MATH325 - ODEs

A Course on Ordinary Differential Equations

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Contents

1 Introduction

1.1 Definitions

↪ **Definition 1.1: Diffferential equation**

A *diffferential equation* (DE) is an equation with derivatives. *Ordinary* DE's (ODE) will be covered in this course; other types (PDE's, SDE's, DDE's, FDE's, etc.) exist as well but won't be discussed. ODE's only have one independent variable (typically, $y = f(x)$ or $y = f(t)$).

⊛ **Example 1.1: A Trivial Example**

$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x$. Integrating both sides:

$$\int \frac{\mathrm{d}y}{\mathrm{d}x} \, \mathrm{d}x = \int 6x \, \mathrm{d}x \implies y(x) = 3x^2 + C.$$

⊛ **Example 1.2: Another One**

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = 0 \implies y = at + b.$$

↪ **Definition 1.2: Order**

The order of a differential equation is defined as the order of the highest derivative in the equation.

1.2 Initival Values

Remark 1.1. *Note the existence of arbitrary constants in the previous examples, indicating infinite solutions. We often desire unique solutions by fixing these coefficients. For first order ODEs, we simply specify a single initial condition (say, some $y(x_0) = \alpha_0$). For higher order ODEs of degree n, we can either specify $n-1$ initial conditions for $n-1$ derivatives (say, $y(x_0) = \alpha_0$, $y'(x_0) = \beta_0$), or boundary conditions (say, $y(x_0) = \alpha_0, y(x_1) = \alpha_1$) where values for the solution itself are specified.*

⊛ **Example 1.3: A Less Trivial Example**

$\frac{\mathrm{d}y}{\mathrm{d}x} = y$. We cannot simply integrate both sides as before, as we have no way to know what $\int y \, \mathrm{d}x$ (the RHS) is equal to. We can fairly easily guess that $y = e^x$ is a solution; its derivative is equal to itself, hence it does indeed solve the equation. This is not the only solution; indeed, given $y = ce^x$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ce^x = y = ce^x.$$

Luckily, we were rather limited in how many places constants could appear; this doesn't always hold.

1.3 Physical Applications

⊛ **Example 1.4: Simple Pendulum**

Let θ be the angle of a pendulum of mass m from vertical and length l. Then, we have the equation of motion

$$ml\ddot{\theta} = -mg\sin\theta \implies \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \implies \ddot{\theta} + \omega^2\sin\theta = 0.$$

Take θ small, then, $\sin\theta \approx \theta$. Then, $\ddot{\theta} + \omega^2\theta = 0$. This is linear simple harmonic motion, and has periodic solutions; how do we know this is a valid solution to the non-linear model?

↪ *Lecture 01; Last Updated: Thu Jan 4 15:16:18 EST 2024*

⊛ **Example 1.5: Lorenz Equations**

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \sigma(y-x) \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= rx - y - xz \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= xy - bz
\end{aligned}$$

These are a famous set of equations originally derived from atmospheric modeling, known for its chaotic behavior for particular parameters. This is a nonlinear system of de's, and beyond the scope of this class (indeed, it is not solvable exactly).

1.4 Uniqueness

Given an ODE of the general form $y^{(n)} = f(t, y, y', \dots, y^{n-1})$, if we wish to determine $y^{(n)}(t_0)$ uniquely, we need to specify the initial conditions

$$y(t_0), y'(t_0), \dots, y^{(n-1)}(t_0).$$

Moreover, this not only determines uniqueness of $y^{(n)}(t_0)$, byt the uniqueness of solution y for $t \in I$ for some "interval of validity" I.

↪ **Definition 1.3: Autonomous/Nonautonomous**

An ODE of the form

$$y^{(n)} = f(y, y', \dots, y^{(n-1)})$$

is called *autonomous*; that is, if it has no explicit dependence on the independent variable. Otherwise, the system is called *nonautonomous*.

↪ **Definition 1.4: Linear/Nonlinear**

Linear ODEs of dimension n have a solution space which is a vector space of dimension n. As a result, solutions can be written as a linear combination of n basis solutions (or "fundamental set of solutions"). Solutions to nonlinear ODEs cannot be written this way (except locally).

Alternatively (but equivalently), if we can write an nth order ODE in the form

$$a_n(t)y^n(t) + \cdots a_1(t)y'(t) + a_0(t)y(t) = g(t),$$

or equivalently,

$$\sum_{i=0}^{n} a_i(t)y^i(t) = g(t), \quad \circledast$$

where each $a_i(t)$ and $g(t)$ are known functions of t, then we say that the ODE is linear. Otherwise, it is nonlinear.

⊛ **Example 1.6**

The pendulum

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

is autonomous and linear;

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

is autonomous and nonlinear, due to the $\sin\theta$ term (indeed, this is a nonlinear oscillator equation); a damped-forced oscillator

$$\ddot{\theta} + k^2\dot{\theta} + \omega^2\theta = A\sin(\mu t)$$

is nonautonomous and linear.

Remark 1.2. *Note that the following definitions apply only to linear ODEs.*

↪ **Definition 1.5: Homogeneous/Nonhomogeneous**

A linear ODE of the form ⊛ is *homogeneous* if $g(t) = 0$; otherwise it is *nonhomogeneous*.

↪ **Definition 1.6: Constant/Variable**

A linear ODE of the form $*$ is *constant coefficient* if $a_j(t) =$ constant $\forall j$; if at least one a_j not constant, it is *non-constant* or *variable coefficient*.

Remark 1.3. *Note that while we define linearity of ODEs in terms of the form of $y^{(n)} = f(t, y, \ldots)$, this more "helpfully" relates to the form of the solution of such an ODE, which is indeed linear.*

1.5 Solutions

Given an n order ODE $y^{(n)} = f(t, y, \dots)$, and assuming f continuous, then for $y(t)$ to be a solution, we need y to be n-times differentiable; hence, $y, \dots, y^{(n-1)}$ must all exist and be continuous. Then, $y^{(n)}$, being a continuous function of continuous functions, is, itself, continuous.

↪ **Definition 1.7: Solution**

The function $y(t) : I \to \mathbb{R}$ is a solution to an ODE on an interval $I \subseteq \mathbb{R}$ if it is n-times differentiable on I, and satisfies the ODE on this interval.

Given an well-defined IVP with $n - 1$ initial values defined at t_0, then $y(t)$ is a solution if $t_0 \in I$, y satisfies the initial values, and $y(t)$ is a solution on the interval.

↪ **Definition 1.8: Interval of Validity**

The largest I on which $y(t) : I \to \mathbb{R}$ solves an ODE is called the *interval of validity* of the problem.

↪ *Lecture 02; Last Updated: Thu Jan 11 11:05:26 EST 2024*

2 First Order ODEs

2.1 Separable ODEs

↪ **Definition 2.1: Separable ODE**

An ODE of the form

$$y' = P(t)Q(y)$$

is called *separable*. We solve them:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= P(t)Q(y) \\ \implies \int \frac{1}{Q(y)} \,\mathrm{d}y &= \int P(t) \,\mathrm{d}t \,. \end{aligned}$$

Finish by evaluating both sides.

⊛ **Example 2.1**

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ty \tag{1}$$
$$\implies \frac{1}{y}\,\mathrm{d}y = t\,\mathrm{d}t \tag{2}$$
$$\implies \ln|y| = \frac{t^2}{2} + C \tag{3}$$
$$\implies |y| = Ke^{\frac{t^2}{2}} \quad \text{where } K = e^C \tag{4}$$
$$\implies y = Be^{\frac{t^2}{2}} \quad \text{where } B = \pm K = \pm e^C \tag{5}$$

Note that we call line (3) an *implicit solution*. In this case, we could easily turn this into an explicit solution by solving for $y(t)$; this won't always be possible.

Note that it would appear, based on the definition, that $B \neq 0$ (as $e^{\cdots} \neq 0$); however, plugging $y = 0$ into (1) shows that this is indeed a solution. It is quite easy to verify that (5) is a valid solution;

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Be^{\frac{t^2}{2}}\right) = Bte^{\frac{t^2}{2}} = t \cdot y,$$

as desired; this holds $\forall\, B \in \mathbb{R}$.

