

# MATH578 - Numerical Analysis 1

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## §1 POLYNOMIAL INTERPOLATION

In general, the goal of interpolation is, given a function  $f(x)$  on  $[a, b]$  and a series of distinct ordered points (often called *nodes* or *collocation points*)  $\{x_j\}_{j=1}^n \subseteq [a, b]$ , to find a polynomial  $P(x)$  such that  $f(x_j) = P(x_j)$  for each  $j$ .

↪ **Theorem 1.1** (Existence and Uniqueness of Lagrange Polynomial): Let  $f \in C[a, b]$  and  $\{x_j\}$  a set of  $n$  distinct points. Then, there exists a unique  $P(x) \in \mathbb{P}_{n-1}$ , the space of  $n - 1$ -degree polynomials, such that  $P(x_j) = f(x_j)$  for each  $j$ .

We call such a  $P$  the *Lagrange polynomial* associated to the points  $\{x_j\}$  for  $f$ .

PROOF. We define the following  $n - 1$  degree “fundamental polynomials” associated to  $\{x_j\}$ ,

$$\ell_j(x) \equiv \prod_{\substack{1 \leq i \leq n \\ i \neq j}} \frac{x - x_i}{x_j - x_i}, \quad j = 1, \dots, n.$$

Then, one readily verifies  $\ell_j(x_i) = \delta_{ij}$ , and that the distinctness of the nodes guarantees the denominator in each term of the product is nonzero. Define

$$P(x) = \sum_{j=1}^n f(x_j) \ell_j(x),$$

which, being a linear combination of  $n - 1$  degree polynomials is also in  $\mathbb{P}_{n-1}$ . Moreover,

$$P(x_i) = \sum f(x_j) \delta_{i,j} = f(x_i),$$

as desired.

For uniqueness, suppose  $\bar{P}$  another  $n - 1$  degree polynomial satisfying the conditions of the theorem. Then,  $q(x) \equiv P(x) - \bar{P}(x)$  is also a degree  $n - 1$  polynomial with  $q(x_i) = 0$  for each  $i = 1, \dots, n$ . Hence,  $q$  a polynomial with more distinct roots than its degree, and thus it must be identically zero, hence  $P = \bar{P}$ , proving uniqueness. ■

↪ **Theorem 1.2** (Interpolation Error): Suppose  $f \in C^n[a, b]$ , and let  $P(x)$  be the Lagrange polynomial for a set of  $n$  points  $\{x_j\}$ , with  $x_1 = a, x_n = b$ . Then, for each  $x \in [a, b]$ , there is a  $\xi \in [a, b]$  such that

$$f(x) - P(x) = \frac{f^{(n)}(\xi)}{n!} (x - x_1) \cdots (x - x_n).$$

Moreover, if we put  $h := \max_i (x_{i+1} - x_i)$ , then

$$\|f - P\|_\infty \leq \frac{h^n}{4n} \|f^{(n)}\|_\infty.$$

PROOF. We prove the first identity, and leave the second “Moreover” as a homework problem. Notice that it holds trivially for  $x = x_j$  for any  $j$ , so assume  $x \neq x_j$  for any  $j$ , and define the function

$$g(t) := f(t) - P(t) - \omega(t) \frac{f(x) - P(x)}{\omega(x)}, \quad \omega(t) := (t - x_1) \dots (t - x_n) \in \mathbb{P}_n[t].$$

Then, we observe the following:

- $g \in C^n[a, b]$
- $g(x) = 0$
- $g(t = x_j) = 0$  for each  $j$

Recall that by Rolle’s Theorem, if a  $C^1$  function has  $\geq m$  roots, then its derivative has  $\geq m - 1$  roots. Thus, applying this principle inductively to  $g(t)$ , we conclude that  $g^{(n)}(t)$  has at least one root. Take  $\zeta$  to be such a root. Then, one readily verifies that  $P^{(n)} \equiv 0$  and  $\omega^{(n)} \equiv n!$  (using polynomial properties), from which we may use the fact that  $g^{(n)}(\zeta) = 0$  to simplify to the required identity. ■

**Remark 1.1:** In general, larger  $n$  leads to smaller maximum step size  $h$ . However, it is *not* true that  $n \rightarrow \infty$  implies  $P \rightarrow f$  in  $L^\infty$ . From the previous theorem, one would need to guarantee  $\|f^{(n)}\| \rightarrow 0$  (or at least, doesn’t grow faster than  $\frac{h^n}{4n}$ ), which certainly won’t hold in general; we have no control on the  $n$ th-derivative of an arbitrarily given function. However, we can try to optimize our choice of points  $\{x_j\}$  for a given  $j$ .

