MATH455 - Analysis 4

Abstract Metric, Topological Spaces; Functional Analysis.

Based on lectures from Winter 2025 by Prof. Jessica Lin.
Notes by Louis Meunier

Contents

§1 Abstract Metric and Topological Spaces

§1.1 Review of Metric Spaces

Throughout fix X a nonempty set.

↪**Definition 1.1** (Metric): $\rho : X \times X \to \mathbb{R}$ is called a *metric*, and thus (X, ρ) a *metric space*, if for all $x, y, z \in X$,
- $\rho(x, y) \geq 0$,
- $\rho(x, y) = 0 \Leftrightarrow x = y$,
- $\rho(x, y) = \rho(y, x)$, and
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

↪**Definition 1.2** (Norm): Let X a linear space. A function $\| \cdot \| : X \to [0, \infty)$ is called a *norm* if for all $u, v \in X$ and $\alpha \in \mathbb{R}$,
- $\|u\| = 0 \Leftrightarrow u = 0$,
- $\|u + v\| \leq \|u\| + \|v\|$, and
- $\|\alpha u\| = |\alpha| \, \|u\|$.

Remark 1.1: A norm induces a metric by $\rho(x, y) := \|x - y\|$.

↪**Definition 1.3**: Given two metrics ρ, σ on X, we say they are *equivalent* if $\exists\, C > 0$ such that $\frac{1}{C}\sigma(x, y) \leq \rho(x, y) \leq C\sigma(x, y)$ for every $x, y \in X$. A similar definition follows for equivalence of norms.

Given a metric space (X, ρ), then, we have the notion of
- open balls $B(x, r) = \{y \in X : \rho(x, y) < r\}$,
- open sets (subsets of X with the property that for every $x \in X$, there is a constant $r > 0$ such that $B(x, r) \subseteq X$), closed sets, closures, and
- *convergence*.

↪**Definition 1.4** (Convergence): $\{x_n\} \subseteq X$ converges to $x \in X$ if $\lim_{n \to \infty} \rho(x_n, x) = 0$.

We have several (equivalent) notions, then, of continuity; via sequences, $\varepsilon - \delta$ definition, and by pullbacks (inverse images of open sets are open).

↪**Definition 1.5** (Uniform Continuity): $f : (X, \rho) \to (Y, \sigma)$ uniformly continuous if f has a "modulus of continuity", i.e. there is a continuous function $\omega : [0, \infty) \to [0, \infty)$ such that

$$\sigma(f(x_1), f(x_2)) \leq \omega(\rho(x_1, x_2))$$

for every $x_1, x_2 \in X$.

Remark 1.2: For instance, we say f Lipschitz continuous if there is a constant $C > 0$ such that $\omega(\cdot) = C(\cdot)$. Let $\alpha \in (0, 1)$. We say f α-Holder continuous if $\omega(\cdot) = C(\cdot)^{\alpha}$ for some constant C.

↪**Definition 1.6** (Completeness): We say (X, ρ) *complete* if every cauchy sequence in (X, ρ) converges to a point in X.

Remark 1.3: If (X, ρ) complete and $E \subseteq X$, then (E, ρ) is complete iff E closed in X.

§1.2 Compactness, Separability

↪**Definition 1.7** (Open Cover, Compactness): $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$, where X_λ open in X and Λ an arbitrary index set, an *open cover* of X if for every $x \in X$, $\exists \lambda \in \Lambda$ such that $x \in X_\lambda$.

X is *compact* if every open cover of X admits a compact subcover. We say $E \subseteq X$ compact if (E, ρ) compact.

↪**Definition 1.8** (Totally Bounded, ε-nets): (X, ρ) *totally bounded* if $\forall \varepsilon > 0$, there is a finite cover of X of balls of radius ε. If $E \subseteq X$, an ε-*net* of E is a collection $\{B(x_i, \varepsilon)\}_{i=1}^N$ such that $E \subseteq \bigcup_{i=1}^N B(x_i, \varepsilon)$ and $x_i \in X$ (note that x_i need not be in E).

↪**Definition 1.9** (Sequentially Compact): (X, ρ) *sequentially compact* if every sequence in X has a convergent subsequence whose limit is in X.

↪**Definition 1.10** (Relatively/Pre- Compact): $E \subseteq X$ *relatively compact* if $\overline{E}$ compact.

↪**Theorem 1.1**: TFAE:

1. X complete and totally bounded;
2. X compact;
3. X sequentially compact.

Remark 1.4: $E \subseteq X$ relatively compact if every sequence in E has a convergent subsequence.

Let $f : (X, \rho) \to (Y, \sigma)$ continuous with (X, ρ) compact. Then,

- $f(X)$ compact in Y;
- if $Y = \mathbb{R}$, the max and min of f over X are achieved;
- f is uniformly continuous.

Let $C(X) := \{f : X \to \mathbb{R} \mid f \text{ continuous}\}$ and $\|f\|_\infty := \max_{x \in X}|f(x)|$ the sup (max, in this case) norm. Then,

↪**Theorem 1.2**: Let (X, ρ) compact. Then, $(C(X), \|\cdot\|_\infty)$ is complete.

Proof. Let $\{f_n\} \subseteq C(X)$ Cauchy with respect to $\|\cdot\|_\infty$. Then, there exists a subsequence $\left\{f_{n_k}\right\}$ such that for each $k \geq 1$, $\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq 2^{-k}$ (to construct this subsequence, let $n_1 \geq 1$ be such that $\|f_n - f_{n_1}\|_\infty < \frac{1}{2}$ for all $n \geq n_1$, which exists since $\{f_n\}$ Cauchy. Then, for each $k \geq 1$, define inductively n_{k+1} such that $n_{k+1} > n_k$ and $\|f_n - f_{n_{k+1}}\|_\infty < \frac{1}{2^{k+1}}$ for each $n \geq n_{k+1}$. Then, for any $k \geq 1$, $\|f_{n_{k+1}} - f_{n_k}\|_\infty < 2^{-k}$, since $n_{k+1} > n_k$.).

Let $j \in \mathbb{N}$. Then, for any $k \geq 1$,

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell} 2^{-\ell}$$

and hence for each $x \in X$, with $c_k := f_{n_k}(x)$,

$$|c_{k+j} - c_k| \leq \sum_{\ell=k}^{\infty} 2^{-\ell}.$$

The RHS is the tail of a converging series, and thus $|c_{k+j} - c_k| \to 0$ as $k \to \infty$ i.e. $\{c_k\}$ a Cauchy sequence, in $\mathbb{R}$. $(\mathbb{R}, |\cdot|)$ complete, so $\lim_{k\to\infty} c_k =: f(x)$ exists for each $x \in X$. So, for each $x \in X$, we find

$$|f_{n_k}(x) - f(x)| \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

and since the RHS is independent of x, we may pass to the sup norm, and find

$$\|f_{n_k} - f\|_\infty \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

with the RHS $\to 0$ as $k \to \infty$. Hence, $f_{n_k} \to f$ in $C(X)$ as $k \to \infty$. In other words, we have uniform convergence of $\left\{f_{n_k}\right\}$. Each $\left\{f_{n_k}\right\}$ continuous, and thus f also continuous, and thus $f \in C(X)$.

It remains to show convergence along the whole sequence. Suppose otherwise. Then, there is some $\alpha > 0$ and a subsequence $\left\{f_{n_j}\right\} \subseteq \{f_n\}$ such that $\|f_{n_j} - f\|_\infty >$

$\alpha > 0$ for every $j \geq 1$. Then, let k be sufficiently large such that $\|f - f_{n_k}\|_\infty \leq \frac{\alpha}{2}$. Then, for every $j \geq 1$ and k sufficiently large,

$$\begin{aligned}\|f_{n_j} - f_{n_k}\|_\infty &\geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty \\ &> \alpha - \frac{\alpha}{2} > 0,\end{aligned}$$

which contradicts the Cauchy-ness of $\{f_n\}$, completing the proof. ■

↪**Definition 1.11** (Density/Separability): A set $D \subseteq X$ is called *dense* in X if for every nonempty open subset $A \subseteq X$, $D \cap A \neq \varnothing$. We say X *separable* if there is a countable dense subset of X.

Remark 1.5: If A dense in X, then $\overline{A} = X$.

↪**Proposition 1.1**: If X compact, X separable.

Proof. Since X compact, it is totally bounded. So, for $n \in \mathbb{N}$, there is some K_n and $\{x_i\} \subseteq X$ such that $X \subseteq \bigcup_{i=1}^{K_n} B\left(x_i, \frac{1}{n}\right)$. Then, $D = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{K_n} \{x_i\}$ countable and dense in X. ■

§1.3 Arzelà-Ascoli

The goal in this section is to find conditions for a sequence of functions $\{f_n\} \subseteq C(X)$ to be precompact, namely, to have a uniformly convergent subsequence.

↪**Corollary 1.1**: Any Cauchy sequence converges if it has a convergent subsequence.

Proof. Let $\{x_n\}$ be a Cauchy sequence in a metric space (X, ρ) with convergent subsequence $\left\{x_{n_k}\right\}$ which converges to some $x \in X$. Fix $\varepsilon > 0$. Let $N \geq 1$ be such that if $m, n \geq N$, $\rho(x_n, x_m) < \frac{\varepsilon}{2}$. Let $K \geq 1$ be such that if $k \geq K$, $\rho\left(x_{n_k}, x\right) < \frac{\varepsilon}{2}$. Let $n, n_k \geq \max\{N, K\}$, then

$$\rho(x, x_n) \leq \rho\left(x, x_{n_k}\right) + \rho\left(x_{n_k}, x_n\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

↪**Definition 1.12** (Equicontinuous): A family $\mathcal{F} \subseteq C(X)$ is called *equicontinuous* at $x \in X$ if $\forall\, \varepsilon > 0$ there exists a $\delta = \delta(x, \varepsilon) > 0$ such that if $\rho(x, x') < \delta$ then $|f(x) - f(x')| < \varepsilon$ for every $f \in \mathcal{F}$.

Remark 1.6: $\mathcal{F}$ equicontinuous at x iff every $f \in \mathcal{F}$ share the same modulus of continuity.

↪**Definition 1.13** (Pointwise/uniformly bounded): $\{f_n\}$ pointwise bounded if $\forall\, x \in X$, $\exists\, M(x) > 0$ such that $|f_n(x)| \le M(x)\, \forall\, n$, and uniformly bounded if such an M exists independent of x.

