MATH455 - Analysis 4

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Contents

§1 Abstract Metric and Topological Spaces

§1.1 Review of Metric Spaces

Throughout fix X a nonempty set.

↪**Definition 1.1** (Metric): $\rho : X \times X \to \mathbb{R}$ is called a *metric*, and thus (X, ρ) a *metric space*, if for all $x, y, z \in X$,
- $\rho(x, y) \geq 0$,
- $\rho(x, y) = 0 \Leftrightarrow x = y$,
- $\rho(x, y) = \rho(y, x)$, and
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

↪**Definition 1.2** (Norm): Let X a linear space. A function $\| \cdot \| : X \to [0, \infty)$ is called a *norm* if for all $u, v \in X$ and $\alpha \in \mathbb{R}$,
- $\|u\| = 0 \Leftrightarrow u = 0$,
- $\|u + v\| \leq \|u\| + \|v\|$, and
- $\|\alpha u\| = |\alpha| \ \|u\|$.

Remark 1.1: A norm induces a metric by $\rho(x, y) := \|x - y\|$.

↪**Definition 1.3**: Given two metrics ρ, σ on X, we say they are *equivalent* if $\exists\, C > 0$ such that $\frac{1}{C}\sigma(x, y) \leq \rho(x, y) \leq C\sigma(x, y)$ for every $x, y \in X$. A similar definition follows for equivalence of norms.

Given a metric space (X, ρ), then, we have the notion of
- open balls $B(x, r) = \{y \in X : \rho(x, y) < r\}$,
- open sets (subsets of X with the property that for every $x \in X$, there is a constant $r > 0$ such that $B(x, r) \subseteq X$), closed sets, closures, and
- *convergence*.

↪**Definition 1.4** (Convergence): $\{x_n\} \subseteq X$ converges to $x \in X$ if $\lim_{n \to \infty} \rho(x_n, x) = 0$.

We have several (equivalent) notions, then, of continuity; via sequences, $\varepsilon - \delta$ definition, and by pullbacks (inverse images of open sets are open).

↪**Definition 1.5** (Uniform Continuity): $f : (X, \rho) \to (Y, \sigma)$ uniformly continuous if f has a "modulus of continuity", i.e. there is a continuous function $\omega : [0, \infty) \to [0, \infty)$ such that

$$\sigma(f(x_1), f(x_2)) \leq \omega(\rho(x_1, x_2))$$

for every $x_1, x_2 \in X$.

Remark 1.2: For instance, we say f Lipschitz continuous if there is a constant $C > 0$ such that $\omega(\cdot) = C(\cdot)$. Let $\alpha \in (0, 1)$. We say f α-Holder continuous if $\omega(\cdot) = C(\cdot)^\alpha$ for some constant C.

↪**Definition 1.6** (Completeness): We say (X, ρ) *complete* if every cauchy sequence in (X, ρ) converges to a point in X.

Remark 1.3: If (X, ρ) complete and $E \subseteq X$, then (E, ρ) is complete iff E closed in X.

§1.2 Compactness, Separability

↪**Definition 1.7** (Open Cover, Compactness): $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$, where X_λ open in X and Λ an arbitrary index set, an *open cover* of X if for every $x \in X$, $\exists\, \lambda \in \Lambda$ such that $x \in X_\lambda$.

X is *compact* if every open cover of X admits a compact subcover. We say $E \subseteq X$ compact if (E, ρ) compact.

↪**Definition 1.8** (Totally Bounded, ε-nets): (X, ρ) *totally bounded* if $\forall\, \varepsilon > 0$, there is a finite cover of X of balls of radius ε. If $E \subseteq X$, an *ε-net* of E is a collection $\{B(x_i, \varepsilon)\}_{i=1}^N$ such that $E \subseteq \bigcup_{i=1}^N B(x_i, \varepsilon)$ and $x_i \in X$ (note that x_i need not be in E).

↪**Definition 1.9** (Sequentially Compact): (X, ρ) *sequentially compact* if every sequence in X has a convergence subsequence whose limit is in X.

↪**Definition 1.10** (Relatively/Pre- Compact): $E \subseteq X$ *relatively compact* if $\overline{E}$ compact.

↪**Theorem 1.1**: TFAE:
- X complete and totally bounded;
- X compact;
- X sequentially compact.

Remark 1.4: $E \subseteq X$ relatively compact if every sequence in E has a convergent subsequence.

Let $f : (X, \rho) \to (Y, \sigma)$ continuous with (X, ρ) compact. Then,

- $f(X)$ compact in Y;
- if $Y = \mathbb{R}$, the max and min of f over X are achieved;
- f is uniformly continuous.

Let $C(X) := \{f : X \to \mathbb{R} \mid f \text{ continuous}\}$ and $\|f\|_\infty := \max_{x \in X}|f(x)|$ the sup (max, in this case) norm. Then,

↪**Theorem 1.2**: Let (X, ρ) compact. Then, $(C(X), \|\cdot\|_\infty)$ is complete.

Proof. Let $\{f_n\} \subseteq C(X)$ Cauchy with respect to $\|\cdot\|_\infty$. Then, there exists a subsequence $\{f_{n_k}\}$ such that for each $k \geq 1$, $\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq 2^{-k}$ (to construct this subsequence, let $n_1 \geq 1$ be such that $\|f_n - f_{n_1}\|_\infty < \frac{1}{2}$ for all $n \geq n_1$, which exists since $\{f_n\}$ Cauchy. Then, for each $k \geq 1$, define inductively n_{k+1} such that $n_{k+1} > n_k$ and $\|f_n - f_{n_{k+1}}\|_\infty < \frac{1}{2^{k+1}}$ for each $n \geq n_{k+1}$. Then, for any $k \geq 1$, $\|f_{n_{k+1}} - f_{n_k}\|_\infty < 2^{-k}$, since $n_{k+1} > n_k$.).

