

# MATH455 - Analysis 4

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## §1 ABSTRACT METRIC AND TOPOLOGICAL SPACES

### §1.1 Review of Metric Spaces

Throughout fix  $X$  a nonempty set.

↪ **Definition 1.1** (Metric):  $\rho : X \times X \rightarrow \mathbb{R}$  is called a *metric*, and thus  $(X, \rho)$  a *metric space*, if for all  $x, y, z \in X$ ,

- $\rho(x, y) \geq 0$ ,
- $\rho(x, y) = 0 \Leftrightarrow x = y$ ,
- $\rho(x, y) = \rho(y, x)$ , and
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ .

↪ **Definition 1.2** (Norm): Let  $X$  a linear space. A function  $\| \cdot \| : X \rightarrow [0, \infty)$  is called a *norm* if for all  $u, v \in X$  and  $\alpha \in \mathbb{R}$ ,

- $\|u\| = 0 \Leftrightarrow u = 0$ ,
- $\|u + v\| \leq \|u\| + \|v\|$ , and
- $\|\alpha u\| = |\alpha| \|u\|$ .

**Remark 1.1:** A norm induces a metric by  $\rho(x, y) := \|x - y\|$ .

↪ **Definition 1.3:** Given two metrics  $\rho, \sigma$  on  $X$ , we say they are *equivalent* if  $\exists C > 0$  such that  $\frac{1}{C}\sigma(x, y) \leq \rho(x, y) \leq C\sigma(x, y)$  for every  $x, y \in X$ . A similar definition follows for equivalence of norms.

Given a metric space  $(X, \rho)$ , then, we have the notion of

- open balls  $B(x, r) = \{y \in X : \rho(x, y) < r\}$ ,
- open sets (subsets of  $X$  with the property that for every  $x \in X$ , there is a constant  $r > 0$  such that  $B(x, r) \subseteq X$ ), closed sets, closures, and
- *convergence*.

↪ **Definition 1.4** (Convergence):  $\{x_n\} \subseteq X$  converges to  $x \in X$  if  $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$ .

We have several (equivalent) notions, then, of continuity; via sequences,  $\varepsilon - \delta$  definition, and by pullbacks (inverse images of open sets are open).

↪ **Definition 1.5** (Uniform Continuity):  $f : (X, \rho) \rightarrow (Y, \sigma)$  uniformly continuous if  $f$  has a “modulus of continuity”, i.e. there is a continuous function  $\omega : [0, \infty) \rightarrow [0, \infty)$  such that

$$\sigma(f(x_1), f(x_2)) \leq \omega(\rho(x_1, x_2))$$

for every  $x_1, x_2 \in X$ .

**Remark 1.2:** For instance, we say  $f$  Lipschitz continuous if there is a constant  $C > 0$  such that  $\omega(\cdot) = C(\cdot)$ . Let  $\alpha \in (0, 1)$ . We say  $f$   $\alpha$ -Holder continuous if  $\omega(\cdot) = C(\cdot)^\alpha$  for some constant  $C$ .

↪ **Definition 1.6** (Completeness): We say  $(X, \rho)$  *complete* if every cauchy sequence in  $(X, \rho)$  converges to a point in  $X$ .

**Remark 1.3:** If  $(X, \rho)$  complete and  $E \subseteq X$ , then  $(E, \rho)$  is complete iff  $E$  closed in  $X$ .

## §1.2 Compactness, Separability

↪ **Definition 1.7** (Open Cover, Compactness):  $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$ , where  $X_\lambda$  open in  $X$  and  $\Lambda$  an arbitrary index set, an *open cover* of  $X$  if for every  $x \in X$ ,  $\exists \lambda \in \Lambda$  such that  $x \in X_\lambda$ .

$X$  is *compact* if every open cover of  $X$  admits a compact subcover. We say  $E \subseteq X$  compact if  $(E, \rho)$  compact.

↪ **Definition 1.8** (Totally Bounded,  $\varepsilon$ -nets):  $(X, \rho)$  *totally bounded* if  $\forall \varepsilon > 0$ , there is a finite cover of  $X$  of balls of radius  $\varepsilon$ . If  $E \subseteq X$ , an  $\varepsilon$ -*net* of  $E$  is a collection  $\{B(x_i, \varepsilon)\}_{i=1}^N$  such that  $E \subseteq \bigcup_{i=1}^N B(x_i, \varepsilon)$  and  $x_i \in X$  (note that  $x_i$  need not be in  $E$ ).

↪ **Definition 1.9** (Sequentially Compact):  $(X, \rho)$  *sequentially compact* if every sequence in  $X$  has a convergent subsequence whose limit is in  $X$ .

↪ **Definition 1.10** (Relatively / Pre- Compact):  $E \subseteq X$  *relatively compact* if  $\overline{E}$  compact.

↪ **Theorem 1.1:** TFAE:

1.  $X$  complete and totally bounded;
2.  $X$  compact;
3.  $X$  sequentially compact.

**Remark 1.4:**  $E \subseteq X$  relatively compact if every sequence in  $E$  has a convergent subsequence.

Let  $f : (X, \rho) \rightarrow (Y, \sigma)$  continuous with  $(X, \rho)$  compact. Then,

- $f(X)$  compact in  $Y$ ;
- if  $Y = \mathbb{R}$ , the max and min of  $f$  over  $X$  are achieved;
- $f$  is uniformly continuous.

Let  $C(X) := \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous}\}$  and  $\|f\|_\infty := \max_{x \in X} |f(x)|$  the sup (max, in this case) norm. Then,

**→ Theorem 1.2:** Let  $(X, \rho)$  compact. Then,  $(C(X), \|\cdot\|_\infty)$  is complete.

PROOF. Let  $\{f_n\} \subseteq C(X)$  Cauchy with respect to  $\|\cdot\|_\infty$ . Then, there exists a subsequence  $\{f_{n_k}\}$  such that for each  $k \geq 1$ ,  $\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq 2^{-k}$  (to construct this subsequence, let  $n_1 \geq 1$  be such that  $\|f_n - f_{n_1}\|_\infty < \frac{1}{2}$  for all  $n \geq n_1$ , which exists since  $\{f_n\}$  Cauchy. Then, for each  $k \geq 1$ , define inductively  $n_{k+1}$  such that  $n_{k+1} > n_k$  and  $\|f_n - f_{n_{k+1}}\|_\infty < \frac{1}{2^{k+1}}$  for each  $n \geq n_{k+1}$ . Then, for any  $k \geq 1$ ,  $\|f_{n_{k+1}} - f_{n_k}\|_\infty < 2^{-k}$ , since  $n_{k+1} > n_k$ ).

Let  $j \in \mathbb{N}$ . Then, for any  $k \geq 1$ ,

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell=k}^{\infty} 2^{-\ell}$$

and hence for each  $x \in X$ , with  $c_k := f_{n_k}(x)$ ,

$$|c_{k+j} - c_k| \leq \sum_{\ell=k}^{\infty} 2^{-\ell}.$$

The RHS is the tail of a converging series, and thus  $|c_{k+j} - c_k| \rightarrow 0$  as  $k \rightarrow \infty$  i.e.  $\{c_k\}$  a Cauchy sequence, in  $\mathbb{R}$ .  $(\mathbb{R}, |\cdot|)$  complete, so  $\lim_{k \rightarrow \infty} c_k =: f(x)$  exists for each  $x \in X$ . So, for each  $x \in X$ , we find

$$|f_{n_k}(x) - f(x)| \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

and since the RHS is independent of  $x$ , we may pass to the sup norm, and find

$$\|f_{n_k} - f\|_\infty \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

with the RHS  $\rightarrow 0$  as  $k \rightarrow \infty$ . Hence,  $f_{n_k} \rightarrow f$  in  $C(X)$  as  $k \rightarrow \infty$ . In other words, we have uniform convergence of  $\{f_{n_k}\}$ . Each  $\{f_{n_k}\}$  continuous, and thus  $f$  also continuous, and thus  $f \in C(X)$ .

It remains to show convergence along the whole sequence. Suppose otherwise. Then, there is some  $\alpha > 0$  and a subsequence  $\{f_{n_j}\} \subseteq \{f_n\}$  such that  $\|f_{n_j} - f\|_\infty >$

$\alpha > 0$  for every  $j \geq 1$ . Then, let  $k$  be sufficiently large such that  $\|f - f_{n_k}\|_\infty \leq \frac{\alpha}{2}$ . Then, for every  $j \geq 1$  and  $k$  sufficiently large,

$$\begin{aligned}\|f_{n_j} - f_{n_k}\|_\infty &\geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty \\ &> \alpha - \frac{\alpha}{2} > 0,\end{aligned}$$

which contradicts the Cauchy-ness of  $\{f_n\}$ , completing the proof. ■

↪ **Definition 1.11** (Density/Separability): A set  $D \subseteq X$  is called *dense* in  $X$  if for every nonempty open subset  $A \subseteq X$ ,  $D \cap A \neq \emptyset$ . We say  $X$  *separable* if there is a countable dense subset of  $X$ .

**Remark 1.5:** If  $A$  dense in  $X$ , then  $\overline{A} = X$ .

↪ **Proposition 1.1:** If  $X$  compact,  $X$  separable.

PROOF. Since  $X$  compact, it is totally bounded. So, for  $n \in \mathbb{N}$ , there is some  $K_n$  and  $\{x_i\} \subseteq X$  such that  $X \subseteq \bigcup_{i=1}^{K_n} B(x_i, \frac{1}{n})$ . Then,  $D = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{K_n} \{x_i\}$  countable and dense in  $X$ . ■

### §1.3 Arzelà-Ascoli

The goal in this section is to find conditions for a sequence of functions  $\{f_n\} \subseteq C(X)$  to be precompact, namely, to have a uniformly convergent subsequence.

↪ **Corollary 1.1:** Any Cauchy sequence converges if it has a convergent subsequence.

PROOF. Let  $\{x_n\}$  be a Cauchy sequence in a metric space  $(X, \rho)$  with convergent subsequence  $\{x_{n_k}\}$  which converges to some  $x \in X$ . Fix  $\varepsilon > 0$ . Let  $N \geq 1$  be such that if  $m, n \geq N$ ,  $\rho(x_n, x_m) < \frac{\varepsilon}{2}$ . Let  $K \geq 1$  be such that if  $k \geq K$ ,  $\rho(x_{n_k}, x) < \frac{\varepsilon}{2}$ . Let  $n, n_k \geq \max\{N, K\}$ , then

$$\rho(x, x_n) \leq \rho(x, x_{n_k}) + \rho(x_{n_k}, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

↪ **Definition 1.12** (Equicontinuous): A family  $\mathcal{F} \subseteq C(X)$  is called *equicontinuous* at  $x \in X$  if  $\forall \varepsilon > 0$  there exists a  $\delta = \delta(x, \varepsilon) > 0$  such that if  $\rho(x, x') < \delta$  then  $|f(x) - f(x')| < \varepsilon$  for every  $f \in \mathcal{F}$ .

**Remark 1.6:**  $\mathcal{F}$  equicontinuous at  $x$  iff every  $f \in \mathcal{F}$  share the same modulus of continuity.

