

Course Outline:

Based on Lectures from Winter, 2024 by Prof. Anush Tserunyan.

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1 Introduction

Remark 1.1. This course is about vector spaces and linear transformations between them; a vector space involves multiplication by scalars, where the scalars come from some field. We recall first examples of fields, then vector spaces, as a motivation, before presenting a formal definition.

1.1 Vector Spaces

Remark 1.2. Much of this is recall from *Algebra 1*.

⊗ Example 1.1: Examples of Fields

1. \mathbb{Q} ; the field of rational numbers.
2. \mathbb{R} ; the field of real numbers; $\mathbb{Q} \subseteq \mathbb{R}$.
3. \mathbb{C} ; the field of complex numbers; $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.
4. $\mathbb{F}_p \equiv \mathbb{Z}/p\mathbb{Z} \equiv \{0, 1, \dots, p-1\}$; the (unique) field of p elements, where p prime.^a
 - (a) $p = 2$; $\mathbb{F}_2 \equiv \{0, 1\}$.
 - (b) $p = 3$; $\mathbb{F}_3 \equiv \{0, 1, 2\}$.
 - (c) \dots

^awhere $a +_p b := \text{remainder of } \frac{a+b}{p}$, $a \cdot_p b := \text{remainder of } \frac{a \cdot b}{p}$.

Remark 1.3. Throughout the course, we will denote an abstract field as \mathbb{F} .

⊗ Example 1.2: Examples of Vector Spaces

1. $\mathbb{R}^3 := \{(x, y, z) : x, y, z \in \mathbb{R}\}$. We can add elements in \mathbb{R}^3 , and multiply them by real scalars.
2. $\mathbb{F}^n := \underbrace{\mathbb{F} \times \mathbb{F} \times \dots \times \mathbb{F}}_{n \text{ times}} := \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{F}\}$, where $n \in \mathbb{N}^+$; this is a generalization of the previous example, where we took $n = 3$, $\mathbb{F} = \mathbb{R}$. Operations follow identically; addition:

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) := (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and, taking a scalar $\lambda \in \mathbb{F}$, multiplication:

$$\lambda \cdot (a_1, a_2, \dots, a_n) := (\lambda \cdot a_1, \lambda \cdot a_2, \dots, \lambda \cdot a_n).$$

We refer to these elements (a_1, \dots, a_n) as *vectors* in \mathbb{F}^n ; the vector for which

$a_i = 0 \forall i$ is the 0 *vector*, and is the additive identity, making \mathbb{F}^n an abelian group under addition, that admits multiplication by scalars from \mathbb{F} .

3. $C(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{R} : f \text{ continuous}\}$. Here, we have the constant zero function as our additive identity ($x \mapsto 0 \forall x$), and addition/scalar multiplication of two continuous real functions are continuous.

4. $\mathbb{F}[t] := \{a_0 + a_1t + a_2t^2 + \cdots + a_nt^n : a_i \in \mathbb{F} \forall i, n \in \mathbb{N}\}$, ie, the set of all polynomials in t with coefficients from \mathbb{F} . Here, we can add two polynomials;

$$(a_0 + a_1t + \cdots + a_nt^n) + (b_0 + b_1t + \cdots + b_mt^m) := \sum_{i=0}^{\max\{n,m\}} (a_i + b_i)t^i,$$

(where we “take” undefined a_i/b_i ’s as 0; that is, if $m > n$, then $a_{m-n}, a_{m-n+1}, \dots, a_m$ are taken to be 0). Scalar multiplication is defined

$$\lambda \cdot (a_0 + a_1t + a_2t^2 + \cdots + a_nt^n) := \lambda a_0 + \lambda a_1t + \lambda a_2t^2 + \cdots + \lambda a_nt^n.$$

Here, the zero polynomial is simply 0 (that is, $a_i = 0 \forall i$).

↪ **Definition 1.1: Vector Space**

A *vector space* V over a field \mathbb{F} is an *abelian group* with an operation denoted $+$ (or $+_V$) and identity element² denoted 0_V , equipped with *scalar multiplication* for each scalar $\lambda \in \mathbb{F}$ satisfying the following axioms:

1. $1 \cdot v = v$ for $1 \in \mathbb{F}, \forall v \in V$.
2. $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta)v, \forall \alpha, \beta \in \mathbb{F}, v \in V$.
3. $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v, \forall \alpha, \beta \in \mathbb{F}, v \in V$.
4. $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v, \forall \alpha \in \mathbb{F}, u, v \in V$.

