

Course Outline:

Based on Lectures from Winter, 2024 by Prof. Dmitry Jakobson.

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1 Introduction

1.1 Metric Spaces

↪ Definition 1.1: Metric Space

A set X is a *metric space* with distance d if

1. (symmetric) $d(x, y) = d(y, x) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. (triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$

Remark 1.1. If 1., 3. are satisfied but not 2., d can be called a “pseudo-distance”.

↪ Definition 1.2: Open Metric Space

Let (X, d) be a metric space. A subset $A \subseteq X$ is open $\iff \forall x \in A, \exists r = r(x) > 0$ s.t. $B(x, r(x)) \subseteq A$.

↪ Definition 1.3: Normed Space

Let X be a vector space over \mathbb{R} . The norm on X , denoted $\|x\| \in \mathbb{R}$, is a function that satisfies

1. $\|x\| \geq 0$
2. $\|x\| = 0 \iff x = 0$
3. $\|c \cdot x\| = |c| \cdot \|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

If X is a normed vector space over \mathbb{R} , we can define a distance d on X by $d(x, y) = \|x - y\|$.

↪ Proposition 1.1

If X is a normed vector space over \mathbb{R} , a distance d on X by $d(x, y) = \|x - y\|$ makes (X, d) a metric space.

Proof. 1. $d(x, y) = \|x - y\| \geq 0$

2. $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$

3. $d(x, y) + d(y, z) = \|x - y\| + \|y - z\| \geq \|(x - y) + (y - z)\| = \|x - z\| := d(x, z)$

■

⊗ **Example 1.1: L^p distance in \mathbb{R}^n**

Let $\bar{x} \in \mathbb{R}^n, x = (x_1, x_2, \dots, x_n)$. The L^p norm is defined

$$\|x\|_p := (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{\frac{1}{p}}.$$

In the case $p = 2, n = 2$, we simply have the standard Euclidean distance over \mathbb{R}^2 .

Unit Balls: consider when $\|x\|_p \leq 1$, over \mathbb{R}^2 .

- $p = 1 : |x_1| + |x_2| \leq 1$; this forms a “diamond ball” in the plane.
- $p = 2 : \sqrt{|x_1|^2 + |x_2|^2} \leq 1$; this forms a circle of radius 1. Clearly, this surrounds a larger area than in $p = 2$.

A natural question that follows is what happens as $p \rightarrow \infty$? Assuming $|x_1| \geq |x_2|$:

$$\begin{aligned} \|x\|_p &= (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \\ &= \left[|x_1|^p \left(1 + \left| \frac{x_2}{x_1} \right|^p \right) \right]^{\frac{1}{p}} \\ &= |x_1| \left(1 + \left| \frac{x_2}{x_1} \right|^p \right)^{\frac{1}{p}} \end{aligned}$$

If $|x_1| > |x_2|$, this goes to $|x_1|$. If they are instead equal, then $\|x\|_p = |x_1| \cdot 2^{\frac{1}{p}} \rightarrow |x_1| \cdot 1$ as well. Hence, $\lim_{p \rightarrow \infty} \|x\|_p = \max\{|x_1|, |x_2|\}$. Thus, the unit ball will approach $\max\{|x_1|, |x_2|\} \leq 1$, that is, the unit square.

↪ **Proposition 1.2**

Let $x \in \mathbb{R}^n$. Then, $\|x\|_p \rightarrow \max\{|x_1|, \dots, |x_n|\}$ as $p \rightarrow \infty$.

Remark 1.2. This is an extension of the previous example to arbitrary real space; the proof follows nearly identically.

↪ **Definition 1.4: Convex Set**

Let X be a normed space, and take $x, y \in X$. The line segment from x to y is the set

$$\{t \cdot x + (1 - t) \cdot y : 0 \leq t \leq 1\}.$$

Let $A \subseteq X$. A is *convex* $\iff \forall x, y \in A$, we have that

$$(t \cdot x + (1 - t) \cdot y) \in A \forall 0 \leq t \leq 1.$$

Remark 1.3. Think of this as saying “a set is convex iff every point on a line segment connected any two points is in the set”.