↪ **Definition 1.13** (Pointwise/uniformly bounded):  $\{f_n\}$  pointwise bounded if  $\forall x \in X$ ,  $\exists M(x) > 0$  such that  $|f_n(x)| \leq M(x) \forall n$ , and uniformly bounded if such an  $M$  exists independent of  $x$ .

↪ **Lemma 1.1** (Arzelà-Ascoli Lemma): Let  $X$  separable and let  $\{f_n\} \subseteq C(X)$  be pointwise bounded and equicontinuous. Then, there is a subsequence  $\{f_{n_k}\}$  and a function  $f$  which converges pointwise to  $f$  on all of  $X$ .

PROOF. Let  $D = \{x_j\}_{j=1}^{\infty} \subseteq X$  be a countable dense subset of  $X$ . Since  $\{f_n\}$  p.w. bounded,  $\{f_n(x_1)\}$  as a sequence of real numbers is bounded and so by the Bolzano-Weierstrass (BW) Theorem there is a convergent subsequence  $\{f_{n(1,k)}(x_1)\}_k$  that converges to some  $a_1 \in \mathbb{R}$ . Consider now  $\{f_{n(1,k)}(x_2)\}_k$ , which is again a bounded sequence of  $\mathbb{R}$  and so has a convergent subsequence, call it  $\{f_{n(2,k)}(x_2)\}_k$  which converges to some  $a_2 \in \mathbb{R}$ . Note that  $\{f_{n(2,k)}\} \subseteq \{f_{n(1,k)}\}$ , so also  $f_{n(2,k)}(x_1) \rightarrow a_1$  as  $k \rightarrow \infty$ . We can repeat this procedure, producing a sequence of real numbers  $\{a_\ell\}$ , and for each  $j \in \mathbb{N}$  a subsequence  $\{f_{n(j,k)}\}_k \subseteq \{f_n\}$  such that  $f_{n(j,k)}(x_\ell) \rightarrow a_\ell$  for each  $1 \leq \ell \leq j$ . Define then

$$f : D \rightarrow \mathbb{R}, f(x_j) := a_j.$$

Consider now

$$f_{n_k} := f_{n(k,k)}, k \geq 1,$$

the “diagonal sequence”, and remark that  $f_{n_k}(x_j) \rightarrow a_j = f(x_j)$  as  $k \rightarrow \infty$  for every  $j \geq 1$ . Hence,  $\{f_{n_k}\}_k$  converges to  $f$  on  $D$ , pointwise.

We claim now that  $\{f_{n_k}\}$  converges on all of  $X$  to some function  $f : X \rightarrow \mathbb{R}$ , pointwise. Put  $g_k := f_{n_k}$  for notational convenience. Fix  $x_0 \in X$ ,  $\varepsilon > 0$ , and let  $\delta > 0$  be such that if  $x \in X$  such that  $\rho(x, x_0) < \delta$ ,  $|g_k(x) - g_k(x_0)| < \frac{\varepsilon}{3}$  for every  $k \geq 1$ , which exists by equicontinuity. Since  $D$  dense in  $X$ , there is some  $x_j \in D$  such that  $\rho(x_j, x_0) < \delta$ . Then, since  $g_k(x_j) \rightarrow f(x_j)$  (pointwise),  $\{g_k(x_j)\}_k$  is Cauchy and so there is some  $K \geq 1$  such that for every  $k, \ell \geq K$ ,  $|g_\ell(x_j) - g_k(x_j)| < \frac{\varepsilon}{3}$ . And hence, for every  $k, \ell \geq K$ ,

$$|g_k(x_0) - g_\ell(x_0)| \leq |g_k(x_0) - g_k(x_j)| + |g_k(x_j) - g_\ell(x_j)| + |g_\ell(x_j) - g_\ell(x_0)| < \varepsilon,$$

so namely  $\{g_k(x_0)\}_k$  Cauchy as a sequence in  $\mathbb{R}$ . Since  $\mathbb{R}$  complete, then  $\{g_k(x_0)\}_k$  also converges, to, say,  $f(x_0) \in \mathbb{R}$ . Since  $x_0$  was arbitrary, this means there is some function  $f : X \rightarrow \mathbb{R}$  such that  $g_k \rightarrow f$  pointwise on  $X$  as we aimed to show. ■

↪ **Definition 1.14** (Uniformly Equicontinuous):  $\mathcal{F} \subseteq C(X)$  is said to be uniformly equicontinuous if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $\forall x, y \in X$  with  $\rho(x, y) < \delta$ ,  $|f(x) - f(y)| < \varepsilon$  for every  $f \in \mathcal{F}$ . That is, every function in  $\mathcal{F}$  has the same modulus of continuity.

↪ **Proposition 1.2** (Sufficient Conditions for Uniform Equicontinuity):

1.  $\mathcal{F} \subseteq C(X)$  uniformly Lipschitz
2.  $\mathcal{F} \subseteq C(X) \cap C^1(X)$  has a uniform  $L^\infty$  bound on the first derivative
3.  $\mathcal{F} \subseteq C(X)$  uniformly Holder continuous
4.  $(X, \rho)$  compact and  $\mathcal{F}$  equicontinuous

↪ **Theorem 1.3** (Arzelà-Ascoli): Let  $(X, \rho)$  a compact metric space and  $\{f_n\} \subseteq C(X)$  be a uniformly bounded and (uniformly) equicontinuous family of functions. Then,  $\{f_n\}$  is pre-compact in  $C(X)$ , i.e. there exists  $\{f_{n_k}\} \subseteq \{f_n\}$  such that  $f_{n_k}$  is uniformly convergent on  $X$ .

PROOF. Since  $(X, \rho)$  compact it is separable and so by the lemma there is a subsequence  $\{f_{n_k}\}$  that converges pointwise on  $X$ . Denote by  $g_k := f_{n_k}$  for notational convenience.

We claim  $\{g_k\}$  uniformly Cauchy. Let  $\varepsilon > 0$ . By uniform equicontinuity, there is a  $\delta > 0$  such that  $\rho(x, y) < \delta \Rightarrow |g_k(x) - g_k(y)| < \frac{\varepsilon}{3}$ . Since  $X$  compact it is totally bounded so there exists  $\{x_i\}_{i=1}^N$  such that  $X \subseteq \bigcup_{i=1}^N B(x_i, \delta)$ . For every  $1 \leq i \leq N$ ,  $\{g_k(x_i)\}$  converges by the lemma hence is Cauchy in  $\mathbb{R}$ . So, there exists a  $K_i$  such that for every  $k, \ell \geq K_i$   $|g_k(x_i) - g_\ell(x_i)| \leq \frac{\varepsilon}{3}$ . Let  $K := \max\{K_i\}$ . Then for every  $\ell, k \leq K$ ,  $|g_k(x_i) - g_\ell(x_i)| \leq \frac{\varepsilon}{3}$  for every  $i = 1, \dots, N$ . So, for all  $x \in X$ , there is some  $x_i$  such that  $\rho(x, x_i) < \delta$ , and so for every  $k, \ell \geq K$ ,

$$\begin{aligned} |g_k(x) - g_\ell(x)| &\leq |g_k(x) - g_k(x_i)| \\ &\quad + |g_k(x_i) - g_\ell(x_i)| \\ &\quad + |g_\ell(x_i) - g_\ell(x)| < \varepsilon, \end{aligned}$$

the first and last follow by the equicontinuity and the second from the lemma. This holds for every  $x$  and thus  $\|g_k - g_\ell\|_\infty < \varepsilon$ , so  $\{g_k\}$  Cauchy in  $C(X)$ . But  $C(X)$  complete so converges in the space. ■

**Remark 1.7:** If  $K \subseteq X$  a compact set, then  $K$  bounded and closed.

↪ **Theorem 1.4:** Let  $(X, \rho)$  compact and  $\mathcal{F} \subseteq C(X)$ . Then,  $\mathcal{F}$  a compact subspace of  $C(X)$  iff  $\mathcal{F}$  closed, uniformly bounded, and (uniformly) equicontinuous.

PROOF. ( $\Leftarrow$ ) Let  $\{f_n\} \subseteq \mathcal{F}$ . By Arzelà-Ascoli Theorem, there exists a subsequence  $\{f_{n_k}\}$  that converges uniformly to some  $f \in C(X)$ . Since  $\mathcal{F}$  closed,  $f \in \mathcal{F}$  and so  $\mathcal{F}$  sequentially compact hence compact.

( $\Rightarrow$ )  $\mathcal{F}$  compact so closed and bounded in  $C(X)$ . To prove equicontinuous, we argue by contradiction. Suppose otherwise, that  $\mathcal{F}$  not-equicontinuous at some  $x \in X$ . Then, there is some  $\varepsilon_0 > 0$  and  $\{f_n\} \subseteq \mathcal{F}$  and  $\{x_n\} \subseteq X$  such that  $|f_n(x_n) - f_n(x)| \geq \varepsilon_0$  while  $\rho(x, x_n) < \frac{1}{n}$ . Since  $\{f_n\}$  bounded and  $\mathcal{F}$  compact, there is a subsequence  $\{f_{n_k}\}$  that converges to  $f$  uniformly. Let  $K$  be such that  $\forall k \geq K, \|f_{n_k} - f\|_\infty \leq \frac{\varepsilon_0}{3}$ . Then,

$$\begin{aligned} |f(x_{n_k}) - f| &\geq |f(x_{n_k}) - f_{n_k}(x_{n_k})| - |f_{n_k}(x_{n_k}) - f_{n_k}(x)| - |f_{n_k}(x) - f(x)| \\ &\geq \frac{\varepsilon_0}{3}, \end{aligned}$$

while  $\rho(x_{n_k}, x) \leq \frac{1}{n_k}$ , so  $f$  cannot be continuous at  $x$ , a contradiction. ■

#### §1.4 Baire Category Theorem

We'll say a set  $E \subseteq X$  *hollow* if  $\text{int } E = \emptyset$ , or equivalently if  $E^c$  dense in  $X$ .

↪ **Theorem 1.5** (Baire Category Theorem): Let  $X$  be a complete metric space.

- (a) Let  $\{F_n\}$  a collection of closed hollow sets. Then,  $\bigcup_{n=1}^{\infty} F_n$  also hollow.
- (b) Let  $\{\mathcal{O}_n\}$  a collection of open dense sets. Then,  $\bigcap_{n=1}^{\infty} \mathcal{O}_n$  also dense.

PROOF. Notice that (a)  $\Leftrightarrow$  (b) by taking complements. We prove (b).

Put  $G := \bigcap_{n=1}^{\infty} \mathcal{O}_n$ . Fix  $x \in X$  and  $r > 0$ , then to show density of  $G$  is to show  $G \cap B(x, r) \neq \emptyset$ .

Since  $\mathcal{O}_1$  dense, then  $\mathcal{O}_1 \cap B(x, r)$  nonempty and in particular open. So, let  $x_1 \in X$  and  $r_1 < \frac{1}{2}$  such that  $\overline{B}(x, r_1) \subseteq B(x, 2r_1) \subseteq \mathcal{O}_1 \cap B(x, r)$ .

Similarly, since  $\mathcal{O}_2$  dense,  $\mathcal{O}_2 \cap B(x_1, r_1)$  open and nonempty so there exists  $x_2 \in X$  and  $r_2 < 2^{-2}$  such that  $\overline{B}(x_2, r_2) \subseteq \mathcal{O}_2 \cap B(x_1, r_1)$ .