We refer to elements $v \in V$ as *vectors*.

↪ **Proposition 1.1**

For a vector space V over a field \mathbb{F} , the following holds:

1. $0 \cdot v = 0_V, \forall v \in V$ (where $0 := 0_{\mathbb{F}}$)
2. $-1 \cdot v = -v, \forall v \in V$ (where $1 := 1_{\mathbb{F}}$)³

¹Where we take $0 \in \mathbb{N}$, for sake of consistency. Moreover, by convention, we define \mathbb{F}^0 (that is, when $n = 0$) to be $\{0\}$; the trivial vector space.

²The “zero vector”.

$$3. \alpha \cdot 0_V = 0_V, \forall \alpha \in \mathbb{F}$$

³NB: “additive inverse”

Proof. 1. $0 \cdot v = (0 + 0) \cdot v = 0 \cdot v + 0 \cdot v \implies 0 \cdot v = 0_V$ (by “cancelling” one of the $0 \cdot v$ terms on each side).

$$2. v + (-1 \cdot v) = (1 \cdot v + (-1) \cdot v) = (1 - 1) \cdot v = 0 \cdot v = 0_V \implies (-1 \cdot v) = -v.$$

$$3. \alpha \cdot 0_V = \alpha \cdot (0_V + 0_V) = \alpha \cdot 0_V + \alpha \cdot 0_V \implies \alpha \cdot 0_V = 0_V \text{ (by, again, cancelling a term on each side).}$$

■

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1.2 Creating Spaces from Other Spaces

↪ [Definition 1.2: Product/Direct Sum of Vector Spaces](#)

For vector spaces U, V over the same field \mathbb{F} , we define their *product* (or *direct sum*) as the set

$$U \times V = \{(u, v) : u \in U, v \in V\},$$

with the operations:

$$(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$$

$$\lambda \cdot (u, v) := (\lambda \cdot u, \lambda \cdot v)$$

⊗ Example 1.3: \mathbb{F}

$\mathbb{F}^2 = \mathbb{F} \times \mathbb{F}$, where \mathbb{F} is considered as the vector space over \mathbb{F} (itself).

↪ **Definition 1.3: Subspace**

For a vector space V over a field \mathbb{F} , a *subspace* of V is a subset $W \subseteq V$ s.t.

1. $0_V \in W$ ⁴
2. $u + v \in W \forall u, v \in W$ (closed under addition)
3. $\alpha \cdot u \in W \forall u \in W, \alpha \in \mathbb{F}$ ⁵

Then, W is a vector space in its own right.

⊗ **Example 1.4: Examples of Subspaces**

1. Let $V := \mathbb{F}^n$.

- $W := \{(x_1, x_2, \dots, x_n) \in \mathbb{F}^n : x_1 = 0\} = \{(0, x_2, x_3, \dots, x_n) : x_i \in \mathbb{F}\}$.
- $W := \{(x_1, x_2, \dots, x_n) \in \mathbb{F}^n : x_1 + 2 \cdot x_2 = 0\}$

Proof. Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in W$. Then, $x + y = (x_1 + y_1, \dots, x_n + y_n)$, and $x_1 + y_1 + 2 \cdot (x_2 + y_2) = x_1 + 2 \cdot x_2 + y_1 + 2 \cdot y_2 = 0 + 0 = 0 \implies x + y \in W$. Similar logic follows for axioms 2., 3. ■

- (More generally)

$$W := \{(x_1, \dots, x_n) \in \mathbb{F}^n : \begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + \dots + a_{2n}x_n = 0 \\ \vdots \\ a_{k1}x_1 + \dots + a_{kn}x_n = 0 \end{array} \},$$

that is, a linear combination of homogenous “conditions” on each term.

- $W^* := \{(x_1, \dots, x_n) : x_1 + x_2 = 1\}$ is *not* a subspace; it is not closed under addition, nor under scalar multiplication.

2. Let $\mathbb{F}[t]_n := \{a_0 + a_1t + \dots + a_nt^n : a_i \in \mathbb{F}\}$. Then, $\mathbb{F}[t]_n$ is a subspace of $\mathbb{F}[t]$, the more general polynomial space. However, the set of all polynomials of degree *exactly* n (all axioms fail, in fact) is not a subspace of $\mathbb{F}[t]_n$.