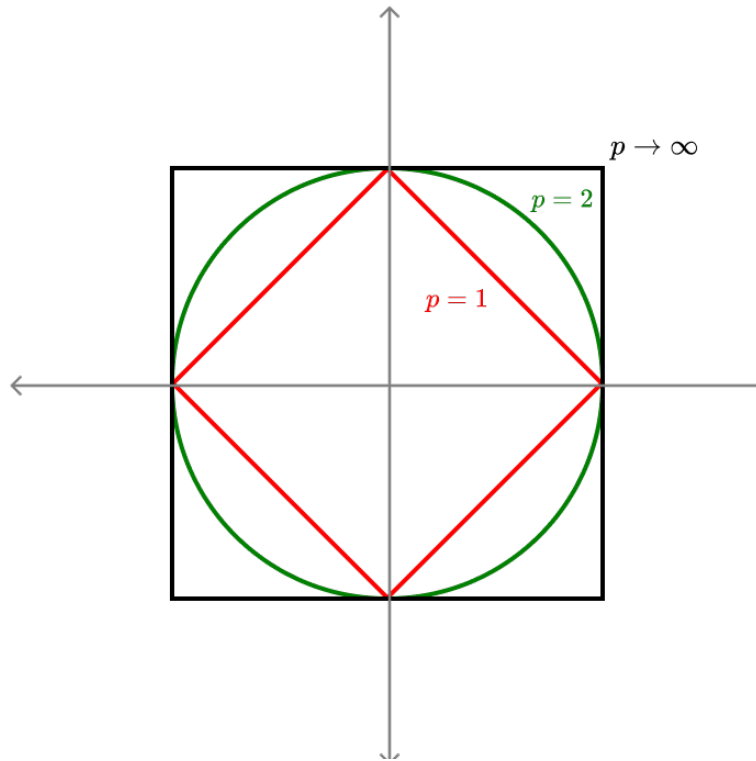


Figure 1: Regions of \mathbb{R}^2 where $\|x\|_p \leq 1$ for various values of p .

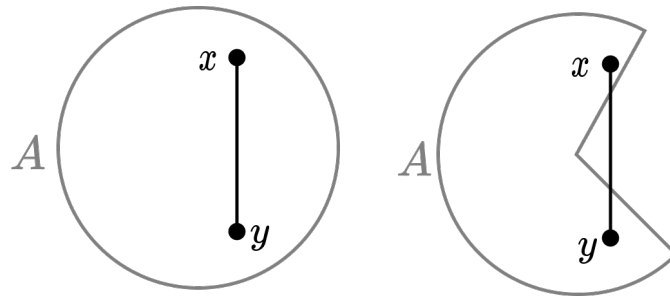


Figure 2: Convex (left) versus not convex (right) sets.

↪ **Definition 1.5:** ℓ_p

The space ℓ_p of sequences is defined as

$$\{x = (x_1, x_2, \dots, x_n, \dots) : \sum_{n=1}^{\infty} |x_n|^p < +\infty\} \quad *.$$

Then, $*$ defines the ℓ^p norm on the space of sequences; that is, $\|x\|_p := (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$.

⊛ **Example 1.2:** $\ell_p, x_n = \frac{1}{n}$

. Let $x_n = \frac{1}{n}$. For which p is $x \in \ell_p$? We have, raising the norm to the power of p for ease:

$$\begin{aligned} \|x\|_p^p &= |x_1|^p + |x_2|^p + \cdots + |x_n|^p + \cdots \\ &= 1^p + \left(\frac{1}{2}\right)^p + \cdots < \infty \iff p > 1. \end{aligned}$$

In the case that $p = 1$, this becomes a harmonic sum, which diverges.

⊛ **Example 1.3: L^p space of functions**

Let $f(x)$ be a continuous function. We define the norm of f over an interval $[a, b]$

$$\|f\|_p = \left[\int_a^b |f(x)|^p dx \right]^{\frac{1}{p}}.$$

Remark 1.4. Triangle inequality for $\|x\|_p$ or $\|f\|_p$ is called *Minkowski inequality*; $\|x\|_p + \|y\|_p \geq \|x + y\|_p$. This will be discussed further.

⊛ **Example 1.4: Distances between sets in \mathbb{R}^2**

Let A, B be bounded, closed, “nice” sets in \mathbb{R}^2 . We define

$$d(A, B) := \text{Area}(A \triangle B),$$

where

$$A \triangle B : (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

It can be shown that this is a “valid” distance.

Remark 1.5. \triangle denotes the “symmetric difference” of two sets.

⊛ **Example 1.5: p -adic distance**

Let p be a prime number. Let $x = \frac{a}{b} \in \mathbb{Q}$, and write $x = p^k \cdot \left(\frac{c}{d}\right)$, where c, d are not divisible by p . Then, the p -adic norm is defined $\|x\|_p := p^{-k}$. It can be shown that this is a norm.

Suppose $p = 2$, $x = 28 = 4 \cdot 7 = 2^2 \cdot 7$. Then, $\|28\|_2 = 2^{-2} = \frac{1}{4}$; similarly, $\|1024\|_2 = \|2^{10}\|_2 = 2^{-10}$.

More generally, we have that $\|2^k\|_2 = 2^{-k}$; conversely, $\|2^{-k}\| = 2^k$. That is, the closer to 0, the larger the distance, and vice versa, contrary to our notion of Euclidean distance.

↪ **Proposition 1.3**

$\|x\|_p$ as defined above is a well-defined norm over \mathbb{Q} .

Proof.

2 Point-Set Topology

2.1 Definitions

↪ [Definition 2.1: Topological space](#)

A set X is a topological space if we have a collection of subsets τ of X called *open sets* s.t.