↪**Lemma 1.1** (Arzelà-Ascoli Lemma): Let X separable and let $\{f_n\} \subseteq C(X)$ be pointwise bounded and equicontinuous. Then, there is a subsequence $\left\{f_{n_k}\right\}$ and a function f which converges pointwise to f on all of X.

Proof. Let $D = \left\{x_j\right\}_{j=1}^{\infty} \subseteq X$ be a countable dense subset of X. Since $\{f_n\}$ p.w. bounded, $\{f_n(x_1)\}$ as a sequence of real numbers is bounded and so by the Bolzano-Weierstrass (BW) Theorem there is a convergent subsequence $\left\{f_{n(1,k)}(x_1)\right\}_k$ that converges to some $a_1 \in \mathbb{R}$. Consider now $\left\{f_{n(1,k)}(x_2)\right\}_k$, which is again a bounded sequence of $\mathbb{R}$ and so has a convergent subsequence, call it $\left\{f_{n(2,k)}(x_2)\right\}_k$ which converges to some $a_2 \in \mathbb{R}$. Note that $\left\{f_{n(2,k)}\right\} \subseteq \left\{f_{n(1,k)}\right\}$, so also $f_{n(2,k)}(x_1) \to a_1$ as $k \to \infty$. We can repeat this procedure, producing a sequence of real numbers $\{a_\ell\}$, and for each $j \in \mathbb{N}$ a subsequence $\left\{f_{n(j,k)}\right\}_k \subseteq \{f_n\}$ such that $f_{n(j,k)}(x_\ell) \to a_\ell$ for each $1 \le \ell \le j$. Define then

$$f : D \to \mathbb{R}, f(x_j) := a_j.$$

Consider now

$$f_{n_k} := f_{n(k,k)}, k \ge 1,$$

the "diagonal sequence", and remark that $f_{n_k}(x_j) \to a_j = f(x_j)$ as $k \to \infty$ for every $j \ge 1$. Hence, $\left\{f_{n_k}\right\}_k$ converges to f on D, pointwise.

We claim now that $\left\{f_{n_k}\right\}$ converges on all of X to some function $f : X \to \mathbb{R}$, pointwise. Put $g_k := f_{n_k}$ for notational convenience. Fix $x_0 \in X$, $\varepsilon > 0$, and let $\delta > 0$ be such that if $x \in X$ such that $\rho(x, x_0) < \delta$, $|g_k(x) - g_k(x_0)| < \frac{\varepsilon}{3}$ for every $k \ge 1$, which exists by equicontinuity. Since D dense in X, there is some $x_j \in D$ such that $\rho(x_j, x_0) < \delta$. Then, since $g_k(x_j) \to f(x_j)$ (pointwise), $\{g_k(x_j)\}_k$ is Cauchy and so there is some $K \ge 1$ such that for every $k, \ell \ge K$, $|g_\ell(x_j) - g_k(x_j)| < \frac{\varepsilon}{3}$. And hence, for every $k, \ell \ge K$,

$$|g_k(x_0) - g_\ell(x_0)| \le |g_k(x_0) - g_k(x_j)| + |g_k(x_j) - g_\ell(x_j)| + |g_\ell(x_j) - g_\ell(x_0)| < \varepsilon,$$

so namely $\{g_k(x_0)\}_k$ Cauchy as a sequence in $\mathbb{R}$. Since $\mathbb{R}$ complete, then $\{g_k(x_0)\}_k$ also converges, to, say, $f(x_0) \in \mathbb{R}$. Since x_0 was arbitrary, this means there is some function $f : X \to \mathbb{R}$ such that $g_k \to f$ pointwise on X as we aimed to show. ■

↪**Definition 1.14** (Uniformly Equicontinuous): $\mathcal{F} \subseteq C(X)$ is said to be uniformly equicontinuous if for every $\varepsilon < 0$, there exists a $\delta > 0$ such that $\forall\, x, y \in X$ with $\rho(x, y) < \delta$, $|f(x) - f(y)| < \varepsilon$ for every $f \in \mathcal{F}$. That is, every function in $\mathcal{F}$ has the same modulus of continuity.

↪**Proposition 1.2** (Sufficient Conditions for Uniform Equicontinuity):
1. $\mathcal{F} \subseteq C(X)$ uniformly Lipschitz
2. $\mathcal{F} \subseteq C(X) \cap C^1(X)$ has a uniform L^∞ bound on the first derivative
3. $\mathcal{F} \subseteq C(X)$ uniformly Holder continuous
4. (X, ρ) compact and $\mathcal{F}$ equicontinuous

↪**Theorem 1.3** (Arzelà-Ascoli): Let (X, ρ) a compact metric space and $\{f_n\} \subseteq C(X)$ be a uniformly bounded and (uniformly) equicontinuous family of functions. Then, $\{f_n\}$ is pre-compact in $C(X)$, i.e. there exists $\left\{f_{n_k}\right\} \subseteq \{f_n\}$ such that f_{n_k} is uniformly convergent on X.

Proof. Since (X, ρ) compact it is separable and so by the lemma there is a subsequence $\left\{f_{n_k}\right\}$ that converges pointwise on X. Denote by $g_k := f_{n_k}$ for notational convenience.

We claim $\{g_k\}$ uniformly Cauchy. Let $\varepsilon > 0$. By uniform equicontinuity, there is a $\delta > 0$ such that $\rho(x, y) < \delta \Rightarrow |g_k(x) - g_k(y)| < \frac{\varepsilon}{3}$. Since X compact it is totally bounded so there exists $\{x_i\}_{i=1}^N$ such that $X \subseteq \bigcup_{i=1}^N B(x_i, \delta)$. For every $1 \leq i \leq N$, $\{g_k(x_i)\}$ converges by the lemma hence is Cauchy in $\mathbb{R}$. So, there exists a K_i such that for every $k, \ell \geq K_i$ $|g_k(x_i) - g_\ell(x_i)| \leq \frac{\varepsilon}{3}$. Let $K := \max\{K_i\}$. Then for every $\ell, k \leq K$, $|g_k(x_i) - g_\ell(x_i)| \leq \frac{\varepsilon}{3}$ for every $i = 1, ..., N$. So, for all $x \in X$, there is some x_i such that $\rho(x, x_i) < \delta$, and so for every $k, \ell \geq K$,

$$\begin{aligned}
|g_k(x) - g_\ell(x)| \leq\ & |g_k(x) - g_k(x_i)| \\
&+ |g_k(x_i) - g_\ell(x_i)| \\
&+ |g_\ell(x_i) - g_\ell(x)| < \varepsilon,
\end{aligned}$$

the first and last follow by the equicontinuity and the second from the lemma. This holds for every x and thus $\|g_k - g_\ell\|_\infty < \varepsilon$, so $\{g_k\}$ Cauchy in $C(X)$. But $C(X)$ complete so converges in the space. ■

Remark 1.7: If $K \subseteq X$ a compact set, then K bounded and closed.

↪**Theorem 1.4**: Let (X, ρ) compact and $\mathcal{F} \subseteq C(X)$. Then, $\mathcal{F}$ a compact subspace of $C(X)$ iff $\mathcal{F}$ closed, uniformly bounded, and (uniformly) equicontinuous.

PROOF. ($\Leftarrow$) Let $\{f_n\} \subseteq \mathcal{F}$. By Arzelà-Ascoli Theorem, there exists a subsequence $\{f_{n_k}\}$ that converges uniformly to some $f \in C(X)$. Since $\mathcal{F}$ closed, $f \in \mathcal{F}$ and so $\mathcal{F}$ sequentially compact hence compact.

($\Rightarrow$) $\mathcal{F}$ compact so closed and bounded in $C(X)$. To prove equicontinuous, we argue by contradiction. Suppose otherwise, that $\mathcal{F}$ not-equicontinuous at some $x \in X$. Then, there is some $\varepsilon_0 > 0$ and $\{f_n\} \subseteq \mathcal{F}$ and $\{x_n\} \subseteq X$ such that $|f_n(x_n) - f_n(x)| \geq \varepsilon_0$ while $\rho(x, x_n) < \frac{1}{n}$. Since $\{f_n\}$ bounded and $\mathcal{F}$ compact, there is a subsequence $\{f_{n_k}\}$ that converges to f uniformly. Let K be such that $\forall\, k \geq K$, $\|f_{n_k} - f\|_\infty \leq \frac{\varepsilon_0}{3}$. Then,

$$\begin{aligned} |f(x_{n_k}) - f\ | &\geq |\ |f(x_{n_k}) - f_{n_k}(x_{n_k})| - |f_{n_k}(x_{n_k}) - f_{n_k}(x)| - |f_{n_k}(x) - f(x)|\ | \\ &\geq \frac{\varepsilon_0}{3}, \end{aligned}$$

while $\rho(x_{n_k}, x) \leq \frac{1}{n_k}$, so f cannot be continuous at x, a contradiction. ■

§1.4 Baire Category Theorem

We'll say a set $E \subseteq X$ *hollow* if int $E = \varnothing$, or equivalently if E^c dense in X.

↪**Theorem 1.5** (Baire Category Theorem): Let X be a complete metric space.

(a) Let $\{F_n\}$ a collection of closed hollow sets. Then, $\bigcup_{n=1}^\infty F_n$ also hollow.

(b) Let $\{\mathcal{O}_n\}$ a collection of open dense sets. Then, $\bigcap_{n=1}^\infty \mathcal{O}_n$ also dense.

PROOF. Notice that $(a) \Leftrightarrow (b)$ by taking complements. We prove (b).

Put $G := \bigcap_{n=1}^\infty \mathcal{O}_n$. Fix $x \in X$ and $r > 0$, then to show density of G is to show $G \cap B(x,r) \neq \varnothing$.

Since $\mathcal{O}_1$ dense, then $\mathcal{O}_1 \cap B(x,r)$ nonempty and in particular open. So, let $x_1 \in X$ and $r_1 < \frac{1}{2}$ such that $\overline{B}(x, r_1) \subseteq B(x, 2r_1) \subseteq \mathcal{O}_1 \cap B(x,r)$.

Similarly, since $\mathcal{O}_2$ dense, $\mathcal{O}_2 \cap B(x_1, r_1)$ open and nonempty so there exists $x_2 \in X$ and $r_2 < 2^{-2}$ such that $\overline{B}(x_2, r_2) \subseteq \mathcal{O}_2 \cap B(x_1, r_1)$.