We switch notation for convention’s sake to  $n + 1$  points  $x_j$ . Our goal is the optimization problem

$$\min_{x_j} \max_{x \in [a, b]} \left| \prod_j (x - x_j) \right|,$$

the only term in the error bound above that we have control over. Remark that we can expand the product term:

$$\prod_j (x - x_j) = x^{n+1} - r(x),$$

where  $r(x) \in \mathbb{P}_n$ . So, really, we equivalently want to solve the problem

$$\min_{r \in \mathbb{P}_n} \|x^{n+1} - r(x)\|_\infty,$$

namely, what  $n$ -degree polynomial minimizes the max difference between  $x^{n+1}$ ?

↪ **Theorem 1.3** (De la Vallé-Poussin Oscillation Theorem): Let  $f \in C([a, b])$ , and suppose  $r \in \mathbb{P}_n$  for which there exists  $n + 2$  distinct points  $\{x_j\}$  such that  $a \leq x_0 < \dots < x_{n+1} \leq b$  at which the error  $f(x) - r(x)$  “oscillate” sign, i.e.

$$\text{sign}(f(x_j) - r(x_j)) = -\text{sign}(f(x_{j+1}) - r(x_{j+1})).$$

Then,

$$\min_{P \in \mathbb{P}_n} \|f - P\|_\infty \geq \min_{0 \leq j \leq n+1} |f(x_j) - r(x_j)|.$$

↪ **Definition 1.1** (Chebyshev Polynomial): The *degree  $n$  Chebyshev polynomial*, defined on  $[-1, 1]$ , is defined by

$$T_n(x) := \cos(n \cos^{-1}(x)).$$

**Remark 1.2:** The fact that  $T_n$  actually is a polynomial follows from the double angle formula for  $\cos$ , which says

$$\cos((n+1)\theta) = 2\cos(\theta)\cos(n\theta) - \cos((n-1)\theta).$$

In the context of  $T_n$ , this implies that for any  $n \geq 1$ , the recursive formula

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

This formula with a simple induction argument proves that each  $T_n$  a polynomial, with for instance  $T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1$  and so on.

↪ **Proposition 1.1:**  $\{T_n\}$  are orthogonal with respect to the inner product given by

$$(f, g) := \int_{-1}^1 f(x)g(x)\omega_2(x) \, dx,$$

where  $\omega_2(x) := (1 - x^2)^{1/2}$ .

**Remark 1.3:** Defining similar *weight* functions by  $\omega_n(x) := (1 - x^n)^{1/n}$ , one can derive a more general class of polynomials called *Geigenbauer polynomials*, which are respectively orthogonal with respect to  $\int \cdot \cdot \omega_n$ .

↪ **Proposition 1.2** (Some Properties of  $T_n$ ):

- $|T_n(x)| \leq 1$  on  $[-1, 1]$
- The roots of  $T_n(x)$  are the  $n$  points

$$\xi_j := \cos\left(\frac{(2j-1)\pi}{2n}\right), \quad j = 1, \dots, n.$$

- For  $n \geq 1$ ,  $|T_n(x)|$  is maximal on  $[-1, 1]$  at the  $n+1$  points

$$\eta_j := \cos\left(\frac{j\pi}{n}\right), \quad j = 0, \dots, n,$$

with  $T_n(\eta_j) = (-1)^j$ .

Note too that  $T_{n+1}(x)$  has leading coefficient  $2^n$ , which can be seen by the recursive formula above; define the *normalized* Chebyshev polynomials by  $\hat{T}_{n+1}(x) := 2^{-n}T_{n+1}(x)$ . Thus, we may write

$$\hat{T}_{n+1}(x) = x^{n+1} - r_n(x),$$

with  $r_n(x) \in \mathbb{P}_n$ . It follows for one that

$$\max_{x \in [-1, 1]} |x^{n+1} - r_n(x)| = 2^{-n}.$$

Moreover, we know that at the  $n+2$  points  $\eta_j$ , we have

$$\hat{T}_{n+1}(\eta_j) = 2^{-j}(-1)^j = \eta_j^{n+1} - r_n(\eta_j).$$

Namely, because of the inclusion of  $(-1)^j$  term, this means that  $\hat{T}_{n+1}(x)$  oscillates sign between the  $\eta_j$  points, which fulfils the condition stated in the Oscillation Theorem. Thus, these observations readily imply the following result, settling our original question on optimizing locations of interpolation points for Lagrange interpolation:

↪ **Theorem 1.4** (Optimal Approximation of  $x^{n+1}$  in  $\mathbb{P}_n$ ): The optimal approximation of  $x^{n+1}$  in  $\mathbb{P}_n$  on  $[-1, 1]$  with respect to the  $L^\infty$  norm is given by

$$r_n(x) := x^{n+1} - 2^{-n}T_{n+1}(x).$$

Thus, the optimal Lagrange interpolation points are the  $n+1$  roots of  $x^{n+1} - r_n(x)$ , namely  $\xi_j = \cos\left(\frac{(2j+1)\pi}{2n+2}\right)$  for  $j = 0, \dots, n$ .

**Remark 1.4:** This, and previous results, were stated over  $[-1, 1]$ . A linear change of coordinates transforming any closed interval to  $[-1, 1]$  readily leads to analogous results.

## §2 FOURIER TRANSFORM

Recall that the Fourier transform of a (Lebesgue) measurable function  $u(x)$  on  $\mathbb{R}$  is defined

$$(\mathcal{F}u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} u(x) \, dx.$$

↪ **Theorem 2.1:** Let  $u \in L^2(\mathbb{R})$ . Then,

1.  $\hat{u} \in L^2$
2. the *inversion* formula holds, ie  $u(x) = \int_{\mathbb{R}} \hat{u}(\xi) e^{i\xi x} dx = (\mathcal{F}^{-1}u)(x)$
3.  $\|\hat{u}\|_2 = \sqrt{2\pi}\|u\|_2$
4. for  $u \in L^2, v \in L^1, u * v \in L^2$ , and  $\widehat{u * v} = \hat{u}\hat{v}$ .

↪ **Theorem 2.2** (Further Properties of Fourier Transform): Let  $u, v \in L^2$ . Then,

1.  $\mathcal{F}$  is linear over  $\mathbb{R}$
2.  $\mathcal{F}(u(\cdot + x_0))(\xi) = e^{i\xi x_0} \hat{u}(\xi)$
3.  $\mathcal{F}(e^{i\xi_0 x} u(x))(\xi) = \hat{u}(\xi - \xi_0)$
4. If  $c \neq 0$ ,  $\mathcal{F}(u(c \cdot))(\xi) = \frac{\hat{u}(\frac{\xi}{c})}{c}$
5.  $\mathcal{F}(\overline{u})(\xi) = \overline{\hat{u}(-\xi)}$
6. if  $u_x$  exists and is in  $L^2$ , then

$$\mathcal{F}(u_x)(\xi) = i\xi \hat{u}(\xi).$$

By extension, if  $\partial_\alpha u \in L^2$ , then  $\widehat{\partial_\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$

7.  $(\mathcal{F}^{-1}u)(\xi) = \frac{1}{2\pi} \hat{u}(-\xi)$ .

In a sense, 6. implies a duality between the smoothness of  $u(x)$  and rapid decay (as  $|\xi| \rightarrow \infty$ ) of  $\hat{u}(\xi)$ ; 7. indicates that the same analogy holds switching the roles of  $u$  and  $\hat{u}$ . We make this more precise.

↪ **Definition 2.1** (Bounded Variation): We say a function  $u$  on  $\mathbb{R}$  is of *bounded variation* or write  $u \in BV$  if there exists a constant  $M$  such that for any finite integer  $m$  and collection of points  $x_0 < x_1 < \dots < x_m$ ,

$$\sum_{j=1}^m |u(x_j) - u(x_{j-1})| \leq M.$$

In a sense, this notion of BV captures a notion of “limited oscillation”.

↪ **Theorem 2.3:** Let  $u \in L^2$ . Then:

1. If  $u$  has  $p - 1$  continuous derivatives in  $L^2$  and its  $p$ th derivative is in BV, then

$$\hat{u}(\xi) = O(|\xi|^{-p-1}).$$

2. If  $u$  has infinitely many derivatives all in  $L^2$ , then

$$\hat{u}(\xi) = O(|\xi|^{-M}), \quad \forall M \geq 1.$$

## §2.1 Discrete Fourier Transform

Let  $h > 0$  be a *step size*. Let  $x_j = jh$  for  $j \in \mathbb{Z}$ . We write  $v = \{v_j\}_{j \in \mathbb{Z}}$  for discrete approximations of a function  $u$  on the grid  $\{x_j\}_{j \in \mathbb{Z}}$ , i.e.  $v_j \approx u(x_j)$ .