1. $\emptyset \in \tau, X \in \tau$
2. Consider $\{A_\alpha\}_{\alpha \in I}$ where A_α an open set for any α ; then, $\bigcup_{\alpha \in I} A_\alpha \in \tau$, that is, it is also an open set.
3. If J is a finite set, and A_β open for all $\beta \in J$, then $\bigcap_{\beta \in J} A_\beta \in \tau$ is also open.

In other words, 2.: arbitrary unions of open sets are open, and 3.: finite intersections of open sets are open.

↪ [Definition 2.2: Closed sets](#)

Closed sets are complements of open sets; hence, axioms for closed sets follow appropriately;

- 1.* X, \emptyset closed;
- 2.* B_α closed $\forall \alpha \in I \implies \bigcap_{\alpha \in I} B_\alpha$ closed.
- 3.* B_β closed $\forall \beta \in J, J$ finite, then $\bigcup_{\beta \in J} B_\beta$ also closed.

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↪ [Definition 2.3: Equivalence of Metrics](#)

Suppose we have a metric space X with two distances d_1, d_2 ; will these necessarily admit the same topology?

A sufficient condition is that, if $\forall x \neq y \in X, \exists 1 < C < +\infty$ s.t.

$$\frac{1}{C} < \frac{d_1(x, y)}{d_2(x, y)} < C.$$

That is, the distances are equivalent, up to multiplication by a constant.

Indeed, this condition gives that $d_2 < C d_1$ and $d_2 > \frac{d_1}{C}$; this gives

$$B_{d_1}\left(x, \frac{r}{C}\right) \subseteq B_{d_2}(x, r) \subseteq B_{d_1}(x, C \cdot r).$$

Hence, d_1, d_2 define the same open/closed sets on X thus admitting the same topologies. We write $d_1 \asymp d_2$.

Remark 2.1. If $d_1 \asymp d_2$ and $d_2 \asymp d_3$, then also $d_1 \asymp d_3$. Moreover, clearly, $d_1 \asymp d_1$ and $d_1 \asymp d_2 \implies d_2 \asymp d_1$, hence this is a well-defined equivalence relation.

Hence, it's enough to show that $\forall 1 < p < +\infty$, we have $\|x\|_p \asymp \|x\|_\infty$ to show that any $\|x\|_q$ norm are equivalent for all q on \mathbb{R}^n .

↪ **Definition 2.4: Interior, Boundary of a Topological Set**

Let X be a topological space, $A \subseteq X$ and let $x \in X$. We have the following possibilities

1. $\exists U$ -open : $x \in U \subseteq A$. In this case, we say $x \in$ the *interior* of A , denoted

$$x \in \text{Int}(A).$$

2. $\exists V$ -open : $x \in V \subseteq X \setminus A = A^C$. In this case, we write

$$x \in \text{Int}(X^C).$$

3. $\forall U$ -open : $x \in U$, $U \cap A \neq \emptyset$ AND $U \cap A^C \neq \emptyset$. In this case, we say x is in the *boundary* of A , and denote

$$x \in \partial A.$$

↪ **Definition 2.5: Closure**

$x \in \text{Int}(A)$ or $x \in \partial A$ (that is, $x \in \text{Int}(A) \cup \partial A$) \iff every open set U that contains x intersects A .¹ Such points are called *limit points* of A . The set of all limit points of A is called the *closure* of A , denoted \overline{A} .

Remark 2.2. We have that

$$\text{Int}(A) \subseteq A \subseteq \overline{A} = \text{Int}(A) \cup \partial A.$$

↪ **Proposition 2.1: Properties of $\text{Int}(A)$**

$\text{Int}(A)$ is *open*, and it is the largest open set contained in A . It is the union of all U -open s.t. $U \subseteq A$. Moreover, we have that

$$\text{Int}(\text{Int}(A)) = \text{Int}(A).$$

↪ **Proposition 2.2: Properties of \overline{A}**

\overline{A} is *closed*; \overline{A} is the smallest closed set that contains A , that is, $\overline{A} = \bigcap B$ where B closed and $A \subseteq B$. We have too that

$$\overline{(\overline{A})} = \overline{A}.$$

¹"Requires" proof.

↪ **Proposition 2.3**

1. A is open $\iff A = \text{Int}(A)$
2. A is closed $\iff A = \overline{A}$

2.2 Basis

↪ **Definition 2.6: Basis for a Topology**

Let τ be a topology on X . Let $\mathcal{B} \subseteq \tau$ be a collection of open sets in X such that every open set is a union of open sets in \mathcal{B} .

⊗ **Example 2.1: Example Basis**

$X = \mathbb{R}$, and $\mathcal{B} = \{\text{all open intervals } (a, b) : -\infty < a < b < +\infty\}$.