Repeat in this manner to find $x_n \in X$ with $r_n < 2^{-n}$ such that $\overline{B}(x_n, r_n) \subseteq \mathcal{O}_n \cap B(x_{n-1}, r_{n-1})$ for any $n \in \mathbb{N}$. This creates a sequence of sets

$$\overline{B}(x_1, r_1) \supseteq \overline{B}(x_2, r_2) \supseteq \cdots,$$

with $r_n \to 0$. Hence, the sequence of points $\{x_n\}$ Cauchy and since X complete, $x_j \to x_0 \in X$, so in particular

$$\{x_0\} = \bigcap_{n=1}^\infty \overline{B}(x_n, r_n),$$

hence $x_0 \in \mathcal{O}_n$ for every n and thus $G \cap B(x,r)$ nonempty. ■

↪**Corollary 1.2**: Let X complete and $\{F_n\}$ a sequence of closed sets in X. If $X = \bigcup_{n\geq 1} F_n$, there is some n_0 such that $\operatorname{int}(F_{n_0}) \neq \varnothing$.

PROOF. If not, violates BCT since X is not hollow in itself. ■

↪**Corollary 1.3**: Let X complete and $\{F_n\}$ a sequence of closed sets in X. Then, $\bigcup_{n=1}^{\infty} \partial F_n$ hollow.

PROOF. We claim $\text{int}(\partial F_n) = \varnothing$. Suppose not, then there exists some $B(x_0, r) \subseteq \partial F_n$. Then $x_0 \in \partial F_n$ but $B(x_0, r) \cap F_n^c = \varnothing$, a contradiction. So, since ∂F_n closed and $\partial F_n \cap B(x_0, r) = \varnothing$ for every such ball, by BCT $\bigcup_{n=1}^{\infty} \partial F_n$ must be hollow. ■

1.4.1 Applications of Baire Category Theorem

↪**Theorem 1.6**: Let $\mathcal{F} \subset C(X)$ where X complete. Suppose $\mathcal{F}$ pointwise bounded. Then, there exists a nonempty, open set $\mathcal{O} \subseteq X$ such that $\mathcal{F}$ uniformly bounded on $\mathcal{O}$.

PROOF. Let

$$\begin{aligned} E_n &:= \{x \in X : |f(x)| \leq n \,\forall\, f \in \mathcal{F}\} \\ &= \bigcap_{f \in \mathcal{F}} \underbrace{\{x : |f(x)| \leq n\}}_{\text{closed}}. \end{aligned}$$

Since $\mathcal{F}$ pointwise bounded, for every $x \in X$ there is some $M_x > 0$ such that $|f(x)| \leq M_x$ for every $f \in \mathcal{F}$. Hence, for every $n \in \mathbb{N}$ such that $n \geq M_x$, $x \in E_n$ and thus $X = \bigcup_{n=1}^{\infty} E_n$.

E_n closed and hence by the previous corollaries there is some n_0 such that $\text{int}\left(E_{n_0}\right) \neq \varnothing$ and hence there is some $r > 0$ and $x_0 \in X$ such that $B(x_0, r) \subseteq E_{n_0}$. Then, for every $x \in B(x_0, r)$, $|f(x)| \leq n_0$ for every $f \in \mathcal{F}$, which gives our desired nonempty open set upon which $\mathcal{F}$ uniformly bounded. ■

↪**Theorem 1.7**: Let X complete, and $\{f_n\} \subseteq C(X)$ such that $f_n \to f$ pointwise on X. Then, there exists a dense subset $D \subseteq X$ such that $\{f_n\}$ equicontinuous on D and f continuous on D.

PROOF. For $m, n \in \mathbb{N}$, let

$$\begin{aligned} E(m, n) &:= \left\{x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \,\forall\, j, k \geq n\right\} \\ &= \bigcap_{j,k \geq n} \left\{x : |f_j(x) - f_k(x)| \leq \frac{1}{m}\right\}. \end{aligned}$$

The union of the boundaries of these sets are hollow, hence $D := \left(\bigcup_{m,n \geq 1} \partial E(m, n)\right)^c$ is dense. Then, if $x \in D \cap E(m, n)$, then $x \in (\partial E(m, n))^c$ implies $x \in \text{int}(E(m, n))$.

We claim $\{f_n\}$ equicontinuous on D. Let $x_0 \in D$ and $\varepsilon > 0$. Let $\frac{1}{m} \leq \frac{\varepsilon}{4}$. Then, since $\{f_n(x_0)\}$ convergent it is therefore Cauchy (in $\mathbb{R}$). Hence, there is some N such that

$|f_j(x_0) - f_k(x_0)| \leq \frac{1}{m}$ for every $j, k \geq N$, so $x_0 \in D \cap E(m, N)$ hence $x_0 \in \operatorname{int}(E(m, N))$.

Let $B(x_0, r) \subseteq E(m, N)$. Since f_N continuous at x_0 there is some $\delta > 0$ such that $\delta < r$ and

$$|f_N(x) - f_N(x_0)| < \frac{1}{m} \,\forall\, x \in B(x_0, \delta),$$

and hence

$$\begin{aligned}|f_j(x) - f_j(x_0)| &\leq |f_j(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f_j(x_0)| \\ &\leq \frac{3}{m} \leq \frac{3}{4}\varepsilon,\end{aligned}$$

for every $x \in B(x_0, \delta)$ and $j \geq N$, where the first, last bounds come from Cauchy and the middle from continuity of f_N. Hence, we've show $\{f_n\}$ equicontinuous at x_0 since δ was independent of f.

In particular, this also gives for every $x \in B(x_0, \delta)$ the limit

$$\frac{3}{4}\varepsilon > \lim_{j \to \infty} |f_j(x) - f_j(x_0)| = |f(x) - f(x_0)|,$$

so f continuous on D. ■

§1.5 Topological Spaces

Throughout, assume $X \neq \varnothing$.

↪**Definition 1.15** (Topology): Let $X \neq \varnothing$. A *topology* $\mathcal{T}$ on X is a collection of subsets of X, called *open sets*, such that

- $X, \varnothing \in \mathcal{T}$;
- If $\{E_n\} \subseteq \mathcal{T}$, $\bigcap_{n=1}^{N} E_n \in \mathcal{T}$ (closed under *finite* intersections);
- If $\{E_n\} \subseteq \mathcal{T}$, $\bigcup_n E_n \in \mathcal{T}$ (closed under *arbitrary* unions).

If $x \in X$, a set $E \in \mathcal{T}$ containing x is called a neighborhood of x.

↪**Proposition 1.3**: $E \subseteq X$ open $\Leftrightarrow$ for every $x \in E$, there is a neighborhood of x contained in E.

Proof. $\Rightarrow$ is trivial by taking the neighborhood to be E itself. $\Leftarrow$ follows from the fact that, if for each x we let $\mathcal{U}_x$ a neighborhood of x contained in E, then

$$E = \bigcup_{x \in E} \mathcal{U}_x,$$

so E open being a union of open sets. ■

⊛ **Example 1.1**: Every metric space induces a natural topology given by open sets under the metric. The *discrete topology* is given by $\mathcal{T} = 2^X$ (and is actually induced by the discrete metric), and is the largest topology. The *trivial topology* $\{\varnothing, X\}$ is the smallest. The *relative topology* defined on a subset $Y \subseteq X$ is given by $\mathcal{T}_Y := \{E \cap Y : E \in \mathcal{T}\}$.

↪**Definition 1.16** (Base): Given a topological space $(X, \mathcal{T})$, let $x \in X$. A collection $\mathcal{B}_x$ of neighborhoods of x is called a *base* of $\mathcal{T}$ at x if for every neighborhood $\mathcal{U}$ of x, there is a set $B \in \mathcal{B}_x$ such that $B \subseteq \mathcal{U}$.

We say a collection $\mathcal{B}$ a base for all of $\mathcal{T}$ if for every $x \in X$, there is a base for x, $\mathcal{B}_x \subseteq \mathcal{B}$.

↪**Proposition 1.4**: If $(X, \mathcal{T})$ a topological space, then $\mathcal{B} \subseteq \mathcal{T}$ a base for $\mathcal{T}$ ⇔ every nonempty open set $\mathcal{U} \in \mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.

Proof. ⇒ If $\mathcal{U}$ open, then for $x \in \mathcal{U}$ there is some basis element B_x contained in $\mathcal{U}$. So in particular $\mathcal{U} = \bigcup_{x \in \mathcal{U}} B_x$.

⇐ Let $x \in \mathcal{U}$ and $\mathcal{B}_x := \{B \in \mathcal{B} \mid x \in B\}$. Then, for every neighborhood of x, there is some B in $\mathcal{B}_x$ such that $B \subseteq \mathcal{U}$ so $\mathcal{B}_x$ a base for $\mathcal{T}$ at x. ■

Remark 1.8: A base $\mathcal{B}$ defines a unique topology, $\{\varnothing, \cup \mathcal{B}_x\}$.

↪**Proposition 1.5**: $\mathcal{B} \subseteq 2^X$ a base for a topology on X ⇔

- $X = \bigcup_{B \in \mathcal{B}} B$
- If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

Proof. (⇒) If $\mathcal{B}$ a base, then X open so $X = \cup_B B$. If $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2$ open so there must exist some $B \subseteq B_1 \cap B_2$ in $\mathcal{B}$.

(⇐) Let

$$\mathcal{T} = \{\mathcal{U} \mid \forall\, x \in \mathcal{U}, \exists\, B \in \mathcal{B} \text{ with } x \in B \subseteq \mathcal{U}\}.$$

One can show this a topology on X with $\mathcal{B}$ as a base. ■

↪**Definition 1.17**: If $\mathcal{T}_1 \subsetneq \mathcal{T}_2$, we say $\mathcal{T}_1$ *weaker/coarser* and $\mathcal{T}_2$ *stronger/finer*.

Given a subset $S \subseteq 2^X$, define

$$\mathcal{T}(S) = \bigcap \text{all topologies containing } S = \text{unique weakest topology containing } S$$

to be the topology *generated* by S.

↪**Proposition 1.6**: If $S \subseteq 2^X$,

$$\mathcal{T}(S) = \bigcup\{\text{finite intersections of elts of } S\}.$$

We call S a "subbase" for $\mathcal{T}(S)$ (namely, we allow finite intersections of elements in S to serve as a base for $\mathcal{T}(S)$).

Proof. Let $\mathcal{B} := \{X, \text{finite intersections of elements of } S\}$. We claim this a base for $\mathcal{T}(S)$. ■

↪**Definition 1.18** (Point of closure/accumulation point): If $E \subseteq X, x \in X$, x is called a *point of closure* if $\forall\, \mathcal{U}_x, \mathcal{U}_x \cap E \neq \varnothing$. The collection of all such sets is called the *closure* of E, denote $\overline{E}$. We say E *closed* if $E = \overline{E}$.