- $W := \{p(t) \in \mathbb{F}[t]_n : p(1) = 0\}$.
- $W := \{p(t) \in \mathbb{F}[t]_n : p''(t) + p'(t) + 2p(t) = 0\}$.

3. Let $V := C(\mathbb{R})$ be the space of continuous function $\mathbb{R} \rightarrow \mathbb{R}$.

⁴This is equivalent to requiring that $W \neq \emptyset$; stated this way, axiom 3. would necessitate that $0 \cdot w = 0_V \in W$.

⁵Note that these axioms are equivalent to saying that W is a subgroup of V with respect to vector addition; 2. ensures closed under addition, and 3. ensures the existence of additive inverses (as per $-1 \cdot v = -v$).

- $W := \{f \in C(\mathbb{R}) : f(\pi) + 7f(\sqrt{2}) = 0\}.$
- $W := C^1(\mathbb{R}) := \text{everywhere differentiable functions}.$
- $W := \{f \in C(\mathbb{R}) : \int_0^1 f \, dx = 0\}.$

→ **Proposition 1.2**

Let W_1, W_2 be subspaces of a vector space V over \mathbb{F} . Then, define the following:

1. $W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$
2. $W_1 \cap W_2 := \{w \in V : w \in W_1 \wedge w \in W_2\}$

These are both subspaces of V .

- Proof.
1. (a) $0_V \in W_1$ and $0_V \in W_2 \implies 0_V = 0_V + 0_V \in W_1 + W_2.$
 (b) $(u_1 + u_2) + (v_1 + v_2) = (u_1 + v_1) + (u_2 + v_2) \in W_1 + W_2.$
 (c) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v \in W_1 + W_2$
 2. (a) $0_V \in W_1$ and $0_V \in W_2 \implies 0_V = 0_V + 0_V \in W_1 \cap W_2.$
 (b) $u, v \in W_1 \cap W_2 \implies u + v \in W_1 \wedge u + v \in W_2 \implies u + v \in W_1 \cap W_2.$
 (c) $\alpha \cdot u \in W_1 \wedge \alpha \cdot u \in W_2 \implies \alpha \cdot u \in W_1 \cap W_2.$

■

1.3 Linear Combinations and Space

→ **Definition 1.4: Linear Combination**

Let V be a vector space over a field \mathbb{F} . For finitely many vectors v_1, v_2, \dots, v_n , their *linear combination* is a sum of the form

$$\sum_{i=1}^n a_i v_i = a_1 \cdot v_1 + \dots + a_n \cdot v_n,$$

where $a_i \in \mathbb{F} \forall i$.

A linear combination is called *trivial* if $a_i = 0 \forall i$, that is, all coefficients are 0.

If $n = 0$ (ie, we are “summing up” 0 vectors), we define the sum as the zero vector;
 $\sum_{i=1}^0 a_i v_i := 0_V.$

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↪ **Definition 1.5: A More General Definition of Linear Combination**

For a (possible infinite) set S of vectors from V , a *linear combination* of vectors in S is a linear combination of $a_1v_1 + \cdots a_nv_n$ for some finite subset $\{v_1, \dots, v_n\} \subseteq S$.⁶

⁶That is, we do not allow infinite sums.

↪ **Definition 1.6: Span**

For a subset $S \subseteq V$, we define its *span* as

$$\text{Span}(S) := \text{set of all linear combinations of } S := \{a_1v_1 + \cdots a_nv_n : a_i \in \mathbb{F}, v_i \in S\}.$$

By convention, we set $\text{Span}(\emptyset) = \{0_V\}$.

⊗ **Example 1.5**

Let $S := \{(1, 0, -1), (0, 1, -1), (1, 1, -2)\} \subseteq \mathbb{R}^3$. Then,

$$0_{\mathbb{R}^3} = (0, 0, 0) = 1 \cdot (1, 0, -1) + 1 \cdot (0, 1, -1) + -1 \cdot (1, 1, -2).$$

We claim, moreover, that $\text{Span}(S) = U := \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ (a plane through the origin).

