MATH454 - Analysis 3

Measure spaces; Integration.

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§1 Sigma Algebras and Measures

§1.1 A Review of Riemann Integration

Let $f : \mathbb{R} \to \mathbb{R}$ and $[a, b] \subset \mathbb{R}$. Define a **partition** of $[a, b]$ as the set

$$\mathrm{part}([a, b]) := \{a =: x_0 < x_1 < \cdots < x_N := b\}.$$

We can then define the upper and lower Riemann integrals of f over the region $[a, b]$ as

$$\text{upper:} \qquad \overline{\int_a^b} f(x)\,\mathrm{d}x := \inf_{\mathrm{part}([a,b])} \left\{ \sum_{\{i=1\}}^{N} \sup_{x\in[x_{i-1},x_i]} f(x) \cdot (x_i - x_{i-1}) \right\}$$

$$\text{lower:} \qquad \underline{\int_a^b} f(x)\,\mathrm{d}x := \sup_{\mathrm{part}([a,b])} \left\{ \sum_{\{i=1\}}^{N} \inf_{x\in[x_{i-1},x_i]} f(x) \cdot (x_i - x_{i-1}) \right\}.$$

We then say f **Riemann integrable** if these two quantities are equal, and denote this value by $\int_a^b f(x)\,\mathrm{d}x$.

Many "nice-enough" (continuous, monotonic, etc.) functions are Riemann integrable, but many that we would like to be able to "integrate" are simply not, for instance Dirichlet's function $x \mapsto \begin{cases} 1 x\in\mathbb{Q}\setminus[a,b] \\ 0 x\in\mathbb{Q}^c\setminus[a,b] \end{cases}$. Hence, we need a more general notion of integration.

§1.2 Sigma Algebras

↪**Definition 1.1** (Sigma algebra): Let X be a *space* (a nonempty set) and $\mathcal{F}$ a collection of subsets of X. $\mathcal{F}$ a *sigma algebra* or simply σ-algebra of X if the following hold:

1. $X \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ (closed under complement)
3. $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ (closed under countable unions)

↪**Proposition 1.1**:

4. $\varnothing \in \mathcal{F}$
5. $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
6. $A_1, ..., A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
7. $A, B \in \mathcal{F} \Rightarrow A \setminus B, B \setminus A \in \mathcal{F}$

⊛ **Example 1.1**: The "largest" sigma algebra of a set X is the power set 2^X, the smallest the trivial $\{\varnothing, X\}$.

Given a set $A \subset X$, the set $\mathcal{F}_A := \{\varnothing, X, A, A^c\}$ is a sigma algebra; given two disjoint sets $A, B \subset X$, then $\mathcal{F}_{A,B} := \{\varnothing, X, A, A^c, B, B^c, A \cup B, A^c \cap B^c\}$ a sigma algebra.

↪**Definition 1.2** (Generating a sigma algebra): Let X be a nonempty set, and $\mathcal{C}$ a collection of subsets of X. Then, the σ-algebra *generated* by $\mathcal{C}$, denoted $\sigma(\mathcal{C})$, is such that

1. $\sigma(\mathcal{C})$ a sigma algebra with $\mathcal{C} \subseteq \sigma(\mathcal{C})$
2. if $\mathcal{F}'$ a sigma algebra with $\mathcal{C} \subseteq \mathcal{F}'$, then $\mathcal{F}' \supseteq \sigma(\mathcal{C})$

Namely, $\sigma(\mathcal{C})$ is the smallest sigma algebra "containing" (as a subset) $\mathcal{C}$.

↪**Proposition 1.2**:

1. $\sigma(\mathcal{C}) = \bigcap\{\mathcal{F} : \mathcal{F} \text{ a sigma algebra containing } \mathcal{C}\}$
2. if $\mathcal{C}$ itself a sigma algebra, then $\sigma(\mathcal{C}) = \mathcal{C}$
3. if $\mathcal{C}_1, \mathcal{C}_2$ are two collections of subsets of X such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_2)$

↪**Definition 1.3** (The Borel sigma-algebra): The *Borel σ-algebra*, denoted $\mathfrak{B}_{\mathbb{R}}$, on the real line is given by

$$\mathfrak{B}_{\mathbb{R}} := \sigma(\{\text{open subsets of } \mathbb{R}\}).$$

We call sets in $\mathfrak{B}_{\mathbb{R}}$ *Borel sets*.

↪**Proposition 1.3**: $\mathfrak{B}_{\mathbb{R}}$ is also generated by the sets

- $\{(a, b) : a < b \in \mathbb{R}\}$
- $\{(a, b] : a < b \in \mathbb{R}\}$
- $\{[a, b] : a < b \in \mathbb{R}\}$
- $\{[a, b) : a < b \in \mathbb{R}\}$ $\circledast$
- $\{(-\infty, c) : c \in \mathbb{R}\}$
- $\{(-\infty, c] : c \in \mathbb{R}\}$
- etc.

Proof. We prove just $\circledast$. It suffices to show that the generating sets of each σ-algebra is contained in the other σ-algebra. Let $a < b \in \mathbb{R}$. Then,

$$(a, b) = \bigcup_{n=1}^{\infty} \underbrace{\left[a + \frac{1}{n}, b\right)}_{\in \circledast} \in \sigma(\{[a, b)\}) \Rightarrow \mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[a, b)\}).$$

Conversely,

$$[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right) \in \mathfrak{B}_{\mathbb{R}}.$$

■

↪**Proposition 1.4**: All intervals (open, closed, half open, half closed, finite, etc) are Borel sets; any set obtained from countable set operations of intervals are Borel; all singletons are Borel; any finite and countable sets are Borel.

§1.3 Measures

↪**Definition 1.4** (Measurable Space): Let X be a space and $\mathcal{F}$ a σ-algebra. We call the tuple $(X, \mathcal{F})$ a *measurable space*.

↪**Definition 1.5** (Measure): Let $(X, \mathcal{F})$ be a measurable space. A *measure* is a function $\mu : \mathcal{F} \to [0, \infty]$ satisfying

(i) $\mu(\varnothing) = 0$;

(ii) if $\{A_n\} \subseteq \mathcal{F}$ a sequence of (pairwise) disjoint sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

i.e. μ is *countably additive*. We further call μ

- *finite* if $\mu(X) < \infty$,
- a *probability measure* if $\mu(X) = 1$,
- σ-finite if $\exists \{A_n\} \subseteq \mathcal{F}$ such that $X = \bigcup_{n=1}^{\infty} A_n$ with $\mu(A_n) < \infty \; \forall \, n \geq 1$,

and call the triple $(X, \mathcal{F}, \mu)$ a *measure space*.

⊛ **Example 1.2**: The measure on $\mathfrak{B}_{\mathbb{R}}$ given by

$$A \mapsto \begin{cases} |A| \text{ if } A \text{ finite} \\ \infty \quad \text{else} \end{cases}$$

is called the *counting measure*.

Fix $x_0 \in \mathbb{R}$, then the measure on $\mathfrak{B}_{\mathbb{R}}$ given by

$$A \mapsto \begin{cases} 1 \text{ if } x_0 \in A \\ 0 \text{ else} \end{cases}$$

is called the *point mass at x_0*.

↪**Theorem 1.1** (Properties of Measures): Fix a measure space $(X, \mathcal{F}, \mu)$. The following properties hold:

1. (finite additivity) For any sequence $\{A_n\}_{n=1}^N \subseteq \mathcal{F}$ of disjoint sets,
$$\mu\left(\bigcup_{n=1}^N A_n\right) = \sum_{n=1}^N \mu(A_n).$$
2. (monotonicity) For any $A \subseteq B \in \mathcal{F}$, then $\mu(A) \leq \mu(B)$.
3. (countable/finite subadditivity) For any sequence $\{A_n\} \subseteq \mathcal{F}$ (**not** necessarily disjoint),
$$\mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n),$$
an analogous statement holding for a finite collection of sets $A_1, ..., A_N$.
4. (continuity from below) For $\{A_n\} \subseteq \mathcal{F}$ such that $A_n \subseteq A_{n+1}\ \forall\, n \geq 1$ (in which case we say $\{A_n\}$ "increasing" and write $A_n \uparrow$) we have
$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n\to\infty} \mu(A_n).$$
5. (continuity from above) For $\{A_n\} \subseteq \mathcal{F}, A_n \supseteq A_{n+1}\ \forall\, n \geq 1$ (we write $A_n \downarrow$) we have that **if** $\mu(A_1) < \infty$,
$$\mu\left(\bigcap_{n=1}^\infty A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Remark 1.1: In 4., note that since A_n increasing, that the union $\bigcup_{n=1}^\infty A_n \supseteq A_m$ for any arbitrarily large m; indeed, one could logically right $\lim_{n\to\infty} A_n = \bigcup_{n=1}^\infty A_n$. This this notation, then, 4. simply states that we may interchange limit and measure. A similar argument can be viewed for 5. (how?).

Remark 1.2: The finiteness condition in 5. may be slightly modified such as to state that $\mu(A_n) < \infty$ for some n; remark why this would suffice to ensure the entire rest of the sequence has finite measure.

Proof.

1. Extend $A_1, ..., A_N$ to an infinite sequence by $A_n := \emptyset$ for $n > N$. Then this simply follows from countable additivity and $\mu(\emptyset) = 0$.
2. We may write $B = A \cup (B \setminus A)$; this is a disjoint union of sets. By finite additivity, then,
$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A),$$
since the measure is positive.
3. We prove only for a countable union; use the technique from 1. to extend to finite. We first "disjointify" the sequence such that we can use the countable additivity

axiom. Let $B_1 = A_1, B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$ for $n \geq 2$. Remark then that $\{B_n\} \subseteq \mathcal{F}$ is a disjoint sequence of sets, and that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$. By countable additivity and subadditivity,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

4. We again "disjointify" the sequence $\{A_n\}$. Put $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$ for all $n \geq 2$ (remark that this is equivalent to the construction from the previous proof because the sets are increasing). Then, again, $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$, and in particular, for all $N \geq 1$, $\bigcup_{n=1}^{N} B_n = A_N$. Then

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \sum_{n=1}^{\infty} \mu(B_n) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n) \\ &= \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} B_n\right) = \lim_{N\to\infty} \mu(A_N).\end{aligned}$$

5. We yet again disjointify, backwards (in a way) from the previous case. Put $B_n = A_1 \setminus A_n$ for all $n \geq 1$. Then, $\{B_n\} \subseteq \mathcal{F}$, B_n increasing, and $\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \bigcap_{n=1}^{\infty} A_n$. Then, by continuity from below,

$$\mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(A_1 \setminus A_n)$$

and also

$$\begin{aligned}\mu(A_1) &= \mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) + \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \\ &= \mu(A_1 \setminus A_n) + \mu(A_n),\end{aligned}$$

and combining these two equalities yields the desired result.

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§1.4 Constructing the Lebesgue Measure on $\mathbb{R}$

↪**Definition 1.6** (Lebesgue outer measure): For all $A \subseteq \mathbb{R}$, define

$$m^*(A) := \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n, I_n \text{ open intervals}\right\},$$

called the *Lebesgue outer measure* of A (where $\ell(I)$ is the length of interval I, i.e. the absolute value of the difference of its endpoints, if finite, or ∞ if not).

↪**Proposition 1.5**: The following properties of m^* hold:

1. $m^*(A) \geq 0$ for all $A \subseteq \mathbb{R}$, and $m^*(\varnothing) = 0$.
2. (monotonicity) For $A \subseteq B$, $m^*(A) \leq m^*(B)$.
3. (countable subadditivity) For $\{A_n\}, A_n \subseteq \mathbb{R}$, $m^*\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty m^*(A_n)$.[1]
4. If $I \subseteq \mathbb{R}$ an interval, then $m^*(I) = \ell(I)$.
5. m^* is translation invariant; for any $A \subseteq R, x \in \mathbb{R}$, $m^*(A) = m^*(A+x)$ where $A + x := \{a + x : a \in A\}$.
6. For all $A \subseteq \mathbb{R}$, $m^*(A) = \inf\{m^*(B) : A \subseteq B \subseteq R, B - \text{open}\}$.
7. If $A = A_1 \cup A_2 \subseteq \mathbb{R}$ with $d(A_1, A_2) > 0$,[2] then $m^*(A_1) + m^*(A_2) = m^*(A)$.
8. If $A = \bigcup_{k=1}^\infty J_k$ where J_k's are "almost disjoint intervals" (i.e. share at most endpoints), then $m^*(A) = \sum_{k=1}^\infty m^*(J_k) = \sum_{k=1}^\infty \ell(J_k)$.

Proof.

3. If $m^*(A_n) = \infty$, for any n, we are done, so assume wlog $m^*(A_n) < \infty$ for all n. Then, for each n and $\varepsilon > 0$, one can choose open intervals $\{I_{n,i}\}_{i\geq 1}$ such that $A_n \subseteq \bigcup_{i=1}^\infty I_{n,i}$ and $\sum_{i=1}^\infty \ell(I_{n,i}) \leq m^*(A_n) + \frac{\varepsilon}{2^n}$. Hence

$$\bigcup_{n=1}^\infty A_n \subseteq \bigcup_{n=1,i=1}^\infty I_{n,i}$$

$$\Rightarrow m^*\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n,i=1}^\infty \ell(I_{n,i}) = \sum_{n=1}^\infty \sum_{i=1}^\infty \ell(I_{n,i}) \leq \sum_{n=1}^\infty \left(m^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^\infty m^*(A_n) + \varepsilon,$$

and as ε arbitrary, the statement follows.

4. We prove first for $I = [a, b]$. For any $\varepsilon > 0$, set $I_1 = (a - \varepsilon, b + \varepsilon)$; then $I \subseteq I_1$ so $m^*(I) \leq \ell(I_1) = (b-1) + 2\varepsilon$ hence $m^*(I) \leq b - a = \ell(I)$. Conversely, let $\{I_n\}$ be any open-interval convering of I (wlog, each of finite length; else the statement holds trivially). Since I compact, it can be covered by finitely many of the I_n's, say $\{I_n\}_{n=1}^N$, denoting $I_n = (a_n, b_n)$ (with relabelling, etc). Moreover, we can pick the a_n, b_n's such that $a_1 < a, b_N > b$, and generally $a_n < b_{n-1}$ $\forall\, 2 \leq n \leq N$. Then,

$$\begin{aligned}\sum_{n=1}^\infty \ell(I_n) \geq \sum_{n=1}^N \ell(I_n) &= b_1 - a_1 + \sum_{n=2}^N (b_n - a_n) \\ &\geq b_1 - a_1 + \sum_{n=2}^N (b_n - b_{n-1}) \\ &= b_N - a_1 \geq b - 1 = \ell(I),\end{aligned}$$

hence since the cover was arbitrary, $m^*(A) \geq \ell(I)$, and equality holds.

Now, suppose I finite, with endpoints $a < b$. Then for any $\frac{b-a}{2} > \varepsilon > 0$, then

$$[a + \varepsilon, b - \varepsilon] \subseteq I \subseteq [a - \varepsilon, b + \varepsilon],$$

[1]More generally, any set function on $2^{\mathbb{R}}$ that satisfies 1., 2., and 3. is called an *outer measure*.
[2]Remark: this is a stronger requirement than disjointness!

hence by monotonicity and the previous part of this proof

$$m^*([a+\varepsilon, b-\varepsilon]) = b-a-2\varepsilon \le m^*(I) \le b-a+2\varepsilon = m^*([a-\varepsilon, b+\varepsilon]),$$

from which it follows that $m^*(I) = b-a = \ell(I)$.

Finally, suppose I infinite. Then, $\forall\, M \ge 0, \exists$ closed, finite interval I_M with $I_M \subseteq I$ and $\ell(I_M) \ge M$. Hence, $m^*(I) \ge m^*(I_M) \ge M$ and thus as M arbitrary it must be that $m^*(I) = \infty = \ell(I)$.

6. Denote $\tilde{m}(A) := \inf\{m^*(B) : A \subseteq B \subseteq R, B - \text{open}\}$. For any $A \subseteq B \subseteq \mathbb{R}$ with B open, monotonicity gives that $m^*(A) \le m^*(B)$, hence $m^*(A) \le \tilde{m}(A)$. Conversely, assuming wlog $m^*(A) < \infty$ (else holds trivially), then for all $\varepsilon > 0$, there exists $\{I_n\}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ with $\sum_{n=1}^{\infty} \ell(I_n) \le m^*(A) + \varepsilon$. Setting $B := \bigcup_{n=1}^{\infty} I_n$, we have that $A \subseteq B$ and $m^*(B) = m^*(\bigcup I_n) \le$ (by finite subadditivity) $\sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \ell(I_n) \le m^*(A) + \varepsilon$ hence $m^*(B) \le m^*(A)$ for all B. Thus $m^*(A) \ge \tilde{m}(A)$ and equality holds.

7. Put $\delta := d(A_1, A_2) > 0$. Clearly $m^*(A) \le m^*(A_1) + m^*(A_2)$ by finite subadditivity. wlog, $m^*(A) < \infty$ (and hence $m^*(A_i) < \infty, i = 1, 2$) (else holds trivially). Then $\forall\, \varepsilon > 0, \exists \{I_n\} : A \subseteq \bigcup I_n$ and $\sum \ell(I_n) \le m^*(A) + \varepsilon$. Then, for all n, we consider a "refinement" of I_n; namely, let $\{I_{n,i}\}_{i \ge 1}$ such that $I_n \subseteq \bigcup_i I_{n,i}$ and $\ell(I_{n,i}) < \delta$ and $\sum_i \ell(I_{n,i}) \le \ell(I_n) + \frac{\varepsilon}{2^n}$. Relabel $\{I_{n,i} : n, i \ge 1\} \rightsquigarrow \{J_m : m \ge 1\}$ (both are countable). Then, $\{J_m\}$ defines an open-interval cover of A, and since $\ell(J_m) < \delta$ for each m, J_m intersects at most one A_i. For each m and $p = 1, 2$, put

$$M_p := \left\{m : J_m \cap A_p \neq \varnothing\right\},$$

noting that $M_1 \cap M_2 = \varnothing$. Then $\left\{J_m : m \in M_p\right\}$ is an open covereing of A_p, and so

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\le \sum_{m \in M_1} \ell(J_m) + \sum_{m \in M_2} \ell(J_m) \\ &\le \sum_{m=1}^{\infty} \ell(J_m) = \sum_{n,i=1}^{\infty} \ell(I_n, i) \\ &\le \sum_n \left(\ell(I_n) + \frac{\varepsilon}{2^n}\right) \\ &= \sum_n \ell(I_n) + \varepsilon \\ &\le m^*(A) + 2\varepsilon, \end{aligned}$$

and hence equality follows.

8. If $\ell(J_k) = \infty$ for some k, then since $J_k \subseteq A$, subadditivity gives us that $m^*(J_k) \le m^*(A)$ and so $m^*(A) = \infty = \sum_{k=1}^{\infty} \ell(J_k)$ (since if any J_k infinite, the sum of the lengths of all of them will also be infinite).

