

# MATH455 - Analysis 4

Abstract Metric, Topological Spaces; Functional Analysis.

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## §1 ABSTRACT METRIC AND TOPOLOGICAL SPACES

### §1.1 Review of Metric Spaces

Throughout fix  $X$  a nonempty set.

↪ **Definition 1.1** (Metric):  $\rho : X \times X \rightarrow \mathbb{R}$  is called a *metric*, and thus  $(X, \rho)$  a *metric space*, if for all  $x, y, z \in X$ ,

- $\rho(x, y) \geq 0$ ,
- $\rho(x, y) = 0 \Leftrightarrow x = y$ ,
- $\rho(x, y) = \rho(y, x)$ , and
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ .

↪ **Definition 1.2** (Norm): Let  $X$  a linear space. A function  $\| \cdot \| : X \rightarrow [0, \infty)$  is called a *norm* if for all  $u, v \in X$  and  $\alpha \in \mathbb{R}$ ,

- $\|u\| = 0 \Leftrightarrow u = 0$ ,
- $\|u + v\| \leq \|u\| + \|v\|$ , and
- $\|\alpha u\| = |\alpha| \|u\|$ .

**Remark 1.1:** A norm induces a metric by  $\rho(x, y) := \|x - y\|$ .

↪ **Definition 1.3:** Given two metrics  $\rho, \sigma$  on  $X$ , we say they are *equivalent* if  $\exists C > 0$  such that  $\frac{1}{C}\sigma(x, y) \leq \rho(x, y) \leq C\sigma(x, y)$  for every  $x, y \in X$ . A similar definition follows for equivalence of norms.

Given a metric space  $(X, \rho)$ , then, we have the notion of

- open balls  $B(x, r) = \{y \in X : \rho(x, y) < r\}$ ,
- open sets (subsets of  $X$  with the property that for every  $x \in X$ , there is a constant  $r > 0$  such that  $B(x, r) \subseteq X$ ), closed sets, closures, and
- *convergence*.

↪ **Definition 1.4** (Convergence):  $\{x_n\} \subseteq X$  converges to  $x \in X$  if  $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$ .

We have several (equivalent) notions, then, of continuity; via sequences,  $\varepsilon - \delta$  definition, and by pullbacks (inverse images of open sets are open).

↪ **Definition 1.5** (Uniform Continuity):  $f : (X, \rho) \rightarrow (Y, \sigma)$  uniformly continuous if  $f$  has a “modulus of continuity”, i.e. there is a continuous function  $\omega : [0, \infty) \rightarrow [0, \infty)$  such that

$$\sigma(f(x_1), f(x_2)) \leq \omega(\rho(x_1, x_2))$$

for every  $x_1, x_2 \in X$ .

**Remark 1.2:** For instance, we say  $f$  Lipschitz continuous if there is a constant  $C > 0$  such that  $\omega(\cdot) = C(\cdot)$ . Let  $\alpha \in (0, 1)$ . We say  $f$   $\alpha$ -Holder continuous if  $\omega(\cdot) = C(\cdot)^\alpha$  for some constant  $C$ .

↪ **Definition 1.6** (Completeness): We say  $(X, \rho)$  *complete* if every Cauchy sequence in  $(X, \rho)$  converges to a point in  $X$ .

**Remark 1.3:** If  $(X, \rho)$  complete and  $E \subseteq X$ , then  $(E, \rho)$  is complete iff  $E$  closed in  $X$ .

## §1.2 Compactness, Separability

↪ **Definition 1.7** (Open Cover, Compactness):  $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$ , where  $X_\lambda$  open in  $X$  and  $\Lambda$  an arbitrary index set, an *open cover* of  $X$  if for every  $x \in X$ ,  $\exists \lambda \in \Lambda$  such that  $x \in X_\lambda$ .

$X$  is *compact* if every open cover of  $X$  admits a compact subcover. We say  $E \subseteq X$  compact if  $(E, \rho)$  compact.

↪ **Definition 1.8** (Totally Bounded,  $\varepsilon$ -nets):  $(X, \rho)$  *totally bounded* if  $\forall \varepsilon > 0$ , there is a finite cover of  $X$  of balls of radius  $\varepsilon$ . If  $E \subseteq X$ , an  $\varepsilon$ -net of  $E$  is a collection  $\{B(x_i, \varepsilon)\}_{i=1}^N$  such that  $E \subseteq \bigcup_{i=1}^N B(x_i, \varepsilon)$  and  $x_i \in X$  (note that  $x_i$  need not be in  $E$ ).

↪ **Definition 1.9** (Sequentially Compact):  $(X, \rho)$  *sequentially compact* if every sequence in  $X$  has a convergent subsequence whose limit is in  $X$ .

↪ **Definition 1.10** (Relatively/Pre- Compact):  $E \subseteq X$  *relatively compact* if  $\overline{E}$  compact.

↪ **Theorem 1.1:** TFAE:

1.  $X$  complete and totally bounded;
2.  $X$  compact;
3.  $X$  sequentially compact.

**Remark 1.4:**  $E \subseteq X$  relatively compact if every sequence in  $E$  has a convergent subsequence.

Let  $f : (X, \rho) \rightarrow (Y, \sigma)$  continuous with  $(X, \rho)$  compact. Then,

- $f(X)$  compact in  $Y$ ;
- if  $Y = \mathbb{R}$ , the max and min of  $f$  over  $X$  are achieved;
- $f$  is uniformly continuous.

Let  $C(X) := \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous}\}$  and  $\|f\|_\infty := \max_{x \in X} |f(x)|$  the sup (max, in this case) norm. Then,

**→ Theorem 1.2:** Let  $(X, \rho)$  compact. Then,  $(C(X), \|\cdot\|_\infty)$  is complete.

PROOF. Let  $\{f_n\} \subseteq C(X)$  Cauchy with respect to  $\|\cdot\|_\infty$ . Then, there exists a subsequence  $\{f_{n_k}\}$  such that for each  $k \geq 1$ ,  $\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq 2^{-k}$  (to construct this subsequence, let  $n_1 \geq 1$  be such that  $\|f_n - f_{n_1}\|_\infty < \frac{1}{2}$  for all  $n \geq n_1$ , which exists since  $\{f_n\}$  Cauchy. Then, for each  $k \geq 1$ , define inductively  $n_{k+1}$  such that  $n_{k+1} > n_k$  and  $\|f_n - f_{n_{k+1}}\|_\infty < \frac{1}{2^{k+1}}$  for each  $n \geq n_{k+1}$ . Then, for any  $k \geq 1$ ,  $\|f_{n_{k+1}} - f_{n_k}\|_\infty < 2^{-k}$ , since  $n_{k+1} > n_k$ ).

Let  $j \in \mathbb{N}$ . Then, for any  $k \geq 1$ ,

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell=k}^{\infty} 2^{-\ell}$$

and hence for each  $x \in X$ , with  $c_k := f_{n_k}(x)$ ,

$$|c_{k+j} - c_k| \leq \sum_{\ell=k}^{\infty} 2^{-\ell}.$$

The RHS is the tail of a converging series, and thus  $|c_{k+j} - c_k| \rightarrow 0$  as  $k \rightarrow \infty$  i.e.  $\{c_k\}$  a Cauchy sequence, in  $\mathbb{R}$ .  $(\mathbb{R}, |\cdot|)$  complete, so  $\lim_{k \rightarrow \infty} c_k =: f(x)$  exists for each  $x \in X$ . So, for each  $x \in X$ , we find

$$|f_{n_k}(x) - f(x)| \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

and since the RHS is independent of  $x$ , we may pass to the sup norm, and find

$$\|f_{n_k} - f\|_\infty \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

with the RHS  $\rightarrow 0$  as  $k \rightarrow \infty$ . Hence,  $f_{n_k} \rightarrow f$  in  $C(X)$  as  $k \rightarrow \infty$ . In other words, we have uniform convergence of  $\{f_{n_k}\}$ . Each  $\{f_{n_k}\}$  continuous, and thus  $f$  also continuous, and thus  $f \in C(X)$ .

It remains to show convergence along the whole sequence. Suppose otherwise. Then, there is some  $\alpha > 0$  and a subsequence  $\{f_{n_j}\} \subseteq \{f_n\}$  such that  $\|f_{n_j} - f\|_\infty >$

$\alpha > 0$  for every  $j \geq 1$ . Then, let  $k$  be sufficiently large such that  $\|f - f_{n_k}\|_\infty \leq \frac{\alpha}{2}$ . Then, for every  $j \geq 1$  and  $k$  sufficiently large,

$$\begin{aligned}\|f_{n_j} - f_{n_k}\|_\infty &\geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty \\ &> \alpha - \frac{\alpha}{2} > 0,\end{aligned}$$

which contradicts the Cauchy-ness of  $\{f_n\}$ , completing the proof. ■

↪ **Definition 1.11** (Density/Separability): A set  $D \subseteq X$  is called *dense* in  $X$  if for every nonempty open subset  $A \subseteq X$ ,  $D \cap A \neq \emptyset$ . We say  $X$  *separable* if there is a countable dense subset of  $X$ .

**Remark 1.5:** If  $A$  dense in  $X$ , then  $\overline{A} = X$ .

↪ **Proposition 1.1:** If  $X$  compact,  $X$  separable.

PROOF. Since  $X$  compact, it is totally bounded. So, for  $n \in \mathbb{N}$ , there is some  $K_n$  and  $\{x_i\} \subseteq X$  such that  $X \subseteq \bigcup_{i=1}^{K_n} B(x_i, \frac{1}{n})$ . Then,  $D = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{K_n} \{x_i\}$  countable and dense in  $X$ . ■

### §1.3 Arzelà-Ascoli

The goal in this section is to find conditions for a sequence of functions  $\{f_n\} \subseteq C(X)$  to be precompact, namely, to have a uniformly convergent subsequence.

↪ **Corollary 1.1:** Any Cauchy sequence converges if it has a convergent subsequence.

PROOF. Let  $\{x_n\}$  be a Cauchy sequence in a metric space  $(X, \rho)$  with convergent subsequence  $\{x_{n_k}\}$  which converges to some  $x \in X$ . Fix  $\varepsilon > 0$ . Let  $N \geq 1$  be such that if  $m, n \geq N$ ,  $\rho(x_n, x_m) < \frac{\varepsilon}{2}$ . Let  $K \geq 1$  be such that if  $k \geq K$ ,  $\rho(x_{n_k}, x) < \frac{\varepsilon}{2}$ . Let  $n, n_k \geq \max\{N, K\}$ , then

$$\rho(x, x_n) \leq \rho(x, x_{n_k}) + \rho(x_{n_k}, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

↪ **Definition 1.12** (Equicontinuous): A family  $\mathcal{F} \subseteq C(X)$  is called *equicontinuous* at  $x \in X$  if  $\forall \varepsilon > 0$  there exists a  $\delta = \delta(x, \varepsilon) > 0$  such that if  $\rho(x, x') < \delta$  then  $|f(x) - f(x')| < \varepsilon$  for every  $f \in \mathcal{F}$ .

**Remark 1.6:**  $\mathcal{F}$  equicontinuous at  $x$  iff every  $f \in \mathcal{F}$  share the same modulus of continuity.

↪ **Definition 1.13** (Pointwise/uniformly bounded):  $\{f_n\}$  pointwise bounded if  $\forall x \in X$ ,  $\exists M(x) > 0$  such that  $|f_n(x)| \leq M(x) \forall n$ , and uniformly bounded if such an  $M$  exists independent of  $x$ .

↪ **Lemma 1.1** (Arzelà-Ascoli Lemma): Let  $X$  separable and let  $\{f_n\} \subseteq C(X)$  be pointwise bounded and equicontinuous. Then, there is a function  $f$  and a subsequence  $\{f_{n_k}\}$  which converges pointwise to  $f$  on all of  $X$ .

PROOF. Let  $D = \{x_j\}_{j=1}^{\infty} \subseteq X$  be a countable dense subset of  $X$ . Since  $\{f_n\}$  p.w. bounded,  $\{f_n(x_1)\}$  as a sequence of real numbers is bounded and so by the Bolzano-Weierstrass (BW) Theorem there is a convergent subsequence  $\{f_{n(1,k)}(x_1)\}_k$  that converges to some  $a_1 \in \mathbb{R}$ . Consider now  $\{f_{n(1,k)}(x_2)\}_k$ , which is again a bounded sequence of  $\mathbb{R}$  and so has a convergent subsequence, call it  $\{f_{n(2,k)}(x_2)\}_k$  which converges to some  $a_2 \in \mathbb{R}$ . Note that  $\{f_{n(2,k)}\} \subseteq \{f_{n(1,k)}\}$ , so also  $f_{n(2,k)}(x_1) \rightarrow a_1$  as  $k \rightarrow \infty$ . We can repeat this procedure, producing a sequence of real numbers  $\{a_\ell\}$ , and for each  $j \in \mathbb{N}$  a subsequence  $\{f_{n(j,k)}\}_k \subseteq \{f_n\}$  such that  $f_{n(j,k)}(x_\ell) \rightarrow a_\ell$  for each  $1 \leq \ell \leq j$ . Define then

$$f : D \rightarrow \mathbb{R}, f(x_j) := a_j.$$

Consider now

$$f_{n_k} := f_{n(k,k)}, k \geq 1,$$

the “diagonal sequence”, and remark that  $f_{n_k}(x_j) \rightarrow a_j = f(x_j)$  as  $k \rightarrow \infty$  for every  $j \geq 1$ . Hence,  $\{f_{n_k}\}_k$  converges to  $f$  on  $D$ , pointwise.

We claim now that  $\{f_{n_k}\}$  converges on all of  $X$  to some function  $f : X \rightarrow \mathbb{R}$ , pointwise. Put  $g_k := f_{n_k}$  for notational convenience. Fix  $x_0 \in X$ ,  $\varepsilon > 0$ , and let  $\delta > 0$  be such that if  $x \in X$  such that  $\rho(x, x_0) < \delta$ ,  $|g_k(x) - g_k(x_0)| < \frac{\varepsilon}{3}$  for every  $k \geq 1$ , which exists by equicontinuity. Since  $D$  dense in  $X$ , there is some  $x_j \in D$  such that  $\rho(x_j, x_0) < \delta$ . Then, since  $g_k(x_j) \rightarrow f(x_j)$  (pointwise),  $\{g_k(x_j)\}_k$  is Cauchy and so there is some  $K \geq 1$  such that for every  $k, \ell \geq K$ ,  $|g_\ell(x_j) - g_k(x_j)| < \frac{\varepsilon}{3}$ . And hence, for every  $k, \ell \geq K$ ,

$$|g_k(x_0) - g_\ell(x_0)| \leq |g_k(x_0) - g_k(x_j)| + |g_k(x_j) - g_\ell(x_j)| + |g_\ell(x_j) - g_\ell(x_0)| < \varepsilon,$$

so namely  $\{g_k(x_0)\}_k$  Cauchy as a sequence in  $\mathbb{R}$ . Since  $\mathbb{R}$  complete, then  $\{g_k(x_0)\}_k$  also converges, to, say,  $f(x_0) \in \mathbb{R}$ . Since  $x_0$  was arbitrary, this means there is some function  $f : X \rightarrow \mathbb{R}$  such that  $g_k \rightarrow f$  pointwise on  $X$  as we aimed to show. ■

↪ **Definition 1.14** (Uniformly Equicontinuous):  $\mathcal{F} \subseteq C(X)$  is said to be uniformly equicontinuous if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $\forall x, y \in X$  with  $\rho(x, y) < \delta$ ,  $|f(x) - f(y)| < \varepsilon$  for every  $f \in \mathcal{F}$ . That is, every function in  $\mathcal{F}$  has the same modulus of continuity.

↪ **Proposition 1.2** (Sufficient Conditions for Uniform Equicontinuity):

1.  $\mathcal{F} \subseteq C(X)$  uniformly Lipschitz
2.  $\mathcal{F} \subseteq C(X) \cap C^1(X)$  has a uniform  $L^\infty$  bound on the first derivative
3.  $\mathcal{F} \subseteq C(X)$  uniformly Hölder continuous
4.  $(X, \rho)$  compact and  $\mathcal{F}$  equicontinuous

PROOF.

1. If  $C > 0$  is such that  $|f(x) - f(y)| \leq C\rho(x, y)$  for every  $x, y \in X$  and  $f \in \mathcal{F}$ , then for  $\varepsilon > 0$ , let  $\delta = \frac{\varepsilon}{C}$ , then if  $\rho(x, y) \leq \delta$ ,  $|f(x) - f(y)| \leq C\delta < \varepsilon$ , and  $\delta$  independent of  $x$  (and  $f$ ) since it only depends on  $C$  which is independent of  $x, y, f$ , etc.
3. Akin to 1.

■

↪ **Theorem 1.3** (Arzelà-Ascoli): Let  $(X, \rho)$  a compact metric space and  $\{f_n\} \subseteq C(X)$  be a uniformly bounded and (uniformly) equicontinuous family of functions. Then,  $\{f_n\}$  is pre-compact in  $C(X)$ , i.e. there exists  $\{f_{n_k}\} \subseteq \{f_n\}$  such that  $f_{n_k}$  is uniformly convergent on  $X$ .

PROOF. Since  $(X, \rho)$  compact it is separable and so by the lemma there is a subsequence  $\{f_{n_k}\}$  that converges pointwise on  $X$ . Denote by  $g_k := f_{n_k}$  for notational convenience.

We claim  $\{g_k\}$  uniformly Cauchy. Let  $\varepsilon > 0$ . By uniform equicontinuity, there is a  $\delta > 0$  such that  $\rho(x, y) < \delta \Rightarrow |g_k(x) - g_k(y)| < \frac{\varepsilon}{3}$ . Since  $X$  compact it is totally bounded so there exists  $\{x_i\}_{i=1}^N$  such that  $X \subseteq \bigcup_{i=1}^N B(x_i, \delta)$ . For every  $1 \leq i \leq N$ ,  $\{g_k(x_i)\}$  converges by the lemma hence is Cauchy in  $\mathbb{R}$ . So, there exists a  $K_i$  such that for every  $k, \ell \geq K_i$   $|g_k(x_i) - g_\ell(x_i)| \leq \frac{\varepsilon}{3}$ . Let  $K := \max\{K_i\}$ . Then for every  $\ell, k \leq K$ ,  $|g_k(x_i) - g_\ell(x_i)| \leq \frac{\varepsilon}{3}$  for every  $i = 1, \dots, N$ . So, for all  $x \in X$ , there is some  $x_i$  such that  $\rho(x, x_i) < \delta$ , and so for every  $k, \ell \geq K$ ,

$$\begin{aligned} |g_k(x) - g_\ell(x)| &\leq |g_k(x) - g_k(x_i)| \\ &\quad + |g_k(x_i) - g_\ell(x_i)| \\ &\quad + |g_\ell(x_i) - g_\ell(x)| < \varepsilon, \end{aligned}$$

the first and last follow by the equicontinuity and the second from the lemma. This holds for every  $x$  and thus  $\|g_k - g_\ell\|_\infty < \varepsilon$ , so  $\{g_k\}$  Cauchy in  $C(X)$ . But  $C(X)$  complete so converges in the space.

■

**Remark 1.7:** If  $K \subseteq X$  a compact set, then  $K$  bounded and closed.

↪ **Theorem 1.4:** Let  $(X, \rho)$  compact and  $\mathcal{F} \subseteq C(X)$ . Then,  $\mathcal{F}$  a compact subspace of  $C(X)$  iff  $\mathcal{F}$  closed, uniformly bounded, and (uniformly) equicontinuous.

PROOF. ( $\Leftarrow$ ) Let  $\{f_n\} \subseteq \mathcal{F}$ . By Arzelà-Ascoli Theorem, there exists a subsequence  $\{f_{n_k}\}$  that converges uniformly to some  $f \in C(X)$ . Since  $\mathcal{F}$  closed,  $f \in \mathcal{F}$  and so  $\mathcal{F}$  sequentially compact hence compact.

( $\Rightarrow$ )  $\mathcal{F}$  compact so closed and bounded in  $C(X)$ . To prove equicontinuous, we argue by contradiction. Suppose otherwise, that  $\mathcal{F}$  not-equicontinuous at some  $x \in X$ . Then, there is some  $\varepsilon_0 > 0$  and  $\{f_n\} \subseteq \mathcal{F}$  and  $\{x_n\} \subseteq X$  such that  $|f_n(x_n) - f_n(x)| \geq \varepsilon_0$  while  $\rho(x, x_n) < \frac{1}{n}$ . Since  $\{f_n\}$  bounded and  $\mathcal{F}$  compact, there is a subsequence  $\{f_{n_k}\}$  that converges to  $f$  uniformly. Let  $K$  be such that  $\forall k \geq K, \|f_{n_k} - f\|_\infty \leq \frac{\varepsilon_0}{3}$ . Then,

$$\begin{aligned} |f(x_{n_k}) - f| &\geq |f(x_{n_k}) - f_{n_k}(x_{n_k})| - |f_{n_k}(x_{n_k}) - f_{n_k}(x)| - |f_{n_k}(x) - f(x)| \\ &\geq \frac{\varepsilon_0}{3}, \end{aligned}$$

while  $\rho(x_{n_k}, x) \leq \frac{1}{n_k}$ , so  $f$  cannot be continuous at  $x$ , a contradiction. ■

## §1.4 Baire Category Theorem

↪ **Definition 1.15** (Hollow/Nowhere Dense): We say a set  $E \subseteq X$  *hollow* if  $\text{int}(E) = \emptyset$ . We say a set  $E \subseteq X$  *nowhere dense* if its closure is hollow, i.e.  $\text{int}(\overline{E}) = \emptyset$ .

**Remark 1.8:** Notice that  $E$  hollow  $\Leftrightarrow E^c$  dense, since  $\text{int}(E) = \emptyset \Rightarrow (\text{int}(E))^c = \overline{E^c} = X$ .

↪ **Theorem 1.5** (Baire Category Theorem): Let  $X$  be a complete metric space.

- (a) Let  $\{F_n\}$  a collection of closed hollow sets. Then,  $\bigcup_{n=1}^{\infty} F_n$  also hollow.
- (b) Let  $\{\mathcal{O}_n\}$  a collection of open dense sets. Then,  $\bigcap_{n=1}^{\infty} \mathcal{O}_n$  also dense.

PROOF. Notice that (a)  $\Leftrightarrow$  (b) by taking complements. We prove (b).

Put  $G := \bigcap_{n=1}^{\infty} \mathcal{O}_n$ . Fix  $x \in X$  and  $r > 0$ , then to show density of  $G$  is to show  $G \cap B(x, r) \neq \emptyset$ .

Since  $\mathcal{O}_1$  dense, then  $\mathcal{O}_1 \cap B(x, r)$  nonempty and in particular open. So, let  $x_1 \in X$  and  $r_1 < \frac{1}{2}$  such that  $\overline{B}(x_1, r_1) \subseteq B(x, r) \subseteq \mathcal{O}_1 \cap B(x, r)$ .

Similarly, since  $\mathcal{O}_2$  dense,  $\mathcal{O}_2 \cap B(x_1, r_1)$  open and nonempty so there exists  $x_2 \in X$  and  $r_2 < 2^{-2}$  such that  $\overline{B}(x_2, r_2) \subseteq \mathcal{O}_2 \cap B(x_1, r_1)$ .



Repeat in this manner to find  $x_n \in X$  with  $r_n < 2^{-n}$  such that  $\overline{B}(x_n, r_n) \subseteq \mathcal{O}_n \cap B(x_{n-1}, r_{n-1})$  for any  $n \in \mathbb{N}$ . This creates a sequence of sets

$$\overline{B}(x_1, r_1) \supseteq \overline{B}(x_2, r_2) \supseteq \cdots,$$

with  $r_n \rightarrow 0$ . Hence, the sequence of points  $\{x_n\}$  is Cauchy and since  $X$  is complete,  $x_j \rightarrow x_0 \in X$ , so in particular

$$\{x_0\} = \bigcap_{n=1}^{\infty} \overline{B}(x_n, r_n),$$

hence  $x_0 \in \mathcal{O}_n$  for every  $n$  and thus  $G \cap B(x, r)$  is nonempty. ■

**Corollary 1.2:** Let  $X$  be complete and  $\{F_n\}$  a sequence of closed sets in  $X$ . If  $X = \bigcup_{n \geq 1} F_n$ , there is some  $n_0$  such that  $\text{int}(F_{n_0}) \neq \emptyset$ .

PROOF. If not, it violates BCT since  $X$  is not hollow in itself;  $\text{int}(X) = X$ . ■

**Corollary 1.3:** Let  $X$  be complete and  $\{F_n\}$  a sequence of closed sets in  $X$ . Then,  $\bigcup_{n=1}^{\infty} \partial F_n$  is hollow.

PROOF. We claim  $\text{int}(\partial F_n) = \emptyset$ . Suppose not, then there exists some  $B(x_0, r) \subseteq \partial F_n$ . Then  $x_0 \in \partial F_n$  but  $B(x_0, r) \cap F_n^c = \emptyset$ , a contradiction. So, since  $\partial F_n$  is closed and  $\partial F_n \cap B(x_0, r) = \emptyset$  for every such ball, by BCT  $\bigcup_{n=1}^{\infty} \partial F_n$  must be hollow. ■

### 1.4.1 Applications of Baire Category Theorem

**Theorem 1.6:** Let  $\mathcal{F} \subset C(X)$  where  $X$  is complete. Suppose  $\mathcal{F}$  is pointwise bounded. Then, there exists a nonempty, open set  $\mathcal{O} \subseteq X$  such that  $\mathcal{F}$  is uniformly bounded on  $\mathcal{O}$ .

PROOF. Let

$$\begin{aligned} E_n &:= \{x \in X : |f(x)| \leq n \forall f \in \mathcal{F}\} \\ &= \bigcap_{f \in \mathcal{F}} \underbrace{\{x : |f(x)| \leq n\}}_{\text{closed}}. \end{aligned}$$

Since  $\mathcal{F}$  is pointwise bounded, for every  $x \in X$  there is some  $M_x > 0$  such that  $|f(x)| \leq M_x$  for every  $f \in \mathcal{F}$ . Hence, for every  $n \in \mathbb{N}$  such that  $n \geq M_x$ ,  $x \in E_n$  and thus  $X = \bigcup_{n=1}^{\infty} E_n$ .

$E_n$  is closed and hence by the previous corollaries there is some  $n_0$  such that  $\text{int}(E_{n_0}) \neq \emptyset$  and hence there is some  $r > 0$  and  $x_0 \in X$  such that  $B(x_0, r) \subseteq E_{n_0}$ . Then, for every  $x \in B(x_0, r)$ ,  $|f(x)| \leq n_0$  for every  $f \in \mathcal{F}$ , which gives our desired non-empty open set upon which  $\mathcal{F}$  is uniformly bounded. ■

↪ **Theorem 1.7:** Let  $X$  complete, and  $\{f_n\} \subseteq C(X)$  such that  $f_n \rightarrow f$  pointwise on  $X$ . Then, there exists a dense subset  $D \subseteq X$  such that  $\{f_n\}$  equicontinuous on  $D$  and  $f$  continuous on  $D$ .

PROOF. For  $m, n \in \mathbb{N}$ , let

$$\begin{aligned} E(m, n) &:= \left\{ x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \forall j, k \geq n \right\} \\ &= \bigcap_{j, k \geq n} \left\{ x : |f_j(x) - f_k(x)| \leq \frac{1}{m} \right\}. \end{aligned}$$

The union of the boundaries of these sets are hollow, hence  $D := \left( \bigcup_{m, n \geq 1} \partial E(m, n) \right)^c$  is dense. Then, if  $x \in D \cap E(m, n)$ , then  $x \in (\partial E(m, n))^c$  implies  $x \in \text{int}(E(m, n))$ .

We claim  $\{f_n\}$  equicontinuous on  $D$ . Let  $x_0 \in D$  and  $\varepsilon > 0$ . Let  $\frac{1}{m} \leq \frac{\varepsilon}{4}$ . Then, since  $\{f_n(x_0)\}$  convergent it is therefore Cauchy (in  $\mathbb{R}$ ). Hence, there is some  $N$  such that  $|f_j(x_0) - f_k(x_0)| \leq \frac{1}{m}$  for every  $j, k \geq N$ , so  $x_0 \in D \cap E(m, N)$  hence  $x_0 \in \text{int}(E(m, N))$ .