Proof. Note that $S \subseteq U$, hence $S \subseteq \text{Span } S \subseteq U$. OTOH, if $(x, y, z) \in U$, we have $z = -x - y$, and so

$$(x, y, z) = (x, y, -x - y) = x \cdot (1, 0, -1) + y \cdot (0, 1, -1) \in \text{Span}(S)$$

hence $U \subseteq \text{Span}(S)$ and thus $\text{Span}(S) = U$. ■

Remark 1.4. We implicitly used the following claim in the proof above; we prove it more generally.

↪ **Proposition 1.3**

Let V be a vector space over \mathbb{F} and let $S \subseteq V$. Then, $\text{Span}(S)$ is always a subspace. Moreover, it is the smallest (minimal) subspace containing S (that is, for any subspace $U \supseteq S$, we have that $U \supseteq \text{Span } S$).

Proof. Because adding/scalar multiplying linear combinations of elements of S again results in a linear combination of elements of S , and $0_V \in \text{Span}(S)$ by definition, we have that $\text{Span}(S)$ is indeed a subspace.

If $U \supset S$ is a subspace of V containing S , then by definition U is closed under addition, that is, taking linear combinations of its elements (in particular, of elements of S); hence, $U \supseteq \text{Span}(S)$. ■

↪ **Lemma 1.1**

For $S \subseteq V$ and $v \in V$, $v \in \text{Span}(S) \iff \text{Span}(S \cup \{v\}) = \text{Span}(S)$.

Proof. (\implies) Let $v \in \text{Span}(S) \implies v = a_1v_1 + \cdots + a_nv_n, a_i \in \mathbb{F}, v_i \in V$. Then, for any linear combination

$$b_1u_1 + \cdots + b_mu_m + b \cdot v = b_1u_1 + \cdots + b_mu_m + b(a_1v_1 + \cdots + a_nv_n)$$

is a linear combination of vectors in $S \cup \{v\}$ (first equality) or equivalently, a combination of vectors in S (second equality) and thus $\text{Span}(S \cup \{v\}) \subseteq \text{Span } S$. The reverse inclusion follows trivially.

(\impliedby) $\text{Span}(S \cup \{v\}) = \text{Span } S \implies v \in \text{Span}(S)$. ■

⊗ **Example 1.6**

(From the above example) We have

$$\text{Span}(\{(1, 0, -1), (0, 1, -1)\} \cup \{(1, 1, -2)\}) = \text{Span}(\{(1, 0, -1), (0, 1, -1)\}),$$

since $(1, 1, -2) \in \text{Span}(\{(1, 0, -1), (0, 1, -1)\})$ (it was redundant, as it could be generated by the other two vectors).

↪ **Definition 1.7: Spanning Set**

Let V be a vector space over a field \mathbb{F} . We call $S \subseteq V$ a *spanning set* for V if $\text{Span}(S) = V$.

We call such a spanning set *minimal* if no proper subset of S is a spanning set ($\nexists v \in S$ s.t. $S \setminus \{v\}$ spanning).

Remark 1.5. Note that any $S \subseteq V$ is a spanning for $\text{Span}(S)$. But, S may not be minimal; indeed, consider the previous example. We were able to remove a vector from S while having the same span.

⊗ **Example 1.7**

For \mathbb{F}^n as a vector space over \mathbb{F} , the *standard spanning set*

$$\text{St}_n := \{\underbrace{(1, \dots, 0)}_{:=e_1}, \underbrace{(0, 1, 0, \dots, 0)}_{:=e_2}, \dots, \underbrace{(0, \dots, 1)}_{e_n}\}.$$

Given any $x := (x_1, \dots, x_n) \in \mathbb{F}^n$, we can write

$$x = x_1 \cdot e_1 + \cdots + x_n \cdot e_n.$$

This is clearly minimal; removing any e_i would then result in a 0 in the i th “coordinate”

of a vector, hence $\text{St} \setminus \{e_i\}$ would span only vectors whose i th coordinate is 0.

↪ **Definition 1.8: Linear Dependence**

Let V be a vector space over a field \mathbb{F} . A set $S \subseteq V$ is said to be *linearly dependent* if there is a nontrivial linear combination of vectors in S that is equal to 0_V .

Conversely, S is called *linearly independent* if there is no nontrivial linear combination of vectors in S that is equal to 0_V ; all linear combinations of vectors in S that equal 0_V are trivial.

