theorem 1.2**: Let (X, ρ) compact. Then, $(C(X), \|\cdot\|_\infty)$ is complete.

Proof. Let $\{f_n\} \subseteq C(X)$ Cauchy with respect to $\|\cdot\|_\infty$. Then, there exists a subsequence $\left\{f_{n_k}\right\}$ such that for each $k \geq 1$, $\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq 2^{-k}$ (to construct this subsequence, let $n_1 \geq 1$ be such that $\|f_n - f_{n_1}\|_\infty < \frac{1}{2}$ for all $n \geq n_1$, which exists since $\{f_n\}$ Cauchy. Then, for each $k \geq 1$, define inductively n_{k+1} such that $n_{k+1} > n_k$ and $\|f_n - f_{n_{k+1}}\|_\infty < \frac{1}{2^{k+1}}$ for each $n \geq n_{k+1}$. Then, for any $k \geq 1$, $\|f_{n_{k+1}} - f_{n_k}\|_\infty < 2^{-k}$, since $n_{k+1} > n_k$.).

Let $j \in \mathbb{N}$. Then, for any $k \geq 1$,

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell} 2^{-\ell}$$

and hence for each $x \in X$, with $c_k := f_{n_k}(x)$,

$$|c_{k+j} - c_k| \leq \sum_{\ell=k}^{\infty} 2^{-\ell}.$$

The RHS is the tail of a converging series, and thus $|c_{k+j} - c_k| \to 0$ as $k \to \infty$ i.e. $\{c_k\}$ a Cauchy sequence, in $\mathbb{R}$. $(\mathbb{R}, |\cdot|)$ complete, so $\lim_{k\to\infty} c_k =: f(x)$ exists for each $x \in X$. So, for each $x \in X$, we find

$$|f_{n_k}(x) - f(x)| \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

and since the RHS is independent of x, we may pass to the sup norm, and find

$$\|f_{n_k} - f\|_\infty \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

with the RHS $\to 0$ as $k \to \infty$. Hence, $f_{n_k} \to f$ in $C(X)$ as $k \to \infty$. In other words, we have uniform convergence of $\left\{f_{n_k}\right\}$. Each $\left\{f_{n_k}\right\}$ continuous, and thus f also continuous, and thus $f \in C(X)$.

It remains to show convergence along the whole sequence. Suppose otherwise. Then, there is some $\alpha > 0$ and a subsequence $\left\{f_{n_j}\right\} \subseteq \{f_n\}$ such that $\|f_{n_j} - f\|_\infty > \alpha > 0$ for every $j \geq 1$. Then, let k be sufficiently large such that $\|f - f_{n_k}\|_\infty \leq \frac{\alpha}{2}$. Then, for every $j \geq 1$ and k sufficiently large,

$$\begin{aligned}\|f_{n_j} - f_{n_k}\|_\infty &\geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty \\ &> \alpha - \frac{\alpha}{2} > 0,\end{aligned}$$

which contradicts the Cauchy-ness of $\{f_n\}$, completing the proof. ∎

§1.3 Arzelà-Ascoli

The goal in this section is to find conditions for a sequence of functions $\{f_n\} \subseteq C(X)$ to be precompact, namely, to have a uniformly convergent subsequence.

↪**Corollary 1.1**: Any Cauchy sequence converges if it has a convergent subsequence.

Proof. Let $\{x_n\}$ be a Cauchy sequence in a metric space (X, ρ) with convergent subsequence $\{x_{n_k}\}$ which converges to some $x \in X$. Fix $\varepsilon > 0$. Let $N \geq 1$ be such that if $m, n \geq N$, $\rho(x_n, x_m) < \frac{\varepsilon}{2}$. Let $K \geq 1$ be such that if $k \geq K$, $\rho\left(x_{n_k}, x\right) < \frac{\varepsilon}{2}$. Let $n, n_k \geq \max\{N, K\}$, then

$$\rho(x, x_n) \leq \rho\left(x, x_{n_k}\right) + \rho\left(x_{n_k}, x_n\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

∎

↪**Definition 1.11** (Equicontinuous): A family $\mathcal{F} \subseteq C(X)$ is called *equicontinuous* at $x \in X$ if $\forall\, \varepsilon > 0$ there exists a $\delta = \delta(x, \varepsilon) > 0$ such that if $\rho(x, x') < \delta$ then $|f(x) - f(x')| < \varepsilon$ for every $f \in \mathcal{F}$.

Remark 1.5: $\mathcal{F}$ equicontinuous at x iff every $f \in \mathcal{F}$ share the same modulus of continuity.

↪**Definition 1.12** (Pointwise/uniformly bounded): $\{f_n\}$ pointwise bounded if $\forall\, x \in X$, $\exists\, M(x) > 0$ such that $|f_n(x)| \leq M(x)\ \forall\, n$, and uniformly bounded if such an M exists independent of x.

↪**Lemma 1.1** (Arzelà-Ascoli Lemma): Let X separable and let $\{f_n\} \subseteq C(X)$ be pointwise bounded and equicontinuous. Then, there is a subsequence $\{f_{n_k}\}$ and a function f which converges pointwise to f on all of X.