Repeat in this manner to find  $x_n \in X$  with  $r_n < 2^{-n}$  such that  $\overline{B}(x_n, r_n) \subseteq \mathcal{O}_n \cap B(x_{n-1}, r_{n-1})$  for any  $n \in \mathbb{N}$ . This creates a sequence of sets

$$\overline{B}(x_1, r_1) \supseteq \overline{B}(x_2, r_2) \supseteq \cdots,$$

with  $r_n \rightarrow 0$ . Hence, the sequence of points  $\{x_n\}$  Cauchy and since  $X$  complete,  $x_j \rightarrow x_0 \in X$ , so in particular

$$\{x_0\} = \bigcap_{n=1}^{\infty} \overline{B}(x_n, r_n),$$

hence  $x_0 \in \mathcal{O}_n$  for every  $n$  and thus  $G \cap B(x, r)$  nonempty. ■

↪ **Corollary 1.2**: Let  $X$  complete and  $\{F_n\}$  a sequence of closed sets in  $X$ . If  $X = \bigcup_{n \geq 1} F_n$ , there is some  $n_0$  such that  $\text{int}(F_{n_0}) \neq \emptyset$ .



PROOF. If not, violates BCT since  $X$  is not hollow in itself. ■

↪**Corollary 1.3:** Let  $X$  complete and  $\{F_n\}$  a sequence of closed sets in  $X$ . Then,  $\bigcup_{n=1}^{\infty} \partial F_n$  hollow.

PROOF. We claim  $\text{int}(\partial F_n) = \emptyset$ . Suppose not, then there exists some  $B(x_0, r) \subseteq \partial F_n$ . Then  $x_0 \in \partial F_n$  but  $B(x_0, r) \cap F_n^c = \emptyset$ , a contradiction. So, since  $\partial F_n$  closed and  $\partial F_n \cap B(x_0, r) = \emptyset$  for every such ball, by BCT  $\bigcup_{n=1}^{\infty} \partial F_n$  must be hollow. ■

### 1.4.1 Applications of Baire Category Theorem

↪**Theorem 1.6:** Let  $\mathcal{F} \subset C(X)$  where  $X$  complete. Suppose  $\mathcal{F}$  pointwise bounded. Then, there exists a nonempty, open set  $\mathcal{O} \subseteq X$  such that  $\mathcal{F}$  uniformly bounded on  $\mathcal{O}$ .

PROOF. Let

$$\begin{aligned} E_n &:= \{x \in X : |f(x)| \leq n \forall f \in \mathcal{F}\} \\ &= \bigcap_{f \in \mathcal{F}} \underbrace{\{x : |f(x)| \leq n\}}_{\text{closed}}. \end{aligned}$$

Since  $\mathcal{F}$  pointwise bounded, for every  $x \in X$  there is some  $M_x > 0$  such that  $|f(x)| \leq M_x$  for every  $f \in \mathcal{F}$ . Hence, for every  $n \in \mathbb{N}$  such that  $n \geq M_x$ ,  $x \in E_n$  and thus  $X = \bigcup_{n=1}^{\infty} E_n$ .

$E_n$  closed and hence by the previous corollaries there is some  $n_0$  such that  $\text{int}(E_{n_0}) \neq \emptyset$  and hence there is some  $r > 0$  and  $x_0 \in X$  such that  $B(x_0, r) \subseteq E_{n_0}$ . Then, for every  $x \in B(x_0, r)$ ,  $|f(x)| \leq n_0$  for every  $f \in \mathcal{F}$ , which gives our desired non-empty open set upon which  $\mathcal{F}$  uniformly bounded. ■

↪**Theorem 1.7:** Let  $X$  complete, and  $\{f_n\} \subseteq C(X)$  such that  $f_n \rightarrow f$  pointwise on  $X$ . Then, there exists a dense subset  $D \subseteq X$  such that  $\{f_n\}$  equicontinuous on  $D$  and  $f$  continuous on  $D$ .

PROOF. For  $m, n \in \mathbb{N}$ , let

$$\begin{aligned} E(m, n) &:= \left\{x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \forall j, k \geq n\right\} \\ &= \bigcap_{j, k \geq n} \left\{x : |f_j(x) - f_k(x)| \leq \frac{1}{m}\right\}. \end{aligned}$$

The union of the boundaries of these sets are hollow, hence  $D := \left(\bigcup_{m, n \geq 1} \partial E(m, n)\right)^c$  is dense. Then, if  $x \in D \cap E(m, n)$ , then  $x \in (\partial E(m, n))^c$  implies  $x \in \text{int}(E(m, n))$ .

We claim  $\{f_n\}$  equicontinuous on  $D$ . Let  $x_0 \in D$  and  $\varepsilon > 0$ . Let  $\frac{1}{m} \leq \frac{\varepsilon}{4}$ . Then, since  $\{f_n(x_0)\}$  convergent it is therefore Cauchy (in  $\mathbb{R}$ ). Hence, there is some  $N$  such that

$|f_j(x_0) - f_k(x_0)| \leq \frac{1}{m}$  for every  $j, k \geq N$ , so  $x_0 \in D \cap E(m, N)$  hence  $x_0 \in \text{int}(E(m, N))$ .

Let  $B(x_0, r) \subseteq E(m, N)$ . Since  $f_N$  continuous at  $x_0$  there is some  $\delta > 0$  such that  $\delta < r$  and

$$|f_N(x) - f_N(x_0)| < \frac{1}{m} \quad \forall x \in B(x_0, \delta),$$

and hence

$$\begin{aligned} |f_j(x) - f_j(x_0)| &\leq |f_j(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f_j(x_0)| \\ &\leq \frac{3}{m} \leq \frac{3}{4}\varepsilon, \end{aligned}$$

for every  $x \in B(x_0, \delta)$  and  $j \geq N$ , where the first, last bounds come from Cauchy and the middle from continuity of  $f_N$ . Hence, we've show  $\{f_n\}$  equicontinuous at  $x_0$  since  $\delta$  was independent of  $f$ .

In particular, this also gives for every  $x \in B(x_0, \delta)$  the limit

$$\frac{3}{4}\varepsilon > \lim_{j \rightarrow \infty} |f_j(x) - f_j(x_0)| = |f(x) - f(x_0)|,$$

so  $f$  continuous on  $D$ . ■

## §1.5 Topological Spaces

Throughout, assume  $X \neq \emptyset$ .

**Definition 1.15** (Topology): Let  $X \neq \emptyset$ . A *topology*  $\mathcal{T}$  on  $X$  is a collection of subsets of  $X$ , called *open sets*, such that

- $X, \emptyset \in \mathcal{T}$ ;
- If  $\{E_n\} \subseteq \mathcal{T}$ ,  $\bigcap_{n=1}^N E_n \in \mathcal{T}$  (closed under *finite* intersections);
- If  $\{E_n\} \subseteq \mathcal{T}$ ,  $\bigcup_n E_n \in \mathcal{T}$  (closed under *arbitrary* unions).

If  $x \in X$ , a set  $E \in \mathcal{T}$  containing  $x$  is called a *neighborhood* of  $x$ .

**Proposition 1.3**:  $E \subseteq X$  open  $\Leftrightarrow$  for every  $x \in E$ , there is a neighborhood of  $x$  contained in  $E$ .

PROOF.  $\Rightarrow$  is trivial by taking the neighborhood to be  $E$  itself.  $\Leftarrow$  follows from the fact that, if for each  $x$  we let  $\mathcal{U}_x$  a neighborhood of  $x$  contained in  $E$ , then

$$E = \bigcup_{x \in E} \mathcal{U}_x,$$

so  $E$  open being a union of open sets. ■

⊗ **Example 1.1:** Every metric space induces a natural topology given by open sets under the metric. The *discrete topology* is given by  $\mathcal{T} = 2^X$  (and is actually induced by the discrete metric), and is the largest topology. The *trivial topology*  $\{\emptyset, X\}$  is the smallest. The *relative topology* defined on a subset  $Y \subseteq X$  is given by  $\mathcal{T}_Y := \{E \cap Y : E \in \mathcal{T}\}$ .

↪ **Definition 1.16** (Base): Given a topological space  $(X, \mathcal{T})$ , let  $x \in X$ . A collection  $\mathcal{B}_x$  of neighborhoods of  $x$  is called a *base* of  $\mathcal{T}$  at  $x$  if for every neighborhood  $\mathcal{U}$  of  $x$ , there is a set  $B \in \mathcal{B}_x$  such that  $B \subseteq \mathcal{U}$ .

We say a collection  $\mathcal{B}$  a base for all of  $\mathcal{T}$  if for every  $x \in X$ , there is a base for  $x$ ,  $\mathcal{B}_x \subseteq \mathcal{B}$ .

↪ **Proposition 1.4:** If  $(X, \mathcal{T})$  a topological space, then  $\mathcal{B} \subseteq \mathcal{T}$  a base for  $\mathcal{T} \Leftrightarrow$  every nonempty open set  $\mathcal{U} \in \mathcal{T}$  can be written as a union of elements of  $\mathcal{B}$ .

PROOF.  $\Rightarrow$  If  $\mathcal{U}$  open, then for  $x \in \mathcal{U}$  there is some basis element  $B_x$  contained in  $\mathcal{U}$ . So in particular  $\mathcal{U} = \bigcup_{x \in \mathcal{U}} B_x$ .

$\Leftarrow$  Let  $x \in \mathcal{U}$  and  $\mathcal{B}_x := \{B \in \mathcal{B} \mid x \in B\}$ . Then, for every neighborhood of  $x$ , there is some  $B$  in  $\mathcal{B}_x$  such that  $B \subseteq \mathcal{U}$  so  $\mathcal{B}_x$  a base for  $\mathcal{T}$  at  $x$ . ■

**Remark 1.8:** A base  $\mathcal{B}$  defines a unique topology,  $\{\emptyset, \cup \mathcal{B}_x\}$ .

↪ **Proposition 1.5:**  $\mathcal{B} \subseteq 2^X$  a base for a topology on  $X \Leftrightarrow$

- $X = \bigcup_{B \in \mathcal{B}} B$
- If  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , then there is a  $B \in \mathcal{B}$  such that  $x \in B \subseteq B_1 \cap B_2$ .

PROOF.  $(\Rightarrow)$  If  $\mathcal{B}$  a base, then  $X$  open so  $X = \cup_B B$ . If  $B_1, B_2 \in \mathcal{B}$ , then  $B_1 \cap B_2$  open so there must exist some  $B \subseteq B_1 \cap B_2$  in  $\mathcal{B}$ .

$(\Leftarrow)$  Let

$$\mathcal{T} = \{\mathcal{U} \mid \forall x \in \mathcal{U}, \exists B \in \mathcal{B} \text{ with } x \in B \subseteq \mathcal{U}\}.$$

One can show this a topology on  $X$  with  $\mathcal{B}$  as a base. ■

↪ **Definition 1.17:** If  $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ , we say  $\mathcal{T}_1$  *weaker/coarser* and  $\mathcal{T}_2$  *stronger/finer*.

Given a subset  $S \subseteq 2^X$ , define

$$\mathcal{T}(S) = \bigcap \text{all topologies containing } S = \text{unique weakest topology containing } S$$

to be the topology *generated* by  $S$ .