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⊗ **Example 1.8**

1. The empty set \emptyset is linearly independent; there are no non-trivial linear combinations that equal 0_V (there are no linear combinations at all).
2. For $v \in V$, the set $\{v\}$ is linearly dependent iff $v = 0_V$.
3. $S := \{(1, 0, -1), (0, 1, -1), (1, 1, -2)\} := \{v_1, v_2, v_3\}$; S is linearly dependent ($v_1 + v_2 - v_3 = (0, 0, 0)$).
4. $V := \mathbb{F}^3$; $S := \{(1, 0, -1), (0, 1, -1), (0, 0, 1)\} = \{v_1, v_2, v_3\}$ is linearly independent.

Proof. Suppose

$$\begin{aligned} a_1 v_1 + a_2 v_2 + a_3 v_3 &= 0_V \\ \implies a_1 = 0 \wedge a_2 = 0 \wedge -a_1 - a_2 + a_3 &= 0 \implies a_3 = 0 \\ \implies a_1 = a_2 = a_3 &= 0 \end{aligned}$$

Hence only a trivial linear combination is possible. ■

5. St_n is linearly independent.

Proof.

$$\sum_{i=1}^n a_i e_i = 0_{\mathbb{F}^n} \implies a_i = 0 \forall i$$

■

→ **Lemma 1.2**

Let V be a vector space over a field \mathbb{F} , and $S \subseteq V$ (possibly infinite).

1. S is linearly dependent \iff there is a finite subset $S_0 \subseteq S$ that is linearly dependent.
2. S is linearly independent \iff all finite subsets of S are linearly independent.

Proof. 2. follows from the negation of 1.

(\Leftarrow) Trivial.

(\Rightarrow) Suppose S linearly dependent. Then, $0_V =$ some nontrivial linear combination of vectors v_1, \dots, v_n in S . Let $S_0 = \{v_1, \dots, v_n\}$, then, S_0 is linearly dependent itself. ■

1.4 Linear Dependence and Span

→ **Proposition 1.4**

Let V be a vector space over a field \mathbb{F} and $S \subseteq V$.

1. S linearly dependent $\iff \exists v \in \text{Span}(S \setminus \{v\})$.
2. S linearly independent \iff there is no $v \in \text{Span}(S \setminus \{v\})$.

Proof. 2. follows from the negation of 1.

(\Rightarrow) Suppose S linearly dependent. Then, $0_V = \sum_{i=1}^n a_i v_i$ for some nontrivial linear combination of distinct vectors S . At least one of $a_i \neq 0$; we can assume wlog (reindexing) $a_1 \neq 0$. Then,

$$a_1 v_1 = - \sum_{i=2}^n a_i v_i \implies v_1 = (-a_1^{-1}) \sum_{i=2}^n a_i v_i = \sum_{i=2}^n (-a_1^{-1} a_i) v_i,$$

hence, $v_1 \in \text{Span}(\{v_2, \dots, v_n\}) \subseteq \text{Span}(S \setminus \{v\})$

(\Leftarrow) Suppose $v \in \text{Span}(S \setminus \{v\})$, then $v = a_1 v_1 + \dots + a_n v_n$, with $v_1, \dots, v_n \in S \setminus \{v\}$, thus

$$0_V = a_1 v_1 + \dots + a_n v_n - v,$$

which is not a trivial combination (-1 on the v ; v cannot “merge” with the other vectors), hence S is linearly dependent. ■

→ **Corollary 1.1**

$S \subseteq V$ is linearly independent $\iff S$ a minimal spanning set of $\text{Span } S$.

Proof. Follows from proposition 1.4, 2. ■

↪ **Definition 1.9: Maximally Independent**

Let V be a vector space over a field \mathbb{F} . A set $S \subseteq V$ is called *maximally independent* if S is linearly independent and $\nexists v \in V \setminus S$ s.t. $S \cup \{v\}$ is still linearly independent.

In other words, there is no proper supset $\tilde{S} \supsetneq S$ that is still independent.

↪ **Lemma 1.3**

If $S \subseteq V$ maximally independent, then S is spanning for V .

