MATH580 - Advanced PDEs 1

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§1 Local Existence Theory

§1.1 Terminology

↪**Definition 1.1** (Multiindex): We'll use *multiindex* notation throughout; if working in $\mathbb{R}^n$, we have a multiindex

$$\alpha \equiv (\alpha_1, ..., \alpha_n), \qquad \alpha_i \in \mathbb{Z}_+.$$

The *length* of a multiindex is given

$$|\alpha| \equiv \sum_i \alpha_i,$$

and we'll also write, for $x \in \mathbb{R}^n$,

$$x^\alpha \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Finally, we'll write

$$\partial^\alpha \equiv \partial_{x_1}^{\alpha_1} \circ \cdots \circ \partial_{x_n}^{\alpha_n}$$

for higher-order partial derivatives in mixed directions.

Thus, the most general form of a k-th order PDE in independent variables $x \in \Omega \subset \mathbb{R}^n$ can be written succinctly by

$$F\left(x, (\partial^\alpha u)_{|\alpha| \leq k}\right)) = 0, \qquad F : \Omega \times \mathbb{R}^{N(k)} \to \mathbb{R}, \qquad (\dagger)$$

with $N(k) \equiv \#\{\alpha \mid |\alpha| \leq k\}$.

↪**Definition 1.2** (Solution): We'll define a *(classical/strong) solution* to $(\dagger)$ to be a C^k-map $u : \Omega \to \mathbb{R}$ for which $(\dagger)$ is satisfied for all $x \in \Omega$.

↪**Definition 1.3** (Linearity/Quasilinearity): We say $(\dagger)$ is *linear* if F is affine-linear in $\partial^\alpha u$ for each multiindex, i.e. we may write equivalently

$$L[u] \coloneqq \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha u = f(x),$$

where $L[u] = f(x) \Leftrightarrow F[x, u] = 0$. Similarly, $(\dagger)$ is said to be *quasilinear* if F is affine-linear in the highest order derivatives, i.e. $\partial^\alpha u$ for $|\alpha| = k$. An equivalent form is given by

$$\sum_{|\alpha|=k} a_\alpha\left(x, (\partial^\beta u)_{|\beta| \leq k-1}\right) \partial^\alpha u = b\left(x, (\partial^\beta u)_{|\beta| \leq k-1}\right).$$

↪**Definition 1.4** (Weak Solution): A *weak solution* to a linear PDE $L[u] = f$ is a function $u : \Omega \to \mathbb{R}$ such that

$$\sum_{|\alpha| \leq k} (-1)^{|\alpha|} \langle u, \partial^\alpha a_\alpha \varphi \rangle = \langle f, \varphi \rangle \qquad \forall \varphi \in C_c^\infty(\Omega),$$

with $\langle \cdot, \cdot \rangle$ the regular $L^2(\Omega)$-inner product.

Remark 1.1: Such a notation allows for non-C^k "solutions" to $(\dagger)$ which still have certain properties akin to those described by F. For a motivation of the definition, one need only integrate by parts $L[u] = f$ multiple times, hitting against $\varphi \in C_c^\infty(\Omega)$; if u were a strong solution, one would find the above equation as a result.

↪**Definition 1.5** (Characteristics): Let L be a linear operator associated to a kth-order linear PDE. The *characteristic form* of L is the kth degree homogeneous polynomial defined by

$$\chi_L(x, \xi) \coloneqq \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha.$$

The *characteristic variety* is defined, for a fixed x, as the set of ξ for whcih χ_L vanishes, i.e.

$$\mathrm{char}_x(L) \coloneqq \{\xi \neq 0 \mid \chi_L(x, \xi) = 0\}.$$

Remark 1.2: Suppose $\overline{\xi} = \xi_j e_j \neq 0 \in \mathrm{char}_x(L)$; then since

$$\chi_L\left(x, \overline{\xi}\right) = a_{\overline{\alpha}} \partial_{x_j}^k \xi_j, \qquad \overline{\alpha} \equiv k e_j,$$

then it must be that $a_{\overline{\alpha}} = 0$ at x. Heuristically, one has that L is not "genuinely" kth order in the direction of $\overline{\xi}$.

↪**Definition 1.6** (Elliptic): We say L is *elliptic* at x if $\operatorname{char}_x(L) = \emptyset$.

↪**Proposition 1.1**: $\operatorname{char}_x(L)$ is independent of choice of coordinates.

§1.2 First Order Scalar PDEs

We consider the quasilinear first-order PDE of the form

$$\sum_{i=1}^{n} a_i(x,u)\partial_i u = b(x,u), \qquad (*)$$

subject to the initial condition $u|_S = \varphi$ where $S \subseteq \mathbb{R}^n$ some hypersurface with φ given. We assume a_i, b C^1 in all arguments.