Let $j \in \mathbb{N}$. Then, for any $k \geq 1$,

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell} 2^{-\ell}$$

and hence for each $x \in X$, with $c_k := f_{n_k}(x)$,

$$|c_{k+j} - c_k| \leq \sum_{\ell=k}^{\infty} 2^{-\ell}.$$

The RHS is the tail of a converging series, and thus $|c_{k+j} - c_k| \to 0$ as $k \to \infty$ i.e. $\{c_k\}$ a Cauchy sequence, in $\mathbb{R}$. $(\mathbb{R}, |\cdot|)$ complete, so $\lim_{k\to\infty} c_k =: f(x)$ exists for each $x \in X$. So, for each $x \in X$, we find

$$|f_{n_k}(x) - f(x)| \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

and since the RHS is independent of x, we may pass to the sup norm, and find

$$\|f_{n_k} - f\|_\infty \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

with the RHS $\to 0$ as $k \to \infty$. Hence, $f_{n_k} \to f$ in $C(X)$ as $k \to \infty$. In other words, we have uniform convergence of $\{f_{n_k}\}$. Each $\{f_{n_k}\}$ continuous, and thus f also continuous, and thus $f \in C(X)$.

It remains to show convergence along the whole sequence. Suppose otherwise. Then, there is some $\alpha > 0$ and a subsequence $\{f_{n_j}\} \subseteq \{f_n\}$ such that $\|f_{n_j} - f\|_\infty >$

$\alpha > 0$ for every $j \geq 1$. Then, let k be sufficiently large such that $\|f - f_{n_k}\|_\infty \leq \frac{\alpha}{2}$. Then, for every $j \geq 1$ and k sufficiently large,

$$\begin{aligned}\|f_{n_j} - f_{n_k}\|_\infty &\geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty \\ &> \alpha - \frac{\alpha}{2} > 0,\end{aligned}$$

which contradicts the Cauchy-ness of $\{f_n\}$, completing the proof. ■

§1.3 Arzelà-Ascoli

The goal in this section is to find conditions for a sequence of functions $\{f_n\} \subseteq C(X)$ to be precompact, namely, to have a uniformly convergent subsequence.

↪**Corollary 1.1**: Any Cauchy sequence converges if it has a convergent subsequence.

Proof. Let $\{x_n\}$ be a Cauchy sequence in a metric space (X, ρ) with convergent subsequence $\left\{x_{n_k}\right\}$ which converges to some $x \in X$. Fix $\varepsilon > 0$. Let $N \geq 1$ be such that if $m, n \geq N$, $\rho(x_n, x_m) < \frac{\varepsilon}{2}$. Let $K \geq 1$ be such that if $k \geq K$, $\rho\left(x_{n_k}, x\right) < \frac{\varepsilon}{2}$. Let $n, n_k \geq \max\{N, K\}$, then

$$\rho(x, x_n) \leq \rho\left(x, x_{n_k}\right) + \rho\left(x_{n_k}, x_n\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

↪**Definition 1.11** (Equicontinuous): A family $\mathcal{F} \subseteq C(X)$ is called *equicontinuous* at $x \in X$ if $\forall\, \varepsilon > 0$ there exists a $\delta = \delta(x, \varepsilon) > 0$ such that if $\rho(x, x') < \delta$ then $|f(x) - f(x')| < \varepsilon$ for every $f \in \mathcal{F}$.

Remark 1.5: $\mathcal{F}$ equicontinuous at x iff every $f \in \mathcal{F}$ share the same modulus of continuity.

↪**Definition 1.12** (Pointwise/uniformly bounded): $\{f_n\}$ pointwise bounded if $\forall\, x \in X$, $\exists\, M(x) > 0$ such that $|f_n(x)| \leq M(x)\, \forall\, n$, and uniformly bounded if such an M exists independent of x.

↪**Lemma 1.1** (Arzelà-Ascoli Lemma): Let X separable and let $\{f_n\} \subseteq C(X)$ be pointwise bounded and equicontinuous. Then, there is a subsequence $\left\{f_{n_k}\right\}$ and a function f which converges pointwise to f on all of X.

Proof. Let $D = \{x_j\}_{j=1}^{\infty} \subseteq X$ be a countable dense subset of X. Since $\{f_n\}$ p.w. bounded, $\{f_n(x_1)\}$ as a sequence of real numbers is bounded and so by the Bolzano-Weierstrass (BW) Theorem there is a convergent subsequence $\left\{f_{n(1,k)}(x_1)\right\}_k$ that converges to some $a_1 \in \mathbb{R}$. Consider now $\left\{f_{n(1,k)}(x_2)\right\}_k$, which is again a bounded sequence of $\mathbb{R}$ and so has a convergent subsequence, call it $\left\{f_{n(2,k)}(x_2)\right\}_k$ which converges to some $a_2 \in \mathbb{R}$. Note that $\left\{f_{n(2,k)}\right\} \subseteq \left\{f_{n(1,k)}\right\}$, so also $f_{n(2,k)}(x_1) \to a_1$ as $k \to \infty$. We can repeat this procedure, producing a sequence of real numbers $\{a_\ell\}$, and for each $j \in \mathbb{N}$ a subsequence $\left\{f_{n(j,k)}\right\}_k \subseteq \{f_n\}$ such that $f_{n(j,k)}(x_\ell) \to a_\ell$ for each $1 \leq \ell \leq j$. Define then

$$f : D \to \mathbb{R}, f(x_j) := a_j.$$

Consider now

$$f_{n_k} := f_{n(k,k)}, k \geq 1,$$

the "diagonal sequence", and remark that $f_{n_k}(x_j) \to a_j = f(x_j)$ as $k \to \infty$ for every $j \geq 1$. Hence, $\left\{f_{n_k}\right\}_k$ converges to f on D, pointwise.