↪ **Proposition 1.6:** If  $S \subseteq 2^X$ ,

$$\mathcal{T}(S) = \bigcup \{\text{finite intersections of elts of } S\}.$$

We call  $S$  a “subbase” for  $\mathcal{T}(S)$  (namely, we allow finite intersections of elements in  $S$  to serve as a base for  $\mathcal{T}(S)$ ).

PROOF. Let  $\mathcal{B} := \{X, \text{finite intersections of elements of } S\}$ . We claim this a base for  $\mathcal{T}(S)$ . ■

↪ **Definition 1.18** (Point of closure/accumulation point): If  $E \subseteq X, x \in X$ ,  $x$  is called a *point of closure* if  $\forall \mathcal{U}_x, \mathcal{U}_x \cap E \neq \emptyset$ . The collection of all such sets is called the *closure* of  $E$ , denote  $\overline{E}$ . We say  $E$  *closed* if  $E = \overline{E}$ .

↪ **Proposition 1.7:** Let  $E \subseteq X$ , then

- $\overline{E}$  closed,
- $\overline{E}$  is the smallest closed set containing  $E$ ,
- $E$  open  $\Leftrightarrow E^c$  closed.

## §1.6 Separation, Countability, Separability

↪ **Definition 1.19:** A neighborhood of a set  $K \subseteq X$  is any open set containing  $K$ .

↪ **Definition 1.20** (Notions of Separation): We say  $(X, \mathcal{T})$ :

- *Tychonoff Separable* if  $\forall x, y \in X, \exists \mathcal{U}_x, \mathcal{U}_y$  such that  $y \notin \mathcal{U}_x, x \notin \mathcal{U}_y$
- *Hausdorff Separable* if  $\forall x, y \in X$  can be separated by two disjoint open sets i.e.  $\exists \mathcal{U}_x \cap \mathcal{U}_y = \emptyset$
- *Normal* if Tychonoff and in addition any 2 disjoint closed sets can be separated by disjoint neighborhoods.

**Remark 1.9:** Metric space  $\subseteq$  normal space  $\subseteq$  Hausdorff space  $\subseteq$  Tychonoff space.

↪ **Proposition 1.8:** Tychonoff  $\Leftrightarrow \forall x \in X, \{x\}$  closed.

PROOF. For every  $x \in X$ ,

$$\begin{aligned} \{x\} \text{ closed} &\Leftrightarrow \{x\}^c \text{ open} \\ &\Leftrightarrow \forall y \in \{x\}^c, \exists \mathcal{U}_y \subseteq \{x\}^c \\ &\Leftrightarrow \forall y \neq x, \exists \mathcal{U}_y \text{ s.t. } x \notin \mathcal{U}_y, \end{aligned}$$

and since this holds for every  $x$ ,  $X$  Tychonoff. ■

↪ **Proposition 1.9:** Every metric space normal.

↪ **Proposition 1.10:** Let  $X$  Tychonoff. Then  $X$  normal  $\Leftrightarrow \forall F \subseteq X$  closed and neighborhood  $\mathcal{U}$  of  $F$ , there exists an open set  $\mathcal{O}$  such that

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}.$$

This is called the “nested neighborhood property” of normal spaces.

PROOF. ( $\Rightarrow$ ) Let  $F$  closed and  $\mathcal{U}$  a neighborhood of  $F$ . Then,  $F$  and  $\mathcal{U}^c$  closed disjoint sets so by normality there exists  $\mathcal{O}, \mathcal{V}$  disjoint open neighborhoods of  $F, \mathcal{U}^c$  respectively. So,  $\mathcal{O} \subseteq \mathcal{V}^c$  hence  $\overline{\mathcal{O}} \subseteq \overline{\mathcal{V}^c} = \mathcal{V}^c$  and thus

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{V}^c \subseteq \mathcal{U}.$$

( $\Leftarrow$ ) Let  $A, B$  be disjoint closed sets. Then,  $B^c$  open and moreover  $A \subseteq B^c$ . Hence, there exists some open set  $\mathcal{O}$  such that  $A \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq B^c$ , and thus  $B \subseteq \overline{\mathcal{O}}^c$ . Then,  $\mathcal{O}$  and  $\overline{\mathcal{O}}^c$  are disjoint open neighborhoods of  $A, B$  respectively so  $X$  normal. ■

↪ **Definition 1.21** (Separable): A space  $X$  is called *separable* if it contains a countable dense subset.

↪ **Definition 1.22** (1st, 2nd Countable): A topological space  $(X, \mathcal{T})$  is called

- *1st countable* if there is a countable base at each point
- *2nd countable* if there is a countable base for all of  $\mathcal{T}$ .

⊗ **Example 1.2:** Every metric space is first countable.

↪ **Proposition 1.11:** Every 2nd countable space is separable.

↪ **Definition 1.23** (Convergence): Let  $\{x_n\} \subseteq X$ . Then, we say  $x_n \rightarrow x$  in  $\mathcal{T}$  if for every neighborhood  $\mathcal{U}_x$ , there exists an  $N$  such that  $\forall n \geq N, x_n \in \mathcal{U}_x$ .

**Remark 1.10:** In general spaces, such a limit may not be unique. For instance, under the trivial topology, the only nonempty neighborhood is the whole space, so every sequence converges to every point in the space.

↪ **Proposition 1.12:** Let  $(X, \mathcal{T})$  be Hausdorff. Then, all limits are unique.

PROOF. Suppose otherwise, that  $x_n \rightarrow$  both  $x$  and  $y$ . If  $x \neq y$ , then since  $X$  Hausdorff there are disjoint neighborhoods  $\mathcal{U}_x, \mathcal{U}_y$  containing  $x, y$ . But then  $x_n$  cannot be on both  $\mathcal{U}_x$  and  $\mathcal{U}_y$  for sufficiently large  $n$ , contradiction. ■

↪ **Proposition 1.13:** Let  $X$  be 1st countable and  $E \subseteq X$ . Then,  $x \in \overline{E} \Leftrightarrow$  there exists  $\{x_j\} \subseteq E$  such that  $x_j \rightarrow x$ .

PROOF.  $(\Rightarrow)$  Let  $\mathcal{B}_x = \{B_j\}$  be a base for  $X$  at  $x \in \overline{E}$ . Wlog,  $B_j \supseteq B_{j+1}$  for every  $j \geq 1$  (by replacing with intersections, etc if necessary). Hence,  $B_j \cap E \neq \emptyset$  for every  $j$ . Let  $x_j \in B_j \cap E$ , then by the nesting property  $x_j \rightarrow x$  in  $\mathcal{T}$ .

$(\Leftarrow)$  Suppose otherwise, that  $x \notin \overline{E}$ . Let  $\{x_j\} \in E_j$ . Then,  $\overline{E}^c$  open, and contains  $x$ . Then,  $\overline{E}^c$  a neighborhood of  $x$  but does not contain any  $x_j$  so  $x_j \nrightarrow x$ . ■

## §1.7 Continuity and Compactness

↪ **Definition 1.24:** Let  $(X, \mathcal{T}), (Y, \mathcal{S})$  be two topological spaces. Then, a function  $f : X \rightarrow Y$  is said to be continuous at  $x_0$  if for every neighborhood  $\mathcal{O}$  of  $f(x_0)$  there exists a neighborhood  $\mathcal{U}(x_0)$  such that  $f(\mathcal{U}) \subseteq \mathcal{O}$ . We say  $f$  continuous on  $X$  if it is continuous at every point in  $X$ .

↪ **Proposition 1.14:**  $f$  continuous  $\Leftrightarrow \forall \mathcal{O}$  open in  $Y, f^{-1}(\mathcal{O})$  open in  $X$ .

↪ **Definition 1.25 (Weak Topology):** Consider  $\mathcal{F} := \{f_\lambda : X \rightarrow X_\lambda\}_{\lambda \in \Lambda}$  where  $X, X_\lambda$  topological spaces. Then, let

$$S := \{f_\lambda^{-1}(\mathcal{O}_\lambda) \mid f_\lambda \in \mathcal{F}, \mathcal{O}_\lambda \in \mathcal{T}_\lambda\} \subseteq X.$$

We say that the topology  $\mathcal{T}(S)$  generated by  $S$  is the *weak topology* for  $X$  induced by the family  $\mathcal{F}$ .

↪ **Proposition 1.15:** The weak topology is the weakest topology in which each  $f_\lambda$  continuous on  $X$ .

⊗ **Example 1.3:** The key example of the weak topology is given by the product topology. Consider  $\{X_\lambda\}_{\lambda \in \Lambda}$  a collection of topological spaces. We can define a “natural” topology on the product  $X := \prod_{\lambda \in \Lambda} X_\lambda$  by consider the weak topology induced by the family of projection maps, namely, if  $\pi_\lambda : X \rightarrow X_\lambda$  a coordinate-wise projection and  $\mathcal{F} = \{\pi_\lambda : \lambda \in \Lambda\}$ , then we say the weak topology induced by  $\mathcal{F}$  is the *product topology* on  $X$ . In particular, a base for this topology is given, by previous discussions,

$$\mathcal{B} = \left\{ \bigcap_{j=1}^n \pi_{\lambda_j}^{-1}(\mathcal{O}_{x_j}) \right\} = \left\{ \prod_{\lambda \in \Lambda} \mathcal{U}_\lambda : \mathcal{U}_\lambda \text{ open and all by finitely many } \mathcal{U}_\lambda \neq X_\lambda \right\}.$$

↪ **Definition 1.26** (Compactness): A space  $X$  is said to be *compact* if every open cover of  $X$  admits a finite subcover.

↪ **Proposition 1.16:**

- Closed subsets of compact spaces are compact
- $X$  compact  $\Leftrightarrow$  if  $\{F_k\} \subseteq X$ -nested and closed,  $\bigcap_{k=1}^\infty F_k \neq \emptyset$ .
- Continuous images of compact sets are compact
- Continuous real-valued functions on a compact topological space achieve their min, max.

↪ **Proposition 1.17:** Let  $K$  compact be contained in a Hausdorff space  $X$ . Then,  $K$  closed in  $X$ .

PROOF. We show  $K^c$  open. Let  $y \in K^c$ . Then for every  $x \in K$ , there exists disjoint open sets  $\mathcal{U}_{xy}, \mathcal{O}_{xy}$  containing  $y, x$  respectively. Then, it follows that  $\{\mathcal{O}_{xy}\}_{x \in K}$  an open cover of  $K$ , and since  $K$  compact there must exist some finite subcover,  $K \subseteq \bigcup_{i=1}^N \mathcal{O}_{x_i y}$ . Let  $E := \bigcap_{i=1}^N \mathcal{U}_{x_i y}$ . Then,  $E$  is an open neighborhood of  $y$  with  $E \cap \mathcal{O}_{x_i y} = \emptyset$  for every  $i = 1, \dots, N$ . Thus,  $E \subseteq \bigcap_{i=1}^N \mathcal{O}_{x_i y}^c = \left( \bigcup_{i=1}^N \mathcal{O}_{x_i y} \right)^c \subseteq K^c$  so since  $y$  was arbitrary  $K^c$  open. ■

↪ **Definition 1.27** (Sequential Compactness): We say  $(X, \mathcal{T})$  *sequentially compact* if every sequence in  $X$  has a converging subsequence with limit contained in  $X$ .