Proof. Let $S \subseteq V$ be maximally independent. Let $v \in V$; supposing $v \notin S$ (in the case that $v \in S$, then $v \in \text{Span}(S)$ trivially). By maximality, $S \cup \{v\}$ is linearly dependent, hence there exists a nontrivial linear combination that equals 0_V . Since S independent, this combination must include v , with a nonzero coefficient. We can write

$$\begin{aligned} av + \sum_{i=1}^n a_i v_i &= 0_V \quad a \neq 0, v_i \in S \\ \implies v &= \sum_{i=1}^n (-a^{-1}a_i)v_i \in \text{Span } S. \end{aligned}$$

■

↪ **Theorem 1.1**

Let V be a vector space over a field \mathbb{F} and let $S \subseteq V$. TFAE:

1. S is a minimal spanning set;
2. S is linearly independent and spanning;
3. S is a maximally linearly independent set;
4. Every vector in V is equal to *unique* linear combination of vectors in S .

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Proof. (1. \implies 2.) Suppose S is spanning for V and is minimal. Then, by corollary 1.1, we have that S is linearly independent, and is thus both linearly independent and spanning.

(2. \implies 3.) Suppose S is linearly independent and spanning. Let $v \in V \setminus S$; S is spanning, hence $v \in \text{Span } S$, that is, there exists a linear combination of vectors in S that is equal to v :

$$v = a_1 v_1 + \cdots + a_n v_n, a_i \in \mathbb{F}, v_i \in S.$$

Thus, $0_V = a_1 v_1 + \cdots + a_n v_n - v$, thus $S \cup \{v\}$ is linearly dependent, and so S is maximally linearly independent.

(3. \implies 1.) Suppose S is maximally linearly independent. By lemma 1.3, S is spanning, and since S is linearly independent, by corollary 1.1, S is minimally spanning for $\text{Span } S$.

(2. \implies 4.) Suppose S is linearly independent and spans V , and let $v \in V$. We have that $v \in \text{Span } S$ and hence is equal to a linear combination of vectors in S . This gives existence; we now need to prove uniqueness.

Suppose there exist two linear combinations that equal v ,

$$v = a_1v_1 + \cdots + a_nv_n = b_1u_1 + \cdots + b_mu_m,$$

$a_i, b_j \in \mathbb{F}$, $v_i, u_j \in S$. With appropriate reindexing/relabelling and allowing certain scalars to equal 0, we can assume that the combinations use the same vectors (with potentially different coefficients), that is,

$$v = a_1w_1 + \cdots + a_kw_k = b_1w_1 + \cdots + b_kw_k.$$

This implies, then,

$$(a_1 - b_1)w_1 + \cdots + (a_k - b_k)w_k = 0_V,$$

and by the assumed linear independent of S , each coefficient $(a_i - b_i) = 0 \forall i \implies a_i = b_i \forall i$, hence, these are indeed the same representations, and thus this representation is unique.

(4. \implies 2.) Suppose every vector in V admits a unique linear combination of vectors in S . Clearly, then, S is spanning. It remains to show S is linearly independent. Suppose

$$0_V = a_1v_1 + \cdots + a_nv_n$$

for $v_i \in S$. But we have that every vector has a unique representation, and we know that $a_i = 0 \forall i$ is a (valid) linear combination that gives 0_V ; hence, this must be the unique combination, $a_i = 0 \forall i$, and the linear combination above is trivial. Hence, S is linearly independent and spanning. ■

\hookrightarrow **Definition 1.10: Basis**

If any (hence all) of the above statements hold, we call S a *basis* for V .

In the words of 4., we call the unique linear combination of vectors in S that is equal to v the *unique representation of v in S* . Its coefficients are called the *Fourier coefficients of v in S* .

\circledast **Example 1.9**

1. $\text{St}_n = \{e_i : 1 \leq i \leq n\}$ is a basis for \mathbb{F}^n .

2. In \mathbb{F}^3 , the set

$$\{(1, 0, -1), (0, 1, -1), (0, 0, 1)\}$$

is a basis; it is linearly independent and spanning.

3. For $\mathbb{F}[t]_n$, the standard basis is

$$\{1, t, t^2, \dots, t^n\}.$$

4. For $\mathbb{F}[t]$, the standard basis is

$$S := \{1, t, t^2, \dots\} = \{t^n : n \in \mathbb{N}\}.$$

5. Let $\mathbb{F}[[t]]$ denote the space of all formal power series $\sum_{n \in \mathbb{N}} a_n t^n$; polynomials are an example, but with only finite nonzero coefficients. Note that, then, the set S defined above is not a basis for this “extended” set. We *can* in fact find a basis for this set; we need more tools first.

↪ **Theorem 1.2**

Every vector space has a basis.

Remark 1.6. *This theorem relies on assuming the Axiom of Choice.*

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