Remark 2.1. *Is it valid to split the differentials like this?*

$$\frac{1}{Q(y)}\frac{\mathrm{d}y}{\mathrm{d}t} = P(t)$$
$$\implies \int \frac{1}{Q(t)}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t = \int P(t)\,\mathrm{d}t$$

Let $g(y) = \frac{1}{Q}(y)$ *and* $G(y) = \int g(y)\,\mathrm{d}y$. *By the chain rule,*

$$\frac{\mathrm{d}}{\mathrm{d}t}(G(y(t))) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}}{\mathrm{d}y}G(y(t)) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot g(y(t)) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{1}{Q(y(t))}.$$

Integrating both sides with respect to time, we have

$$G(y(t)) = \int \frac{1}{Q(y(t))}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t = \int P(t)\,\mathrm{d}t + C$$
$$\implies \int g(y)\,\mathrm{d}y = \int P(t)\,\mathrm{d}t + C$$
$$\implies \int \frac{1}{Q(y)}\,\mathrm{d}y = \int P(t)\,\mathrm{d}t + C$$

This was our original expression obtaining by "splitting", hence it is indeed "valid".

⊛ **Example 2.2**

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2}{1-y^2} \\
\implies \int (1-y^2)\,\mathrm{d}y &= \int x^2\,\mathrm{d}x \\
\implies y - \frac{y^3}{3} &= \frac{x^3}{x} + C \\
\implies y - \frac{1}{3}(y^3 + x^3) &= C
\end{aligned}
$$

Suppose we have the same ODE but now with an IVP $y(0) = 4$. Then, plugging this into our implicit solution:

$$4 - \frac{1}{3}(64 + 0) = C \implies C = 4 - \frac{64}{3} = -\frac{52}{3},$$

so our IVP solution is

$$y - \frac{1}{3}(y^3 + x^3) = -\frac{52}{3}.$$

2.2 Linear First Order ODEs

↪ Definition 2.2: Integrating Factor

A linear first order ODE of the form

$$\begin{aligned}
a_1(t)y'(t) + a_0(t)y(t) &= g(t) \\
\implies y' + \frac{a_0}{a_1}y &= \frac{g}{a_1} \\
\implies y' + p(t)y &= q(t).
\end{aligned}$$

To solve, we multiply by some integrating factor $\mu(t)$;

$$\mu(t)y'(t) + p(t)\mu(t)y(t) = \mu(t)q(t)$$

It would be quite convenient if $p(t)\mu(t) = \mu'(t)$; in this case, we'd have

$$\begin{aligned}
\mu(t)y' + \mu'(t)y &= \mu(t)q(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)y(t)) &= \mu(t)q(t) \\
\implies \mu(t)y(t) &= \int \mu(t)q(t)\,\mathrm{d}t + C \\
\implies y(t) &= \frac{1}{\mu(t)}\int \mu(t)q(t)\,\mathrm{d}t + \frac{C}{\mu(t)}
\end{aligned}$$

Now, what is $\mu(t)$? We required that

$$\begin{aligned}
\mu'(t) &= p(t)\mu \\
\frac{\mathrm{d}\mu}{\mathrm{d}t} &= p(t)\mu \\
\implies \int \frac{\mathrm{d}\mu}{\mu} = \int p(t)\,\mathrm{d}t \implies \ln|\mu| &= \int p(t)\,\mathrm{d}t \\
\implies \mu(t) &= Ke^{\int p(t)\mathrm{d}t}
\end{aligned}$$

However, note in our whole process earlier, we need only one μ; hence, for convenience, we can disregard any constants of integration and simply take

$$\boxed{\text{Integrating Factor:} \quad \mu(t) := e^{\int p(t)\mathrm{d}t}}$$

Then, our original linear ODE has general solution

$$y(t) = Ce^{-\int p(t)\mathrm{d}t} + e^{-\int p(t)\mathrm{d}t}\int e^{\int p(t)\mathrm{d}t}q(t)\,\mathrm{d}t\,.$$

⊛ **Example 2.3**

$$
\begin{aligned}
ty' + 3y - t^2 &= 0 \\
y' + \frac{3}{t}y &= t \\
&\implies \mu(t) = e^{\int \frac{3}{t}\,\mathrm{d}t} = e^{3\ln|t|} = t^3 \\
&\implies t^3y' + 3t^2y = t^4 \\
&\implies \frac{\mathrm{d}}{\mathrm{d}t}(yt^3) = t^4 \\
&\implies yt^3 = \int t^4 \,\mathrm{d}t \\
&\implies y = \frac{1}{t^3}\cdot\frac{t^5}{5} + \frac{C}{t^3} = \frac{t^2}{5} + \frac{C}{t^3}
\end{aligned}
$$

Note the division by zero issue when $t = 0$; this is not an issue with the solution method, but indeed with the ODE itself. The ODE breaks down when $t = 0$ for the same reason.

Thus, this solution is valid for $t \in (-\infty, 0) \cup (0, \infty) =: I_1 \cup I_2$; if we are given an IVP $y(t_0) = y_0$, if $t_0 < 0$, then the interval of validity is I_1, and if $t_0 > 0$, the interval of validity is I_2.

2.3 Exact Equations

↪ **Definition 2.3: Exact Equations**

A first order ODE of the form

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \iff \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{M(x,y)}{N(x,y)}
$$

is said to be *exact* if

$$
\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y) \iff M_y(x,y) = N_x(x,y).
$$

Suppose we have a solution $f(x, y(x)) = C$. Then,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(f(x,y(x))) &= 0 \\
\implies \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\implies \frac{f_x}{f_y} &= -\frac{\mathrm{d}y}{\mathrm{d}x}
\end{aligned}
$$

Now, with $f_x(x,y) = M(x,y)$ and $f_y = N(x,y)$, then $M_y(x,y) = f_{xy}(x,y)$ and $N_x = f_{yx}(x,y)$. Assuming f continuous with existing, continuous partial derivatives, then $f_{xy} = f_{yx}$ and hence $M_y(x,y) = N_x(x,y)$. Thus, a function f such that $f_x = M$ and $f_y = N$ yields a solution to the ODE.

⊛ **Example 2.4**

$$
\begin{aligned}
2xy^2\,\mathrm{d}x + 2x^2y\,\mathrm{d}y = 0 &\equiv M\,\mathrm{d}x + N\,\mathrm{d}y = 0 \\
\implies M_y = 4xy, &\quad \implies N_x = 4xy \\
f_x = M = 2xy^2 \implies f(x,y) &= x^2y^2 + C + F(y) \\
f_y = N = 2x^2y \implies f(x,y) &= x^2y^2 + C + F(x) \\
\implies f(x,y) &= x^2y^2 + C = K
\end{aligned}
$$

We can rearrange this as an explicit solution

$$y = \frac{k}{x}$$

for some constant k.

↪ *Lecture 03; Last Updated: Tue Jan 16 10:10:00 EST 2024*

↪ **Theorem 2.1**

This technique works generally.

Proof. Given an exact ODE of the form $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$, we need to show that $\exists f(x,y)$ s.t. $f(x,y) = c$ solves the ODE. Let

$$f(x,y) = \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + g(y)$$

for some function $g(y)$ to be chosen such that $f_y = N$. But we have

$$
\begin{aligned}
N(x,y) = f_y(x,y) &= \frac{\partial}{\partial y}\left[\int_{x_0}^{x} M(s,y)\,\mathrm{d}s + g(y)\right] \\
&= g'(y) + \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s \\
&\implies g'(y) = N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\,.
\end{aligned}
$$

But the LHS is a function of y only, while the RHS depends explicitly on x; hence, this technique will only work if the entire expression is actually independent of x. To show this, we take the partial of the RHS with respect to x:

$$
\begin{aligned}
\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right] &= N_x(x,y) - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s \\
&= N_x(x,y) - \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right] \\
&= N_x(x,y) - \frac{\partial}{\partial y}\left[M(x,y)\right] \\
&= N_x - M_y = 0,
\end{aligned}
$$

as the ODE is exact. Hence, the RHS is indeed a function of y alone. So, integrating both sides with respect to y:

$$g(y) = \int \left[N(x,y) - \frac{\partial}{\partial y} \int_{x_0}^{x} M(s,y)\,\mathrm{d}s \right] \mathrm{d}y\,,$$

which gives us a $f(x,y)$ of

$$\begin{aligned} f(x,y) &= \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + \int \left[N(x,y) - \frac{\partial}{\partial y} \int_{x_0}^{x} M(s,y)\,\mathrm{d}s \right] \mathrm{d}y\,, \\ \implies f(x,y) &= \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + \int_{y_0}^{y} N(x,t)\,\mathrm{d}t - \int_{y_0}^{y} \int_{x_0}^{x} M_y(s,t)\,\mathrm{d}s\,\mathrm{d}t \qquad \star \end{aligned}$$

which satisfies $f_x = M$ and $f_y = N$. Then, for $f(x,y) = C$, we have

$$\frac{\partial f}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial f}{\partial y} = M + \frac{\mathrm{d}y}{\mathrm{d}x} N = 0 \implies M\,\mathrm{d}x + N\,\mathrm{d}y = 0,$$

as desired.