Proof. Let $D = \{x_j\}_{j=1}^{\infty} \subseteq X$ be a countable dense subset of X. Since $\{f_n\}$ p.w. bounded, $\{f_n(x_1)\}$ as a sequence of real numbers is bounded and so by the Bolzano-Weierstrass (BW) Theorem there is a convergent subsequence $\left\{f_{n(1,k)}(x_1)\right\}_k$ that converges to some $a_1 \in \mathbb{R}$. Consider now $\left\{f_{n(1,k)}(x_2)\right\}_k$, which is again a bounded sequence of $\mathbb{R}$ and so has a convergent subsequence, call it $\left\{f_{n(2,k)}(x_2)\right\}_k$ which converges to some $a_2 \in \mathbb{R}$. Note that $\left\{f_{n(2,k)}\right\} \subseteq \left\{f_{n(1,k)}\right\}$, so also $f_{n(2,k)}(x_1) \to a_1$ as

$k \to \infty$. We can repeat this procedure, producing a sequence of real numbers $\{a_\ell\}$, and for each $j \in \mathbb{N}$ a subsequence $\left\{f_{n(j,k)}\right\}_k \subseteq \{f_n\}$ such that $f_{n(j,k)}(x_\ell) \to a_\ell$ for each $1 \leq \ell \leq j$. Define then

$$f : D \to \mathbb{R}, f\left(x_j\right) := a_j.$$

Consider now

$$f_{n_k} := f_{n(k,k)}, k \geq 1,$$

the "diagonal sequence", and remark that $f_{n_k}\left(x_j\right) \to a_j = f\left(x_j\right)$ as $k \to \infty$ for every $j \geq 1$. Hence, $\left\{f_{n_k}\right\}_k$ converges to f on D, pointwise.

We claim now that $\left\{f_{n_k}\right\}$ converges on all of X to some function $f : X \to \mathbb{R}$, pointwise. Put $g_k := f_{n_k}$ for notational convenience. Fix $x_0 \in X$, $\varepsilon > 0$, and let $\delta > 0$ be such that if $x \in X$ such that $\rho(x, x_0) < \delta$, $|g_k(x) - g_k(x_0)| < \frac{\varepsilon}{3}$ for every $k \geq 1$, which exists by equicontinuity. Since D dense in X, there is some $x_j \in D$ such that $\rho\left(x_j, x_0\right) < \delta$. Then, since $g_k\left(x_j\right) \to f\left(x_j\right)$ (pointwise), $\left\{g_k\left(x_j\right)\right\}_k$ is Cauchy and so there is some $K \geq 1$ such that for every $k, \ell \geq K$, $|g_\ell\left(x_j\right) - g_k\left(x_j\right)| < \frac{\varepsilon}{3}$. And hence, for every $k, \ell \geq K$,

$$|g_k(x_0) - g_\ell(x_0)| \leq |g_k(x_0) - g_k\left(x_j\right)| + |g_k\left(x_j\right) - g_\ell\left(x_j\right)| + |g_\ell\left(x_j\right) - g_\ell(x_0)| < \varepsilon,$$

so namely $\{g_k(x_0)\}_k$ Cauchy as a sequence in $\mathbb{R}$. Since $\mathbb{R}$ complete, then $\{g_k(x_0)\}_k$ also converges, to, say, $f(x_0) \in \mathbb{R}$. Since x_0 was arbitrary, this means there is some function $f : X \to \mathbb{R}$ such that $g_k \to f$ pointwise on X as we aimed to show. ■

↪**Definition 1.13** (Uniformly Equicontinuous): $\mathcal{F} \subseteq C(X)$ is said to be uniformly equicontinuous if for every $\varepsilon < 0$, there exists a $\delta > 0$ such that $\forall x, y \in X$ with $\rho(x, y) < \delta$, $|f(x) - f(y)| < \varepsilon$ for every $f \in \mathcal{F}$. That is, every function in $\mathcal{F}$ has the same modulus of continuity.

↪**Proposition 1.1** (Sufficient Conditions for Uniform Equicontinuity):
1. $\mathcal{F} \subseteq C(X)$ uniformly Lipschitz
2. $\mathcal{F} \subseteq C(X) \cap C^1(X)$ has a uniform L^∞ bound on the first derivative
3. $\mathcal{F} \subseteq C(X)$ uniformly Holder continuous
4. (X, ρ) compact and $\mathcal{F}$ equicontinuous

↪**Theorem 1.3** (Arzelà-Ascoli): Let (X, ρ) a compact metric space and $\{f_n\} \subseteq C(X)$ be a uniformly bounded and (uniformly) equicontinuous family of functions. Then, $\{f_n\}$ is pre-compact in $C(X)$, i.e. there exists $\left\{f_{n_k}\right\} \subseteq \{f_n\}$ such that f_{n_k} is uniformly convergent on X.

Remark 1.6: If $K \subseteq X$ a compact set, then K bounded and closed.

↪**Theorem 1.4**: Let (X, ρ) compact and $\mathcal{F} \subseteq C(X)$. Then, $\mathcal{F}$ a compact subspace of $C(X)$ iff $\mathcal{F}$ closed, uniformly bounded, and (uniformly) equicontinuous.

§1.4 Baire Category Theorem

We'll say a set $E \subseteq X$ *hollow* if int $E = \varnothing$, or equivalently if E^c dense in X.

↪**Theorem 1.5**: Let X be a complete metric space.

(a) Let $\{F_n\}$ a collection of closed hollow sets. Then, $\bigcup_{n=1}^{\infty} F_n$ also hollow.

(b) Let $\{O_n\}$ a collection of open dense sets. Then, $\bigcap_{n=1}^{\infty} O_n$ also dense.

↪**Corollary 1.2**: Let X complete and $\{F_n\}$ a sequence of closed sets in X. If $X = \bigcup_{n \geq 1} F_n$, there is some n_0 such that $\operatorname{int}\left(F_{n_0}\right) \neq \varnothing$.

↪**Corollary 1.3**: Let X complete and $\{F_n\}$ a sequence of closed sets in X. Then, $\bigcup_{n=1}^{\infty} \partial F_n$ hollow.