The  $\ell_h^2$  norm is defined for such  $v$  by

$$\|v\|_2 := \left[ h \sum_{j \in \mathbb{Z}} |v_j|^2 \right]^{1/2}.$$

Then,  $\ell_h^2$  is defined as the space of such sequences  $v$  such that this norm is finite. analogous definitions hold for other  $\ell_h^p$  spaces and norms.

↪ **Proposition 2.1** (Nesting):  $\ell_h^p \subset \ell_h^q$  for each  $q \geq p$ .

**Remark 2.1:** Note that the analogous result to this does *not* hold for  $L^p$  spaces (unless restricted to a compact domain).

We define the convolution of two sequences  $v, w$  by the new sequence  $v * w$  with entries

$$(v * w)_m = h \sum_{j \in \mathbb{Z}} v_j w_{m-j} = h \sum_{j \in \mathbb{Z}} v_{m-j} w_j.$$

For any  $v \in \ell_h^2$ , we define too the *semi-discrete Fourier transform* of  $v$  by

$$\hat{v}(\xi) = (\mathcal{F}_h v)(\xi) = h \sum_{j \in \mathbb{Z}} e^{-i\xi x_j} v_j, \quad \xi \in \left[ -\frac{\pi}{h}, \frac{\pi}{h} \right],$$

where we remark that  $\hat{v}(\xi)$   $\frac{2\pi}{h}$ -periodic (hence the domain restriction) and continuous.

We define the norm of  $\hat{v}$  by the usual  $L^2$ -norm, restricted to the appropriate domain:

$$\|\hat{v}\|_2 := \left( \int_{-\pi/h}^{\pi/h} |\hat{v}(\xi)|^2 d\xi \right)^{1/2},$$

and  $L_h^2$  the space of such functions with finite norm.

↪ **Theorem 2.4:** If  $v \in \ell_h^2$ , then  $\hat{v} \in L_h^2$ , and we can recover  $v$  from  $\hat{v}$  by the “inverse semi-discrete Fourier transform”, i.e.

$$v_j = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{i\xi x_j} \hat{v}(\xi) d\xi.$$

Also, Parseval’s identity holds, i.e.  $\|\hat{v}\|_2 = \sqrt{2\pi} \|v\|_2$ , as does the expected convolution identity (for  $v \in \ell_h^2, w \in \ell_h^1$  for instance).

**Remark 2.2:** Note that each wave number  $\xi$  is indistinguishable from  $\xi + \frac{2\pi j}{h}$  for  $j \in \mathbb{Z}$  on  $h\mathbb{Z}$ ; this is called *aliasing*. The cutoff  $\frac{\pi}{h}$  is called the *Nyquist Wave Number*.

↪ **Theorem 2.5:** Let  $u \in L^2$ , sufficiently smooth, with  $v \in \ell_h^2$  be a restriction of  $u$  to  $h\mathbb{Z}$ . Then,

$$\hat{v}(\xi) = \sum_{j \in \mathbb{Z}} \hat{u}\left(\xi + \frac{2\pi j}{h}\right), \quad \xi \in \left[ -\frac{\pi}{h}, \frac{\pi}{h} \right].$$

## §2.2 Wavelet Transform

The heuristic idea of the wavelet transform is to construct a basis of functions which effectively compromise between localization in space and frequency; indeed, the issues related

to aliasing in the discrete case are linked to the fact that localization of a function simultaneously in physical and fourier space is impossible except for the zero function.

More precisely, we dictate that a wavelet  $\psi$  should have:

1. non-negligible values in a limited range of space *and* frequency;
2. finite energy, by which we mean

$$\int_0^\infty |\hat{\psi}(\omega)|^2 \frac{d\omega}{|\omega|} < \infty.$$

3. zero mean, i.e.  $\int_{-\infty}^\infty \psi(t) dt = 0$ .

Note that 2., 3., imply that  $\psi$  has actual “frequency content” and zero mean, so  $\psi$  satisfying these properties must oscillate.