Suppose then $\ell(J_k) < \infty$ for all k. Fix $\varepsilon > 0$. Then for all $k \ge 1$, choose $I_k \subseteq J_k$ such that $\ell(J_k) \le \ell(I_k) + \frac{\varepsilon}{2^k}$. For any $N \ge 1$, we can choose a subset $\{I_1, ..., I_N\}$ of intervals such that all are disjoint, with strictly positive distance between them, and so

$$\bigcup_{k=1}^{N} I_k \subseteq \bigcup_{k=1}^{N} I_k \subseteq A$$

$$\begin{aligned}
\Rightarrow m^*(A) \geq m^*\left(\bigcup_{k=1}^{N} I_k\right) &\geq \sum_{k=1}^{N} \ell(I_k) \\
&\geq \sum_{k=1}^{N} \left(\ell(J_k) - \frac{\varepsilon}{2^k}\right) \\
&\geq \sum_{k=1}^{N} \ell(J_k) - \varepsilon
\end{aligned}$$

$$\Rightarrow m^*(A) \geq \sum_{k=1}^{\infty} \ell(J_k),$$

the second inequality following from finite subadditivity. The converse of the final inequality holds trivially. ■

§1.5 Lebesgue-Measurable Sets

↪**Definition 1.7**: $A \subseteq \mathbb{R}$ is m^*-*measurable* if $\forall\, B \subseteq \mathbb{R}$,

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c).$$

Remark 1.3: By subadditivity, $\leq$ always holds in the definition above.

↪**Theorem 1.2** (Carathéodary's Theorem): Let

$$\mathcal{M} := \{A \subseteq \mathbb{R} : A \;\; m^* - \text{measurable}\}.$$

Then, $\mathcal{M}$ is a σ-algebra of subsets of $\mathbb{R}$.

Define $m : \mathcal{M} \to [0, \infty]$, $m(A) = m^*(A)$. Then, m is a measure on $\mathcal{M}$, called the *Lebesgue measure* on $\mathbb{R}$. We call sets in $\mathcal{M}$ *Lebesgue-measurable* or simply *measurable* (if clear from context) accordingly. We call $(\mathbb{R}, \mathcal{M}, m)$ the *Lebesgue measure space*.

Proof. The first two σ-algebra axioms are easy. We have for any $B \subseteq \mathbb{R}$ that

$$m^*(B \cap \mathbb{R}) + m^*(B \cap \mathbb{R}^c) = m^*(B) + m^*(B \cap \varnothing) = m^*(B)$$

so $\mathbb{R} \in \mathcal{M}$. Further, $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$ by the symmetry of the requirement for sets to be in $\mathcal{M}$.

The final axiom takes more work. We show first $\mathcal{M}$ closed under finite unions; by induction it suffices to show for 2 sets. Let $A_1, A_2 \in \mathcal{M}$. Then, for all $B \subseteq \mathbb{R}$,

$$\begin{aligned}
m^*(B) &= m^*(B \cap A_1) + m^*(B \cap A_1^c) \\
&= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*(B \cap A_1^c \cap A_2^c) \\
&= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*\left(B \cap (A_1 \cup A_2)^c\right)
\end{aligned}$$

Note that $(B \cap A_1) \cup (B \cap A_1^c \cap A_2^c) = B \cap (A_1 \cup A_2)$, hence by subadditivity,

$$m^*(B) \geq m^*(B \cap (A_1 \cup A_2)) + m^*\big(B \cap (A_1 \cup A_2)^c\big),$$

and since the other direction of the inequality comes for free, we conclude $A_1 \cup A_2 \in \mathcal{M}$.

Let now $\{A_n\} \subseteq \mathcal{M}$. We "disjointify" $\{A_n\}$; put $B_1 := A_1$, $B_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i$, $n \geq 2$, noting $\bigcup_n A_n = \bigcup_n B_n$, and each $B_n \in \mathcal{M}$, as each is but a finite number of set operations applied to the A_n's, and thus in $\mathcal{M}$ as demonstrated above. Put $E_n := \bigcup_{i=1}^n B_i$, noting again $E_n \in \mathcal{M}$. Then, for all $B \subseteq \mathbb{R}$,

$$\begin{aligned}
m^*(B) &= m^*\left(\underbrace{B \cap E_n}_{\text{chop up } B_n}\right) + m^*\left(\underbrace{B \cap E_n^c}_{E_n \subseteq \cup B_n \Rightarrow E_n^c \supseteq (\cup B_n)^c}\right) \\
&\geq m^*\left(B \cap \underbrace{E_n \cap B_n}_{=B_n}\right) + m^*\left(B \cap \underbrace{E_n \cap B_n^c}_{=E_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\
&\geq m^*(B \cap B_n) + m^*\left(\underbrace{B \cap E_{n-1}}_{\text{chop up } B_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\
&\geq m^*(B \cap B_n) + m^*(B \cap E_{n-1} \cap B_{n-1}) \\
&\qquad + m^*(B \cap E_{n-1} \cap B_{n-1}^c) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right).
\end{aligned}$$

Notice that the last line is essentially the second applied to B_{n-1}; hence, we have a repeating (essentially, "descending") pattern in this manner, which we repeat until $n \to 1$. We have, thus, that

$$m^*(B) \geq \sum_{i=1}^{n} [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right),$$

so taking $n \to \infty$,

$$\begin{aligned}
m^*(B) &\geq \sum_{i=1}^{\infty} [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\
&\geq m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right).
\end{aligned}$$

As usual, the inverse inequality comes for free, and thus we can conclude $\bigcup_{n=1}^{\infty} B_n$ also m^*-measurable, and thus so is $\bigcup_{n=1}^{\infty} A_n$. This proves $\mathcal{M}$ a σ-algebra.

We show now m a measure. By previous propositions, we have that $m \geq 0$ and $m(\varnothing) = 0$ (since $m = m^* |_{\mathcal{M}}$), so it remains to prove countable subadditivity.

Let $\{A_n\} \subseteq \mathcal{M}$-disjoint. Following precisely the same argument as above, used to prove that $\mathcal{M}$ closed under countable unions, shows that for any $n \geq 1$

$$m\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(A_i),$$

that is, finite additivity holds, and thus by subadditivity

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq m\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(A_i),$$

and so taking the limit of $n \to \infty$, we have

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} m(A_i),$$

with the converse inequality coming for free. Thus, m indeed a measure on $\mathcal{M}$. ■

↪**Proposition 1.6**: $\mathcal{M}, m$ translation invariant; for all $A \in \mathcal{M}, x \in \mathbb{R}, x + A = \{x + a : a \in A\} \in \mathcal{M}$ and $m(A) = m(A + x)$.

Remark 1.4: We would like this to hold, heuristically, since if we shift sets on the real line, we should expect their length to remain constant.

PROOF. For all $B \subseteq \mathbb{R}$, we have (since m^* translation invariant)

$$\begin{aligned} m^*(B) = m^*(B - x) = m^*\left(\underbrace{(B - x) \cap A}_{=B\cap(A+x)}\right) + m^*\left(\underbrace{(B - x) \cap A^c}_{=B\cap(A^c+x)=B\cap(A+x)^c}\right) \\ = m^*(B \cap (A + x)) + m^*\left(B \cap (A + x)^c\right), \end{aligned}$$

thus $A + x \in \mathcal{M}$, and since m^* translation invariant, it follows that m is. ■

↪**Theorem 1.3**: $\forall\, a, b \in \mathbb{R}$ with $a < b$, $(a, b) \in \mathcal{M}$, and $m((a, b)) = b - a$.

Remark 1.5: Again, we'd like this to hold, heuristically, since we would like the measure of an interval to simply be its length; we'd moreover like to be able to measure intervals, i.e. have intervals be contained in $\mathcal{M}$.

↪**Corollary 1.1**: $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M}$

PROOF. $\mathfrak{B}_{\mathbb{R}}$ is generated by open intervals of the form (a, b). All such intervals are in $\mathcal{M}$ by the previous theorem, and hence the proof. ■

§1.6 Properties of the Lebesgue Measure

↪**Proposition 1.7** (Regularity Properties of m): For all $A \in \mathcal{M}$, the following hold.

- For all $\varepsilon > 0, \exists G$ open such that $A \subseteq G$ and $m(G \setminus A) < \varepsilon$.
- For all $\varepsilon > 0$, $\exists F$-closed such that $F \subseteq A$ and $m(A \setminus F) \leq \varepsilon$.
- $m(A) = \inf\{m(G) : G \text{ open}, G \supseteq A\}$.
- $m(A) = \sup\{m(K) : K \text{ compact}, K \subseteq A\}$.
- If $m(A) < \infty$, then for all $\varepsilon > 0$, $\exists K \subseteq A$ compact, such that $m(A \setminus K) < \varepsilon$.
- If $m(A) < \infty$, then for all $\varepsilon \geq 0$, $\exists$ finite collection of open intervals $I_1, ..., I_N$ such that $m\left(A \triangle \left(\bigcup_{n=1}^N I_n\right)\right) \leq \varepsilon$.

↪**Proposition 1.8** (Completeness of m): $(\mathbb{R}, \mathcal{M}, m)$ is *complete*, in the sense that for all $A \subseteq \mathbb{R}$, if $\exists B \in \mathcal{M}$ such that $A \subseteq B$ and $m(B) = 0$, then $A \in \mathcal{M}$ and $m(A) = 0$.

Equivalently, any subset of a null set is again a null set.

Remark 1.6: In general, $A \in \mathcal{F}, B \subseteq A \not\Rightarrow B \in \mathcal{F}$.

↪**Proposition 1.9**: Up to rescaling, m is the unique, nontrivial measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ that is finite on compact sets and is translation invariant, i.e. if μ another such measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ with $\mu = c \cdot m$ for $c > 0$, then $\mu = m$.

Remark 1.7: Such a c is simply $c = \mu((0, 1))$.

To prove this proposition, we first introduce some helpful tooling:

↪**Theorem 1.4** (Dynkin's π-d): Given a space X, let $\mathcal{C}$ be a collection of subsets of X. $\mathcal{C}$ is called a *π-system* if $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$ (that is, it is closed under finite intersections).

Let $\mathcal{F} = \sigma(\mathcal{C})$, and suppose μ_1, μ_2 are two finite measures on $(X, \mathcal{F})$ such that $\mu_1(X) = \mu_2(X)$ and $\mu_1 = \mu_2$ when restricted to $\mathcal{C}$. Then, $\mu_1 = \mu_2$ on all of $\mathcal{F}$.

↪**Proposition 1.10**: $\{\varnothing\} \cup \{(a, b) : a < b \in \mathbb{R}\}$ a π-system.

↪**Proposition 1.11**: If μ a measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ such that for all intervals I, $\mu(I) = \ell(I)$, then $\mu = m$.

Proof. Consider for all $n \geq 1$ $\mu|_{\mathfrak{B}_{[-n,n]}}$. Clearly, $\mu([-n, n]) = m([-n, n]) = 2n$, and for all $a, b \in \mathbb{R}$, $\mu((a, b) \cap [-n, n]) = \ell((a, b) \cap [-n, n]) = m((a, b) \cap [-n, n])$. Thus, by the previous theorem, μ must match m on all of $\mathfrak{B}_{[-n,n]}$.

Let now $A \in \mathfrak{B}_{\mathbb{R}}$. Let $A_n := A \cap [-n, n] \in \mathfrak{B}_{[-n,n]}$. By continuity of m from below,

$$\begin{aligned}\mu(A) &= \lim_{n\to\infty} \mu(A_n) \\ &= \lim_{n\to\infty} m(A_n) \\ &= m(A),\end{aligned}$$

hence $\mu = m$. ■

↪**Proposition 1.12**: If μ a measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ assigning finite values to compact sets and is translation invariant, then $\mu = cm$ for some $c > 0$.

Remark 1.8: This proposition is also tacitly stating that $\mathfrak{B}_{\mathbb{R}}$ translation invariant; this needs to be shown.

↪**Lemma 1.1**: $\mathfrak{B}_{\mathbb{R}}$ translation invariant; for any $A \in \mathfrak{B}_{\mathbb{R}}, x \in \mathbb{R}, A + x \in \mathfrak{B}_{\mathbb{R}}$.

Proof. We employ the "good set strategy"; fix some $x \in \mathbb{R}$ and let

$$\Sigma := \{B \in \mathfrak{B}_{\mathbb{R}} : B + x \in \mathfrak{B}_{\mathbb{R}}\}.$$

One can check that Σ a σ-algebra, and so $\Sigma \subseteq \mathfrak{B}_{\mathbb{R}}$. But in addition, its easy to see that $\{(a, b) : a < b \in \mathbb{R}\} \subseteq \Sigma$, since a translated interval is just another interval, and since these sets generate $\mathfrak{B}_{\mathbb{R}}$, it must be further that $\mathfrak{B}_{\mathbb{R}} \subseteq \Sigma$, completing the proof. ■

Proof. (of the proposition) Let $c = \mu((0, 1])$, noting that $c > 0$ (why? Consider what would happen if $c = 0$).

This implies that $\forall\, n \geq 1, \mu\left(\left(0, \frac{1}{n}\right]\right) = \frac{c}{n}$ (obtained by "chopping up" $(0, 1]$ into n disjoint intervals); from here we can draw many further conclusions:

$$\begin{aligned}&\forall\, m = 1, ..., n-1, \mu\left(\left(0, \frac{m}{n}\right]\right) = \frac{m}{n}c \\ &\Rightarrow \forall\, q \in \mathbb{Q} \cap (0, 1], \mu((0, q]) = qc \\ &\Rightarrow \forall\, q \in \mathbb{Q}^+, \mu((0, q]) = q \cdot c \text{ (translate)} \\ &\Rightarrow \forall\, a \in \mathbb{R}, \mu((a, a+q]) = q \cdot c \\ &\Rightarrow \forall \text{ intervals } I, \mu(I) = c \cdot \ell(I) \text{ (continuity)} \\ &\Rightarrow \forall\, n \geq 1, a, b \in \mathbb{R}, \mu((a, b) \cap [-n, n]) = c \cdot \ell((a, b) \cap [-n, n]) = c \cdot m((a, b) \cap [-n, n]),\end{aligned}$$

but then, $\mu = c \cdot m$ on $\mathfrak{B}_{\mathbb{R}[-n,n]}$, and by appealing again the Dynkin's, $\mu = c \cdot m$ on all of $\mathfrak{B}_{\mathbb{R}}$. ■

↪**Proposition 1.13** (Scaling): m has the *scaling property* that $\forall\, A \in \mathcal{M}, c \in \mathbb{R}, c \cdot A = \{cx : x \in A\} \in \mathcal{M}$, and $m(c \cdot A) = |c|\, m(A)$.

PROOF. Assume $c \neq 0$. Given $A \subseteq \mathbb{R}$, remark that $\{I_n\}$ an open interval cover of A iff $\{cI_n\}$ and open interval cover of cA, and $\ell(cI_n) = |c|\,\ell(I_n)$, and thus $m^*(cA) = |c|\,m^*(A)$.

Now, suppose $A \in \mathcal{M}$. Then, we have for any $B \subseteq \mathbb{R}$,

$$\begin{aligned} m^*(B) = |c|\,m^*\left(\frac{1}{c}B\right) &= |c|\,m^*\left(\frac{1}{c}B \cap A\right) + |c|\,m^*\left(\frac{1}{c}B \cap A^c\right) \\ &= m^*(B \cap cA) + m^*\big(B \cap (cA)^c\big), \end{aligned}$$

so $cA \in \mathcal{M}$. ■

§1.7 Relationship between $\mathfrak{B}_{\mathbb{R}}$ and $\mathcal{M}$

↪**Definition 1.8**: Given $(X, \mathcal{F}, \mu)$, consider the following collection of subsets of X,

$$\mathcal{N} := \{B \subseteq X : \exists\, A \in \mathcal{F} \text{ s.t. } \mu(A) = 0, B \subseteq A\}.$$

Put $\overline{\mathcal{F}} := \sigma(\mathcal{F} \cup \mathcal{N})$; this is called the *completion* of $\mathcal{F}$ with respect to μ.

↪**Proposition 1.14**: $\overline{\mathcal{F}} = \{F \subseteq X : \exists\, E, G \in \mathcal{F} \text{ s.t. } \exists\, E \subseteq F \subseteq G \text{ and } m(G \setminus E) = 0\}$.

PROOF. Put $\mathcal{G}$ the set on the right; one can check $\mathcal{G}$ a σ-algebra. Since $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{N} \subseteq \mathcal{G}$, we have $\overline{\mathcal{F}} \subseteq \mathcal{G}$.

Conversely, for any $F \in \mathcal{G}$, we have $E, G \in \mathcal{F}$ such that $E \subseteq F \subseteq G$ with $m(G \setminus E) = 0$. We can rewrite

$$F = \underbrace{E}_{\in \mathcal{F}} \cup \underbrace{(F \setminus E)}_{\substack{\subseteq G \setminus E \\ \Rightarrow \mu(F \setminus E) = 0 \\ \Rightarrow G \setminus E \in \mathcal{N}}},$$

hence $F \in \mathcal{F} \cup \mathcal{N}$ and thus in $\mathcal{F}$, and equality holds. ■

↪**Definition 1.9**: Given $(X, \mathcal{F}, \mu)$, μ can be *extended* to $\overline{\mathcal{F}}$ by, for each $F \in \overline{\mathcal{F}}$ with $E \subseteq F \subseteq G$ s.t. $\mu(G \setminus E) = 0$, put

$$\mu(F) = \mu(E) = \mu(G).$$

We call then $(X, \mathcal{F}, \mu)$ a *complete measure space*.

Remark 1.9: It isn't obvious that this is well defined a priori; in particular, the E, G sets are certainly not guaranteed to be unique in general, so one must check that this definition is valid regardless of choice of "sandwich sets".

↪**Theorem 1.5**: $(\mathbb{R}, \mathcal{M}, m)$ is the completion of $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, m)$.

Proof. Given $A \in \mathcal{M}$, then $\forall\, n \geq 1, \exists\, G_n$-open with $A \subseteq G_n$ s.t. $m^*(G_n \setminus A) \leq \frac{1}{n}$ and $\exists\, F_n$-closed with $F_n \subseteq A$ s.t. $m^*(A \setminus F_n) \leq \frac{1}{n}$.

Put $C := \bigcap_{n=1}^{\infty} G_n$, $B := \bigcap_{n=1}^{\infty} F_n$, remarking that $C, B \in \mathfrak{B}_{\mathbb{R}}$, $B \subseteq A \subseteq C$, and moreover

$$m(C \setminus A) \leq \frac{1}{n}, m(A \setminus B) \leq \frac{1}{n}$$

$$\Rightarrow m(C \setminus B) = m(C \setminus A) + m(A \setminus B) \leq \frac{2}{n},$$

but n can be arbitrarily large, hence $m(C \setminus B) = 0$; in short, given a measurable set, we can "sandwich it" arbitrarily closely with Borel sets. Thus, $A \in \overline{\mathfrak{B}_{\mathbb{R}}} \Rightarrow \mathcal{M} \subseteq \overline{\mathfrak{B}_{\mathbb{R}}}$. But recall that $\mathcal{M}$ complete, so $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M} \Rightarrow \overline{\mathfrak{B}_{\mathbb{R}}} \subseteq \overline{\mathcal{M}} = \mathcal{M}$, and thus $\overline{\mathfrak{B}_{\mathbb{R}}} = \mathcal{M}$ indeed.