Note that $\star$ is evaluated over a rectangle $[x_0, x] \times [y_0, y]$, but holds for any connected domain containing (x_0, y_0) and (x, y).

Also note that, as described, $g(y)$ is not a function of x; hence, we can pick x arbitrarily. Suppose we take $x = x_0$, then

$$f(x,y) = \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + \int_{y_0}^{y} N(x_0,t)\,\mathrm{d}t\,.$$

■

Remark 2.2. *We could have taken $g(x)$ and started from $f_y = N$. Then, we would have had the formula*

$$f(x,y) = \int_{y_0}^{y} N(x,t)\,\mathrm{d}t + \int_{x_0}^{x} M(s,y_0)\,\mathrm{d}y\,.$$

⊛ **Example 2.5**

$$2xy\,\mathrm{d}x + (x^2-1)\,\mathrm{d}y = 0.$$

We have $M(x,y) = 2xy$ and $N(x,y) = x^2-1$, so $M_y = 2x = N_y$ and the ODE is exact; hence, a solution exists of the form $f(x,y) = c$ where $f_x = M, f_y = N$.

$$\begin{aligned} f(x,y) &= \int M(x,y)\,\mathrm{d}x = \int 2xy\,\mathrm{d}x = x^2y + k_1(y) \\ f(x,y) &= \int N(x,y)\,\mathrm{d}y = \int (x^2-1)\,\mathrm{d}y = x^2y - y + k_2(x) \end{aligned}$$

Hence $k_1(y) = -y$ and $k_2(x) = 0$, so

$$f(x,y) = x^2y - y = y(x^2-1),$$

so solutions to the original ODE are

$$y(x^2-1) = C \implies y = \frac{C}{x^2-1}.$$

2.4 Exact ODEs Via Integrating Factors

Suppose

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$

but $M_y \neq N_x$, that is, the ODE is not exact. Can we find an integrating factor $\mu(x,y)$ s.t.

$$[\mu(x,y)M(x,y)]\,\mathrm{d}x + [\mu(x,y)N(x,y)]\,\mathrm{d}y = 0$$

is exact? If so, such a μ must satisfy

$$\begin{aligned} &\frac{\partial}{\partial y}[\mu(x,y)M(x,y)] = \frac{\partial}{\partial x}[\mu(x,y)N(x,y)] \\ &\implies \mu_y M + \mu M_y = \mu_x N + \mu N_x \\ &\implies N\mu_x - M\mu_y = (M_y - N_x)\,\mu \quad ⊛ \end{aligned}$$

This is not a generally easily soluble PDE; we will consider cases where μ is a function of only one independent variable, which greatly simplifies the expression; this could be simply $\mu(x), \mu(y)$, or even $\mu(x \cdot y)$.

Suppose $\mu = \mu(x) \implies \mu_y = 0$. Then, ⊛ becomes

$$N\mu' = (M_y - N_x)\mu \implies \mu' = \left(\frac{M_y - N_x}{N}\right)\mu.$$

This is valid, provided the expression $\left(\frac{M_y-N_x}{N}\right)$ is a function solely of x. In this case, this becomes a linear first order

ODE, with solution

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} \mathrm{d}x}.$$

OTOH, if $\mu = \mu(y)$, we can similarly derive

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} \mathrm{d}y},$$

with a similar stipulation on the expression $\left(\frac{N_x - M_y}{M}\right)$ being a function of y solely.

⊛ **Example 2.6**

$$xy \,\mathrm{d}x + (2x^2 + 3y^2 - 20) \,\mathrm{d}y = 0,$$

with $M(x,y) = xy \implies M_y = x$ and $N(x,y) = 2x^2 + 3y^2 - 20 \implies N_x = 4x$. We have $M_y - N_x = x - 4x = -3x$ (so the ODE is not exact). We write

$$\frac{M_y - N_x}{M} = \frac{-3x}{xy} = \frac{-3}{y},$$

which is a function solely of y; hence, can find a $\mu(y)$:

$$\mu(y) = e^{-\int \frac{M_y - N_x}{M} \mathrm{d}y} = e^{-\int -\frac{3}{y} \mathrm{d}y} = e^{3 \ln y} = y^3,$$

noting that we, as before, do not care about any integrating factors; we are seeking a single function that works. Multiplying this into our original ODE:

$$\underbrace{xy^4}_{:=\tilde{M}} \,\mathrm{d}x + \underbrace{(2x^2 + 3y^2 - 20)y^3}_{:=\tilde{N}} \,\mathrm{d}y = 0.$$

And indeed, we have

$$\tilde{M}_y = 4xy^3; \quad \tilde{N}_x = 4xy^3 \implies \tilde{M}_y = \tilde{N}_x,$$

as desired.

↪ *Lecture 04; Last Updated: Tue Jan 23 10:02:55 EST 2024*

↪ *Lecture 05; Last Updated: Tue Jan 23 10:23:37 EST 2024*

2.5 Qualitative Methods and Theory

Remark 2.3. *Read the first few chapters of Strogatz's Nonlinear Dynamics and Chaos book and you should be all good.*

⊛ **Example 2.7**

Show that $y' = y^{\frac{1}{3}}$ with $y(0) = 0$ has infinite solutions.

↪ *Lecture 06; Last Updated: Tue Jan 23 11:21:04 EST 2024*

2.6 Existence and Uniqueness

↪ **Definition 2.4: Lipschitz Continuity**

A function $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$ is said to be *Lipschitz continuous* in y on the rectangle $R = \{(x,y) : x \in [a,b], y \in [c,d]\} = [a,b] \times [c,d]$ if there exists a constant $L > 0$ s.t.

$$|f(x,y_1) - f(x,y_2)| \leq L\,|y_1 - y_2|\,, \quad \forall\,(x,y_1),(x,y_2) \in R.$$

L is called the *Lipschitz constant.*

Remark 2.4. *Note that we define in terms on continuity in y; the x variable in each coordinate is kept constant.*

↪ **Lemma 2.1**

If $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $f(x,y)$ and $\frac{\partial f}{\partial y}$ are both continuous in x, y in the rectangle R, then f is Lipschitz in y on R.

Remark 2.5. *This result gives Differentiable $\implies$ Lipschitz Continuous $\implies$ Continuous.*

Proof. Using FTC, we have

$$\begin{aligned} f(x,y_2) &= f(x,y_1) + \int_{y_1}^{y_2} f_y(x,y)\,\mathrm{d}y \\ \implies |f(x,y_2) - f(x,y_1)| &= \left|\int_{y_1}^{y_2} f_y(x,y)\right| \leq \int_{y_1}^{y_2} |f_y(x,y)|\,\mathrm{d}y \\ &\leq |y_2 - y_1| \cdot \max_{(x,y)\in R} |f_y(x,y)|\,, \end{aligned}$$

noting that this maximum exists, and is attained, because f_y is continuous on a compact set. This gives, then, that f is Lipschitz in y with $L = \max_{(x,y)\in\mathbb{R}} |f_y(x,y)|$. ■

$\hookrightarrow$ **Theorem 2.2: Existence and Uniqueness for Scalar First Order IVPs**

If $f(t,y)$, $f_y(t,y)$ are continuous in t and y on a rectangle $R = \{(t,y) : t \in [t_0-a, t_0+a], y \in [y_0-b, y_0+b]\} = [t_0-a, t_0+a] \times [y_0-b, y_0+b]$, then $\exists h \in (0,a]$ s.t. the IVP

$$y' = f(t,y), y(t_0) = y_0$$

has a unique solution, defined for $t \in [t_0-h, t_0+h]$. Moreover, this solution satisfies $y(t) \in [y_0-b, y_0+b] \,\forall\, t \in [t_0-h, t_0+h]$.