↪**Proposition 1.7**: Let $E \subseteq X$, then

- $\overline{E}$ closed,
- $\overline{E}$ is the smallest closed set containing E,
- E open $\Leftrightarrow E^c$ closed.

§1.6 Separation, Countability, Separability

↪**Definition 1.19**: A neighborhood of a set $K \subseteq X$ is any open set containing K.

↪**Definition 1.20** (Notions of Separation): We say $(X, \mathcal{T})$:

- *Tychonoff Separable* if $\forall\, x, y \in X, \exists\, \mathcal{U}_x, \mathcal{U}_y$ such that $y \notin \mathcal{U}_x, x \notin \mathcal{U}_y$
- *Hausdorff Separable* if $\forall\, x, y \in X$ can be separated by two disjoint open sets i.e. $\exists\, \mathcal{U}_x \cap \mathcal{U}_y = \varnothing$
- *Normal* if Tychonoff and in addition any 2 disjoint closed sets can be separated by disjoint neighborhoods.

Remark 1.9: Metric space $\subseteq$ normal space $\subseteq$ Hausdorff space $\subseteq$ Tychonoff space.

↪**Proposition 1.8**: Tychonoff $\Leftrightarrow \forall\, x \in X$, $\{x\}$ closed.

Proof. For every $x \in X$,

$$\begin{aligned}\{x\} \text{ closed} &\Leftrightarrow \{x\}^c \text{ open}\\ &\Leftrightarrow \forall\, y \in \{x\}^c, \exists\, \mathcal{U}_y \subseteq \{x\}^c\\ &\Leftrightarrow \forall\, y \neq x, \exists\, \mathcal{U}_y \text{ s.t. } x \notin \mathcal{U}_y,\end{aligned}$$

and since this holds for every x, X Tychonoff. ■

↪**Proposition 1.9**: Every metric space normal.

Proof. Define, for $F \subseteq X$, the function

$$\operatorname{dist}(F, x) := \inf\{\rho(x, x') \mid x' \in F\}.$$

Notice that if F closed and $x \notin F$, then $\operatorname{dist}(F, x) > 0$ (since F^c open so there exists some $B(x, \varepsilon) \subseteq F^c$ so $\rho(x, x') \geq \varepsilon$ for every $x' \in F$). Let F_1, F_2 be closed disjoint sets, and define

$$\begin{aligned} \mathcal{O}_1 &:= \{x \in X \mid \operatorname{dist}(F_1, x) < \operatorname{dist}(F_2, x)\}, \\ \mathcal{O}_2 &:= \{x \in X \mid \operatorname{dist}(F_1, x) > \operatorname{dist}(F_2, x)\}. \end{aligned}$$

Then, $F_1 \subseteq \mathcal{O}_1, F_2 \subseteq \mathcal{O}_2$, and $\mathcal{O}_1 \cap \mathcal{O}_2 = \varnothing$. If we show $\mathcal{O}_1, \mathcal{O}_2$ open, we'll be done.

Let $x \in \mathcal{O}_1$ and $\varepsilon > 0$ such that $\operatorname{dist}(F_1, x) + \varepsilon \leq \operatorname{dist}(F_2, x)$. I claim that $B\left(x, \frac{\varepsilon}{5}\right) \subseteq \mathcal{O}_1$. Let $y \in B\left(x, \frac{\varepsilon}{5}\right)$. Then,

$$\begin{aligned} \operatorname{dist}(F_2, y) &\geq \rho(y, y') - \frac{\varepsilon}{5} && \text{for some } y' \in F_2 \\ &\geq \rho(x, y') - \rho(x, y) + \frac{\varepsilon}{5} && \text{reverse triangle inequality} \\ &\geq \operatorname{dist}(F_2, x) - \frac{2\varepsilon}{5} \\ &\geq \operatorname{dist}(F_1, x) + \varepsilon - \frac{2\varepsilon}{5} \\ &\geq \rho(x, \tilde{y}) + \frac{2\varepsilon}{5} && \text{for some } \tilde{y} \in F_1 \\ &\geq \rho(y, \tilde{y}) - \rho(y, x) + \frac{2\varepsilon}{5} && \text{reverse triangle inequality} \\ &\geq \rho(y, \tilde{y}) - \frac{\varepsilon}{5} + \frac{2\varepsilon}{5} \\ &\geq \operatorname{dist}(F_1, y) + \frac{\varepsilon}{5} > \operatorname{dist}(F_1, y), \end{aligned}$$

hence, $y \in \mathcal{O}_1$ and thus $\mathcal{O}_1$ open. Similar proof follows for $\mathcal{O}_2$. ■

↪**Proposition 1.10**: Let X Tychonoff. Then X normal $\Leftrightarrow \forall\, F \subseteq X$ closed and neighborhood $\mathcal{U}$ of F, there exists an open set $\mathcal{O}$ such that

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}.$$

This is called the "nested neighborhood property" of normal spaces.

Proof. ($\Rightarrow$) Let F closed and $\mathcal{U}$ a neighborhood of F. Then, F and $\mathcal{U}^c$ closed disjoint sets so by normality there exists $\mathcal{O}, \mathcal{V}$ disjoint open neighborhoods of $F, \mathcal{U}^c$ respectively. So, $\mathcal{O} \subseteq \mathcal{V}^c$ hence $\overline{\mathcal{O}} \subseteq \overline{\mathcal{V}}^c = \mathcal{V}^c$ and thus

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{V}^c \subseteq \mathcal{U}.$$

($\Leftarrow$) Let A, B be disjoint closed sets. Then, B^c open and moreover $A \subseteq B^c$. Hence, there exists some open set $\mathcal{O}$ such that $A \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq B^c$, and thus $B \subseteq \overline{\mathcal{O}}^c$. Then, $\mathcal{O}$ and $\overline{\mathcal{O}}^c$ are disjoint open neighborhoods of A, B respectively so X normal. ■

↪**Definition 1.21** (Separable): A space X is called *separable* if it contains a countable dense subset.

↪**Definition 1.22** (1st, 2nd Countable): A topological space $(X, \mathcal{T})$ is called

- *1st countable* if there is a countable base at each point
- *2nd countable* if there is a countable base for all of $\mathcal{T}$.

⊛ **Example 1.2**: Every metric space is first countable; for $x \in X$ let $\mathcal{B}_x = \left\{B\left(x, \frac{1}{n}\right) \mid n \in \mathbb{N}\right\}$.

↪**Proposition 1.11**: Every 2nd countable space is separable.

↪**Definition 1.23** (Convergence): Let $\{x_n\} \subseteq X$. Then, we say $x_n \to x$ in $\mathcal{T}$ if for every neighborhood $\mathcal{U}_x$, there exists an N such that $\forall\, n \geq N$, $x_n \in \mathcal{U}_x$.

Remark 1.10: In general spaces, such a limit may not be unique. For instance, under the trivial topology, the only nonempty neighborhood is the whole space, so every sequence converges to every point in the space.

↪**Proposition 1.12**: Let $(X, \mathcal{T})$ be Hausdorff. Then, all limits are unique.

Proof. Suppose otherwise, that $x_n \to$ both x and y. If $x \neq y$, then since X Hausdorff there are disjoint neighborhoods $\mathcal{U}_x, \mathcal{U}_y$ containing x, y. But then x_n cannot be on both $\mathcal{U}_x$ and $\mathcal{U}_y$ for sufficiently large n, contradiction. ■

↪**Proposition 1.13**: Let X be 1st countable and $E \subseteq X$. Then, $x \in \overline{E} \Leftrightarrow$ there exists $\{x_j\} \subseteq E$ such that $x_j \to x$.

Proof. ($\Rightarrow$) Let $\mathcal{B}_x = \{B_j\}$ be a base for X at $x \in \overline{E}$. Wlog, $B_j \supseteq B_{j+1}$ for every $j \geq 1$ (by replacing with intersections, etc if necessary). Hence, $B_j \cap E \neq \varnothing$ for every j. Let $x_j \in B_j \cap E$, then by the nesting property $x_j \to x$ in $\mathcal{T}$.

($\Leftarrow$) Suppose otherwise, that $x \notin \overline{E}$. Let $\{x_j\} \in E_j$. Then, $\overline{E}^c$ open, and contains x. Then, $\overline{E}^c$ a neighborhood of x but does not contain any x_j so $x_j \not\to x$. ■

§1.7 Continuity and Compactness

↪**Definition 1.24**: Let $(X, \mathcal{T}), (Y, \mathcal{S})$ be two topological spaces. Then, a function $f : X \to Y$ is said to be continuous at x_0 if for every neighborhood $\mathcal{O}$ of $f(x_0)$ there exists a neighborhood $\mathcal{U}(x_0)$ such that $f(\mathcal{U}) \subseteq \mathcal{O}$. We say f continuous on X if it is continuous at every point in X.

↪**Proposition 1.14**: f continuous $\Leftrightarrow$ $\forall\, \mathcal{O}$ open in Y, $f^{-1}(\mathcal{O})$ open in X.

↪**Definition 1.25** (Weak Topology): Consider $\mathcal{F} := \{f_\lambda : X \to X_\lambda\}_{\lambda \in \Lambda}$ where X, X_λ topological spaces. Then, let

$$S := \{f_\lambda^{-1}(\mathcal{O}_\lambda) \mid f_\lambda \in \mathcal{F}, \mathcal{O}_\lambda \in X_\lambda\} \subseteq X.$$

We say that the topology $\mathcal{T}(S)$ generated by S is the *weak topology* for X induced by the family $\mathcal{F}$.

↪**Proposition 1.15**: The weak topology is the weakest topology in which each f_λ continuous on X.

⊛ **Example 1.3**: The key example of the weak topology is given by the product topology. Consider $\{X_\lambda\}_{\lambda \in \Lambda}$ a collection of topological spaces. We can defined a "natural" topology on the product $X := \prod_{\lambda \in \Lambda} X_\lambda$ by consider the weak topology induced by the family of projection maps, namely, if $\pi_\lambda : X \to X_\lambda$ a coordinate-wise projection and $\mathcal{F} = \{\pi_\lambda : \lambda \in \Lambda\}$, then we say the weak topology induced by $\mathcal{F}$ is the *product topology* on X. In particular, a base for this topology is given, by previous discussions,

$$\mathcal{B} = \left\{ \bigcap_{j=1}^{n} \pi_{\lambda_j}^{-1}(\mathcal{O}_j) \right\} = \left\{ \prod_{\lambda \in \Lambda} \mathcal{U}_\lambda : \mathcal{U}_\lambda \text{ open and all by finitely many } U_\lambda\text{'}s = X_\lambda \right\}.$$

↪**Definition 1.26** (Compactness): A space X is said to be *compact* if every open cover of X admits a finite subcover.