↪ **Proposition 2.4**

Let \mathcal{B} be a collection of open sets in X . Then, \mathcal{B} is a basis \iff

1. $\forall x \in X, \exists U\text{-open} \in \mathcal{B} \text{ s.t. } x \in U$.
2. If $U_1 \in \mathcal{B}$ and $U_2 \in \mathcal{B}$, and $x \in U_1 \cap U_2$, then $\exists U_3 \in \mathcal{B} \text{ s.t. } x \in U_3 \subseteq U_1 \cap U_2$.

⊗ **Example 2.2**

Consider $X = \mathbb{R}$. Requirement 1. follows from taking $U = (x - \varepsilon, x + \varepsilon)$ for any $\varepsilon > 0$. For 2., suppose $x \in (a, b) \cap (c, d) =: U_1 \cap U_2$. Let $U_3 = (\max\{a, c\}, \min\{b, d\})$; then, we have that $U_3 \subseteq U_1 \cap U_2$, while clearly $x \in U_3$.

↪ **Proposition 2.5**

In a metric space, a basis for a topology is a collection of open balls,

$$\{B(x, r) : x \in X, r > 0\} = \{\{y \in X : d(x, y) < r\} : x \in X, r > 0\}.$$

Proof. We prove via proposition 2.4. Property 1. holds clearly; $x \in B(x, \varepsilon)\text{-open} \subseteq \mathcal{B}$.

For property 2., let $x \in B(y_1, r_1) \cap B(y_2, r_2)$, that is, $d(x, y_1) < r_1$ and $d(x, y_2) < r_2$. Let

$$\delta := \min\{r_1 - d(x, y_1), r_2 - d(x, y_2)\}.$$

We claim that $B(x, \delta) \subseteq U_1 \cap U_2$.

Let $z \in B(x, \delta)$. Then,

$$d(z, y_1) \stackrel{\Delta \neq}{\leq} d(z, x) + d(x, y_1) < \delta + d(x, y_1) \leq r_1 - d(x, y_1) + d(x, y_1) = r_1,$$

hence, as $d(z, y_1) < r_1 \implies z \in B(y_1, r_1) = U_1$. Replacing each occurrence of y_1, r_1 with y_2, r_2 respectively gives identically that $z \in B(y_2, r_2) = U_2$. Hence, we have that $B(x, \delta) \subseteq U_1 \cap U_2$ and 2. holds. ■

2.3 Subspaces

↪ Definition 2.7

Let X be a topological space and let $Y \subseteq X$. We define the subspace topology on Y :

1. Open sets in $Y = \{Y \cap \text{open sets in } X\}$

↪ Proposition 2.6: Consequences of Subspace Topologies

Suppose \mathcal{B} is a basis for a topology in X . Then, $\{U \cap Y : U \in \mathcal{B}\}$ forms a basis for the subspace $Y \subseteq X$.

Suppose X a metric space. Then, Y is also a metric space, with the same distance.

↪ Proposition 2.7

Let $Y \subseteq X$ - a metric space. Then, the metric space topology for (Y, d) is the same as the subspace topology.

Proof. (Sketch) A basis for the open sets in X can be written $\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)$; hence

$$Y \cap \left(\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha) \right) = \bigcup_{\alpha \in I} (Y \cap B(x_\alpha, r_\alpha))$$

is an open set topology for Y . ■

↪ Lemma 2.1

Let $A \subseteq X$ -open, $B \subseteq A$; B -open in subspace topology for $A \iff B$ -open in X .

↪ Lemma 2.2

Let $Y \subseteq X$, $A \subseteq Y$. Then, \overline{A} in $Y = Y \cap \overline{A}$ in X . We can denote this

$$\overline{A}_Y = \overline{A}_X \cap Y.$$

2.4 Continuous Functions

↪ Definition 2.8: Continuous Function

Let X, Y be topological spaces. Let $f : X \rightarrow Y$. f is *continuous* $\iff \forall$ open $V \in Y$, $f^{-1}(V)$ -open in X .

↪ Proposition 2.8

This definition is consistent with the normal ε - δ definition on the real line.

Proof. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, continuous; that is, $\forall \varepsilon > 0, \forall x \in \mathbb{R} \exists \delta > 0$ s.t. $|x_1 - x| < \delta$, then $|f(x_1) - f(x)| < \varepsilon$.

Let $V \subseteq \mathbb{R}$ open. Let $y \in V$. Then, $\exists \varepsilon : (y - \varepsilon, y + \varepsilon) \subseteq V$. Let $y = f(x)$, hence $y \in f^{-1}(V)$. Now, if $d(x, x_1) < \delta$, we have that $d(f(x_1), f(x)) < \varepsilon$ (by continuity of f), hence $f(x_1) \in (y - \varepsilon, y + \varepsilon) \subseteq V$; moreover, $(x - \delta, x + \delta) \subseteq f^{-1}(V)$, thus $f^{-1}(V)$ is open as required.

The inverse of this proof follows identically. ■

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↪ Proposition 2.9

Suppose \mathcal{B} forms a basis of topology for Y . Then, $f : X \rightarrow Y$ is continuous if $f^{-1}(U)$ open $\forall U \in \mathcal{B}$.