↪ **Proposition 1.18:** Let  $(X, \mathcal{T})$  second countable. Then,  $X$  compact  $\Leftrightarrow$  sequentially compact.

PROOF. ( $\Rightarrow$ ) Let  $\{x_k\} \subseteq X$  and put  $F_n := \overline{\{x_k \mid k \geq n\}}$ . Then,  $\{F_n\}$  defines a sequence of closed and nested subsets of  $X$  and, since  $X$  compact,  $\bigcap_{n=1}^\infty F_n$  nonempty. Let  $x_0$  in this intersection. Since  $X$  2nd and so in particular 1st countable, let  $\{B_j\}$  a (wlog nested) countable base at  $x_0$ .  $x_0 \in F_n$  for every  $n \geq 1$  so each  $B_j$  must intersect some

$F_n$ . Let  $n_j$  be an index such that  $x_{n_j} \in B_j$ . Then, if  $\mathcal{U}$  a neighborhood of  $x_0$ , there exists some  $N$  such that  $B_j \subseteq \mathcal{U}$  for every  $j \geq N$  and thus  $\{x_{n_j}\} \subseteq B_N \subseteq \mathcal{U}$ , so  $x_{n_j} \rightarrow x_0$  in  $X$ .

( $\Leftarrow$ ) Remark that since  $X$  second countable, every open cover of  $X$  certainly has a countable subcover by intersecting a given cover with our countable basis. So, assume we have a countable cover  $X \subseteq \bigcup_{n=1}^{\infty} \mathcal{O}_n$  and suppose towards a contradiction that no finite subcover exists. Then, for every  $n \geq 1$ , there exists some  $m(n) \geq n$  such that  $\mathcal{O}_{m(n)} \setminus \bigcup_{i=1}^n \mathcal{O}_i \neq \emptyset$ . Let  $x_n$  in this set for every  $n \geq 1$ . Since  $X$  sequentially compact, there exists a convergent subsequence  $\{x_{n_k}\} \subseteq \{x_n\}$  such that  $x_{n_k} \rightarrow x_0$  in  $X$ , so there exists some  $\mathcal{O}_N$  such that  $x_0 \in \mathcal{O}_N$ . But by construction,  $x_{n_k} \notin \mathcal{O}_N$  if  $n_k \geq N$ , and we have a contradiction. ■

↪ **Theorem 1.8:** If  $X$  compact and Hausdorff,  $X$  normal.

PROOF. We show that any closed set  $F$  and any point  $x \notin F$  can be separated by disjoint open sets. Then, the proof in the more general case follows.

For each  $y \in F$ ,  $X$  is Hausdorff so there exists disjoint open neighborhoods  $\mathcal{O}_{xy}$  and  $\mathcal{U}_{xy}$  of  $x, y$  respectively. Then,  $\{\mathcal{U}_{xy} \mid y \in F\}$  defines an open cover of  $F$ . Since  $F$  closed and thus, being a subset of a compact space, compact, there exists a finite subcover  $F \subseteq \bigcup_{i=1}^N \mathcal{U}_{xy_i}$ . Put  $\mathcal{N} := \bigcap_{i=1}^N \mathcal{O}_{xy_i}$ . This is an open set containing  $x$ , with  $\mathcal{N} \cap \bigcup_{i=1}^N \mathcal{U}_{xy_i} = \emptyset$  hence  $F$  and  $x$  separated by  $\mathcal{N}, \bigcup_{i=1}^N \mathcal{U}_{xy_i}$ . ■

## §1.8 Connected Topological Spaces

↪ **Definition 1.28** (Separate): 2 non-empty sets  $\mathcal{O}_1, \mathcal{O}_2$  separate  $X$  if  $\mathcal{O}_1, \mathcal{O}_2$  disjoint and  $X = \mathcal{O}_1 \cup \mathcal{O}_2$ .

↪ **Definition 1.29** (Connected): We say  $X$  connected if it cannot be separated.

**Remark 1.11:** Note that if  $X$  can be separated, then  $\mathcal{O}_1, \mathcal{O}_2$  are closed as well as open, being complements of each other.

↪ **Proposition 1.19:** Let  $f : X \rightarrow Y$  continuous. Then, if  $X$  connected, so is  $f(X)$ .

PROOF. Suppose otherwise, that  $f(X) = \mathcal{O}_1 \sqcup \mathcal{O}_2$  for nonempty, open, disjoint  $\mathcal{O}_1, \mathcal{O}_2$ . Then,  $X = f^{-1}(\mathcal{O}_1) \sqcup f^{-1}(\mathcal{O}_2)$ , and each of these inverse images remain nonempty and open in  $X$ , so this a contradiction to the connectedness of  $X$ . ■

**Remark 1.12:** On  $\mathbb{R}$ ,  $C \subseteq \mathbb{R}$  connected  $\Leftrightarrow$  an interval  $\Leftrightarrow$  convex.



↪ **Definition 1.30** (Intermediate Value Property): We say  $X$  has the intermediate value property (IVP) if  $\forall f \in C(X)$ ,  $f(X)$  an interval.

↪ **Proposition 1.20**:  $X$  has IVP  $\Leftrightarrow X$  connected.

PROOF. ( $\Leftarrow$ ) If  $X$  connected,  $f(X)$  connected in  $\mathbb{R}$  hence an interval.

( $\Rightarrow$ ) Suppose otherwise, that  $X = \mathcal{O}_1 \sqcup \mathcal{O}_2$ . Then define the function  $f : X \rightarrow \mathbb{R}$  by  $x \mapsto \begin{cases} 1 & \text{if } x \in \mathcal{O}_2 \\ 0 & \text{if } x \in \mathcal{O}_1 \end{cases}$ . Then, for every  $A \subseteq \mathbb{R}$ ,

$$f^{-1}(A) = \begin{cases} \emptyset & \text{if } \{0, 1\} \not\subseteq A \\ \mathcal{O}_1 & \text{if } 0 \in A \\ \mathcal{O}_2 & \text{if } 1 \in A \\ X & \text{if } \{0, 1\} \subseteq A \end{cases},$$

which are all open sets, hence  $f$  continuous. But  $f(X) = \{0, 1\}$  which is not an interval, hence the IVP fails and so  $X$  must be connected. ■

↪ **Definition 1.31** (Arcwise/Path Connected):  $X$  arc connected/path connected if  $\forall x, y \in X$ , there exists a continuous function  $f : [0, 1] \rightarrow X$  such that  $f(0) = x, f(1) = y$ .

↪ **Proposition 1.21**: Arc connected  $\Rightarrow$  connected.

PROOF. Suppose otherwise,  $X = \mathcal{O}_1 \sqcup \mathcal{O}_2$ . Let  $x \in \mathcal{O}_1, y \in \mathcal{O}_2$  and define a continuous function  $f : [0, 1] \rightarrow X$  such that  $f(0) = x$  and  $f(1) = y$ . Then,  $f^{-1}(\mathcal{O}_i)$  each open, nonempty and disjoint for  $i = 1, 2$ , but

$$f^{-1}(\mathcal{O}_1) \sqcup f^{-1}(\mathcal{O}_2) = [0, 1],$$

a contradiction to the connectedness of  $[0, 1]$ . ■

## §1.9 Urysohn's Lemma and Urysohn's Metrization Theorem

We present the main lemma of this section first, but need more tools before proving it.

↪ **Lemma 1.2** (Urysohn's): Let  $A, B \subseteq X$  closed and disjoint subsets of a normal space  $X$ . Then,  $\forall [a, b] \subseteq \mathbb{R}$ , there exists a continuous function  $f : [a, b] \rightarrow \mathbb{R}$  such that  $f(X) \subseteq [a, b]$ ,  $f|_A = a$  and  $f|_B = b$ .

**Remark 1.13**: We have a partial converse of this statement as well:

↪ **Proposition 1.22:** Let  $X$  Tychonoff and suppose  $X$  satisfies the properties of Urysohn's Lemma. Then,  $X$  normal.

PROOF. Let  $A, B$  be closed nonempty disjoint subsets. Let  $f : X \rightarrow \mathbb{R}$  continuous such that  $f|_A = 0$ ,  $f|_B = 1$  and  $0 \leq f \leq 1$ . Let  $I_1, I_2$  be two disjoint open intervals in  $\mathbb{R}$  with  $0 \in I_1$  and  $1 \in I_2$ . Then,  $f^{-1}(I_1)$  open and contains  $A$ , and  $f^{-1}(I_2)$  open and contains  $B$ . Moreover,  $f^{-1}(I_1) \cap f^{-1}(I_2) = \emptyset$ ; hence,  $f^{-1}(I_1), f^{-1}(I_2)$  disjoint open neighborhoods of  $A, B$  respectively, so indeed  $X$  normal. ■

↪ **Definition 1.32** (Normally Ascending): Let  $(X, \mathcal{T})$  a topological space and  $\Lambda \subseteq \mathbb{R}$ . A collection of open sets  $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$  is said to be *normally ascending* if  $\forall \lambda_1, \lambda_2 \in \Lambda$ ,

$$\overline{\mathcal{O}_{\lambda_1}} \subseteq \mathcal{O}_{\lambda_2} \text{ if } \lambda_1 < \lambda_2.$$

↪ **Lemma 1.3:** Let  $\Lambda \subseteq (a, b)$  a dense subset, and let  $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$  a normally ascending collection of subsets of  $X$ . Let  $f : X \rightarrow \mathbb{R}$  defined such that

$$f(x) = \begin{cases} b & \text{if } x \in \left(\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda\right)^c \\ \inf\{\lambda \in \Lambda \mid x \in \mathcal{O}_\lambda\} & \text{else} \end{cases}.$$

Then,  $f$  continuous.

PROOF. We claim  $f^{-1}(-\infty, c)$  and  $f^{-1}(c, \infty)$  open for every  $c \in \mathbb{R}$ . Since such sets define a subbase for  $\mathbb{R}$ , it suffices to prove continuity on these sets. We show just the first for convenience. Notice that since  $f(x) \in [a, b]$ , if  $c \in (a, b)$  then  $f^{-1}(-\infty, c) = f^{-1}[a, c)$ , so really it suffices to show that  $f^{-1}[a, c)$  open to complete the proof.

Suppose  $x \in f^{-1}([a, c])$  so  $a \leq f(x) < c$ . Let  $\lambda \in \Lambda$  be such that  $a < \lambda < f(x)$ . Then,  $x \notin \mathcal{O}_\lambda$ . Let also  $\lambda' \in \Lambda$  such that  $f(x) < \lambda' < c$ . By density of  $\Lambda$ , there exists a  $\varepsilon > 0$  such that  $f(x) + \varepsilon \in \Lambda$ , so in particular

$$\overline{\mathcal{O}_{f(x)+\varepsilon}} \subseteq \mathcal{O}_{\lambda'} \Rightarrow x \in \mathcal{O}_{\lambda'},$$

by nesting. So, repeating this procedure, we find

$$f^{-1}([a, c)) \subseteq \bigcup_{a \leq \lambda < \lambda' < c} \mathcal{O}_{\lambda'} \setminus \overline{\mathcal{O}_\lambda},$$

noticing the set on the right is open. By similar reasoning, the opposite inclusion holds and we have equality. Hence,  $f$  continuous. ■

↪ **Lemma 1.4:** Let  $X$  normal,  $F \subseteq X$  closed, and  $\mathcal{U}$  a neighborhood of  $F$ . Then, for any  $(a, b) \subseteq \mathbb{R}$ , there exists a dense subset  $\Lambda \subseteq (a, b)$  and a normally ascending collection  $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$  such that

$$F \subseteq \mathcal{O}_\lambda \subseteq \overline{\mathcal{O}_\lambda} \subseteq \mathcal{U}, \quad \forall \lambda \in \Lambda.$$

**Remark 1.14:** This is essentially a generalization of the nested neighborhood property, and indeed the proof essentially just uses this property repeatedly to construct the collection  $\{\mathcal{O}_\lambda\}$ .