We claim now that $\left\{f_{n_k}\right\}$ converges on all of X to some function $f : X \to \mathbb{R}$, pointwise. Put $g_k := f_{n_k}$ for notational convenience. Fix $x_0 \in X$, $\varepsilon > 0$, and let $\delta > 0$ be such that if $x \in X$ such that $\rho(x, x_0) < \delta$, $|g_k(x) - g_k(x_0)| < \frac{\varepsilon}{3}$ for every $k \geq 1$, which exists by equicontinuity. Since D dense in X, there is some $x_j \in D$ such that $\rho(x_j, x_0) < \delta$. Then, since $g_k(x_j) \to f(x_j)$ (pointwise), $\{g_k(x_j)\}_k$ is Cauchy and so there is some $K \geq 1$ such that for every $k, \ell \geq K$, $|g_\ell(x_j) - g_k(x_j)| < \frac{\varepsilon}{3}$. And hence, for every $k, \ell \geq K$,

$$|g_k(x_0) - g_\ell(x_0)| \leq |g_k(x_0) - g_k(x_j)| + |g_k(x_j) - g_\ell(x_j)| + |g_\ell(x_j) - g_\ell(x_0)| < \varepsilon,$$

so namely $\{g_k(x_0)\}_k$ Cauchy as a sequence in $\mathbb{R}$. Since $\mathbb{R}$ complete, then $\{g_k(x_0)\}_k$ also converges, to, say, $f(x_0) \in \mathbb{R}$. Since x_0 was arbitrary, this means there is some function $f : X \to \mathbb{R}$ such that $g_k \to f$ pointwise on X as we aimed to show. ■

↪**Definition 1.13** (Uniformly Equicontinuous): $\mathcal{F} \subseteq C(X)$ is said to be uniformly equicontinuous if for every $\varepsilon < 0$, there exists a $\delta > 0$ such that $\forall\, x, y \in X$ with $\rho(x, y) < \delta$, $|f(x) - f(y)| < \varepsilon$ for every $f \in \mathcal{F}$. That is, every function in $\mathcal{F}$ has the same modulus of continuity.

↪**Proposition 1.1** (Sufficient Conditions for Uniform Equicontinuity):
1. $\mathcal{F} \subseteq C(X)$ uniformly Lipschitz
2. $\mathcal{F} \subseteq C(X) \cap C^1(X)$ has a uniform L^∞ bound on the first derivative
3. $\mathcal{F} \subseteq C(X)$ uniformly Holder continuous
4. (X, ρ) compact and $\mathcal{F}$ equicontinuous

↪**Theorem 1.3** (Arzelà-Ascoli): Let (X, ρ) a compact metric space and $\{f_n\} \subseteq C(X)$ be a uniformly bounded and (uniformly) equicontinuous family of functions. Then, $\{f_n\}$ is pre-compact in $C(X)$, i.e. there exists $\left\{f_{n_k}\right\} \subseteq \{f_n\}$ such that f_{n_k} is uniformly convergent on X.

Remark 1.6: If $K \subseteq X$ a compact set, then K bounded and closed.

↪**Theorem 1.4**: Let (X, ρ) compact and $\mathcal{F} \subseteq C(X)$. Then, $\mathcal{F}$ a compact subspace of $C(X)$ iff $\mathcal{F}$ closed, uniformly bounded, and (uniformly) equicontinuous.

§1.4 Baire Category Theorem

We'll say a set $E \subseteq X$ *hollow* if $\operatorname{int} E = \varnothing$, or equivalently if E^c dense in X.

↪**Theorem 1.5** (Baire Category Theorem): Let X be a complete metric space.

(a) Let $\{F_n\}$ a collection of closed hollow sets. Then, $\bigcup_{n=1}^{\infty} F_n$ also hollow.

(b) Let $\{O_n\}$ a collection of open dense sets. Then, $\bigcap_{n=1}^{\infty} O_n$ also dense.

↪**Corollary 1.2**: Let X complete and $\{F_n\}$ a sequence of closed sets in X. If $X = \bigcup_{n \geq 1} F_n$, there is some n_0 such that $\operatorname{int}\left(F_{n_0}\right) \neq \varnothing$.

↪**Corollary 1.3**: Let X complete and $\{F_n\}$ a sequence of closed sets in X. Then, $\bigcup_{n=1}^{\infty} \partial F_n$ hollow.

1.4.1 Applications of Baire Category Theorem

↪**Theorem 1.6**: Let $\mathcal{F} \subset C(X)$ where X complete. Suppose $\mathcal{F}$ pointwise bounded. Then, there exists a nonempty, open set $\mathcal{O} \subseteq X$ such that $\mathcal{F}$ uniformly bounded on $\mathcal{O}$.

↪**Theorem 1.7**: Let X complete, and $\{f_n\} \subseteq C(X)$ such that $f_n \to f$ pointwise on X. Then, there exists a dense subset $D \subseteq X$ such that $\{f_n\}$ equicontinuous on D and f continuous on D.

§1.5 Topological Spaces

Throughout, assume $X \neq \varnothing$.

↪**Definition 1.14** (Topology): Let $X \neq \varnothing$. A *topology* $\mathcal{T}$ on X is a collection of subsets of X, called *open sets*, such that

- $X, \varnothing \in \mathcal{T}$;
- If $\{E_n\} \subseteq \mathcal{T}$, $\bigcap_{n=1}^{N} E_n \in \mathcal{T}$ (closed under *finite* intersections);
- If $\{E_n\} \subseteq \mathcal{T}$, $\bigcup_{n} E_n \in \mathcal{T}$ (closed under *arbitrary* unions).