Proof. If U -open set in Y , then $\exists I$ -index set and a collection of open sets $\{A_\alpha\}_{\alpha \in I}$, $A_\alpha \in \mathcal{B}$, s.t. $U = \cup_{\alpha \in I} A_\alpha$. Then, we have

$$f^{-1}(U) = f^{-1}(\cup_{\alpha \in I} (A_\alpha)) = \cup_{\alpha \in I} \underbrace{f^{-1}(A_\alpha)}$$

Hence, if each $f^{-1}(A_\alpha)$ open, then $\cup_{\alpha \in I} f^{-1}(A_\alpha)$ open; hence it suffices to check if $f^{-1}(U) \forall U$ -open in V is open to see if f continuous. ■

↪ Theorem 2.1: Continuity of Composition

If $f : X \rightarrow Y$ continuous and $g : Y \rightarrow Z$ continuous, then $g \circ f$ continuous as well.

Proof. Let U -open in Z . Then

$$(g \circ f)^{-1}(U) = f^{-1}(\underbrace{g^{-1}(U)}_{\text{open in } Y})$$

$\underbrace{\hspace{10em}}_{\text{open in } X}$

↪ **Proposition 2.10**

If $f : X \rightarrow Y$ continuous and $A \subseteq X$, A has subspace topology, then $f|_A : A \rightarrow Y$ is also continuous.²

Proof. Let U -open in Y . Then

$$(f|_A)^{-1}(U) = \underbrace{f^{-1}(U)}_{\text{open}} \cap \underbrace{A}_{\text{open}}$$

By the definition of subspace topology, this is an open set and hence $f|_A$ is continuous. ■

2.5 Product Spaces

↪ **Definition 2.9: Finite Product Spaces**

Let X_1, \dots, X_n be topological spaces. We define

$$(X_1 \times X_2 \times \cdots \times X_n),$$

and aim to define a *product topology*; a basis of which consists of cylinder sets.

↪ **Definition 2.10: Cylinder Set**

A *cylinder set* has the form

$$A_1 \times A_2 \times \cdots \times A_n$$

where each A_j -open in X_j .

⊛ **Example 2.3**

Given an open interval $(a_1, b_1), (a_2, b_2) \subset \mathbb{R}$, the set $(a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$ is a basis for the topology on \mathbb{R}^2 .

↪ **Definition 2.11: Projection**

Let $X_1 \times X_2 \times \cdots \times X_n =: X$. The *projection* $\pi_j : X \rightarrow X_j$ maps $(x_1, \dots, x_n) \rightarrow x_j \in X_j$.

Remark 2.3. One can show π_j continuous.

↪ **Definition 2.12: Coordinate Function**

²We denote $f|_A$ as the restriction of the domain of f to A .

Given a function $f : Y \rightarrow X_1 \times \cdots \times X_n = (x_1(y), x_2(y), \dots, x_n(y))$. The *coordinate function* is

$$f_j = \pi_j \circ f; \quad f_j = x_j(y).$$

↪ **Proposition 2.11**

$f : Y \rightarrow X = X_1 \times \cdots \times X_n$ continuous $\iff f_j : Y \rightarrow X_j$ continuous.

Proof. Its enough to show that $\forall U \in \mathcal{B}$ -basis for X -product space, $f^{-1}(U)$ -open in Y . Take $U = A_1 \times \cdots \times A_n$ -open. Then, we claim that

$$f^{-1}(U) = f^{-1}(A_1 \times \cdots \times A_n) = f_1^{-1}(A_1) \cap f_2^{-1}(A_2) \cap \cdots \cap f_n^{-1}(A_n). \quad \star$$

If this holds, then as each f_i continuous (being a composition of continuous functions) and each A_i open in X_i , then each $f_i^{-1}(A_i)$ open in Y and hence \star , being the finite intersection of open sets in Y , is itself open in Y . ■

⊗ **Example 2.4: Fourier Transform: Motivation for Infinite Product Topologies**

Let $f \in C([0, 2\pi])$ is real-valued. We write the n th Fourier coefficients

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos(nx) dx - i \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin(nx) dx. \end{aligned}$$

And the Fourier transform of f as the infinite product

$$f(x) \mapsto (\dots, \hat{f}(-n), \hat{f}(-n+1), \dots, \hat{f}(-1), \hat{f}(0), \hat{f}(1), \dots, \hat{f}(n), \dots) \in \prod_{n \in \mathbb{Z}} (\mathbb{C})_n.$$

Hence, this is an (countably, as indexed by integers) infinite product space.

Now, let $f : \mathbb{R} \rightarrow \mathbb{R}$. Suppose $f(x) \rightarrow 0$ “fast enough” as $|x| \rightarrow \infty$ and f continuous. Then, we can define the Fourier coefficients

$$\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-itx} dx,$$

where $t \in \mathbb{R}$. We then have the transform

$$f \mapsto \{\hat{f}(t)\}_{t \in \mathbb{R}}.$$

In this case, our index set is \mathbb{R} is (uncountably) infinite.