Remark 2.6. *A stronger theorem also holds with a weakened condition on f that requires only f Lipschitz. Clearly, f_y continuous $\implies$ f Lipschitz, so we will use this fact to prove the statement, but won't prove it for the only Lipschitz case for sake of conciseness.*

Proof. Rewrite the IVP as

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s)) \,\mathrm{d}s\,.$$

We will show this form has a unique solution, using an iteration method (namely, Picard Iteration).

We will begin by guessing a solution of the IVP, $y_0(t) = y_0$, $\forall\, t \in [t_0-a, t_0+a]$. This clearly satisfies the initial condition, but not the ODE itself.

Now, given $y_n(t)$, we define

$$y_{n+1}(t) = y(t_0) + \int_{t_0}^{t} f(s, y_n(s)) \,\mathrm{d}s\,.$$

If this terminates, that is, $y_{n+1}(t) = y_n(t) \,\forall\, t \in [t_0-a, t_0+a]$, then $y_n(t)$ solves the IVP.

We now show that this iteration is both well-defined, and converges to unique solution.

By construction, $y_0 : [t_0-a, t_0+a] \to [y_0-b, y_0+b]$, and is continuous. As a bounded function on a bounded interval, it is integrable, and the first step of our step is well-defined.

Now suppose $y_n(t) : [t_0-a, t_0+a] \to [y_0-b, y_0+b]$ is continuous and integrable. Then,

$$y_{n+1}(t) = y(t_0) + \int_{t_0}^{t} f(s, y_n(s)) \,\mathrm{d}s$$

is continuous as well, as f is continuous and $y_n(s)$ is as well. It is not guaranteed to be restricted to $[y_0-b, y_0+b]$, however.

Since f continuous and attains its maximum on R, let

$$M := \max_{(t,y)\in R} |f(t,y)| < \infty.$$

We have, then, that

$$
\begin{aligned}
y_{n+1}(t) - y(t_0) &= \int_{t_0}^{t} f(s, y_n(s)) \, \mathrm{d}s \\
\implies |y_{n+1}(t) - y(t_0)| &\le |t - t_0| \, M
\end{aligned}
$$

Hence, if we choose $h : Mh \le b$, and then $y_{n+1}(t) : [t_0 - h, t_0 + h] \to [y_0 - b, y_0 + b]$ and we can iterative inductively, $y_n(t) : [t_0 - h, t_0 + h] \to [y_0 - b, y_0 + b] \, \forall \, n$. Here, we take $h = \min\{\frac{b}{M}, a\}$.

Now, let $I = [t_0 - h, t_0 + h]$, then $y_n(t) : I \to [y_0 - b, y_0 + b]$ for all n. Each iterate satisfies $y_n(t_0) = y(t_0) = y_0$; it remains to show that the iteration converges.

Let $C(I, [y_0 - b, y_0 + b])$ be the space of continuous functions $f : I \to [y_0 - b, y_0 + b]$, noting that $y_n \in C \, \forall \, n$. We define a mapping on C, $T : C \to C$ by

$$
v = Tu, v(t) = y_0(t_0) + \int_{t_0}^{t} f(s, u(s)) \, \mathrm{d}s \, .
$$

Then, $y_{n+1} = Ty_n$. We aim to show that this iteration converges uniquely; we will do this by showing T is a contraction mapping.

For $y \in C$ define the norm $||y||_\infty$ by $||y||_\infty := \max_{t \in I} |y(t)|$. This is a norm;

1. $\forall \, k \in \mathbb{R}, ||ky||_\infty = |k| \, ||y||_\infty$.
2. $||y||_\infty = 0 \iff \max_{t \in I} |y(t)| = 0 \iff y(t) = 0 \, \forall \, t \in I$.
3. $||y_1 + y_2||_\infty = \max_{t \in I} |y_1 + y_2| \le \max_{t \in I}(|y_1| + |y_2|) \le \max_{t \in I} |y_1| + \max_{t \in I} |y_2| = ||y_1||_\infty + ||y_2||_\infty$.

Now let $u, v \in C$. Then,

$$
\begin{aligned}
||Tu - Tv||_\infty &= \max_{t \in I} |Tu(t) - Tv(t)| \\
&= \max_{t \in I} \left| y(t_0) + \int_{t_0}^{t} f(s, u(s) \, \mathrm{d}s) - y_0 + \int_{t_0}^{t} f(s, v(s)) \, \mathrm{d}s \right| \\
&= \max_{t \in I} \left| \int_{t_0}^{t} f(s, u(s)) - f(s, v(s)) \, \mathrm{d}s \right| \\
&\le \max_{t \in I} \int_{t_0}^{t} |f(s, u(s)) - f(s, v(s))| \, \mathrm{d}s \\
&\le \max_{t \in I} |t - t_0| \cdot \max_{s \in I} |f(s, u(s)) - f(s, v(s))| \\
&\le hL \cdot \max_{s \in I} |u(s) - v(s)| \\
&= hL \cdot ||u - v||_\infty,
\end{aligned}
$$

hence, we have a contraction mapping if $hL < 1$; if $hL \ge 1$, let $h < \min\{a, \frac{b}{m}, \frac{1}{L}\} > 0$. With such an h, $\exists \mu \in (0, 1) : hL \le \mu < 1$, and $||Tu - Tv||_\infty \le \mu ||u - v||_\infty$, hence, a contraction mapping.

The contractive mapping theorem, which will not be proven, states that any contraction mapping has a unique fixed point $y = Ty$; moreover, for any $y_0 \in C$, the iteration $y_{n+1} = Ty_n$ converges to y.

To see this, suppose $u = Tn, v = Tv$ are two solutions of our IVP. Then, by the contraction quality,

$$||u - v||_\infty = ||Tu - Tv||_\infty \leq \mu ||u - v||_\infty,$$

a contradiction unless $||u - v||_\infty = 0 \iff u = v$, hence, we have uniqueness of our solution; that is, our IVP has at most one solution. It remains to show that this solution exists.

Consider a sequence y_n, with $y_{n+1} = Ty_n$. Then,

$$\sum_{i=0}^{N} ||y_{i+1} - y_i||_\infty \leq \mu^N ||y_1 - y_0||_\infty,$$

by the contractive property, thus,

$$\sum_{i=0}^{\infty} ||y_{i+1} - y_i|| \leq (\sum_{i=0}^{\infty} \mu^j) ||y_1 - y_0||_\infty = \frac{1}{1-\mu} ||y_1 - y_0||_\infty = R_0,$$

for some radius (real number) R_0. Similarly, looking only at the tail of the series,

$$\sum_{j=n}^{\infty} ||y_{j+1} - y_j||_\infty \leq \frac{\mu^n}{1-\mu} ||y_1 - y_0||_\infty = \mu^n R_0,$$

that is, a "smaller" radius. We could, but won't, show that this sequence is Cauchy, and space C we are working in is complete and hence this sequence converges to some limit in the space; moreover, the limit of this sequence satisfies the IVP by construction. This is beyond the scope of this course. ∎

↪ Lecture 07; Last Updated: Tue Jan 30 10:23:03 EST 2024

⊛ **Example 2.8: Using Picard Iteration**

$$y' = 2t(1+y) =: f(t,y), \quad y(0) = 0.$$

This ODE is linear and separable, and has solution $y(t) = e^{t^2} - 1$ (solving whichever way you like). We can alternatively solve this using Picard Iteration.

Let $y_0(t) = 0 \,\forall\, t$, noting that the IC is satisfied. We define

$$y_{n+1}(t) = \cancelto{0}{y(0)} + \int_{\cancelto{0}{t_0}}^{t} f(s, y_n(s))\,\mathrm{d}s\,,$$

where $f(s, y_n(s)) = 2s(1 + y(s))$. This gives

$$\begin{aligned}
y_{n+1}(t) &= \int_0^t 2s(1 + y_n(s))\,\mathrm{d}s\,. \\
&\implies y_1(t) = \int_0^t 2s(1 + y_0(s))\,\mathrm{d}s = \int_0^t 2s\,\mathrm{d}s = t^2 \\
&\implies y_2(t) = \int_0^t 2s(1 + s^2)\,\mathrm{d}s = t^2 + \frac{1}{2}t^4 \\
&\implies y_3(t) = \cdots = t^2 + \frac{1}{2!}t^4 + \frac{1}{3!}t^6 \\
&\cdots \implies y_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!} \\
&\implies \lim_{n\to\infty} y_n(t) = \sum_{k=1}^{\infty} \frac{(t^2)^k}{k!} = e^{t^2} - 1,
\end{aligned}$$

the same solution as previously shown.