PROOF. Without loss of generality, we assume  $(a, b) = (0, 1)$ , for the two intervals are homeomorphic, i.e. the function  $f : (0, 1) \rightarrow \mathbb{R}, f(x) := a(1 - x) + bx$  is continuous, invertible with continuous inverse and with  $f(0) = a, f(1) = b$  so a homeomorphism.

Let

$$\Lambda := \left\{ \frac{m}{2^n} \mid m, n \in \mathbb{N} \mid 1 \leq m \leq 2^{n-1} \right\} = \bigcup_{n \in \mathbb{N}} \underbrace{\left\{ \frac{m}{2^n} \mid m \in \mathbb{N} 1 \leq m \leq 2^{n-1} \right\}}_{=: \Lambda_n},$$

which is clearly dense in  $(0, 1)$ . We need now to define our normally ascending collection. We do so by defining on each  $\Lambda_1$  and proceeding inductively.

For  $\Lambda_1$ , since  $X$  normal, let  $\mathcal{O}_{1/2}$  be such that  $F \subseteq \mathcal{O}_{1/2} \subseteq \overline{\mathcal{O}_{1/2}} \subseteq \mathcal{U}$ , which exists by the nested neighborhood property.

For  $\Lambda_2 = \{\frac{1}{4}, \frac{3}{4}\}$ , we use the nested neighborhood property again, but first with  $F$  as the closed set and  $\mathcal{O}_{1/2}$  an open neighborhood of it, and then with  $\overline{\mathcal{O}_{1/2}}$  as the closed set and  $\mathcal{U}$  an open neighborhood of it. In this way, we find

$$\underbrace{F \subseteq \mathcal{O}_{1/4} \subseteq \overline{\mathcal{O}_{1/4}} \subseteq \mathcal{O}_{1/2}}_{\text{nested nbhd}} \subseteq \overbrace{\overline{\mathcal{O}_{1/2}} \subseteq \mathcal{O}_{3/4} \subseteq \overline{\mathcal{O}_{3/4}}}^{\text{nested nbhd}} \subseteq \mathcal{U}.$$

We repeat in this manner over all of  $\Lambda$ , in the end defining a normally ascending collection  $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ . ■

PROOF (Of Urysohn's Lemma, [Lem. 1.2](#)). Let  $F = A$  and  $\mathcal{U} = B^c$  as in the previous lemma [Lem. 1.4](#). Then, there is some dense subset  $\Lambda \subseteq (a, b)$  and a normally ascending collection  $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$  such that  $A \subseteq \mathcal{O}_\lambda \subseteq \overline{\mathcal{O}_\lambda} \subseteq B^c$  for every  $\lambda \in \Lambda$ . Let  $f(x)$  as in the previous lemma, [Lem. 1.3](#). Then, if  $x \in B$ ,  $B \subseteq \left(\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda\right)^c$  and so  $f(x) = b$ . Otherwise if  $x \in A$ , then  $x \in \bigcap_{\lambda \in \Lambda} \mathcal{O}_\lambda$  and thus  $f(x) = \inf\{\lambda \in \Lambda\} = a$ . By the first lemma,  $f$  continuous, so we are done. ■

↪ **Theorem 1.9** (Urysohn's Metrization Theorem): Let  $X$  be a second countable topological space. Then,  $X$  is metrizable (that is, there exists a metric on  $X$  that induces the topology) if and only if  $X$  normal.

PROOF. ( $\Rightarrow$ ) We have already showed, every metric space is normal.

( $\Leftarrow$ ) Let  $\{\mathcal{U}_n\}$  be a countable basis for  $\mathcal{T}$  and put

$$A := \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid \overline{\mathcal{U}_n} \subseteq \mathcal{U}_m\}.$$

By Urysohn's lemma, for each  $(n, m) \in A$  there is some continuous function  $f_{n,m} : X \rightarrow \mathbb{R}$  such that  $f_{n,m}$  is 1 on  $\mathcal{U}_m^c$  and 0 on  $\overline{\mathcal{U}_n}$  (these are disjoint closed sets). For  $x, y \in X$ , define

$$\rho(x, y) := \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)|.$$

The absolute valued term is  $\leq 2$ , so this function will always be finite. Moreover, one can verify that it is indeed a metric on  $X$ . It remains to show that it induces the same topology; it suffices to compare bases of the two.

Let  $x \in \mathcal{U}_m$ . We wish to show there exists  $B_\rho(x, \varepsilon) \subseteq \mathcal{U}_m$ .  $\{x\}$  is closed in  $X$  being normal, so there exists some  $n$  such that

$$\{x\} \subseteq \mathcal{U}_n \subseteq \overline{\mathcal{U}_n} \subseteq \mathcal{U}_m,$$

so  $(n, m) \in A$  and so  $f_{n,m}(x) = 0$ . Let  $\varepsilon = \frac{1}{2^{n+m}}$ . Then, if  $\rho(x, y) < \varepsilon$ , it must be

$$\begin{aligned} \frac{1}{2^{n+m}} &> \sum_{(n',m') \in A} \frac{1}{2^{n'+m'}} |f_{n',m'}(x) - f_{n',m'}(y)| \\ &\geq \frac{1}{2^{n+m}} \underbrace{|f_{n,m}(x) - f_{n,m}(y)|}_{=0} \\ &= \frac{1}{2^{n+m}} |f_{n,m}(y)|, \end{aligned}$$

so  $|f_{n,m}(y)| < 1$  and thus  $y \notin \mathcal{U}_m^c$  so  $y \in \mathcal{U}_m$ . It follow that  $B_\rho(x, \varepsilon) \subseteq \mathcal{U}_m$ , and so every open set in  $X$  is open with respect to the metric topology.

Conversely, if  $B_\rho(x, \varepsilon)$  some open ball in the metric topology, then notice that  $y \mapsto \rho(x, y)$  for fixed  $y$  a continuous function, and thus  $(\rho(x, \cdot))^{-1}(-\varepsilon, \varepsilon)$  an open set in  $\mathcal{T}$  containing  $x$ . But this set also just equal to  $B_\rho(x, \varepsilon)$ , hence  $B_\rho(x, \varepsilon)$  open in  $\mathcal{T}$ . We conclude the two topologies are equal, completing the proof. ■

**Remark 1.15:** Recall metric  $\Rightarrow$  first countable hence not first countable  $\Rightarrow$  not metrizable.

## §1.10 Stone-Weierstrass Theorem

We need to use the following theorem, which we'll prove later.

↪ **Theorem 1.10** (Weierstrass Approximation Theorem): Let  $f : [a, b] \rightarrow \mathbb{R}$  continuous. Then, for every  $\varepsilon > 0$ , there exists a polynomial  $p(x)$  such that  $\|f - p\|_\infty < \varepsilon$ .

↪ **Definition 1.33** (Algebra, Separation of Points): We call a subset  $\mathcal{A} \subseteq C(X)$  an *algebra* if it is a linear subspace that is closed under multiplication (that is,  $f, g \in \mathcal{A} \Rightarrow f \cdot g \in \mathcal{A}$ ).

We say  $\mathcal{A}$  *separates points* in  $X$  if for every  $x, y \in X$ , there exists an  $f \in \mathcal{A}$  such that  $f(x) \neq f(y)$ .

↪ **Theorem 1.11** (Stone-Weierstrass): Let  $X$  be a compact Hausdorff space. Suppose  $\mathcal{A} \subseteq C(X)$  an algebra that separates points and contains constant functions. Then,  $\mathcal{A}$  dense in  $C(X)$ .

We tacitly assume the conditions of the theorem in the following lemmas.

↪ **Lemma 1.5**: For every  $F \subseteq X$  closed, and every  $x_0 \in F^c$ , there exists a neighborhood  $\mathcal{U}(x_0)$  such that  $F \cap \mathcal{U} = \emptyset$  and  $\forall \varepsilon > 0$  there is some  $h \in \mathcal{A}$  such that  $h < \varepsilon$  on  $\mathcal{U}$ ,  $h > 1 - \varepsilon$  on  $F$ , and  $0 \leq h \leq 1$  on  $X$ .

In particular,  $\mathcal{U}$  is *independent* of choice of  $\varepsilon$ .

PROOF.

■

↪ **Lemma 1.6**: For every disjoint closed set  $A, B$  and  $\varepsilon > 0$ , there exists  $h \in \mathcal{A}$  such that  $h|_A < \varepsilon$ ,  $h|_B > 1 - \varepsilon$ , and  $0 \leq h \leq 1$  on  $X$ .

PROOF. (Of Stone-Weierstrass) WLOG, assume  $f \in C(X)$ ,  $0 \leq f \leq 1$ , by replacing with

$$\tilde{f}(x) = \frac{f(x) + \|f\|_\infty}{\|f\|_\infty + \|f\|_\infty}$$

if necessary, since if there exists a  $\tilde{g} \in \mathcal{A}$  such that  $\|\tilde{f} - \tilde{g}\|_\infty < \varepsilon$ , then using the properties of  $\mathcal{A}$  we can find some appropriate  $g \in \mathcal{A}$  such that  $\|f - g\|_\infty < \varepsilon$ .

Fix  $n \in \mathbb{N}$ , and consider the set  $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$ , and let for  $1 \leq j \leq n$

$$A_j := \left\{ x \in X \mid f(x) \leq \frac{j-1}{n} \right\}, \quad B_j := \left\{ x \in X \mid f(x) \geq \frac{j}{n} \right\},$$

which are both closed and disjoint. By the lemma, there exists  $g_j \in \mathcal{A}$  such that

$$g_j|_{A_j} < \frac{1}{n}, \quad g_j|_{B_j} > 1 - \frac{1}{n},$$

with  $0 \leq g_j \leq 1$ . Let then

$$g(x) := \frac{1}{n} \sum_{j=1}^n g_j(x) \in \mathcal{A}.$$

We claim then  $\|f - g\|_\infty \leq \frac{3}{n}$ , which proves the claim by taking  $n$  sufficiently large.