↪**Proposition 1.16**:
- Closed subsets of compact spaces are compact
- X compact $\Leftrightarrow$ if $\{F_k\} \subseteq X$-nested and closed, $\cap_{k=1}^{\infty} F_k \neq \varnothing$.
- Continuous images of compact sets are compact
- Continuous real-valued functions on a compact topological space achieve their min, max.

↪**Proposition 1.17**: Let K compact be contained in a Hausdorff space X. Then, K closed in X.

Proof. We show K^c open. Let $y \in K^c$. Then for every $x \in K$, there exists disjoint open sets $\mathcal{U}_{xy}, \mathcal{O}_{xy}$ containing y, x respectively. Then, it follows that $\{\mathcal{O}_{xy}\}_{x \in K}$ an open cover of K, and since K compact there must exist some finite subcover, $K \subseteq \bigcup_{i=1}^{N} \mathcal{O}_{x_i y}$. Let $E := \bigcap_{i=1}^{N} \mathcal{U}_{x_i y}$. Then, E is an open neighborhood of y with $E \cap \mathcal{O}_{x_i y} = \varnothing$ for every $i = 1, \ldots N$. Thus, $E \subseteq \bigcap_{i=1}^{N} \mathcal{O}_{x_i y}^c = \left(\bigcup_{i=1}^{N} \mathcal{O}_{x_i y}\right)^c \subseteq K^c$ so since y was arbitrary K^c open. ■

↪**Definition 1.27** (Sequential Compactness): We say $(X, \mathcal{T})$ *sequentially compact* if every sequence in X has a converging subsequence with limit contained in X.

↪**Proposition 1.18**: Let $(X, \mathcal{T})$ second countable. Then, X compact $\Leftrightarrow$ sequentially compact.

Proof. ($\Rightarrow$) Let $\{x_k\} \subseteq X$ and put $F_n := \overline{\{x_k \mid k \geq n\}}$. Then, $\{F_n\}$ defines a sequence of closed and nested subsets of X and, since X compact, $\bigcap_{n=1}^{\infty} F_n$ nonempty. Let x_0 in this intersection. Since X 2nd and so in particular 1st countable, let $\{B_j\}$ a (wlog nested) countable base at x_0. $x_0 \in F_n$ for every $n \geq 1$ so each B_j must intersect some F_n. Let n_j be an index such that $x_{n_j} \in B_j$. Then, if $\mathcal{U}$ a neighborhood of x_0, there exists some N such that $B_j \subseteq \mathcal{U}$ for every $j \geq N$ and thus $\left\{x_{n_j}\right\} \subseteq B_N \subseteq \mathcal{U}$, so $x_{n_j} \to x_0$ in X.

($\Leftarrow$) Remark that since X second countable, every open cover of X certainly has a countable subcover by intersecting a given cover with our countable basis. So, assume we have a countable cover $X \subseteq \bigcup_{n=1}^{\infty} \mathcal{O}_n$ and suppose towards a contradiction that no finite subcover exists. Then, for every $n \geq 1$, there exists some $m(n) \geq n$ such that $\mathcal{O}_{m(n)} \setminus \bigcup_{i=1}^{n} \mathcal{O}_i \neq \varnothing$. Let x_n in this set for every $n \geq 1$. Since X sequentially compact, there exists a convergent subsequence $\left\{x_{n_k}\right\} \subseteq \{x_n\}$ such that $x_{n_k} \to x_0$ in X, so there exists some $\mathcal{O}_N$ such that $x_0 \in \mathcal{O}_N$. But by construction, $x_{n_k} \notin \mathcal{O}_N$ if $n_k \geq N$, and we have a contradiction. ■

↪**Theorem 1.8**: If X compact and Hausdorff, X normal.

Proof. We show that any closed set F and any point $x \notin F$ can be separated by disjoint open sets. Then, the proof in the more general case follows.

For each $y \in X$, X is Hausdorff so there exists disjoint open neighborhoods $\mathcal{O}_{xy}$ and $\mathcal{U}_{xy}$ of x, y respectively. Then, $\{\mathcal{U}_{xy} \mid y \in F\}$ defines an open cover of F. Since F closed and thus, being a subset of a compact space, compact, there exists a finite subcover $F \subseteq \bigcup_{i=1}^{N} \mathcal{U}_{xy_i}$. Put $\mathcal{N} := \bigcap_{i=1}^{N} \mathcal{O}_{xy_i}$. This is an open set containing x, with $\mathcal{N} \cap \bigcup_{i=1}^{N} \mathcal{U}_{xy_i} = \varnothing$ hence F and x separated by $\mathcal{N}, \bigcup_{i=1}^{N} \mathcal{U}_{xy_i}$. ■

§1.8 Connected Topological Spaces

↪**Definition 1.28** (Separate): 2 non-empty sets $\mathcal{O}_1, \mathcal{O}_2$ *separate* X if $\mathcal{O}_1, \mathcal{O}_2$ disjoint and $X = \mathcal{O}_1 \cup \mathcal{O}_2$.

↪**Definition 1.29** (Connected): We say X *connected* if it cannot be separated.

Remark 1.11: Note that if X can be separated, then $\mathcal{O}_1, \mathcal{O}_2$ are closed as well as open, being complements of each other.

↪**Proposition 1.19**: Let $f : X \to Y$ continuous. Then, if X connected, so is $f(X)$.

Proof. Suppose otherwise, that $f(X) = \mathcal{O}_1 \sqcup \mathcal{O}_2$ for nonempty, open, disjoint $\mathcal{O}_1, \mathcal{O}_2$. Then, $X = f^{-1}(\mathcal{O}_1) \sqcup f^{-1}(\mathcal{O}_2)$, and each of these inverse images remain nonempty and open in X, so this a contradiction to the connectedness of X. ■

Remark 1.12: On $\mathbb{R}$, $C \subseteq \mathbb{R}$ connected $\Leftrightarrow$ an interval $\Leftrightarrow$ convex.

↪**Definition 1.30** (Intermediate Value Property): We say X has the intermediate value property (IVP) if $\forall f \in C(X)$, $f(X)$ an interval.

↪**Proposition 1.20**: X has IVP $\Leftrightarrow$ X connected.

Proof. ($\Leftarrow$) If X connected, $f(X)$ connected in $\mathbb{R}$ hence an interval.

($\Rightarrow$) Suppose otherwise, that $X = \mathcal{O}_1 \sqcup \mathcal{O}_2$. Then define the function $f : X \to \mathbb{R}$ by $x \mapsto \begin{cases} 1 \text{ if } x \in \mathcal{O}_2 \\ 0 \text{ if } x \in \mathcal{O}_1 \end{cases}$. Then, for every $A \subseteq \mathbb{R}$,

$$f^{-1}(A) = \begin{cases} \varnothing & \text{if } \{0,1\} \not\subseteq A \\ \mathcal{O}_1 & \text{if } 0 \in A \\ \mathcal{O}_2 & \text{if } 1 \in A \\ X & \text{if } \{0,1\} \subseteq A \end{cases},$$

which are all open sets, hence f continuous. But $f(X) = \{0,1\}$ which is not an interval, hence the IVP fails and so X must be connected. ■

↪**Definition 1.31** (Arcwise/Path Connected): X *arc connected/path connected* if $\forall\, x, y \in X$, there exists a continuous function $f : [0,1] \to X$ such that $f(0) = x$, $f(1) = y$.

↪**Proposition 1.21**: Arc connected $\Rightarrow$ connected.

Proof. Suppose otherwise, $X = \mathcal{O}_1 \sqcup \mathcal{O}_2$. Let $x \in \mathcal{O}_1, y \in \mathcal{O}_2$ and define a continuous function $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Then, $f^{-1}(\mathcal{O}_i)$ each open, nonempty and disjoint for $i = 1, 2$, but

$$f^{-1}(\mathcal{O}_1) \sqcup f^{-1}(\mathcal{O}_2) = [0,1],$$

a contradiction to the connectedness of $[0,1]$. ■

§1.9 Urysohn's Lemma and Urysohn's Metrization Theorem

We present the main lemma of this section first, but need more tools before proving it.

↪**Lemma 1.2** (Urysohn's): Let $A, B \subseteq X$ closed and disjoint subsets of a normal space X. Then, $\forall\, [a,b] \subseteq \mathbb{R}$, there exists a continuous function $f : [a,b] \to \mathbb{R}$ such that $f(X) \subseteq [a,b]$, $f|_A = a$ and $f|_B = b$.

Remark 1.13: We have a partial converse of this statement as well:

↪**Proposition 1.22**: Let X Tychonoff and suppose X satisfies the properties of Urysohn's Lemma. Then, X normal.

Proof. Let A, B be closed nonempty disjoint subsets. Let $f : X \to \mathbb{R}$ continuous such that $f|_A = 0$, $f|_B = 1$ and $0 \leq f \leq 1$. Let I_1, I_2 be two disjoint open intervals in $\mathbb{R}$ with $0 \in I_1$ and $1 \in I_2$. Then, $f^{-1}(I_1)$ open and contains A, and $f^{-1}(I_2)$ open and contains B. Moreover, $f^{-1}(I_1) \cap f^{-1}(I_2) = \varnothing$; hence, $f^{-1}(I_1), f^{-1}(I_2)$ disjoint open neighborhoods of A, B respectively, so indeed X normal. ■

↪**Definition 1.32** (Normally Ascending): Let $(X, \mathcal{T})$ a topological space and $\Lambda \subseteq \mathbb{R}$. A collection of open sets $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ is said to be *normally ascending* if $\forall\, \lambda_1, \lambda_2 \in \Lambda$,

$$\overline{\mathcal{O}_{\lambda_1}} \subseteq \mathcal{O}_{\lambda_2} \text{ if } \lambda_1 < \lambda_2.$$

↪**Lemma 1.3**: Let $\Lambda \subseteq (a, b)$ a dense subset, and let $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ a normally ascending collection of subsets of X. Let $f : X \to \mathbb{R}$ defined such that

$$f(x) = \begin{cases} b & \text{if } x \in \left(\bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda\right)^c \\ \inf\{\lambda \in \Lambda \mid x \in \mathcal{O}_\lambda\} & \text{else} \end{cases}.$$

Then, f continuous.

Proof. We claim $f^{-1}(-\infty, c)$ and $f^{-1}(c, \infty)$ open for every $c \in \mathbb{R}$. Since such sets define a subbase for $\mathbb{R}$, it suffices to prove continuity on these sets. We show just the first for convenience. Notice that since $f(x) \in [a, b]$, if $c \in (a, b)$ then $f^{-1}(-\infty, c) = f^{-1}[a, c)$, so really it suffices to show that $f^{-1}[a, c)$ open to complete the proof.