Heuristically, this means that any measurable set is "different" from a Borel set by at most a null set. ■

§1.8 Some Special Sets

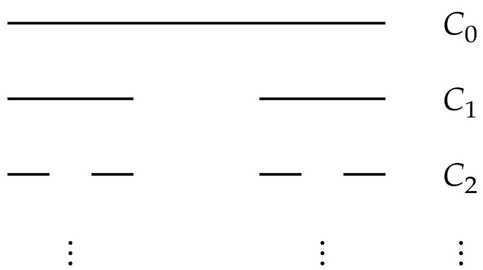
1.8.1 Uncountable Null Set?

Remark that for any countable set $A \in \mathcal{M}$, $m(A) = 0$; indeed, one may write $A = \bigcup_{n=1}^{\infty} \{a_n\}$ for singleton sets $\{a_n\}$, and so

$$m(A) = \sum_{n=1}^{\infty} m(a_n) = 0.$$

One naturally asks the opposite question, does there exist a measurable, *uncountable* set with measure 0? We construct a particular one here, the Cantor set, C.

This requires an "inductive" construction. Define $C_0 = [0, 1]$, and define C_k to be C_{k-1} after removing the middle third from each of its disjoint components. For instance $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, then $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, and so on. This may be clearest graphically:



Remark that the $C_n \downarrow$. Put finally

$$C := \bigcap_{n=1}^{\infty} C_n.$$

↪**Proposition 1.15**: The following hold for the Cantor set C:

1. C is closed (and thus $C \in \mathfrak{B}_{\mathbb{R}}$);
2. $m(C) = 0$;
3. C is uncountable.

Proof.

1. For each n, C_n is the countable (indeed, finite) union of 2^n-many disjoint, closed intervals, hence each C_n closed. C is thus a countable intersection of closed sets, and is thus itself closed.
2. For each n, each of the 2^n disjoint closed intervals in C_n has length $\frac{1}{3^n}$, hence
$$m(C_n) = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$
Since $\{C_n\} \downarrow$, by continuity of m we have
$$m(C) = \lim_{n \Rightarrow \infty} m(C_n) = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0.$$
3. This part is a little trickier. Notice that for any $x \in [0,1]$, we can define a sequence (a_n) where each $a_n \in \{0,1,2\}$, and such that
$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n};$$
in particular, this is just the base-3 representation of x, which we denote $(x)_3 = (a_1 a_2 \cdots)$.

I claim now that
$$C = \{x \in [0,1] : (x)_3 \text{ has no 1's}\}.$$
Indeed, at each stage n of the construction of the Cantor set, we get rid of the segment of the real line that would correspond to the $a_n = 1$. One should note that $(x)_3$ not necessarily unique; for instance $\left(\frac{1}{3}\right)_3 = (1,0,0,\ldots) = (0,2,2,\ldots)$, but if we specifically consider all x such that there *exists* a base three representation with no 1's, i.e. like $\frac{1}{3}$, then C indeed captures all the desired numbers.

Thus, we have that
$$\mathrm{card}(C) = \mathrm{card}(\{\{a_n\} : a_n = 0, 2\}).$$
Define now the function
$$f : C \to [0,1], \quad x \mapsto \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot \frac{1}{2^n}, \text{where } (x)_3 = (a_n)$$
i.e., we "squish" the base-3 representation into a base-2 representation of a number. This is surjective; for any $y \in [0,1]$, $(b_n) := (y)_2$ contains only 0's and 1's, hence $(2b_n)$

contains only 0's and 1's, so let x be the number such that $(x)_3 = (2b_n)$. This necessarily exists, indeed, we simply take our definitions backwards:

$$x := \sum_{n=1}^{\infty} \frac{2b_n}{3^n},$$

which maps to y under f and is contained in C. Hence, $\text{card}(C) \geq \text{card}([0,1])$; but $[0,1]$ uncountable, and thus so is C. ■

We can naturally extend the function f used here to map the entire interval $[0,1] \to [0,1]$ as follows

$$f(x) := \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot \frac{1}{2^n} \text{ if } x \in C, (x)_3 = (a_n) \\ f(a) \text{ if } x \notin C \text{ then } x \in (a,b) \text{ s.t. } (a,b) \text{ removed from } [0,1] \end{cases}.$$

This function is often called the *Devil's Staircase* or *Cantor-Lebesgue function*.

↪**Proposition 1.16**:
1. $f(0) = 0, f(1) = 1, f \equiv \frac{1}{2}$ on $\left(\frac{1}{3}, \frac{2}{3}\right), f \equiv \frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9}\right)$
2. $f : [0,1] \to [0,1]$ a surjection
3. f is nondecreasing
4. f is continuous

Proof. 1., 2., clear from construction.

For 3., let $x_1 < x_2 \in C$, and suppose $(x_1)_3 = (a_n), (x_2)_3 = (b_n)$. Then, since $x_1 < x_2$, it must be that a_n, b_n can only be equal up to some finite N; then the next $0 = a_{N+1} < b_{N+1} = 2$. Hence, it follows that the "modified binary expansion" that arises from f gives directly that $f(x_1) \leq f(x_2)$.

For 4., f is clearly continuous on $[0,1] - C$, since it is piecewise-constant here. Also, f is "one-sided continuous" at each of the "boundary points" $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \ldots$. If $x \in C$, for any $n \geq 1$, there must be x_n, x_n' such that $x_n < x < x_n'$ (if $x = 0$, only need x_n', if $x = 1$, only need x_n) and $f(x_n') - f(x_n) \leq \frac{1}{2^n}$. Then, f is continuous at x by monotonicity of f. ■

1.8.2 Non-Measurable Sets?

We've shown then that there is indeed an uncountable set of measure 0. Another question we may ask ourselves is, is there a $A \subseteq \mathbb{R}$ that is non-measurable? The answer to this turns out to be yes, but the construction requires invoking the axiom of choice:

Axiom 1 (Of Choice): If Σ a collection of nonempty sets, then $\exists$ a function

$$S : \Sigma \to \bigcup_{A \in \Sigma} A,$$

such that $A \in \sigma$, $S(A) \in A$. Such a function is called a *selection function*, and $S(A)$ a *representative* of A.

We construct now a non-measurable set, assuming the above. Consider $[0, 1]$, and define an equivalence relation $\sim$ on $[0, 1]$ by

$$a \sim b \Leftrightarrow a - b \in \mathbb{Q}.$$

Its easy to check that this is indeed an equivalence relation. Denote by E_a the equivalence class containing a, and set $\Sigma = \{E_a : a \in [0, 1]\}$. Note that for any $E_a \in \Sigma, E_a \neq \varnothing$.

Invoking the axiom of choice, we can select exactly one element S_a from E_a for each $E_a \in \Sigma$. Set

$$N := \{S_a : S_a \text{ is a representative of } E_a, E_a \in \Sigma\}.$$

↪**Proposition 1.17**: N, called a *Vitali set*, is non-measurable.

Proof. Assume towards a contradiction that N indeed measurable, $N \in \mathcal{M}$. Consider $[-1, 1] \cap \mathbb{Q}$; this is countable, so we can enumerate it $\{q_k\}$, $k \geq 1$. For each k, put

$$N_k := N + q_k.$$

By the assumption of measurability and translation invariance of m, it must be that each N_k measurable and has the same measure as N.

We claim each N_k disjoint. Assume not, then $\exists k \neq \ell$ (i.e. $q_k \neq q_\ell$) and $S_a, S_b \in N$ such that $S_a + q_k = S_b + q_\ell$. But then $S_a - S_b = q_\ell - q_k \in \mathbb{Q}$, hence $S_a \sim S_b$. But we constructed N to have only one representative from each equivalence class, hence it must be that $S_a = S_b$, and so $S_a + q_k = S_a + q_\ell \Rightarrow q_k = q_\ell$, contradicting the assumed distinctness of the q's; hence, the N_k's indeed disjoint.

We claim next that $[0, 1] \subseteq \bigcup_{k=1}^{\infty} N_k$. Let $x \in [0, 1]$. Then, $x \sim S_a$ for some unique $S_a \in N$ and so $x - S_a \in \mathbb{Q}$. But also, $x, S_a \in [0, 1]$, hence $x - S_a \in [-1, 1]$ (moreover, $x - S_a \in [-1, 1] \cap \mathbb{Q}$) and there must exist a k such that $x - S_a = q_k$, since the q_k's enumerate the entire $[-1, 1] \cap \mathbb{Q}$. Thus, $x \in N_k$ by the construction of the N_k's. Thus, $[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k$ indeed.

On the other hand, $\bigcup_{k=1}^{\infty} N_k \subseteq [-1, 2]$ and so we have the "bound"

$$[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k \subseteq [-1, 2].$$

Taking the measure of all sides then, we have the bound

$$1 \leq \mu\left(\bigcup_{n=1}^{\infty} N_k\right) \leq 3.$$

Invoking the disjointness of the N_k's, we can also use countable additivity to write

$$\mu\left(\bigcup_{n=1}^{\infty} N_k\right) = \sum_{k=1}^{\infty} m(N_k) = \sum_{k=1}^{\infty} m(N),$$

but this final line is a sequence of positive, constant real numbers; hence, it is impossible for it to be within 1 and 3, and we have a contradiction. Hence, N indeed not measurable.

Remark that this proof also shows that $m^*(N_k) > 0$ so $m^*(N) > 0$ (given the interval bound on N we've found). ■

↪**Proposition 1.18**: For every $A \in \mathcal{M}$ such that $m(A) > 0$, there exists $B \subseteq A$ such that B is non-measurable.

Proof. Assume otherwise, that there is a $A \in \mathcal{M}$ with $m(A) > 0$ such that any subset B of A is also measurable.

Remark that $A \subseteq \bigcup_{n \in \mathbb{Z}} A \cap [n, n+1]$. Then, there exists an n such that $m(A \cap [n, n+1]) > 0$ and thus, translating $A' := A \cap [n, n+1] - n)$, $m(A') > 0$, noting that $A' \subseteq [0, 1]$. Now, for any $B' \subseteq A'$, $B' + n \subseteq A$. By assumption, then $B' + n$ must be measurable so B' measurable.

In summary, then, we have $A' \subseteq [0, 1]$ with $m(A') > 0$ such that (by assumption) B' measurable for all $B' \subseteq A'$.

Let $N, \{q_k\}, N_k$ be as in the previous proof. Set

$$A_k{}' := A' \cap N_k, k \geq 1.$$

Then, $A_k{}'$ disjoint, and

$$A' = [0, 1] \cap A' \subseteq \bigcup_{k=1}^{\infty} (N_k \cap A') = \bigcup_{k=1}^{\infty} A_{k'}.$$

Since $m(A') > 0$, there exists a k such that $m(A_k{}') > 0$. Set, for this k,

$$L := \{\ell \geq 1 : q_\ell + q_k \in [-1, 1]\}.$$

This set is again countably infinite. We translate, obtaining a disjoint sequence of sets $\{q_\ell + A_k{}' : \ell \in L\}$; since $q_\ell + q_k \in [-1, 1] \cap \mathbb{Q}$, then $q_\ell + q_k = q_m$ for some unique m, and so $q_\ell + A_k{}' = q_\ell + A' \cap (N + q_k) \subseteq N_m$. Hence, we have on the one hand that by countable additivity

$$\bigcup_{\ell \in L} (q_\ell + A_k{}') \subseteq [-1, 2] \Rightarrow \sum_{\ell \in L} m(q_\ell + A_k{}') \leq 3,$$

and so it must be that $m(q_\ell + A_k{}') = m(A_k{}') = 0$ (else the series couldn't be finite), contradicting the finiteness assumption on $m(A_k{}')$. ■

1.8.3 Non-Borel Measurable Set?

We may ask, is there $A \in \mathcal{M}$ such that $A \notin \mathfrak{B}_{\mathbb{R}}$?

Let $f : [0,1] \to [0,1]$ be the Cantor-Lebesgue function, and put $g(x) = f(x) + x$; note that g is continuous and strictly increasing, and is defined $g : [0,1] \to [0,2]$. Remark that g bijective; the strictly increasing gives injective, and moreover $g(0) = 0, g(1) = 2$ hence by intermediate value theorem it is surjective. Hence, $g^{-1} : [0,2] \to [0,1]$ exists, and is also continuous, so in short g is a homeomorphism; it maps open to open, closed to closed. In particular, if $A \in \mathfrak{B}_{\mathbb{R}}$, then $g(A) \in \mathfrak{B}_{\mathbb{R}}$.

Recall that if (a,b) an open interval that gets removed from the construction of C, then f is constant and so g will map (a,b) to another open interval of the same length $b-a$. Thus,

$$m(g([0,1] \setminus C)) = m([0,1] \setminus C) = 1.$$

Hence, $m(g(C)) = 2 - 1 = 1 > 0$, since $g(C \cup [0,1] \setminus C) = [0,2]$. Hence, there exists a $B \subseteq g(C)$ such that $B \notin \mathcal{M}$, as per the previous proposition.

Let $A := g^{-1}(B)$; then $A \subseteq g^{-1}(g(C)) = C$. Since $m(C) = 0$, $A \in \mathcal{M}$ and $m(A) = 0$. But, $A \notin \mathfrak{B}_{\mathbb{R}}$; if it were, then $g(A) = B \in \mathfrak{B}_{\mathbb{R}}$, since g “maintains” Borel sets, but B is not even Lebesgue measurable and so this is a contradiction).

§2 Integration Theory

§2.1 Measurable Functions

We will be considering functions f defined on $\mathbb{R}$ or some subset of $\mathbb{R}$ that could take positive or negative infinity as its value i.e.

$$f : \mathbb{R} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\},$$

where $\overline{\mathbb{R}}$ the *extended real line*; we say f is $\overline{\mathbb{R}}$-valued. If f never takes $\infty, -\infty$ for any $x \in \mathbb{R}$, we say f finite-valued, or just $\mathbb{R}$-valued.

For all $a \in \mathbb{R}$, we consider inverse images

$$f^{-1}([-\infty, a)) := \{x \in \mathbb{R} : f(x) \in [-\infty, a)\} = \{f < a\},$$

remarking the inclusion of $-\infty$; similarly

$$f^{-1}((a, \infty]) := \{x \in \mathbb{R} : f(x) \in (a, \infty]\} = \{f > a\},$$

and so on, for any $B \subseteq \mathbb{R}$,

$$f^{-1}(B) := \{x \in \mathbb{R} : f(x) \in B\} = \{f \in B\}.$$

Remark that

$$\begin{aligned} f^{-1}(B^c) &= \left(f^{-1}(B)\right)^c \\ f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B) \\ f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B), \end{aligned}$$

which extend naturally for countable unions/intersections.

↪**Definition 2.1** (Measurable Function): $f : \mathbb{R} \to \overline{\mathbb{R}}$ is *measurable* if $\forall\, a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) \in \mathcal{M}.$$

↪**Proposition 2.1** (Equivalent Definitions of Measurability):

$$\begin{aligned} f \text{ is measurable} &\Leftrightarrow \forall\, a \in \mathbb{R}, f^{-1}([a,\infty]) \in \mathcal{M} \\ &\Leftrightarrow \forall\, a \in \mathbb{R}, f^{-1}((a,\infty]) \in \mathcal{M} \\ &\Leftrightarrow \forall\, a \in \mathbb{R}, f^{-1}([-\infty,a]) \in \mathcal{M} \end{aligned}$$

PROOF. We prove just the last equivalence. Notice that $\forall\, a \in \mathbb{R}$, we can use the commuting of inverse images with countable unions, intersections, complement to write

$$f^{-1}([-\infty, a)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a - \frac{1}{n}\right)\right)$$

and

$$f^{-1}([-\infty, a]) = \bigcap_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a + \frac{1}{n}\right)\right).$$

■

↪**Proposition 2.2**: If f finite-valued, Then

$$\begin{aligned} f \text{ is measurable} &\Leftrightarrow \forall\, a < b \in \mathbb{R}, f^{-1}((a,b)) \in \mathcal{M} \\ &\Leftrightarrow \quad \cdots \quad f^{-1}((a,b]) \in \mathcal{M} \\ &\Leftrightarrow \quad \cdots \quad f^{-1}([a,b)) \in \mathcal{M} \\ &\Leftrightarrow \quad \cdots \quad f^{-1}([a,b]) \in \mathcal{M}. \end{aligned}$$

↪**Definition 2.2** (Extended Borel Sigma Algebra): Define the Borel "extended" algebra $\mathfrak{B}_{\overline{\mathbb{R}}}$ of subsets of $\overline{\mathbb{R}}$, defined by

$$\mathfrak{B}_{\overline{\mathbb{R}}} := \sigma(\mathfrak{B}_{\mathbb{R}} \cup \{\{-\infty\}, \{\infty\}\}).$$

↪**Proposition 2.3**: $\mathfrak{B}_{\overline{\mathbb{R}}} = \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$.

PROOF. For every $a \in \mathbb{R}$, we may write

$$[-\infty, a) = \underbrace{(-\infty, a)}_{\in \mathfrak{B}_{\mathbb{R}}} \cup \{-\infty\} \in \mathfrak{B}_{\overline{\mathbb{R}}},$$

so $\sigma(\{[-\infty, a) : a \in \mathbb{R}\}) \subseteq \mathfrak{B}_{\overline{\mathbb{R}}}$.

Conversely, notice that

$$\{-\infty\} = \bigcap_{n=1}^{\infty} [-\infty, -n),$$

and

$$\{\infty\} = \overline{\mathbb{R}} - \left(\bigcup_{n=1}^{\infty} [-\infty, n)\right),$$

so $\{-\infty\}, \{\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$. Hence, for any $a \in \mathbb{R}$,

$$(-\infty, a) = [-\infty, a) - \{-\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\}),$$

and so $\mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$. $\{-\infty\}, \{\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ already, and thus $\mathfrak{B}_{\overline{\mathbb{R}}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$. ■

↪**Proposition 2.4**: $f : \mathbb{R} \to \overline{\mathbb{R}}$ measurable $\Leftrightarrow$ for all $B \in \mathfrak{B}_{\overline{\mathbb{R}}}, f^{-1}(B) \in \mathcal{M}$.

Proof. $\Leftarrow$ is immediate. For $\Rightarrow$, let $\mathcal{C}$ be a collection of subsets of $\overline{\mathbb{R}}$, then put

$$f^{-1}(\mathcal{C}) := \{f^{-1}(B) : B \in \mathcal{C}\}.$$

By an assignment question (2.6),

$$f^{-1}(\sigma(\mathcal{C})) = \sigma\left(f^{-1}(\mathcal{C})\right).$$

Take $\mathcal{C} = \{[-\infty, a) : a \in \mathbb{R}\}$. Then,

$$f^{-1}(\sigma(\mathcal{C})) = f^{-1}\left(\mathfrak{B}_{\overline{\mathbb{R}}}\right) = \sigma\left(f^{-1}(\{[-\infty, a) : a \in \mathbb{R}\})\right).$$

But f measurable, so $f^{-1}([-\infty, a)) \in \mathcal{M}$ for each $a \in \mathbb{R}$, hence sigma $\left(f^{-1}(\{[-\infty, a) : a \in \mathbb{R}\})\right) \subseteq \mathcal{M}$ and so $f^{-1}(\sigma(\mathcal{C})) \subseteq \mathcal{M}$ completing the proof. ■

↪**Corollary 2.1**: If f finite-valued, then f is measurable $\Leftrightarrow$ for every $B \in \mathfrak{B}_{\mathbb{R}}, f^{-1}(B) \in \mathcal{M}$.