Suppose  $k \in [1, n]$ . If  $f(x) \leq \frac{k}{n}$ , then

$$g_j(x) = \begin{cases} < \frac{1}{n} & \text{if } j-1 \geq k \\ \leq 1 & \text{else} \end{cases},$$

so

$$g(x) = \frac{1}{n} \sum_{j=1}^n g_j(x) = \frac{1}{n} \left[ \sum_{j=1}^k g_j(x) + \sum_{j=k+1}^n g_j(x) \right] \leq \frac{1}{n} \left[ k + \frac{n-k}{n} \right] \leq \frac{k}{n} + \frac{n-k}{n^2} \leq \frac{k+1}{n}.$$

Similarly if  $f(x) \geq \frac{k-1}{n}$ , then

$$g_j(x) = \begin{cases} > 1 - \frac{1}{n} & \text{if } j \leq k-1 \\ \geq 0 & \text{else} \end{cases},$$

so

$$g(x) \geq \frac{1}{n} \sum_{j=1}^{k-1} \left( 1 - \frac{1}{n} \right) \geq \frac{1}{n} (k-1) \left( 1 - \frac{1}{n} \right) = \frac{k-1}{n} - \frac{k-1}{n^2} \geq \frac{k-2}{n}.$$

So, we've show that if  $\frac{k-1}{n} \leq f(x) \leq \frac{k}{n}$ , then  $\frac{k-2}{n} \leq g(x) \leq \frac{k+1}{n}$ , and so repeating this argument and applying triangle inequality we conclude  $\|f - g\|_\infty \leq \frac{3}{n}$ . ■

↪ **Theorem 1.12** (Borsuk):  $X$  compact, Hausdorff and  $C(X)$  separable  $\Leftrightarrow X$  is metrizable.

## §2 FUNCTIONAL ANALYSIS

Here, we will primarily work with a normed vector space (nvs). Moreover, we usually work in:

↪ **Definition 2.1** (Banach Space): A normed vector space  $(X, \|\cdot\|)$  is a *Banach space* if it is complete as a metric space under the norm-induced metric.

### §2.1 Introduction to Linear Operators

↪ **Definition 2.2** (Linear Operator, Operator Norm): Let  $X, Y$  be vector spaces. Then, a map  $T : X \rightarrow Y$  is called *linear* if  $\forall x, y \in X, \alpha, \beta \in \mathbb{R}, T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ .

If  $X, Y$  normed vector spaces, we say  $T$  is a bounded linear operator if  $T$  linear and the operator norm

$$\|T\| = \|T\|_{\mathcal{L}(X, Y)} = \sup_{\substack{x \in X, \\ \|x\|_X \leq 1}} \|Tx\|_Y < \infty$$

is finite. Then, we put

$$\mathcal{L}(X, Y) := \{\text{bounded linear operators } X \rightarrow Y\}.$$

↪ **Theorem 2.1** (Bounded iff Continuous): If  $X, Y$  are nvs,  $T \in \mathcal{L}(X, Y)$  iff and only if  $T$  is continuous, i.e. if  $x_n \rightarrow x$  in  $X$ , then  $Tx_n \rightarrow Tx$  in  $Y$ .

PROOF. If  $T \in \mathcal{L}(X, Y)$ ,

$$\begin{aligned} \|Tx_n - Tx\|_Y &= \|T(x_n - x)\|_Y \\ &= \|x_n - x\|_X \cdot \left\| \frac{T(x_n - x)}{\|x_n - x\|_X} \right\|_Y \\ &\leq \underbrace{\|T\|}_{< \infty} \|x_n - x\|_X \rightarrow 0, \end{aligned}$$

hence  $T$  continuous. Conversely, if  $T$  continuous, then by linearity  $T0 = 0$ , so by continuity, there is some  $\delta > 0$  such that  $\|Tx\|_Y < 1$  if  $\|x\|_X < \delta$ . For  $x \in X$  nonzero, let  $\lambda = \frac{\delta}{\|x\|_X}$ . Then,  $\|\lambda x\|_X \leq \delta$  so  $\|T(\lambda x)\|_Y < 1$ , i.e.  $\frac{\|T(x)\|_Y \delta}{\|x\|_X} < 1$ . Hence,

$$\|T\| = \sup_{x \in X: x \neq 0} \frac{\|T(x)\|_Y}{\|x\|_X} \leq \frac{1}{\delta},$$

so  $T \in \mathcal{L}(X, Y)$ . ■

↪ **Proposition 2.1** (Properties of  $\mathcal{L}(X, Y)$ ): If  $X, Y$  nvs,  $\mathcal{L}(X, Y)$  a nvs, and if  $X, Y$  Banach, then so is  $\mathcal{L}(X, Y)$ .

PROOF. (a) For  $T, S \in \mathcal{L}(X, Y), \alpha, \beta \in \mathbb{R}$ , and  $x \in X$ , then

$$\begin{aligned} \|(\alpha T + \beta S)(x)\|_Y &\leq |\alpha| \|Tx\|_Y + |\beta| \|Sx\|_Y \\ &\leq |\alpha| \|T\| \|x\|_X + |\beta| \|S\| \|x\|_X. \end{aligned}$$

Dividing both sides by  $\|x\|$ , we find  $\|\alpha T + \beta S\| < \infty$ . The same argument gives the triangle inequality on  $\|\cdot\|$ . Finally,  $T = 0$  iff  $\|Tx\|_Y = 0$  for every  $x \in X$  iff  $\|T\| = 0$ .

(b) Let  $\{T_n\} \subseteq \mathcal{L}(X, Y)$  be a Cauchy sequence. We have that

$$\|T_n x - T_m x\|_Y \leq \|T_n - T_m\| \|x\|_X,$$

so in particular the sequence  $\{T_n(x)\}$  a Cauchy sequence in  $Y$  for any  $x \in X$ .  $Y$  complete so this sequence converges, say  $T_n(x) \rightarrow y^*$  in  $Y$ . Let  $T(x) := y^*$  for each  $x$ . We claim that  $T \in \mathcal{L}(X, Y)$  and that  $T_n \rightarrow T$  in the operator norm. We check:

$$\begin{aligned}\alpha T(x_1) + \beta T(x_2) &= \lim_{n \rightarrow \infty} \alpha T_n(x_1) + \lim_{n \rightarrow \infty} \beta T_n(x_2) \\ &= \lim_{n \rightarrow \infty} [T_n(\alpha x_1) + T_n(\beta x_2)] \\ &= \lim_{n \rightarrow \infty} T_n(\alpha x_1 + \beta x_2) \\ &= T(\alpha x_1 + \beta x_2),\end{aligned}$$

so  $T$  linear.

Let now  $\varepsilon > 0$  and  $N$  such that for every  $n \geq N$  and  $k \geq 1$  such that  $\|T_n - T_{n+k}\| < \frac{\varepsilon}{2}$ . Then,

$$\begin{aligned}\|T_n(x) - T_{n+k}(x)\|_Y &= \|(T_n - T_{n+k})(x)\|_Y \\ &\leq \|T_n - T_{n+k}\| \|x\|_X \\ &< \frac{\varepsilon}{2} \|x\|_X.\end{aligned}$$

Letting  $k \rightarrow \infty$ , we find that

$$\|T_n(x) - T(x)\|_Y < \frac{\varepsilon}{2} \|x\|_X,$$

so normalizing both sides by  $\|x\|_X$ , we find  $\|T_n - T\| < \frac{\varepsilon}{2}$ , and we have convergence. ■

↪ **Definition 2.3** (Isomorphism): We say  $T \in \mathcal{L}(X, Y)$  an *isomorphism* if  $T$  is bijective and  $T^{-1} \in \mathcal{L}(Y, X)$ . In this case we write  $X \simeq Y$ , and say  $X, Y$  isomorphic.

## §2.2 Finite versus Infinite Dimensional

If  $X$  a nvs, then we can look for a basis  $\beta$  such that  $\text{span}(\beta) = X$ . If  $\beta = \{e_1, \dots, e_n\}$  has no proper subset spanning  $X$ , then we say  $\dim(X) = n$ .

As we saw on homework, any two norms on a finite dimensional space are equivalent.

↪ **Corollary 2.1**: (a) Any two nvs of the same finite dimension are isomorphic.

(b) Any finite dimensional space is complete, and so any finite dimensional subspace is closed.

(c)  $\overline{B}(0, 1)$  is compact in a finite dimensional space.

PROOF. (a) Let  $(X, \|\cdot\|)$  have finite dimension  $n$ . Then, we claim  $(X, \|\cdot\|) \simeq (\mathbb{R}^n, |\cdot|)$ . Let  $\{e_1, \dots, e_n\}$  be a basis for  $X$ . Let  $T : \mathbb{R}^n \rightarrow X$  given by

$$T(x) = \sum_{i=1}^n x_i e_i,$$



where  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , which is clearly linear. Moreover,

$$Tx = 0 \Leftrightarrow \sum_{i=1}^n x_i e_i = 0 \Leftrightarrow x = 0,$$

so  $T$  injective, and so being linear between two spaces of the same dimension gives  $T$  surjective. It remains to check boundedness.

First, we claim  $x \mapsto \|T(x)\|$  is a norm on  $\mathbb{R}^n$ .  $\|T(x)\| = 0 \Leftrightarrow x = 0$  by the injectivity of  $T$ , and the properties  $\|T(\lambda x)\| = |\lambda| \|Tx\|$  and  $\|T(x + y)\| \leq \|Tx\| + \|Ty\|$  follow from linearity of  $T$  and the fact that  $\|\cdot\|$  already a norm. Hence,  $\|T(\cdot)\|$  a norm on  $\mathbb{R}^n$  and so equivalent to  $|\cdot|$ , i.e. there exists constants  $C_1, C_2 > 0$  such that

$$C_1|x| \leq \|T(x)\| \leq C_2|x|,$$

for every  $x \in X$ . It follows that  $\|T\|$  (operator norm now) is bounded.

Letting  $T(x) = y$ , we find similarly

$$C_1'\|y\| \leq |T^{-1}(y)| \leq C_2'\|y\|,$$

so  $\|T^{-1}\|$  also bounded. Hence, we've shown any  $n$ -dimensional space is isomorphic to  $\mathbb{R}^n$ , so by transitivity of isomorphism any two  $n$ -dimensional spaces are isomorphic.

(b) The property of completeness is preserved under isomorphism, so this follows from the previous statement since  $\mathbb{R}^n$  complete.

(c) Consider  $\overline{B}(0, 1) \subseteq X$ . Let  $T$  be an isomorphism  $X \rightarrow \mathbb{R}^n$ . Then, for  $x \in \overline{B}(0, 1)$ ,  $\|Tx\| \leq \|T\| < \infty$ , so  $T(\overline{B}(0, 1))$  is a bounded subset of  $\mathbb{R}^n$ , and since  $T$  and its inverse continuous,  $T(\overline{B}(0, 1))$  closed in  $\mathbb{R}^n$ . Hence,  $T(\overline{B}(0, 1))$  closed and bounded hence compact in  $\mathbb{R}^n$ , so since  $T^{-1}$  continuous  $T^{-1}(T(\overline{B}(0, 1))) = \overline{B}(0, 1)$  also compact, in  $X$ . ■

↪ **Theorem 2.2** (Riesz's): If  $X$  is an nvs, then  $\overline{B}(0, 1)$  is compact if and only if  $X$  is finite dimensional.