We call such a  $\psi$  a the *model wavelet*, from which we will generate our desired basis by translating and scaling:

$$\psi_{s,\tau}(t) := \frac{1}{\sqrt{s}} \psi\left(\frac{t-\tau}{s}\right).$$

From this, we define

$$\gamma(s, \tau) = \int f(t) \psi_{s,\tau}^*(t) dt.$$

Then, one can retrieve  $f$  (with appropriate properties) by the *inverse wavelet transform*

$$f(t) = \int_{\mathbb{R}^+} \int_{\mathbb{R}} \gamma(s, \tau) \psi_{s,\tau}(t) d\tau ds.$$

In a sense,  $\gamma(s, t)$  provides a *compromise* between space (i.e.  $\tau$ ) and frequency/scale (i.e.  $s$ ) and energy localization.

More precisely, we’d like a quantitative decay of  $\gamma(s, t)$  for small  $s$ , i.e. small frequencies, which are the problematic range. If we Taylor expand  $f$  in the definition of  $\gamma(s, \tau)$  about  $s = 0$  (and taking  $\tau = 0$  for convenience), one notices that

$$\gamma(s, 0) = \frac{1}{\sqrt{s}} \left[ \sum_{p=0}^n f^{(p)}(0) \int \frac{t^p}{p!} \psi(t/s) dt + O(n+1) \right].$$

If we define  $M_p := \int t^p \psi(t) dt$  to be the  $p$ th moment of  $\psi$ , then one clearly sees that if the first  $n$  moments of  $\psi$  are identically 0, then

$$\psi(s, \tau) = O(s^{n+2}),$$

those providing a qualitative decay rate for these coefficients. Thus, we generally want such vanishing moments in designing “good” wavelets.

### §3 FINITE DIFFERENCE (FD) APPROXIMATION

Given  $u \in C^\ell$ , our goal is to approximate derivatives of  $u$  by a combination of finitely many function values, i.e.



$$\frac{\partial^k u}{\partial x^k} \Big|_{x_0} = \sum_{i=0}^m \alpha_i u(x_i), \quad k \leq \ell.$$

The vector  $\alpha = (\alpha_i)$  is called the *finite difference stencil*. Such schemes are found by Taylor expanding about  $x_0$ :

$$u(x) = u(x_0) + u_x(x_0)(x - x_0) + \frac{1}{2}u_{xx}(x_0)(x - x_0)^2 + O(|x - x_0|^3).$$

So assuming we are given a grid of points  $x_i, i = 0, \dots, m$ , put  $\bar{x}_i := x_i - x_0$ ; summing the above line over  $i$  with  $x$  evaluated on each  $x_i$  gives

$$\sum_{i=0}^m \alpha_i u(x_i) = u(x_0) \left( \sum_{i=0}^m \alpha_i \right) + u_x(x_0) \left( \sum_{i=0}^m \alpha_i \bar{x}_i \right) + \frac{1}{2}u_{xx}(x_0) \left( \sum_{i=0}^m \alpha_i \bar{x}_i^2 \right) + O\left(\sum \bar{x}_i^2\right).$$

So, suppose we want an approximation of  $u_x(x_0)$ ; then, we need to cancel the first and third paranthesed terms and set the second to 1;

$$\sum \alpha_i = 0, \quad \sum \alpha_i \bar{x}_i = 1, \quad \sum \alpha_i \bar{x}_i^2 = 0.$$

(Alternatively, we can just restrict this last term to be  $O(|x|^2)$ , or some similar consistency result.) To discuss existence/uniqueness of such schemes, we define first the  $k \times m$ -Vandermonde matrix associated to a set of points  $\{x_0, \dots, x_m\}$ ,

$$V(x_0, \dots, x_m) := \begin{pmatrix} 1 & \dots & 1 \\ \bar{x}_0 & \dots & \bar{x}_m \\ \bar{x}_0^2 & \dots & \bar{x}_m^2 \\ \vdots & & \vdots \\ \bar{x}_0^k & \dots & \bar{x}_m^k \end{pmatrix}.$$

For  $k = 1$ , notice that

$$V\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

has solution identical to our first scheme for the first derivative. Similarly, for  $k = 2$ ,

$$V\alpha = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

gives a stencil for the second derivative.

If  $m = k$ ,  $V$  is square and so has unique solution (assuming invertibility). If  $m > k$ , there are multiple solutions necessarily; we can specify a solution by adding more constraints to make it square.