↪**Proposition 2.5**: Given $f : \mathbb{R} \to \overline{\mathbb{R}}$, define the *finite valued component* of f given by

$$f_{\mathbb{R}}(x) := \begin{cases} f(x) : -\infty < f(x) < \infty \\ 0 \text{ otherwise} \end{cases}.$$

Then, f measurable $\Leftrightarrow$ $\forall\, B \in \mathfrak{B}_{\mathbb{R}}, f_{\mathbb{R}}^{-1}(B) \in \mathcal{M}$ AND $\{f = \infty\}, \{f = -\infty\}$ both in $\mathcal{M}$.

Proof. ($\Leftarrow$) For any $a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) = \{f = -\infty\} \cup f^{-1}((-\infty, a)) = \{f = -\infty\} \cup f_{\mathbb{R}}^{-1}((-\infty, a)),$$

a union of measurable sets and hence is itself measurable.

($\Rightarrow$) Remark that $\{f = \infty\}, \{f = -\infty\} \in \mathcal{M}$ automatically. For any $B \in \mathfrak{B}_{\mathbb{R}}$, we have

$f_{\mathbb{R}}^{-1}(B) = \{x \in \mathbb{R} : f_{\mathbb{R}}(x) \in B\} = \{x \in \mathbb{R} : f(x) \in B, -\infty < f < \infty\} \cup \{x \in \mathbb{R} : 0 \in B, f(x) = \pm\infty\} \in \mathcal{M}$.

■

↪**Definition 2.3**: If a statement is true for every $x \in A$ where $A \in \mathcal{M}$ s.t. $m(A^c) = 0$, then we say the statement is true a.e. (almost everywhere).

↪**Proposition 2.6**: If $f : \mathbb{R} \to \overline{\mathbb{R}}$ is measurable and $f = g$ a.e. then g is measurable.

↪**Corollary 2.2**: If f is finite-valued a.e., then f is measurable $\Leftrightarrow$ $f_{\mathbb{R}}$ is measurable $\Leftrightarrow$ $\forall\, a < b \in \mathbb{R}, f^{-1}((a,b)) \in \mathcal{M}$.

↪**Proposition 2.7**: If $f \equiv c$ then f measurable.

If $f = \mathbb{1}_A$ for some $A \subseteq \mathbb{R}$, then f is measurable $\Leftrightarrow$ $A \in \mathcal{M}$.

Proof. Assume $f \equiv c$. Then

$$f^{-1}([-\infty, a)) = \begin{cases} \mathbb{R} \text{ if } c < a \\ \varnothing \text{ if } c \geq a \end{cases} \in \mathcal{M}.$$

Assume now $f = \mathbb{1}_A$. For all $a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) = \begin{cases} \mathbb{R} \text{ if } a > 1 \\ A^c \text{ if } 0 < a \leq 1 \\ \varnothing \text{ if } a \leq 0 \end{cases} \in \mathcal{M} \Leftrightarrow A \in \mathcal{M}.$$

■

↪**Proposition 2.8**: If f is (finite-valued) continuous, then f is measurable.

Proof. $f : \mathbb{R} \to \mathbb{R}$ continuous $\Leftrightarrow$ for all $G \subseteq \mathbb{R}$ open, $f^{-1}(G)$ open. For all $a < b \in \mathbb{R}$, then $f^{-1}((a,b))$ open so $f^{-1}((a,b)) \in \mathcal{M}$ so f measurable.

In fact, if $f : \mathbb{R} \to \mathbb{R}$ continuous, then for all $B \in \mathfrak{B}_{\mathbb{R}}, f^{-1}(B) \in \mathfrak{B}_{\mathbb{R}}$;

$$f^{-1}(\mathfrak{B}_{\mathbb{R}}) = f^{-1}(\sigma(\{\text{open sets}\})) = \sigma\left(\underbrace{f^{-1}(\{\text{open sets}\})}_{\text{all open}}\right) \subseteq \sigma(\{\text{open sets}\}) = \mathfrak{B}_{\mathbb{R}}.$$

Moreover, if f^{-1} (inverse) exists and is continuous, then for any $B \in \mathfrak{B}_{\mathbb{R}}, f(B) \in \mathfrak{B}_{\mathbb{R}}$. ■

↪**Proposition 2.9**: If $f : \mathbb{R} \to \mathbb{R}$ is measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g \circ f$ is measurable.

Remark 2.1: The order matters! The converse doesn't hold in general.

PROOF. For all $a \in \mathbb{R}$,

$$
\begin{aligned}
(g \circ f)^{-1}((-\infty, a)) &= \{x \in \mathbb{R} : g(f(x)) < a\} \\
&= \left\{x \in \mathbb{R} : f(x) \in g^{-1}([-\infty, a))\right\} \\
&= f^{-1}\left(g^{-1}([-\infty, a))\right) \in \mathcal{M}.
\end{aligned}
$$

■

↪**Proposition 2.10**: If $f : \mathbb{R} \to \overline{\mathbb{R}}$ is measurable, then:

1. for every $c \in \mathbb{R}$, cf is measurable (in particular $-f$ measurable);
2. $|f|$ is measurable;
3. for every $k \in \mathbb{N}$, f^k is a measurable.

PROOF. We prove just 3. If $k = 0$ this is trivial. For any $a \in \mathbb{R}$,

$$
\left(f^k\right)^{-1}([-\infty, a)) = \begin{cases} f^{-1}\left([-\infty, a^{\frac{1}{k}})\right) & \text{if } k \text{ is odd} \\ \varnothing & \text{if } k \text{ is even and } a \leq 0 \in \mathcal{M}. \\ f^{-1}\left([-a^{\frac{1}{k}}, a^{\frac{1}{k}})\right) & \text{if } k \text{ is even and } a > 0 \end{cases}
$$

■

↪**Proposition 2.11**: If f, g are two finite-valued measurable functions, then $f + g, f \cdot g, f \vee g := \max\{f, g\}, f \wedge g := \min\{f, g\}$ are measurable functions, where

$$
(f \vee g)(x) = \max\{f(x), g(x)\}.
$$

PROOF. For all $a \in \mathbb{R}$,

$$
\begin{aligned}
(f + g)^{-1}([-\infty, a) &= \{x \in \mathbb{R} : f(x) + g(x) < a\} \\
&= \{x \in \mathbb{R} : f(x) < a - g(x)\} \\
&= \bigcup_{q \in \mathbb{Q}} \{x \in \mathbb{R} : f(x) < q < a - g(x)\} \\
&= \bigcup_{q \in \mathbb{Q}} \underbrace{\{x \in \mathbb{R} : f(x) < q\}}_{\in \mathcal{M}} \cap \underbrace{\{x \in \mathbb{R} : g(x) < a - q\}}_{\in \mathcal{M}} \in \mathcal{M}.
\end{aligned}
$$

This implies, then, that $f - g$ measurable, as are $(f + g)^2$ and $(f - g)^2$, and thus

$$fg = \frac{1}{4}\Big[(f+g)^2 - (f-g)^2\Big]$$

is measurable.

We have too that

$$f \vee g = \frac{1}{2}(|f-g| + (f+g))$$

and so is measurable, and so

$$f \wedge g = -\max\{-f, -g\} = -(-f \vee -g)$$

is measurable. ■

↪**Corollary 2.3**: If f is measurable, then $f^+ := f \vee 0 = \max\{f, 0\}$ and $f^- := -(f \wedge 0) = \max\{-f, 0\}$ are measurable, as is $f \wedge k$ for any $k \in \mathbb{R}$.

Remark 2.2: Notice that $f = f^+ - f^-$, even with "infinities", and $|f| = f^+ + f^-$.

↪**Proposition 2.12**: Let $\{f_n\}$ be a sequence of measurable functions. Then, $\sup_n f_n$, $\inf_n f_n$, $\limsup_{n\to\infty} f_n$, and $\liminf_{n\to\infty} f_n$ are all measurable (where $(\limsup_{n\to\infty} f_n)(x) := \limsup_{n\to\infty} f_n(x) = \inf_{m\geq 1} \sup_{n\geq m} f_n(x) = \lim_{m\to\infty} \sup_{n\geq m} f_n(x)$).

Proof. To show $\sup_n f_n$ measurable, we will show for all $a \in \mathbb{R}$ $\{\sup_n f_n \leq a\} \in \mathcal{M}$.

$$x \in \left\{\sup_n f_n \leq a\right\} \Leftrightarrow \sup_n f_n(x) \leq a \Leftrightarrow f_n(x) \leq a \,\forall\, n \geq 1 \Leftrightarrow x \in \bigcap_{n=1}^{\infty} \{f_n \leq a\},$$

hence $\{\sup_n f_n \leq a\} = \bigcap_{n=1}^{\infty} \underbrace{\{f_n \leq a\}}_{\in \mathcal{M}} \in \mathcal{M}$ and hence $\sup_n f_n$ is measurable. Note that using $\leq$ was important; $\{\sup_n f_n < a\} \subsetneq \bigcap_{n=1}^{\infty} \{f_n < a\}$, since the $\sup_n f_n$ could equal a. We could say the following, however:

$$\left\{\sup_n f_n < a\right\} = \bigcup_{k=1}^{\infty} \left\{\sup_n f_n \leq a - \frac{1}{k}\right\} = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{f_n \leq a - \frac{1}{k}\right\} \in \mathcal{M}.$$

Next, we have $\inf_n f_n = -\sup_n(-f_n)$ so we are done.

For lim sup, lim inf, we have

$$\limsup_n f_n = \inf_{m\geq 1} \underbrace{\sup_{n\geq m} f_n}_{:= g_m}.$$

g_m is measurable for each $m \geq 1$, hence $\inf_m g_m$ is measurable, hence $\limsup_n f_n$ is measurable. Similar logic follows for lim inf.

We could have show, more directly, that

$$\left\{\limsup_n f_n < a\right\} = \left\{\inf_{m\geq 1}\sup_{n\geq m} f_n < a\right\}$$
$$= \bigcup_{m=1}^{\infty}\left\{\sup_{n\geq m} f_n < a\right\}$$
$$= \bigcup_{m=1}^{\infty}\bigcup_{k=1}^{\infty}\left\{\sup_{n\geq m} f_n \leq a - \frac{1}{k}\right\}$$
$$= \bigcup_{m=1}^{\infty}\bigcup_{k=1}^{\infty}\bigcap_{n=m}^{\infty}\left\{f_n \leq a - \frac{1}{k}\right\}.$$

■

↪**Proposition 2.13**: Let $\{f_n\}$ be a sequence of measurable functions. Then, all of the following sets are also measurable:

$$\left\{x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\right\} =: \left\{\lim_{n\to\infty} f_n \text{ exists in } \mathbb{R}\right\},$$
$$\{\lim f_n = \infty\}, \{\lim f_n = -\infty\}, \{\lim f_n = c \in \mathbb{R}\}.$$

Moreover, if $\lim_{n\to\infty} f_n$ exists (in $\mathbb{R}$ or as $\pm\infty$) a.e. with $f = \lim_{n\to\infty} f_n$ a.e. then f is measurable.

PROOF. We have

$$\{\lim f_n \text{ exists in } \mathbb{R}\} = \{\limsup f_n = \liminf f_n \text{ and } -\infty < \limsup f_n < \infty\}$$
$$= \{-\infty < \liminf f_n < \infty\} \cap \{-\infty < \limsup f_n < \infty\} \cap \{\limsup f_n - \liminf f_n = 0\} \in \mathcal{M}.$$

Similarly,

$$\{\lim f_n = c\} = \left\{x \in \mathbb{R} : \forall\, k \geq 1, \exists\, n \geq 1 \text{ s.t. } \forall\, m \geq n, |f_n(x) - c| \leq \frac{1}{k}\right\}$$
$$= \underbrace{\bigcap_{k=1}^{\infty}}_{\forall\, \varepsilon = \frac{1}{k} > 0} \underbrace{\bigcup_{n=1}^{\infty}}_{\exists\, n\geq 1} \underbrace{\bigcap_{m=n}^{\infty}}_{\forall\, m\geq n} \left\{|f_n(x) - c| \leq \frac{1}{k}\right\}.$$

■

§2.2 Approximation by Simple Functions

Given a function $f : \mathbb{R} \to \overline{\mathbb{R}}$, measurable, we may write

$$f = f^+ - f^-,$$

where f^+, f^- are non-negative measurable functions; so, it suffices to study non-negative measurable functions. For any $n \geq 1$, we have

$$f_n^+ := (f^+ \wedge n) \cdot \mathbb{1}_{[-n,n]},$$

i.e., we cap f^+ at n, and disregard values of f^+ outside of $[-n, n]$; hence we limit our view to a $2n \times n$ "box". Then, f_n^+ is non-negative, measurable, bounded (by n), compactly supported (zero outside a bounded set), and in particular $f_n^+ \uparrow$, with limit

$$\lim_{n\to\infty} f_n^+ = f^+.$$

An identical construction follows for f^- with

$$f_n^- := (f^- \wedge n)\mathbb{1}_{[-n,n]},$$

with $f_n^- \uparrow$ and

$$\lim_{n\to\infty} f_n^- = f^-.$$

Fix some n and consider f_n^+. For $k = 0, 1, 2, ..., 2^n n$, define

$$A_{n,k} := \left\{x \in [-n,n] : \frac{k}{2^n} \leq f_n^+(x) < \frac{k+1}{2^n}\right\} = \left\{\frac{k}{2^n} \leq f_n^+ < \frac{k+1}{2^n}\right\} \cap [-n,n] \in \mathcal{M},$$

noting that $A_{n,k} \cap A_{n,\ell} = \varnothing$ if $k \neq \ell$. Set now

$$\varphi_n := \sum_{k=0}^{n\cdot 2^n} \mathbb{1}_{A_{n,k}} \frac{k}{2^n} = \sum_{k=0}^{n\cdot 2^n} \begin{cases} \frac{k}{2^n} \text{ if in } A_{n,k} \\ 0 \text{ else} \end{cases}.$$

We call φ_n a "simple function"; more generally:

↪**Definition 2.4**: φ is a *simple function* if $\varphi = \sum_{k=1}^{L} \mathbb{1}_{E_k} \cdot a_k$ where L a positive integer, a_k's are constant, E_k's are measurable sets of finite measure.

Moreover, note that $\varphi_n \uparrow$; at each new stage $n \to n+1$, the regions are cut in two, $A_{n,k} = A_{n+1,2k} \cup A_{n+1,2k+1}$. In addition, we have $\varphi_n \leq f_n^+ \leq f^+$ for all n. Moreover, we have the following:

↪**Proposition 2.14**:

$$\lim_{n\to\infty} \varphi_n(x) = f^+(x)$$

for all $x \in \mathbb{R}$.

Proof. For all $x \in \mathbb{R}$, for sufficiently large n we have that $x \in [-n,n]$ and so $f^+(x) = f^+(x)\mathbb{1}_{[-n,n]}(x)$. Assume for now $f^+ < \infty$. Then, for sufficiently large(r?) n, we can ensure $f^+(x) < n$ and so $f^+(x) = f_n^+(x)$ for such an x. Further, we have that $x \in A_{n,k}$ for some k so $\varphi_n(x) = \frac{k}{2^n}$ and $f_n^+(x) < \frac{k+1}{2^n}$ and thus

$$0 \leq f_n^+(x) - \varphi_n(x) < \frac{k+1}{2^n} - \frac{k}{2^n} = 2^{-n}$$

by construction and so $0 \leq f^+(x) - \varphi_n(x) \leq 2^{-n}$ and thus $\lim_{n\to\infty} \varphi_n(x) = f^+(x)$.

In the case that $f^+(x) = \infty$, then $\varphi_n(x) = n$ for all sufficiently large n hence

$$\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} n = \infty = f^+(x).$$

■

↪**Theorem 2.1**: If g is measurable and non-negative, there exists a sequence of simply functions $\{\varphi_n\}$ such that $\varphi_n \uparrow$ and $\lim_{n\to\infty} \varphi_n(x) = g(x)$ for every $x \in \mathbb{R}$.

We can repeat this same construction and proof for f^- with a sequence $\widetilde{\varphi_n}$. Even better:

↪**Theorem 2.2**: If f is measurable, then $\exists$ a sequence of simple functions $\{\psi_n\}$ such that $|\psi_n| \uparrow$ and $|\psi_n| \leq |f|$ for all n and for all $x \in \mathbb{R}$, $\lim_{n\to\infty} \psi_n(x) = f(x)$.

Proof. Take $\psi_n = \varphi_n - \widetilde{\varphi_n}$ as above; then for all $x \in \mathbb{R}$, at least one of $\varphi_n(x), \widetilde{\varphi_n}(x)$ equals zero. Then

$$|\psi_n| = \varphi_n + \widetilde{\varphi_n} < f^+ + f^- = |f|,$$

and

$$\lim_{n\to\infty} \psi_n(x) = \lim_{n\to\infty} \varphi_n(x) - \lim_{n\to\infty} \widetilde{\varphi_n}(x) = f^+ - f^- = f.$$

■

↪**Definition 2.5** (Step Function): θ a *step function* if it takes the form

$$\theta(x) = \sum_{k=1}^{L} a_k \mathbb{1}_{I_k}(x),$$

where $L \in \mathbb{N}, a_k$'s constant, and I_k finite, open intervals.

↪**Theorem 2.3**: If f is measurable, then there exists a sequence of step functions $\{\theta_n\}$ such that

$$\lim_{n\to\infty} \theta_n(x) = f(x) \text{ for } \textbf{almost every } x \in \mathbb{R}.$$

In particular, we do not have pointwise convergence as for general simple functions, but we have convergence outside a zero-measure set.