↪ **Lemma 2.1** (Riesz's): Let  $Y \subsetneq X$  be a closed nvs (and  $X$  a nvs). Then for every  $\varepsilon > 0$ , there exists  $x_0 \in X$  with  $\|x_0\| = 1$  and such that

$$\|x_0 - y\|_X > \varepsilon \quad \forall y \in Y.$$

PROOF. Fix  $\varepsilon > 0$ . Since  $Y \subsetneq X$ , let  $x \in Y^c$ .  $Y$  closed so  $Y^c$  open and hence there exists some  $r > 0$  such that  $B(x, r) \cap Y = \emptyset$ . In other words,

$$\inf\{\|x - y'\| \mid y' \in Y\} > r > 0.$$

Let then  $y' \in Y$  be such that

$$r < \|x - y_1\| < \varepsilon^{-1}r,$$

and take

$$x_0 := \frac{x - y_1}{\|x - y_1\|_X}.$$

Then,  $x_0$  a unit vector, and for every  $y \in Y$ ,

$$\begin{aligned} x_0 - y &= \frac{x - y_1}{\|x - y_1\|} - y \\ &= \frac{1}{\|x - y_1\|} [x - y_1 - y \|x - y_1\|] \\ &= \frac{1}{\|x - y_1\|} [x - y'], \end{aligned}$$

where  $y' = y_1 + y \|x - y_1\| \in Y$ , since it is closed under vector addition. Hence

$$\|x_0 - y\| = \frac{1}{\|x - y_1\|} \|x - y'\| > \frac{\varepsilon}{r} \|x - y'\| > \varepsilon,$$

for every  $y \in Y$ . ■

PROOF. (Of [Thm. 2.2](#)) ( $\Leftarrow$ ) By the previous corollary.

( $\Rightarrow$ ) Suppose  $X$  infinite dimensional. We will show  $B := \overline{B}(0, 1)$  not compact.

*Claim:* there exists  $\{x_i\}_{i=1}^\infty \subseteq B$  such that  $\|x_i - x_j\| > \frac{1}{2}$  if  $i \neq j$ .

We proceed by induction. Let  $x_1 \in B$ . Suppose  $\{x_1, \dots, x_n\} \subseteq B$  are such that  $\|x_i - x_j\| > \frac{1}{2}$ . Let  $X_n = \text{span}\{x_1, \dots, x_n\}$ , so  $X_n$  finite dimensional hence  $X_n \subsetneq X$ . By the previous lemma (taking  $\varepsilon = \frac{1}{2}$ ) there is then some  $x_{n+1} \in B$  such that  $\|x_1 - x_{n+1}\| > \frac{1}{2}$  for every  $i = 1, \dots, n$ . We can thus inductively build such a sequence  $\{x_i\}_{i=1}^\infty$ . Then, every subsequence of this sequence cannot be Cauchy so  $B$  is not sequentially compact and thus  $B$  is not compact. ■

### §2.3 Open Mapping and Closed Graph Theorems

**Definition 2.4** ( $T$  open): If  $X, Y$  topological spaces and  $T : X \rightarrow Y$  a linear operator,  $T$  is said to be *open* if for every  $\mathcal{U} \subseteq X$  open,  $T(\mathcal{U})$  open in  $Y$ .

In particular if  $X, Y$  are metric spaces (or nvs), then  $T$  is open iff the image of every open ball in  $X$  contains an open ball in  $Y$ , i.e.  $\forall x \in X, r > 0$  there exists  $r' > 0$  such that  $T(B_X(x, r)) \supseteq B_Y(Tx, r')$ . Moreover, by translating/scaling appropriately, it suffices to prove for  $x = 0, r = 1$ .

**Theorem 2.3** (Open Mapping Theorem): Let  $X, Y$  be Banach spaces and  $T : X \rightarrow Y$  a bounded linear operator. If  $T$  is surjective, then  $T$  is open.

PROOF. Its enough to show that there is some  $r > 0$  such that  $T(B_X(0, 1)) \supseteq B_Y(0, r)$ .

*Claim:*  $\exists c > 0$  such that  $\overline{T(B_X(0, 1))} \supseteq B_Y(0, 2c)$ .

Put  $E_n = n \cdot \overline{T(B_X(0, 1))}$  for  $n \in \mathbb{N}$ . Since  $T$  surjective,  $\bigcup_{n=1}^{\infty} E_n = Y$ . Each  $E_n$  closed, so by the Baire Category Theorem there exists some index  $n_0$  such that  $E_{n_0}$  has nonempty interior, i.e.

$$\text{int}(\overline{T(B_X(0, 1))}) \neq \emptyset,$$

where we drop the index by homogeneity. Pick then  $c > 0$  and  $y_0 \in Y$  such that  $B_Y(y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$ . We claim then that  $B_Y(-y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$  as well. Indeed, if  $B_Y(y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$ , then  $\forall \tilde{y} \in Y$  with  $\|y_0 - \tilde{y}\|_Y < 4c$ , Then,  $\| -y_0 + \tilde{y}\|_Y < 4c$  so  $-\tilde{y} \in B_Y(-y_0, 4c)$ . But  $\tilde{y} = \lim_{n \rightarrow \infty} T(x_n)$  and so  $-\tilde{y} = \lim_{n \rightarrow \infty} T(-x_n)$ . Since  $\{-x_n\} \subseteq B_X(0, 1)$ , this implies  $-\tilde{y} \in \overline{T(B_X(0, 1))}$  hence the “subclaim” holds.

Now, for any  $\tilde{y} \in B_Y(0, 4c)$ ,  $\|\tilde{y}\| \leq 4c$  so

$$\tilde{y} = y_0 \underbrace{-y_0 + \tilde{y}}_{\in B_Y(-y_0, 4c)} = \overbrace{y_0 + \tilde{y}}^{\in B(y_0, 4c)} - y_0.$$

Therefore,

$$\begin{aligned} B_Y(0, 4c) &= B_Y(y_0 - y_0, 4c) \\ &\subseteq B_Y(y_0, 4c) + B_Y(-y_0, 4c) \\ &\subseteq \overline{T(B_X(0, 1))} + \overline{T(B_X(0, 1))} = \overline{2T(B_X(0, 1))}, \end{aligned}$$

(where summation of two sets is the vector addition of all the elements in the sets), hence  $B_Y(0, 2c) \subseteq \overline{T(B_X(0, 1))}$ .

We claim next that  $T(B_X(0, 1)) \supseteq B_Y(0, c)$ . Choose  $y \in Y$  with  $\|y\|_Y < c$ . By the first claim,  $B_Y(0, c) \subseteq \overline{T(B_X(0, \frac{1}{2}))}$ , so for every  $\varepsilon > 0$  there is some  $z \in X$  with  $\|z\|_X < \frac{1}{2}$  and  $\|y - Tz\|_Y < \varepsilon$ . Let  $\varepsilon = \frac{c}{2}$  and  $z_1 \in X$  such that  $\|z_1\|_X < \frac{1}{2}$  and  $\|y - Tz_1\|_Y < \frac{c}{2}$ . But the first claim can also be written as  $B_Y(0, \frac{c}{2}) \subseteq \overline{T(B_X(0, \frac{1}{4}))}$  so if  $\varepsilon = \frac{c}{4}$ , let  $z_2 \in X$  such that  $\|z_2\|_X < \frac{1}{4}$  and  $\|(y - Tz_1) - Tz_2\|_Y < \frac{c}{4}$ . Continuing in this manner we find that

$$B_Y\left(0, \frac{c}{2^k}\right) \subseteq \overline{T\left(B_X\left(0, \frac{1}{2^{k+1}}\right)\right)},$$

so exists  $z_k \in X$  such that  $\|z_k\|_X < \frac{1}{2^k}$  and  $\|y - T(z_1 + \dots + z_k)\|_Y < \frac{c}{2^k}$ . Let  $x_n = z_1 + \dots + z_n \in X$ . Then  $\{x_n\}$  is Cauchy in  $X$ , since

$$\|x_n - x_m\|_X \leq \sum_{k=m}^n \|z_k\|_X < \sum_{k=m}^n \frac{1}{2^k} \rightarrow 0.$$

Since  $X$  a Banach space,  $x_n \rightarrow \bar{x}$  and in particular  $\|\bar{x}\| \leq \sum_{k=1}^{\infty} \|z_k\|_X < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ , so  $\bar{x} \in B_X(0, 1)$ . Since  $T$  bounded it is continuous, so  $Tx_n \rightarrow T\bar{x}$ , so  $y = T\bar{x}$  and thus  $B_Y(0, c) \subseteq T(B_X(0, 1))$ . ■

↪ **Corollary 2.2:** Let  $X, Y$  Banach and  $T : X \rightarrow Y$  be bounded, linear and bijective. Then,  $T^{-1}$  continuous.

↪ **Corollary 2.3:** Let  $(X, \|\cdot\|_1), (X, \|\cdot\|_2)$  be Banach spaces. Suppose there exists  $c > 0$  such that  $\|x\|_2 \leq C \|x\|_1$  for every  $x \in X$ . Then,  $\|\cdot\|_1, \|\cdot\|_2$  are equivalent.

PROOF. Let  $T$  be the identity linear operator and use the previous corollary. ■

↪ **Definition 2.5** ( $T$  closed): If  $X, Y$  are nvs and  $T$  is linear, the *graph* of  $T$  is the set

$$G(T) = \{(x, Tx) \mid x \in X\} \subseteq X \times Y.$$

We then say  $T$  is *closed* if  $G(T)$  closed in  $X \times Y$ .

**Remark 2.1:** Since  $X, Y$  are nvs, they are metric spaces so first countable, hence closed  $\leftrightarrow$  contains all limit points.

What norm do we put on  $X \times Y$ ?  $\|(x, y)\|_2 = \|x\| + \|y\|$ . Then if  $(x_n, y_n) \rightarrow (x, y)$  in the product topology, then since the projection maps are continuous  $x_n \rightarrow x, y_n \rightarrow y$  in the respective topologies on  $X, Y$ . On the other hand if  $(x_n, y_n)$

So, to prove  $G(T)$  is closed we just need to prove that if  $x_n \rightarrow x$  in  $X$  and  $Tx_n \rightarrow y$  in  $Y$ , then  $y = Tx$ .

**Remark 2.2:** If  $T$  is continuous, then  $G(T)$  is closed. The converse?

↪ **Theorem 2.4** (Closed Graph Theorem): Let  $X, Y$  be Banach spaces and  $T : X \rightarrow Y$  linear. Then,  $T$  is continuous iff  $T$  is closed.

PROOF. ( $\Rightarrow$ ) Immediate.

( $\Leftarrow$ ) Suppose  $T$  closed and consider the function  $x \rightarrow \|x\|_*$ , where  $\|x\|_* := \|x\|_X + \|Tx\|_Y$ . Then,  $T$  closed implies  $(X, \|\cdot\|_*)$  is complete, i.e.  $x_n \rightarrow x$  under  $\|\cdot\|_*$  iff  $x_n \rightarrow x$  and  $Tx_n \rightarrow Tx$ . However,  $\|\cdot\|_X \leq \|\cdot\|_*$  on  $X$ , and so since  $(X, \|\cdot\|_X), (X, \|\cdot\|_*)$  are both Banach spaces, by the corollary, there is  $c > 0$  such that  $\|\cdot\|_* \leq C \|\cdot\|_X$ , hence

$$\|x\|_X + \|Tx\|_Y \leq C \|x\|_X,$$

and hence

$$\|Tx\|_Y \leq \|x\|_X + \|Tx\|_Y \leq C \|x\|_X,$$

so  $T$  is bounded and thus  $T$  continuous. ■