Extending to higher-dimensions is similar, where the Taylor-expansion logic leads to extra cross terms being involved. So, for instance, in 2 dimensions,

$$V = \begin{pmatrix} 1 & \dots & 1 \\ \bar{x}_0 & \dots & \bar{x}_m \\ \bar{y}_0 & \dots & \bar{y}_m \\ \bar{x}_0^2 & \dots & \bar{x}_m^2 \\ \bar{y}_0^2 & \dots & \bar{y}_m^2 \\ \bar{x}_0\bar{y}_0 & \dots & \bar{x}_m\bar{y}_m \\ \vdots & & \vdots \end{pmatrix}.$$

Of course, we aren't just restricting to approximation single derivatives in this way; for instance with the above  $V$  and RHS set to  $(0, 0, 0, 2, 2, 0)^t$ , we obtain an approximation of the Laplacian.

To apply this to numerically solving an ODE, we consider the 1-dimensional Poisson's equation with mixed boundary conditions,

$$-u_{xx} = f(x), x \in (0, 1) \quad u(0) = a, u_x(1) = c.$$

We discretize with a uniform grid  $\bar{x} = (0, h, \dots, nh, 1)$ . On the interior points, we'll use a centered difference scheme;

$$f(x_i) = -u_{xx}(x_i) = -\frac{1}{h^2}(u_{i-1} - 2u_i + u_{i+1}) + O(h^2), \quad 1 \leq i \leq n.$$

We know  $u(x_0) = u_0 = a$  from the Dirichlet boundary condition, so on the left-hand-most-point we obtain

$$-\frac{a - 2u_1 + u_2}{h^2} = f(x_1).$$

For the Neumann boundary condition, we don't have access to information about " $u_{n+2}$ ", so we can't directly use the centered scheme above. One idea would be to naively use a backward difference scheme, putting

$$c = u_x(1) = \frac{u_{n+1} - u_n}{h},$$

but this is of order  $O(h)$ , so not amazing. However, suppose we use a centered difference approximation of the first derivative here, then we'd have

$$O(h^2) + \frac{u_{n+2} - u_n}{2h} = c.$$

Again,  $u_{n+2}$  is not defined. However, we can employ what is called a "ghost" point; assume there is some point  $x_{n+2}$ , and solve for what it should be using the interior scheme. Namely, if we put  $u_{n+2} = 2hc + u_n$ , then the interior scheme would say

$$-\frac{u_n - 2u_{n+1} + u_{n+2}}{h^2} = f(x_{n+1}),$$

which implies

$$-2\frac{u_n - 2(u_{n+1} + 1)}{h^2} = f(x_{n+1}) + \frac{2c}{h},$$

which gives now a second-order approximation.

### §3.1 Error, Consistency and Stability

We'll discuss the results here for the specific instance of the Poisson equation,  $u'' = f$ , for sake of concreteness, but they hold in a more general setting. Let  $U$  be a discrete approximation of  $u$  (i.e.,  $U_i \approx u(x_i)$ ) and  $A$  a matrix for which  $AU = F \approx f$ .

↪ **Definition 3.1** (Local Truncation Error (LTE)): Plug the “true” solution  $\hat{u} = (u_i)$  into the FD scheme, and put  $\tau$  for the difference vector, i.e.

$$\tau := A\hat{u} - F.$$

↪ **Definition 3.2** (Global Truncation Error (GTE)): Put  $E := U - \hat{u}$ , called the GTE. Note that  $AE = -\tau$ .

↪ **Definition 3.3** (Stability): Suppose we have a parametrized family of discretizations by some (maximum, say) grid size  $h$ , so  $A^h, E^h, \tau^h$  are all given. Following from the above, we know that

$$\|E^h\| \leq \|(A^h)^{-1}\| \|\tau^h\|.$$

we say the parametrized scheme is stable if  $\|(A^h)^{-1}\|$  is bounded from above uniformly in  $h$  for sufficiently small  $h$ .

↪ **Definition 3.4** (Consistency): The scheme above is consistent if  $\|\tau^h\| \rightarrow 0$  as  $h \rightarrow 0$ .

↪ **Definition 3.5** (Convergence): The scheme above converges if  $\|E^h\| \rightarrow 0$  as  $h \rightarrow 0$ .

↪ **Theorem 3.1** (Lax Equivalence): A scheme is Consistent and stable  $\Leftrightarrow$  it is convergent.

## §4 SPECTRAL METHODS

The previous section lead to schemes that were of  $O(h^p)$  error for some fixed  $p$ . Our schemes here lead to  $O(h^p)$  error for all  $p$  if  $u \in C^\infty$ . Such *spectral* methods have limited domain of application (namely linear equations, simple boundary conditions, and smooth functions), but for such problems they are very good.

## §5 SOME BACKGROUND ON PDEs