Proof. Assume, wlog, that f non-negative (by the previous construction, we can "split" f if not and approximate its positive, negative parts). Given $A \in \mathcal{M}$ with finite measure, recall that for every $\varepsilon > 0$, there exists finitely many finite open intervals $I_1, \ldots, I_N$ such that

$$m\left(A \triangle \left(\bigcup_{i=1}^{N} I_i\right)\right) < \varepsilon.$$

By renaming/rearranging I_i's if necessary, we may assume that I_i's are disjoint; hence

$$\mathbb{1}_{\bigcup_{i=1}^N I_i} = \sum_{i=1}^{N} \mathbb{1}_{I_i}.$$

Put

$$\theta_A := \sum_{i=1}^{N} \mathbb{1}_{I_i},$$

noting this is indeed a step function as the name suggests. Then, remark that

$$m\underbrace{(\{x \in \mathbb{R} : \mathbb{1}_A(x) \neq \theta_A(x)\})}_{=A\triangle\left(\bigcup_{n=1}^{N} I_i\right)} < \varepsilon.$$

Since f measurable and non-negative, $\exists\,\{\varphi_n\}$ sequence of simple functions with limit f. In particular,

$$\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}}.$$

Applying our above analysis to each $A_{n,k}$, then, we have that for any $n \geq 1$ and $k = 0, 1, \ldots, n2^n$ we can find a step function $\theta_{n,k}$ such that

$$m\Big(\Big\{x \in \mathbb{R} : \mathbb{1}_{A_{n,k}} \neq \theta_{n,k}(x)\Big\}\Big) < \frac{1}{2^n(n2^n+1)} \quad ("=\varepsilon").$$

Put then

$$\theta_n := \sum_{k=0}^{n2^n} \frac{k}{2^n} \theta_{n,k},$$

which is itself a step function. Put

$$E_n := \{x \in \mathbb{R} : \theta_n(x) \neq \varphi_n(x)\}.$$

Then,

$$m(E_n) \leq m\left(\bigcup_{k=0}^{n2^n} \Big\{\theta_{n,k} \neq \mathbb{1}_{A_{n,k}}\Big\}\right) \leq \sum_{k=0}^{n2^n} m\Big(\Big\{\theta_{n,k} \neq \mathbb{1}_{A_{n,k}}\Big\}\Big) \leq 2^{-n}.$$

The φ_n's are chosen such that $\forall\, x \in \mathbb{R}$, $|\varphi_n(x) - f_n(x)| \leq \frac{1}{2^n}$. Putting

$$F_n := \{x \in \mathbb{R} : |\theta_n(x) - f_n(x)\,| > 2^{-n}\},$$

then remark that $F_n \subseteq E_n$ so $m(F_n) \leq \frac{1}{2^n}$.

We claim now that for a.e. $x \in \mathbb{R}$, $\exists\, m \geq 1$ such that $\forall\, n \geq m$, $|\theta_n(x) - f_n(x)| \leq \frac{1}{2^n}$, remarking that such an m is *dependent* on x. Consider the complement of this statement; if this set has measure 0, we are done. The logical negation would be "for every $m \geq 1$, exist $n \geq m$ such that $|\theta_n(x) - f_n(x)\,| > 2^{-n}$", which is equivalent to the set

$$\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in \mathbb{R} : |\theta_n(x) - f_n(x)| > 2^{-n}\} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n.$$

Let $B_m := \bigcup_{n=m}^{\infty} F_n$; notice $B_m \downarrow$. Then, by continuity from above ****

$$m\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}F_n\right) = \lim_{m\to\infty} m(B_m) \le \lim_{m\to\infty}\sum_{n=m}^{\infty} m(F_n) \le \lim_{m\to\infty}\sum_{n=m}^{\infty}\frac{1}{2^n} = 0,$$

since the tail of a convergent series must converge to zero. Hence, the set has measure 0 as desired so for almost every $x \in \mathbb{R}$ there exists $m \ge 1$ such that for all $n \ge m$, $|\theta_n - f_n| \le \frac{1}{2^n}$, hence almost every where $\lim_{n\to\infty}(\theta_n - f_n) = 0$. Therefore, almost everywhere,

$$\theta_n = (\theta_n - f_n) + f_n \overset{n\to\infty}{\longrightarrow} f.$$

■

In this proof, we have proven (and then used) more generally:

↪**Lemma 2.1** (Borel-Cantelli Lemma): If $\{F_n\} \subseteq \mathcal{M}$ such that $\sum_{n=1}^{\infty} m(F_n) < \infty$, then

$$m\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}F_n\right) = 0.$$

§2.3 Convergence Almost Everywhere vs Convergence in Measure

↪**Definition 2.6** (Convergence Almost Everywhere): For measurable functions $\{f_n\}, f$ we say f_n converges to f a.e. and write $f_n \to f$ a.e. if for almost every $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Similarly, we say $f_n \to f$ a.e. on A if $\exists\, B \subseteq A$ with $m(B) = 0$ such that $\forall\, x \in A - B$, $\lim_{n\to\infty} f_n(x) = f(x)$.

↪**Definition 2.7** (Convergence in Measure): For measurable, finite-valued functions $\{f_n\}, f$ we say f_n converges to f in measure and write $f_n \to f$ in measure if for every $\delta > 0$,

$$\lim_{n\to\infty} m(\{x \in \mathbb{R} : |f_n(x) - f(x)| \ge \delta\}) = 0.$$

Similarly, we say $f_n \to f$ in measure on A if $\forall\, \delta > 0$, $\lim_{n\to\infty} m(\{x \in A : |f_n(x) - f(0)| \ge \delta\}) = 0$.

↪**Proposition 2.15**: Given finite-valued measurable functions $\{f_n\}, f$ and $A \in \mathcal{M}$ with finite measure, then if $f_n \to f$ a.e. on A, then $f_n \to f$ in measure on A.

Proof. For all $\delta > 0$,

$$\bigcap_{m=1}^{\infty}\bigcup_{n=m}\{x \in A : |f_n(x) - f(x)| > \delta\} \subseteq \left\{x \in A : \lim_{n\to\infty} f_n(x) \ne f(x)\right\}.$$

The set on the RHS has measure zero and thus so does the left one. Then,

$$\lim_{m\to\infty} m\left(\bigcup_{n=m}\{x \in A : |f_n(x) - f(x)| > \delta\}\right) = 0$$

by continuity, and

$$\{|f_m - f| > \delta\} \subseteq \bigcup_{n=m}^{\infty} \{|f_n - f| > \delta\}$$

hence $m(\{|f_m - f| > \delta\}) \leq m\left(\bigcup_{n=m}^{\infty}\{|f_n - f| > \delta\}\right) \xrightarrow{m\to\infty} 0$. ■

⊛ **Example 2.1**: We give an example of why the assumption that $m(A) < \infty$ is necessary. Let, $f_n = \mathbb{1}_{[n,\infty)}$ and $f \equiv 0$. Then, $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \mathbb{R}$. But $m(\{x \in \mathbb{R} : |f_n(x) - f(x)| = 1\}) = m([n,\infty)) = \infty$.

In general, the converse statement $f_n \to f$ in measure does *not* imply that $f_n \to f$ almost everywhere, even on finite measure sets. Put $\varphi_{1,1} = \mathbb{1}_{[0,1)}$, $\varphi_{2,1} = \mathbb{1}_{\left[0,\frac{1}{2}\right)}$, $\varphi_{2,2} = \mathbb{1}_{\left[\frac{1}{2},1\right)}$, $\varphi_{3,1} = \mathbb{1}_{\left[0,\frac{1}{3}\right)}$, $\varphi_{3,2} = \mathbb{1}_{\left[\frac{1}{3},\frac{2}{3}\right)}$, $\varphi_{3,3} = \mathbb{1}_{\left[\frac{2}{3},1\right)}$, or in general $\varphi_{k,j} = \mathbb{1}_{\left[\frac{j-1}{k},\frac{j}{k}\right)}$ for $j = 1, \ldots, k$. Reorder $\varphi_{k,j}$ "lexicographically" into $\{f_n\}$. Then, we claim $f_n \to 0$ in measure on $[0,1)$; for any $\delta \in (0,1)$,

$$m(\{|f_n - 0| > \delta\}) = \frac{1}{k(n)} \to 0,$$

where $k(n)$ the "row" that f_n comes from. Hence, f_n converges in measure. However, f_n does not converge almost eveywhere on $[0,1)$. Indeed, for each $x \in \mathbb{R}$ and $k \geq 1$, there exists a *unique* j such that $x \in \left[\frac{j-1}{k}, \frac{j}{k}\right]$ hence $\varphi_{k,j}(x) = 1$, so in other notation there always exists an n such that $f_n(x) = 1$, and so precisely $f_n(x) = 1$ for infinitely many n. Hence, we do not have convergence everywhere (in fact, anywhere).

↪**Proposition 2.16**: Given $\{f_n\}, f$ measurable, finite-valued functions, if $f_n \to f$ in measure, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. as $k \to \infty$.

Proof. Assume $f_n \to f$ in measure, that is for every $\delta > 0$, $m(\{|f_n - f| > \delta\}) \to 0$. Hence, for all $k \geq 1$, with $\delta = \frac{1}{k}$, we have that for some sufficiently large n_k, we have that $m\left(\underbrace{\left\{|f_{n_k} - f| > \frac{1}{k}\right\}}_{:=A_k}\right) \leq \frac{1}{k^2}$, hence $\sum_{k=1}^{\infty} m(A_k) < \infty$. Hence,

$$m\left(\bigcap_{\ell=1}^{\infty}\bigcup_{k=\ell}^{\infty} A_k\right) = \lim_{\ell\to\infty} m\left(\bigcup_{k=\ell}^{\infty} A_k\right) \leq \lim_{\ell\to\infty}\sum_{k=\ell}^{\infty} m(A_k) = 0,$$

since $\sum_{k=\ell}^{\infty} m(A_k)$ the tail of a converging series. Hence, complementing the above, a.e. there $\exists\,\ell$ such that for every $k \geq \ell$, $|f_{n_k} - f| \leq \frac{1}{k}$ and so $\lim_{k\to\infty}|f_{n_k} - f| = 0$ almost everywhere, and so $f_{n_k} \to f$ a.e. (as $k \to \infty$). ■

↪**Proposition 2.17** (Subsequence Test): Given $\{f_n\}, f$ measurable, finite-valued functions, $f_n \to f$ in measure $\Leftrightarrow$ for every subsequence $\{n_k\}$, there exists a subsubsequence $\{n_{k_\ell}\} \subset \{n_k\}$ such that $f_{n_{k_\ell}} \to f$ in measure as $\ell \to \infty$.

PROOF. $\Rightarrow$ is clear. For $\Leftarrow$, suppose towards a contradiction that $f_n \nrightarrow f$ in measure. Then, $\exists\, \delta > 0$ and subsequence $\{n_k\}$ $m\left(\left\{|f_{n_k} - f| > \delta\right\}\right) > \delta$ for every k. By the assumption of the RHS, there exists a further subsequence $\left\{n_{k_\ell}\right\}$ such that $f_{n_{k_\ell}} \to f$ in measure. This is a contradiction. ■

⊛ **Example 2.2** (Assignment Exercise): Prove that if $f_n \to f$ in measure and $g_n \to g$ in measure, $f_n g_n \to fg$ in measure (everything finite valued, measurable).

§2.4 Egorov's Theorem and Lusin's Theorem

Recall that if f is measurable, then $\exists\, \{\theta_n\}$ sequence of step functions such that $\theta_n \to f$ almost everywhere.

↪**Theorem 2.4** (Egorov's): Given $\{f_n\}, f$ measurable functions and $A \in \mathcal{M}$ with $m(A) < \infty$, if $f_n \to f$ a.e. on A, then $\forall\, \varepsilon > 0$, there exists a closed subset $A_\varepsilon \subseteq A$ with $m(A \setminus A_\varepsilon) \leq \varepsilon$ such that $f_n \to f$ uniformly on A_ε.

PROOF. We assume first f is finite-valued on A (otherwise, replace A with $A \cap \{-\infty < f < \infty\}$; we'll deal with $\{f = \pm\infty\}$ later). We want to show that $\forall\, \varepsilon > 0, \exists$ closed $A_\varepsilon \subseteq A$ s.t. $m(A \setminus A_\varepsilon) < \varepsilon$ *and* $\sup_{x \in A_\varepsilon} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

For each $k \geq 1$ and $n \geq 1$, put

$$E_n^{(k)} := \left\{x \in A : |f_j(x) - f(x)| \leq \frac{1}{k}\ \forall\, j \geq n\right\}.$$

For fixed k, remark that $E_n^{(k)} \subseteq E_{n+1}^{(k)}$, i.e. $E_n^{(k)}$ increasing (wrt n), so we may consider

$$\bigcup_{n=1}^{\infty} E_n^{(k)} = \left\{x \in A : \exists\, n \geq 1 \text{ s.t.} \forall\, j \geq n, |f_j(x) - f(x)| \leq \frac{1}{k}\right\} \supseteq \left\{x \in A : \lim_{n\to\infty} f_n(x) = f(x)\right\} =: A'.$$

By assumption, $m(A') = m(A)$, so by continuity and the superset relation above, $m(A) = m(A') \leq m\left(\bigcup_{n=1}^{\infty} E_n^{(k)}\right) = \lim_{n\to\infty} m\left(E_n^{(k)}\right) \leq m(A)$, and thus $\lim_{n\to\infty} m\left(E_n^{(k)}\right) = m(A)$ for every $k \geq 1$.

Given, then, any $\varepsilon > 0$, there exists a n_k such that $m\left(A \setminus E_{n_k}^{(k)}\right) = m(A) - m\left(E_{n_k}^{(k)}\right) < \frac{1}{2^k}\frac{\varepsilon}{2}$. Set

$$B := A \setminus \left(\bigcap_{k=1}^{\infty} E_{n_k}^{(k)}\right),$$

then

$$m(B) = m\left(\bigcup_{k=1}^{\infty} A \setminus E_{n_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} m\left(A \setminus E_{n_k}^{(k)}\right) \leq \frac{\varepsilon}{2}.$$

Put

$$\tilde{A} := A \setminus B = \bigcap_{k=1}^{\infty} E_{n_k}^{(k)}.$$

Then, if $x \in \tilde{A}$, then $x \in E_{n_k}^{(k)}$ for every k, and hence for every $k \geq 1$ and $j \geq n_k$, $|f_j(x) - f(x)| \leq \frac{1}{k}$. This shows then that $f_n \to f$ uniformly on $\tilde{A}$. By regularity of m, there exists a closed $A_\varepsilon \subseteq \tilde{A}$ such that $m(\tilde{A} \setminus A_\varepsilon) \leq \frac{\varepsilon}{2}$. Then, $f_n \to f$ uniformly on A_ε, and $m(A \setminus A_\varepsilon) = m(A \setminus \tilde{A}) + m(\tilde{A} \setminus A_\varepsilon) < \varepsilon$.

Now, if $f = \infty / -\infty$ on A, then $A = A^\infty \cup A^{-\infty} \cup A^{\mathbb{R}}$ (with $A^\bullet := \{f = \bullet\} \cap A$). The last case is done. For A^∞ (similar construction for $A^{-\infty}$), define for every $k, n \geq 1$,

$$E_n^{(k)} := \left\{x \in A : f_j(x) > k \, \forall \, j \geq n\right\}.$$

Then, the remainder of the proof follows precisely the same for the sequence of sets $E_n^{(k)}$. ■

Remark 2.3:

1. The assumption $m(A) < \infty$ is necessary. For instance $f_n = \mathbb{1}_{[n,\infty)} \to 0$ pointwise, but for any $a \in \mathbb{R}$, f_n does not converge to 0 uniformly on (a, ∞).
2. In general, Egorov's $\not\Rightarrow f_n \to f$ uniformly a.e.. For instance, on $[0,1]$, let $f_n(x) = x^n$ and $f(x) \equiv 0$. For every $x \in [0,1)$, $f_n(x) \to f(x)$ as $n \to \infty$. Hence, $f_n \to f$ a.e. on $[0,1]$ (the only point that doesn't converge, indeed, is at 1). If $A \subseteq [0,1]$ is closed such that $1 \in A$, then $f_n \nrightarrow f$ uniformly on A. To see this, let $\{x_m\} \subseteq A$ such that $x_m \uparrow$ and $\lim_{m\to\infty} x_m = 1$. Then, for any fixed n,

$$\sup_{x\in A} |f_n(x) - f(x)| \geq \sup_m |f_n(x_m) - f(x_m)| = \sup_m x_m^n = 1,$$

hence f_n does not converge uniformly on A.

↪**Theorem 2.5** (Lusin's Theorem): Given f measurable and finite-valued and $A \in \mathcal{M}$ with $m(A) < \infty$, for all $\varepsilon > 0$, there exists a closed $A_\varepsilon \subseteq A$ with $m(A \setminus A_\varepsilon) < \varepsilon$ such that $f|_{A_\varepsilon}$ is continuous.

Remark 2.4: Lusin's Theorem states that $f|_{A_\varepsilon}$ is continuous as a function on ε, which is *not* the same as saying f as a function of A is continuous at points in A_ε.

For instance, $f = \mathbb{1}_{\mathbb{Q}\cap[0,1]}$ is not continuous anywhere on $[0,1]$. However, $f|_{\mathbb{Q}\cap[0,1]}$ is constant and therefore continuous *on* $\mathbb{Q} \cap [0,1]$.

Proof. Let $\{\theta_n\}$ be a sequence of step functions such that $\theta_n \to f$ a.e. on A. Note that θ_n piecewise constant and hence piecewise continuous. Given $\varepsilon > 0$ and $n \geq 1$, we can find an open set E_n such that $\theta_n|_{E_n^c}$ is continuous and $m(E_n) \leq \frac{\varepsilon}{2}\frac{1}{2^n}$. Meanwhile, Egorov's implies that there exists a closed $B \subseteq A$ such that $m(A \setminus B) \leq \frac{\varepsilon}{2}$ such that $\theta_n \to f$ uniformly on B. Set

$$A_\varepsilon = B \setminus \bigcup_{n=1}^{\infty} E_n,$$

noting that $A_\varepsilon \subset A$ closed and

$$m(A \setminus A_\varepsilon) = m(A \setminus B) + m\left(\bigcup_{n=1}^{\infty} E_n\right) = \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} m(E_m) \leq \varepsilon.$$

Finally, on A_ε, $\theta_n \to f$ uniformly and $\theta_n|_{A_\varepsilon}$ continuous, and hence $f|_{A_\varepsilon}$ continuous (uniform limit of continuous functions is continuous). ■

Remark 2.5:

1. Lusin's Theorem $\not\Rightarrow f$ is continuous almost everywhere in general. For instance, recall that fat Cantor set $\tilde{C}$, with $m(\tilde{C}) = \frac{1}{2}$. Let $f = \mathbb{1}_{\tilde{C}}$. f is NOT continuous a.e. on $[0,1]$, i.e. $\forall\, B \subseteq [0,1]$ with $m(B) = 1$, $f|_B$ is NOT continuous. To see this, let $\tilde{D} = [0,1] \setminus \tilde{C}$. Since $m(B) = 1$, then $m(\tilde{C} \cap B) = m(\tilde{D} \cap B) = \frac{1}{2}$. Then for any $x \in \tilde{C} \cap B$, $f|_B$ is NOT continuous at x. If it were at say some $x_0 \in \tilde{C} \cap B$, then there must exist some $\delta > 0$ such that for any $x \in (x_0 - \delta, x_0 + \delta) \cap B$, $|f(x) - f(x_0)| < \frac{1}{2}$. Hence, for any $x \in (x_0 - \delta, x_0 + \delta) \cap B$, $\frac{1}{2} \leq f(x) \leq \frac{3}{2}$. However, $m((x_0 - \delta, x_0 + \delta) \cap B \cap \tilde{D}) > 0$ so it must be that $\exists\, y \in (x_0 - \delta, x_0 + \delta) \cap B \cap \tilde{D} \Rightarrow f(y) = 0$, a contradiction. How, then, does one apply Lusin's; that is, $\forall\, \varepsilon > 0$, there must exist some $A_\varepsilon \subseteq [0,1]$ such that $m([0,1] \setminus A_\varepsilon) < \varepsilon$ and $f|_{A_\varepsilon} < \varepsilon$ (exercise)?
2. (Exercise) The $\{\theta_n\}$'s are not continuous on $\mathbb{R}$, but you can choose a sequence $\{\widetilde{\theta_n}\}$ to be continuous on $\mathbb{R}$ such that $\widetilde{\theta_n} \to f$ a.e..
3. Lusin's Theorem $\Rightarrow$ $\forall\, k$ sufficiently large, $\exists\, A_k \subseteq A$ closed such that $m(A \setminus A_k) \leq \frac{1}{k}$ and $f|_{A_k}$ continuous on A_k. In fact, we can construct them such that $A_k \uparrow$ (otherwise replace A_k with $\bigcup_{i=1}^{k} A_i$).