Suppose $x \in f^{-1}([a, c])$ so $a \leq f(x) < c$. Let $\lambda \in \Lambda$ be such that $a < \lambda < f(x)$. Then, $x \notin \mathcal{O}_\lambda$. Let also $\lambda' \in \Lambda$ such that $f(x) < \lambda' < c$. By density of Λ, there exists a $\varepsilon > 0$ such that $f(x) + \varepsilon \in \Lambda$, so in particular

$$\overline{\mathcal{O}}_{f(x)+\varepsilon} \subseteq \mathcal{O}_{\lambda'} \Rightarrow x \in \mathcal{O}_{\lambda'},$$

by nesting. So, repeating this procedure, we find

$$f^{-1}([a, c)) \subseteq \bigcup_{a\leq\lambda<\lambda'<c} \mathcal{O}_{\lambda'} \setminus \overline{\mathcal{O}}_\lambda,$$

noticing the set on the right is open. By similar reasoning, the opposite inclusion holds and we have equality. Hence, f continuous. ■

↪**Lemma 1.4**: Let X normal, $F \subseteq X$ closed, and $\mathcal{U}$ a neighborhood of F. Then, for any $(a, b) \subseteq \mathbb{R}$, there exists a dense subset $\Lambda \subseteq (a, b)$ and a normally ascending collection $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ such that

$$F \subseteq \mathcal{O}_\lambda \subseteq \overline{\mathcal{O}}_\lambda \subseteq \mathcal{U}, \qquad \forall\, \lambda \in \Lambda.$$

Remark 1.14: This is essentially a generalization of the nested neighborhood property, and indeed the proof essentially just uses this property repeatedly to construct the collection $\{\mathcal{O}_\lambda\}$.

Proof. Without loss of generality, we assume $(a, b) = (0, 1)$, for the two intervals are homeomorphic, i.e. the function $f : (0, 1) \to \mathbb{R}$, $f(x) := a(1 - x) + bx$ is continuous, invertible with continuous inverse and with $f(0) = a$, $f(1) = b$ so a homeomorphism.

Let

$$\Lambda := \left\{ \frac{m}{2^n} \mid m, n \in \mathbb{N} \mid 1 \le m \le 2^{n-1} \right\} = \bigcup_{n \in \mathbb{N}} \underbrace{\left\{ \frac{m}{2^n} \mid m \in \mathbb{N}, 1 \le m \le 2^{n-1} \right\}}_{=:\Lambda_n},$$

which is clearly dense in $(0, 1)$. We need now to define our normally ascending collection. We do so by defining on each Λ_1 and proceding inductively.

For Λ_1, since X normal, let $\mathcal{O}_{1/2}$ be such that $F \subseteq \mathcal{O}_{1/2} \subseteq \overline{\mathcal{O}}_{1/2} \subseteq \mathcal{U}$, which exists by the nested neighborhood property.

For $\Lambda_2 = \{\frac{1}{4}, \frac{3}{4}\}$, we use the nested neighborhood property again, but first with F as the closed set and $\mathcal{O}_{1/2}$ an open neighborhood of it, and then with $\overline{\mathcal{O}}_{1/2}$ as the closed set and $\mathcal{U}$ an open neighborhood of it. In this way, we find

$$\underbrace{F \subseteq \mathcal{O}_{1/4} \subseteq \overline{\mathcal{O}}_{1/4} \subseteq \mathcal{O}_{1/2}}_{\text{nested nbhd}} \subseteq \overbrace{\overline{\mathcal{O}}_{1/2} \subseteq \mathcal{O}_{3/4} \subseteq \overline{\mathcal{O}}_{3/4} \subseteq \mathcal{U}}^{\text{nested nbhd}}.$$

We repeat in this manner over all of Λ, in the end defining a normally ascending collection $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$. ■

Proof *(Of Urysohn's Lemma, Lem. 1.2)*. Let $F = A$ and $\mathcal{U} = B^c$ as in the previous lemma Lem. 1.4. Then, there is some dense subset $\Lambda \subseteq (a, b)$ and a normally ascending collection $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ such that $A \subseteq \mathcal{O}_\lambda \subseteq \overline{\mathcal{O}}_\lambda \subseteq B^c$ for every $\lambda \in \Lambda$. Let $f(x)$ as in the previous previous lemma, Lem. 1.3. Then, if $x \in B$, $B \subseteq \left(\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda\right)^c$ and so $f(x) = b$. Otherwise if $x \in A$, then $x \in \bigcap_{\lambda \in \Lambda} \mathcal{O}_\lambda$ and thus $f(x) = \inf\{\lambda \in \Lambda\} = a$. By the first lemma, f continuous, so we are done. ■

↪**Theorem 1.9** (Urysohn's Metrization Theorem): Let X be a second countable topological space. Then, X is metrizable (that is, there exists a metric on X that induces the topology) if and only if X normal.

Proof. $(\Rightarrow)$ We have already showed, every metric space is normal.

$(\Leftarrow)$ Let $\{\mathcal{U}_n\}$ be a countable basis for $\mathcal{T}$ and put

$$A := \left\{ (n, m) \in \mathbb{N} \times \mathbb{N} \mid \overline{\mathcal{U}}_n \subseteq \mathcal{U}_m \right\}.$$

By Urysohn's lemma, for each $(n, m) \in A$ there is some continuous function $f_{n,m} : X \to \mathbb{R}$ such that $f_{n,m}$ is 1 on $\mathcal{U}_m^c$ and 0 on $\overline{\mathcal{U}}_n$ (these are disjoint closed sets). For $x, y \in X$, define

$$\rho(x, y) := \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)|.$$

The absolute valued term is ≤ 2, so this function will always be finite. Moreover, one can verify that it is indeed a metric on X. It remains to show that it induces the same topology; it suffices to compare bases of the two.

Let $x \in \mathcal{U}_m$. We wish to show there exists $B_\rho(x, \varepsilon) \subseteq \mathcal{U}_m$. $\{x\}$ is closed in X being normal, so there exists some n such that

$$\{x\} \subseteq \mathcal{U}_n \subseteq \overline{\mathcal{U}}_n \subseteq \mathcal{U}_m,$$

so $(n, m) \in A$ and so $f_{n,m}(x) = 0$. Let $\varepsilon = \frac{1}{2^{n+m}}$. Then, if $\rho(x, y) < \varepsilon$, it must be

$$\begin{aligned}
\frac{1}{2^{n+m}} &> \sum_{(n',m') \in A} \frac{1}{2^{n'+m'}} \left|f_{n',m'}(x) - f_{n',m'}(y)\right| \\
&\geq \frac{1}{2^{n+m}} \left|\underbrace{f_{n,m}(x)}_{=0} - f_{n,m}(y)\right| \\
&= \frac{1}{2^{n+m}} \left|f_{n,m}(y)\right|,
\end{aligned}$$

so $\left|f_{n,m}(y)\right| < 1$ and thus $y \notin \mathcal{U}_m^c$ so $y \in \mathcal{U}_m$. It follow that $B_\rho(x, \varepsilon) \subseteq \mathcal{U}_m$, and so every open set in X is open with respect to the metric topology.

Conversely, if $B_\rho(x, \varepsilon)$ some open ball in the metric topology, then notice that $y \mapsto \rho(x, y)$ for fixed y a continuous function, and thus $(\rho(x, \cdot))^{-1}(-\varepsilon, \varepsilon)$ an open set in $\mathcal{T}$ containing x. But this set also just equal to $B_\rho(x, \varepsilon)$, hence $B_\rho(x, \varepsilon)$ open in $\mathcal{T}$. We conclude the two topologies are equal, completing the proof. ■

Remark 1.15: Recall metric $\Rightarrow$ first countable hence not first countable $\Rightarrow$ not metrizable.

§1.10 Stone-Weierstrass Theorem

We need to use the following theorem, which we'll prove later.

↪**Theorem 1.10** (Weierstrass Approximation Theorem): Let $f : [a, b] \to \mathbb{R}$ continuous. Then, for every $\varepsilon > 0$, there exists a polynomial $p(x)$ such that $\|f - p\|_\infty < \varepsilon$.

↪**Definition 1.33** (Algebra, Separation of Points): We call a subset $\mathcal{A} \subseteq C(X)$ an *algebra* if it is a linear subspace that is closed under multiplication (that is, $f, g \in \mathcal{A} \Rightarrow f \cdot g \in \mathcal{A}$).

We say $\mathcal{A}$ *separates points* in X if for every $x, y \in X$, there exists an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

↪**Theorem 1.11** (Stone-Weierstrass): Let X be a compact Hausdorff space. Suppose $\mathcal{A} \subseteq C(X)$ an algebra that separates points and contains constant functions. Then, $\mathcal{A}$ dense in $C(X)$.

We tacitly assume the conditions of the theorem in the following lemmas as as not to restate them.

↪**Lemma 1.5**: For every $F \subseteq X$ closed, and every $x_0 \in F^c$, there exists a neighborhood $\mathcal{U}(x_0)$ such that $F \cap \mathcal{U} = \varnothing$ and $\forall\, \varepsilon > 0$ there is some $h \in \mathcal{A}$ such that $h < \varepsilon$ on $\mathcal{U}$, $h > 1 - \varepsilon$ on F, and $0 \leq h \leq 1$ on X.

In particular, $\mathcal{U}$ is *independent* of choice of ε.