If $x \in X$, a set $E \in \mathcal{T}$ containing x is called a neighborhood of x.

↪**Proposition 1.2**: $E \subseteq X$ open $\Leftrightarrow$ for every $x \in X$, there is a neighborhood of x contained E.

⊛ **Example 1.1**: Every metric space induces a natural topology given by open sets under the metric. The *discrete topology* is given by $\mathcal{T} = 2^X$ (and is actually induced by the discrete metric), and is the largest topology. The *trivial topology* $\{\varnothing, X\}$ is the smallest. The *relative topology* defined on a subset $Y \subseteq X$ is given by $\mathcal{T}_Y := \{E \cap Y : E \in \mathcal{T}\}$.

↪**Definition 1.15** (Base): Given a topological space $(X, \mathcal{T})$, let $x \in X$. A collection $\mathcal{B}_x$ of neighborhoods of x is called a *base* of $\mathcal{T}$ at x if for every neighborhood $\mathcal{U}$ of x, there is a set $B \in \mathcal{B}_x$ such that $B \subseteq \mathcal{U}$.

We say a collection $\mathcal{B}$ a base for all of $\mathcal{T}$ if for every $x \in X$, there is a base for x, $\mathcal{B}_x \subseteq \mathcal{B}$.

↪**Proposition 1.3**: If $(X, \mathcal{T})$ a topological space, then $\mathcal{B} \subseteq \mathcal{T}$ a base for $\mathcal{T} \Leftrightarrow$ every nonempty open set $\mathcal{U} \in \mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.

↪**Proposition 1.4**: $\mathcal{B} \subseteq \mathcal{T}$ a base $\Leftrightarrow$
- $X = \bigcup_{B \in \mathcal{B}} B$
- If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

↪**Definition 1.16**: If $\mathcal{T}_1 \subsetneq \mathcal{T}_2$, we say $\mathcal{T}_1$ *weaker/coarser* and $\mathcal{T}_2$ *stronger/finer*.

Given a subset $S \subseteq 2^X$, define

$$\mathcal{T}(S) = \bigcap \text{all topologies containing } S = \text{unique weakest topology containing } S$$

to be the topology *generated* by S.

↪**Proposition 1.5**: If $S \subseteq 2^X$,

$$\mathcal{T}(S) = \bigcup \{\text{finite intersection of elts of } S\}.$$

↪**Definition 1.17** (Point of closure/accumulation point): If $E \subseteq X, x \in X$, x is called a *point of closure* if $\forall\, \mathcal{U}_x, \mathcal{U}_x \cap E \neq \varnothing$. The collection of all such sets is called the *closure* of E, denote $\overline{E}$. We say E *closed* if $E = \overline{E}$.

↪**Proposition 1.6**: Let $E \subseteq X$, then

- $\overline{E}$ closed,
- $\overline{E}$ is the smallest closed set containing E,
- E open $\Leftrightarrow E^c$ closed.

§1.6 Separation, Countability, Separability

↪**Definition 1.18**: A neighborhood of a set $K \subseteq X$ is any open set containing K.

↪**Definition 1.19** (Notions of Separation): We say $(X, \mathcal{T})$:

- *Tychonoff Separable* if $\forall\, x, y \in X, \exists\, \mathcal{U}_x, \mathcal{U}_y$ such that $y \notin \mathcal{U}_x, x \notin \mathcal{U}_y$
- *Hausdorff Separable* if $\forall\, x, y \in X$ can be separated by two disjoint open sets i.e. $\exists\, \mathcal{U}_x \cap \mathcal{U}_y = \varnothing$
- *Normal* if Tychonoff and in addition any 2 disjoint closed sets can be separated by disjoint neighborhoods.

Remark 1.7: Metric space $\subseteq$ normal space $\subseteq$ Hausdorff space $\subseteq$ Tychonoff space.

↪**Proposition 1.7**: Tychonoff $\Leftrightarrow\ \forall\, x \in X$, $\{x\}$ closed.

↪**Proposition 1.8**: Every metric space normal.

↪**Proposition 1.9**: Let X Tychonoff. Then X normal $\Leftrightarrow\ \forall\, F \subseteq X$ closed and neighborhood $\mathcal{U}$ of F, there exists an open set $\mathcal{O}$ such that

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}.$$

This is called the "nested neighborhood property" of normal spaces.

↪**Definition 1.20** (Separable): A space X is called *separable* if it contains a countable dense subset.

↪**Definition 1.21** (1st, 2nd Countable): A topological space $(X, \mathcal{T})$ is called

- *1st countable* if there is a countable base at each point
- *2nd countable* if there is a countable base for all of $\mathcal{T}$.

⊛ **Example 1.2**: Every metric space is first countable.

↪**Definition 1.22** (Convergence): Let $\{x_n\} \subseteq X$. Then, we say $x_n \to x$ in $\mathcal{T}$ if for every neighborhood $\mathcal{U}_x$, there exists an N such that $\forall\, n \geq N$, $x_n \in \mathcal{U}_x$.

Remark 1.8: In general spaces, such a limit may not be unique. For instance, under the trivial topology, the only nonempty neighborhood is the whole space, so every sequence converges to every point in the space.

↪**Proposition 1.10**: Let $(X, \mathcal{T})$ be Hausdorff. Then, all limits are unique.

↪**Proposition 1.11**: Let X be 1st countable and $E \subseteq X$. Then, $x \in \overline{E} \Leftrightarrow$ there exists $\{x_j\} \subseteq E$ such that $x_j \to x$.