Remark 2.7. *The previous example worked nicely due to $y_n(t)$ always being a simple polynomial with a familiar convergence. This is not always (nor often) the case.*

Remark 2.8. *Recall the example $y' = y^{\frac{1}{3}}$ with multiple solutions. In the language of the theorem, $f(t,y) = y^{\frac{1}{3}}$ is continuous, but $f_1(t,y) = \frac{1}{3}y^{-\frac{2}{3}}$ becomes unbounded as $y \to 0$, and the function is thus not Lipschitz in a neighborhood of $y = 0$.*

Remark 2.9. *Recall that this theorem guarantees solutions in a closed rectangular region; it is possible, under certain conditions, to extend the solution beyond the bounds. But how far?*

⊛ **Example 2.9**

$$y' = y^2, \quad y(0) = 1.$$

This has a solution $y(t) = \frac{1}{c-t} = \frac{1}{1-t}$ (with IC). Notice that $y(t) \to +\infty$ as $t \to 1$. By this observation, we have that, if we were to repeat Picard iteration for increasing time t, the rectangular domains of our validity of each piecewise solution would be bounded by 1.

↪ **Corollary 2.1**

If $f(t,y)$ and $f_y(t,y)$ are continuous for all $t,y \in \mathbb{R}$, then $\exists t_- < t_0 < t_+$ such that the IVP

$$y' = f(t,y), \quad y(t_0) = y_0$$

has a unique solution $y(t)\,\forall\, t \in (t_-, t_+)$, and moreover, either $t_+ = +\infty$ or $\lim_{t\to t_+} |y(t)| = \infty$, and either $t_- = -\infty$ or $\lim_{t\to t_-} |y(t)| = \infty$.

Remark 2.10. *Finding t_-, t_+ requires the solution. In example 2.9, $t_- = -\infty$, $t_+ = 1$. Changing the IC will naturally change these values.*

↪ **Theorem 2.3**

If $p(t), g(t)$ continuous on an open interval $I = (\alpha, \beta)$ and $t_0 \in I$, then the IVP

$$y'(t) + p(t)y = g(t), \quad y(t_0) = y_0$$

has a unique solution $y(t) : I \to \mathbb{R}$.

Remark 2.11. *In other words, this is a special case of the corollary above for linear ODEs; any "misbehavior" of the solutions would be solely due to discontinuities in the defining ODE.*

3 Second Order ODEs

3.1 Introduction

Second Order ODEs are of the form

$$y'' = f(t, y, y').$$

There is no general technique to solving these; we will be looking at special classes throughout.

Specifically in the case of nonlinear odes, there are two special cases we can solve,

1. f does not depend on y; ie $y'' = f(t, y')$. A substitution $u = y'$ yields $u' = f(t, u)$, hence this is just a first order ODE, with corresponding $y(t) = k_1 + \int u(t)\,\mathrm{d}t$.
2. f does not depend on t; ie $y'' = f(y, y')$. Let $u = y'$, so $u' = y'' = f(y, u)$. Consider $u = u(y(t))$, then,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = u\frac{\mathrm{d}u}{\mathrm{d}y},$$

and so

$$u\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{\mathrm{d}u}{\mathrm{d}t} = f(y,u) \implies \frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{y}f(y,u),$$

which again yields a first order ODE, in $u = u(y)$.

⊛ **Example 3.1: Of Case 2.**

$$y'' + \omega^2 y = 0^a$$

Rewrite this as $y'' = -\omega^2 y = f(y, y')$, and let $u = y'$, then $\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{u} f(y, u) = \frac{1}{u}[-\omega^2 y]$. This is a separable equation:

$$\begin{aligned} u \,\mathrm{d}u &= -\omega^2 y \,\mathrm{d}y \\ \frac{1}{2}u^2 &= -\frac{1}{2}\omega^2 y^2 + c \\ \implies u^2 &= -\omega^2 y^2 + c' \\ \implies u = \pm\sqrt{k^2 - \omega^2 y^2} \implies \frac{\mathrm{d}y}{\mathrm{d}t} &= \pm\sqrt{k^2 - \omega^2 y^2} \end{aligned}$$

Which is just another separable equation[b]:

$$\begin{aligned} \pm \int \mathrm{d}t &= \frac{1}{\omega} \int \frac{\mathrm{d}y}{\sqrt{\frac{k^2}{\omega^2} - y^2}} \\ \implies \frac{1}{\omega} \arcsin\left(\frac{\omega y}{k}\right) &= \pm t + C \\ \implies \frac{\omega y}{k} = \sin\left(\pm\omega t \pm \omega\tilde{C}\right) &= \pm \sin\left(\omega t + \omega\tilde{C}\right) \\ \implies y(t) &= \pm\frac{k}{\omega}\sin\left(\omega t + \omega\tilde{C}\right) \\ \implies y(t) &= K\sin(\omega t + C), \end{aligned}$$

which can be rewritten $y(t) = k_1 \sin(\omega t) + k_2 \cos(\omega t)$ with the appropriate substitutions.

[a]This is the equation for a simple harmonic oscillator.

[b]Please excuse the sloppy use of constants, it doesn't really matter.

Remark 3.1. *This is not the easiest way to solve this equation. More generally, this technique can lead to intractable integrals.*

⊛ **Example 3.2: Nonlinear Pendulum**

$$y'' + \omega^2 \sin y = 0.$$

Making the same substitution as before, $u = y'$, we have

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}y} &= -\frac{1}{u}\omega^2 \sin y \\
\int u \,\mathrm{d}u &= \int -\omega^2 \sin y \,\mathrm{d}y \\
\frac{1}{2}u^2 &= \omega^2 \cos y + c_1 \\
\frac{1}{2}(y')^2 &= \omega^2 \cos y + c_1 \\
y' &= \pm\sqrt{2c_1 + 2\omega^2 \cos y} \\
\pm \int \mathrm{d}t &= \int \frac{\mathrm{d}y}{\sqrt{2c + 2\omega^2 \cos y}},
\end{aligned}$$

where the integral on the RHS is some type of elliptic integral.

3.2 Linear, Homogeneous

We will solve a general form

$$a(t)y'' + b(t)y' + c(t)y = 0 \quad \circledast .$$

3.2.1 Principle of Superposition

↪ **Theorem 3.1: Superposition of Solutions to Linear Second Order ODEs**

If $y_1(t), y_2(t)$ solve ⊛ for $t \in I$-interval, then $y(t) = k_1 y_1(t) + k_2 y_2(t)$, for constants k_1, k_2 solves ⊛ on I as well. In other words, linear combinations of solutions are themselves solutions.

Remark 3.2. *This can be extended quite naturally to any linear order of ODE.*

Proof. This is clear by just plugging into the problem; let $y(t) = k_1 y_1(t) + k_2 y_2(t)$. Then:

$$\begin{aligned}
a(t)y''(t) + b(t)y'(t) + c(t)y(t) &= a(t)(k_1 y_1'' + k_2 y_2'') + b(t)(k_1 y_1' + k_2 y_2') + c(t)(k_1 y_1 + k_2 y_2) \\
&= k_1(ay_1'' + by_1' + cy_1) + k_2(ay_2'' + by_2' + cy_2) \\
&= k_1 \cdot 0 + k_2 \cdot 0 = 0,
\end{aligned}$$

as desired. ■

↪ *Lecture 08; Last Updated: Thu Feb 1 10:07:59 EST 2024*

↪ **Definition 3.1: Linear Independence of Functions**

If $y_1(t), y_2(t)$ are defined $\forall\, t \in I$ for some interval $I \subseteq \mathbb{R}$, then $y_1(t), y_2(t)$ are *linearly dependent on* I if $\exists k_1, k_2$ constants (not both zero) so that $k_1 \cdot y_1(t) + k_2 \cdot y_2(t) = 0 \,\forall\, t \in I$.