Proof. Our first claim is that for every $y \in F$, there is a $g_y \in \mathcal{L}$ such that $g_y(x_0) = 0$ and $g_y(y) > 0$, and moreover $0 \leq g_y \leq 1$. Since $\mathcal{A}$ separates points, there is an $f \in \mathcal{A}$ such that $f(x_0) \neq f(y)$. Then, let

$$g_y(x) := \left[\frac{f(x) - f(x_0)}{\|f - f(x)\|_\infty}\right]^2.$$

Then, every operation used in this new function keeps $g_y \in \mathcal{A}$. Moreover one readily verifies it satisfies the desired qualities. In particular since g_y continuous, there is a neighborhood $\mathcal{O}_y$ such that $g_y|_{\mathcal{O}_y} > 0$. Hence, we know that $F \subseteq \bigcup_{y \in F} \mathcal{O}_y$, but F closed and so compact, hence there exists a finite subcover i.e. some $n \geq 1$ and finite sequence $\{y_i\}_{i=1}^n$ such that $F \subseteq \bigcup_{i=1}^n \mathcal{O}_{y_i}$. Let for each y_i $g_{y_i} \in \mathcal{A}$ with the properties from above, and consider the "averaged" function

$$g(x) := \frac{1}{n}\sum_{i=1}^n g_{y_i}(x) \in \mathcal{A}.$$

Then, $g(x_0) = 0$, $g > 0$ on F and $0 \leq g \leq 1$ on all of X. Hence, there is some $1 > c > 0$ such that $g \geq c$ on F, and since g continuous at x_0 there exists some $\mathcal{U}(x_0)$ such that $g < \frac{c}{2}$ on $\mathcal{U}$, with $\mathcal{U} \cap F = \varnothing$. So, $0 \leq g|_{\mathcal{U}} < \frac{c}{2}$, and $1 \geq g|_F \geq c$. To complete the proof, we need $\left(0, \frac{c}{2}\right) \leftrightarrow (0, \varepsilon)$ and $(c, 1) \Leftrightarrow (1 - \varepsilon, 1)$. By the Weierstrass Approximation Theorem, there exists some polynomial p such that $p|_{[0, \frac{c}{2}]} < \varepsilon$ and $p|_{[c,1]} > 1 - \varepsilon$. Then if we let $h(x) := (p \circ g)(x)$, this is just a polynomial of g hence remains if $\mathcal{A}$, and we find

$$h|_{\mathcal{U}} < \varepsilon, \qquad h|_F > 1 - \varepsilon, \qquad 0 \leq h \leq 1.$$

■

↪**Lemma 1.6**: For every disjoint closed set A, B and $\varepsilon > 0$, there exists $h \in \mathcal{A}$ such that $h|_A < \varepsilon$, $h|_B > 1 - \varepsilon$, and $0 \leq h \leq 1$ on X.

Proof. Let $F = B$ as in the last lemma. Let $x \in A$, then there exists $\mathcal{U}_x \cap B = \varnothing$ and for every $\varepsilon > 0$, $h|_{\mathcal{U}_x} < \varepsilon$ and $h|_B > 1 - \varepsilon$ and $0 \leq h \leq 1$. Then $A \subseteq \bigcup_{x \in A} \mathcal{U}_x$. Since A closed so compact, $A \subseteq \bigcup_{i=1}^N \mathcal{U}_{x_i}$. Let $\varepsilon_0 < \varepsilon$ such that $\left(1 - \frac{\varepsilon_0}{N}\right)^N > 1 - \varepsilon$. For each i, let $h_i \in \mathcal{A}$ such that $h_i|_{\mathcal{U}_{x_i}} < \frac{\varepsilon_0}{N}$, $h_i|_B > 1 - \frac{\varepsilon_0}{N}$ and $0 \leq h_i \leq 1$. Then, put

$$h(x) = h_1(x) \cdot h_2(x) \cdots h_N(x) \in \mathcal{A}.$$

Then, $0 \leq h \leq 1$ and $h|_B > \left(1 - \frac{\varepsilon_0}{N}\right)^N > 1 - \varepsilon$. Then, for every $x \in A$, $x \in \mathcal{U}_{x_i}$ so $h_i(x) < \frac{\varepsilon_0}{N}$ and $h_i(x) \leq i$ so $h(x) < \frac{\varepsilon_0}{N}$ so $h|A < \frac{\varepsilon_0}{N} < \varepsilon$. ■

Proof. (Of Stone-Weierstrass) WLOG, assume $f \in C(X)$, $0 \leq f \leq 1$, by replacing with

$$\tilde{f}(x) = \frac{f(x) + \|f\|_\infty}{\|f + \|f\|_\infty\|_\infty}$$

if necessary, since if there exists a $\tilde{g} \in \mathcal{A}$ such that $\|\tilde{f} - \tilde{g}\|_\infty < \varepsilon$, then using the properties of $\mathcal{A}$ we can find some appropriate $g \in \mathcal{A}$ such that $\|f - g\|_\infty < \varepsilon$.

Fix $n \in \mathbb{N}$, and consider the set $\left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$, and let for $1 \leq j \leq n$

$$A_j := \left\{x \in X \mid f(x) \leq \frac{j-1}{n}\right\}, \qquad B_j := \left\{x \in X \mid f(x) \geq \frac{j}{n}\right\},$$

which are both closed and disjoint. By the lemma, there exists $g_j \in \mathcal{A}$ such that

$$g_j|_{A_j} < \frac{1}{n}, \qquad g_j|_{B_j} > 1 - \frac{1}{n},$$

with $0 \leq g_j \leq 1$. Let then

$$g(x) := \frac{1}{n}\sum_{j=1}^{n} g_j(x) \in \mathcal{A}.$$

We claim then $\|f - g\|_\infty \leq \frac{3}{n}$, which proves the claim by taking n sufficiently large.

Suppose $k \in [1, n]$. If $f(x) \leq \frac{k}{n}$, then

$$g_j(x) = \begin{cases} < \frac{1}{n} \text{ if } j - 1 \geq k \\ \leq 1 \ \text{ else} \end{cases},$$

so

$$g(x) = \frac{1}{n}\sum_{j=1}^{n} g_j(x) = \frac{1}{n}\left[\sum_{j=1}^{k} g_j(x) + \sum_{j=k+1}^{n} g_j(x)\right] \leq \frac{1}{n}\left[k + \frac{n-k}{n}\right] \leq \frac{k}{n} + \frac{n-k}{n^2} \leq \frac{k+1}{n}.$$

Similarly if $f(x) \geq \frac{k-1}{n}$, then

$$g_j(x) = \begin{cases} > 1 - \frac{1}{n} \text{ if } j \leq k - 1 \\ \geq 0 \qquad \text{else} \end{cases},$$

so

$$g(x) \geq \frac{1}{n}\sum_{j=1}^{k-1}\left(1 - \frac{1}{n}\right) \geq \frac{1}{n}(k-1)\left(1 - \frac{1}{n}\right) = \frac{k-1}{n} - \frac{k-1}{n^2} \geq \frac{k-2}{n}.$$

So, we've show that if $\frac{k-1}{n} \leq f(x) \leq \frac{k}{n}$, then $\frac{k-2}{n} \leq g(x) \leq \frac{k+1}{n}$, and so repeating this argument and applying triangle inequality we conclude $\|f - g\|_\infty \leq \frac{3}{n}$. ■

↪**Theorem 1.12** (Borsuk): X compact, Hausdorff and $C(X)$ separable $\Leftrightarrow$ X is metrizable.

§2 Functional Analysis

Here, we will primarily work with a normed vector space (nvs). Moreover, we usually work in:

↪**Definition 2.1** (Banach Space): A normed vector space $(X, \|\cdot\|)$ is a *Banach space* if it is complete as a metric space under the norm-induced metric.

§2.1 Introduction to Linear Operators

↪**Definition 2.2** (Linear Operator, Operator Norm): Let X, Y be vector spaces. Then, a map $T : X \to Y$ is called *linear* if $\forall\, x, y \in X, \alpha, \beta \in \mathbb{R}$, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

If X, Y normed vector spaces, we say T is a bounded linear operator if T linear and the *operator norm*

$$\|T\| = \|T\|_{\mathcal{L}(X,Y)} = \sup_{\substack{x \in X, \\ \|x\|_X \leq 1}} \|Tx\|_Y < \infty$$

is finite. Then, we put

$$\mathcal{L}(X, Y) := \{\text{bounded linear operators } X \to Y\}.$$

↪**Theorem 2.1** (Bounded iff Continuous): If X, Y are nvs, $T \in \mathcal{L}(X, Y)$ iff and only if T is continuous, i.e. if $x_n \to x$ in X, then $Tx_n \to Tx$ in Y.

Proof. If $T \in \mathcal{L}(X, Y)$,

$$\begin{aligned}\|Tx_n - Tx\|_Y &= \|T(x_n - x)\|_Y \\ &= \|x_n - x\|_X \cdot \|\frac{T(x_n - x)}{\|x_n - x\|_X}\|_Y \\ &\leq \underbrace{\|T\|}_{<\infty} \|x_n - x\|_X \to 0,\end{aligned}$$

hence T continuous. Conversely, if T continuous, then by linearity $T0 = 0$, so by continuity, there is some $\delta > 0$ such that $\|Tx\|_Y < 1$ if $\|x\|_X < \delta$. For $x \in X$ nonzero, let $\lambda = \frac{\delta}{\|x\|_X}$. Then, $\|\lambda x\|_X \leq \delta$ so $\|T(\lambda x)\|_Y < 1$, i.e. $\frac{\|T(x)\|_Y \delta}{\|x\|_X} < 1$. Hence,

$$\|T\| = \sup_{x \in X : x \neq 0} \frac{\|T(x)\|_Y}{\|x\|_X} \leq \frac{1}{\delta},$$

so $T \in \mathcal{L}(X, Y)$. ■

↪**Proposition 2.1** (Properties of $\mathcal{L}(X, Y)$): If X, Y nvs, $\mathcal{L}(X, Y)$ a nvs, and if X, Y Banach, then so is $\mathcal{L}(X, Y)$.

PROOF. (a) For $T, S \in \mathcal{L}(X, Y)$, $\alpha, \beta \in \mathbb{R}$, and $x \in X$, then

$$\begin{aligned}\|(\alpha T + \beta S)(x)\|_Y &\leq |\alpha|\ \|Tx\|_Y + |\beta|\ \|Sx\|_Y \\ &\leq |\alpha|\ \|T\|\ \|x\|_X + |\beta|\ \|T\|\ \|x\|_X.\end{aligned}$$

Dividing both sides by $\|x\|$, we find $\|\alpha T + \beta S\| < \infty$. The same argument gives the triangle inequality on $\|\cdot\|$. Finally, $T = 0$ iff $\|Tx\|_Y = 0$ for every $x \in X$ iff $\|T\| = 0$.

(b) Let $\{T_n\} \subseteq \mathcal{L}(X, Y)$ be a Cauchy sequence. We have that

$$\|T_n x - T_m x\|_Y \leq \|T_n - T_m\|\ \|x\|_X,$$

so in particular the sequence $\{T_n(x)\}$ a Cauchy sequence in Y for any $x \in X$. Y complete so this sequence converges, say $T_n(x) \to y^*$ in Y. Let $T(x) := y^*$ for each x. We claim that $T \in \mathcal{L}(X, Y)$ and that $T_n \to T$ in the operator norm. We check:

$$\begin{aligned}\alpha T(x_1) + \beta T(x_2) &= \lim_{n\to\infty} \alpha T_n(x_1) + \lim_{n\to\infty} \beta T_n(x_2) \\ &= \lim_{n\to\infty} [T_n(\alpha x_1) + T_n(\beta x_2)] \\ &= \lim_{n\to\infty} T_n(\alpha x_1 + \beta x_2) \\ &= T(\alpha x_1 + \beta x_2),\end{aligned}$$

so T linear.