§2.5 Construction of Integrals

2.5.1 Integral of Simple Functions

↪**Definition 2.8**: Given a simple function $\varphi = \sum_{k=1}^{L} a_k \mathbb{1}_{E_k}$, the *(Lebesgue) integral* of φ is defined as

$$\int_{\mathbb{R}} \varphi(x)\, \mathrm{d}x = \int_{\mathbb{R}} \varphi := \sum_{k=1}^{L} a_k \cdot m(E_k).$$

For any $A \in \mathcal{M}$, $\mathbb{1}_A \varphi$ is again a simple function and we define

$$\int_A \varphi := \int_{\mathbb{R}} \mathbb{1}_A \varphi.$$

↪**Proposition 2.18** (Properties of $\int_{\mathbb{R}} \varphi$):

1. (Well-definedness) The written representation of φ is not necessarily unique, but if $\varphi = \sum_{k=1}^{L} a_k \mathbb{1}_{E_k} = \sum_{\ell=1}^{M} b_\ell \mathbb{1}_{F_\ell}$, then
$$\sum_{k=1}^{L} a_k m(E_k) = \sum_{\ell=1}^{M} b_\ell m(F_\ell).$$
2. (Linearity) If φ, ψ two simple functions and $a, b \in \mathbb{R}$, then $a\varphi + b\psi$ a simple function, and
$$\int_{\mathbb{R}} a\varphi + b\psi = a \cdot \int_{\mathbb{R}} \varphi + b \cdot \int_{\mathbb{R}} \psi.$$
3. (Finite Additivity) If φ a simple function, $A, B \in \mathcal{M}$ with $A \cap B = \varnothing$, then
$$\int_{A \cup B} \varphi = \int_A \varphi + \int_B \varphi.$$
4. (Monotonicity) If φ, ψ are two simple functions with $\varphi \leq \psi$, then $\int_{\mathbb{R}} \varphi \leq \int_{\mathbb{R}} \psi$.
5. If φ a simple function then so is $|\varphi|$ and $|\int_{\mathbb{R}} \varphi| \leq \int_{\mathbb{R}} |\varphi|$.

Proof.

1. wlog, we may assume E_k and F_ℓ are respectively disjoint. Set $a_0 = b_0 = 0$, $E_0 := \left(\bigcup_{k=1}^{L} E_k\right)^c, F_0 := \left(\bigcup_{\ell=1}^{M} F_\ell\right)^c$ for convenience. Now, $\{E_0, \ldots, E_L\}, \{F_0, \ldots, F_M\}$ are two partitions of $\mathbb{R}$. In particular, then, for each k, $\mathbb{1}_{E_k} = \sum_{\ell=0}^{M} \mathbb{1}_{E_k \cap F_\ell}$, since $E_k = \bigsqcup_{\ell=0}^{M} (E_k \cap F_\ell)$. Now, we have
$$\varphi = \sum_{k=0}^{L} a_k \mathbb{1}_{E_k} = \sum_{k=0}^{L} \sum_{\ell=0}^{M} a_k \mathbb{1}_{E_k \cap F_\ell}.$$
Similarly partitioning, we have
$$\varphi = \sum_{\ell=0}^{M} b_\ell \mathbb{1}_{F_\ell} = \sum_{\ell=0}^{M} \sum_{k=0}^{L} b_\ell \mathbb{1}_{E_k \cap F_\ell}.$$
If $E_k \cap F_\ell \neq \varnothing$, then $a_k = b_\ell$, and thus on the one hand
$$\int_{\mathbb{R}} \varphi = \sum_{k=0}^{L} \sum_{\ell=0}^{M} a_k m(E_k \cap F_\ell)$$
and on the other
$$\int_{\mathbb{R}} \varphi = \sum_{\ell=0}^{M} \sum_{k=0}^{L} b_\ell m(E_k \cap F_\ell),$$
(with summation convention $0 \cdot \infty = 0$). If $m(E_k \cap F_\ell) > 0$, then $E_k \cap F_\ell \neq \varnothing$ and so $a_k = b_\ell$ and so the two sums agree.
4. Assume $\varphi = \sum_{k=1}^{L} a_k \mathbb{1}_{E_k}, \psi = \sum_{\ell=1}^{M} b_\ell \mathbb{1}_{F_\ell}$. Repeat the partitioning/rewriting steps from part 1, then note that since $\varphi \leq \psi$, if $E_k \cap F_\ell \neq \varnothing$, it must be that $a_k \leq b_\ell$, so if $m(E_k \cap F_\ell) > 0$ $a_k \leq b_\ell$ and thus the monotonicity follows.

■

2.5.2 Integral of Non-Negative Functions

↪**Definition 2.9**: If f a non-negative, measurable function then the integral of f is given by

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x = \int_{\mathbb{R}} f := \sup\left\{\int_{\mathbb{R}} \varphi : \varphi \text{ is simple and } \varphi \leq f\right\}.$$

↪**Proposition 2.19**: The definition above agrees with that for simple functions that are also non-negative, namely this definition is consistent with the previous.

PROOF. Let φ be non-negative. Then $\varphi \leq \varphi$ certainly so the first definition $\int_{\mathbb{R}} \varphi \leq \sup\{\cdots\}$. Conversely, it suffices to show that for any non-negative simple $\psi \leq \varphi$, $\int_{\mathbb{R}} \psi \leq \int_{\mathbb{R}} \varphi$, using the first definition. But this simply follows from monotonicity of $\int$, and we are done. ■

Remark 2.6: Given $f \geq 0$ and measurable, this definition implies that there exists a sequence $\{\varphi_n\}$ of simple functions such that $\varphi_n \leq f$ and $\lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n = \int_{\mathbb{R}} f$. We would like to show that, in some sense, the choice of $\{\varphi_n\}$ is arbitrary.

↪**Theorem 2.6**: Suppose $f \geq 0$ and measurable. If $\{\varphi_n\}$ a sequence of simple functions such that $\varphi_n \uparrow$ and $\lim_{n\to\infty} \varphi_n = f$ pointwise, then

$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n = \int_{\mathbb{R}} f.$$

PROOF. Since $\varphi_n \leq f$ for all $n \geq 1$, then $\int_{\mathbb{R}} \varphi_n \leq \int_{\mathbb{R}} f$ and so $\lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n \leq \int_{\mathbb{R}} f$ (nothing the limit on the LHS necessarily always exists by monotonicity). On the other hand, it suffices to show that $\forall\, \psi \leq f$ simple, that $\int_{\mathbb{R}} \psi \leq \lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n$. Assume $\psi = \sum_{k=1}^{L} a_k \mathbb{1}_{E_k} = \sum_{k=0}^{L} a_k \mathbb{1}_{E_k}$ where $\{E_0, \ldots, E_L\}$ forms a partition of $\mathbb{R}$. Since

$$\int_{\mathbb{R}} \psi = \sum_{k=0}^{L} a_k m(E_k)$$

and

$$\int_{\mathbb{R}} \varphi_n = \sum_{k=0}^{L} \int_{E_k} \varphi_n$$

by finite additivity. It suffices to show then that for each $k = 0, \ldots, L$, $a_k m(E_k) \leq \lim_{n\to\infty} \int_{E_k} \varphi_n$.

First, if $a_k = 0$ or $m(E_k) = 0$, then we are done. Assume $a_k, m(E_k) > 0$. For each fixed k, $\lim_{n\to\infty} \varphi_n = f \geq \psi$ so for every $x \in E_k$, $\lim_{n\to\infty} \varphi_n(x) \geq \psi(x) = a_k$. For any $\varepsilon > 0$, put

$$C_n^\varepsilon := \{x \in E_k : \varphi_n(x) \geq (1-\varepsilon)a_k\}.$$

Since $\varphi_n \leq \varphi_{n+1}$, $C_n^\varepsilon \uparrow$ wrt n. Then note

$$\bigcup_{n=1}^{\infty} C_n^\varepsilon = E_k.$$

Then,

$$\lim_{n\to\infty} \int_{E_k} \varphi_n = \lim_{n\to\infty} \int_{\mathbb{R}} \mathbb{1}_{E_k}\varphi_n \geq \lim_{n\to\infty} \int_{\mathbb{R}} \mathbb{1}_{C_n^\varepsilon}\varphi_n \geq \lim_{n\to\infty} (1-\varepsilon)a_k m(C_n^\varepsilon) = (1-\varepsilon)a_k m(E_k),$$

where we use the fact that $\mathbb{1}_{E_k}\varphi_n \geq \mathbb{1}_{C_n^\varepsilon}\varphi_n \geq (1-\varepsilon)a_k\mathbb{1}_{C_k^\varepsilon}$ and $\lim_{n\to\infty} m(C_n^\varepsilon) = m\left(\bigcup_{n=1}^{\infty} C_n^\varepsilon\right) = m(E_k)$. Since ε arbitrary, then

$$\lim_{n\to\infty} \int_{E_k} \varphi_n \geq a_k m(E_k),$$

and we are done. ■

↪**Corollary 2.4**: For any $f \geq 0$ measurable, if $\forall\, n \geq 1, k = 0, 1, \dots, n2^n$ with $A_{n,k} := \left\{\frac{k}{2^n} \leq f < \frac{k+1}{2^n}\right\}$, then

$$\int_{\mathbb{R}} f = \lim_{n\to\infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} m(A_{n,k}).$$

PROOF. Let $\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n}\mathbb{1}_{A_{n,k}}$, then $\varphi_n \uparrow$ and $\varphi_n \to f$. ■

↪**Proposition 2.20** (Properties of Integral of Non-Negative Functions):

1. (Well-definedness) If $f, g \geq 0$ measurable such that $f = g$ a.e., then $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$.
2. (Linearity) For any $f, g \geq 0$ measurable and $a, b \geq 0$, then $\int_{\mathbb{R}} (af + bg) = a\int_{\mathbb{R}} f + b\int_{\mathbb{R}} g$.
3. (Monotonicity) If $f, g \geq 0$ measurable and $f \leq g$ a.e., then $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$.
4. i. Let $f \geq 0$ measurable, then $\int_{\mathbb{R}} f = 0 \Leftrightarrow f \equiv 0$ a.e.
   ii. Let $f \geq 0$ measurable, $A \in \mathcal{M}$. Then $\int_A f = 0 \Leftrightarrow$ either $f \equiv 0$ a.e. on A or $m(A) = 0$.
   iii. Let $f \geq 0$ measurable, then if $\int_{\mathbb{R}} f < \infty$ then f is finite valued a.e.
5. (Markov's Inequality) Let $f \geq 0$ measurable and $0 < a < \infty$. Then, $m(\{f > a\}) \leq \frac{1}{a}\int_{\mathbb{R}} f$. In particular, if the RHS is finite, $\lim_{\{a\to\infty\}} m(\{f > a\}) = 0$, in fact in $\mathcal{O}\left(\frac{1}{a}\right)$.

PROOF.

1. Let $\{\varphi_n\}, \{\psi_n\}$ sequences of simple functions such that both are monotonically increasing with $\varphi_n \to f, \psi_n \to g$. Put $h_n := \varphi_n\mathbb{1}_{\{f=g\}} + \psi_n\mathbb{1}_{\{f\neq g\}}$; then h_n again simple, $h_n \uparrow$, and $h_n \to g$ everywhere. Then,

$$\int_{\mathbb{R}} g = \lim_n \int_{\mathbb{R}} h_n = \lim_n\left(\int_{\{f=g\}} \varphi_n + \int_{\{f\neq g\}} \psi_n\right) = \lim_n \int_{\{f=g\}} \varphi_n.$$

Meanwhile,

$$\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} \varphi_n = \lim_n \left(\int_{\{f=g\}} \varphi_n + \int_{\{f\neq g\}} \varphi_n \right) = \lim_n \int_{\{f=g\}} \varphi_n,$$

and so $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$.

2. Take $\{\varphi_n\}, \{\psi_n\}$ as in the previous proof. Then $\{h_n : a\varphi_n + b\psi_n\}$ again a sequence of monotonically increasing simple functions with limit $af + bg$. Then

$$\int_{\mathbb{R}} (af + bg) = \lim_n \int_{\mathbb{R}} h_n = \lim_n \int_{\mathbb{R}} (a\varphi_n + b\psi_n) = \lim_n \left(a \int_{\mathbb{R}} \varphi_n + b \int_{\mathbb{R}} \psi_n \right) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g.$$

3. wlog, assume that $f \leq g$ everywhere by replacing f with $f\mathbb{1}_{\{f\leq g\}}$. Then, $\{\varphi :$ simple, $\varphi \leq f\} \subseteq \{\varphi :$ simple, $\varphi \leq g\}$ and so $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$.

4. i. $\Leftarrow$ clear. Conversely, we would like to prove that if $A = \{f > 0\}, m(A) = 0$. Put $A_n := \left\{ f \geq \frac{1}{n} \right\}$ for $n \geq 1$. Then, $A_n \uparrow$ and $\bigcup_{n=1}^{\infty} A_n = A$. By continuity of m,

$$m(A) = \lim_n m(A_n).$$

Suppose towards a contradiction that $m(A) = \delta > 0$. Then, $\delta = \lim_n m(A_n)$, and so must exist $N \geq 1$ such that $m(A_N) \geq \frac{\delta}{2}$. Since $f \geq f\mathbb{1}_{A_N} \geq \frac{1}{N}\mathbb{1}_{A_N}$. By monotonicity, $\int_{\mathbb{R}} f \geq \int_{\mathbb{R}} \frac{1}{N}\mathbb{1}_{A_N} = \frac{1}{N} m(A_N) \geq \frac{1}{N}\frac{\delta}{2} > 0$, a contradiction.
ii. By i., $\int_A f = 0 \Leftrightarrow \mathbb{1}_A f \equiv 0$ a.e. on $\mathbb{R}$. If $m(A) = 0$, then $\mathbb{1}_A \equiv 0$ a.e. so $\mathbb{1}_A f \equiv 0$ a.e.. Else, if $m(A) > 0$, then $f \equiv 0$ a.e. on A.
iii. Put $A := \{f = \infty\}$. Assume towards a contradiction that $m(A) = \delta > 0$. Then, for every $n \geq 1, f \geq f\mathbb{1}_A \geq n\mathbb{1}_A$ and so $\int_{\mathbb{R}} f \geq \int_{\mathbb{R}} n\mathbb{1}_A = nm(A) = n\delta$. But this holds for any arbitrary n, so $\int_{\mathbb{R}} f = \infty$, a contradiction.

5. Put $A_a := \{f > a\}$. Then $f \geq f\mathbb{1}_{A_a} > a\mathbb{1}_{A_a}$ so $\int_{\mathbb{R}} f \geq am(A_a)$.

■

2.5.3 Integral of General Measurable, Integrable Functions

↪**Definition 2.10**: For f measurable, $\int_{\mathbb{R}} f := \int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-$, provided that at least one of $\int_{\mathbb{R}} f^+$, $\int_{\mathbb{R}} f^-$ is finite; in particular, $\int_{\mathbb{R}} f$ may be finite or infinite.

Remark 2.7: Only having $\int_{\mathbb{R}} f$ being defined is not sufficient for the desirable properties (linearity, monotonicity) to hold.

↪**Definition 2.11** (Integrable): A measurable function f is called *integrable*, denoted $f \in L^1(\mathbb{R})$, if *both* $\int_{\mathbb{R}} f^+ < \infty$ and $\int_{\mathbb{R}} f^- < \infty$. Note that

$$\begin{aligned} f \in L^1(\mathbb{R}) &\Leftrightarrow \int_{\mathbb{R}} |f| < \infty \text{ (since } \int_{\mathbb{R}} |f| = \int_{\mathbb{R}} f^+ + \int_{\mathbb{R}} f^-) \\ &\Leftrightarrow \int_{\mathbb{R}} f \text{ finite valued.} \end{aligned}$$

↪**Proposition 2.21** (Properties of Integrals of Integrable Functions):

1. $|\int_{\mathbb{R}} f| \leq \int_{\mathbb{R}} |f|$
2. $f \in L^1(\mathbb{R}) \Rightarrow f$ is finite valued a.e.
3. (Linearity) For $f, g \in L^1(\mathbb{R})$ and $a, b \in \mathbb{R}$, $af + bg \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}}(af + bg) = a\int_{\mathbb{R}} f + b\int_{\mathbb{R}} g$
4. If $f \in L^1(\mathbb{R})$ and $A \in \mathcal{M}$ and $m(A) = 0$ then $\int_A f = 0$; in particular if $f, g \in L^1(\mathbb{R})$ with $f = g$ a.e. then $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$
5. (Monotonicity) If $f, g \in L^1(\mathbb{R})$ with $f \leq g$ a.e., then $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$

Proof.

1. $-\int_{\mathbb{R}} f^- \leq \int_{\mathbb{R}} f \leq \int_{\mathbb{R}} f^+$ and $\int_{\mathbb{R}} f^{\pm} \leq \int_{\mathbb{R}} |f|$.
2. We know $\int_{\mathbb{R}} |f| < \infty$ so $|f| < \infty$ a.e. by properties of integrals of non-negative functions so $m(\{f = \pm\infty\}) = 0$
3. $|af| \leq |a|\,|f|$ so by monotonicity of non-negative functions, $\int_{\mathbb{R}} |af| \leq |a| \int_{\mathbb{R}} |f| < \infty$ so af in $L^1(\mathbb{R})$. Note then that

$$(af)^+ = \begin{cases} af^+ \text{ if } a \geq 0 \\ -af^- \text{ if } a < 0 \end{cases}, \qquad (af)^- = \begin{cases} af^- \text{ if } a \geq 0 \\ -af^+ \text{ if } a < 0 \end{cases}$$

so

$$\begin{aligned} \int_{\mathbb{R}} af &= \int_{\mathbb{R}} (af)^+ - \int_{\mathbb{R}} (af)^- \\ &= \begin{cases} \int_{\mathbb{R}} af^+ - \int_{\mathbb{R}} af^- \text{ if } a \geq 0 \\ \int_{\mathbb{R}} (-a)f^- - \int_{\mathbb{R}} (-a)f^+ \text{ if } a < 0 \end{cases} \\ &= \begin{cases} a\left(\int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-\right) \text{ if } a \geq 0 \\ (-a)\left(\int_{\mathbb{R}} f^- - \int_{\mathbb{R}} f^+\right) \text{ if } a < 0 \end{cases} = a\int_{\mathbb{R}} f. \end{aligned}$$

   By the same argument $bg \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}}(bg) = b\int_{\mathbb{R}} g$. wlog, $a = b = 1$. We want to show $f + g \in L^1(\mathbb{R})$; clearly $|f + g| \leq |f| + |g| < \infty$ so it must be $f + g \in L^1(\mathbb{R})$. Set $h := f + g$ then $|h, f, g| < \infty$ a.e. and each of the integrals of $|h, f, g| < \infty$. Then, $h^+ - h^- = f^+ - f^- + g^+ - g^-$. Then $h^+ + f^- + g^- = f^+ + g^+ + h^-$, where now both sides are non-negative functions. By linearity of integrals of non-negative functions and since all terms finite a.e.,

$$\begin{aligned} \int h^+ + \int f^- + \int g^- &= \int f^+ + \int g^+ + \int h^- \\ &\Rightarrow \int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^- \\ &\Rightarrow \int (f + g) = \int h = \int f + \int g. \end{aligned}$$

4. $|\int_A f| \leq \int_A |f| = 0$.
5. Put $h = g - f$ (valid since $f, g \in L^1(\mathbb{R})$) then $h \geq 0$ a.e. Then $\int_{\mathbb{R}} h \geq 0$ so by linearity $\int_{\mathbb{R}}(g - f) = \int_{\mathbb{R}} g - \int_{\mathbb{R}} f \geq 0$.