Let  $B(x_0, r) \subseteq E(m, N)$ . Since  $f_N$  continuous at  $x_0$  there is some  $\delta > 0$  such that  $\delta < r$  and

$$|f_N(x) - f_N(x_0)| < \frac{1}{m} \forall x \in B(x_0, \delta),$$

and hence

$$\begin{aligned} |f_j(x) - f_j(x_0)| &\leq |f_j(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f_j(x_0)| \\ &\leq \frac{3}{m} \leq \frac{3}{4}\varepsilon, \end{aligned}$$

for every  $x \in B(x_0, \delta)$  and  $j \geq N$ , where the first, last bounds come from Cauchy and the middle from continuity of  $f_N$ . Hence, we've show  $\{f_n\}$  equicontinuous at  $x_0$  since  $\delta$  was independent of  $f$ .

In particular, this also gives for every  $x \in B(x_0, \delta)$  the limit

$$\frac{3}{4}\varepsilon > \lim_{j \rightarrow \infty} |f_j(x) - f_j(x_0)| = |f(x) - f(x_0)|,$$

so  $f$  continuous on  $D$ . ■

## §1.5 Topological Spaces

Throughout, assume  $X \neq \emptyset$ .

↪ **Definition 1.16** (Topology): Let  $X \neq \emptyset$ . A *topology*  $\mathcal{T}$  on  $X$  is a collection of subsets of  $X$ , called *open sets*, such that

- $X, \emptyset \in \mathcal{T}$ ;
- If  $\{E_n\} \subseteq \mathcal{T}$ ,  $\bigcap_{n=1}^N E_n \in \mathcal{T}$  (closed under *finite* intersections);
- If  $\{E_n\} \subseteq \mathcal{T}$ ,  $\bigcup_n E_n \in \mathcal{T}$  (closed under *arbitrary* unions).

If  $x \in X$ , a set  $E \in \mathcal{T}$  containing  $x$  is called a neighborhood of  $x$ .

↪ **Proposition 1.3**:  $E \subseteq X$  open  $\Leftrightarrow$  for every  $x \in E$ , there is a neighborhood of  $x$  contained in  $E$ .

PROOF.  $\Rightarrow$  is trivial by taking the neighborhood to be  $E$  itself.  $\Leftarrow$  follows from the fact that, if for each  $x$  we let  $\mathcal{U}_x$  a neighborhood of  $x$  contained in  $E$ , then

$$E = \bigcup_{x \in E} \mathcal{U}_x,$$

so  $E$  open being a union of open sets. ■

⊗ **Example 1.1**: Every metric space induces a natural topology given by open sets under the metric. The *discrete topology* is given by  $\mathcal{T} = 2^X$  (and is actually induced by the discrete metric), and is the largest topology. The *trivial topology*  $\{\emptyset, X\}$  is the smallest. The *relative topology* defined on a subset  $Y \subseteq X$  is given by  $\mathcal{T}_Y := \{E \cap Y : E \in \mathcal{T}\}$ .

↪ **Definition 1.17** (Base): Given a topological space  $(X, \mathcal{T})$ , let  $x \in X$ . A collection  $\mathcal{B}_x$  of neighborhoods of  $x$  is called a *base* of  $\mathcal{T}$  at  $x$  if for every neighborhood  $\mathcal{U}$  of  $x$ , there is a set  $B \in \mathcal{B}_x$  such that  $B \subseteq \mathcal{U}$ .

We say a collection  $\mathcal{B}$  a base for all of  $\mathcal{T}$  if for every  $x \in X$ , there is a base for  $x$ ,  $\mathcal{B}_x \subseteq \mathcal{B}$ .

↪ **Proposition 1.4**: If  $(X, \mathcal{T})$  a topological space, then  $\mathcal{B} \subseteq \mathcal{T}$  a base for  $\mathcal{T}$   $\Leftrightarrow$  every nonempty open set  $\mathcal{U} \in \mathcal{T}$  can be written as a union of elements of  $\mathcal{B}$ .

PROOF.  $\Rightarrow$  If  $\mathcal{U}$  open, then for  $x \in \mathcal{U}$  there is some basis element  $B_x$  contained in  $\mathcal{U}$ . So in particular  $\mathcal{U} = \bigcup_{x \in \mathcal{U}} B_x$ .

$\Leftarrow$  Let  $x \in \mathcal{U}$  and  $\mathcal{B}_x := \{B \in \mathcal{B} \mid x \in B\}$ . Then, for every neighborhood of  $x$ , there is some  $B$  in  $\mathcal{B}_x$  such that  $B \subseteq \mathcal{U}$  so  $\mathcal{B}_x$  a base for  $\mathcal{T}$  at  $x$ . ■

**Remark 1.9**: A base  $\mathcal{B}$  defines a unique topology,  $\{\emptyset, \cup \mathcal{B}_x\}$ .

↪ **Proposition 1.5:**  $\mathcal{B} \subseteq 2^X$  a base for a topology on  $X \Leftrightarrow$

- $X = \bigcup_{B \in \mathcal{B}} B$
- If  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , then there is a  $B \in \mathcal{B}$  such that  $x \in B \subseteq B_1 \cap B_2$ .

PROOF. ( $\Rightarrow$ ) If  $\mathcal{B}$  a base, then  $X$  open so  $X = \bigcup_B B$ . If  $B_1, B_2 \in \mathcal{B}$ , then  $B_1 \cap B_2$  open so there must exist some  $B \subseteq B_1 \cap B_2$  in  $\mathcal{B}$ .

( $\Leftarrow$ ) Let

$$\mathcal{T} = \{\mathcal{U} \mid \forall x \in \mathcal{U}, \exists B \in \mathcal{B} \text{ with } x \in B \subseteq \mathcal{U}\}.$$

One can show this a topology on  $X$  with  $\mathcal{B}$  as a base. ■

↪ **Definition 1.18:** If  $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ , we say  $\mathcal{T}_1$  *weaker/coarser* and  $\mathcal{T}_2$  *stronger/finer*.

Given a subset  $S \subseteq 2^X$ , define

$$\mathcal{T}(S) = \bigcap \text{all topologies containing } S = \text{unique weakest topology containing } S$$

to be the topology *generated* by  $S$ .

↪ **Proposition 1.6:** If  $S \subseteq 2^X$ ,

$$\mathcal{T}(S) = \bigcup \{\text{finite intersections of elts of } S\}.$$

We call  $S$  a “subbase” for  $\mathcal{T}(S)$  (namely, we allow finite intersections of elements in  $S$  to serve as a base for  $\mathcal{T}(S)$ ).

PROOF. Let  $\mathcal{B} := \{X, \text{finite intersections of elements of } S\}$ . We claim this a base for  $\mathcal{T}(S)$ . ■

↪ **Definition 1.19** (Point of closure/accumulation point): If  $E \subseteq X, x \in X$ ,  $x$  is called a *point of closure* if  $\forall \mathcal{U}_x, \mathcal{U}_x \cap E \neq \emptyset$ . The collection of all such sets is called the *closure* of  $E$ , denoted  $\overline{E}$ . We say  $E$  *closed* if  $E = \overline{E}$ .

↪ **Proposition 1.7:** Let  $E \subseteq X$ , then

- $\overline{E}$  closed,
- $\overline{E}$  is the smallest closed set containing  $E$ ,
- $E$  open  $\Leftrightarrow E^c$  closed.

## §1.6 Separation, Countability, Separability

↪ **Definition 1.20:** A neighborhood of a set  $K \subseteq X$  is any open set containing  $K$ .

↪ **Definition 1.21** (Notions of Separation): We say  $(X, \mathcal{T})$ :

- *Tychonoff Separable* if  $\forall x, y \in X, \exists \mathcal{U}_x, \mathcal{U}_y$  such that  $y \notin \mathcal{U}_x, x \notin \mathcal{U}_y$
- *Hausdorff Separable* if  $\forall x, y \in X$  can be separated by two disjoint open sets i.e.  $\exists \mathcal{U}_x \cap \mathcal{U}_y = \emptyset$
- *Normal* if Tychonoff and in addition any 2 disjoint closed sets can be separated by disjoint neighborhoods.

**Remark 1.10:** Metric space  $\subseteq$  normal space  $\subseteq$  Hausdorff space  $\subseteq$  Tychonoff space.

↪ **Proposition 1.8:** Tychonoff  $\Leftrightarrow \forall x \in X, \{x\}$  closed.

PROOF. For every  $x \in X$ ,

$$\begin{aligned} \{x\} \text{ closed} &\Leftrightarrow \{x\}^c \text{ open} \\ &\Leftrightarrow \forall y \in \{x\}^c, \exists \mathcal{U}_y \subseteq \{x\}^c \\ &\Leftrightarrow \forall y \neq x, \exists \mathcal{U}_y \text{ s.t. } x \notin \mathcal{U}_y, \end{aligned}$$

and since this holds for every  $x$ ,  $X$  Tychonoff. ■

↪ **Proposition 1.9:** Every metric space normal.

PROOF. Define, for  $F \subseteq X$ , the function

$$\text{dist}(F, x) := \inf\{\rho(x, x') \mid x' \in F\}.$$

Notice that if  $F$  closed and  $x \notin F$ , then  $\text{dist}(F, x) > 0$  (since  $F^c$  open so there exists some  $B(x, \varepsilon) \subseteq F^c$  so  $\rho(x, x') \geq \varepsilon$  for every  $x' \in F$ ). Let  $F_1, F_2$  be closed disjoint sets, and define

$$\begin{aligned} \mathcal{O}_1 &:= \{x \in X \mid \text{dist}(F_1, x) < \text{dist}(F_2, x)\}, \\ \mathcal{O}_2 &:= \{x \in X \mid \text{dist}(F_1, x) > \text{dist}(F_2, x)\}. \end{aligned}$$

Then,  $F_1 \subseteq \mathcal{O}_1, F_2 \subseteq \mathcal{O}_2$ , and  $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$ . If we show  $\mathcal{O}_1, \mathcal{O}_2$  open, we'll be done.

Let  $x \in \mathcal{O}_1$  and  $\varepsilon > 0$  such that  $\text{dist}(F_1, x) + \varepsilon \leq \text{dist}(F_2, x)$ . I claim that  $B(x, \frac{\varepsilon}{5}) \subseteq \mathcal{O}_1$ . Let  $y \in B(x, \frac{\varepsilon}{5})$ . Then,

$$\begin{aligned}
\text{dist}(F_2, y) &\geq \rho(y, y') - \frac{\varepsilon}{5} && \text{for some } y' \in F_2 \\
&\geq \rho(x, y') - \rho(x, y) + \frac{\varepsilon}{5} && \text{reverse triangle inequality} \\
&\geq \text{dist}(F_2, x) - \frac{2\varepsilon}{5} \\
&\geq \text{dist}(F_1, x) + \varepsilon - \frac{2\varepsilon}{5} \\
&\geq \rho(x, \tilde{y}) + \frac{2\varepsilon}{5} && \text{for some } \tilde{y} \in F_1 \\
&\geq \rho(y, \tilde{y}) - \rho(y, x) + \frac{2\varepsilon}{5} && \text{reverse triangle inequality} \\
&\geq \rho(y, \tilde{y}) - \frac{\varepsilon}{5} + \frac{2\varepsilon}{5} \\
&\geq \text{dist}(F_1, y) + \frac{\varepsilon}{5} > \text{dist}(F_1, y),
\end{aligned}$$

hence,  $y \in \mathcal{O}_1$  and thus  $\mathcal{O}_1$  open. Similar proof follows for  $\mathcal{O}_2$ . ■

↪ **Proposition 1.10:** Let  $X$  Tychonoff. Then  $X$  normal  $\Leftrightarrow \forall F \subseteq X$  closed and neighborhood  $\mathcal{U}$  of  $F$ , there exists an open set  $\mathcal{O}$  such that

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}.$$

This is called the “nested neighborhood property” of normal spaces.

PROOF. ( $\Rightarrow$ ) Let  $F$  closed and  $\mathcal{U}$  a neighborhood of  $F$ . Then,  $F$  and  $\mathcal{U}^c$  closed disjoint sets so by normality there exists  $\mathcal{O}, \mathcal{V}$  disjoint open neighborhoods of  $F, \mathcal{U}^c$  respectively. So,  $\mathcal{O} \subseteq \mathcal{V}^c$  hence  $\overline{\mathcal{O}} \subseteq \overline{\mathcal{V}^c} = \mathcal{V}^c$  and thus

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{V}^c \subseteq \mathcal{U}.$$

( $\Leftarrow$ ) Let  $A, B$  be disjoint closed sets. Then,  $B^c$  open and moreover  $A \subseteq B^c$ . Hence, there exists some open set  $\mathcal{O}$  such that  $A \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq B^c$ , and thus  $B \subseteq \overline{\mathcal{O}}^c$ . Then,  $\mathcal{O}$  and  $\overline{\mathcal{O}}^c$  are disjoint open neighborhoods of  $A, B$  respectively so  $X$  normal. ■

↪ **Definition 1.22** (Separable): A space  $X$  is called *separable* if it contains a countable dense subset.

↪ **Definition 1.23** (1st, 2nd Countable): A topological space  $(X, \mathcal{T})$  is called

- *1st countable* if there is a countable base at each point; and
- *2nd countable* if there is a countable base for all of  $\mathcal{T}$ .

⊗ **Example 1.2:** Every metric space is first countable; for  $x \in X$  let  $\mathcal{B}_x = \{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$ .

↪ **Proposition 1.11:** Every 2nd countable space is separable.

↪ **Definition 1.24** (Convergence): Let  $\{x_n\} \subseteq X$ . Then, we say  $x_n \rightarrow x$  in  $\mathcal{T}$  if for every neighborhood  $\mathcal{U}_x$ , there exists an  $N$  such that  $\forall n \geq N, x_n \in \mathcal{U}_x$ .

**Remark 1.11:** In general spaces, such a limit may not be unique. For instance, under the trivial topology, the only nonempty neighborhood is the whole space, so every sequence converges to every point in the space.

↪ **Proposition 1.12:** Let  $(X, \mathcal{T})$  be Hausdorff. Then, all limits are unique.

PROOF. Suppose otherwise, that  $x_n \rightarrow$  both  $x$  and  $y$ . If  $x \neq y$ , then since  $X$  Hausdorff there are disjoint neighborhoods  $\mathcal{U}_x, \mathcal{U}_y$  containing  $x, y$ . But then  $x_n$  cannot be on both  $\mathcal{U}_x$  and  $\mathcal{U}_y$  for sufficiently large  $n$ , contradiction. ■

↪ **Proposition 1.13:** Let  $X$  be 1st countable and  $E \subseteq X$ . Then,  $x \in \overline{E} \Leftrightarrow$  there exists  $\{x_j\} \subseteq E$  such that  $x_j \rightarrow x$ .

PROOF. ( $\Rightarrow$ ) Let  $\mathcal{B}_x = \{B_j\}$  be a base for  $X$  at  $x \in \overline{E}$ . Wlog,  $B_j \supseteq B_{j+1}$  for every  $j \geq 1$  (by replacing with intersections, etc if necessary). Hence,  $B_j \cap E \neq \emptyset$  for every  $j$ . Let  $x_j \in B_j \cap E$ , then by the nesting property  $x_j \rightarrow x$  in  $\mathcal{T}$ .

( $\Leftarrow$ ) Suppose otherwise, that  $x \notin \overline{E}$ . Let  $\{x_j\} \in E_j$ . Then,  $\overline{E}^c$  open, and contains  $x$ . Then,  $\overline{E}^c$  a neighborhood of  $x$  but does not contain any  $x_j$  so  $x_j \not\rightarrow x$ . ■

## §1.7 Continuity and Compactness

↪ **Definition 1.25:** Let  $(X, \mathcal{T}), (Y, \mathcal{S})$  be two topological spaces. Then, a function  $f : X \rightarrow Y$  is said to be continuous at  $x_0$  if for every neighborhood  $\mathcal{O}$  of  $f(x_0)$  there exists a neighborhood  $\mathcal{U}(x_0)$  such that  $f(\mathcal{U}) \subseteq \mathcal{O}$ . We say  $f$  continuous on  $X$  if it is continuous at every point in  $X$ .

↪ **Proposition 1.14:**  $f$  continuous  $\Leftrightarrow \forall \mathcal{O}$  open in  $Y, f^{-1}(\mathcal{O})$  open in  $X$ .

↪ **Definition 1.26** (Weak Topology): Consider  $\mathcal{F} := \{f_\lambda : X \rightarrow X_\lambda\}_{\lambda \in \Lambda}$  where  $X, X_\lambda$  topological spaces. Then, let

$$S := \{f_\lambda^{-1}(\mathcal{O}_\lambda) \mid f_\lambda \in \mathcal{F}, \mathcal{O}_\lambda \in X_\lambda\} \subseteq X.$$

We say that the topology  $\mathcal{T}(S)$  generated by  $S$  is the *weak topology* for  $X$  induced by the family  $\mathcal{F}$ .

↪ **Proposition 1.15:** The weak topology is the weakest topology in which each  $f_\lambda$  continuous on  $X$ .

⊗ **Example 1.3:** The key example of the weak topology is given by the product topology. Consider  $\{X_\lambda\}_{\lambda \in \Lambda}$  a collection of topological spaces. We can defined a “natural” topology on the product  $X := \prod_{\lambda \in \Lambda} X_\lambda$  by consider the weak topology induced by the family of projection maps, namely, if  $\pi_\lambda : X \rightarrow X_\lambda$  a coordinate-wise projection and  $\mathcal{F} = \{\pi_\lambda : \lambda \in \Lambda\}$ , then we say the weak topology induced by  $\mathcal{F}$  is the *product topology* on  $X$ . In particular, a base for this topology is given, by previous discussions,

$$\mathcal{B} = \left\{ \bigcap_{j=1}^n \pi_{\lambda_j}^{-1}(\mathcal{O}_j) \right\} = \left\{ \prod_{\lambda \in \Lambda} \mathcal{U}_\lambda : \mathcal{U}_\lambda \text{ open and all by finitely many } \mathcal{U}_\lambda' \text{'s} = X_\lambda \right\}.$$

↪ **Definition 1.27** (Compactness): A space  $X$  is said to be *compact* if every open cover of  $X$  admits a finite subcover.

↪ **Proposition 1.16:**

- Closed subsets of compact spaces are compact
- $X$  compact  $\Leftrightarrow$  if  $\{F_k\} \subseteq X$ -nested and closed,  $\bigcap_{k=1}^\infty F_k \neq \emptyset$ .
- Continuous images of compact sets are compact
- Continuous real-valued functions on a compact topological space achieve their min, max.

↪ **Proposition 1.17:** Let  $K$  compact be contained in a Hausdorff space  $X$ . Then,  $K$  closed in  $X$ .

PROOF. We show  $K^c$  open. Let  $y \in K^c$ . Then for every  $x \in K$ , there exists disjoint open sets  $\mathcal{U}_{xy}, \mathcal{O}_{xy}$  containing  $y, x$  respectively. Then, it follows that  $\{\mathcal{O}_{xy}\}_{x \in K}$  an open cover of  $K$ , and since  $K$  compact there must exist some finite subcover,  $K \subseteq \bigcup_{i=1}^N \mathcal{O}_{x_i y}$ . Let  $E := \bigcap_{i=1}^N \mathcal{U}_{x_i y}$ . Then,  $E$  is an open neighborhood of  $y$  with  $E \cap \mathcal{O}_{x_i y} = \emptyset$  for every



$i = 1, \dots, N$ . Thus,  $E \subseteq \bigcap_{i=1}^N \mathcal{O}_{x_i y}^c = \left( \bigcup_{i=1}^N \mathcal{O}_{x_i y} \right)^c \subseteq K^c$  so since  $y$  was arbitrary  $K^c$  open. ■

↪ **Definition 1.28** (Sequential Compactness): We say  $(X, \mathcal{T})$  *sequentially compact* if every sequence in  $X$  has a converging subsequence with limit contained in  $X$ .

↪ **Proposition 1.18**: Let  $(X, \mathcal{T})$  second countable. Then,  $X$  compact  $\Leftrightarrow$  sequentially compact.

PROOF. ( $\Rightarrow$ ) Let  $\{x_k\} \subseteq X$  and put  $F_n := \overline{\{x_k \mid k \geq n\}}$ . Then,  $\{F_n\}$  defines a sequence of closed and nested subsets of  $X$  and, since  $X$  compact,  $\bigcap_{n=1}^{\infty} F_n$  nonempty. Let  $x_0$  in this intersection. Since  $X$  2nd and so in particular 1st countable, let  $\{B_j\}$  a (wlog nested) countable base at  $x_0$ .  $x_0 \in F_n$  for every  $n \geq 1$  so each  $B_j$  must intersect some  $F_n$ . Let  $n_j$  be an index such that  $x_{n_j} \in B_j$ . Then, if  $\mathcal{U}$  a neighborhood of  $x_0$ , there exists some  $N$  such that  $B_j \subseteq \mathcal{U}$  for every  $j \geq N$  and thus  $\{x_{n_j}\} \subseteq B_N \subseteq \mathcal{U}$ , so  $x_{n_j} \rightarrow x_0$  in  $X$ .

( $\Leftarrow$ ) Remark that since  $X$  second countable, every open cover of  $X$  certainly has a countable subcover by intersecting a given cover with our countable basis. So, assume we have a countable cover  $X \subseteq \bigcup_{n=1}^{\infty} \mathcal{O}_n$  and suppose towards a contradiction that no finite subcover exists. Then, for every  $n \geq 1$ , there exists some  $m(n) \geq n$  such that  $\mathcal{O}_{m(n)} \setminus \bigcup_{i=1}^n \mathcal{O}_i \neq \emptyset$ . Let  $x_n$  in this set for every  $n \geq 1$ . Since  $X$  sequentially compact, there exists a convergent subsequence  $\{x_{n_k}\} \subseteq \{x_n\}$  such that  $x_{n_k} \rightarrow x_0$  in  $X$ , so there exists some  $\mathcal{O}_N$  such that  $x_0 \in \mathcal{O}_N$ . But by construction,  $x_{n_k} \notin \mathcal{O}_N$  if  $n_k \geq N$ , and we have a contradiction. ■

↪ **Theorem 1.8**: If  $X$  compact and Hausdorff,  $X$  normal.

PROOF. We show that any closed set  $F$  and any point  $x \notin F$  can be separated by disjoint open sets. Then, the proof in the more general case follows.

For each  $y \in X$ ,  $X$  is Hausdorff so there exists disjoint open neighborhoods  $\mathcal{O}_{xy}$  and  $\mathcal{U}_{xy}$  of  $x, y$  respectively. Then,  $\{\mathcal{U}_{xy} \mid y \in F\}$  defines an open cover of  $F$ . Since  $F$  closed and thus, being a subset of a compact space, compact, there exists a finite subcover  $F \subseteq \bigcup_{i=1}^N \mathcal{U}_{xy_i}$ . Put  $\mathcal{N} := \bigcap_{i=1}^N \mathcal{O}_{xy_i}$ . This is an open set containing  $x$ , with  $\mathcal{N} \cap \bigcup_{i=1}^N \mathcal{U}_{xy_i} = \emptyset$  hence  $F$  and  $x$  separated by  $\mathcal{N}, \bigcup_{i=1}^N \mathcal{U}_{xy_i}$ . ■

## §1.8 Connected Topological Spaces

↪ **Definition 1.29** (Separate): 2 non-empty sets  $\mathcal{O}_1, \mathcal{O}_2$  *separate*  $X$  if  $\mathcal{O}_1, \mathcal{O}_2$  disjoint and  $X = \mathcal{O}_1 \cup \mathcal{O}_2$ .

↪ **Definition 1.30** (Connected): We say  $X$  *connected* if it cannot be separated.

**Remark 1.12:** Note that if  $X$  can be separated, then  $\mathcal{O}_1, \mathcal{O}_2$  are closed as well as open, being complements of each other.

↪ **Proposition 1.19:** Let  $f : X \rightarrow Y$  continuous. Then, if  $X$  connected, so is  $f(X)$ .

PROOF. Suppose otherwise, that  $f(X) = \mathcal{O}_1 \sqcup \mathcal{O}_2$  for nonempty, open, disjoint  $\mathcal{O}_1, \mathcal{O}_2$ . Then,  $X = f^{-1}(\mathcal{O}_1) \sqcup f^{-1}(\mathcal{O}_2)$ , and each of these inverse images remain nonempty and open in  $X$ , so this a contradiction to the connectedness of  $X$ . ■

**Remark 1.13:** On  $\mathbb{R}$ ,  $C \subseteq \mathbb{R}$  connected  $\Leftrightarrow$  an interval  $\Leftrightarrow$  convex.

↪ **Definition 1.31** (Intermediate Value Property): We say  $X$  has the intermediate value property (IVP) if  $\forall f \in C(X)$ ,  $f(X)$  an interval.

↪ **Proposition 1.20:**  $X$  has IVP  $\Leftrightarrow X$  connected.

PROOF. ( $\Leftarrow$ ) If  $X$  connected,  $f(X)$  connected in  $\mathbb{R}$  hence an interval.

( $\Rightarrow$ ) Suppose otherwise, that  $X = \mathcal{O}_1 \sqcup \mathcal{O}_2$ . Then define the function  $f : X \rightarrow \mathbb{R}$  by  $x \mapsto \begin{cases} 1 & \text{if } x \in \mathcal{O}_2 \\ 0 & \text{if } x \in \mathcal{O}_1 \end{cases}$ . Then, for every  $A \subseteq \mathbb{R}$ ,

$$f^{-1}(A) = \begin{cases} \emptyset & \text{if } \{0, 1\} \not\subseteq A \\ \mathcal{O}_1 & \text{if } 0 \in A \\ \mathcal{O}_2 & \text{if } 1 \in A \\ X & \text{if } \{0, 1\} \subseteq A \end{cases},$$

which are all open sets, hence  $f$  continuous. But  $f(X) = \{0, 1\}$  which is not an interval, hence the IVP fails and so  $X$  must be connected. ■

↪ **Definition 1.32** (Arcwise/Path Connected):  $X$  arc connected/path connected if  $\forall x, y \in X$ , there exists a continuous function  $f : [0, 1] \rightarrow X$  such that  $f(0) = x, f(1) = y$ .

↪ **Proposition 1.21:** Arc connected  $\Rightarrow$  connected.

PROOF. Suppose otherwise,  $X = \mathcal{O}_1 \sqcup \mathcal{O}_2$ . Let  $x \in \mathcal{O}_1, y \in \mathcal{O}_2$  and define a continuous function  $f : [0, 1] \rightarrow X$  such that  $f(0) = x$  and  $f(1) = y$ . Then,  $f^{-1}(\mathcal{O}_i)$  each open, nonempty and disjoint for  $i = 1, 2$ , but

$$f^{-1}(\mathcal{O}_1) \sqcup f^{-1}(\mathcal{O}_2) = [0, 1],$$

a contradiction to the connectedness of  $[0, 1]$ . ■

## §1.9 Urysohn's Lemma and Urysohn's Metrization Theorem

We present the main lemma of this section first, but need more tools before proving it.

↪ **Lemma 1.2** (Urysohn's): Let  $A, B \subseteq X$  closed and disjoint subsets of a normal space  $X$ . Then,  $\forall [a, b] \subseteq \mathbb{R}$ , there exists a continuous function  $f : [a, b] \rightarrow \mathbb{R}$  such that  $f(X) \subseteq [a, b]$ ,  $f|_A = a$  and  $f|_B = b$ .

**Remark 1.14:** We have a partial converse of this statement as well:

↪ **Proposition 1.22:** Let  $X$  Tychonoff and suppose  $X$  satisfies the properties of Urysohn's Lemma. Then,  $X$  normal.

PROOF. Let  $A, B$  be closed nonempty disjoint subsets. Let  $f : X \rightarrow \mathbb{R}$  continuous such that  $f|_A = 0$ ,  $f|_B = 1$  and  $0 \leq f \leq 1$ . Let  $I_1, I_2$  be two disjoint open intervals in  $\mathbb{R}$  with  $0 \in I_1$  and  $1 \in I_2$ . Then,  $f^{-1}(I_1)$  open and contains  $A$ , and  $f^{-1}(I_2)$  open and contains  $B$ . Moreover,  $f^{-1}(I_1) \cap f^{-1}(I_2) = \emptyset$ ; hence,  $f^{-1}(I_1), f^{-1}(I_2)$  disjoint open neighborhoods of  $A, B$  respectively, so indeed  $X$  normal. ■

↪ **Definition 1.33** (Normally Ascending): Let  $(X, \mathcal{T})$  a topological space and  $\Lambda \subseteq \mathbb{R}$ . A collection of open sets  $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$  is said to be *normally ascending* if  $\forall \lambda_1, \lambda_2 \in \Lambda$ ,

$$\overline{\mathcal{O}_{\lambda_1}} \subseteq \mathcal{O}_{\lambda_2} \text{ if } \lambda_1 < \lambda_2.$$

↪ **Lemma 1.3:** Let  $\Lambda \subseteq (a, b)$  a dense subset, and let  $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$  a normally ascending collection of subsets of  $X$ . Let  $f : X \rightarrow \mathbb{R}$  defined such that

$$f(x) = \begin{cases} b & \text{if } x \in \left(\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda\right)^c \\ \inf\{\lambda \in \Lambda \mid x \in \mathcal{O}_\lambda\} & \text{else} \end{cases}.$$

Then,  $f$  continuous.