↪ **Definition 2.13: Product Topology/Cylinder Sets for ∞ Products**

Let $X = \prod_{\alpha \in I} X_\alpha$. Then, a basis for X is given by cylinder sets of the form $A = \prod_{\alpha \in I} A_\alpha$ where A_α -open in X_α , AND $A_\alpha = X_\alpha$ except for finitely many indices α .

That is, there exists a finite set $J = (\alpha_1, \dots, \alpha_k) \subseteq I$, such that we can write $A = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$ (where A_α open in X_α).

↪ **Proposition 2.12**

Given $f : Y \rightarrow \prod_{\alpha \in I} X_\alpha = X$, then (taking $f_\alpha = \pi_\alpha \circ f$ as before) we have that f is continuous in $X \iff f_\alpha : Y \rightarrow X_\alpha$ continuous in $X_\alpha \forall \alpha \in I$.

Remark 2.4. Extension of proposition 2.11 to infinite product space.

Proof. Write $U = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$. Then,

$$f^{-1}(U) = \bigcap_{\alpha \in J} f_\alpha^{-1}(A_\alpha)$$

which is open in Y , hence f continuous. ■

Remark 2.5. The intersection of the entire spaces give no restriction.

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2.6 Metrizable

↪ **Proposition 2.13**

Different metrics can define the same topology.

⊛ **Example 2.5**

1. Different ℓ_p metrics in \mathbb{R}^n (PSET 1)
2. Let (X, d) be a metric space. Then,

$$\tilde{d}(x, y) := \frac{d(x, y)}{d(x, y) + 1}$$

is also a metric (the first two axioms are trivial), and defines the same topology. Note, moreover, that $\tilde{d}(x, y) \leq 1 \forall x, y$; this distance is bounded, and can often be more convenient to work with in particular contexts.

↪ **Question 2.1**

Suppose (X_k, d_k) are metric spaces $\forall k \geq 1$. Then, we can define the product topology τ on

$$X := \prod_{k=1}^{\infty} X_k.$$

Does the product topology τ come from a metric? That is, is τ metrizable?

Remark 2.6. *There do indeed exist examples of non-metrizable topological spaces; this question is indeed well-founded.*

Answer. Let $\underline{x} = (x_1, x_2, \dots, x_n, \dots), \underline{y} = (y_1, y_2, \dots, y_n, \dots) \in \prod_{k=1}^{\infty} (X_k)$ (where $x_i, y_i \in X_i$) be infinite sequences of elements. Then, for each metric space X_k take the metric

$$\tilde{d}_k(x_k, y_k) = \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)}$$

(as in the example above). Then, we define

$$D(\underline{x}, \underline{y}) = \sum_{k=1}^{\infty} \frac{\tilde{d}_k(x_k, y_k)}{2^k},$$

noting that $D(\underline{x}, \underline{y}) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ (by our construction, “normalizing” each metric), hence this is a valid, *converging* metric (which wouldn’t otherwise be guaranteed if we didn’t normalize the metrics). It remains to show whether this metric omits the same topology as τ . ■

2.7 Compactness, Connectedness

↪ Definition 2.14: Compact

A set A in a topological space is said to be *compact* if every cover has a finite subcover. That is, if

$$A \subseteq \bigcup_{\alpha \in I} U_{\alpha} - \text{open},$$

then $\exists \{\alpha_1, \dots, \alpha_n \in I\}$ such that $A \subseteq \bigcup_{i=1}^n U_{\alpha_i}$.

↪ Proposition 2.14

A closed interval $[a, b]$ is compact.

Proof. If $a = b$, this is clear. Suppose $a < b$. Then, suppose

$$[a, b] \subseteq \bigcup_{\alpha \in I} U_{\alpha} - \text{open}.$$

Then, it must be that $a \in U_{\alpha_1}$ for some α_1 ; hence, $\exists \varepsilon > 0$ $(a - \varepsilon, a + \varepsilon) \subseteq U_{\alpha_1}$. Let $c = a + \frac{\varepsilon}{2}$, then $[a, c] \subseteq U_{\alpha_1}$. Then, $[a, c]$ has a finite subcover; it is covered by the single open set U_{α_1} .