■

§2.6 Convergence Theorems of Integral

↪**Theorem 2.7** (Monotone Covergence Theorem (MON)): Assume $\{f_n\}, f$ are non-negative, measurable functions. If $f_n \uparrow$ and $\lim_{n\to\infty} f_n = f$, then

$$\int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n.$$

Remark 2.8: When we write $\lim_n f_n = f$, we mean pointwise convergence; however, one can replace these statements with convergence a.e. and obtain an equivalent, more general result wlog.

Proof. By monotonicity of non-negative functions, $\lim_{n\to\infty} \int_{\mathbb{R}} f_n$ exists, forming an increasing sequence. Since $f_n \leq f$, then we know too that $\lim_{n\to\infty} \int_{\mathbb{R}} f_n \leq \int_{\mathbb{R}} f$.

Conversely, for every n, let $\{\varphi_{n,k}\}_{k\in\mathbb{N}}$ be a sequence of simple functions such that $\varphi_{n,k} \uparrow$ w.r.t k and $\varphi_{n,k} \to f_n$ as $k \to \infty$;

$$\begin{array}{cccccccc}
f_1 & f_2 & \cdots & f_k & f_{k+1} & \cdots & \to f \\
\vdots & \vdots & \iddots & \vdots & \vdots & & \\
\varphi_{1,k} & \varphi_{2,k} & \iddots & \varphi_{k,k} & \varphi_{k+1,k} & \cdots & \\
\vdots & \vdots & \iddots & \vdots & \vdots & \cdots & \\
\varphi_{1,2} & \varphi_{2,2} & \iddots & \varphi_{k,2} & \varphi_{k+1,2} & \cdots & \\
\varphi_{1,1} & \varphi_{2,1} & \cdots & \varphi_{k,1} & \varphi_{k+1,1} & \cdots &
\end{array}$$

For each $k \geq 1$, let

$$g_k := \max\{\varphi_{1,k}, \varphi_{2,k}, \ldots, \varphi_{k,k}\}.$$

Then, g_k simple for each k, and $g_k \uparrow$ and $g_k \leq f$. So, $\lim_{k\to\infty} g_k$ exists. Then, for all $n \geq 1$, $\lim_{k\to\infty} g_k \geq \lim_{k\to\infty} \varphi_{n,k} = f_n$ so $\lim_{k\to\infty} g_k \geq \lim_{n\to\infty} f_n = f$. Thus, $\lim_{k\to\infty} \int_{\mathbb{R}} g_k = \int_{\mathbb{R}} f$ by a previous theorem. Since $\forall\, k \geq 1$, $\varphi_{1,k}, \varphi_{2,k}, \cdots, \varphi_{k,k} \leq f_k$, $g_k \leq f_k$ and thus by monotonicity $\int_{\mathbb{R}} g_k \leq \int_{\mathbb{R}} f_k \Rightarrow \int_{\mathbb{R}} f = \lim_{k\to\infty} \int_{\mathbb{R}} g_k \leq \lim_{k\to\infty} \int_{\mathbb{R}} f_k$ as desired. ■

↪**Corollary 2.5**: If $\{f_n\}, f$ measurable functions such that $f_n \uparrow$ and $\lim_n f_n = f$ and $\int_{\mathbb{R}} f_1^- < \infty$, then $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$.

Proof. Since $f_n \uparrow, f_n \geq f_1$ so $f \geq f_1$. Then, $f_n^- \leq f_1^-, f^- \leq f_1^-$, all of these are finite valued a.e., and $\int_{\mathbb{R}} f_n^- \leq \int_{\mathbb{R}} f_1^- < \infty$ and $\int_{\mathbb{R}} f^- \leq \int_{\mathbb{R}} f_1^- < \infty$. For each $n \geq 1$, set $\tilde{f}_n := f_n + f_1^- = f_n^+ - f_n^- + f_1^- \geq 0$, and $\tilde{f}_n \uparrow$ with $\lim_n \tilde{f}_n = f + f_1^- =: \tilde{f} \geq 0$. By MON, $\int_{\mathbb{R}} \tilde{f} = \lim_n \int_{\mathbb{R}} \tilde{f}_n$ so $\int_{\mathbb{R}} (f + f_1^-) = \lim_n \int_{\mathbb{R}} (f_n + f_1^-)$.

We have that $\tilde{f}_n = f_n + f_1^- = f_n^+ - f_n^- + f_1^- \Rightarrow \tilde{f}_n + f_n^- = f_n^+ + f_1^-$, which is valid since $f_n^- < \infty$ a.e.. By linearity, then,

$$\int_{\mathbb{R}} \tilde{f}_n + \int_{\mathbb{R}} f_n^- = \int_{\mathbb{R}} f_n^+ + \int_{\mathbb{R}} f_1^-$$
$$\Rightarrow \int_{\mathbb{R}} \tilde{f}_n = \int_{\mathbb{R}} f_n^+ - \int_{\mathbb{R}} f_n^- + \int_{\mathbb{R}} f_1^- \qquad \text{because } \int_{\mathbb{R}} f_n^- < \infty$$
$$\Rightarrow \int_{\mathbb{R}} \tilde{f}_n = \int_{\mathbb{R}} f_n + \int_{\mathbb{R}} f_1^-.$$

Similar work gives $\int_{\mathbb{R}} \tilde{f} = \int_{\mathbb{R}} f + \int_{\mathbb{R}} f_1^-$, and taking limits and using $\lim_n \int_{\mathbb{R}} (f_n + f_1^-) = \int_{\mathbb{R}} (f + f_1^-)$ completes the proof. ■

↪**Theorem 2.8** (Reverse MON): Assume $\{f_n\}$, measurable such that $f_n \downarrow$ and $\lim_{n\to\infty} f_n = f$. If $\int_{\mathbb{R}} f_1^+ < \infty$, then $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$.

Proof. Consider $\{-f_n\}$ and use the previous corollary. ■

↪**Theorem 2.9** (Fatou's Lemma): Assume $\{f_n\}$ non-negative, measurable. Then

$$\int_{\mathbb{R}} \left(\liminf_{n\to\infty} f_n\right) \leq \liminf_{n\to\infty} \left(\int_{\mathbb{R}} f_n\right).$$

Proof. For every $m \geq 1$, set $g_m := \inf_{n\geq m} f_n$. Then, g_m non-negative and $g_m \uparrow$, with $\lim_m g_m = \liminf_n f_n$. By MON, $\int_{\mathbb{R}} \liminf_n f_n = \lim_{m\to\infty} \left(\int_{\mathbb{R}} g_m\right)$. For every $n \geq m$, $g_m \leq f_n$, so by monotonicity, $\int_{\mathbb{R}} g_m \leq \int_{\mathbb{R}} f_n$ for every $n \geq m$, so $\int_{\mathbb{R}} g_m \leq \inf_{n\geq m} \int_{\mathbb{R}} f_n$, and hence $\lim_{m\to\infty} \int_{\mathbb{R}} g_m \leq \lim_{m\to\infty} \inf_{n\geq m} \int_{\mathbb{R}} f_n = \liminf_n \left(\int_{\mathbb{R}} f_n\right)$, and the proof follows. ■

↪**Corollary 2.6**: Assume $\{f_n\}$ measurable and there exists a measurable function g such that $\int_{\mathbb{R}} g^- < \infty$ and $f_n \geq g$ for every n. Then,

$$\int_{\mathbb{R}} \left(\liminf_{n} f_n\right) \leq \liminf_{n} \left(\int_{\mathbb{R}} f_n\right).$$

Proof. Since $f_n \geq g$ for all $n \geq 1$, $f_n^- \leq g^-$ so $f_n^- < \infty$ a.e. and $\int_{\mathbb{R}} f_n^- < \infty$. Set $\tilde{f}_n := f_n + g^- \geq 0$. Then, apply Fatou to get $\int_{\mathbb{R}} \liminf_n \tilde{f}_n \leq \liminf_n \int_{\mathbb{R}} \tilde{f}_n$, then it suffices to check linearity. ■

↪**Theorem 2.10** (Reverse Fatou): Assume $\{f_n\}$ measurable and there exists a g measurable such that $\int_{\mathbb{R}} g^+ < \infty$ and $f_n \leq g$ for all $n \geq 1$. Then,

$$\int_{\mathbb{R}} \left(\limsup_{n} f_n\right) \geq \limsup_{n} \left(\int_{\mathbb{R}} f_n\right).$$

Proof. Apply previous proof to $\{-f_n\}$. ■

Remark 2.9: The "floor" g is necessary. Let $f_n(x) := \begin{cases} -1 & \text{if } x \geq n \\ 0 & \text{if } x < n \end{cases}$. Then, $f_n \uparrow$, and $\lim_n f_n = 0$ while $\int_{\mathbb{R}} f_n = -\infty$ for every n, so MON doesn't apply.

↪**Theorem 2.11** (Dominated Convergence Theorem (DOM)): Assume $\{f_n\}, f$ measurable with $\lim_n f_n = f$. If there exists a $g \in L^1(\mathbb{R})$ such that $|f_n| \leq |g|$ for all n, then $f_n \to f$ in $L^1(\mathbb{R})$ i.e. $\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - f| = 0$. In particular, $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$.

Proof. Since $|f_n| \leq |g|$ and $f = \lim_{n\to\infty} f_n$, then $|f| \leq |g|$. So, $\int_{\mathbb{R}} |f_n| \leq \int_{\mathbb{R}} |g| < \infty$ and similarly $\int_{\mathbb{R}} |f| \leq \int_{\mathbb{R}} |g| < \infty$ so $|f_n|, f \in L^1(\mathbb{R})$.

Observe that $|f_n - f| \leq 2\,|g|$, and $\int_{\mathbb{R}} (2\,|g|) < \infty$. Applying Reverse Fatou to $\{|f_n - f|\}_{n\in\mathbb{N}}$, we find

$$\int_{\mathbb{R}} \left(\underbrace{\limsup_n (|f_n - f|)}_{0} \right) \geq \limsup_n \left(\int_{\mathbb{R}} |f_n - f| \right)$$

$$\Rightarrow \lim_{n\to\infty} \int_{\mathbb{R}} |f_n - f| = 0,$$

so in particular

$$\left| \int_{\mathbb{R}} f_n - \int_{\mathbb{R}} f \right| = \left| \int_{\mathbb{R}} (f_n - f) \right| \leq \int_{\mathbb{R}} |f_n - f| \to 0$$

so $\lim_n \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$. ■

Remark 2.10: We must find $g \in L^1(\mathbb{R})$ to dominate $|g| \geq |f_n|$ irrespective of n. For instance, if $f_n = \mathbb{1}_{[n,2n]}$, then $\lim_n f_n = 0$, but $\int_{\mathbb{R}} f_n = n$ for all $n \geq 1$. DOM doesn't apply, since we would need a constant 1 function to dominate all f_n, which is not integrable.

↪**Proposition 2.22**: Assume $f \in L^1(\mathbb{R})$, $\{h_n\}$ a sequence of measurable functions that are uniformly bounded, i.e. $\exists\, M > 0$ such that $|h_n| \leq M$ a.e. for all $n \geq 1$. If $h_n \to h$ a.e. for some measurable function h, then

$$\lim_n \int_{\mathbb{R}} (f h_n) = \int_{\mathbb{R}} (f h).$$

Proof. For every n, $|f \cdot h_n| \leq M\,|f| \in L_1(\mathbb{R})$. The conclusion follows from DOM. ■

↪**Corollary 2.7**: If $f \in L^1(\mathbb{R})$ then for all $\varepsilon > 0$, there exists a compact set $K \subseteq \mathbb{R}$ such that $\int_{K^c} |f| \leq \varepsilon$.

Proof. If $h_n := \mathbb{1}_{[-n,n]}$, the $\lim_n \int_{\mathbb{R}} f h_n = \lim_n \int_{[-n,n]} f = \int_{\mathbb{R}} f$, and also $\lim_n \int_{\{\mathbb{R}-[-n,n]\}} f = 0$. ■

↪**Corollary 2.8**: If $f \in L^1(\mathbb{R})$, then for all $\varepsilon > 0$, $\exists N \geq 1$ such that $\int_{\{|f|>N\}} |f| \leq \varepsilon$.

Proof. Let $h_n = \mathbb{1}_{\{|f|>n\}}$ then $\lim_{n\to\infty} \int_{\{|f|>n\}} f = 0$. ■

↪**Corollary 2.9**: If $\{A_n\} \subseteq \mathcal{M}$ such that $A_n \uparrow$, then $\int_{\bigcup_{n=1}^{\infty} A_n} f = \lim_{n\to\infty} \int_{A_n} f$ $(\mathbb{1}_{A_n} f \to \mathbb{1}_{\bigcup_{n=1}^{\infty} A_n} f)$.

↪**Corollary 2.10** (Countable Additivity): If $\{B_n\} \subseteq \mathcal{M}$ are disjoint, then $\int_{\bigcup_{n=1}^{\infty} B_n} f = \sum_{n=1}^{\infty} \int_{B_n} f$.

↪**Corollary 2.11**: If $\{A_n\} \subseteq \mathcal{M}$ such that $A_n \downarrow$, then $\int_{\bigcap_{n=1}^{\infty} A_n} f = \lim_{n\to\infty} \int_{A_n} f$.

↪**Proposition 2.23**: Assume f is non-negative, measurable, and finite-valued a.e.. Then, for every $k \in \mathbb{Z}$, put $A_k := \left\{x \in \mathbb{R} : 2^k \leq f(x) < 2^{k+1}\right\}$. Then,

$$f \text{ integrable} \Leftrightarrow \int_{\mathbb{R}} f < \infty \Leftrightarrow \sum_{k\in\mathbb{Z}} 2^k m(A_k) < \infty.$$

Proof. $(\Rightarrow)$ Note that the A_k's disjoint and $\bigcup_{k\in\mathbb{Z}} A_k = \{0 < f < \infty\}$. So,

$$\int_{\mathbb{R}} f = \underbrace{\int_{\{f=0\}} f}_{=0 \text{ since } f=0} + \int_{\{0<f<\infty\}} + \underbrace{\int_{\{f=\infty\}} f}_{=0 \text{ since } f<\infty \text{ a.e.}} = \sum_{k\in\mathbb{Z}} \int_{A_k} f.$$

For each $k \in \mathbb{Z}$, for every $x \in A_k$, $2^k \leq f(x) < 2^{k+1}$ so $2^k m(A_k) \leq \int_{A_k} f(x) < 2^{k+1} m(A_k)$. Hence,

$$\sum_{k\in\mathbb{Z}} 2^k m(A_k) \leq \sum_{k\in\mathbb{Z}} \int_{A_k} f = \int_{\mathbb{R}} f < \infty.$$

$(\Leftarrow)$ Suppose $\sum_{k\in\mathbb{Z}} 2^k m(A_k) < \infty$. We know again

$$\int_{\mathbb{R}} f = \int_{\{0<f<\infty\}} f \underset{\text{By MON}}{=} \sum_{k\in\mathbb{Z}} \int_{A_k} f < \sum_{k\in\mathbb{Z}} 2^{k+1} m(A_k) = 2 \sum_{k\in\mathbb{Z}} 2^k m(A_k) < \infty.$$

■

⊛ **Example 2.3**: Let $f(x) = |x|^{-\alpha} \mathbb{1}_{[-1,1]}(x)$, with $f(0) = \infty$ and $\alpha > 0$; f finite-valued a.e.. For every $k \in \mathbb{Z}$, put $A_k := \left\{2^k \leq f < 2^{k+1}\right\} = \left\{x \in [-1,1] : 2^k \leq |x|^{-\alpha} < 2^{k+1}\right\}$. By definition, $|f| \geq 1$, so

$$A_k = \left[-2^{-\frac{k}{\alpha}}, -2^{\frac{-(k+1)}{\alpha}}\right) \cup \left(2^{\frac{-(k+1)}{\alpha}}, 2^{-\frac{k}{\alpha}}\right] \text{ for } k \geq 0, \qquad A_k = \varnothing \text{ if } k < 0.$$

Hence,

$$\sum_{k\in\mathbb{Z}} 2^k m(A_k) = \sum_{k=0}^{\infty} 2^k \cdot 2 \cdot \left(1 - 2^{-\frac{1}{\alpha}}\right) 2^{-\frac{k}{\alpha}} = 2\left(1 - 2^{-\frac{1}{\alpha}}\right) \sum_{k=0}^{\infty} 2^{k\left(1-\frac{1}{\alpha}\right)}.$$

Hence, the series $< \infty \Leftrightarrow \alpha < 1$, and thus $\int_{[-1,1]} |x|^{-\alpha} \, \mathrm{d}x < \infty \Leftrightarrow \alpha < 1$.