PROOF. We claim  $f^{-1}(-\infty, c)$  and  $f^{-1}(c, \infty)$  open for every  $c \in \mathbb{R}$ . Since such sets define a subbase for  $\mathbb{R}$ , it suffices to prove continuity on these sets. We show just the first for convenience. Notice that since  $f(x) \in [a, b]$ , if  $c \in (a, b)$  then  $f^{-1}(-\infty, c) = f^{-1}[a, c)$ , so really it suffices to show that  $f^{-1}[a, c)$  open to complete the proof.

Suppose  $x \in f^{-1}([a, c])$  so  $a \leq f(x) < c$ . Let  $\lambda \in \Lambda$  be such that  $a < \lambda < f(x)$ . Then,  $x \notin \mathcal{O}_\lambda$ . Let also  $\lambda' \in \Lambda$  such that  $f(x) < \lambda' < c$ . By density of  $\Lambda$ , there exists a  $\varepsilon > 0$  such that  $f(x) + \varepsilon \in \Lambda$ , so in particular

$$\overline{\mathcal{O}_{f(x)+\varepsilon}} \subseteq \mathcal{O}_{\lambda'} \Rightarrow x \in \mathcal{O}_{\lambda'},$$

by nesting. So, repeating this procedure, we find

$$f^{-1}([a, c)) \subseteq \bigcup_{a \leq \lambda < \lambda' < c} \mathcal{O}_{\lambda'} \setminus \overline{\mathcal{O}}_{\lambda},$$

noticing the set on the right is open. By similar reasoning, the opposite inclusion holds and we have equality. Hence,  $f$  continuous.  $\blacksquare$

**Lemma 1.4:** Let  $X$  normal,  $F \subseteq X$  closed, and  $\mathcal{U}$  a neighborhood of  $F$ . Then, for any  $(a, b) \subseteq \mathbb{R}$ , there exists a dense subset  $\Lambda \subseteq (a, b)$  and a normally ascending collection  $\{\mathcal{O}_{\lambda}\}_{\lambda \in \Lambda}$  such that

$$F \subseteq \mathcal{O}_{\lambda} \subseteq \overline{\mathcal{O}}_{\lambda} \subseteq \mathcal{U}, \quad \forall \lambda \in \Lambda.$$

**Remark 1.15:** This is essentially a generalization of the nested neighborhood property, and indeed the proof essentially just uses this property repeatedly to construct the collection  $\{\mathcal{O}_{\lambda}\}$ .

**PROOF.** Without loss of generality, we assume  $(a, b) = (0, 1)$ , for the two intervals are homeomorphic, i.e. the function  $f : (0, 1) \rightarrow \mathbb{R}$ ,  $f(x) := a(1 - x) + bx$  is continuous, invertible with continuous inverse and with  $f(0) = a$ ,  $f(1) = b$  so a homeomorphism.

Let

$$\Lambda := \left\{ \frac{m}{2^n} \mid m, n \in \mathbb{N} \mid 1 \leq m \leq 2^{n-1} \right\} = \bigcup_{n \in \mathbb{N}} \underbrace{\left\{ \frac{m}{2^n} \mid m \in \mathbb{N}, 1 \leq m \leq 2^{n-1} \right\}}_{=: \Lambda_n},$$

which is clearly dense in  $(0, 1)$ . We need now to define our normally ascending collection. We do so by defining on each  $\Lambda_1$  and proceeding inductively.

For  $\Lambda_1$ , since  $X$  normal, let  $\mathcal{O}_{1/2}$  be such that  $F \subseteq \mathcal{O}_{1/2} \subseteq \overline{\mathcal{O}}_{1/2} \subseteq \mathcal{U}$ , which exists by the nested neighborhood property.

For  $\Lambda_2 = \{\frac{1}{4}, \frac{3}{4}\}$ , we use the nested neighborhood property again, but first with  $F$  as the closed set and  $\mathcal{O}_{1/2}$  an open neighborhood of it, and then with  $\overline{\mathcal{O}}_{1/2}$  as the closed set and  $\mathcal{U}$  an open neighborhood of it. In this way, we find

$$\underbrace{F \subseteq \mathcal{O}_{1/4} \subseteq \overline{\mathcal{O}}_{1/4} \subseteq \mathcal{O}_{1/2}}_{\text{nested nbhd}} \subseteq \overbrace{\overline{\mathcal{O}}_{1/2} \subseteq \mathcal{O}_{3/4} \subseteq \overline{\mathcal{O}}_{3/4}}^{\text{nested nbhd}} \subseteq \mathcal{U}.$$

We repeat in this manner over all of  $\Lambda$ , in the end defining a normally ascending collection  $\{\mathcal{O}_{\lambda}\}_{\lambda \in \Lambda}$ .  $\blacksquare$

**PROOF (Of Urysohn's Lemma, [Lem. 1.2](#)).** Let  $F = A$  and  $\mathcal{U} = B^c$  as in the previous lemma [Lem. 1.4](#). Then, there is some dense subset  $\Lambda \subseteq (a, b)$  and a normally ascending collection  $\{\mathcal{O}_{\lambda}\}_{\lambda \in \Lambda}$  such that  $A \subseteq \mathcal{O}_{\lambda} \subseteq \overline{\mathcal{O}}_{\lambda} \subseteq B^c$  for every  $\lambda \in \Lambda$ . Let  $f(x)$  as in the previous lemma, [Lem. 1.3](#). Then, if  $x \in B$ ,  $B \subseteq \left( \bigcup_{\lambda \in \Lambda} \mathcal{O}_{\lambda} \right)^c$  and so  $f(x) = b$ .

Otherwise if  $x \in A$ , then  $x \in \bigcap_{\lambda \in \Lambda} \mathcal{O}_\lambda$  and thus  $f(x) = \inf\{\lambda \in \Lambda\} = a$ . By the first lemma,  $f$  continuous, so we are done. ■

↪ **Theorem 1.9** (Urysohn's Metrization Theorem): Let  $X$  be a second countable topological space. Then,  $X$  is metrizable (that is, there exists a metric on  $X$  that induces the topology) if and only if  $X$  normal.

PROOF. ( $\Rightarrow$ ) We have already showed, every metric space is normal.

( $\Leftarrow$ ) Let  $\{\mathcal{U}_n\}$  be a countable basis for  $\mathcal{T}$  and put

$$A := \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid \overline{\mathcal{U}_n} \subseteq \mathcal{U}_m\}.$$

By Urysohn's lemma, for each  $(n, m) \in A$  there is some continuous function  $f_{n,m} : X \rightarrow \mathbb{R}$  such that  $f_{n,m}$  is 1 on  $\mathcal{U}_m^c$  and 0 on  $\overline{\mathcal{U}_n}$  (these are disjoint closed sets). For  $x, y \in X$ , define

$$\rho(x, y) := \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)|.$$

The absolute valued term is  $\leq 2$ , so this function will always be finite. Moreover, one can verify that it is indeed a metric on  $X$ . It remains to show that it induces the same topology; it suffices to compare bases of the two.

Let  $x \in \mathcal{U}_m$ . We wish to show there exists  $B_\rho(x, \varepsilon) \subseteq \mathcal{U}_m$ .  $\{x\}$  is closed in  $X$  being normal, so there exists some  $n$  such that

$$\{x\} \subseteq \mathcal{U}_n \subseteq \overline{\mathcal{U}_n} \subseteq \mathcal{U}_m,$$

so  $(n, m) \in A$  and so  $f_{n,m}(x) = 0$ . Let  $\varepsilon = \frac{1}{2^{n+m}}$ . Then, if  $\rho(x, y) < \varepsilon$ , it must be

$$\begin{aligned} \frac{1}{2^{n+m}} &> \sum_{(n',m') \in A} \frac{1}{2^{n'+m'}} |f_{n',m'}(x) - f_{n',m'}(y)| \\ &\geq \frac{1}{2^{n+m}} \underbrace{|f_{n,m}(x) - f_{n,m}(y)|}_{=0} \\ &= \frac{1}{2^{n+m}} |f_{n,m}(y)|, \end{aligned}$$

so  $|f_{n,m}(y)| < 1$  and thus  $y \notin \mathcal{U}_m^c$  so  $y \in \mathcal{U}_m$ . It follow that  $B_\rho(x, \varepsilon) \subseteq \mathcal{U}_m$ , and so every open set in  $X$  is open with respect to the metric topology.

Conversely, if  $B_\rho(x, \varepsilon)$  some open ball in the metric topology, then notice that  $y \mapsto \rho(x, y)$  for fixed  $x$  a continuous function, and thus  $(\rho(x, \cdot))^{-1}(-\varepsilon, \varepsilon)$  an open set in  $\mathcal{T}$  containing  $x$ . But this set also just equal to  $B_\rho(x, \varepsilon)$ , hence  $B_\rho(x, \varepsilon)$  open in  $\mathcal{T}$ . We conclude the two topologies are equal, completing the proof. ■

**Remark 1.16:** Recall metric  $\Rightarrow$  first countable hence not first countable  $\Rightarrow$  not metrizable.

## §1.10 Stone-Weierstrass Theorem

We need to use the following theorem, which we'll prove later.

↪ **Theorem 1.10** (Weierstrass Approximation Theorem): Let  $f : [a, b] \rightarrow \mathbb{R}$  continuous. Then, for every  $\varepsilon > 0$ , there exists a polynomial  $p(x)$  such that  $\|f - p\|_\infty < \varepsilon$ .

↪ **Definition 1.34** (Algebra, Separation of Points): We call a subset  $\mathcal{A} \subseteq C(X)$  an *algebra* if it is a linear subspace that is closed under multiplication (that is,  $f, g \in \mathcal{A} \Rightarrow f \cdot g \in \mathcal{A}$ ).

We say  $\mathcal{A}$  *separates points* in  $X$  if for every  $x, y \in X$ , there exists an  $f \in \mathcal{A}$  such that  $f(x) \neq f(y)$ .

↪ **Theorem 1.11** (Stone-Weierstrass): Let  $X$  be a compact Hausdorff space. Suppose  $\mathcal{A} \subseteq C(X)$  an algebra that separates points and contains constant functions. Then,  $\mathcal{A}$  dense in  $C(X)$ .

We tacitly assume the conditions of the theorem in the following lemmas as not to restate them.

↪ **Lemma 1.5**: For every  $F \subseteq X$  closed, and every  $x_0 \in F^c$ , there exists a neighborhood  $\mathcal{U}(x_0)$  such that  $F \cap \mathcal{U} = \emptyset$  and  $\forall \varepsilon > 0$  there is some  $h \in \mathcal{A}$  such that  $h < \varepsilon$  on  $\mathcal{U}$ ,  $h > 1 - \varepsilon$  on  $F$ , and  $0 \leq h \leq 1$  on  $X$ .

In particular,  $\mathcal{U}$  is *independent* of choice of  $\varepsilon$ .

PROOF. Our first claim is that for every  $y \in F$ , there is a  $g_y \in \mathcal{A}$  such that  $g_y(x_0) = 0$  and  $g_y(y) > 0$ , and moreover  $0 \leq g_y \leq 1$ . Since  $\mathcal{A}$  separates points, there is an  $f \in \mathcal{A}$  such that  $f(x_0) \neq f(y)$ . Then, let

$$g_y(x) := \left[ \frac{f(x) - f(x_0)}{\|f - f(x_0)\|_\infty} \right]^2.$$

Then, every operation used in this new function keeps  $g_y \in \mathcal{A}$ . Moreover one readily verifies it satisfies the desired qualities. In particular since  $g_y$  continuous, there is a neighborhood  $\mathcal{O}_y$  such that  $g_y|_{\mathcal{O}_y} > 0$ . Hence, we know that  $F \subseteq \bigcup_{y \in F} \mathcal{O}_y$ , but  $F$  closed and so compact, hence there exists a finite subcover i.e. some  $n \geq 1$  and finite sequence  $\{y_i\}_{i=1}^n$  such that  $F \subseteq \bigcup_{i=1}^n \mathcal{O}_{y_i}$ . Let for each  $y_i$   $g_{y_i} \in \mathcal{A}$  with the properties from above, and consider the “averaged” function

$$g(x) := \frac{1}{n} \sum_{i=1}^n g_{y_i}(x) \in \mathcal{A}.$$

Then,  $g(x_0) = 0$ ,  $g > 0$  on  $F$  and  $0 \leq g \leq 1$  on all of  $X$ . Hence, there is some  $1 > c > 0$  such that  $g \geq c$  on  $F$ , and since  $g$  continuous at  $x_0$  there exists some  $\mathcal{U}(x_0)$  such that  $g < \frac{c}{2}$  on  $\mathcal{U}$ , with  $\mathcal{U} \cap F = \emptyset$ . So,  $0 \leq g|_{\mathcal{U}} < \frac{c}{2}$ , and  $1 \geq g|_F \geq c$ . To complete the proof, we need  $(0, \frac{c}{2}) \leftrightarrow (0, \varepsilon)$  and  $(c, 1) \leftrightarrow (1 - \varepsilon, 1)$ . By the Weierstrass Approximation Theorem, there exists some polynomial  $p$  such that  $p|_{[0, \frac{c}{2}]} < \varepsilon$  and  $p|_{[c, 1]} > 1 - \varepsilon$ . Then if we let  $h(x) := (p \circ g)(x)$ , this is just a polynomial of  $g$  hence remains in  $\mathcal{A}$ , and we find

$$h|_{\mathcal{U}} < \varepsilon, \quad h|_F > 1 - \varepsilon, \quad 0 \leq h \leq 1.$$

■

↪ **Lemma 1.6:** For every disjoint closed set  $A, B$  and  $\varepsilon > 0$ , there exists  $h \in \mathcal{A}$  such that  $h|_A < \varepsilon$ ,  $h|_B > 1 - \varepsilon$ , and  $0 \leq h \leq 1$  on  $X$ .

PROOF. Let  $F = B$  as in the last lemma. Let  $x \in A$ , then there exists  $\mathcal{U}_x \cap B = \emptyset$  and for every  $\varepsilon > 0$ ,  $h|_{\mathcal{U}_x} < \varepsilon$  and  $h|_B > 1 - \varepsilon$  and  $0 \leq h \leq 1$ . Then  $A \subseteq \bigcup_{x \in A} \mathcal{U}_x$ . Since  $A$  closed so compact,  $A \subseteq \bigcup_{i=1}^N \mathcal{U}_{x_i}$ . Let  $\varepsilon_0 < \varepsilon$  such that  $(1 - \frac{\varepsilon_0}{N})^N > 1 - \varepsilon$ . For each  $i$ , let  $h_i \in \mathcal{A}$  such that  $h_i|_{\mathcal{U}_{x_i}} < \frac{\varepsilon_0}{N}$ ,  $h_i|_B > 1 - \frac{\varepsilon_0}{N}$  and  $0 \leq h_i \leq 1$ . Then, put

$$h(x) = h_1(x) \cdot h_2(x) \cdots h_N(x) \in \mathcal{A}.$$

Then,  $0 \leq h \leq 1$  and  $h|_B > (1 - \frac{\varepsilon_0}{N})^N > 1 - \varepsilon$ . Then, for every  $x \in A$ ,  $x \in \mathcal{U}_{x_i}$  so  $h_i(x) < \frac{\varepsilon_0}{N}$  and  $h_i(x) \leq i$  so  $h(x) < \frac{\varepsilon_0}{N}$  so  $h|_A < \frac{\varepsilon_0}{N} < \varepsilon$ . ■

PROOF. (Of Stone-Weierstrass) WLOG, assume  $f \in C(X)$ ,  $0 \leq f \leq 1$ , by replacing with

$$\tilde{f}(x) = \frac{f(x) + \|f\|_{\infty}}{\|f\|_{\infty} + \|f\|_{\infty}}$$

if necessary, since if there exists a  $\tilde{g} \in \mathcal{A}$  such that  $\|\tilde{f} - \tilde{g}\|_{\infty} < \varepsilon$ , then using the properties of  $\mathcal{A}$  we can find some appropriate  $g \in \mathcal{A}$  such that  $\|f - g\|_{\infty} < \varepsilon$ .

Fix  $n \in \mathbb{N}$ , and consider the set  $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$ , and let for  $1 \leq j \leq n$

$$A_j := \left\{ x \in X \mid f(x) \leq \frac{j-1}{n} \right\}, \quad B_j := \left\{ x \in X \mid f(x) \geq \frac{j}{n} \right\},$$

which are both closed and disjoint. By the lemma, there exists  $g_j \in \mathcal{A}$  such that

$$g_j|_{A_j} < \frac{1}{n}, \quad g_j|_{B_j} > 1 - \frac{1}{n},$$

with  $0 \leq g_j \leq 1$ . Let then

$$g(x) := \frac{1}{n} \sum_{j=1}^n g_j(x) \in \mathcal{A}.$$

We claim then  $\|f - g\|_{\infty} \leq \frac{3}{n}$ , which proves the claim by taking  $n$  sufficiently large.

Suppose  $k \in [1, n]$ . If  $f(x) \leq \frac{k}{n}$ , then

$$g_j(x) = \begin{cases} < \frac{1}{n} & \text{if } j-1 \geq k, \\ \leq 1 & \text{else} \end{cases},$$

so

$$g(x) = \frac{1}{n} \sum_{j=1}^n g_j(x) = \frac{1}{n} \left[ \sum_{j=1}^k g_j(x) + \sum_{j=k+1}^n g_j(x) \right] \leq \frac{1}{n} \left[ k + \frac{n-k}{n} \right] \leq \frac{k}{n} + \frac{n-k}{n^2} \leq \frac{k+1}{n}.$$

Similarly if  $f(x) \geq \frac{k-1}{n}$ , then

$$g_j(x) = \begin{cases} > 1 - \frac{1}{n} & \text{if } j \leq k-1, \\ \geq 0 & \text{else} \end{cases},$$

so

$$g(x) \geq \frac{1}{n} \sum_{j=1}^{k-1} \left( 1 - \frac{1}{n} \right) \geq \frac{1}{n} (k-1) \left( 1 - \frac{1}{n} \right) = \frac{k-1}{n} - \frac{k-1}{n^2} \geq \frac{k-2}{n}.$$

So, we've show that if  $\frac{k-1}{n} \leq f(x) \leq \frac{k}{n}$ , then  $\frac{k-2}{n} \leq g(x) \leq \frac{k+1}{n}$ , and so repeating this argument and applying triangle inequality we conclude  $\|f - g\|_\infty \leq \frac{3}{n}$ . ■

↪ **Theorem 1.12** (Borsuk):  $X$  compact, Hausdorff and  $C(X)$  separable  $\Leftrightarrow X$  is metrizable.

## §2 FUNCTIONAL ANALYSIS

Here, we will primarily work with a normed vector space (nvs). Moreover, we usually work in:

↪ **Definition 2.1** (Banach Space): A normed vector space  $(X, \|\cdot\|)$  is a *Banach space* if it is complete as a metric space under the norm-induced metric.

### §2.1 Introduction to Linear Operators

↪ **Definition 2.2** (Linear Operator, Operator Norm): Let  $X, Y$  be vector spaces. Then, a map  $T : X \rightarrow Y$  is called *linear* if  $\forall x, y \in X, \alpha, \beta \in \mathbb{R}, T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ .

If  $X, Y$  normed vector spaces, we say  $T$  is a bounded linear operator if  $T$  linear and the *operator norm*

$$\|T\| = \|T\|_{\mathcal{L}(X, Y)} = \sup_{\substack{x \in X, \\ \|x\|_X \leq 1}} \|Tx\|_Y < \infty$$

is finite. Then, we put

$$\mathcal{L}(X, Y) := \{\text{bounded linear operators } X \rightarrow Y\}.$$

We'll also write  $\mathcal{L}(X) := \mathcal{L}(X, X)$ .



↪ **Theorem 2.1** (Bounded iff Continuous): If  $X, Y$  are nvs,  $T \in \mathcal{L}(X, Y)$  iff and only if  $T$  is continuous, i.e. if  $x_n \rightarrow x$  in  $X$ , then  $Tx_n \rightarrow Tx$  in  $Y$ .

PROOF. If  $T \in \mathcal{L}(X, Y)$ ,

$$\begin{aligned}\|Tx_n - Tx\|_Y &= \|T(x_n - x)\|_Y \\ &= \|x_n - x\|_X \cdot \left\| \frac{T(x_n - x)}{\|x_n - x\|_X} \right\|_Y \\ &\leq \underbrace{\|T\|}_{< \infty} \|x_n - x\|_X \rightarrow 0,\end{aligned}$$

hence  $T$  continuous. Conversely, if  $T$  continuous, then by linearity  $T0 = 0$ , so by continuity, there is some  $\delta > 0$  such that  $\|Tx\|_Y < 1$  if  $\|x\|_X < \delta$ . For  $x \in X$  nonzero, let  $\lambda = \frac{\delta}{\|x\|_X}$ . Then,  $\|\lambda x\|_X \leq \delta$  so  $\|T(\lambda x)\|_Y < 1$ , i.e.  $\frac{\|T(x)\|_Y \delta}{\|x\|_X} < 1$ . Hence,

$$\|T\| = \sup_{x \in X: x \neq 0} \frac{\|T(x)\|_Y}{\|x\|_X} \leq \frac{1}{\delta},$$

so  $T \in \mathcal{L}(X, Y)$ . ■

↪ **Proposition 2.1** (Properties of  $\mathcal{L}(X, Y)$ ): If  $X, Y$  nvs,  $\mathcal{L}(X, Y)$  a nvs, and if  $X, Y$  Banach, then so is  $\mathcal{L}(X, Y)$ .

PROOF. (a) For  $T, S \in \mathcal{L}(X, Y)$ ,  $\alpha, \beta \in \mathbb{R}$ , and  $x \in X$ , then

$$\begin{aligned}\|(\alpha T + \beta S)(x)\|_Y &\leq |\alpha| \|Tx\|_Y + |\beta| \|Sx\|_Y \\ &\leq |\alpha| \|T\| \|x\|_X + |\beta| \|S\| \|x\|_X.\end{aligned}$$

Dividing both sides by  $\|x\|$ , we find  $\|\alpha T + \beta S\| < \infty$ . The same argument gives the triangle inequality on  $\|\cdot\|$ . Finally,  $T = 0$  iff  $\|Tx\|_Y = 0$  for every  $x \in X$  iff  $\|T\| = 0$ .

(b) Let  $\{T_n\} \subseteq \mathcal{L}(X, Y)$  be a Cauchy sequence. We have that

$$\|T_n x - T_m x\|_Y \leq \|T_n - T_m\| \|x\|_X,$$

so in particular the sequence  $\{T_n(x)\}$  a Cauchy sequence in  $Y$  for any  $x \in X$ .  $Y$  complete so this sequence converges, say  $T_n(x) \rightarrow y^*$  in  $Y$ . Let  $T(x) := y^*$  for each  $x$ . We claim that  $T \in \mathcal{L}(X, Y)$  and that  $T_n \rightarrow T$  in the operator norm. We check:

$$\begin{aligned}\alpha T(x_1) + \beta T(x_2) &= \lim_{n \rightarrow \infty} \alpha T_n(x_1) + \lim_{n \rightarrow \infty} \beta T_n(x_2) \\ &= \lim_{n \rightarrow \infty} [T_n(\alpha x_1) + T_n(\beta x_2)] \\ &= \lim_{n \rightarrow \infty} T_n(\alpha x_1 + \beta x_2) \\ &= T(\alpha x_1 + \beta x_2),\end{aligned}$$

so  $T$  linear.

Let now  $\varepsilon > 0$  and  $N$  such that for every  $n \geq N$  and  $k \geq 1$  such that  $\|T_n - T_{n+k}\| < \frac{\varepsilon}{2}$ . Then,

$$\begin{aligned}\|T_n(x) - T_{n+k}(x)\|_Y &= \|(T_n - T_{n+k})(x)\|_Y \\ &\leq \|T_n - T_{n+k}\| \|x\|_X \\ &< \frac{\varepsilon}{2} \|x\|_X.\end{aligned}$$

Letting  $k \rightarrow \infty$ , we find that

$$\|T_n(x) - T(x)\|_Y < \frac{\varepsilon}{2} \|x\|_X,$$

so normalizing both sides by  $\|x\|_X$ , we find  $\|T_n - T\| < \frac{\varepsilon}{2}$ , and we have convergence. ■

↪ **Definition 2.3** (Isomorphism): We say  $T \in \mathcal{L}(X, Y)$  an *isomorphism* if  $T$  is bijective and  $T^{-1} \in \mathcal{L}(Y, X)$ . In this case we write  $X \simeq Y$ , and say  $X, Y$  isomorphic.

## §2.2 Finite versus Infinite Dimensional

If  $X$  a nvs, then we can look for a basis  $\beta$  such that  $\text{span}(\beta) = X$ . If  $\beta = \{e_1, \dots, e_n\}$  has no proper subset spanning  $X$ , then we say  $\dim(X) = n$ .

As we saw on homework, any two norms on a finite dimensional space are equivalent.

↪ **Corollary 2.1:** (a) Any two nvs of the same finite dimension are isomorphic.

(b) Any finite dimensional space is complete, and so any finite dimensional subspace is closed.

(c)  $\overline{B}(0, 1)$  is compact in a finite dimensional space.

PROOF. (a) Let  $(X, \|\cdot\|)$  have finite dimension  $n$ . Then, we claim  $(X, \|\cdot\|) \simeq (\mathbb{R}^n, |\cdot|)$ . Let  $\{e_1, \dots, e_n\}$  be a basis for  $X$ . Let  $T : \mathbb{R}^n \rightarrow X$  given by

$$T(x) = \sum_{i=1}^n x_i e_i,$$

where  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , which is clearly linear. Moreover,

$$Tx = 0 \Leftrightarrow \sum_{i=1}^n x_i e_i = 0 \Leftrightarrow x = 0,$$

so  $T$  injective, and so being linear between two spaces of the same dimension gives  $T$  surjective. It remains to check boundedness.

First, we claim  $x \mapsto \|T(x)\|$  is a norm on  $\mathbb{R}^n$ .  $\|T(x)\| = 0 \Leftrightarrow x = 0$  by the injectivity of  $T$ , and the properties  $\|T(\lambda x)\| = |\lambda| \|Tx\|$  and  $\|T(x + y)\| \leq \|Tx\| + \|Ty\|$  follow from linearity of  $T$  and the fact that  $\|\cdot\|$  already a norm. Hence,  $\|T(\cdot)\|$  a norm on  $\mathbb{R}^n$  and so equivalent to  $|\cdot|$ , i.e. there exists constants  $C_1, C_2 > 0$  such that

$$C_1|x| \leq \|T(x)\| \leq C_2|x|,$$

for every  $x \in X$ . It follows that  $\|T\|$  (operator norm now) is bounded.

Letting  $T(x) = y$ , we find similarly

$$C_1'\|y\| \leq |T^{-1}(y)| \leq C_2' \|y\|,$$

so  $\|T^{-1}\|$  also bounded. Hence, we've shown any  $n$ -dimensional space is isomorphic to  $\mathbb{R}^n$ , so by transitivity of isomorphism any two  $n$ -dimensional spaces are isomorphic.

(b) The property of completeness is preserved under isomorphism, so this follows from the previous statement since  $\mathbb{R}^n$  complete.

(c) Consider  $\overline{B}(0, 1) \subseteq X$ . Let  $T$  be an isomorphism  $X \rightarrow \mathbb{R}^n$ . Then, for  $x \in \overline{B}(0, 1)$ ,  $\|Tx\| \leq \|T\| < \infty$ , so  $T(\overline{B}(0, 1))$  is a bounded subset of  $\mathbb{R}^n$ , and since  $T$  and its inverse continuous,  $T(\overline{B}(0, 1))$  closed in  $\mathbb{R}^n$ . Hence,  $T(\overline{B}(0, 1))$  closed and bounded hence compact in  $\mathbb{R}^n$ , so since  $T^{-1}$  continuous  $T^{-1}(T(\overline{B}(0, 1))) = \overline{B}(0, 1)$  also compact, in  $X$ . ■

↪ **Theorem 2.2** (Riesz's): If  $X$  is an nvs, then  $\overline{B}(0, 1)$  is compact if and only if  $X$  if finite dimensional.