If the only constants which solve this are $k_1 = k_2 = 0$, then $y_1(t), y_2(t)$ are linearly independent on I.

Remark 3.3. *If $y_j(t)$ is the zero function, then take $k_j = 1$ and the other constant zero; ie, the zero function is always linearly dependent.*

3.3 Reduction of Order

Suppose $y_1(t)$ solves the homogeneous ODE $0 = a(t)y'' + b(t)y' + c(t)y$. Let $y(t) = u(t)y_1(t)$ for some unknown $u(t)$, and assume it solves the ODE. Then:

$$y = uy_1 \implies y' = u'y_1 + uy_1' \implies y'' = u''y_1 + u'y_1' + u'y_1' + uy_1'' = uy_1'' + 2u'y_1' + u''y_1.$$

Substituting this into the original ODE:

$$\begin{aligned} 0 =& a(u''y_1 + 2u'y_1' + uy_1'') + b(u'y_1 + uy_1') + c(uy_1') \\ &= [ay_1]u'' + [2ay_1' + by_1]u' + \underbrace{[ay_1'' + by_1' + cy_1]}_{=0} u \end{aligned}$$

Let $v = u' \implies v' = u''$, and we have reduced to a first-order ODE

$$0 = [ay_1]v' + [2ay_1' + by_1]v$$

which we can solve for v, then conclude by integrating v to solve for u.

3.4 Constant Coefficient Linear Homogeneous Second Order ODEs

We consider the case

$$ay'' + by' + cy = 0,$$

where a, b, c are constants. If $a = 0$, this is simply first order with an exponential solution; so, suppose (guess) that this ODE has solution $y = e^{rt}$ for $a \neq 0$. This gives

$$\begin{aligned} a(e^{rt})'' + b(e^{rt})' + c(e^{rt}) = 0 \\ \implies ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0 \\ \implies ar^2 + br + c = 0 \implies r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

and we thus have just to solve a quadratic equation. We call this the *auxiliary equation* or *characteristic equation* for the ODE.

We thus have three cases to consider:

1. $b^2 > 4ac$: r has two real roots, giving two real solutions to the original ODE of the form

$$y_1(t) = e^{r_+ t}, \quad y_2(t) = e^{r_- t},$$

where $r_\pm := r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Note that $\frac{y_2}{y_2} = e^{(r_- - r_+)t}$ is non-constant hence these solutions are independent. It follows that we have a general solution

$$y(t) = k_1 e^{r_+ t} + k_2 e^{r_- t}$$

for arbitrary constants k_1, k_2.

2. $b^2 = 4ac$: r has one real (repeated) solution, $r = \frac{-b}{2a}$. This gives only one solution $y_1 = e^{r_1 t}$: we find another by reduction of order. Let $y = uy_1 = ue^{r_1 t} = ue^{\frac{-bt}{2a}}$. We have:

$$\begin{aligned}
0 &= ay'' + by' + cy \\
0 &= a(u''y_1 + 2u'y_1' + uy_1'') + b(u'y_1 + uy_1') + cuy' \\
0 &= ay_1u'' + (2ay_1' + by_1)u' + \cancelto{0}{(ay_1'' + by_1' + cy_1)}u \\
0 &= ae^{rt}u'' + (2are^{rt} + be^{rt})u' \\
0 &= au'' + (2ar + b)u' \\
0 &= au'' + (-\frac{2ab}{2a} + b)u' \\
0 &= au'' \\
0 &= u'' \implies u' = k_1 \implies u = k_1 t + k_2,
\end{aligned}$$

and so we have another solution $y = uy_1 = (k_1 t + k_2)e^{rt}$; these constants k_1, k_2 are arbitrary (as long as $k_1 \neq 0$, which would just give a linearly dependent solution to the original), so take $k_1 = 1, k_2 = 0$. This gives a general solution

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} = (c_1 + c_2 t)e^{rt},$$

which is actually just the "second" solution we found before (and thus this one was indeed the general solution by itself).

3. $b^2 < 4ac$: r has two complex conjugate roots $r_\pm = -\frac{b}{2a} \pm \frac{\sqrt{4ac-b^2}}{2a} i := \alpha \pm i\beta$. This gives solutions

$$y_+ = e^{(\alpha + i\beta)t}, \quad y_- = e^{(\alpha - i\beta)t}.$$

While valid, these complex solutions are not necessarily useful in this form; we can rewrite them using Euler's formula to take only the real parts.

$$\begin{aligned}
y_+ &= e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] \\
y_- &= e^{(\alpha-i\beta)t} = e^{\alpha t}e^{-i\beta t} = e^{\alpha t}[\cos(-\beta t) + i\sin(-\beta t)] = e^{\alpha t}[\cos(\beta t) - i\sin(\beta t)]
\end{aligned}$$

Let $y_1 = \frac{1}{2}(y_+ + y_-) = e^{\alpha t}\cos(\beta t)$; this is a first, purely real solution to our ODE. To find a second, we could use reduction of order, or just take another linear combination of y_+, y_-

$$y_2 = \frac{1}{2i}[y_+ - y_-] = e^{\alpha t}\sin(\beta t).$$

y_1, y_2 are linearly independent, since $\frac{y_2}{y_1} = \tan(\beta t) = 0\,\forall\, t \iff \beta = 0$, which we assumed was not the case (otherwise, we'd be in case 2.). Together, we have a general, purely real solution

$$y(t) = e^{\alpha t}(k_1 \sin(\beta t) + k_2 \cos(\beta t)),$$

where k_1, k_2 arbitrary and $r = \alpha \pm i\beta$.

Harding once said: that "there is no permanent place in the world for ugly mathematics"; that means that there is a temporary place in the world for ugly mathematics. Make it pretty later.

⊛ **Example 3.3**

1. $y'' - 3y' + 2y = 0$

   This gives an auxiliary equation $r^2 - 3r + 2 = 0$ with solution $r = \frac{3\pm\sqrt{9-8}}{2} = 2, 1$. These are both positive and real, and we thus have a general solution

$$y(t) = k_1 e^t + k_2 e^{2t}.$$

2. $y'' - 2y' + y = 0$

$$\begin{aligned} r^2 - 2r + 1 = 0 &\implies (r-1)(r-1) = 0 \implies r = 1 \\ &\implies y(t) = (k_1 t + k_2)e^t \end{aligned}$$

3. $y'' + 4y' + 7y = 0$

$$\begin{aligned} r^2 + 4r + 7 = 0 &\implies r = \frac{-4 \pm \sqrt{16 - 28}}{2} = -2 \pm i\frac{1}{2}\sqrt{12} = -2 \pm i\sqrt{3} \\ &\implies y(t) = e^{-2t}\left(k_1 \sin\left(\sqrt{3}t\right) + k_2 \cos\left(\sqrt{3}t\right)\right) \end{aligned}$$

↪ *Lecture 09; Last Updated: Tue Feb 6 10:07:06 EST 2024*

3.5 Nonhomogeneous Second Order ODEs

We consider equations of the form

$$a(t)y'' + b(t)y' + cy = g(t).$$

Let's look for solutions of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t),$$

where y_1, y_2 are linearly independent solutions of the homogenous equation ($g = 0$) and y_p is a particular solution to the ODE. Plugging this into the original equation:

$$\begin{aligned} ay'' + by' + cy &= a(c_1y_1'' + c_2y_2'' + y_p'') + b(c_1y_1' + c_2y_2' + y_p') + c(c_1y_1 + c_2y_2 + y_p) \\ &= c_1\cancelto{0}{(ay_1'' + by_1' + cy_1)} + c_2\cancelto{0}{(ay_2'' + by_2' + cy_2)} + \cancelto{g}{ay_p'' + by_p' + cy_p} \\ &= g, \end{aligned}$$

as desired. Indeed, all solutions are of this form; we will show this later.