§1.7 Continuity and Compactness

↪**Definition 1.23**: Let $(X, \mathcal{T}), (Y, \mathcal{S})$ be two topological spaces. Then, a function $f : X \to Y$ is said to be continuous at x_0 if for every neighborhood $\mathcal{O}$ of $f(x_0)$ there exists a neighborhood $\mathcal{U}(x_0)$ such that $f(\mathcal{U}) \subseteq \mathcal{O}$. We say f continuous on X if it is continuous at every point in X.

↪**Proposition 1.12**: f continuous $\Leftrightarrow \forall\, \mathcal{O}$ open in Y, $f^{-1}(\mathcal{O})$ open in X.

↪**Definition 1.24** (Weak Topology): Consider $\mathcal{F} := \{f_\lambda : X \to X_\lambda\}_{\lambda \in \Lambda}$ where X, X_λ topological spaces. Then, let

$$S := \{f_\lambda^{-1}(\mathcal{O}_\lambda) \mid f_\lambda \in \mathcal{F}, \mathcal{O}_\lambda \in X_\lambda\} \subseteq X.$$

We say that the topology $\mathcal{T}(S)$ generated by S is the *weak topology* for X induced by the family $\mathcal{F}$.

↪**Proposition 1.13**: The weak topology is the weakest topology in which each f_λ continuous on X.

⊛ **Example 1.3**: The key example of the weak topology is given by the product topology. Consider $\{X_\lambda\}_{\lambda\in\Lambda}$ a collection of topological spaces. We can defined a "natural" topology on the product $X := \prod_{\lambda\in\Lambda} X_\lambda$ by consider the weak topology induced by the family of projection maps, namely, if $\pi_\lambda : X \to X_\lambda$ a coordinate-wise projection and $\mathcal{F} = \{\pi_\lambda : \lambda \in \Lambda\}$, then we say the weak topology induced by $\mathcal{F}$ is the *product topology* on X. In particular, a base for this topology is given, by previous discussions,

$$\mathcal{B} = \left\{\bigcap_{j=1}^{n} \pi_{\lambda_j}^{-1}\left(\mathcal{O}_{\mathcal{J}}\right)\right\} = \left\{\prod_{\lambda\in\Lambda} \mathcal{U}_\lambda : \mathcal{U}_\lambda \text{ open and all by finitely many } U_{\lambda'}s = X_\lambda\right\}.$$

↪**Definition 1.25** (Compactness): A space X is said to be *compact* if every open cover of X admits a finite subcover.

↪**Proposition 1.14**:

- Closed subsets of compact spaces are compact
- X compact ⇔ if $\{F_k\} \subseteq X$-nested and closed, $\cap_{k=1}^{\infty} F_k \neq \emptyset$.
- Continuous images of compact sets are compact
- Continuous real-valued functions on a compact topological space achieve their min, max.

↪**Proposition 1.15**: Let K compact be contained in a Hausdorff space X. Then, K closed in X.

↪**Definition 1.26** (Sequential Compactness): We say $(X, \mathcal{T})$ *sequentially compact* if every sequence in X has a converging subsequence with limit contained in X.

↪**Proposition 1.16**: Let $(X, \mathcal{T})$ second countable. Then, X compact ⇔ sequentially compact.

↪**Theorem 1.8**: If X compact and Hausdorff, X normal.

§1.8 Connected Topological Spaces

↪**Definition 1.27** (Separate): 2 non-empty sets $\mathcal{O}_1, \mathcal{O}_2$ *separate* X if $\mathcal{O}_1$, $\mathcal{O}_2$ disjoint and $X = \mathcal{O}_1 \cup \mathcal{O}_2$.

↪**Definition 1.28** (Connected): We say X *connected* if it cannot be separated.

Remark 1.9: Note that if X can be separated, then $\mathcal{O}_1, \mathcal{O}_2$ are closed as well as open, being complements of each other.

↪**Proposition 1.17**: Let $f : X \to Y$ continuous. Then, if X connected, so is $f(X)$.

Remark 1.10: On $\mathbb{R}$, $C \subseteq \mathbb{R}$ connected $\Leftrightarrow$ an interval $\Leftrightarrow$ convex.

↪**Definition 1.29** (Intermediate Value Property): We say X has the intermediate value property (IVP) if $\forall f \in C(X)$, $f(X)$ an interval.

↪**Proposition 1.18**: X has IVP $\Leftrightarrow$ X connected.

↪**Definition 1.30** (Arcwise/Path Connected): X *arc connected/path connected* if $\forall x, y \in X$, there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x$, $f(1) = y$.

↪**Proposition 1.19**: Arc connected $\Rightarrow$ connected.

§1.9 Urysohn's Lemma and Urysohn's Metrization Theorem

↪**Lemma 1.2** (Urysohn's): Let $A, B \subseteq X$ closed and disjoint subsets of a normal space X. Then, $\forall [a,b] \subseteq \mathbb{R}$, there exists a continuous functions $f : [a,b] \to \mathbb{R}$ such that $f(X) \subseteq [a,b]$, $f|_A = a$ and $f|_B = b$.

Remark 1.11: We have a partial converse of this statement as well:

↪**Proposition 1.20**: Let X Tychonoff and suppose X satisfies the properties of Urysohn's Lemma. Then, X normal.