↪ **Lemma 2.1** (Riesz's): Let  $Y \subsetneq X$  be a closed nvs (and  $X$  a nvs). Then for every  $\varepsilon > 0$ , there exists  $x_0 \in X$  with  $\|x_0\| = 1$  and such that

$$\|x_0 - y\|_X > \varepsilon \forall y \in Y.$$

PROOF. Fix  $\varepsilon > 0$ . Since  $Y \subsetneq X$ , let  $x \in Y^c$ .  $Y$  closed so  $Y^c$  open and hence there exists some  $r > 0$  such that  $B(x, r) \cap Y = \emptyset$ . In other words,

$$\inf\{\|x - y'\| \mid y' \in Y\} > r > 0.$$

Let then  $y_1 \in Y$  be such that

$$r < \|x - y_1\| < \varepsilon^{-1}r,$$

and take

$$x_0 := \frac{x - y_1}{\|x - y_1\|_X}.$$

Then,  $x_0$  a unit vector, and for every  $y \in Y$ ,

$$\begin{aligned} x_0 - y &= \frac{x - y_1}{\|x - y_1\|} - y \\ &= \frac{1}{\|x - y_1\|} [x - y_1 - y \|x - y_1\|] \\ &= \frac{1}{\|x - y_1\|} [x - y'], \end{aligned}$$

where  $y' = y_1 + y$   $\|x - y_1\| \in Y$ , since it is closed under vector addition. Hence

$$\|x_0 - y\| = \frac{1}{\|x - y_1\|} \|x - y'\| > \frac{\varepsilon}{r} \|x - y'\| > \varepsilon,$$

for every  $y \in Y$ . ■

PROOF. (Of [Thm. 2.2](#)) ( $\Leftarrow$ ) By the previous corollary.

( $\Rightarrow$ ) Suppose  $X$  infinite dimensional. We will show  $B := \overline{B}(0, 1)$  not compact.

*Claim:* there exists  $\{x_i\}_{i=1}^{\infty} \subseteq B$  such that  $\|x_i - x_j\| > \frac{1}{2}$  if  $i \neq j$ .

We proceed by induction. Let  $x_1 \in B$ . Suppose  $\{x_1, \dots, x_n\} \subseteq B$  are such that  $\|x_i - x_j\| > \frac{1}{2}$ . Let  $X_n = \text{span}\{x_1, \dots, x_n\}$ , so  $X_n$  finite dimensional hence  $X_n \subsetneq X$ . By the previous lemma (taking  $\varepsilon = \frac{1}{2}$ ) there is then some  $x_{n+1} \in B$  such that  $\|x_1 - x_{n+1}\| > \frac{1}{2}$  for every  $i = 1, \dots, n$ . We can thus inductively build such a sequence  $\{x_i\}_{i=1}^{\infty}$ . Then, every subsequence of this sequence cannot be Cauchy so  $B$  is not sequentially compact and thus  $B$  is not compact. ■

### §2.3 Open Mapping and Closed Graph Theorems

$\hookrightarrow$  **Definition 2.4** ( $T$  open): If  $X, Y$  topological spaces and  $T : X \rightarrow Y$  a linear operator,  $T$  is said to be *open* if for every  $\mathcal{U} \subseteq X$  open,  $T(\mathcal{U})$  open in  $Y$ .

In particular if  $X, Y$  are metric spaces (or nvs), then  $T$  is open iff the image of every open ball in  $X$  contains an open ball in  $Y$ , i.e.  $\forall x \in X, r > 0$  there exists  $r' > 0$  such that  $T(B_X(x, r)) \supseteq B_Y(Tx, r')$ . Moreover, by translating/scaling appropriately, it suffices to prove for  $x = 0, r = 1$ .

$\hookrightarrow$  **Theorem 2.3** (Open Mapping Theorem): Let  $X, Y$  be Banach spaces and  $T : X \rightarrow Y$  a bounded linear operator. If  $T$  is surjective, then  $T$  is open.

PROOF. Its enough to show that there is some  $r > 0$  such that  $T(B_X(0, 1)) \supseteq B_Y(0, r)$ .

*Claim:*  $\exists c > 0$  such that  $\overline{T(B_X(0, 1))} \supseteq B_Y(0, 2c)$ .

Put  $E_n = n \cdot \overline{T(B_X(0, 1))}$  for  $n \in \mathbb{N}$ . Since  $T$  surjective,  $\bigcup_{n=1}^{\infty} E_n = Y$ . Each  $E_n$  closed, so by the Baire Category Theorem there exists some index  $n_0$  such that  $E_{n_0}$  has nonempty interior, i.e.

$$\text{int}(\overline{T(B_X(0, 1))}) \neq \emptyset,$$

where we drop the index by homogeneity. Pick then  $c > 0$  and  $y_0 \in Y$  such that  $B_Y(y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$ . We claim then that  $B_Y(-y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$  as well. Indeed, if  $B_Y(y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$ , then  $\forall \tilde{y} \in Y$  with  $\|y_0 - \tilde{y}\|_Y < 4c$ , Then,  $\| -y_0 + \tilde{y}\|_Y < 4c$  so  $-\tilde{y} \in B_Y(-y_0, 4c)$ . But  $\tilde{y} = \lim_{n \rightarrow \infty} T(x_n)$  and so  $-\tilde{y} = \lim_{n \rightarrow \infty} T(-x_n)$ . Since  $\{-x_n\} \subseteq B_X(0, 1)$ , this implies  $-\tilde{y} \in \overline{T(B_X(0, 1))}$  hence the “subclaim” holds.

Now, for any  $\tilde{y} \in B_Y(0, 4c)$ ,  $\|\tilde{y}\| \leq 4c$  so

$$\tilde{y} = y_0 - \underbrace{y_0 + \tilde{y}}_{\in B_Y(-y_0, 4c)} = \overbrace{y_0 + \tilde{y}}^{\in B(y_0, 4c)} - y_0.$$

Therefore,

$$\begin{aligned} B_Y(0, 4c) &= B_Y(y_0 - y_0, 4c) \\ &\subseteq B_Y(y_0, 4c) + B_Y(-y_0, 4c) \\ \overline{T(B_X(0, 1))} + \overline{T(B_X(0, 1))} &= 2\overline{T(B_X(0, 1))}, \end{aligned}$$

(where summation of two sets is the vector addition of all the elements in the sets), hence  $B_Y(0, 2c) \subseteq \overline{T(B_X(0, 1))}$ .

We claim next that  $T(B_X(0, 1)) \supseteq B_Y(0, c)$ . Choose  $y \in Y$  with  $\|y\|_Y < c$ . By the first claim,  $B_Y(0, c) \subseteq \overline{T(B_X(0, \frac{1}{2}))}$ , so for every  $\varepsilon > 0$  there is some  $z \in X$  with  $\|z\|_X < \frac{1}{2}$  and  $\|y - Tz\|_Y < \varepsilon$ . Let  $\varepsilon = \frac{c}{2}$  and  $z_1 \in X$  such that  $\|z_1\|_X < \frac{1}{2}$  and  $\|y - Tz_1\|_Y < \frac{c}{2}$ . But the first claim can also be written as  $B_Y(0, \frac{c}{2}) \subseteq \overline{T(B_X(0, \frac{1}{4}))}$  so if  $\varepsilon = \frac{c}{4}$ , let  $z_2 \in X$  such that  $\|z_2\|_X < \frac{1}{4}$  and  $\|(y - Tz_1) - Tz_2\|_Y < \frac{c}{4}$ . Continuing in this manner we find that

$$B_Y\left(0, \frac{c}{2^k}\right) \subseteq \overline{T\left(B_X\left(0, \frac{1}{2^{k+1}}\right)\right)},$$

so exists  $z_k \in X$  such that  $\|z_k\|_X < \frac{1}{2^k}$  and  $\|y - T(z_1 + \dots + z_k)\|_Y < \frac{c}{2^k}$ . Let  $x_n = z_1 + \dots + z_n \in X$ . Then  $\{x_n\}$  is Cauchy in  $X$ , since

$$\|x_n - x_m\|_X \leq \sum_{k=m}^n \|z_k\|_X < \sum_{k=m}^n \frac{1}{2^k} \rightarrow 0.$$

Since  $X$  a Banach space,  $x_n \rightarrow \bar{x}$  and in particular  $\|\bar{x}\| \leq \sum_{k=1}^{\infty} \|z_k\|_X < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ , so  $\bar{x} \in B_X(0, 1)$ . Since  $T$  bounded it is continuous, so  $Tx_n \rightarrow T\bar{x}$ , so  $y = T\bar{x}$  and thus  $B_Y(0, c) \subseteq T(B_X(0, 1))$ . ■

↪ **Corollary 2.2:** Let  $X, Y$  Banach and  $T : X \rightarrow Y$  be bounded, linear and bijective. Then,  $T^{-1}$  continuous.

↪ **Corollary 2.3:** Let  $(X, \|\cdot\|_1), (X, \|\cdot\|_2)$  be Banach spaces. Suppose there exists  $c > 0$  such that  $\|x\|_2 \leq C\|x\|_1$  for every  $x \in X$ . Then,  $\|\cdot\|_1, \|\cdot\|_2$  are equivalent.

PROOF. Let  $T$  be the identity linear operator and use the previous corollary. ■

↪ **Definition 2.5** ( $T$  closed): If  $X, Y$  are nvs and  $T$  is linear, the *graph* of  $T$  is the set

$$G(T) = \{(x, Tx) \mid x \in X\} \subseteq X \times Y.$$

We then say  $T$  is *closed* if  $G(T)$  closed in  $X \times Y$ .

**Remark 2.1:** Since  $X, Y$  are nvs, they are metric spaces so first countable, hence closed  $\leftrightarrow$  contains all limit points.

In the product topology, a countable base for  $X \times Y$  at  $(x, y)$  is given by

$$\left\{ B_X\left(x, \frac{1}{n}\right) \times B\left(y, \frac{1}{m}\right) \right\}_{n,m \in \mathbb{N}}.$$

Then,  $G(T)$  closed iff  $G(T)$  contains all limit points. How can we put a norm on  $X \times Y$  that generates this product topology? Let

$$\|(x, y)\|_1 := \|x\|_X + \|y\|_Y.$$

If  $(x_n, y_n) \rightarrow (x, y)$  in the product topology, then since  $\Pi_1, \Pi_2$  continuous maps,  $(x_n, y_n) \rightarrow (x, y)$  in the  $\|\cdot\|_1$  topology. On the other hand if  $(x_n, y_n) \rightarrow (x, y)$  in the  $\|\cdot\|_1$  norm, then

$$\|x_n - x\|_X \leq \|(x_n, y_n) - (x, y)\|_1,$$

hence since the RHS  $\rightarrow 0$  so does the LHS and so  $x_n \rightarrow x$  in  $\|\cdot\|_X$ ; similar gives  $y_n \rightarrow y$  in  $\|\cdot\|_Y$ . From here it follows that  $(x_n, y_n) \rightarrow (x, y)$  in the product topology.

So, to prove  $G(T)$  closed, we just need to prove that if  $x_n \rightarrow x$  in  $X$  and  $Tx_n \rightarrow y$ , then  $y = Tx_n$ .

$\hookrightarrow$  **Theorem 2.4** (Closed Graph Theorem): Let  $X, Y$  be Banach spaces and  $T : X \rightarrow Y$  linear. Then,  $T$  is continuous iff  $T$  is closed.

PROOF.  $(\Rightarrow)$  Immediate from the above remark.

$(\Leftarrow)$  Consider the function

$$x \mapsto \|x\|_* := \|x\|_X + \|Tx\|_Y.$$

So by the above,  $T$  closed implies  $(X, \|\cdot\|_*)$  is complete, i.e. if  $x_n \rightarrow x$  in  $\|\cdot\|_*$  in  $X$  iff  $x_n \rightarrow x$  in  $\|\cdot\|_X$  and  $Tx_n \rightarrow Tx$  in  $\|\cdot\|_Y$ . However,  $\|\cdot\|_X \leq \|\cdot\|_*$ , hence since  $(X, \|\cdot\|_X)$  and  $(X, \|\cdot\|_*)$  are Banach spaces, by the corollary, there is some  $C > 0$  such that  $\|\cdot\|_* \leq C\|\cdot\|_X$ . So,

$$\|x\|_X + \|Tx\|_Y \leq C\|x\|_X,$$

so

$$\|Tx\|_Y \leq \|x\|_X + \|Tx\|_Y \leq C\|x\|_X,$$

so  $T$  bounded and thus continuous. ■

**Remark 2.2:** The Closed Graph Theorem simplifies proving continuity of  $T$ . It tells us we can assume if  $x_n \rightarrow x$ ,  $\{Tx_n\}$  Cauchy so  $\exists y$  such that  $Tx_n \rightarrow y$  since  $Y$  is Banach. So, it suffices to check that  $y = Tx$  to check continuity; we don't need to check convergence of  $Tx_n$ .

## §2.4 Uniform Boundedness Principle

Recall the following consequence of the Baire Category Theorem:

↪ **Theorem 2.5:** Let  $\mathcal{F} \subseteq C(X)$  where  $(X, \rho)$  a complete metric space. Suppose  $\mathcal{F}$  pointwise bounded. Then, there exists a nonempty open set  $\mathcal{O} \subseteq X$  such that there is some  $M > 0$  such that  $|f(x)| \leq M$  for every  $x \in \mathcal{O}, f \in \mathcal{F}$ .

This leads to the following result:

↪ **Theorem 2.6** (Uniform Boundedness Principle): Let  $X$  a Banach space and  $Y$  a nvs. Consider  $\mathcal{F} \subseteq \mathcal{L}(X, Y)$ . Suppose  $\mathcal{F}$  is pointwise bounded, i.e. for every  $x \in X$ , there is some  $M_x > 0$  such that

$$\|Tx\|_Y \leq M_x, \forall T \in \mathcal{F}.$$

Then,  $\mathcal{F}$  is uniformly bounded, i.e.  $\exists M > 0$  such that

$$\|T\|_Y \leq M, \forall T \in \mathcal{F}.$$

PROOF. For every  $T \in \mathcal{F}$ , let  $f_T : X \rightarrow \mathbb{R}$  be given by

$$f_T(x) = \|Tx\|_Y.$$

Since  $T \in \mathcal{L}(X, Y)$ ,  $T$  is continuous, so  $x_n \xrightarrow{X} x \Rightarrow Tx_n \xrightarrow{Y} Tx$ , hence  $\|Tx_n\|_Y \rightarrow \|Tx\|_Y$  so  $f_T$  continuous for each  $T$  i.e.  $f_T \in C(X)$ , so  $\{f_T\} \subseteq C(X)$  pointwise bounded. So by the previous theorem, there is some ball  $B(x_0, r) \subseteq X$  and some  $K > 0$  such that  $\|Tx\| \leq K$  for every  $x \in B(x_0, r)$  and  $T \in \mathcal{F}$ . Thus, for every  $x \in B(0, r)$ ,

$$\begin{aligned} \|Tx\| &= \|T(x - x_0 + x_0)\| \\ &\leq \left\| \underbrace{T(x - x_0)}_{\in B(x_0, r)} \right\| + \|Tx_0\| \\ &\leq K + M_{x_0}, \quad \forall x \in B(0, r), T \in \mathcal{F}. \end{aligned}$$

Thus, for every  $x \in B(0, 1)$ ,

$$\|Tx\| = \frac{1}{r} \left\| T \left( \underbrace{rx}_{\in B(0, r)} \right) \right\| \leq \frac{1}{r} (K + M_{x_0}) =: M,$$

so its clear  $\|T\| \leq M$  for every  $T \in \mathcal{F}$ . ■

↪ **Theorem 2.7** (Banach-Saks-Steinhaus): Let  $X$  a Banach space and  $Y$  a nvs. Let  $\{T_n\} \subseteq \mathcal{L}(X, Y)$ . Suppose for every  $x \in X$ ,  $\lim_{n \rightarrow \infty} T_n(x)$  exists in  $Y$ . Then,

- $\{T_n\}$  are uniformly bounded in  $\mathcal{L}(X, Y)$ ;
- For  $T : X \rightarrow Y$  defined by

$$T(x) := \lim_{n \rightarrow \infty} T_n(x),$$

we have  $T \in \mathcal{L}(X, Y)$ ;

- $\|T\| \leq \liminf_{n \rightarrow \infty} \|T_n\|$  (*lower semicontinuity result*).

PROOF. (a) For every  $x \in X$ ,  $T_n(x) \rightarrow T(x)$  so  $\|Tx\| < \infty$  hence  $\sup_n \|T_n x\| < \infty$ . By uniform boundedness, then, we find  $\sup_n \|T_n\| =: C < \infty$ .

(b)  $T$  is linear (by linearity of  $T_n$ ). By (a),

$$\|T_n x\| \leq C \|x\|,$$

for every  $n, x$ , so

$$\|Tx\| \leq C \|x\| \forall x \in X,$$

so  $T$  bounded.

(c) We know

$$\|T_n x\| \leq \|T_n\| \|x\| \forall x \in X,$$

so

$$\frac{\|T_n x\|}{\|x\|} \leq \|T_n\|,$$

so

$$\liminf_n \frac{\|T_n x\|}{\|x\|} = \frac{\|Tx\|}{\|x\|} \leq \liminf_n \|T_n\|,$$

so by “suping” both sides,

$$\|T\| \leq \liminf_n \|T_n\|.$$

■

### Remark 2.3:

- We do not have  $T_n \rightarrow T$  in  $\mathcal{L}(X, Y)$  i.e. with respect to the operator norm.
- If  $Y$  is a Banach space, then  $\lim_{n \rightarrow \infty} T_n(x)$  exists in  $Y \Leftrightarrow \{T_n x\}$  Cauchy in  $Y$  for every  $x \in X$ .

## §2.5 Introduction to Hilbert Spaces



↪ **Definition 2.6** (Inner Product): An *inner product* on a vector space  $X$  is a map  $(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$  such that for every  $\lambda, \mu \in \mathbb{R}$  and  $x, y, z \in X$ ,

- $(\lambda x + \mu y, z) = \lambda(x, z) + \mu(y, z)$ ;
- $(x, y) = (y, x)$ ;
- $(x, x) \geq 0$  and  $(x, x) = 0 \Leftrightarrow x = 0$ .

**Remark 2.4:** The first and second conditions combined imply that  $(\cdot, \cdot)$  actually *bilinear*, namely, linear in both coordinates.

**Remark 2.5:** An inner product induces a norm on a vector space by

$$\|x\| := (x, x)^{\frac{1}{2}}.$$

↪ **Proposition 2.2** (Cauchy-Schwarz Inequality): Any inner product satisfies Cauchy-Schwarz, namely,

$$|(x, y)| \leq \|x\| \|y\|,$$

for every  $x, y \in X$ .

PROOF. Suppose first  $y = 0$ . Then, the right hand side is clearly 0, and by linearity  $(x, y) = 0$ , hence we have  $0 \leq 0$  and are done. Suppose then  $y \neq 0$ . Then, let  $z = x - \frac{(x, y)}{(y, y)}y$  where  $y \neq 0$ . Then,

$$\begin{aligned} 0 \leq \|z\|^2 &= \left( x - \frac{(x, y)}{(y, y)}y, x - \frac{(x, y)}{(y, y)}y \right) \\ &= (x, x) - \frac{(x, y)}{(y, y)}(x, y) - \frac{(x, y)}{(y, y)}(y, x) + \frac{(x, y)^2}{(y, y)^2}(y, y) \\ &= (x, x) - \frac{2((x, y))^2}{(y, y)} + \frac{(x, y)^2}{(y, y)} \\ &= \|x\|^2 - \frac{(x, y)^2}{(y, y)} \\ &\Rightarrow \frac{(x, y)^2}{(y, y)} \leq \|x\|^2 \Rightarrow (x, y)^2 \leq \|x\|^2 \|y\|^2 \\ &\Rightarrow |(x, y)| \leq \|x\| \|y\|. \end{aligned}$$

■

↪ **Corollary 2.4:** The function  $\|x\| := (x, x)^{\frac{1}{2}}$  is actually a norm on  $X$ .

PROOF. By definition,  $\|x\| \geq 0$  and equal to zero only when  $x = 0$ . Also,

$$\|\lambda x\| = (\lambda x, \lambda x)^{\frac{1}{2}} = |\lambda|(x, x)^{\frac{1}{2}} = |\lambda|\|x\|.$$

Finally,

$$\begin{aligned} \|x + y\|^2 &= (x + y, x + y) \\ &= (x, x) + 2(x, y) + (y, y) \\ &= \|x\|^2 + \|y\|^2 + 2(x, y) \\ \text{by Cauchy-Schwarz} \quad &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &= (\|x\| + \|y\|)^2, \end{aligned}$$

hence by taking square roots we see  $\|x + y\| \leq \|x\| + \|y\|$  as desired. ■

↪ **Proposition 2.3** (Parallelogram Law): Any inner product space satisfies the following:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

↪ **Corollary 2.5:**  $(\cdot, \cdot)$  is continuous, i.e. if  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , then  $(x_n, y_n) \rightarrow (x, y)$ .

PROOF.

$$\begin{aligned} |(x_n, y_n) - (x, y)| &= |(x_n, y_n) - (x, y_n) + (x, y_n) - (x, y)| \\ &= |(x_n - x, y_n) + (x, y_n - y)| \\ &\leq |(x_n - x, y_n)| + |(x, y_n - y)| \\ \text{(Cauchy-Schwarz)} \quad &\leq \underbrace{\|x_n - x\|}_{\rightarrow 0} \underbrace{\|y_n\|}_{\leq M} + \|x\| \underbrace{\|y_n - y\|}_{\rightarrow 0} \rightarrow 0. \end{aligned}$$
■

↪ **Definition 2.7** (Hilbert Space): A *Hilbert Space*  $H$  is a complete inner product space, namely, it is complete with respect to the norm induced by the inner product.

⊗ **Example 2.1:**

1.  $\ell^2$ , the space of square-summable real-valued sequences, equipped with inner product  $(x, y) = \sum_{i=1}^{\infty} x_i y_i$ .
2.  $L^2$ , with inner product  $(f, g) = \int f(x)g(x) dx$ .

↪ **Definition 2.8** (Orthogonality): We say  $x, y$  *orthogonal* and write  $x \perp y$  if  $(x, y) = 0$ . If  $M \subseteq H$ , then the *orthogonal complement* of  $M$ , denoted  $M^\perp$ , is the set

$$M^\perp = \{y \in H \mid (x, y) = 0, \forall x \in M\}.$$

**Remark 2.6:**  $M^\perp$  is always a closed subspace of  $H$ . If  $y_1, y_2 \in M^\perp$ , then for every  $x \in M$ ,

$$(x, \alpha y_1 + \beta y_2) = \alpha(x, y_1) + \beta(x, y_2) = 0,$$

so  $M^\perp$  a subspace.

If  $y_n \rightarrow y$  in the norm on  $H$  and  $\{y_n\} \subseteq M^\perp$ , then using the continuity of  $(\cdot, \cdot)$ , we know that for every  $x \in M$ ,  $(x, y_n) \rightarrow (x, y)$ . But the  $(x, y_n) = 0$  for every  $n$  and thus  $(x, y) = 0$  so  $y \in M^\perp$ , hence  $M^\perp$  closed.

↪ **Proposition 2.4:** If  $M \subsetneq H$  is a closed subspace, then every  $x \in H$  has a unique decomposition

$$x = u + v, \quad u \in M, v \in M^\perp.$$

Hence, we may write  $H = M \oplus M^\perp$ . Moreover,

$$\|x - u\| = \inf_{y \in M} \|x - y\|, \quad \|x - v\| = \inf_{y \in M^\perp} \|x - y\|.$$

PROOF. Let  $x \in H$ . If  $x \in M$ , we're done with  $u = x, v = 0$ . Else, if  $x \notin M$ , then we claim that there is some  $u \in M$  such that  $\|x - u\| = \inf_{y \in M} \|x - y\| =: \delta > 0$ . By definition of the infimum, there exists a sequence  $\{u_n\} \subseteq M$  such that

$$\|x - u_n\|^2 \leq \delta^2 + \frac{1}{n}.$$

Let  $\bar{x} := u_m - x, \bar{y} = u_n - x$ . By the Parallelogram Law,

$$\|\bar{x} - \bar{y}\|^2 + \|\bar{x} + \bar{y}\|^2 = 2\|\bar{x}\|^2 + 2\|\bar{y}\|^2$$

hence

$$\|u_m - u_n\|^2 + \|u_m + u_n - 2x\|^2 = 2\|u_m - x\|^2 + 2\|u_n - x\|^2.$$

Now, the second term can be written

$$\|u_m + u_n - 2x\|^2 = 4 \left\| \frac{u_m + u_n}{2} - x \right\|^2,$$

hence we find

$$\|u_m - u_n\|^2 = 2\|u_m - x\|^2 + 2\|u_n - x\|^2 - 4 \left\| \frac{u_m + u_n}{2} - x \right\|^2.$$

Recall that  $M$  a subspace, hence  $\frac{1}{2}(u_m + u_n) \in M$  so  $\|x - \frac{1}{2}(u_m + u_n)\| \geq \delta$  as defined before. Thus, we find that by our choice of  $\{u_n\}$ ,

$$\|u_m - u_n\|^2 \leq 2\left(\delta^2 + \frac{1}{m}\right) + 2\left(\delta^2 + \frac{1}{n}\right) - 4\delta^2 = \frac{2}{m} + \frac{2}{n},$$

and thus, by making  $m, n$  sufficiently large we can make  $\|u_m - u_n\|$  arbitrarily small. Hence,  $\{u_n\} \subseteq M$  are Cauchy.  $H$  is complete, hence the  $\{u_n\}$ 's converge, and thus since  $M$  closed,  $u_n \rightarrow u \in M$ . Then, we find

$$\begin{aligned} \|x - u\| &\leq \|x - u_n\| + \|u_n - u\| \\ &\leq \underbrace{\left(\delta^2 + \frac{1}{n}\right)^{\frac{1}{2}}}_{\rightarrow \delta} + \underbrace{\|u_n - u\|}_{\rightarrow 0} \rightarrow \delta. \end{aligned}$$

But also,  $u \in M$  and thus  $\|x - y\| \geq \delta$ , and we conclude  $\|x - u\| = \delta = \inf_{y \in M} \|x - y\|$ .

Next, we claim that if we define  $v = x - y$ , then  $v \in M^\perp$ . Consider  $y \in M, t \in \mathbb{R}$ , then

$$\left\|x - \underbrace{(u - ty)}_{\in M}\right\|^2 = \|v + ty\|^2 = \|v\|^2 + 2t(v, y) + t^2\|y\|^2.$$

Then, notice that the map

$$t \mapsto \|v + ty\|^2$$

is minimized when  $t = 0$ , since  $\|x - z\|$  for  $z \in M$  is minimized when  $z = u$ , as we showed in the previous part, so equivalently  $\|x - (u - ty)\|^2$  minimized when  $t = 0$ . Thus,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \|v + ty\|^2|_{t=0} = \frac{\partial}{\partial t} [\|v\|^2 + 2t(v, y) + t^2\|y\|^2]|_{t=0} \\ &= (2(v, y) + 2t\|y\|^2)|_{t=0} = (v, y) \\ &\Rightarrow (v, y) = 0 \forall y \in M \Rightarrow v \in M^\perp. \end{aligned}$$

So,  $x = u + v$  and  $u \in M, v \in M^\perp$ . For uniqueness, suppose  $x = u_1 + v_1 = u_2 + v_2$ . Then,  $u_1 - u_2 = v_2 - v_1$ , but then

$$\|v_2 - v_1\|^2 = (v_2 - v_1, v_2 - v_1) = (v_2 - v_1, u_2 - u_1) = 0,$$

so  $v_2 = v_1$  so it follows  $u_2 = u_1$  and uniqueness holds. ■

↪ **Definition 2.9** (Dual of  $H$ ): The *dual* of  $H$ , denoted  $H^*$ , is the set

$$H^* := \{f : H \rightarrow \mathbb{R} \mid f \text{ continuous and linear}\}.$$

On this space, we may equip the operator norm

$$\|f\|_{H^*} = \|f\| = \sup_{x \in H} \frac{|f(x)|}{\|x\|_H} = \sup_{\|x\| \leq 1} |f(x)|.$$

⊗ **Example 2.2:** For  $y \in H$ , let  $f_y : H \rightarrow \mathbb{R}$  be given by  $f_y(x) = (x, y)$ . By CS,

$$\|f_y\|_{H^*} = \sup_{\|x\| \leq 1} (x, y) \leq \sup_{\|x\| \leq 1} \|x\| \|y\| \leq \|y\|.$$

Also, if  $y \neq 0$ , then

$$f_y\left(\frac{y}{\|y\|}\right) = \left(\frac{y}{\|y\|}, y\right) = \|y\|.$$

Thus,  $\|f_y\|_{H^*} = \|y\|_H$ . It turns out all such functionals are of this form.