Remark 3.4. *Note that c_1, c_2 are arbitrary constants; y_p is not multiplied by a constant, and should not be.*

Remark 3.5. *y_1, y_2 are called a fundamental set of solutions; $y_c = c_1y_1 + c_2y_2$, the general solution to the homogeneous equation, is called the complementary solution of the nonhomogeneous equation. $y = y_c + y_p$ is the general solution of the nonhomogeneous equation.*

3.5.1 Linear Operator Notation

We denote $C(\mathbb{R})$ to be the space of continuous functions on $\mathbb{R}$. Let $C^p(\mathbb{R})$ be the space of p-times differentiable functions on $\mathbb{R}$; ie, $y \in C^p(\mathbb{R}) \implies y^{(j)} \in C(\mathbb{R}), j = 0, 1, \ldots, p$. Notice that $C^{p+1}(\mathbb{R}) \subsetneq C^p(\mathbb{R})$. It follows that $C^\infty(\mathbb{R}) \subsetneq \cdots \subsetneq C^n(\mathbb{R}) \subsetneq \cdots \subsetneq C(\mathbb{R})$.

Let $D : C^n(\mathbb{R}) \to C^{(n-1)}(\mathbb{R})$ be the differentiation operator, ie $Dy = y'$, noting that Dy less differentiable than y unless $y \in C^\infty(\mathbb{R})$. Its clear that D is a linear operator.

Now, define the operator $L = a(x)D^2 + b(x)D + c(x)$. Then, $L[y] = a(x)y'' + b(x)y' + c(x)y$; hence, $L[y] = 0$ and $L[y] = g$ are equivalent to our homogeneous and nonhomogeneous equations. It is clearly linear.

We explore two methods for finding the particular solution.

3.5.2 Finding y_p: Method of Undetermined Coefficients

This method only applies to ODEs with constant coefficients, and only for certain functions g.

⊛ **Example 3.4**

Consider $g(t) = L[y](t)$. Suppose $g(t) = \mu e^{\gamma t}$. Let's guess that $y_p = Ae^{\gamma t}$. Then:

$$L[y_p] = aA\gamma^2e^{\gamma t} + bA\gamma e^{\gamma t} + cAe^{\gamma t} = (a\gamma^2 + b\gamma + c)Ae^{\gamma t},$$

hence, for $L[y_p] = g = \mu e^{\gamma t}$, we need $\mu = A(a\gamma^2 + b\gamma + c) \implies A = \frac{\mu}{a\gamma^2 + b\gamma + c}$. Provided $a\gamma^2 + b\gamma + c \neq 0 \iff \gamma$ does not solve auxiliary equation, this A as defined will provide y_p.

Remark 3.6. *This example worked* because differentiating the exponential yields another exponential, which cancel nicely. The same idea can be applied for polynomials and trig functions.*

Example 3.5: With trig

Suppose $L[y] = y'' - y' + y = g(t) = 2\sin(3t)$, with auxiliary equation $r^2 - r + 1 = 0 \implies r = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. This gives complementary solution

$$y_c = e^{\frac{t}{2}} \left(k_1 \sin\left(\frac{\sqrt{3}}{2}t\right) + k_2 \cos\left(\frac{\sqrt{3}}{2}t\right)\right).$$

Suppose $y_p = A\sin(3t)$; this would give

$$-9A\sin(3t) - 3A\cos(3t) + A\sin(3t) = 2\sin(3t),$$

which implies $2 = -8A$ and $0 = -3A$, which has no solution. This does not necessarily mean that no y_p exists; at least in this case, we made a wrong guess at the beginning.

Suppose instead that $y_p = A\sin(3t) + B\cos(3t)$. This gives

$$\begin{aligned} -9A\sin(3t) - 9B\cos(3t) - 3A\cos(3t) + 3B\sin(3t) + A\sin(3t) + B\cos(3t) &= 2\sin(3t) \\ 2\sin(3t) &= (-3B - 8A)\sin(3t) + (-8B - 3A)\cos(3t) \\ \implies 2 &= -3B - 8A, \quad 0 = -8B - 3A \end{aligned}$$

Solving this equation gives $A = -\frac{16}{73}$ and $B = \frac{6}{73}$. This gives $y_p = \frac{-16}{73}\sin(3t) + \frac{6}{73}\cos(3t)$.

Example 3.6: With polynomials

Consider $L[y] = y'' + 2y' + y = t^3 = g$. Suppose $y_p = At^3 + Bt^2 + Ct + D$. Then:

$$\begin{gathered} L[y_p] = 6At + 2B + 2(3At^2 + 2Bt + C) + At^3 + Bt^2 + Ct + D = t^3 \\ At^3 + (6A + B)t^2 + (6A + B)t^2 + (6A + 4B + C)t + (2B + C + D) = t^3 \\ \implies \begin{matrix} 1 = A \\ 0 = 6A + B \\ 0 = 6A + 4B + C \\ 0 = 2B + C + D \end{matrix} \implies \begin{matrix} A = 1 \\ B = -6 \\ C = 18 \\ D = -24 \end{matrix}, \end{gathered}$$

so $y_p = t^3 - 6t^2 + 18t - 24$.

⊛ **Example 3.7: Exponential**

Take $L[y] = y'' - 2y' + y = 4e^x$ with homogeneous auxiliary $r^2 - 2r + 1 = 0 \implies (r-1)^2 = 0$ so

$$y_1 = e^x, \quad y_2 = xe^x.$$

If we guessed, $y_p = Ae^x$ then we'd have $L[Ae^x] = AL[e^x] = 0$, so it will not work. The same happens with guessing Axe^x. Suppose, then, that Ax^2e^x. Then:

$$\begin{aligned} L[Ax^2e^x] = A(x^2+4x+2)e^x - 2A(x^2+2x)e^x + Ax^2e^x &= 4e^x \\ 4e^x = 2Ae^x &\implies A = 2. \end{aligned}$$

$y_p = 2x^2e^x$, with general solution $y = (k_1 + k_2 + 2x^2)e^x$.

We now generalize the method:

Let $p(x) = \sum_{j=0}^n a_j x^j$ and $q(x) = \sum_{j=0}^n b_j x^j$ be given polynomials. To solve $L[y](x) = g(x)$ for a constant coefficient ODE, we have the following cases:

$g(x)$ (given)	$y_{p(x)}$ (guess)
$p(x)$	$x^2(A_nx^n + \cdots + A_1x + A_0)$
$e^{\alpha x}$	$x^sAe^{\alpha x}$
$p(x)e^{\alpha x}$	$x^2(A_nx^n + \cdots + A_1x + A_0)e^{\alpha x}$
$p(x)e^{\alpha x}\cos\beta x + q(x)e^{\alpha x}\sin\beta x$	$x^se^{\alpha x}\cos(\beta x)\sum_{i=0}^n A_ix^i + x^se^{\alpha x}\sin(\beta x)\sum_{j=0}^n B_jx^j$.

- $s = 0$ if $\alpha + i\beta$ is not a root of the auxiliary equation.
- $s =$ multiplicity of the root of $\alpha + i\beta$ if it is a root of the equation.

Remark 3.7. *First two cases are just special cases of the third; they are all just special cases of the last one.*

↪ *Lecture 10; Last Updated: Thu Feb 8 10:22:54 EST 2024*

Remark 3.8. *Linear combinations of the g's above can also be solved, ie if $L[y] = g_1 + g_2$, take $y_p = y_{p1} + y_{p2}$ where y_{pi} matches the "proper guess" for g_i.*

Remark 3.9. *The method fails if a, b, c not constants, or if g not of the required form.*

⊛ **Example 3.8**

1. Consider $y'' + y' - 2y = 3e^{2x}$. We have

$$r^2 + r - 2 = 0 \implies (r-1)(r+2) = 0 \implies y_1 = e^x, y_2 = e^{-2x}$$

for the homogeneous equations. Let $y_p = Ae^{2x}$, since e^{2x} does solve the equation.

2. $y'' = 1 - x^2$. $r^2 = 0 \implies y_1 = 1, y_2 = x$. Guess $g(x) = p(x)e^{\alpha x}\cos(\beta x)$ for $\alpha = 0, \beta = 0$, $p(x) = 1 - x^2$. Guessing $y_p = Ax^2 + Bx + C$ won't work; instead, guess $x^2(Ax^2 + Bx + C)$. Forgetting the x^2 would yield an unsolvable equation.

3. $y'' + 4y = 3\cos x$. $r^2 + 4 = 0 \implies r = \pm 2i$ so $y_1 = \cos 2x, y_2 = \sin 2x$. Guess $y_p = A\cos x + B\sin x$. We don't need the $\sin$, since it won't appear in the ODE; this isn't a problem anyways, as this way we'll just find that $B = 0$.