PROOF. Let A, B be closed nonempty disjoint subsets. Let $f : X \to \mathbb{R}$ continuous such that $f|_A = 0$, $f|_B = 1$ and $0 \leq f \leq 1$. Let I_1, I_2 be two disjoint open intervals in $\mathbb{R}$ with $0 \in I_1$ and $1 \in I_2$. Then, $f^{-1}(I_1)$ open and contains A, and $f^{-1}(I_2)$ open and contains B. Moreover, $f^{-1}(I_1) \cap f^{-1}(I_2) = \varnothing$; hence, $f^{-1}(I_1), f^{-1}(I_2)$ disjoint open neighborhoods of A, B respectively, so indeed X normal. ■

↪**Definition 1.31** (Normally Ascending): Let $(X, \mathcal{T})$ a topological space and $\Lambda \subseteq \mathbb{R}$. A collection of open sets $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ is said to be *normally ascending* if $\forall\, \lambda_1, \lambda_2 \in \Lambda$,

$$\overline{\mathcal{O}_{\lambda_1}} \subseteq \mathcal{O}_{\lambda_2} \text{ if } \lambda_1 < \lambda_2.$$

↪**Lemma 1.3**: Let $\Lambda \subseteq (a, b)$ a dense subset, and let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ a normally ascending collection of subsets of X. Let $f : X \to \mathbb{R}$ defined such that

$$f(x) = \begin{cases} b & \text{if } x \in \left(\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda\right)^c \\ \inf\{\lambda \in \Lambda \mid x \in \mathcal{O}_\lambda\} & \text{else} \end{cases}.$$

Then, f continuous.

↪**Lemma 1.4**: Let X normal, $F \subseteq X$ closed, and $\mathcal{U}$ a neighborhood of F. Then, for any $(a, b) \subseteq \mathbb{R}$, there exists a dense subset $\Lambda \subseteq (a, b)$ and a normally ascending collection $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ such that

$$F \subseteq \mathcal{O}_\lambda \subseteq \overline{\mathcal{O}}_\lambda \subseteq \mathcal{U}, \qquad \forall\, \lambda \in \Lambda.$$

Remark 1.12: This is essentially a generalization of the nested neighborhood property, and indeed the proof essentially just uses this property repeatedly to construct the collection $\{\mathcal{O}_\lambda\}$.

Proof *Of Urysohn's*. Let $F = A$ and $\mathcal{U} = B^c$ as in the previous lemma. Then, there is some dense subset $\Lambda \subseteq (a, b)$ and a normally ascending collection $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ such that $A \subseteq \mathcal{O}_\lambda \subseteq \overline{\mathcal{O}}_\lambda \subseteq B^c$ for every $\lambda \in \Lambda$. Let $f(x)$ as in the previous previous lemma. Then, if $x \in B$, $B \subseteq \left(\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda\right)^c$ and so $f(x) = b$. Otherwise if $x \in A$, then $x \in \bigcap_{\lambda \in \Lambda} \mathcal{O}_\lambda$ and thus $f(x) = \inf\{\lambda \in \Lambda\} = a$. By the first lemma, f continuous, so we are done. ■

↪**Theorem 1.9** (Urysohn's Metrization Theorem): Let X be a second countable topological space. Then, X is metrizable (that is, there exists a metric on X that induces the topology) if and only if X normal.

Remark 1.13: Recall metric $\Rightarrow$ first countable hence not first countable $\Rightarrow$ not metrizable.

§1.10 Stone-Weierstrass Theorem

We need to use the following theorem, which we'll prove later.

↪**Theorem 1.10** (Weierstrass Approximation Theorem): Let $f : [a, b] \to \mathbb{R}$ continuous. Then, for every $\varepsilon > 0$, there exists a polynomial $p(x)$ such that $\|f - p\|_\infty < \varepsilon$.

↪**Definition 1.32** (Algebra, Separation of Points): We call a subset $\mathcal{A} \subseteq C(X)$ an *algebra* if it is a linear subspace that is closed under multiplication (that is, $f, g \in \mathcal{A} \Rightarrow f \cdot g \in \mathcal{A}$).

We say $\mathcal{A}$ *separates points* in X if for every $x, y \in X$, there exists an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

↪**Theorem 1.11** (Stone-Weierstrass): Let X be a compact Hausdorff space. Suppose $\mathcal{A} \subseteq C(X)$ an algebra that separates points and contains constant functions. Then, $\mathcal{A}$ dense in $C(X)$.

We tacitly assume the conditions of the theorem in the following lemmas.

↪**Lemma 1.5**: For every $F \subseteq X$ closed, and every $x_0 \in F^c$, there exists a neighborhood $\mathcal{U}(x_0)$ such that $F \cap \mathcal{U} = \varnothing$ and $\forall \varepsilon > 0$ there is some $h \in \mathcal{A}$ such that $h < \varepsilon$ on $\mathcal{U}$, $h > 1 - \varepsilon$ on F, and $0 \leq h \leq 1$ on X.

In particular, $\mathcal{U}$ is *independent* of choice of ε.

↪**Lemma 1.6**: For every disjoint closed set A, B and $\varepsilon > 0$, there exists $h \in \mathcal{A}$ such that $h|_A < \varepsilon$, $h|_B > 1 - \varepsilon$, and $0 \leq h \leq 1$ on X.

Proof. (Of Stone-Weierstrass) WLOG, assume $f \in C(X)$, $0 \leq f \leq 1$, by replacing with

$$\tilde{f}(x) = \frac{f(x) + \|f\|_\infty}{\|f + \|f\|_\infty\|_\infty}$$

if necessary, since if there exists a $\tilde{g} \in \mathcal{A}$ such that $\|\tilde{f} - \tilde{g}\|_\infty < \varepsilon$, then using the properties of $\mathcal{A}$ we can find some appropriate $g \in \mathcal{A}$ such that $\|f - g\|_\infty < \varepsilon$.