↪ **Theorem 2.8** (Riesz Representation for Hilbert Spaces): If  $f \in H^*$ , there exists a unique  $y \in H$  such that  $f(x) = (x, y)$  for every  $x \in X$ .

PROOF. We show first existence. If  $f \equiv 0$ , then  $y = 0$ . Otherwise, let  $M = \{x \in X \mid f(x) = 0\}$ , so  $M \subsetneq H$ .  $f$  linear, so  $M$  a linear subspace.  $f$  is continuous, so in addition  $M$  is closed. By the previous theorem,  $M^\perp \neq \{0\}$ . Let  $z \in M^\perp$  of norm 1.

Fix  $x \in H$ , and define

$$u := f(x)z - f(z)x.$$

Then, notice that by linearity

$$f(u) = f(x)f(z) - f(z)f(x) = 0,$$

so  $u \in M$ . Thus, since  $z \in M^\perp$ ,  $(u, z) = 0$ , so in particular,

$$\begin{aligned} (u, z) = 0 &= (f(x)z - f(z)x - z, z) \\ &= f(x)(z, z) - f(z)(x, z) \\ &= f(x)\|z\|^2 - (x, f(z)z) \\ &= f(x) - (x, f(z)z), \end{aligned}$$

hence, rearranging we find

$$f(x) = (x, f(z)z),$$

and thus letting  $y = f(z)z$  completes the proof of existence, noting  $z$  independent of  $x$ .

For uniqueness, suppose  $(x, y) = (x, y')$  for every  $x \in X$ . Then,  $(x, y - y') = 0$  for every  $x \in X$ , hence letting  $x = y - y'$  we conclude  $(y - y', y - y') = 0$  thus  $y - y' = 0$  so  $y = y'$ , and uniqueness holds. ■

↪ **Definition 2.10** (Orthonormal Set): A collection  $\{e_j\} \subseteq H$  is *orthonormal* if  $(e_i, e_j) = \delta_i^j$ .

**Remark 2.7:** The following section writes notations assuming  $H$  has a countable. However, for more general Hilbert spaces, all countable summations can be replaced with uncountable ones in which only countably many elements are nonzero. The theory is very similar.

↪ **Definition 2.11** (Orthonormal Basis): A collection  $\{e_j\} \subseteq H$  is an *orthonormal basis* for  $H$  if  $\{e_j\}$  is an orthonormal set, and  $x = \sum_{j=1}^{\infty} (x, e_j) e_j$  for every  $x \in H$ , in the sense that

$$\left\| x - \sum_{j=1}^N (x, e_j) e_j \right\| \rightarrow 0, \quad N \rightarrow \infty.$$

↪ **Theorem 2.9** (General Pythagorean Theorem): If  $\{e_j\}_{j=1}^{\infty} \subseteq H$  are orthonormal and  $\{\alpha_i\}_{i=1}^{\infty} \subseteq \mathbb{R}$  are orthonormal, then for any  $N$ ,

$$\left\| \sum_{i=1}^N \alpha_i e_i \right\|^2 = \sum_{i=1}^N |\alpha_i|^2.$$

PROOF.

$$\left\| \sum_{i=1}^N \alpha_i e_i \right\|^2 = \left( \sum_{i=1}^N \alpha_i e_i, \sum_{j=1}^N \alpha_j e_j \right) = \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \underbrace{(e_i, e_j)}_{=\delta_i^j} = \sum_{i=1}^N \alpha_i^2.$$

We can also **Gram-Schmidt** in infinite-dimensional Hilbert spaces. Let  $\{x_i\} \subseteq H$ . Let

$$e_1 = \frac{x_1}{\|x_1\|},$$

and inductively, for any  $n \geq 2$ , define

$$v_N = x_N - \sum_{i=1}^{N-1} (x_N, e_i) e_i.$$

Then, for any  $N$ ,  $\text{span}(v_1, \dots, v_N) = \text{span}(e_1, \dots, e_N)$ , and for any  $j < N$ ,

$$(v_N, e_j) = (x_N, e_j) - \sum_{i=1}^N (x_N, e_i) (e_i, e_j) = (x_N, e_j) - (x_N, e_j) = 0.$$

Let then  $e_N = \frac{v_N}{\|v_N\|}$ . Then,  $\{e_i\}_{i=1}^\infty$  will be orthonormal; we discuss how to establish when this set will actually be a basis to follow.

↪ **Theorem 2.10** (Bessel's Inequality): If  $\{e_i\}_{i=1}^\infty$  are orthonormal, then for any  $x \in H$ ,

$$\sum_{i=1}^{\infty} |(x, e_i)|^2 \leq \|x\|^2.$$

PROOF. We have

$$\begin{aligned} 0 &\leq \left\| x - \sum_{i=1}^N (x, e_i) e_i \right\|^2 \\ &= \left( x - \sum_{i=1}^N (x, e_i) e_i, x - \sum_{j=1}^N (x, e_j) e_j \right) \\ &= \|x\|^2 - 2 \sum_{i=1}^N (x, e_i)^2 + \sum_{i=1}^N (x, e_i)^2 \\ &= \|x\|^2 - \sum_{i=1}^N (x, e_i)^2, \end{aligned}$$

so  $\sum_{i=1}^N (x, e_i)^2 \leq \|x\|^2$ ; letting  $N \rightarrow \infty$  proves the desired inequality, since the RHS is independent of  $N$ . ■

↪ **Theorem 2.11**: If  $\{e_i\}_{i=1}^\infty$  are orthonormal, then TFAE:

- (a) completeness: if  $(x, e_i) = 0$  for every  $i$ , then  $x = 0$ , the zero vector;
- (b) Parseval's identity holds:  $\|x\|^2 = \sum_{i=1}^\infty (x, e_i)^2$  for every  $x \in H$ ;
- (c)  $\{e_i\}_{i=1}^\infty$  form a basis for  $H$ , i.e.  $x = \sum_{i=1}^\infty (x, e_i) e_i$  for every  $x \in H$ .

PROOF. ((a)  $\Rightarrow$  (c)) By Bessel's,  $\sum_{i=1}^\infty (x, e_i)^2 \leq \|x\|^2$ . So, for any  $M \geq N$ ,

$$\left\| \sum_{i=N}^M (x, e_i) e_i \right\|^2 = \sum_{i=N}^M (x, e_i)^2,$$

which must converge to zero as  $N, M \rightarrow \infty$ , since the whole series converges (being bounded). Hence,  $\left\{ \sum_{i=1}^N (x, e_i) e_i \right\}_N$  is Cauchy in  $\|\cdot\|$  and since  $H$  complete,  $\sum_{i=1}^\infty (x, e_i) e_i$  converges in  $H$ . Putting  $y = x - \sum_{i=1}^\infty (x, e_i) e_i$ , we find

$$(y, e_i) = (x, e_i) - (x, e_i) = 0 \quad \forall i,$$

hence by assumption in (a), it follows that  $y = 0$  so  $x = \sum_{i=1}^\infty (x, e_i) e_i$  and thus  $\{e_i\}$  a basis for  $H$  and (c) holds.

((c)  $\Rightarrow$  (b)) Since  $x = \sum_{i=1}^\infty (x, e_i) e_i$ , then,

$$\|x\|^2 - \sum_{i=1}^N (x, e_i)^2 = \left\| x - \sum_{i=1}^N (x, e_i) e_i \right\|^2 \rightarrow 0$$

as  $N \rightarrow \infty$ , hence  $\|x\|^2 = \sum_{i=1}^{\infty} (x, e_i)^2$ .

((b)  $\Rightarrow$  (a)) If  $(x, e_i) = 0$  for every  $i$ , then by Parseval's  $\|x\|^2 = \sum_{i=1}^{\infty} 0 = 0$  so  $x = 0$ . ■

**Remark 2.8:** (a) is equivalent to  $\text{span}(e_1, e_2, \dots)$  is *dense* in  $H$ .

↪ **Theorem 2.12:** Every Hilbert space has an orthonormal basis.

PROOF. Let  $\mathcal{F} = \{\text{orthonormal subsets of } H\}$ .  $\mathcal{F}$  can be (partially) ordered by inclusion, as can be upper bounded by the union over the whole space. By Zorn's Lemma, there is a maximal set in  $\mathcal{F}$ , which implies completeness, (a). ■

↪ **Proposition 2.5:**  $H$  is separable iff  $H$  has a countable basis.

PROOF. ( $\Leftarrow$ ) If  $H$  has a countable basis  $\{e_j\}$ ,  $\text{span}_{\mathbb{Q}}\{e_j\}$  is a countable dense set.

( $\Rightarrow$ ) If  $H$  is separable, let  $\{x_n\}$  be a countable dense set. Use Gram-Schmidt, to produce a countable, orthonormal set, which is dense and hence a (countable) basis for  $H$ . ■

**Remark 2.9:** All this can be extended to uncountable bases.

## §2.6 Adjoints, Duals and Weak Convergence (for Hilbert Spaces)

First consider  $T : H \rightarrow H$  bounded and linear. Fix  $y \in H$ . We claim that the map

$$x \mapsto (T(x), y)$$

belongs to  $H^*$ , namely is bounded and linear. Linearity is clear since  $T$  linear. We know by Cauchy-Schwarz that

$$|(T(x), y)| \leq \|T(x)\| \|y\| \leq \|T\| \|x\| \|y\| \leq C \|x\|,$$

so indeed  $x \mapsto (T(x), y) \in H^*$ . By Riesz Representation Theorem, there is some unique  $z \in H$  such that

$$(T(x), y) = (x, z) \quad \forall x \in H.$$

This motivates the following.

↪ **Definition 2.12** (Adjoint of  $T$ ): Let  $T^* : H \rightarrow H$  be defined by

$$(Tx, y) = (x, T^*y), \quad \forall x, y \in H.$$



**Remark 2.10:** In finite dimensions,  $T$  can be identified with some  $n \times n$  matrix, in which case  $T^* = T^t$ , the transpose of  $T$ ; namely  $Tx \cdot b = x \cdot T^t b$ .

↪ **Proposition 2.6:** If  $T \in \mathcal{L}(H) := \mathcal{L}(H, H)$ , then  $T^* \in \mathcal{L}(H)$  and  $\|T^*\| = \|T\|$ .

PROOF. Linearity of  $T^*$  is clear. Also, for any  $\|y\| \leq 1$ ,

$$\|T^*y\|^2 = (T^*y, T^*y) = (TT^*y, y) \leq \|T\|\|T^*(y)\|\|y\|$$

so  $\|T^*y\| \leq \|T\|$  for all  $\|y\| = 1$ . so  $\|T^*\| \leq \|T\|$  hence  $T^* \in \mathcal{L}(H)$ . But also, if  $x \in H$  with  $\|x\| = 1$ , then symmetrically,

$$\|Tx\|^2 = (Tx, Tx) = (x, T^*Tx) \leq \|T^*\|\|Tx\|$$

so similarly  $\|T\| \leq \|T^*\|$  hence equality holds. ■

↪ **Proposition 2.7:**  $(T^*)^* = T$ .

PROOF. On the one hand,

$$(T^*y, x) = (y, (T^*)^*x) = ((T^*)^*x, y)$$

while also

$$(T^*y, x) = (x, T^*y) = (Tx, y)$$

so  $(Tx, y) = ((T^*)^*x, y)$ , from which it follows that  $T = T^{**}$ . ■

↪ **Proposition 2.8:**  $(T + S)^* = T^* + S^*$ , and  $(T \circ S)^* = S^* \circ T^*$ .

We'll write  $N(T)$  for the nullspace/kernel of  $T$ , and  $R(T)$  for the range/image of  $T$ .

↪ **Proposition 2.9:** Suppose  $T \in \mathcal{L}(H)$ . Then,

- $N(T^*) = R(T)^\perp$  (and hence, if  $R(T)$  closed,  $H = N(T^*) \oplus R(T)$ );
- $N(T) = R(T^*)^\perp$  (and hence, if  $R(T^*)$  closed,  $H = N(T) \oplus R(T^*)$ ).

PROOF.  $N(T^*) = \{y \in H : T^*y = 0\}$ , so if  $y \in N(T^*)$ ,  $(Tx, y) = (x, T^*y) = (x, 0) = 0$ , which holds iff  $y$  orthogonal to  $Tx$ , and since this holds for all  $x \in H$ ,  $y \in R(T)^\perp$ .

Then, if  $R(T)$  closed, the by orthogonal decomposition we'll find  $H = R(T) \oplus R(T)^\perp = R(T) \oplus N(T^*)$ .

The other claim follows similarly. ■

**Remark 2.11:** Recall that  $R(T)^\perp$  is closed; hence

$$(R(T)^\perp)^\perp = \{z \in H \mid (y, z) = 0 \forall y \in R(T)^\perp\},$$

and is also closed; hence  $(R(T)^\perp)^\perp = \overline{R(T)}$  thus equivalently  $N(T^*)^\perp = \overline{R(T)}$ .

**Remark 2.12:** By the Closed Graph Theorem,  $T$  linear and bounded gives  $T$  closed; namely, the graph of  $T$  closed; this is *not* the same as saying the range of  $T$  closed.

⊗ **Example 2.3:** Consider  $C([0, 1]) \subseteq L^2([0, 1])$ , and  $T : C([0, 1]) \rightarrow L^2([0, 1])$  given by the identity,  $Tf = f$ . Then,  $T$  is bounded, but  $R(T) = C([0, 1])$ ; this subspace is *not* closed in  $L^2([0, 1])$ , since there exists sequences of continuous functions that converge to an  $L^2$ , but not continuous, function.

**Remark 2.13:** The prior theorem is key in “solvability”, especially if  $T$  a differential or integral operator. If we wish to find  $u$  such that  $Tu = f$ , we need that  $f \in R(T)$ , hence  $f \in N(T^*)^\perp$ .

⊗ **Example 2.4:** Let  $M \subsetneq H$  a closed linear subspace. Then,  $H = M \oplus M^\perp$ ; define the projection operator

$$P : H \rightarrow H, \quad x = u + v \in M \oplus M^\perp \mapsto u.$$

This means, in particular,  $x = Px + (\text{id} - P)x$ . We claim  $P \in \mathcal{L}(H)$ ,  $\|P\| = 1$ ,  $P^2 = P$ , and  $P^* = P$ .

Linearity is clear. To show  $P^2 = P$ , write  $x = Px + v$ . Then, composing both sides with  $P$ , we find  $Px = P^2x + Pv = P^2x$ , so  $Px = P^2x$  for every  $x \in H$ . To see the norm, we find that for every  $x \in H$ ,

$$\begin{aligned} \|x\|^2 &= (x, x) = (Px + (\text{id} - P)x, Px + (\text{id} - P)x) \\ &= \|Px\|^2 + 2\underbrace{(Px, (\text{id} - P)x)}_{\perp} + \|(\text{id} - P)x\|^2 \\ &= \|Px\|^2 + \|(\text{id} - P)x\|^2 \geq \|Px\|^2 \\ &\Rightarrow \|Px\| \leq \|x\| \Rightarrow \|P\| \leq 1, \end{aligned}$$

and moreover if  $x \in M$ ,  $Px = x$  so  $\|Px\| = \|x\|$  hence  $\|P\| = 1$  indeed.

Finally, to show  $P$  self-adjoint, let  $x, y \in H$ , then,

$$0 = (Px, (\text{id} - P)y) = (Px, y - Py) \Rightarrow (Px, y) = (Px, Py).$$

Symmetrically,  $(x, Py) = (Px, Py)$ , hence  $(Px, y) = (x, Py)$ , and so  $P = P^*$ .

## §2.7 Introduction to Weak Convergence

We let throughout  $X$  be a Banach space.

↪ **Definition 2.13** (Weak convergence): We say  $\{x_n\} \subseteq X$  converges weakly to  $x \in X$ , and write

$$x_n \rightharpoonup x$$

iff for every  $f \in X^* = \{f : X \rightarrow \mathbb{R} \text{ bounded, linear}\}$ ,  $f(x_n) \rightarrow f(x)$ .

↪ **Definition 2.14** (Weak topology  $\sigma(X, X^*)$ ): The weak topology  $\sigma(X, X^*)$  is the weak topology induced by

$$\mathcal{F} = X^*.$$

In particular, this is the smallest topology in which every  $f$  continuous.

Recall that this was defined as being  $\tau(\{f^{-1}(\mathcal{O})\})$  for  $\mathcal{O}$  open in  $\mathbb{R}$ . A base for this topology is given by  $\mathcal{B} = \{\text{finite intersections of } \{f^{-1}(\mathcal{O})\}\}$ . Namely, let  $\mathcal{B}_X := \{B_{\varepsilon, f_1, f_2, \dots, f_n}(x)\}$  where

$$B_{\varepsilon, f_1, f_2, \dots, f_n}(x) = \{x' \in X \mid |f_k(x') - f_k(x)| < \varepsilon, \forall 1 \leq k \leq n\}.$$

So,  $x_n \rightarrow x$  in  $\sigma(X, X^*)$  if for every  $\varepsilon > 0$ , and ball  $B_{\varepsilon, f_1, \dots, f_m}(x)$ , there is an  $N$  such that for every  $n \geq N$ ,  $x_n \in B_{\varepsilon, f_1, \dots, f_m}(x)$ , hence for every  $f \in X^*$ ,  $|f(x_n) - f(x)| < \varepsilon$ .

For Hilbert spaces, by Riesz we know  $f \in H^*$  can always be identified with  $f(x) = (x, y)$  for some  $y \in H$ . So, we find  $x_n \rightharpoonup x$  in  $H$  iff for every  $y \in H$ ,  $(x_n, y) \rightarrow (x, y)$ .

**Remark 2.14:** If  $x_n \rightarrow x$  in  $H$ , then  $(x_n, y) \rightarrow (x, y)$ ; so this normal convergence implies weak convergence.

↪ **Proposition 2.10:** (i) Suppose  $x_n \rightharpoonup x$  in  $H$ . Then,  $\{x_n\}$  are bounded in  $H$ , and  $\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$ .

(ii) If  $y_n \rightarrow y$  (strongly) in  $H$  and  $x_n \rightharpoonup x$  (weakly) in  $H$ , then  $(x_n, y_n) \rightarrow (x, y)$ .

**Remark 2.15:** It does *not* hold, though, that  $x_n \rightharpoonup x$ ,  $y_n \rightarrow y$  gives  $(x_n, y_n) \rightarrow (x, y)$ .

PROOF. (i) If  $x_n \rightharpoonup x$ , then

$$\left(x_n, \frac{x}{\|x\|}\right) \rightarrow \left(x, \frac{x}{\|x\|}\right) = \|x\|.$$

By Cauchy-Schwarz, we also have

$$\left|\left(x_n, \frac{x}{\|x\|}\right)\right| \leq \|x_n\| \left(\frac{\|x\|}{\|x\|}\right) = \|x_n\|,$$

hence we conclude

$$\liminf_{n \rightarrow \infty} \left( x_n, \frac{x}{\|x\|} \right) \leq \liminf_{n \rightarrow \infty} \|x_n\| \Rightarrow \|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|.$$

To argue  $\{x_n\}$  bounded, need the uniform boundedness principle. Let  $\{x_n\} \subseteq H^{**} = H$ . By weak convergence, for every  $f = f_y \in H^*$ ,  $f \mapsto f(x_n) = (x_n, y) \rightarrow (x, y)$ . So,

$$\sup_n f(x_n) \leq C.$$

Thus, the map  $f \mapsto f(x_n)$  a bounded linear operator on  $H^*$ , so by uniform boundedness  $\sup_n \|x_n\| \leq C$ .

(ii) If  $y_n \rightarrow y$  in  $H$ ,

$$\begin{aligned} |(x_n, y_n) - (x, y)| &\leq |(x_n, y_n - y)| + |(x_n - x, y)| \\ &\leq \underbrace{\|x_n\|}_{\text{bounded}} \underbrace{\|y_n - y\|}_{\rightarrow 0} + \underbrace{|(x_n - x, y)|}_{\rightarrow 0 \text{ by weak}} \rightarrow 0. \end{aligned}$$

■

The real help of weak convergence is in the ease of achieving weak compactness;

↪ **Theorem 2.13** (Weak Compactness): Every bounded sequence in  $H$  has a weakly convergent subsequence.

↪ **Theorem 2.14** (Helley's Theorem): Let  $X$  a separable normed vector space and  $\{f_n\} \subseteq X^*$  such that there is a constant  $C > 0$  such that  $|f_n(x)| \leq C\|x\|$  for every  $x \in X$  and  $n \geq 1$ . Then, there exists a subsequence  $\{f_{n_k}\}$  and an  $f \in X^*$  such that  $f_{n_k}(x) \rightarrow f(x)$  for every  $x \in X$ .

PROOF. This is essentially a specialization of the Arzelà-Ascoli lemma. To apply it, we need  $X$  separable (done), the sequence to be pointwise bounded (done), and the sequence to be equicontinuous. To verify this last one, we know that

$$\|f_n(x)\| \leq C\|x\| \Rightarrow \|f_n\| \leq C, \forall n \geq 1,$$

hence by linearity, for any  $x, y \in X$ ,

$$\|f_n(x) - f_n(y)\| \leq C\|x - y\|, \forall n \geq 1,$$

so in particular  $\{f_n\}$  uniformly Lipschitz, thus equicontinuous. ■

PROOF. (Of [Thm. 2.13](#)) Let  $\{x_n\} \subseteq H$  be bounded and let  $H_0 = \overline{\text{span}\{x_1, \dots, x_n, \dots\}}$ , so  $H_0$  is separable, and  $(H_0, (\cdot, \cdot))$  is a Hilbert space (being closed). Let  $f_n \in H_0^*$  be given by

$$f_n(x) = (x_n, x), \forall x \in H_0.$$

Then,

$$|f_n(x)| \leq \|x_n\| \|x\| \leq C\|x\|,$$

since  $\{x_n\}$  bounded by assumption. By Helly's Theorem, then, there is a subsequence  $\{f_{n_k}\}$  such that  $f_{n_k}(x) \rightarrow f(x)$  for every  $x \in H_0$ , where  $f \in H_0^*$ . By Riesz, then,  $f(x) = (x, x_0)$  for some  $x_0 \in H_0^*$ . This implies

$$(x_{n_k}, x) \rightarrow (x_0, x), \forall x \in H_0.$$

Let  $P$  the projection of  $H$  onto  $H_0$ . Then, for every  $x \in H$ ,

$$(x_{n_k}, (\text{id} - P)x) = (x_0, (\text{id} - P)x) = 0$$

so for any  $x \in H$ ,

$$\begin{aligned} \lim_{k \rightarrow \infty} (x_{n_k}, x) &= \lim_{k \rightarrow \infty} (x_{n_k}, Px + (\text{id} - P)x) \\ &= \lim_{k \rightarrow \infty} (x_{n_k}, \underbrace{Px}_{\in H_0}) \\ &= (x_0, Px) = (x_0, Px + (\text{id} - P)x) = (x_0, x), \end{aligned}$$

as we aimed to show. ■

## §2.8 Review of $L^p$ Spaces

We always consider  $\Omega \subseteq \mathbb{R}^d$ .

↪ **Definition 2.15** ( $L^p(\Omega)$ ): For  $1 \leq p < \infty$ , define

$$L^p(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{R} \mid f \text{ measurable and } \int_{\Omega} |f|^p dx < \infty \right\},$$

endowed with the norm

$$\|f\|_{L^p(\Omega)} = \|f\|_p := \left[ \int_{\Omega} |f(x)|^p dx \right]^{\frac{1}{p}}.$$

For  $p = \infty$ , define

$$L^\infty(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ measurable and } \exists C < \infty \text{ s.t. } |f| \leq C \text{ a.e.}\},$$

endowed with the norm

$$\|f\|_{L^\infty(\Omega)} = \|f\|_\infty := \inf\{C : |f| \leq C \text{ a.e.}\}.$$

The following are recalled but not proven here, see [here](#).

↪ **Theorem 2.15** (Holder's Inequality): For  $1 \leq p, q \leq \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , then if  $f \in L^p(\Omega)$ ,  $g \in L^q(\Omega)$ , then  $fg \in L^1(\Omega)$ , and

$$\int |fg| dx \leq \|f\|_p \|g\|_q.$$

↪ **Theorem 2.16** (Minkowski's Inequality): For all  $1 \leq p \leq \infty$ ,  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ . In particular,  $L^p(\Omega)$  is a normed vector space.

↪ **Theorem 2.17** (Riesz-Fischer Theorem):  $L^p(\Omega)$  is a Banach space for every  $1 \leq p \leq \infty$ .

↪ **Theorem 2.18**:  $C_c(\mathbb{R}^d)$ , the space of continuous functions with compact support, simple functions, and step functions are all dense subsets of  $L^p(\mathbb{R}^d)$ , for every  $1 \leq p < \infty$ .

↪ **Theorem 2.19** (Separability of  $L^p(\Omega)$ ):  $L^p$  is separable, for every  $1 \leq p < \infty$ .

PROOF. We prove for  $\Omega = \mathbb{R}^d$ . Let

$$\mathcal{R} := \left\{ \prod_{i=1}^d (a_i, b_i) \mid a_i, b_i \in \mathbb{Q} \right\},$$

and let

$$\mathcal{E} := \{\text{finite linear combinations of } \chi_R \text{ for } R \in \mathcal{R} \text{ with coefficients in } \mathbb{Q}\},$$

where  $\chi_R$  the indicator function of the set  $R$ . Then, we claim  $\mathcal{E}$  dense in  $L^p(\mathbb{R}^d)$ .

Given  $f \in L^p(\mathbb{R}^d)$  and  $\varepsilon > 0$ , by density of  $C_c(\mathbb{R}^d)$  there is some  $f_1$  with  $\|f - f_1\|_p < \varepsilon$ . Let  $\text{supp}(f_1) \subseteq R \in \mathcal{R}$ . Now, let  $\delta > 0$ . Write

$$R = \cup_{i=1}^N R_i, \quad R_i \in \mathcal{R},$$

such that

$$\text{osc}_{R_i}(f_1) := \sup_{R_i} f_1 - \inf_{R_i} f_1 < \delta.$$

Then, let

$$f_2(x) = \sum_{i=1}^N q_i \chi(R_i), \quad q_i \in \mathbb{Q} \text{ s.t. } q_i \approx f_1|_{R_i},$$

so

$$\|f_2 - f_1\|_\infty < \delta.$$

Hence,

$$\begin{aligned} \|f_2 - f_1\|_p &\leq \left( \int_R |f_2(x) - f_1(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq \|f_1 - f_2\|_\infty \cdot m(R)^{\frac{1}{p}} < \delta \cdot m(R)^{\frac{1}{p}}, \end{aligned}$$

where  $m$  the Lebesgue measure on  $\mathbb{R}^d$ .  $\delta$  was arbitrary so we may take it arbitrarily small such that  $\delta m(R)^{\frac{1}{p}} < \varepsilon$ , hence for such a  $\delta$ ,

$$\|f - f_2\|_p \leq \|f - f_1\|_p + \|f_1 - f_2\|_p < 2\varepsilon.$$

Now,  $f_2 \in \mathcal{E}$ , and thus  $\mathcal{E}$  is dense in  $L^p(\mathbb{R}^d)$ , and countable by construction, thus  $L^p(\mathbb{R}^d)$  separable. ■

**Remark 2.16:**  $L^\infty(\Omega)$  is *not* separable, and  $C_c(\mathbb{R}^d)$  is *not* dense in  $L^\infty(\Omega)$ .

**Remark 2.17** (Special Cases):

- If  $\Omega$  has finite measure,  $L^p(\Omega) \subseteq L^{p'}(\Omega)$  for every  $p \leq p'$ .
- $\ell^p := \left\{ a = (a_n)_{n=1}^\infty \mid \sum_{n=1}^\infty |a_n|^p < \infty \right\}$  endowed with the norm  $\|a\|_\ell^p := \left( \sum_{n=1}^\infty |a_n|^p \right)^{1/p}$ .