Let, then, $l = \text{least upper bound for all } c \leq b \text{ s.t. } [a, c] \text{ has a finite subcover. Then, } l \in [a, b] \text{ and so } \exists \beta \in I \text{ such that } l \in U_\beta. \text{ It follows that } \exists \varepsilon > 0 \text{ s.t. } [l - \varepsilon, l] \subseteq U_\beta.$

By our definition of l , $\exists c_j \text{ s.t. } d(l, c_j) < \frac{1}{j} \text{ s.t. } [a, c_j] \text{ is contained in a finite union of } U_\alpha \text{'s. Let } c_j \in [l - \varepsilon, l]. \text{ Then, } [a, c_j] \subseteq \bigcup_{i=1}^k U_{\alpha_i}, \text{ and then, the interval } [a, l] \subseteq \bigcup_{i=1}^k U_{\alpha_i} \cup U_\beta.$

But we can “choose” U_β such that $[l, l + \varepsilon] \subseteq U_\beta$, and so we’d further have that $[a, l + \varepsilon] \subseteq \bigcup_{i=1}^k U_{\alpha_i} \cup U_\beta$; but we have that $L + \varepsilon > l$, and we have a contradiction to l being the least upper bound for which this holds, unless $l = b$ itself. Then, we cannot “increase” further, and hence $l = b$ and so $[a, l] = [a, b] \subseteq \text{a finite union of open sets.}$ ■

Remark 2.7. A similar proof shows that $[a, b]$ is connected; we cannot cover it by two disjoint open sets.

↪ **Theorem 2.2: On Compactness**

Let $A \subseteq \mathbb{R}^n$. Then, $A \text{ compact} \iff A \text{ closed and bounded.}$

↪ **Proposition 2.15**

If X, Y are compact topological spaces, then $X \times Y$ is compact.

Remark 2.8. By induction, if X_1, \dots, X_n compact, so is $\prod_{i=1}^n X_i$.

↪ **Proposition 2.16**

A closed subset of a compact topological space is compact in the subspace topology.

Proof. (Of theorem 2.2)

(\Leftarrow) If $A \subseteq \mathbb{R}^n$ closed and bounded, then $A \subseteq [-R, +R]^n$ for some $R > 0$ (it is contained in some “ n -cube”). Then, we have that $[-R, R]$ is compact, by proposition 2.14, proposition 2.15, and proposition 2.16, A itself compact.

(\Rightarrow) Suppose $A \subseteq \mathbb{R}^n$ is compact. Then, $\bigcup_{x \in A} B(x, \varepsilon)$ for some $\varepsilon > 0$ is an open cover of A . As A compact, there must exist a finite subcover of this cover, $A \subseteq \bigcup_{i=1}^N B(x_i, r_i)$. Let $R := \max_{i=1}^N (||x_i|| + r_i)$. Then, $A \subseteq \overline{B(0, R)}$, that is, A is bounded.

Now, suppose x is a limit point of A . Then, any neighborhood of x contains a point in A , so $\forall r > 0, B(x, r) \cap A \neq \emptyset$, and so $\overline{B(x, r)}$ also contains a point of A for any $r > 0$.

Now, suppose $x \notin A$ (looking for a contradiction). Then,

$$U := \bigcup_{r>0} U_r := \bigcup_{r>0} (\mathbb{R}^n \setminus \overline{B(x, r)}) = \mathbb{R}^n \setminus \{x\}$$

is an open cover for the set A . A being compact implies that U has an finite subcover such that $A \subset U_{r_1} \cup U_{r_2} \cup \dots \cup U_{r_N}$. Let $r_0 = \min_{i=1}^N r_i$. Then, $A \subseteq U_{r_0}$, and $A \cap B(x, r_0) = \emptyset$; but this is a contradiction to the definition of a limit point, hence any limit point x is contained in A and A is thus closed by definition. ■

↪ **Proposition 2.17**

Compact \implies sequentially compact; that is, every sequence in a compact set has a convergent subsequence.

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↪ **Definition 2.15: Connected**

A topological space X is *not connected* if $X = U \cup V$ for two open, nonempty, disjoint sets U, V .

If this does not hold, X is said to be *connected*.

A set $A \subseteq X$ is not connected if A is not connected in the subspace topology $\iff A \subseteq U \cup V$, for U, V -open in X , $(U \cap A) \neq \emptyset$, $(V \cap A) \neq \emptyset$ and $U \cap V = \emptyset$.

↪ **Theorem 2.3**

Let X be a connected topological space. Let $f : X \rightarrow Y$ be a continuous. Then, $f(X)$ is also connected.

Proof. Suppose, seeking a contradiction, that X is connected, but $f(X)$ is not. Then, we can write $f(X) \subseteq Y$ as $f(X) \subseteq U \cup V$, such that U, V open in Y and $U \cap V = \emptyset$. Then,

$$(U \cap f(X)) \cap (V \cap f(X)) = \emptyset.$$

We also have that

$$X \subseteq \underbrace{f^{-1}(U)}_{\text{open in } X, \neq \emptyset} \cup \underbrace{f^{-1}(V)}_{\text{open in } X, \neq \emptyset}.$$

$f^{-1}(U) \cap f^{-1}(V) = \emptyset$ (that is, they are disjoint) by our assumption; this is a contradiction to the connectedness of X , as we are able to write it as a subset of two disjoint open sets. Hence, $f(X)$ is indeed connected. ■

↪ **Lemma 2.3**

Any interval $(a, b), [a, b], [a, b), \dots, \subseteq \mathbb{R}$ is connected.