Fix $n \in \mathbb{N}$, and consider the set $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$, and let for $1 \leq j \leq n$

$$A_j := \left\{x \in X \mid f(x) \leq \frac{j-1}{n}\right\}, \qquad B_j := \left\{x \in X \mid f(x) \geq \frac{j}{n}\right\},$$

which are both closed and disjoint. By the lemma, there exists $g_j \in \mathcal{A}$ such that

$$g_j|_{A_j} < \frac{1}{n}, \qquad g_j|_{B_j} > 1 - \frac{1}{n},$$

with $0 \leq g_j \leq 1$. Let then

$$g(x) := \frac{1}{n}\sum_{j=1}^{n} g_j(x) \in \mathcal{A}.$$

We claim then $\|f-g\|_\infty \leq \frac{3}{n}$, which proves the claim by taking n sufficiently large.

Suppose $k \in [1,n]$. If $f(x) \leq \frac{k}{n}$, then

$$g_j(x) = \begin{cases} < \frac{1}{n} \text{ if } j-1 \geq k \\ \leq 1 \;\text{ else} \end{cases},$$

so

$$g(x) = \frac{1}{n}\sum_{j=1}^{n} g_j(x) = \frac{1}{n}\left[\sum_{j=1}^{k} g_j(x) + \sum_{j=k+1}^{n} g_j(x)\right] \leq \frac{1}{n}\left[k + \frac{n-k}{n}\right] \leq \frac{k}{n} + \frac{n-k}{n^2} \leq \frac{k+1}{n}.$$

Similarly if $f(x) \geq \frac{k-1}{n}$, then

$$g_j(x) = \begin{cases} > 1 - \frac{1}{n} \text{ if } j \leq k-1 \\ \geq 0 \qquad\;\; \text{else} \end{cases},$$

so

$$g(x) \geq \frac{1}{n}\sum_{j=1}^{k-1}\left(1-\frac{1}{n}\right) \geq \frac{1}{n}(k-1)\left(1-\frac{1}{n}\right) = \frac{k-1}{n} - \frac{k-1}{n^2} \geq \frac{k-2}{n}.$$

So, we've show that if $\frac{k-1}{n} \leq f(x) \leq \frac{k}{n}$, then $\frac{k-2}{n} \leq g(x) \leq \frac{k+1}{n}$, and so repeating this argument and applying triangle inequality we conclude $\|f-g\|_\infty \leq \frac{3}{n}$. ■

↪**Theorem 1.12** (Borsuk): X compact, Hausdorff and $C(X)$ separable $\Leftrightarrow$ X is metrizable.

§2 Functional Analysis

Here, we will primarily work with a normed vector space (nvs). Moreover, we usually work in:

↪**Definition 2.1** (Banach Space): A normed vector space $(X, \|\cdot\|)$ is a *Banach space* if it is complete as a metric space under the norm-induced metric.

§2.1 Introduction to Linear Operators

↪**Definition 2.2** (Linear Operator, Operator Norm): Let X, Y be vector spaces. Then, a map $T : X \to Y$ is called *linear* if $\forall\, x, y \in X, \alpha, \beta \in \mathbb{R}$, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

If X, Y normed vector spaces, we say T is a bounded linear operator if T linear and the *operator norm*

$$\|T\| = \|T\|_{\mathcal{L}(X,Y)} = \sup_{\substack{x \in X, \\ \|x\|_X \leq 1}} \|Tx\|_Y < \infty$$

is finite. Then, we put

$$\mathcal{L}(X, Y) := \{\text{bounded linear operators } X \to Y\}.$$

↪**Theorem 2.1** (Bounded iff Continuous): If X, Y are nvs, $T \in \mathcal{L}(X, Y)$ iff and only if T is continuous, i.e. if $x_n \to x$ in X, then $Tx_n \to Tx$ in Y.

Proof. If $T \in \mathcal{L}(X, Y)$,

$$\begin{aligned} \|Tx_n - Tx\|_Y &= \|T(x_n - x)\|_Y \\ &= \|x_n - x\|_X \cdot \|\frac{T(x_n - x)}{\|x_n - x\|_X}\|_Y \\ &\leq \underbrace{\|T\|}_{<\infty} \|x_n - x\|_X \to 0, \end{aligned}$$

hence T continuous. Conversely, if T continuous, then by linearity $T0 = 0$, so by continuity, there is some $\delta > 0$ such that $\|Tx\|_Y < 1$ if $\|x\|_X < \delta$. For $x \in X$ nonzero, let $\lambda = \frac{\delta}{\|x\|_X}$. Then, $\|\lambda x\|_X \leq \delta$ so $\|T(\lambda x)\|_Y < 1$, i.e. $\frac{\| T(x)\|_Y \,\delta}{\|x\|_X} < 1$. Hence,

$$\|T\| = \sup_{x \in X : x \neq 0} \frac{\| T(x)\|_Y}{\|x\|_X} \leq \frac{1}{\delta},$$

so $T \in \mathcal{L}(X, Y)$. ■

↪**Proposition 2.1** (Properties of $\mathcal{L}(X, Y)$): If X, Y nvs, $\mathcal{L}(X, Y)$ a nvs, and if X, Y Banach, then so is $\mathcal{L}(X, Y)$.

Proof. (a) For $T, S \in \mathcal{L}(X, Y)$, $\alpha, \beta \in \mathbb{R}$, and $x \in X$, then

$$\begin{aligned} \|(\alpha T + \beta S)(x)\|_Y &\leq |\alpha|\ \|Tx\|_Y + |\beta|\ \|Sx\|_Y \\ &\leq |\alpha|\ \|T\|\ \|x\|_X + |\beta|\ \|T\|\ \|x\|_X. \end{aligned}$$

Dividing both sides by $\|x\|$, we find $\|\alpha T + \beta S\| < \infty$. The same argument gives the triangle inequality on $\|\cdot\|$. Finally, $T = 0$ iff $\|Tx\|_Y = 0$ for every $x \in X$ iff $\|T\| = 0$.