## §2.9 $(L^p)^*$ : The Riesz Representation Theorem

We are interested in functions  $T : L^p(\Omega) \rightarrow \mathbb{R}$  which is bounded and linear. For instance, let  $g \in L^q(\Omega)$  and  $f \in L^p(\Omega)$  where  $p, q$  conjugates, and define

$$T(f) := \int_{\Omega} f(x)g(x) \, dx.$$

This is clearly linear, and by Holders,

$$|Tf| = \left| \int_{\Omega} fg \right| \leq \|f\|_p \|g\|_q.$$

so

$$\left| T\left(\frac{f}{\|f\|_p}\right) \right| \leq \|g\|_q, \quad \forall f \in L^p(\Omega), \Rightarrow \|T\| \leq \|g\|_q,$$

and thus  $T \in (L^p(\Omega))^*$ . Moreover, if  $1 < p < \infty$ ,  $1 < q < \infty$ , let

$$f(x) = \frac{|g(x)|^{q-2}g(x)}{\|g\|_q^{q-1}}.$$

Then,

$$\begin{aligned} \int_{\Omega} |f(x)|^p \, dx &= \frac{1}{\|g\|_q^{(q-1)p}} \int_{\Omega} |g(x)|^{(q-2)p} |g(x)|^p \, dx \\ &= \frac{1}{\|g\|_q^{(q-1)p}} \int_{\Omega} |g(x)|^{qp-p} \, dx. \end{aligned}$$

Since  $\frac{1}{p} + \frac{1}{q} = 1$ , we have  $q + p = pq$ , so further

$$= \frac{1}{\|g\|_q^q} \int_{\Omega} |g(x)|^q \, dx = \frac{1}{\|g\|_q^q} \cdot \|g\|_q^q = 1,$$

so  $f$  as defined indeed in  $L^p(\Omega)$  and moreover has  $L^p$ -norm of 1. In addition,

$$\begin{aligned} |Tf| &= \frac{1}{\|g\|_q^{q-1}} \int_{\Omega} |g(x)^{q-2}| g(x) g(x) \, dx \\ &= \frac{1}{\|g\|_q^{q-1}} \int_{\Omega} |g(x)|^q \, dx \\ &= \frac{1}{\|g\|_q^{q-1}} \|g\|_q^q = \|g\|_q, \end{aligned}$$

so  $\|T\| = \|g\|_q$  as desired. We have, more generally, akin to the Riesz representation theorem,

↪ **Theorem 2.20** (Riesz-Representation Theorem for  $L^p(\Omega)$ ): Let  $1 \leq p < \infty$ . For any  $T \in (L^p(\Omega))^*$ , there exists a unique  $g \in L^q(\Omega)$  such that  $T(f) = \int_{\Omega} f(x)g(x) \, dx$  with  $\|T\| = \|g\|_q$ .

We'll only prove for  $\Omega \subseteq \mathbb{R}$ . First:

↪ **Proposition 2.11**: Let  $T, S \in (L^p(\Omega))^*$ . If  $T = S$  on a dense subset  $E \subseteq L^p(\Omega)$ , then  $T = S$  everywhere.

PROOF. Let  $f_0 \in L^p(\Omega)$ . By density, there exists  $\{f_n\} \subseteq E$  such that  $f_n \rightarrow f$  in  $L^p(\Omega)$ . By continuity,  $Tf_n \rightarrow Tf_0$  and  $Sf_n \rightarrow Sf_0$ , while  $Tf_n = Sf_n$  for every  $n \geq 1$ , so by uniqueness of limits in  $\mathbb{R}$ ,  $Tf_0 = Sf_0$ . ■

The general outline of the proof of [Thm. 2.20](#) is the following:

- prove the theorem for  $f$  a step function;
- prove the theorem for  $f$  bounded and measurable;
- conclude the full theorem by appealing to the previous proposition.

To do this, we need first to recall the notion of *absolutely continuous functions*. Fix  $[a, b] \subseteq \mathbb{R}$  and  $G : [a, b] \rightarrow \mathbb{R}$ .  $G$  is said to be absolutely continuous on  $[a, b]$  if for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for every disjoint collection  $\{(a_k, b_k)\}_{k=1}^N \subseteq [a, b]$  with  $\sum_{k=1}^N (a_k - b_k) < \delta$ , then  $\sum_{k=1}^N |G(b_k) - G(a_k)| < \varepsilon$ . In particular, we need the following result, proven [here](#):

↪ **Theorem 2.21**: If  $G : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous, then  $g = G'$  exists a.e. on  $[a, b]$ ,  $g \in L^1([a, b])$ , and for every  $x \in [a, b]$ ,

$$G(x) - G(a) = \int_a^x g(t) \, dt.$$

PROOF (Of [Thm. 2.20](#) with  $\Omega = [a, b]$ ). Let  $T \in (L^p([a, b]))^*$ .

Step 1: Let  $f$  a step function. The function  $\chi_{[a, x]} \in L^p([a, b])$ ; define

$$G_T(x) := T(\chi_{[a, x]}).$$



We claim  $G_T$  absolutely continuous. Consider  $\{(a_k, b_k)\}_{k=1}^N$  disjoint. Then, for every  $[c, d] \subseteq [a, b]$ ,  $G_T(d) - G_T(c) = T(\chi_{[a,d]}) - T(\chi_{[a,c]}) = T(\chi_{[a,d]} - \chi_{[a,c]}) = T(\chi_{[c,d]})$ , so

$$\begin{aligned} \sum_{k=1}^N (G_T(b_k) - G_T(a_k)) &= \sum_{k=1}^N c_k \cdot (G_T(b_k) - G_T(a_k)), \quad c_k := \operatorname{sgn}(G_T(b_k) - G_T(a_k)) \\ &= \sum_{k=1}^N c_k \cdot T(\chi_{[a_k, b_k]}) \\ &= T\left(\sum_{k=1}^N c_k \chi_{[a_k, b_k]}\right) \\ &\leq \|T\| \left\| \sum_{k=1}^N c_k \chi_{[a_k, b_k]} \right\|_p. \end{aligned}$$

By the disjointedness of the intervals, we may write

$$\int_a^b \left| \sum_{k=1}^N c_k \chi_{[a_k, b_k]} \right|^p dx \leq \sum_{k=1}^N \int_{a_k}^{b_k} dx = \sum_{k=1}^N (b_k - a_k).$$

So,  $\left\| \sum_{k=1}^N c_k \chi_{[a_k, b_k]} \right\|_p = \left( \sum_{k=1}^N (b_k - a_k) \right)^{\frac{1}{p}}$ , thus

$$\sum_{k=1}^N |G_T(b_k) - G_T(a_k)| \leq \|T\| \cdot \left( \sum_{k=1}^N (b_k - a_k) \right)^{\frac{1}{p}}.$$

Hence, for  $\varepsilon > 0$ , letting  $\delta = \left( \frac{\varepsilon}{\|T\|} \right)^p$  proves absolute continuity of  $G_T$ . Thus,  $g = G'_T$  exists and is such that  $g \in L^1([a, b])$  and

$$G_T(x) = \int_a^x g(t) dt, \quad \forall x \in [a, b].$$

Hence,

$$\begin{aligned} T(\chi_{[c,d]}) &= G_T(d) - G_T(c) = \int_a^d g(t) dt - \int_a^c g(t) dt \\ &= \int_c^d g(t) dt \\ &= \int_a^b g(t) \cdot \chi_{[c,d]}(t) dt. \end{aligned}$$

This proves the theorem for indicator functions; by linearity of  $T$  and linearity of the integral, we can repeat this procedure to find a function  $g$  such that  $Tf = \int_a^b f(t)g(t) dt$  for every step function  $f$ .

*Step 2:* Let  $f$  bounded and measurable. We know that for every step function  $\psi$ ,  $T\psi = \int_a^b \psi(t)g(t) dt$  (with the  $g$  as “found” in step 1). So,

$$\begin{aligned}
\left| Tf - \int_a^b \psi(t)g(t) \right| &= \left| T(f - \psi) - \int_a^b (f(t) - \psi(t))g(t) dt \right| \\
&\leq \|T\| \|f - \psi\|_p + \int_a^b |f(t) - \psi(t)| |g(t)| dt.
\end{aligned}$$

Then, since  $g \in L^1([a, b])$ , for every  $\varepsilon > 0$  there is some  $\delta > 0$  such that if  $E$  a set of measure less than  $\delta$ ,  $\int_E |g(t)| dt < \varepsilon$ . Fix  $\varepsilon > 0$  and  $\delta > 0$  such that this holds; let  $\delta < \varepsilon$  if necessary wlog. Since  $f$  bounded and measurable, there is some step function  $\psi$  such that  $|f - \psi| < \delta$  on  $E \subseteq [a, b]$ , and that  $m(E^c) < \delta$  and  $|\psi| \leq \|f\|_\infty$ . Hence,

$$\begin{aligned}
\|f - \psi\|_p^p &= \int_E |f - \psi|^p + \int_{E^c} |f - \psi|^p \\
&\leq \delta^p \cdot m(E) + (2\|f\|_\infty)^p m(E^c) \\
&\leq \delta^p |b - a| + (2\|f\|_\infty)^p \delta.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_a^b |f - \psi| |g| dt &\leq \int_E \delta \cdot |g| dt + \int_{E^c} 2\|f\|_\infty |g| dt \\
&\leq \delta \|g\|_1 + 2\|f\|_\infty \varepsilon.
\end{aligned}$$

All together then,

$$\begin{aligned}
\left| Tf - \int_a^b f(t)g(t) dt \right| &\leq \|T\| \left( \delta^p |b - a| + (2\|f\|_\infty)^p \delta \right)^{\frac{1}{p}} + \delta \|g\|_1 + 2\|f\|_\infty \varepsilon \\
&< C(\|f\|_\infty, \|g\|_1, a, b, \|T\|) \cdot \varepsilon^{\frac{1}{p}},
\end{aligned}$$

where  $C$  a constant. The LHS does not depend on  $\varepsilon$ , hence taking the limit  $\varepsilon \rightarrow 0^+$ , we conclude

$$Tf = \int_a^b f(t)g(t) dt.$$

Note that all simple functions are bounded and measurable, so the necessary property also holds for  $f$  simple.

We need now to show  $g \in L^q([a, b])$  and  $\|g\| = \|T\|$ .

- Case 1:  $p > 1$  so  $q < \infty$ . Let  $g_n := \begin{cases} g & \text{if } |g| \leq n \\ 0 & \text{o.w.} \end{cases}$  and  $f_n := \begin{cases} |g|^{q-1} \operatorname{sgn}(g) & \text{if } |g| \leq n \\ 0 & \text{o.w.} \end{cases}$ . Then,

$$\begin{aligned}
\|g_n\|_q^q &= \int_{\{|g| \leq n\}} |g|^q \, dt \\
&= \int_{\{|g| \leq n\}} f_n \cdot g_n \, dt \\
&= \int_{\{|g| \leq n\}} f_n g \, dt \\
&= T f_n \leq \|T\| \|f_n\|_p,
\end{aligned}$$

since  $f_n$  bounded and measurable so Step 2 applies. Also,

$$\begin{aligned}
\|f_n\|_p^p &= \int_{\{|g| \leq n\}} |g|^{(q-1)p} \, dt \\
&= \int_{\{|g| \leq n\}} |g|^q \, dt = \|g_n\|_q^q.
\end{aligned}$$

All together then,

$$\|g_n\|_q^q \leq \|T\| \|g_n\|_q^{q/p} \Rightarrow \|g_n\|_q^{q(1-\frac{1}{p})} = \|g_n\|_q \leq \|T\|.$$

By construction,  $|g_n|^q \rightarrow |g|^q$  a.e. and monotonely, so by the monotone convergence theorem,

$$\|g_n\|_q \rightarrow \|g\|_q,$$

so  $\|g\|_q \leq \|T\|$  and so  $g \in L^q([a, b])$ . From here, as in the example at the beginning of this section, one can show equality by choosing  $f$  appropriately.

- Case 2:  $p = 1$  so  $q = \infty$ . We claim that  $\|g\|_\infty = \sup_{\substack{f \text{ bdd} \\ \|f\|_1=1}} \int f g$ . Let  $\varepsilon > 0$  and  $A \subseteq [a, b]$  such that  $|g| \geq \|g\|_\infty - \varepsilon$  on  $A$  where  $m(A) > 0$ . Let

$$f(x) = \frac{\chi_A}{m(A)} \operatorname{sgn}(g).$$

Then,  $f$  bounded and  $\|f\|_1 = 1$ . So,

$$\int f g = \frac{1}{m(A)} \int_A |g| \geq \frac{1}{m(A)} \int_A (\|g\|_\infty - \varepsilon) = \|g\|_\infty - \varepsilon,$$

hence we have proven  $\leq$  of our claim. By Holder,

$$\sup_{\|f\|=1} \int f g \leq \|f\|_1 \|g\|_\infty = \|g\|_\infty,$$

so  $\geq$  holds and the claim is proven. Thus,

$$\|g\|_\infty = \sup_{\substack{\|f\|=1, \\ f \text{ bdd}}} T f \leq \|T\| \|f\|_1 = \|T\|,$$

so in particular  $g \in L^\infty([a, b])$ . For the other inequality,

$$|Tf| = \left| \int fg \, dt \right| \leq \|f\|_1 \|g\|_\infty,$$

hence

$$\|T\| \leq \|g\|_\infty$$

so  $\|g\|_\infty = \|T\|$  as we aimed to show.

*Step 3:* We need to show  $Tf = \int_a^b fg \, dt$  for every  $f \in L^p([a, b])$ . Simple functions are dense in  $L^p([a, b])$ , and since  $Tf = \int_a^b fg \, dt$  for every simple function  $f$ , we conclude  $Tf = \int_a^b fg \, dt$  for every  $f \in L^p([a, b])$  by the previous density lemma.

Moreover,  $g$  is unique because if

$$\int_a^b fg = \int_a^b fg',$$

then

$$\int_a^b f(g - g') = 0,$$

for every  $f \in L^p$ . Let  $f(t) = \text{sgn}(g - g')$ , then

$$0 = \int_a^b |g - g'| \, dt \Rightarrow g = g' \text{ a.e..}$$

So,  $g$  uniquely defined up to a set of measure 0 so  $g = g'$  in  $L^q$ . ■

**PROOF (Of RRT if  $\Omega = \mathbb{R}$ ).** Fix  $T \in (L^p(\mathbb{R}))^*$ . Then,  $T|_{[-N, N]} \in (L^p([-N, N]))^*$  for every  $N \geq 1$ , and  $\|T|_{[-N, N]}\| \leq \|T\|$ . Then, by RRT on  $[-N, N]$ , there is a  $g_N \in L^q([-N, N])$  such that  $Tf = \int_{-N}^N fg_N \, dt$ . By uniqueness,  $g_{N+1}|_{[-N, N]} = g_N$ . Define

$$g(t) := g_N(t), \quad t \in [-N, N].$$

So,  $g_N(t) \rightarrow g(t)$  pointwise and  $|g_N(t)|^q \rightarrow |g(t)|^q$  pointwise and monotonely. By monotone convergence, then,  $\int_{\mathbb{R}} |g_N|^q \, dt \rightarrow \int_{\mathbb{R}} |g|^q \, dt$ . So,  $g \in L^q(\mathbb{R})$  since  $\|g_N\|_{L^q([-N, N])} \leq \|T\|$  for every  $N \geq 1$ . Let  $f_N(t) = f(t)\chi_{[-N, N]}$ . Then,  $f_N \rightarrow f$  in  $L^p(\mathbb{R})$  so  $Tf_N \rightarrow Tf$ . So also

$$Tf_N = \int_{-N}^N f_N g_N = \int_{-N}^N f(t)g_N(t) \, dt = \int_{\mathbb{R}} fg_N \, dt \rightarrow Tf,$$

if we take by convention the  $g_N$ 's to be zero outside of  $[-N, N]$ . But also,  $f \in L^p(\mathbb{R})$  and  $g_N \rightarrow g$  in  $L^q(\mathbb{R})$ , so applying Holder's to the quantity  $\int_{\mathbb{R}} fg_N$ , we know

$$\int_{\mathbb{R}} fg_N \rightarrow \int_{\mathbb{R}} fg,$$

hence equating the two

$$Tf = \int_{\mathbb{R}} fg,$$

for every  $f \in L^p(\mathbb{R})$ . A similar proof to the previous gives the necessary norm identity. ■

**PROOF (Of RRT for general  $\Omega \subseteq \mathbb{R}$ ).** If  $T \in (L^p(\Omega))^*$ , let  $\hat{T} \in (L^p(\mathbb{R}))^*$  given by  $\hat{T}f = T(f|_\Omega)$ . Then by the previous case there is  $\hat{g} \in L^q(\mathbb{R})$  such that  $\hat{T}(f) = \int f \hat{g}$ . Let  $g = \hat{g}|_\Omega$ , then  $Tf = \int_\Omega fg$ . ■

So, RRT gives us that for  $p \in [1, \infty]$ ,  $(L^p(\Omega))^* \sim L^q(\Omega)$ , and that  $\|f\|_p = \sup_{\|g\|_q=1} |\int fg|$ .

In particular, if  $p = 1$ ,

$$\|f\|_{L^1} = \int f \operatorname{sgn} f(x) dx = \sup_{\|g\|_\infty=1} \int fg.$$

What, though, is  $(L^\infty)^*$ . Certainly,  $L^1(\Omega) \subseteq (L^\infty(\Omega))^*$  since for  $f \in L^\infty$ ,  $Tf = \int fg dx$  with  $g \in L^1$ , which is bounded by Hölder's. However, it turns out that this inclusion is a strict one. Consider for instance

$$Tf := f(0), \quad T : L^\infty([-1, 1]) \rightarrow \mathbb{R}.$$

Then, certainly  $|Tf| \leq \|f\|_\infty$  so  $T \in (L^\infty)^*$ . However, there is no function  $g$  such that  $f(0) = \int f(t)g(t) dt$ .

## §2.10 Weak Convergence in $L^p(\Omega)$

↪ **Definition 2.16** (Weak convergence in  $L^p(\Omega)$ ): Let  $\Omega \subset \mathbb{R}^d$ ,  $p \in [1, \infty)$  and  $q$  its conjugate. Then, we say  $f_n \rightarrow f$  *weakly* in  $L^p(\Omega)$ , and write

$$f_n \xrightarrow{L^p(\Omega)} f,$$

if for every  $g \in L^q(\Omega)$ ,

$$\lim_{n \rightarrow \infty} \int_\Omega f_n g dx = \int_\Omega fg dx.$$

**Remark 2.18:** Weak limits are unique; suppose otherwise that  $f_n \rightharpoonup f, \bar{f}$ . Let  $g = \operatorname{sgn}(f - \bar{f}) \cdot |f - \bar{f}|^{p-1}$ , which is in  $L^q(\Omega)$ . So,

$$\lim_n \int g f_n dx = \int g f dx = \int g \bar{f} dx,$$

by assumption, so

$$0 = \int_\Omega g(f - \bar{f}) dx = \int |f - \bar{f}|^p dx,$$

hence  $f = \bar{f}$  a.e. (and so equal as elements of  $L^p(\Omega)$ ).

**Remark 2.19:** Many of the properties of weakly convergent sequences in a Hilbert space carry over to this setting.

↪ **Proposition 2.12:** Let  $\Omega \subseteq \mathbb{R}^d$ .

- (i) If  $p \in (1, \infty)$ ,  $f_n \xrightarrow{L^p(\Omega)} f$ , then  $\{f_n\} \subseteq L^p(\Omega)$  are bounded, and moreover  $\|f\|_p \leq \liminf_n \|f_n\|_p$ .
- (ii) If  $p \in [1, \infty)$  and  $f_n \xrightarrow{L^p(\Omega)} f$ ,  $g_n \xrightarrow{L^p(\Omega)} g$ , then  $\lim_{n \rightarrow \infty} \int g_n f_n dx = \int g f dx$ .

PROOF. Identical to Hilbert space proofs; replace usage of Cauchy-Schwarz with Holder's. ■

**Remark 2.20:** In (i),  $p \in (1, \infty)$ , since  $L^p$  “reflexive” in this case, i.e.  $(L^p)^{**} = L^p$  (just as we had in the Hilbert space case). We don't have this property for  $p = 1$ .

**Remark 2.21:** A related notion of convergence is called *weak\* convergence*, written  $f_n \xrightarrow{L^p(\Omega)^*} f$ ; we say this holds if for every  $g \in L^q(\Omega)$  such that  $(L^q)^* = L^p$ , then  $\int f_n g dx \rightarrow \int f g dx$ . So if  $p \in (1, \infty)$ , weak convergence = weak\* convergence, by Riesz.

**Remark 2.22:** There are many equivalent notions to weak convergence.

↪ **Theorem 2.22** (Equivalent Weak Convergence): Let  $p \in (1, \infty)$ . Suppose  $\{f_n\} \subseteq L^p(\Omega)$  are bounded and  $f \in L^p$ . Then,  $f_n \xrightarrow{L^p(\Omega)} f$  iff

- for any  $g \in G \subseteq L^q(\Omega)$  such that  $\overline{\text{span}(G)} = L^q(\Omega)$ , then  $\lim_{n \rightarrow \infty} \int f_n g = \int f g$ ;
- $\forall A \subseteq \Omega$  measurable with finite measure, then  $\lim_{n \rightarrow \infty} \int_A f_n dx = \int_A f dx$ ;
- if  $d = 1$  and  $\Omega = [a, b]$ , then  $\lim_{n \rightarrow \infty} \int_a^x f_n dx = \int_a^x f dx$  for every  $x \in [a, b]$ .
- $f_n \rightarrow f$  pointwise a.e..

**Remark 2.23:** Some of these notions extend to  $p = 1$ , but we state in the  $p > 1$  case for simplicity.

↪ **Theorem 2.23** (Radon-Riesz): Let  $p \in (1, \infty)$ . Suppose  $f_n \xrightarrow{L^p(\Omega)} f$ , then  $f_n \xrightarrow{L^p(\Omega)} f$  iff  $\lim_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p$ .

Alternatively, there exists a subsequence  $\{f_{n_k}\}$  such that  $f_{n_k} \rightarrow f$  in  $L^p(\Omega)$  iff  $\liminf_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p$ .

PROOF. ( $\Rightarrow$ ) If  $f_n \xrightarrow{L^p(\Omega)} f$  then  $\|f_n\|_p \rightarrow \|f\|_p$  by triangle inequality.

The converse, ( $\Leftarrow$ ), is hard. ■

**↪ Theorem 2.24** (Weak Compactness): Let  $p \in (1, \infty)$ , then every bounded sequence in  $L^p(\Omega)$  has a weakly convergent subsequence, with limit in  $L^p(\Omega)$ .

PROOF. Let  $\{f_n\} \subseteq L^p(\Omega)$  be bounded.  $p \in (1, \infty)$  so so is  $q$ , and in particular  $L^q(\Omega)$  is separable. Let  $T_n \in (L^q(\Omega))^*$  be given by  $T_n(g) := \int f_n g \, dx$  for  $g \in L^q(\Omega)$ . Then,  $\|T_n\| = \|f_n\|_p \leq C$ . So,

$$\sup_n |T_n(g)| \leq \|T_n\| \|g\|_q \leq C \|g\|_q.$$

By Helley's Theorem (Thm. 2.14), there exists a subsequence  $\{T_{n_k}\}$  and  $T \subseteq (L^q(\Omega))^*$  such that  $\lim_{k \rightarrow \infty} T_{n_k}(g) = T(g)$  for every  $g \in L^q(\Omega)$ . By Riesz, there exists some  $f \in L^p(\Omega)$  such that  $T(g) = \int f g \, dx$ , and hence

$$\lim_k \int f_{n_k} g \, dx = \int f g \, dx,$$

for every  $g \in L^q(\Omega)$ , so  $f_{n_k} \xrightarrow{L^p(\Omega)} f$ . ■

## §2.11 Convolution and Mollifiers

**↪ Definition 2.17** (Convolution):

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x - y)g(y) \, dy = \int_{\mathbb{R}^d} f(y)g(x - y) \, dy.$$

**↪ Proposition 2.13** (Properties of Convolution):

- $(f * g) * h = f * (g * h)$  (convolution is associative)
- Let  $\tau_z f(x) := f(x - z)$  be the  $z$ -translate of  $f$  which centers  $f$  at  $z$ . Then,

$$\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g).$$

- $\text{supp}(f * g) \subseteq \overline{\{x + y \mid x \in \text{supp}(f), y \in \text{supp}(g)\}}$ .

PROOF. (a) Assuming all the necessary integrals are finite, we can change order of integration,

$$\begin{aligned}
((f * g) * h)(x) &= \left( \int f(y)g(x-y) \, dy \right) * h(x) \\
&= \int \int f(y)g(x-z-y) \, dy, h(z) \, dz \\
&= \int \int f(y)g(x-y-z)h(z) \, dz \, dy \quad (y' = x-y) \\
&= \int \int f(x-y')g(y'-z)h(z) \, dz \, dy' \\
&= \int f(x-y')(g * h)(y') \, dy' = (f * (g * h))(x).
\end{aligned}$$

(b) For the first equality,

$$\begin{aligned}
\tau_z(f * g)(x) &= \tau_z \int f(x-y)g(y) \, dy \\
&= \int f(x-z-y)g(y) \, dy \\
&= \int (\tau_z f(x-y))g(y) \, dy = ((\tau_z f) * g)(x).
\end{aligned}$$

The second follows from a change of variables in the second line.

(c) We'll show that  $A^c \subseteq (\text{supp}(f * g))^c$  where  $A$  the set as defined in the proposition. Let  $x \in A^c$ , then if  $y \in \text{supp}(g)$ ,  $x-y \notin \text{supp}(f)$  so  $f(x-y) = 0$ ; else if  $y \notin \text{supp}(g)$  it must be  $g(y) = 0$ . So, if  $x \in A^c$ , it must be that

$$\int f(x-y)g(y) \, dy = \int_{\text{supp}(g)} \underbrace{f(x-y)}_{=0} g(y) \, dy + \int_{\text{supp}(g)^c} f(x-y) \underbrace{g(y)}_{=0} \, dy = 0.$$

■

We've been rather loose with finiteness of the convolutions so far. To establish this, we need the following result.

↪ **Theorem 2.25** (Young's Inequality): Let  $f \in L^1(\mathbb{R}^d)$ ,  $g \in L^p(\mathbb{R}^d)$  for any  $p \in [1, \infty]$ . Then,

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p,$$

hence  $f * g \in L^p(\mathbb{R}^d)$ .

PROOF. Suppose first  $p = \infty$ , then

$$(f * g)(x) = \int f(y)g(x-y) \, dy \leq \|g\|_\infty \int |f(y)| \, dy = \|g\|_\infty \|f\|_1,$$

for every  $x \in \mathbb{R}^d$ , so passing to the  $L^\infty$ -norm,

$$\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty.$$

Suppose now  $p = 1$ . Then,



$$\|f * g\|_1 = \int \left| \int f(x-y)g(y) dy \right| dx.$$

Let  $F(x, y) = f(x-y)g(y)$ , then for almost every  $y \in \mathbb{R}^d$ ,

$$\begin{aligned} \int |F(x, y)| dx &= \int |g(y)| |f(x-y)| dx \\ &= |g(y)| \int |f(x-y)| dx \\ &= |g(y)| \|f\|_1. \end{aligned}$$

Applying Tonelli's Theorem, we have then

$$\iint |F(x, y)| dy dx = \iint |F(x, y)| dx dy = \int |g(y)| \|f\|_1 dy = \|f\|_1 \|g\|_1,$$

(so really  $F \in L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$ ), hence all together

$$\|f * g\|_1 = \int \left| \int F(x, y) dy \right| dx \leq \iint |F(x, y)| dy dx = \|f\|_1 \|g\|_1.$$

**Remark 2.24:** It also follows that for a.e.  $x \in \mathbb{R}^d$ ,  $\int |F(x, y)| dy < \infty$ , i.e.  $\int |f(x-y)g(y)| dy < \infty$ . Moreover, since if  $g \in L^p(\Omega)$  then  $|g|^p \in L^1(\Omega)$ , a similar argument gives that for almost every  $x \in \mathbb{R}^d$ ,  $\int |f(x-y)||g(y)|^p dy < \infty$ .