Proof. ■

↪ **Theorem 2.4: “Intermediate Value Theorem”**

Suppose X is connected and $f : X \rightarrow \mathbb{R}$ is a continuous function. Then, f takes intermediate values.

More precisely, let $a = f(x), b = f(y)$ for $x, y \in X$. Assume $a < b$. Then, $\forall a < c < b, \exists z \in X$ s.t. $f(z) = c$.

Proof. Suppose, seeking a contradiction, that $\exists c : a < c < b$ s.t. $c \notin f(X)$ (that is, there exists an intermediate value that is “not reached” by the function).

Let $U = (-\infty, c)$ and $V = (c, +\infty)$; note that these are disjoint open sets. Then, we have that

$$X = f^{-1}(U) \cup f^{-1}(V),$$

by our assumption of $c \notin f(X)$. But this gives that X is not connected, as the union of two open (by continuity), disjoint, nonempty ($f(x) = a \in U \implies x \in f^{-1}(U)$, and $f(y) = b \in V \implies y \in f^{-1}(V)$) sets, a contradiction. ■

↪ **Theorem 2.5**

Suppose X is compact, Y -topological space, $f : X \rightarrow Y$ is a continuous function. Then, $f(X)$ is also compact.

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(X) \subseteq Y$, that is,

$$f(X) \subseteq \bigcup_{\alpha \in I} U_\alpha \implies X \subseteq f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(U_\alpha) =: \bigcup_{\alpha \in I} V_\alpha - \text{open.}$$

Then, this is an open cover of X ; X is compact, thus there exists a finite subcover, that is, indices $\{\alpha_1, \dots, \alpha_n\} \subseteq I$ such that $X = \bigcup_{i=1}^n V_{\alpha_i}$. Thus,

$$f(X) \subseteq \bigcup_{i=1}^n U_{\alpha_i},$$

which is a finite subcover of $f(X)$. Thus, $f(X)$ is compact. ■

Remark 2.9. Recall the “extreme value theorem”: let $f : [a, b] \rightarrow \mathbb{R}$ a continuous function; then, a minimum and maximum is obtained for $f(x)$ on this interval for values in this interval.

↪ **Theorem 2.6**

Let X compact, and $f : X \rightarrow \mathbb{R}$ a continuous function. Then,

$$\max_{x \in X} f(x) \text{ and } \min_{x \in X} f(x)$$

are both attained.

Proof. $f(X) \subseteq \mathbb{R}$ is compact by theorem 2.5, and so by theorem 2.2, $f(X)$ is closed and bounded. Let, then, $m = \inf f(X)$ and $M = \sup f(X)$; these necessarily exist, since $f(X)$ is bounded. Both m and M are limit points of $f(X)$. But $f(X)$ is closed, and hence contains all of its limit points, and thus $m \in f(X)$ and $M \in f(X)$, and thus $\exists y_m : f(y_m) = m$ and $y_M : f(y_M) = M$. ■

↪ **Definition 2.16: Path Connected**

A set $A \subseteq X$ is called *path connected* if $\forall x, y \in A, \exists f : [a, b] \rightarrow X$, continuous, s.t. $f(a) = x, f(b) = y$ and $f([a, b]) \subseteq A$.

The set $\{f(t) : a \leq t \leq b\}$ is called a *path* from x to y .

↪ **Theorem 2.7: Path connected \implies connected**

If $A \subseteq X$ is path connected, then A is connected.

Proof. Suppose, seeking a contradiction, that A is path connected, but not connected. Then, we can write $A \subseteq U \cup V$, for open, disjoint, nonempty subsets $U, V \subseteq X$.

Let $x \in U \cap A$ and $y \in V \cap A$. Then, $\exists f : [a, b] \rightarrow A$ s.t. $f(a) = x, f(b) = y$, and $f([a, b]) \subseteq A$, by the path connectedness of A . Then,

$$[a, b] \subseteq f^{-1}(A) \subseteq \underbrace{f^{-1}(U \cap A)}_{\text{open}} \cup \underbrace{f^{-1}(V \cap A)}_{\text{open}} =: \underbrace{U_1}_{a \in} \cup \underbrace{U_2}_{b \in},$$

that is, $[a, b]$ is contained in a union of open, nonempty, disjoint sets, contradicting $[a, b]$ the connectedness of $[a, b]$ by lemma 2.3. Thus, A is connected. ■

Remark 2.10. A counterexample to the opposite side of the implication is the Topologist's sine curve, the set

$$\{(x, \sin\left(\frac{1}{x}\right)) : x \in (0, 1]\} \cup \{0\} \times [-1, 1].$$

This set is connected in \mathbb{R}^2 , but is not path connected.

↪ Proposition 2.18

For open sets in \mathbb{R}^n , path connected \iff connected.

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