(b) Let $\{T_n\} \subseteq \mathcal{L}(X, Y)$ be a Cauchy sequence. We have that

$$\|T_n x - T_m x\|_Y \leq \|T_n - T_m\|\ \|x\|_X,$$

so in particular the sequence $\{T_n(x)\}$ a Cauchy sequence in Y for any $x \in X$. Y complete so this sequence converges, say $T_n(x) \to y^*$ in Y. Let $T(x) := y^*$ for each x. We claim that $T \in \mathcal{L}(X, Y)$ and that $T_n \to T$ in the operator norm. We check:

$$\begin{aligned} \alpha T(x_1) + \beta T(x_2) &= \lim_{n\to\infty} \alpha T_n(x_1) + \lim_{n\to\infty} \beta T_n(x_2) \\ &= \lim_{n\to\infty} [T_n(\alpha x_1) + T_n(\beta x_2)] \\ &= \lim_{n\to\infty} T_n(\alpha x_1 + \beta x_2) \\ &= T(\alpha x_1 + \beta x_2), \end{aligned}$$

so T linear.

Let now $\varepsilon > 0$ and N such that for every $n \geq N$ and $k \geq 1$ such that $\|T_n - T_{n+k}\| < \frac{\varepsilon}{2}$. Then,

$$\begin{aligned} \|T_n(x) - T_{n+k}(x)\|_Y &= \left\|(T_n - T_{n+k})(x)\right\|_Y \\ &\leq \left\|T_n - T_{n+k}\right\| \|x\|_X \\ &< \frac{\varepsilon}{2} \|x\|_X. \end{aligned}$$

Letting $k \to \infty$, we find that

$$\|T_n(x) - T(x)\|_Y < \frac{\varepsilon}{2} \|x\|_X,$$

so normalizing both sides by $\|x\|_X$, we find $\|T_n - T\| < \frac{\varepsilon}{2}$, and we have convergence. ■

↪**Definition 2.3** (Isomorphism): We say $T \in \mathcal{L}(X, Y)$ an *isomorphism* if T is bijective and $T^{-1} \in \mathcal{L}(Y, X)$. In this case we write $X \simeq Y$, and say X, Y isomorphic.

§2.2 Finite versus Infinite Dimensional

If X a nvs, then we can look for a basis β such that $\text{span}(\beta) = X$. If $\beta = \{e_1, ..., e_n\}$ has no proper subset spanning X, then we say $\dim(X) = n$.

As we saw on homework, any two norms on a finite dimensional space are equivalent.

↪**Corollary 2.1**: (a) Any two nvs of the same finite dimension are isomorphic.

(b) Any finite dimensional space is complete, and so any finite dimensional subspace is closed.

(c) $\overline{B}(0, 1)$ is compact in a finite dimensional space.

Proof. (a) Let $(X, \|\cdot\|)$ have finite dimension n. Then, we claim $(X, \|\cdot\|) \simeq (\mathbb{R}^n, |\cdot|)$. Let $\{e_1, ..., e_n\}$ be a basis for X. Let $T : \mathbb{R}^n \to X$ given by

$$T(x) = \sum_{i=1}^{n} x_i e_i,$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$, which is clearly linear. Moreover,

$$Tx = 0 \Leftrightarrow \sum_{i=1}^{n} x_i e_i = 0 \Leftrightarrow x = 0,$$

so T injective, and so being linear between two spaces of the same dimension gives T surjective. It remains to check boundedness.

First, we claim $x \mapsto \|T(x)\|$ is a norm on $\mathbb{R}^n$. $\|T(x)\| = 0 \Leftrightarrow x = 0$ by the injectivity of T, and the properties $\|T(\lambda x)\| = |\lambda| \ \|Tx\|$ and $\|T(x+y)\| \leq \|Tx\| + \|Ty\|$ follow from linearity of T and the fact that $\| \cdot \|$ already a norm. Hence, $\|T(\cdot)\|$ a norm on $\mathbb{R}^n$ and so equivalent to $| \cdot |$, i.e. there exists constants $C_1, C_2 > 0$ such that

$$C_1 |x| \leq \|T(x)\| \leq C_2 |x|,$$

for every $x \in X$. It follows that $\|T\|$ (operator norm now) is bounded.

Letting $T(x) = y$, we find similarly

$$C_{1'} \|y\| \leq |T^{-1}(y)| \leq C_{2'} \ \|y\|,$$

so $\|T^{-1}\|$ also bounded. Hence, we've shown any n-dimensional space is isomorphic to $\mathbb{R}^n$, so by transitivity of isomorphism any two n-dimensional spaces are isomorphic.

(b) The property of completeness is preserved under isomorphism, so this follows from the previous statement since $\mathbb{R}^n$ complete.

(c) Consider $\overline{B}(0,1) \subseteq X$. Let T be an isomorphism $X \to \mathbb{R}^n$. Then, for $x \in \overline{B}(0,1)$, $\|Tx\| \leq \|T\| < \infty$, so $T\left(\overline{B}(0,1)\right)$ is a bounded subset of $\mathbb{R}^n$, and since T and its inverse continuous, $T\left(\overline{B}(0,1)\right)$ closed in $\mathbb{R}^n$. Hence, $T\left(\overline{B}(0,1)\right)$ closed and bounded hence compact in $\mathbb{R}^n$, so since T^{-1} continuous $T^{-1}\left(T\left(\overline{B}(0,1)\right)\right) = \overline{B}(0,1)$ also compact, in X. ■