Suppose now  $1 < p < \infty$ . For a.e.  $x \in \mathbb{R}^d$ ,  $\int |g(y)|^p |f(x-y)| dy < \infty$ , so  $g \in L^p(\mathbb{R}^d)$  implies for a.e.  $x \in \mathbb{R}^d$ ,  $|g(\cdot)|^p |f(x-\cdot)| \in L^1(\mathbb{R}^d)$  as a function of  $\cdot$ . This further implies  $g(y)f^{\frac{1}{p}}(x-y) \in L^p(\mathbb{R}^d, dy)$ . Also, if  $f \in L^1(\mathbb{R}^d)$ , then  $f^{\frac{1}{q}} \in L^q(\mathbb{R}^d)$ . All together then,

$$\begin{aligned} \int |f(x-y)||g(y)| dy &= \int \overbrace{|f^{\frac{1}{q}}(x-y)|}^q \underbrace{|f^{\frac{1}{p}}(x-y)||g(y)|}_{p} dy \\ \text{Holder's} \quad &\leq \left( \int |f(x-y)| dy \right)^{\frac{1}{q}} \left( \int |f(x-y)||g(y)|^p dy \right)^{\frac{1}{p}}, \end{aligned}$$

hence, raising both sides to the  $p$ ,

$$|(f * g)(x)|^p \leq \|f\|_1^{\frac{p}{q}} \cdot (|f| * |g|^p)(x)$$

and integrating both sides

$$\int |(f * g)(x)|^p dx \leq \|f\|_1^{\frac{p}{q}} \int \left( \underbrace{|f|}_{\in L^1(\mathbb{R}^d)} * \underbrace{|g|^p}_{\in L^1(\mathbb{R}^d)} \right)(x) dx.$$

Hence, we can bound the right-hand term using the previous case for  $p = 1$ , and find

$$\begin{aligned}
\int |(f * g)(x)|^p dx &\leq \|f\|_1^{\frac{p}{q}} \|f\|_1 \|g^p\|_1 \\
&= \|f\|_1^{\frac{p}{q}+1} \|g\|_p^p \\
&= \|f\|_1^{\frac{p+q}{q}} \|g\|_p^p \\
\left(\frac{p+q}{q} = p\right) &= \|f\|_1^p \|g\|_p^p,
\end{aligned}$$

so raising both sides to  $\frac{1}{p}$ , we conclude

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

■

↪ **Proposition 2.14:** If  $f \in L^1(\mathbb{R}^d)$  and  $g \in C^1(\mathbb{R}^d)$  with  $|\partial_{x_i} g| \in L^\infty(\mathbb{R}^d)$  for  $i = 1, \dots, d$ , then  $(f * g) \in C^1(\mathbb{R}^d)$  and moreover

$$\partial_{x_i}(f * g) = f * (\partial_{x_i} g).$$

**Remark 2.25:** There are many different conditions we can place on  $f, g$  to make this true; most basically, we need  $|(\partial_i g) * f| < \infty$ .

PROOF.

$$\frac{\partial}{\partial x_i} \left( \int f(y) g(x-y) dy \right) = \int \underbrace{f(y)}_{\in L^1(\mathbb{R}^d)} \underbrace{\partial_i g(x-y)}_{\in L^\infty(\mathbb{R}^d)} dy < \infty,$$

citing the previous theorem for the finiteness; the dominated convergence theorem allows us to pass the derivative inside. ■

**Remark 2.26:** This also follows for the gradient; namely  $\nabla(f * g) = f * (\nabla g)$  with a component-wise convolution.

Consider the function

$$\rho(x) = \begin{cases} C \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| \leq 1, \\ 0 & \text{o.w.} \end{cases}$$

where  $C = C(d)$  a constant such that  $\int_{\mathbb{R}^d} \rho(x) dx = 1$ . Then, note that  $\rho \in C_c^\infty(\mathbb{R}^d)$  (infinitely differentiable with compact support). Let now

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^d} \rho\left(\frac{x}{\varepsilon}\right).$$

Notice that  $\rho_\varepsilon(x)$  is supported on  $B(0, \varepsilon)$ , but

$$\int_{\mathbb{R}^d} \rho_\varepsilon(x) dx = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \rho\left(\frac{x}{\varepsilon}\right) dx = \frac{1}{\varepsilon^d} \cdot \varepsilon^d \cdot \int_{\mathbb{R}^d} \rho(y) dy = 1,$$

for every  $\varepsilon$ , by making a change of variables  $y = \frac{x}{\varepsilon}$ . We'll be interested in the convolution

$$f_\varepsilon(x) := (\rho_\varepsilon * f)(x)$$

for some function  $f$ .  $\rho_\varepsilon$  is often called a “convolution kernel”. In particular, it is a “good kernel”, namely has the properties:

- $\int_{\mathbb{R}^d} \rho_\varepsilon(y) dy = 1$ ;
- $\int_{\mathbb{R}^d} |\rho_\varepsilon(y)| dy \leq M$  for some finite  $M$ ;
- $\forall \delta > 0, \int_{\{|y|>\delta\}} |\rho_\varepsilon(y)| dy \xrightarrow{\varepsilon \rightarrow 0} 0$ .

The second condition is trivially satisfied in this case since our kernel is nonnegative. The last also follows easily since  $\rho_\varepsilon$  has compact support; more generally, this imposes rapid decay conditions on the tails of good kernels.

Since  $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^d)$ , for “reasonable”  $f$ ,  $f_\varepsilon = \rho_\varepsilon * f \in C^\infty(\mathbb{R}^d)$  by the previous proposition. In fact, we'll see that in many contexts  $f_\varepsilon \rightarrow f$  as  $\varepsilon \rightarrow 0$  in some notion of convergence. So,  $f_\varepsilon$  provides a good, now smooth, approximation to  $f$ .

↪ **Proposition 2.15:** Suppose  $f \in L^\infty(\mathbb{R}^d)$  and  $f_\varepsilon$  is well-defined. Then, if  $f$  is continuous at  $x$ , then  $f_\varepsilon(x) \rightarrow f(x)$  as  $\varepsilon \rightarrow 0$ .

If  $f \in C(\mathbb{R}^d)$ , then  $f_\varepsilon \rightarrow f$  uniformly on compact sets.

PROOF.  $f$  continuous at  $x$  gives that for every  $\eta > 0$  there exists a  $\delta > 0$  such that  $|f(y) - f(x)| < \eta$  whenever  $|x - y| < \delta$ . Then

$$\begin{aligned} |f_\varepsilon(x) - f(x)| &= \left| \int \rho_\varepsilon(y) f(x-y) dy - f(x) \underbrace{\int \rho_\varepsilon(y) dy}_{=1} \right| \\ &= \left| \int \rho_\varepsilon(y) (f(x-y) - f(x)) dy \right| \\ &\leq \int_{\{|y| \leq \delta\}} |f(x-y) - f(x)| |\rho_\varepsilon(y)| dy + \int_{\{|y| > \delta\}} |f(x-y) - f(x)| |\rho_\varepsilon(y)| dy \\ &\stackrel{\substack{\text{(cnty in first argument)} \\ \text{(} L^\infty\text{-bound in second)}}}{\leq} \int_{\{|y| \leq \delta\}} \eta |\rho_\varepsilon(y)| dy + 2\|f\|_\infty \int_{\{|y| > \delta\}} |\rho_\varepsilon(y)| dy \\ &\leq \eta \cdot M + 2\|f\|_\infty \int_{\{|y| > \delta\}} |\rho_\varepsilon| \end{aligned}$$

for  $\varepsilon \rightarrow 0$ , by using the second property of good kernels for the first bound. By the last property, the right-most term  $\rightarrow 0$  as  $\varepsilon \rightarrow 0$ ; moreover, then,

$$\lim_{\varepsilon \rightarrow 0} |f_\varepsilon(x) - f(x)| \leq C\eta$$

for some  $C$  and every  $\eta > 0$ , and thus  $f_\varepsilon(x) \rightarrow f(x)$  as  $\varepsilon \rightarrow 0$ .

Now, if  $f \in C(\mathbb{R}^d)$  fix a subset  $K \subseteq \mathbb{R}^d$  compact. Hence,  $\|f\|_{L^\infty(K)} < \infty$  and  $f$  uniformly continuous on  $K$  since  $K$  compact; so the modulus continuity is uniform for all  $x \in K$ , so for  $\delta > 0$  and for every  $x \in K$ ,

$$\int_{\{|y| \leq \delta\}} |f(x-y) - f(x)| |\rho_\varepsilon(y)| dy \leq C\eta.$$

Also, using the bound on  $f$ , we may write the second integral in the argument above as

$$\int_{\varepsilon > |y| > \delta} |f(x-y) - f(x)| |\rho_\varepsilon(y)| dy \leq \|f\|_{L^\infty(K+B_\varepsilon)} \int_{\{|y| > \delta\}} |\rho_\varepsilon(y)| dy \xrightarrow{\varepsilon \rightarrow 0} 0$$

where we take  $K$  slightly larger as  $K + B_\varepsilon$ , which is still compact. So, since this held for all  $x \in K$ ,

$$\max_{x \in K} |f_\varepsilon(x) - f(x)| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

*Note that we proved the first for general good kernels but the second only in our constructed one.* ■

**Remark 2.27:** This pointwise convergence result is why “good kernels” are called “approximations to the identity”.

**Remark 2.28:** If  $f \in C_c(\mathbb{R}^d)$ , then  $\text{supp}(f_\varepsilon) \subseteq \overline{\text{supp}(f) + B(0, \varepsilon)}$ ; so,  $f_\varepsilon$  is compactly supported if  $f$  is. Hence in this case  $f_\varepsilon \rightarrow f$  uniformly on  $\mathbb{R}^d$ .

**↪ Theorem 2.26** (Weierstrass Approximation Theorem): Let  $[a, b] \subseteq \mathbb{R}$  and let  $f \in C([a, b])$ . Then for every  $\eta > 0$ , there exists a polynomial  $P_N(x)$  of degree  $N$  such that

$$\|P_N - f\|_{L^\infty([a, b])} < \eta.$$

That is, polynomials are dense in  $C([a, b])$ .

**PROOF.** Extend  $f$  to be continuous with compact support on all of  $\mathbb{R}$  in whatever convenient way, such that  $\text{supp}(f) \subseteq [-M, M]$  for some sufficiently large  $M > 0$ . Consider now

$$K_\varepsilon(x) := \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi x^2}{\varepsilon}},$$

noting that

$$\int_{-\infty}^{\infty} K_\varepsilon(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi x^2}{\varepsilon}} dx = 1,$$

which is clear by a change of variables  $y = \frac{\sqrt{2\pi}}{\sqrt{\varepsilon}} x$ . As a consequence,  $\int_{-\infty}^{\infty} |K_\varepsilon(x)| dx = 1 < \infty$ , since  $K_\varepsilon \geq 0$ . Finally,

$$\begin{aligned}
\int_{|x|>\delta} |K_\varepsilon(x)| dx &= \int_{|x|>\delta} \frac{1}{\sqrt{\pi}} e^{-\frac{\pi x^2}{\varepsilon}} dx \\
&= \int_{|y|>\frac{\sqrt{2\pi}}{\sqrt{\varepsilon}}\delta} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\
&\leq \int_{|y|>\frac{\sqrt{2\pi}}{\sqrt{\varepsilon}}\delta} \frac{|y|}{\sqrt{2\pi}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\
&\leq C e^{-\frac{y^2}{2}} \Big|_{\frac{\sqrt{2\pi}}{\sqrt{\varepsilon}}\delta}^{\infty} \xrightarrow{\varepsilon \rightarrow 0} 0.
\end{aligned}$$

since  $|y| \geq 1$  here for suff. small  $\varepsilon$

So,  $K_\varepsilon$  is a good kernel, and so  $(f * K_\varepsilon)(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} f$  uniformly in  $[a, b]$  by our last remark. In particular, for  $\eta > 0$  there is some  $\varepsilon_0 > 0$ ,

$$\|(f * K_{\varepsilon_0}) - f\|_{L^\infty([a, b])} < \frac{\eta}{2}.$$

We claim now that there is a polynomial  $P_N$  such that  $\|P_N - (f * K_{\varepsilon_0})\|_{L^\infty([a, b])} < \frac{\eta}{2}$ . Recall that  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ , which converges uniformly on compact sets. So, there exists a polynomial  $S_N$  (from truncating this sum) such that  $\|K_{\varepsilon_0} - S_N\|_{L^\infty([-M, M])} < \frac{\eta}{4\|f\|_\infty M}$ . Thus,

$$\begin{aligned}
|f * K_{\varepsilon_0}(x) - f * S_N(x)| &\leq \left| \int f(x-y)(K_{\varepsilon_0}(y) - S_N(y)) dy \right| \\
\text{supp}(f) \subset [-M, M] \quad &\leq \int_{-M}^M |f(x-y)| |K_{\varepsilon_0}(y) - S_N(y)| dy \\
&\leq 2M \|f\|_\infty \frac{\eta}{4M\|f\|_\infty} = \frac{\eta}{2},
\end{aligned}$$

for every  $x$ . Let  $P_N(x) = (f * S_N)(x)$ , which we see to be a polynomial. ■

↪ **Theorem 2.27:** Let  $f \in L^p(\mathbb{R}^d)$  with  $p \in [1, \infty)$ . Then  $f_\varepsilon \xrightarrow{L^p(\mathbb{R}^d)} f$ .

PROOF. Since  $f \in L^p(\mathbb{R}^d)$ , for every  $\eta > 0$  there is a  $\tilde{f} \in C_c(\mathbb{R}^d)$  such that  $\|f - \tilde{f}\|_p < \eta$ . Since  $\tilde{f} \in C_c(\mathbb{R}^d)$ , by the previous theorem dealing with mollifiers and uniform convergence,  $\tilde{f}_\varepsilon \rightarrow \tilde{f}$  uniformly. In particular, we have  $\|\tilde{f}_\varepsilon - \tilde{f}\|_p \xrightarrow{p} 0$ , hence

$$\|f - f_\varepsilon\|_p \leq \|f_\varepsilon - \tilde{f}_\varepsilon\|_p + \|\tilde{f}_\varepsilon - \tilde{f}\|_p + \|\tilde{f} - f\|_p.$$

We've dealt with the second two bounds. For the first,

$$\begin{aligned}
\|f_\varepsilon - \tilde{f}_\varepsilon\|_p &= \|(f - \tilde{f}) * \rho_\varepsilon\|_p \\
(\text{Young's}) \quad &\leq \|\rho_\varepsilon\|_1 \|f - \tilde{f}\|_p = \|f - \tilde{f}\|_p,
\end{aligned}$$

so

$$\|f - f_\varepsilon\|_p \leq 2\|f - \tilde{f}\|_p + \|\tilde{f}_\varepsilon - \tilde{f}\|_p < 3\eta.$$

■

↪ **Corollary 2.6:**  $C_c^\infty(\mathbb{R}^d)$  dense in  $L^p(\mathbb{R}^d)$ .

PROOF. We showed  $\tilde{f}_\varepsilon$  approximates  $f$  in  $L^p(\mathbb{R}^d)$ , and by construction  $\tilde{f}_\varepsilon$  is smooth with compact support. ■

### §2.12 Strong Compactness in $L^p(\mathbb{R}^d)$

We saw that for  $p \in (1, \infty)$ ,  $\{f_n\} \subset L^p(\Omega)$ , that any bounded sequence admits a weakly converging subsequence,  $f_{n_k} \xrightarrow{L^p} f$ . In addition, if the norms also converge i.e.  $\lim_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p$ , then we actually have strong convergence  $f_{n_k} \xrightarrow{L^p} f$ .

We provide now a strong compactness result in  $L^p$ , akin to Arzelà-Ascoli.

↪ **Theorem 2.28** (Strong Compactness): Let  $\{f_n\} \subseteq L^p(\mathbb{R}^d)$  for  $p \in [1, \infty)$  s.t.

- i.  $\exists C > 0$  s.t.  $\|f_n\|_p < C \forall n$ , i.e.  $\{f_n\}$  uniformly bounded in  $L^p$ ;
- ii.  $\lim_{|h| \rightarrow 0} \|f_n - \tau_h f_n\|_p = 0$  uniformly in  $n$ , i.e. for every  $\eta > 0$ , there exists  $\delta > 0$  such that if  $|h| < \delta$ ,  $\int |f_n(x) - f_n(x-h)|^p dx < \eta^p$  for every  $n$ ;

Then, for any  $\Omega \subseteq \mathbb{R}^d$  with finite measure, there exists a subsequence  $\{f_{n_k}\}$  such that  $f_{n_k} \xrightarrow{L^p(\Omega)} f$ .

PROOF. Recall that  $L^p(\Omega)$  is a complete metric space, so TFAE:

1. sequential compactness;
2. totally bounded (& complete);
3. compact.

Let  $\mathcal{F} = \{f \in L^p(\mathbb{R}^d) \text{ satisfying i., ii.}\}$  and fix  $\Omega \subseteq \mathbb{R}^d$  with finite measure. We aim to show that  $\mathcal{F}|_\Omega$  is sequentially compact in  $L^p(\Omega)$  (with no regard to whether the limit lives in  $\mathcal{F}_\Omega$ ); equivalently, we wish to show  $\mathcal{F}|_\Omega$  is precompact in  $L^p(\Omega)$  i.e.  $\overline{\mathcal{F}|_\Omega}$  is compact. Since  $\overline{\mathcal{F}|_\Omega}$  is a complete metric space, to prove the result it suffices to show that  $\mathcal{F}|_\Omega$  is totally bounded (recall: for every  $\delta > 0$ ,  $\mathcal{F}|_\Omega \subseteq \bigcup_{i=1}^N B_{L^p(\Omega)}(g_i, \delta)$ ). We'll do this using mollifiers and AA to do this.

*Step 1:* Fix  $\eta, \delta$  as in ii. in the statement of the theorem, and let  $f \in \mathcal{F}$ . Then, for every  $\varepsilon < \delta$ , we claim

$$\|(\rho_\varepsilon * f) - f\|_{L^p(\mathbb{R}^d)} < \eta.$$

We have

$$\begin{aligned}
|(\rho_\varepsilon * f)(x) - f(x)| &= \left| \int_{B_\varepsilon} \rho_\varepsilon(y) f(x-y) \, dy - f(x) \int \rho_\varepsilon(y) \, dy \right| \\
&\leq \int_{B_\varepsilon} \rho_\varepsilon(y) |f(x-y) - f(x)| \, dy \\
&= \int_{B_\varepsilon} \rho_\varepsilon^{\frac{1}{q}}(y) \rho_\varepsilon^{\frac{1}{p}}(y) |f(x-y) - f(x)| \, dy \\
\text{(Holder's)} \quad &\leq \left( \int \rho_\varepsilon(y) |f(x-y) - f(x)|^p \, dy \right)^{1/p} \underbrace{\left( \int \rho_\varepsilon(y) \, dy \right)^{1/q}}_{=1},
\end{aligned}$$

and hence

$$\begin{aligned}
\int |(\rho_\varepsilon * f)(x) - f(x)|^p \, dx &\leq \iint \rho_\varepsilon(y) |f(x-y) - f(x)|^p \, dy \, dx \\
\text{(Tonelli's)} \quad &= \int_{B_\varepsilon} \rho_\varepsilon(y) \underbrace{\int |f(x-y) - f(x)|^p \, dx}_{\varepsilon < \delta \Rightarrow \eta^p} \, dy \\
&< \eta^p \underbrace{\int_{B_\varepsilon} \rho_\varepsilon(y) \, dy}_{=1} = \eta^p,
\end{aligned}$$

hence  $\|(\rho_\varepsilon * f)(x) - f(x)\|_p < \eta$ .

*Step 2:* We first claim that there exists some  $C_\varepsilon$  such that for any  $f \in \mathcal{F}$ ,

$$\|\rho_\varepsilon * f\|_\infty \leq C_\varepsilon \|f\|_p, \quad (1)$$

and that for any  $x_1, x_2 \in \mathbb{R}^d$ ,

$$|(\rho_\varepsilon * f)(x_1) - (\rho_\varepsilon * f)(x_2)| \leq C_\varepsilon \|f\|_p |x_1 - x_2|. \quad (2)$$

In particular, this shows that *for  $\varepsilon$  fixed*,  $(\rho_\varepsilon * f)$  satisfy hypothesis of AA. Remark that the first is a uniform boundedness type condition for  $\rho_\varepsilon * f$ , and the second is a uniform Lipschitz bound.

For the first claim (1),

$$\begin{aligned}
|(\rho_\varepsilon * f)(x)| &= \left| \int \rho_\varepsilon(x-y) f(y) \, dy \right| \\
\text{(Holder's)} \quad &\leq \left( \int \rho_\varepsilon^q(x-y) \, dy \right)^{\frac{1}{q}} \cdot \|f\|_p \\
&= \|\rho_\varepsilon\|_q \|f\|_p,
\end{aligned}$$

so we have the bound with  $C_\varepsilon := \|\rho_\varepsilon\|_q$  since the bound is independent of  $x$ .

**Remark 2.29:** One can explicitly compute  $\|\rho_\varepsilon\|_{q'}$ , and realize that it will in general depend explicitly on  $\varepsilon$ .

For the second statement (2), we find that  $\nabla(\rho_\varepsilon * f) = (\nabla\rho_\varepsilon) * f$  since the RHS is finite, because

$$(\nabla\rho_\varepsilon * f)(x) = \int \nabla\rho_\varepsilon(x-y)f(y) dy \leq \|\nabla\rho_\varepsilon\|_q \|f\|_p.$$

So,

$$\|\nabla(\rho_\varepsilon * f)\|_\infty \leq \underbrace{\|\nabla\rho_\varepsilon\|_q}_{=:C_\varepsilon} \|f\|_p.$$

By the mean-value theorem then, we have all together

$$\begin{aligned} \|(\rho_\varepsilon * f)(x_1) - (\rho_\varepsilon * f)(x_2)\| &\leq \|\nabla(\rho_\varepsilon * f)\|_\infty |x_1 - x_2| \\ &\leq C_\varepsilon \|f\|_p |x_1 - x_2|. \end{aligned}$$

This proves (2).

*Step 3:* Next, we claim that for  $\eta > 0$  and fixed  $\varepsilon < \eta$  and  $\Omega \subseteq \mathbb{R}^d$  with finite measure, there exists  $E \subseteq \Omega \subseteq \mathbb{R}^d$  such that  $E$  is bounded, i.e.  $E \subseteq B(0, M)$  where  $M$  sufficiently large, and moreover that  $\|f\|_{L^p(\Omega \setminus E)} < \eta$  for every  $f \in \mathcal{F}$ .

We have that

$$\|f\|_{L^p(\Omega \setminus E)} \leq \|f - (\rho_\varepsilon * f)\|_{L^p(\mathbb{R}^d)} + \|\rho_\varepsilon * f\|_{L^p(\Omega \setminus E)}.$$

By the very first step of the proof, the first term is  $< \eta$ , so this is bounded by

$$\begin{aligned} &< \eta + \left( \int_{\Omega \setminus E} |\rho_\varepsilon * f|^p dx \right)^{1/p} \\ &< \eta + \|\rho_\varepsilon * f\|_\infty |\Omega \setminus E|^{\frac{1}{p}} \\ &< \eta + C_\varepsilon \|f\|_p |\Omega \setminus E|^{\frac{1}{p}}. \end{aligned}$$

$C_\varepsilon$  finite and  $\|f\|_p$  upper bounded uniformly over  $\mathcal{F}$ , so it suffices to construct  $E$  with the measure of  $\Omega \setminus E$  sufficiently small, so we can get  $\|f\|_{L^p(\Omega \setminus E)} < 2\eta$ .

*Step 4:* Fix  $\eta > 0$ . We claim  $\mathcal{F}|_\Omega$  is totally bounded. Let  $\varepsilon < \delta$  then let

$$\mathcal{H} := (\rho_\varepsilon * \mathcal{F})|_{\overline{E}} = \{\rho_\varepsilon * f|_E : f \in \mathcal{F}\}.$$

$E \subseteq \Omega \subseteq \mathbb{R}^d$  is bounded implies  $\overline{E}$  is compact. So by Step 2., we showed  $(\rho_\varepsilon * \mathcal{F})$  satisfies hypotheses of AA on  $\overline{E}$ . Hence,  $\mathcal{H}$  is precompact in  $C(\overline{E})$ . Thus, since we have uniform convergence we certainly have  $L^p$  convergence thus  $\mathcal{H}$  also precompact in  $L^p(\overline{E})$ . Thus, for  $\eta > 0$ , there exists  $\{\bar{g}_i\} \subseteq L^p(\overline{E})$  such that



$$\mathcal{H} \subseteq \bigcup_{i=1}^N B_{L^p(\overline{E})}(\overline{g}_i, \eta). \quad \star$$

Let  $g_i : \Omega \rightarrow \mathbb{R}$  be given by

$$g_i(x) = \begin{cases} \overline{g}_i & \text{on } E \\ 0 & \text{on } \Omega \setminus E \end{cases}.$$

Then, we claim  $\mathcal{F}|_{\Omega} \subseteq \bigcup_{i=1}^N B_{L^p(\Omega)}(g_i, 3\eta)$ . If  $f \in \mathcal{F}$  by  $\star$ , there is an  $i$  such that  $\|\rho_\varepsilon * f - \overline{g}_i\|_{L^p(\overline{E})} < \eta$ . But also,

$$\begin{aligned} \|f - g_i\|_{L^p(\Omega)}^p &= \int_{\Omega \setminus E} |f|^p dx + \int_{\overline{E}} |f - \overline{g}_i|^p dx \\ &= \|f\|_{L^p(\Omega \setminus E)}^p + \int_{\overline{E}} |f - \overline{g}_i|^p dx \\ (\text{Step 3.}) \quad &\leq \eta^p + \int_{\overline{E}} |f - \overline{g}_i|^p dx. \end{aligned}$$

Recall  $(a + b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}$ . Applying this bound to the above, we find

$$\begin{aligned} \|f - g_i\|_{L^p(\Omega)} &\leq \eta + \|f - \overline{g}_i\|_{L^p(\overline{E})} \\ &\leq \eta + \underbrace{\|f - f * \rho_\varepsilon\|_{L^p(\mathbb{R}^d)}}_{< \eta \text{ by Step 1.}} + \underbrace{\|(f * \rho_\varepsilon) - \overline{g}_i\|_{L^p(\overline{E})}}_{< \eta \text{ by } \star} \\ &\leq 3\eta. \end{aligned}$$

Hence,  $\mathcal{F}|_{\Omega} \subseteq \bigcup_{i=1}^N B(g_i, 3\eta)$ , thus  $\mathcal{F}|_{\Omega}$  is sequentially compact so any sequence in  $\mathcal{F}$  has a converging subsequence, which proves the theorem.  $\blacksquare$

**Remark 2.30:** This can be extended to  $L^p(\mathbb{R}^d)$  with some conditions.

### §3 INTRODUCTION TO FOURIER ANALYSIS

References are Folland, Chapter 8 and *Fourier Analysis* by Stein & Shkarchi.

#### §3.1 Fourier Series

We will denote the torus  $\mathbb{T} = [0, 1) \simeq \mathbb{R}/\mathbb{Z}$  (with 1 identified back with 0), and specifically complex-valued functions on the torus

$$L^2(\mathbb{T}) = \left\{ f : \mathbb{T} \rightarrow \mathbb{C} \mid \int_0^1 |f(x)|^2 dx < \infty \right\},$$

where now  $|\cdot|$  the modulus (i.e.  $|a + bi|^2 = a^2 + b^2$ ). Equivalently,  $f : \mathbb{T} \rightarrow \mathbb{C}$  can be identified with  $\tilde{f} : \mathbb{R} \rightarrow \mathbb{C}$  which is periodic.

↪ **Proposition 3.1:** The function  $L^2(\mathbb{T}) \times L^2(\mathbb{T}) \rightarrow \mathbb{C}$

$$(f, g) = \int_0^1 f(x) \overline{g(x)} dx$$

is an inner product on  $L^2(\mathbb{T})$ . In particular,  $(L^2(\mathbb{T}), (\cdot, \cdot))$  a Hilbert space.

PROOF. For  $\mathbb{C}$ -valued functions, we need to check:

- linearity in the first variable: for  $\alpha \in \mathbb{C}$ ,

$$(\alpha f + h, g) = \int_0^1 (\alpha f + h) \bar{g} dx = \alpha(f, g) + (h, g)$$

by linearity of the integral;

- conjugate symmetry:

$$\begin{aligned} \int_0^1 f(x) \overline{g(x)} dx &= \int_0^1 (\operatorname{Re}(f) + i\operatorname{Im}(f))(\operatorname{Re}(g) - i\operatorname{Im}(g)) dx \\ &=: \int_0^1 (a + ib)(c - id) dx \\ &= \int_0^1 (ac + bd) + i(bc - ad) dx \\ &= \int_0^1 (ac + bd) - i(ad - bc) dx \\ &= \overline{\int_0^1 g \bar{f} dx} = \overline{(g, f)}; \end{aligned}$$

- $f$  inner product with  $f$  properties:

$$(f, f) = \int_0^1 f(x) \overline{f(x)} dx = \int_0^1 |f(x)|^2 dx = \|f\|_{L^2(\mathbb{T})}^2 \geq 0, = 0 \text{ iff } f \equiv 0.$$

We know  $L^2(\mathbb{T})$  is complete, so  $L^2(\mathbb{T})$  a Hilbert space with this inner product since it induces the same norm as the usual norm  $L^2$ -norm. ■

↪ **Theorem 3.1:** Let  $e_n(x) := e^{2\pi i n x}$  for  $n \in \mathbb{Z}$ . Then,  $\{e_n\}_{n \in \mathbb{Z}}$  is an orthonormal basis of  $L^2(\mathbb{T})$ .