↪**Theorem 1.1**: Let $A(x) = (a_1(x,u), ..., a_n(x,u), b(x,u))$ and $S^* = \{(x, \varphi(x)) : x \in S\} \subseteq \mathbb{R}^{n+1}$. Then, if A nowhere tangent to S^*, then for any sufficiently small neighborhood Ω on S, there exists a unique solution to $(*)$ on Ω.

Proof. Locally, S can be parametrized by

$$(s_1, ..., s_{n-1}) \mapsto g(s) = (g_1(s), ..., g_n(s)).$$

Then, the "transversality condition" (about the tangency of A) can equivalently be written as

$$\det \begin{pmatrix} \partial g_1/\partial s_1 & \dots & \partial g_1/\partial s_{n-1} & a_1(g(s)) \\ \vdots & & \vdots & \vdots \\ \partial g_n/\partial s_1 & \dots & \partial g_n/\partial s_{n-1} & a_n(g(s)) \end{pmatrix} \neq 0.$$

Remark 1.3: In the linear case, one sees that this equivalently means that the normal ν of S is not in $\operatorname{char}_x(L)$; in particular, it is independent of the choice of initial data.

Remark that if we write coordinates $(x_1, ..., x_n, y) \in \mathbb{R}^{n+1}$ and define $F(x,y) = u(x) - y$, then the PDE can be written succinctly as the statement $A \cdot \nabla F = 0$, and that the zero set $F = 0$ gives the graph of the solution u; hence, we essentially need that the vector field A everywhere tangent to the graph of any solution. The idea of our solution is to consider A "originating" at S^*, and "flowing" our solution along the integral curves defined by A to obtain a solution locally.

The integral curves of A are defined by the system of ODEs

$$\begin{cases} \frac{\mathrm{d}x_j}{\mathrm{d}t} = a_j(x,y), \frac{\mathrm{d}y}{\mathrm{d}t} = b(x,y) \\ x_j(s,0) = g_j(s), y(s,0) = \varphi(g(s)) \end{cases} \qquad j = 1, ..., n.$$

By existence/uniqueness theory of ODEs, there is a local solution to this ODE, viewing s as a parameter, inducing a map

$$(s,t) \mapsto (x_1(s,t), ..., x_n(s,t)),$$

which is at least C^1 in s, t by smooth dependence on initial data. By the transversality condition, we may apply inverse function theorem to this mapping to find C^1-inverses $s = s(x), t = t(x)$ with $t(x) = 0$ and $g(s(x)) = 0$ whenever $x \in S$. Define now

$$u(x) := y(t(x), s(x)).$$

We claim this a solution. By the inverse function theorem argument, it certainly satisfies the initial condition, and repeated application of the chain rule shows that the solution satisfies the PDE. ■

We briefly discuss, but don't prove in detail, the fully nonlinear case, i.e.

$$F(x, u, \partial u) = 0,$$

where we assume $F \in C^2$. We approach by analogy. Putting $\xi_i := \frac{\partial u}{\partial x_i}$, then we see F as a function $\mathbb{R}^{2n+1} \to \mathbb{R}$. We seek "characteristic" ODEs akin to those found for the integral curves in the quasilinear case. We naturally take, as in the previous, $\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial F}{\partial \xi_i}$. Applying chain rule, we find that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sum_i \frac{\partial u}{\partial x_i}\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_i \xi_i \frac{\partial F}{\partial \xi_i}.$$

Finally, if we differentiate $F = 0$ w.r.t. x_j, we find

$$0 = \frac{\partial F}{\partial x_j} + \xi_j \frac{\partial F}{\partial y} + \sum_k \frac{\partial F}{\partial \xi_k}\frac{\partial \xi_k}{\partial x_j}$$

whence

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}t} = \sum_k \frac{\partial \xi_j}{\partial x_k}\frac{\partial x_k}{\partial t} = -\frac{\partial F}{\partial x_j} - \xi_j \frac{\partial F}{\partial y}.$$

In summary, this gives a system of $2n+1$ ODEs in (x, y, ξ) variables

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = \frac{\partial F}{\partial \xi_j}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \sum_i \xi_i \frac{\partial F}{\partial \xi_i}$$
$$\frac{\mathrm{d}\xi_j}{\mathrm{d}t} = -\frac{\partial F}{\partial x_j} - \xi_j \frac{\partial F}{\partial y}.$$

After imposing a similar (but slightly more complex) transversality requirement, one can show similarly obtain a solution from this system by an inverse function theorem argument.

In terms of initial conditions, if u is specified on some hypersurface S, we need to lift it to $S^{**} \subseteq \mathbb{R}^{2n+1}$ to “encode” the information of the initial values of u and its derivatives on u.

⊛ **Example 1.1**: Show that

$$\partial_1 u \partial_2 u = u, \qquad u(0, x_2) = x_2^2$$

has solution

$$u(x_1, x_2) = \frac{(x_1 + 4x_2)^2}{16}.$$

⊛ **Example 1.2** (Geodesics): For an invertible matrix $g = (g^{ij})$, solve

$$\sum_{ij} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = 0.$$