3.6 Variation of Parameters

This method works for non-constant coefficient ODEs, and (in principle) any g. To use it, we need first to know a fundamental set of solutions y_1, y_2 of the homogeneous equation.

Consider the nonhomogeneous equation

$$L[y](x) = g(x) = a(x)y'' + b(x)y' + c(x)y. \quad ⊛$$

Suppose $L[y_1] = L[y_2] = 0$, so $y_c = k_1 y_1 + k_2 y_2$ solves the homogeneous equation (constants k_i). Replace these k_i's with unknown functions, $u_i(x)$, and assume that $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ solves the ODE.

We have

$$\begin{aligned} y_p' &= [u_1' y_1 + u_2' y_2] + [y_1 u_1' + y_2 u_2'] \\ y_p'' &= [u_1' y_1 + u_2' y_2]' + [u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''] \end{aligned}$$

Substituting this into ⊛, we have that

$$\begin{aligned} g = L[y_p] &= a(x)([u_1' y_1 + u_2' y_2]') + a(x)[u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''] \\ &\quad + b(x)[u_1' y_1 + u_2' y_2] + b(x)[u_1 y_1' + u_2 y_2'] \\ &\quad + c(x)[u_1 y_1 + u_2 y_2] \\ &= u_1[\cancelto{0}{ay_1'' + by_1' + cy_1}] + u_2[\cancelto{0}{ay_2'' + by_2' + cy_2}] \qquad \text{(solve ODE by assumption)} \\ &\quad + a[u_1' y_1 + u_2' y_2]' + a[u_1' y_1' + u_2' y_2'] + b[u_1' y_1 + u_2' y_2]. \end{aligned}$$

But this is a single equation "trying" to define two unknown functions u_1, u_2; it is undetermined. We introduce an extra constraint to make it solvable. Let us state, for convenience, $u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0 \,\forall\, x$, implying

$[u_1'y_1 + u_2'y_2]' = 0\,\forall\, x$.[1] This assumption yields $g = a[u_1'y_1' + u_2'y_2']$, so we write

$$\begin{aligned} f(x) := \frac{g}{a} &= u_1'y_1' + u_2'y_2' \\ 0 &= u_1'y_1 + u_2'y_2, \end{aligned}$$

a system of two differential equations for u_1, u_2. We can solve these:

$$\begin{aligned} \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} &= \begin{pmatrix} 0 \\ f(x) \end{pmatrix} \\ \implies \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} &= \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ f(x) \end{pmatrix} \\ &= \frac{1}{y_1y_2' - y_1'y_2} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix}. \end{aligned}$$

This can be problematic if $y_1y_2' - y_1'y_2 = 0$; define $W(y_1, y_2)(x) := y_1y_2' - y_1'y_2$. Then, assuming $W(y_1, y_2)(x) \neq 0$, we have

$$u_1'(x) = \frac{-y_2(x)f(x)}{W(y_1, y_2)(x)} \quad u_2'(x) = \frac{y_1(x)f(x)}{W(y_1, y_2)(x)},$$

which we can then integrate to find u_1, u_2 appropriately. We call $W(y_1, y_2)(x)$ the *Wronskian* of y_1, y_2 wrt x.

Note that, if y_1, y_2 are linearly dependent with $y_2 = cy_1$, then $W(y_1, y_2)(x) = y_1(cy_1') - y_1'(cy_1) = 0$; that is, a necessary condition for $W(y_1, y_2) \neq 0$ is for y_1, y_2 to be linearly independent; it is not sufficient. However, we'll only use W when y_1, y_2 both solve the same ODE; in this case, it can be shown that $W(y_1, y_2)(x) \neq 0 \iff y_1, y_2$ are linearly independent[2].

[1] This is a "trust me for now" instance.

[2] Abel's Identity

⊛ **Example 3.9**

$4y'' + 36y = \frac{1}{\sin(3x)} \implies y'' + 9y = \frac{1}{4\sin(3x)} = \frac{1}{4}\csc(3x)$.

Solving the homogeneous equation: $r^2+9=0 \implies r = \pm 3i$. This gives us $y_1 = \cos(3x), y_2 = \sin(3x)$. Let $y_p = u_1\cos(3x) + u_2\sin(3x)$. We have $W(y_1, y_2) = (\cos 3x)3\cos(3x) + (3\sin(3x))(\sin(3x)) = 3$, yielding

$$u_1' = \frac{-y_2 f}{W(y_1,y_2)(x)} = \frac{-\sin(3x)\frac{1}{4\sin(3x)}}{3} = -\frac{1}{12} \implies u_1 = -\frac{x}{12}$$
$$u_2' = \frac{\cos(3x)\frac{1}{4\sin(3x)}}{3} = \frac{1}{36}\left(\frac{3\cos(3x)}{\sin(3x)}\right) = \frac{1}{36}\frac{h'}{h} \implies u_1 = \frac{1}{36}\ln(|\sin 3x|)$$

We have

$$y_p = -\frac{x}{12}\cos(3x) + \frac{1}{36}(\ln|\sin 3x|)\sin(3x),$$

with a general solution

$$y(x) = \left(k_1 - \frac{x}{12}\right)\cos(3x) + \sin(3x)\left(k_2 + \frac{1}{36}\ln|\sin(3x)|\right).$$

4 Nth Order ODEs

4.1 A Little Theory

Consider a nonlinear nth order IVP,

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x)) \quad (i)$$
$$y(x_0) = \alpha_1, \ldots, y^{(n-1)}(x_0) = \alpha_n \quad (ii),$$

noting that this is sufficient to specify a unique solution.

↪ **Theorem 4.1**

If $f(x, y_1, y_2, \ldots, y_n)$ and $\frac{\partial f}{\partial y_j}$ are continuous on the box $R = \{(x, y_1, \ldots, y_n) : |x - x_0| \leq a, |y_i - \alpha_i| \leq b, i = 1, \ldots, n\}$, then the initial value problem (i), (ii) has a unique solution $y(x)$ for $x \in [x_0 - h, x + 0 + h]$ for some $h \in (0, a]$, with solution satisfying $|y(x) - \alpha_1| \leq b \,\forall\, x \in [x_0 - h, x_0 + h]$.

Remark 4.1. *The proof is very similar to the case $n = 1$; the key step is to rewrite the nth order ODE as a system of first order ODEs.*

Let $u_1 = y, u_2 = y', \ldots, u_n = y^{(n-1)}$, and define $\underline{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$. The ODE, then, can be written

$$\underline{u}'(t) = \begin{pmatrix} u_1'(t) \\ \vdots \\ u_n'(t) \end{pmatrix} = \begin{pmatrix} y' \\ \vdots \\ y^{(n)} \end{pmatrix} = \begin{pmatrix} u_2 \\ \vdots \\ u_n \end{pmatrix} =: \underline{F}(x, \underline{u}),$$

"vectorally".

4.2 Linear nth Order ODEs

We consider

$$y^{(n)} + \sum_{i=1}^{n} p_i(x) y^{(n-1)} = g(x) =: L[y],$$

with ICs

$$y(x_0) = \alpha_1, \ldots, y^{(n-1)}(x_0)\alpha_n.$$

We would like to show that the general solution is as before with second order ODEs, ie

$$y(x) = \sum_{j=1}^{n} k_j y_j + y_p,$$

where y_p is a particular solution of $L[y] = g$, and $y_1, \ldots, y_n$ a fundamental set of solutions (of $L[y] = 0$, eg). We want to show "both directions" of this equality; this form defines solutions, and any solution is of this form. This implies, then, that the solution space has exactly dimension n.

↪ **Lemma 4.1**

Let $\varphi(x)$ be any solution of the homogeneous ODE $L[y](x) = 0$ on I. Let $u(x) \geq 0$ be defined by $(u(x))^2 = \varphi(x)^2 + \varphi'(x)^2 + \cdots + \varphi^{(n-1)}(x)^2$. Then, $\forall\, x \in I$,

$$u(x_0) e^{-k|x-x_0|} \leq u(x) \leq u(x_0) e^{k|x-x_0|},$$

where $k = 1 + \sum_{i=1}^{n} \beta_i$, $\beta = \max_{x \in I} |p_i(x)|$.

↪ Lecture 11; Last Updated: Thu Feb 8 11:26:20 EST 2024