PROOF. For orthonormality, if  $n \neq m$ ,

$$\begin{aligned}
(e_n, e_m) &= \int_0^1 e^{2\pi i n x} e^{-2\pi i m x} dx \\
&= \int_0^1 e^{2\pi i (n-m)x} dx \\
&= \frac{1}{2\pi i (n-m)} e^{2\pi i (n-m)x} \Big|_0^1 \\
&= \frac{1}{2\pi i (n-m)} [e^{2\pi i (n-m)} - 1] \\
&= \frac{1}{2\pi i (n-m)} \left[ \underbrace{\cos(2\pi(n-m))}_{=1} + \underbrace{i \sin(2\pi(n-m))}_{=0} - 1 \right] = 0,
\end{aligned}$$

and if  $n = m$ ,

$$(e_n, e_n) = \int_0^1 |e^{2\pi i n x}|^2 dx = \int_0^1 1 dx = 1.$$

To prove its a basis, we use Stone-Weierstrass.  $\mathbb{T}$  is compact; let

$$\mathcal{A} := \left\{ \sum_{n=-N}^N \alpha_n e_n : \alpha_n \in \mathbb{C}, N \in \mathbb{N} \right\}.$$

Notice  $e_n e_m = e^{2\pi i (n+m)x} = e_{n+m}$ , and  $e_0 = 1$ , so this family stays closed under multiplication (and clearly addition and scalar multiplication by definition), so is an algebra which contains constant functions. Also, if  $x_1 \neq x_2$  and  $x_1, x_2 \in [0, 1)$ , then if  $n \neq 0$ ,  $e_n(x_1) = e^{2\pi i n x_1} \neq e^{2\pi i n x_2} = e_n(x_2)$ , so  $\mathcal{A}$  separates points. By (complex) Stone-Weierstrass, then we know  $\mathcal{A}$  is dense in  $C(\mathbb{T}, \mathbb{C})$  with respect to  $\|\cdot\|_\infty$ . We know  $C(\mathbb{T}, \mathbb{C})$  is dense in  $L^2(\mathbb{T})$  (by some mollifier argument, for example) with respect to  $\|\cdot\|_{L^2(\mathbb{T})}$ . So,

$$f(x) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \alpha_n e_n(x),$$

with the limit taken in the sense of  $L^2(\mathbb{T})$ . ■

Recall that in Hilbert spaces, TFAE:

- $\{e_n\}$  are a basis, i.e.  $f = \sum_{n=-\infty}^{\infty} \alpha_n e_n = \sum_{n=-\infty}^{\infty} (f, e_n) e_n$ , in  $L^2(\mathbb{T})$ ;
- if  $(f, e_n) = 0$  for every  $n$ ,  $f \equiv 0$  (completeness);
- $\|f\|_{L^2(\mathbb{T})}^2 = \sum_{n=-\infty}^{\infty} |(f, e_n)|^2$  (Parseval's).

With this in mind, we define:

↪ **Definition 3.1** (Fourier Series): Let

$$\hat{f}(n) := (f, e_n) = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Then, the *complex Fourier series* is defined by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

**Remark 3.1:** A Fourier series can be defined for any periodic function, while we only do so for 1-periodic here. If  $f$  were  $L$ -periodic, we'd define

$$\hat{f}_L(n) := \frac{1}{L} \int_0^L f(x) e^{-\frac{2\pi i n x}{L}} dx,$$

with complex Fourier series  $\sum_{n=-\infty}^{\infty} \hat{f}_L(n) e^{\frac{2\pi i n x}{L}}$ .

**Remark 3.2:** We can also make Fourier series to be real-valued, with sines and cosines, of the form

$$A_0 + \sum_{n=1}^{\infty} \left[ A_n \cos\left(\frac{2n\pi x}{L}\right) + B_n \sin\left(\frac{2n\pi x}{L}\right) \right],$$

for some  $A_n, B_n$  also given by inner products.

What conditions do we need on  $f$  to make this series converge? In the general  $L^2$ -theory, we just need  $f \in L^2(\mathbb{T})$ . By Parseval's,

$$\|f\|_{L^2(\mathbb{T})}^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

So, the operator  $\hat{\cdot} : L^2(\mathbb{T}) \rightarrow \ell^2(\mathbb{C})$ . Note that this implies  $\lim_{n \rightarrow \infty} |\hat{f}(n)|^2 = 0$ , so also  $\lim_{n \rightarrow \infty} |\hat{f}(n)| = 0$ . This proves the following proposition:

↪ **Proposition 3.2** (Riemann-Lebesgue Lemma): If  $f \in L^2(\mathbb{T})$ ,

$$\lim_{n \rightarrow \infty} |\hat{f}(n)| = 0.$$

**Remark 3.3:** This result is *very* useful, particularly for the real Fourier Series. In particular, it tells us statements such as

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) \sin(2n\pi x) dx = 0,$$

with similar for the cosine term. These are so-called “oscillatory integrals”.

While the  $L^2(\mathbb{T})$ -theory is very useful for Hilbert space interpretation, we are really concerned with the partial sums

$$S_N(x) = \sum_{n=-N}^N \hat{f}(n) e^{2\pi i n x},$$

and ways it might converge. We may rewrite by definition

$$\begin{aligned} S_N(x) &= \sum_{n=-N}^N \left( \int_0^1 f(y) e^{-2\pi i n y} dy \right) e^{2\pi i n x} \\ \text{(because finite sum)} \quad &= \int_0^1 f(y) \sum_{n=-N}^N e^{2\pi i n(x-y)} dy \\ \text{(* just over } [0, 1]) \quad &= (f * D_N)(x), \quad D_N(x) := \sum_{n=-N}^N e^{2\pi i n x}. \end{aligned}$$

So in short,

$$S_N(x) = (f * D_N)(x),$$

where  $D_N(x)$  is called the *Dirichlet kernel*. Let’s look at some of its properties.

$$D_N(x) = 1 + \sum_{n=1}^N [e^{2\pi i n x} + e^{-2\pi i n x}],$$

so

$$\int_0^1 D_N(x) dx = \int_0^1 1 dx + \underbrace{\sum_{n=1}^N \int_0^1 (\text{some periodic functions})}_{=0} = 1,$$

by periodicity. However,  $D_N(x)$  is not actually a good kernel; one can show that  $\int_0^1 |D_n(x)| dx \geq C \log N$  as  $N \rightarrow \infty$ .

↪ **Theorem 3.2** (Pointwise Convergence): Let  $f \in L^2(\mathbb{T})$  and suppose  $f$  is Lipschitz continuous at  $x_0$ . Then,

$$S_N(x_0) \rightarrow f(x_0).$$

PROOF. Left as an exercise. ■

Note that

$$\begin{aligned}
D_N(x) &= \sum_{n=-N}^N e^{2\pi i n x} \\
&= \sum_{n=0}^{2N} e^{2\pi i (n-N)x} \\
&= e^{-2\pi i N x} \sum_{n=0}^{2N} (e^{2\pi i x})^n \\
&= e^{-2\pi i N x} \left( \frac{1 - e^{2\pi i (2N+1)x}}{1 - e^{2\pi i x}} \right) \quad (\text{geometric series}) \\
&= \frac{e^{-2\pi i N x} - e^{2\pi i (N+1)x}}{1 - e^{2\pi i x}} \cdot \frac{e^{-2\pi i \frac{x}{2}}}{e^{-2\pi i \frac{x}{2}}} \\
&= \frac{e^{-2\pi i (N+\frac{1}{2})x} - e^{2\pi i (N+\frac{1}{2})x}}{e^{-2\pi i \frac{x}{2}} - e^{2\pi i \frac{x}{2}}} \\
&= \frac{\sin(2\pi (N + \frac{1}{2})x)}{\sin(2\pi \frac{x}{2})}.
\end{aligned}$$

↪ **Theorem 3.3** (Uniform convergence): If  $f \in C^2(\mathbb{T})$ , then  $S_N(x) \rightarrow f(x)$  uniformly on  $\mathbb{T}$ .

PROOF. Exercise. ■

**Remark 3.4:** In fact, we see that  $\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$  is well-defined whenever  $f \in L^1(\mathbb{T})$ . So, we can view

$$\hat{\cdot} : L^1(\mathbb{T}) \rightarrow \ell^\infty(\mathbb{Z}).$$

**Remark 3.5:** All the prior results can be extended to  $f \in L^1(\mathbb{T})$ , via density.

### §3.2 Introduction to the Fourier Transform

↪ **Definition 3.2** (Fourier Transform): Let  $f : \mathbb{R} \rightarrow \mathbb{C}$ . Then, for any  $\zeta \in \mathbb{R}$ , define

$$\hat{f}(\zeta) := \int_{\mathbb{R}} f(x) e^{-2\pi i \zeta x} dx.$$

**Remark 3.6:** If  $f \in L^1(\mathbb{R})$ ,

$$|\hat{f}(\zeta)| \leq \int_{-\infty}^{\infty} |f(x)| \underbrace{|e^{-2\pi i \zeta x}|}_{=1} dx = \|f\|_{L^1(\mathbb{R})}$$

so in particular,  $\hat{f} \in L^\infty(\mathbb{R})$ . Moreover,

$$\begin{aligned} |\hat{f}(\zeta + h) - \hat{f}(\zeta)| &= \left| \int_{\mathbb{R}} f(x) e^{-2\pi i(\zeta+h)x} - f(x) e^{-2\pi i \zeta x} dx \right| \\ &= \left| \int_{\mathbb{R}} f(x) e^{-2\pi i \zeta x} (e^{-2\pi i h x} - 1) dx \right| \\ &\leq \int_{\mathbb{R}} |f(x)| |e^{-2\pi i h x} - 1| dx. \end{aligned}$$

We have that

$$\lim_{h \rightarrow 0} |e^{-2\pi i h x} - 1| = 0$$

for a.e.  $x \in \mathbb{R}$ , and

$$\int_{\mathbb{R}} |f(x)| |e^{-2\pi i h x} - 1| dx \leq 2 \int_{\mathbb{R}} |f(x)| dx = 2\|f\|_{L^1(\mathbb{R})},$$

so we can apply dominated convergence theorem to find

$$\lim_{h \rightarrow 0} |\hat{f}(\zeta + h) - \hat{f}(\zeta)| \leq \int_{\mathbb{R}} |f(x)| \underbrace{\lim_{h \rightarrow 0} |e^{-2\pi i h x} - 1|}_{=0} dx = 0,$$

so  $\hat{f} \in C(\mathbb{R})$ .

**Proposition 3.3** (Properties of the Fourier Transform): Let  $f \in L^1(\mathbb{R})$  and  $g \in L^1(\mathbb{R})$ . Then,

a.  $\widehat{\tau_y f}(\zeta) = e^{-2\pi i \zeta y} \hat{f}(\zeta)$ , and  $\tau_\eta \hat{f}(\zeta) = \widehat{e^{2\pi i \eta(\cdot)} f(\cdot)}(\zeta)$ ;

b.  $\widehat{f * g} = \hat{f} \cdot \hat{g}$ ;

c.  $\int_{\mathbb{R}} f(x) \hat{g}(x) dx = \int_{\mathbb{R}} \hat{f}(x) g(x) dx$ .

**PROOF.** a. A change of variables gives

$$\widehat{\tau_y f}(\zeta) = \int_{\mathbb{R}} f(x - y) e^{-2\pi i \zeta x} dx = e^{-2\pi i \zeta y} \int_{\mathbb{R}} f(x) e^{-2\pi i \zeta x} dx = e^{-2\pi i \zeta y} \hat{f}(\zeta).$$

Similarly,

$$\tau_\eta \hat{f}(\zeta) = \int_{\mathbb{R}} f(x) e^{-2\pi i(\zeta - \eta)x} dx = \int_{\mathbb{R}} f(x) e^{2\pi i \eta x} \cdot e^{-2\pi i \zeta x} dx = \widehat{e^{2\pi i \eta(\cdot)} f(\cdot)}(\zeta).$$

b. First, by Young's inequality  $f * g \in L^1(\mathbb{R})$  so this makes sense. Moreover, since  $f, g \in L^1(\mathbb{R})$ , everything we need to be finite is finite, so we can apply Fubini's theorem to find

$$\begin{aligned}\widehat{f * g}(\zeta) &= \int \left( \int f(x-y)g(y) \, dy \right) e^{-2\pi i \zeta x} \, dx \\ &= \int \left( \int f(x-y)e^{-2\pi i \zeta x} \, dx \right) g(y) \, dy \\ &= \int \left( \int f(x-y)e^{-2\pi i \zeta(x-y)} \, dx \right) e^{-2\pi i \zeta y} g(y) \, dy \\ &= \left( \int f(x)e^{-2\pi i \zeta x} \, dx \right) \left( \int g(y)e^{-2\pi i \zeta y} \, dy \right) = \hat{f}(\zeta) \cdot \hat{g}(\zeta),\end{aligned}$$

where we “multiply by 1” in the second to last line to change variables in the appropriate way.

c. We can apply Fubini's again,

$$\begin{aligned}\int f(x)\hat{g}(x) \, dx &= \int f(x) \left( \int g(y)e^{-2\pi i xy} \, dy \right) \, dx \\ &= \int g(y) \left( \int f(x)e^{-2\pi i xy} \, dx \right) \, dy \\ &= \int g(y)\hat{f}(y) \, dy.\end{aligned}$$

■

↪ **Lemma 3.1:** Let  $f(x) = e^{-\pi ax^2}$  for  $a > 0$ . Then,

$$\hat{f}(\zeta) = \frac{1}{\sqrt{a}} e^{-\pi \frac{\zeta^2}{a}}.$$

PROOF. First, note that

$$\begin{aligned}\widehat{\frac{d}{dx}f}(\zeta) &= \int_{-\infty}^{\infty} f'(x)e^{-2\pi i \zeta x} \, dx \\ &= f(x)e^{-2\pi i \zeta x} \Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-2\pi i \zeta)e^{-2\pi i \zeta x} \, dx.\end{aligned}$$

Specifying  $f(x) = e^{-\pi ax^2}$ , this becomes

$$\begin{aligned}&= e^{-\pi ax^2} \cdot e^{-2\pi i \zeta x} \Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-\pi ax^2} (-2\pi i \zeta) e^{-2\pi i \zeta x} \, dx \\ &= 2\pi i \zeta \cdot \int_{-\infty}^{\infty} e^{-\pi ax^2} e^{-2\pi i \zeta x} \, dx = 2\pi i \zeta \cdot \hat{f}(\zeta).\end{aligned}$$

On the other hand,



$$\begin{aligned}\frac{d}{d\zeta}\hat{f}(\zeta) &= \frac{d}{d\zeta}\left(\int_{-\infty}^{\infty} f(x)e^{-2\pi i\zeta x} dx\right) \\ &= \int_{-\infty}^{\infty} f(x)(-2\pi ix)e^{-2\pi i\zeta x} dx,\end{aligned}$$

assuming finiteness; indeed,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} e^{-\pi ax^2}(-2\pi x)e^{-2\pi i\zeta x} dx\right| &\leq C \int_{-\infty}^{\infty} |x|e^{-\pi ax^2} dx \\ &= 2C \int_0^{\infty} xe^{-\pi ax^2} dx = Ce^{-\pi a\zeta^2}\Big|_0^{\infty} < \infty,\end{aligned}$$

so our differentiation was valid. Thus, combining these two,

$$\begin{aligned}\frac{d}{d\zeta}\hat{f}(\zeta) &= \int_{-\infty}^{\infty} -2\pi ix f(x)e^{-2\pi i\zeta x} dx \\ &= \int_{-\infty}^{\infty} i(-2\pi xe^{-\pi ax^2})e^{-2\pi i\zeta x} dx \\ &= \int_{-\infty}^{\infty} \frac{i}{a}f'(x)e^{-2\pi i\zeta x} dx \\ &= \frac{i}{a}2\pi i\zeta\hat{f}(\zeta) \\ &\Rightarrow \frac{d}{d\zeta}\hat{f}(\zeta) = -\frac{2\pi}{a}\zeta\hat{f}(\zeta).\end{aligned}$$

Thus,

$$\frac{d}{d\zeta}\left(e^{\frac{\pi\zeta^2}{a}}\hat{f}(\zeta)\right) = e^{(\pi\zeta^2)/a}\left(-\frac{2\pi}{a}\zeta\hat{f}(\zeta)\right) + \frac{2\pi\zeta}{a}e^{(\pi\zeta^2)/a}\hat{f}(\zeta) = 0,$$

and thus  $e^{\frac{\pi\zeta^2}{a}}\hat{f}(\zeta)$  is constant in  $\zeta$  so  $e^{\frac{\pi\zeta^2}{a}}\hat{f}(\zeta) = \hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi ax^2} dx = \frac{1}{\sqrt{a}}$ . Thus,  $\hat{f}(\zeta) = \frac{1}{\sqrt{a}}e^{-\frac{\pi\zeta^2}{a}}$ . ■

With this, we are ready to define the inverse Fourier transform;

↪ **Definition 3.3** (Inverse Fourier Transform): If  $f \in L^1(\mathbb{R})$ , then

$$\check{f}(x) := \int_{\mathbb{R}} f(\zeta)e^{2\pi i\zeta x} d\zeta = \widehat{f(-\cdot)}(x).$$

**Remark 3.7:** By similar computations to before,  $f \in L^1(\mathbb{R})$  implies  $\check{f} \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ .

**Remark 3.8:** One would hope  $\check{f} = f$ . However, if we check, naively,

$$\check{f}(x) = \int \left( \int f(x) e^{-2\pi i \zeta y} dy \right) e^{2\pi i \zeta x} d\zeta;$$

however the integral may not be finite in general, i.e. we cannot switch the integrals for free. We must be more careful, in short.

**→ Theorem 3.4 (Fourier Inversion):** If  $f \in L^1(\mathbb{R})$  and  $\hat{f} \in L^1(\mathbb{R})$ , then  $f$  agrees almost everywhere with some  $f_0 \in C(\mathbb{R})$ , and  $\check{\check{f}} = \hat{\hat{f}} = f_0$ .

PROOF. Let  $\varepsilon > 0$  and  $x \in \mathbb{R}$ . Let  $\varphi(\zeta) := e^{2\pi i x \zeta} e^{-\pi \varepsilon \zeta^2}$ . Then,

$$\begin{aligned} \hat{f}(y) &= \int \varphi(\zeta) e^{-2\pi i y \zeta} d\zeta \\ &= \int e^{2\pi i x \zeta} e^{-\pi \varepsilon \zeta^2} e^{-2\pi i y \zeta} d\zeta \\ &= e^{2\pi i x(\cdot)} \widehat{e^{-\pi \varepsilon(\cdot)^2}}(y) \\ &= \tau_x \widehat{e^{-\pi \varepsilon(\cdot)^2}}(y) \\ &= \tau_x \left( \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi y^2}{\varepsilon}} \right) \\ &= \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi}{\varepsilon}(y-x)^2}. \end{aligned}$$

Since  $\int f \hat{\varphi} dy = \int \hat{f} \varphi dy$ , we find

$$\int f(y) \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi}{\varepsilon}(x-y)^2} dy = \int \hat{f}(y) \varphi(y) dy.$$

Let  $K_\varepsilon(y) := \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi}{\varepsilon} y^2}$ . Recall that this is the good kernel that we used in the proof of the Weierstrass Approximation Theorem. In particular, the formula equation above can be written

$$(f * K_\varepsilon)(x) = \int \hat{f}(y) e^{2\pi i x y} e^{-\pi \varepsilon y^2} dy. \quad \circledast$$

Recall that if  $f$  is continuous at  $x$  and compactly-supported, then  $\lim_{\varepsilon \rightarrow 0} |(f * K_\varepsilon)(x) - f(x)| = 0$ . This implies that for every  $f \in L^1(\mathbb{R})$   $\lim_{\varepsilon \rightarrow 0} \|f * K_\varepsilon - f\|_1 = 0$ . So, taking  $\varepsilon \rightarrow 0$  in  $\circledast$ ,  $f(x) \stackrel{L^1(\mathbb{R})}{=} \lim_{\varepsilon \rightarrow 0} \int \hat{f}(y) e^{2\pi i x y} e^{-\pi \varepsilon y^2} dy$ .  $\hat{f} \in L^1(\mathbb{R})$ , so by DCT we can pass the limit inside, so

$$f(x) \stackrel{L^1(\mathbb{R})}{=} \int \hat{f}(y) e^{2\pi i x y} dy = \check{\check{f}}(x).$$

This equality in  $L^1(\mathbb{R})$  thus gives  $\check{\check{f}} = f$  a.e.. A similar proof follows for showing  $\hat{\hat{f}} = f$  a.e. by replacing  $e^{2\pi i x}$  with  $e^{-2\pi i x}$  everywhere it appears. Since  $\left(\hat{\check{f}}\right), \left(\check{\hat{f}}\right)$  are

continuous by our remarks earlier, it follows that  $f$  is equal to a continuous function almost everywhere. ■

So far, all we've worked with is  $f \in L^1(\mathbb{R})$ , which results in  $\hat{f} \in L^\infty(\mathbb{R})$ . Really, we'd like to extend the Fourier transform to be  $L^2(\mathbb{R})$ , since this is a nice Hilbert space. To do so, we need the following:

**→ Theorem 3.5** (Plancherel's Theorem): Let  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ . Then  $\hat{f} \in L^2(\mathbb{R})$  and  $\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}$ .

**Remark 3.9:** One can view Plancherel's Theorem as a type of continuous analog of Parseval's identity for Fourier Series.

PROOF. Let  $f(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , and put  $g(x) := \overline{f(-x)}$ , noting that then  $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$  as well. Put

$$w(x) := (f * g)(x).$$

By Young's,

$$\|w\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} < \infty$$

so  $w \in L^1(\mathbb{R})$ .

We claim  $w$  continuous at 0. For  $h$  sufficiently small, we find

$$\begin{aligned} |w(h) - w(0)| &= \left| \int_{\mathbb{R}} f(h-y)g(y) \, dy - \int_{\mathbb{R}} f(-y)g(y) \, dy \right| \\ &= \left| \int_{\mathbb{R}} (f(h-y) - f(-y))g(y) \, dy \right| \\ &\leq \|f(h-\cdot) - f(-\cdot)\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \\ &\leq \|\tau_h f - f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}. \end{aligned}$$

Let  $\tilde{f} \in C_c(\mathbb{R})$  such that  $\|f - \tilde{f}\|_{L^2(\mathbb{R})} < \eta$  for some small  $\eta > 0$ . Then we further bound

$$\|\tau_h f - f\|_{L^2(\mathbb{R})} \leq \|\tau_h f - \tau_h \tilde{f}\|_{L^2(\mathbb{R})} + \|\tau_h \tilde{f} - \tilde{f}\|_{L^2(\mathbb{R})} + \|\tilde{f} - f\|_{L^2(\mathbb{R})}$$

$$\begin{aligned} \text{(since norm translation invariant)} \quad &= 2\|\tilde{f} - f\|_{L^2(\mathbb{R})} + \|\tau_h \tilde{f} - \tilde{f}\|_{L^2(\mathbb{R})} \\ &\leq 2\eta + \|\tau_h \tilde{f} - \tilde{f}\|_{L^2(\mathbb{R})}. \end{aligned}$$

Now,  $\tilde{f} \in C_c(\mathbb{R})$  and thus is uniformly continuous hence  $|\tau_h \tilde{f}(x) - \tilde{f}(x)| \rightarrow 0$  uniformly on  $\mathbb{R}$  hence  $\|\tau_h \tilde{f} - \tilde{f}\|_{L^2(\mathbb{R})} \rightarrow 0$  as well, as  $h \rightarrow 0$ . Finally, since  $\|g\|_{L^2(\mathbb{R})}$  finite, we conclude indeed  $w$  continuous at 0.

Next, notice that  $\hat{w} = \hat{f} \cdot \hat{g}$ , and

$$\begin{aligned}
\hat{g}(\zeta) &= \widehat{f(-\cdot)}(\zeta) = \int_{\mathbb{R}} \overline{f(-x)} e^{-2\pi i x \zeta} dx \\
&= \int_{\mathbb{R}} \overline{f(-x)} e^{2\pi i x \zeta} dx \\
&= \overline{\int_{\mathbb{R}} f(-x) e^{2\pi i x \zeta} dx} \\
&= \overline{\int_{\mathbb{R}} f(x) e^{-2\pi i x \zeta} dx} \\
&= \overline{\hat{f}(\zeta)},
\end{aligned}$$

so

$$\hat{w} = \hat{f} \cdot \overline{\hat{f}} = |\hat{f}|^2 \geq 0.$$

Recall our good kernel from the Weierstrass Approximation Theorem,  $K_\varepsilon(y) = \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi}{\varepsilon} y^2} = \widehat{e^{-\pi \varepsilon (\cdot)^2}}$ . So,

$$\begin{aligned}
\int \hat{w}(y) e^{-\pi \varepsilon y^2} dy &= \int w(y) \frac{1}{\sqrt{\varepsilon}} e^{-(\pi y^2)/\varepsilon} dy \\
&= \int w(y) K_\varepsilon(y) dy \\
(\text{by symmetry}) \quad &= \int w(y) K_\varepsilon(-y) dy \\
&= (w * K_\varepsilon)(0).
\end{aligned}$$

On the LHS,  $\hat{w} \geq 0$  so  $\hat{w}(y) e^{-\pi \varepsilon y^2} \nearrow_{\varepsilon \rightarrow 0^+} \hat{w}(y)$  so by monotone convergence,  $\int \hat{w} e^{-\pi \varepsilon y^2} dy \xrightarrow{\varepsilon \rightarrow 0} \int \hat{w}(y) dy$ . On the other hand, we claim  $(w * K_\varepsilon)(0) \xrightarrow{\varepsilon \rightarrow 0} w(0)$  (this isn't immediate from the fact that  $K_\varepsilon$  is a good kernel because we don't know a priori that  $w$  (essentially) bounded). Supposing this claim holds, this implies  $\int \hat{w}(y) dy = w(0)$ , hence

$$\begin{aligned}
\int \hat{w}(y) dy &= \int |\hat{f}(y)|^2 dy = w(0) \\
&= (f * g)(0) \\
&= \int f(y) \overline{f(0 - (-y))} dy \\
&= \int |f(y)|^2 dy,
\end{aligned}$$

which precisely means  $\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$ .

To prove the claim of  $(K_\varepsilon * w)(0) \rightarrow w(0)$ , let  $\eta > 0$ . Since  $w$  continuous at 0, there is a  $\delta > 0$  such that  $|y| < \delta \Rightarrow |w(y) - w(0)| < \eta$ . Then,

$$\begin{aligned}
\left| \int w(0-y) K_\varepsilon(y) \, dy - w(0) \right| &= \left| \int (w(-y) - w(0)) K_\varepsilon(y) \, dy \right| \\
&\leq \eta \int_{|y| < \delta} K_\varepsilon(y) \, dy + \int_{|y| > \delta} |w(0)| K_\varepsilon(y) \, dy + \int_{|y| > \delta} |w(-y)| K_\varepsilon(y) \, dy \\
&\leq \eta \cdot 1 + \underbrace{|w(0)|}_{\substack{w \text{ cnts at } 0 \\ \text{so this finite}}} \cdot \underbrace{\int_{|y| > \delta} K_\varepsilon(y) \, dy}_{\substack{\rightarrow 0 \text{ since good kernel} \\ \varepsilon \rightarrow 0}} + \int_{|y| > \delta} |w(-y)| K_\varepsilon(y) \, dy.
\end{aligned}$$

It remains to show the last term  $\rightarrow 0$ . We have

$$\begin{aligned}
\int_{|y| > \delta} |w(-y)| K_\varepsilon(y) \, dy &\leq \int_{|y| > \delta} |w(-y)| \frac{1}{\sqrt{\varepsilon}} e^{-\pi \frac{\delta^2}{\varepsilon}} \, dy \\
&\leq \underbrace{\frac{1}{\sqrt{\varepsilon}} e^{-\pi \frac{\delta^2}{\varepsilon}}}_{\rightarrow 0 \text{ as } \varepsilon \rightarrow 0} \cdot \|w\|_{L^1(\mathbb{R})} \rightarrow 0.
\end{aligned}$$

This completes the proof. ■