⊛ **Example 2.4**: Let $g(x) = |x|^{-\beta} \mathbb{1}_{\mathbb{R}-[-1,1]}(x)$ with $\beta > 0$. We have $|g| < 1$; we again put

$$A_k := \left\{2^k \leq g < 2^{k+1}\right\} = \begin{cases} \left[-2^{-\frac{k}{\beta}}, -2^{\frac{-(k+1)}{\beta}}\right) \cup \left(2^{\frac{-(k+1)}{\beta}}, 2^{-\frac{k}{\beta}}\right] & \text{if } k < 0 \\ \varnothing & \text{if } k \geq 0. \end{cases}.$$

So,

$$\int_{\mathbb{R}-[-1,1]} |x|^{-\beta} \, \mathrm{d}x < \infty \Leftrightarrow \sum_{k\in\mathbb{Z}} 2^k m(A_k) < \infty \Leftrightarrow \beta > 1.$$

⊛ **Example 2.5**: Let $f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)$. What is $\lim_{n\to\infty} \int_{(0,\infty)} f_n(x) \, \mathrm{d}x$? We have that for all $x > 0$, $\lim_{n\to\infty} f_n(x) = 0$. We have that since $|\sin\left(\frac{x}{n}\right)| \leq 1$, so

$$|f_n(x)| \leq \left(1 + \frac{x}{n}\right)^{-n} \leq \left(1 + \frac{x}{2}\right)^{-2} \; \forall \, x > 0, \forall \, n \geq 2.$$

Let $g(x) := \left(1 + \frac{x}{2}\right)^{-2}$. We would like to apply DOM, so we need to check that $g \in L^1((0,\infty))$. We have that

$$\int_{(0,\infty)} g = \int_{(0,1]} g + \int_{(1,\infty)} g \leq \int_{(0,1]} 1 + \underbrace{\int_{(1,\infty)} \frac{4}{x^2} \, \mathrm{d}x}_{\beta=2 \text{ of previous example}} < \infty,$$

so indeed $g \in L^1((0,\infty))$. Applying DOM, then, we have that

$$\lim_{n\to\infty} \int_{(0,\infty)} f_n = \int_{(0,\infty)} \lim_{n\to\infty} f_n = 0.$$

⊛ **Example 2.6**: Let $c > 0, f_n(x) = x^{-c}(\cosh x)^{-\frac{1}{n}}$. What is $\lim_n \int_{(1,\infty)} f_n$?

For every $x > 1$, $\cosh x > 1$, so $(\cosh x)^{-\frac{1}{n}} \uparrow$ with respect to n, with $\lim_n (\cosh x)^{-\frac{1}{n}} = 1$, so $\lim_{n\to\infty} f_n(x) = x^{-c}$ for every $x > 1$. Let $g(x) = x^{-c}$, then. By previous examples, when $c > 1$, $g \in L^1((1,\infty))$ so DOM applies and thus

$$\lim_n \int_{(1,\infty)} f_n = \int_{(1,\infty)} \lim_n f_n = \int_{(1,\infty)} x^{-c}\, \mathrm{d}x < \infty.$$

When $0 < c \le 1$, by Fatou,

$$\liminf_n \int_{(1,\infty)} f_n \ge \int_{(1,\infty)} \liminf_n (f_n) = \int_{(1,\infty)} x^{-c}\, \mathrm{d}x,$$

since f_n converges. When $0 < c \le 1$, the RHS $= \infty$, and thus $\lim_{n\to\infty} \int_{(1,\infty)} f_n = \infty$.

⊛ **Example 2.7**: Let $c \ge 0, f_n(x) := \frac{n}{1+n^2x^2}$ for $x \ge 0$. What is $\lim_n \int_{[c,\infty)} f_n$?

We have that

$$\lim_n f_n(x) = \begin{cases} 0 & \text{if } x > 0 \\ \infty & \text{if } x = 0 \end{cases}.$$

On $x \in [1,\infty), f_n(x) \ge f_{n+1}(x)$ for all $n \ge 1$, namely $f_n \downarrow$, and so $f_n(x) \le f_1(x) = \frac{1}{1+x^2}$. $f_1(x) \in L^1(\mathbb{R})$, by comparison with $\frac{1}{x^2}$ $(\alpha = 2)$.

If $x \in (0,1), f_n(x) = \frac{1}{x}\frac{nx}{1+(nx)^2} \le A\frac{1}{x}$, with $A := \sup_{t>0} \frac{t}{1+t^2} < \infty$. But $\frac{A}{x} \notin L^1((0,1))$.

When $c > 0$, for all $x \ge c$ and for all $n \ge 1$,

$$f_n(x) \le \mathbb{1}_{[1,\infty)}(x)\frac{1}{1+x^2} + \mathbb{1}_{[c,1)}\frac{A}{x} \le \mathbb{1}_{[1,\infty)}(x)\frac{1}{1+x^2} + \mathbb{1}_{[c,1)}(x)\frac{A}{c} \in L^1([c,\infty)).$$

Hence, we may apply DOM, so

$$\lim_n \int_{[c,\infty)} f_n = \int_{[c,\infty)} \lim_n f_n = 0,$$

when $c > 0$. However, when $c = 0$, we have no such dominating function; so what is $\int_{[0,\infty)} f_n(x)\, \mathrm{d}x$?

§2.7 Riemann Integral vs Lebesgue Integral

Recall; let f be bounded on $[a,b]$. Then, f is Riemann integrable on $[a,b]$ if

$$\begin{cases} f \text{ is continuous on } [a,b] \\ f \text{ is monotonic on } [a,b] \\ f \text{ is continuous except at possibly finitely many points in } [a,b] \end{cases}.$$

Recall the function $f = \mathbb{1}_{\mathbb{Q}\cap[0,1]}$. f is not Riemann integrable, but is Lebesgue integrable, because $|f| \le \mathbb{1}_{[0,1]} \in L^1(\mathbb{R})$.

Remark 2.11:
1. $\exists$ bounded functions on $[a, b]$ that are not Riemann integrable.
2. In general, g being Riemann integrable and $|f| \leq |g| \not\Rightarrow f$ is Riemann integrable ($\mathbb{1}_{\mathbb{Q}\cap[0,1]} \leq \mathbb{1}_{[0,1]}$).
3. In general, DOM and MON do *not* apply to Riemann integrable. For instance, consider $\{q_n\}$ an enumeration of $\mathbb{Q} \cap [0, 1]$. Define $f_n(x) := \begin{cases} 1 \text{ if } x \in \{q_1, \ldots, q_n\} \\ 0 \text{ else} \end{cases}$. $f_n \uparrow$, with $f_n \to \mathbb{1}_{\mathbb{Q}\cap[0,1]}$. So, MON applies with the Lebesgue integral, but f_n is only discontinuous, for every n, at finitely many points, so f_n Riemann integrable with $\int_0^{1(R)} f_n = 0$, but the limit is not Riemann integrable.

↪**Theorem 2.12**: Assume f is Riemann integrable on $[a, b]$. Then, f is Lebesgue integrable on $[a, b]$, i.e. $f \in L^1([a, b])$. Moreover, $\int_a^{b(R)} f = \int_{[a,b]} f$.

Proof. f is Riemann integrable on $[a, b]$, so there is some $M > 0$ such that $|f| \leq M$ on $[a, b]$. Further, there exist step functions φ_n, ψ_n with $\varphi_n \leq f \leq \psi_n$ on $[a, b]$ and $|\varphi_n|, |\psi_n| \leq M$ for all $n \geq 1$, and

$$\lim_{n\to\infty} \int_a^{b(R)} \varphi_n = \int_a^{b(R)} f = \lim_{n\to\infty} \int_a^{b(R)} \psi_n.$$

Denote $\varphi := \lim_{n\to\infty} \varphi_n, \psi := \lim_{n\to\infty} \psi_n$, which exist by Monotonicity. Since φ_n, ψ_n are step functions, they are measurable hence φ, ψ measurable with $\varphi \leq f \leq \psi$. Observe that the Lebesgue, Riemann integral coincide on step functions. Hence, $\int_a^{b(R)} \varphi_n = \int_{[a,b]} \varphi_n$, same with ψ_n. By DOM, (with M as the dominator)

$$\int_{[a,b]} \varphi = \lim_n \int_{[a,b]} \varphi_n = \lim_n \int_a^{b(R)} \varphi_n = \int_a^{b(R)} (f) = \lim_n \int_a^{b(R)} \psi_n = \lim_n \int_{[a,b]} \psi_n = \int_{[a,b]} \psi.$$

Since $\varphi \leq \psi$ and $\int_{[a,b]} (\psi - \varphi) = 0$, we have that $\psi = \varphi$ a.e. on $[a, b]$ by properties of integrals of non-negative functions, and thus $f = \varphi = \psi$ a.e. on $[a, b]$. In particular, then, f is measurable, being equal a.e. to measurable functions. Thus, since $|f| \leq M$ on $[a, b]$, $f \in L^1([a, b])$, and so since integrals agree on functions that are equal a.e., $\int_{[a,b]} f = \int_{[a,b]} \varphi = \int_a^{b(R)} f$ as desired. ■

⊛ **Example 2.8**: We return to our example of computing $\lim_{n\to\infty} \int_{[0,\infty)} \frac{n}{1+n^2x^2} \,\mathrm{d}x$. We may rewrite

$$\int_{[0,\infty)} \frac{n}{1+n^2x^2} \,\mathrm{d}x = \int_{[0,T]} \frac{n}{1+n^2x^2} \,\mathrm{d}x + \int_{[T,\infty)} \frac{n}{1+n^2x^2} \,\mathrm{d}x$$

where $T > 0$. We know from the previous example that the RHS integral converges to 0 by application of DOM. Now, $\frac{n}{1+n^2x^2}$ is continuous on $[0,T]$ and thus Riemann integrable, and so by the previous theorem

$$\int_{[0,T]} \frac{n}{1+n^2x^2} = \int_{[0,T]}^{(R)} \frac{n}{1+n^2x^2} = \arctan(nT).$$

As $n \to \infty$, $\arctan(nT) \to \frac{\pi}{2}$, and thus the limit of the whole integral indeed exists, and is in fact equal to $\frac{\pi}{2}$.

§2.8 L^p-space

↪**Definition 2.12** (p-integrable): Let f measurable and $1 \leq p < \infty$. We say f is *p-integrable* and write $f \in L^p(\mathbb{R})$ if $\int_{\mathbb{R}} |f|^p < \infty$, i.e. $|f|^p \in L^1(\mathbb{R})$.

For $f \in L^p(\mathbb{R})$, define the p-norm

$$\|f\|_p := \left(\int_{\mathbb{R}} |f|^p\right)^{\frac{1}{p}}.$$

Remark 2.12: When $p = 1$, we see that $\|\cdot\|_1$ a norm fairly clearly from properties of the integral. We need to show this for more general $p > 1$.

Remark 2.13: $\|\cdot\|_p$ also defined when $p = \infty$; given f measurable, we define

$$\|f\|_\infty := \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)| := \inf\left\{a \in \overline{\mathbb{R}} : |f| \leq a \text{ a.e.}\right\}.$$

Then, we define

$$L^\infty(\mathbb{R}) := \{f \text{ measurable s.t. } \|f\|_\infty < \infty\}.$$

One can show that if $f \in L^\infty(\mathbb{R})$, $|f| \leq \|f\|_\infty$ a.e..

↪**Theorem 2.13** (Hölder's Inequality): Let $1 < p < \infty$ and let $q := \frac{p}{p-1}$ (such a q is called the *Hölder Conjugate* of p). If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then $fg \in L^1(\mathbb{R})$, and

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_q.$$

In particular, if $p = q = 2$, then we have the *Cauchy-Schwarz Inequality*.

Remark 2.14: $\frac{1}{p} + \frac{1}{q} = 1$.

PROOF. We will employ "Young's Inequality", which states that for all $a, b \geq 0$, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ where $\frac{1}{p} + \frac{1}{q} = 1$. Since $f \in L^p, g \in L^q$, set $\tilde{f} := \frac{f}{\|f\|_p}$ and $\tilde{g} := \frac{g}{\|g\|_q}$. Then, a.e.

$$|\tilde{f}\tilde{g}| \leq \frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q}.$$

We have

$$\int_{\mathbb{R}} |\tilde{f}\tilde{g}| = \int_{\mathbb{R}} \frac{|fg|}{\|f\|_p \, \|g\|_q}$$

and

$$\int_{\mathbb{R}} \frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q} = \frac{1}{p} \frac{\int_{\mathbb{R}} |f|^p}{\|f\|_p^p} + \frac{1}{q} \frac{\int_{\mathbb{R}} |g|^q}{\|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1,$$

and thus

$$\int_{\mathbb{R}} |fg| = \|fg\|_q \leq \|f\|_p \, \|g\|_q$$

as required. ■

Remark 2.15: This inequality also holds for $p = 1, q = \infty$ (assignment question).

↪**Lemma 2.2**: For all $a, b \geq 0$, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

PROOF. ■

↪**Theorem 2.14** (Minkowski's Inequality): Let $1 \leq p < \infty$ and $f, g \in L^p(\mathbb{R})$. Then, $f + g \in L^p(\mathbb{R})$, and in particular

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

In particular, then, $\|\cdot\|_p$ satisfies the triangle inequality and is indeed a norm on $L^p(\mathbb{R})$.

PROOF. We have $|f + g|^p \leq 2^p(|f|^p + |g|^p)$ hence $f + g \in L^p(\mathbb{R})$ since $|f|^p, |g|^p \in L^1(\mathbb{R})$. Further

$$\begin{aligned}\int_{\mathbb{R}} |f+g|^p = \int_{\mathbb{R}} |f+g|\,|f+g|^{p-1} &\leq \int_{\mathbb{R}} |f|\,|f+g|^{p-1} + \int_{\mathbb{R}} |g|\,|f+g|^{p-1} \\ \text{(Hölder's)} \qquad &\leq \left(\int_{\mathbb{R}} |f|^p\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |f+g|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\int_{\mathbb{R}} |g|^p\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |f+g|^{(p-1)q}\right)^{\frac{1}{q}} \\ &\leq \left(\|f\|_p + \|g\|_p\right)\left(\int_{\mathbb{R}} |f+g|^p\right)^{\frac{1}{q}} \\ &\Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p\end{aligned}$$

■

Remark 2.16: Minkowski's also holds for $p = \infty$.

↪**Lemma 2.3**: Let $1 \leq p < \infty$. If $\{g_k\} \in L^p(\mathbb{R})$ such that $\sum_{k=1}^{\infty} \|g_k\|_p < \infty$, then $\exists\, G \in L^p(\mathbb{R})$ such that $G_m := \sum_{k=1}^{m} g_k \to G$ as $m \to \infty$ a.e. as well as in $L^p(\mathbb{R})$.

Proof. Put $\widetilde{G_m} := \sum_{k=1}^{m} |g_k|$ and $\tilde{G} := \sum_{k=1}^{\infty} |g_k|$. Then, $\widetilde{G_m} \uparrow$ with $\lim_{m\to\infty} \widetilde{G_m} = \tilde{G}$. By MON,

$$\int_{\mathbb{R}} \tilde{G}^p = \lim_{m\to\infty} \int_{\mathbb{R}} \widetilde{G_m}^p = \lim_{m\to\infty} \|\widetilde{G_m}\|_p^p \leq \lim_{m\to\infty} \left(\sum_{k=1}^{m} \|g_k\|_p\right)^p$$

where the final inequality is by Minkowski's. Then,

$$\leq \left(\lim_{m\to\infty} \sum_{k=1}^{m} \|g_k\|_p\right)^p = \left(\sum_{k=1}^{\infty} \|g_k\|_p\right)^p < \infty, \text{by assumption}$$

Hence, $\tilde{G} \in L^p(\mathbb{R})$ and $\|\tilde{G}\|_p \leq \sum_{k=1}^{\infty} \|g_k\|_p$ and thus $\tilde{G}$ finite-valued a.e. and hence $\sum_{k=1}^{\infty} g_k$ absolutely convergent a.e.. Set $G = \lim_{m\to\infty} G_m = \sum_{k=1}^{\infty} g_k$ a.e.. Moreover, we know

$$|G| = \left|\sum_{k=1}^{\infty} g_k\right| \leq \sum_{k=1}^{\infty} |g_k| = \tilde{G} \Rightarrow G \in L^p(\mathbb{R})$$

and

$$|G - G_m| \leq \sum_{k=m+1}^{\infty} |g_k|.$$

Fix $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} \|g_k\|_p < \infty$, exists some $M \geq 1$ such that $\sum_{k=M+1}^{\infty} \|g_k\|_p < \varepsilon$. Then,

$$\begin{aligned}\int_{\mathbb{R}} |G - G_M|^p \le \int_{\mathbb{R}} \left(\sum_{k=M+1}^{\infty} |g_k| \right)^p &= \lim_{L \to \infty} \int_{\mathbb{R}} \left(\sum_{k=M+1}^{L} |g_k| \right)^p \\ \text{(Minkowski)} \qquad &\le \lim_{L \to \infty} \left(\sum_{k=M+1}^{L} \|g_k\|_p \right)^p \\ &= \left(\sum_{k=M+1}^{\infty} \|g_k\|_p \right)^p \le \varepsilon\end{aligned}$$

hence $G_m \to G$ in $L^p(\mathbb{R})$. ■

↪**Theorem 2.15**: Let $1 \le p < \infty$. Then $L^p(\mathbb{R})$ is a complete normed space under the p-norm.

Proof. Let $f_n \in L^p(\mathbb{R})$ be a Cauchy sequence under $\|\cdot\|_p$. We can choose a subsequence $\{n_k\}$ such that for every $k \ge 1$, $\|f_{n_{k+1}} - f_{n_k}\|_p \le 2^{-k}$. Set $g_k := f_{n_{k+1}} - f_{n_k}$. By the lemma, if $G_m := \sum_{k=1}^m g_k$, there exists some $G \in L^p(\mathbb{R})$ such that $G_m \to G$ a.e. and in $L^p(\mathbb{R})$. In fact, we have

$$G_m = \sum_{k=1}^{m} g_k = \sum_{k=1}^{m} \left(f_{n_{k+1}} - f_{n_k}\right) = f_{n_{m+1}} - f_{n_1},$$

hence

$$G = \lim_{m \to \infty} G_m = \left(\lim_{m \to \infty} f_{n_{m+1}}\right) - f_{n_1}.$$

Let $f := G + f_{n_1}$. Then, $f = \lim_{m \to \infty} f_{n_m}$ a.e. and since $G_m \to G$ in L^p, we have that $f_{n_m} \to f$ in L^p as $m \to \infty$. It remains to show convergence in L^p along the whole subsequence.

Fix $\varepsilon > 0$. Let $N \ge 1$ such that $\sup_{k,\ell \ge N} \|f_k - f_\ell\|_p < \varepsilon$ and m sufficiently large such that $n_m > N$ and $\|f_{n_m} - f\|_p \le \varepsilon$. Then,

$$\|f_n - f\|_p \le \underbrace{\|f_n - f_{n_m}\|_p}_{<\varepsilon} + \underbrace{\|f_{n_m} - f\|_p}_{<\varepsilon} < 2\varepsilon,$$

completing the proof. ■

Remark 2.17: L^∞ also complete.

↪**Lemma 2.4**: Bounded and compactly supported functions are dense in $L^p(\mathbb{R})$.

Proof. Given $f \in L^p(\mathbb{R})$, set

$$f_n(x) = \mathbb{1}_{[-n,n]}(x) \cdot f(x) \cdot \mathbb{1}_{\{|f| \le n\}}(x)$$

which are bounded and compactly supported on $[-n, n]$. We claim $f_n \to f$ in $L^p(\mathbb{R})$. We have $\int_{\mathbb{R}} |f_n - f|^p$ nonzero only if $x \notin [-n, n]$ or $|f(x) > n|$. Hence

$$\int_{\mathbb{R}} |f_n - f|^p \le \int_{\mathbb{R} \setminus [-n,n]} |f|^p + \int_{\{|f| > n\}} |f|^p \to 0 \text{ as } n \to \infty.$$

■