Let now $\varepsilon > 0$ and N such that for every $n \geq N$ and $k \geq 1$ such that $\|T_n - T_{n+k}\| < \frac{\varepsilon}{2}$. Then,

$$\begin{aligned}\|T_n(x) - T_{n+k}(x)\|_Y &= \left\|(T_n - T_{n+k})(x)\right\|_Y \\ &\leq \left\|T_n - T_{n+k}\right\| \|x\|_X \\ &< \frac{\varepsilon}{2} \|x\|_X.\end{aligned}$$

Letting $k \to \infty$, we find that

$$\|T_n(x) - T(x)\|_Y < \frac{\varepsilon}{2}\ \|x\|_X,$$

so normalizing both sides by $\|x\|_X$, we find $\|T_n - T\| < \frac{\varepsilon}{2}$, and we have convergence. ∎

↪**Definition 2.3** (Isomorphism): We say $T \in \mathcal{L}(X, Y)$ an *isomorphism* if T is bijective and $T^{-1} \in \mathcal{L}(Y, X)$. In this case we write $X \simeq Y$, and say X, Y isomorphic.

§2.2 Finite versus Infinite Dimensional

If X a nvs, then we can look for a basis β such that $\operatorname{span}(\beta) = X$. If $\beta = \{e_1, ..., e_n\}$ has no proper subset spanning X, then we say $\dim(X) = n$.

As we saw on homework, any two norms on a finite dimensional space are equivalent.

↪**Corollary 2.1**: (a) Any two nvs of the same finite dimension are isomorphic.

(b) Any finite dimensional space is complete, and so any finite dimensional subspace is closed.

(c) $\overline{B}(0,1)$ is compact in a finite dimensional space.

Proof. (a) Let $(X, \|\cdot\|)$ have finite dimension n. Then, we claim $(X, \|\cdot\|) \simeq (\mathbb{R}^n, |\cdot|)$. Let $\{e_1, ..., e_n\}$ be a basis for X. Let $T : \mathbb{R}^n \to X$ given by

$$T(x) = \sum_{i=1}^n x_i e_i,$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$, which is clearly linear. Moreover,

$$Tx = 0 \Leftrightarrow \sum_{i=1}^n x_i e_i = 0 \Leftrightarrow x = 0,$$

so T injective, and so being linear between two spaces of the same dimension gives T surjective. It remains to check boundedness.

First, we claim $x \mapsto \|T(x)\|$ is a norm on $\mathbb{R}^n$. $\|T(x)\| = 0 \Leftrightarrow x = 0$ by the injectivity of T, and the properties $\|T(\lambda x)\| = |\lambda|\ \|Tx\|$ and $\|T(x+y)\| \leq \|Tx\| + \|Ty\|$ follow from linearity of T and the fact that $\|\cdot\|$ already a norm. Hence, $\|T(\cdot)\|$ a norm on $\mathbb{R}^n$ and so equivalent to $|\cdot|$, i.e. there exists constants $C_1, C_2 > 0$ such that

$$C_1|x| \leq \|T(x)\| \leq C_2|x|,$$

for every $x \in X$. It follows that $\|T\|$ (operator norm now) is bounded.

Letting $T(x) = y$, we find similarly

$$C_{1'}\|y\| \leq |T^{-1}(y)| \leq C_{2'}\ \|y\|,$$

so $\|T^{-1}\|$ also bounded. Hence, we've shown any n-dimensional space is isomorphic to $\mathbb{R}^n$, so by transitivity of isomorphism any two n-dimensional spaces are isomorphic.

(b) The property of completeness is preserved under isomorphism, so this follows from the previous statement since $\mathbb{R}^n$ complete.

(c) Consider $\overline{B}(0,1) \subseteq X$. Let T be an isomorphism $X \to \mathbb{R}^n$. Then, for $x \in \overline{B}(0,1)$, $\|Tx\| \leq \|T\| < \infty$, so $T\left(\overline{B}(0,1)\right)$ is a bounded subset of $\mathbb{R}^n$, and since T and its inverse continuous, $T\left(\overline{B}(0,1)\right)$ closed in $\mathbb{R}^n$. Hence, $T\left(\overline{B}(0,1)\right)$ closed and bounded hence compact in $\mathbb{R}^n$, so since T^{-1} continuous $T^{-1}\left(T\left(\overline{B}(0,1)\right)\right) = \overline{B}(0,1)$ also compact, in X. ∎

↪**Theorem 2.2** (Riesz's): If X is an nvs, then $\overline{B}(0,1)$ is compact if and only if X if finite dimensional.

↪**Lemma 2.1** (Riesz's): Let $Y \subsetneq X$ be a closed nvs (and X a nvs). Then for every $\varepsilon > 0$, there exists $x_0 \in X$ with $\|x_0\| = 1$ and such that

$$\|x_0 - y\|_X > \varepsilon \,\forall\, y \in Y.$$

Proof. Fix $\varepsilon > 0$. Since $Y \subsetneq X$, let $x \in Y^c$. Y closed so Y^c open and hence there exists some $r > 0$ such that $B(x, r) \cap Y = \varnothing$. In other words,

$$\inf\{\|x - y'\| \mid y' \in Y\} > r > 0.$$

Let then $y' \in Y$ be such that

$$r < \|x - y_1\| < \varepsilon^{-1} r,$$

and take

$$x_0 := \frac{x - y_1}{\|x - y_1\|_X}.$$

Then, x_0 a unit vector, and for every $y \in Y$,

$$\begin{aligned} x_0 - y &= \frac{x - y_1}{\|x - y_1\|} - y \\ &= \frac{1}{\|x - y_1\|}[x - y_1 - y\,\|x - y_1\|] \\ &= \frac{1}{\|x - y_1\|}[x - y'], \end{aligned}$$

where $y' = y_1 + y\,\|x - y_1\| \in Y$, since it is closed under vector addition. Hence

$$\|x_0 - y\| = \frac{1}{\|x - y_1\|}\,\|x - y'\| > \frac{\varepsilon}{r}\,\|x - y'\| > \varepsilon,$$

for every $y \in Y$. ■

Proof. (Of Thm. 2.2) ($\Leftarrow$) By the previous corollary.

($\Rightarrow$) Suppose X infinite dimensonal. We will show $B := \overline{B}(0, 1)$ not compact.

Claim: there exists $\{x_i\}_{i=1}^{\infty} \subseteq B$ such that $\|x_i - x_j\| > \frac{1}{2}$ if $i \neq j$.

We proceed by induction. Let $x_1 \in B$. Suppose $\{x_1, ..., x_n\} \subseteq B$ are such that $\|x_i - x_j\| > \frac{1}{2}$. Let $X_n = \operatorname{span}\{x_1, ..., x_n\}$, so X_n finite dimensional hence $X_n \subsetneq X$. By the previous lemma (taking $\varepsilon = \frac{1}{2}$) there is then some $x_{n+1} \in B$ such that $\|x_1 - x_{n+1}\| > \frac{1}{2}$ for every $i = 1, ..., n$. We can thus inductively build such a sequence $\{x_i\}_{i=1}^{\infty}$. Then, every subsequence of this sequence cannot be Cauchy so B is not sequentially compact and thus B is not compact. ■

§2.3 Open Mapping and Closed Graph Theorems

↪**Definition 2.4** (T open): If X, Y toplogical spaces and $T : X \to Y$ a linear operator, T is said to be *open* if for every $\mathcal{U} \subseteq X$ open, $T(\mathcal{U})$ open in Y.

In particular if X, Y are metric spaces (or nvs), then T is open iff the image of every open ball in X containes an open ball in Y, i.e. $\forall\, x \in X, r > 0$ there exists $r' > 0$ such that $T(B_X(x, r)) \supseteq B_Y(Tx, r')$. Moreover, by translating/scaling appropriately, it suffices to prove for $x = 0, r = 1$.

↪**Theorem 2.3** (Open Mapping Theorem): Let X, Y be Banach spaces and $T : X \to Y$ a bounded linear operator. If T is surjective, then T is open.

Proof. Its enough to show that there is some $r > 0$ such that $T(B_X(0, 1)) \supseteq B_Y(0, r)$.

Claim: $\exists\, c > 0$ such that $\overline{T(B_X(0, 1))} \supseteq B_Y(0, 2c)$.

Put $E_n = n \cdot \overline{T(B_X(0, 1))}$ for $n \in \mathbb{N}$. Since T surjective, $\bigcup_{n=1}^{\infty} E_n = Y$. Each E_n closed, so by the Baire Category Theorem there exists some index n_0 such that E_{n_0} has nonempty interior, i.e.

$$\operatorname{int}\left(\overline{T(B_X(0, 1))}\right) \neq \varnothing,$$

where we drop the index by homogeneity. Pick then $c > 0$ and $y_0 \in Y$ such that $B_Y(y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$. We claim then that $B_Y(-y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$ as well. Indeed, if $B_Y(y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$, then $\forall\, \tilde{y} \in Y$ with $\|y_0 - \tilde{y}\|_Y < 4c$, Then, $\|-y_0 + \tilde{y}\|_Y < 4c$ so $-\tilde{y} \in B_Y(-y_0, 4c)$. But $\tilde{y} = \lim_{n\to\infty} T(x_n)$ and so $-\tilde{y} = \lim_{n\to\infty} T(-x_n)$. Since $\{-x_n\} \subseteq B_X(0, 1)$, this implies $-\tilde{y} \in \overline{T(B_X(0, 1))}$ hence the "subclaim" holds.

Now, for any $\tilde{y} \in B_Y(0, 4c)$, $\|\tilde{y}\| \leq 4c$ so

$$\tilde{y} = y_0 \underbrace{-y_0 + \tilde{y}}_{\in B_Y(-y_0, 4c)} = \overbrace{y_0 + \tilde{y}}^{\in B(y_0, 4c)} - y_0.$$

Therefore,

$$\begin{aligned} B_Y(0, 4c) &= B_Y(y_0 - y_0, 4c) \\ &\subseteq B_Y(y_0, 4c) + B_Y(-y_0, 4c) \\ &\overline{T(B_X(0, 1))} + \overline{T(B_X(0, 1))} = 2\overline{T(B_X(0, 1))}, \end{aligned}$$

(where summation of two sets is the vector addition of all the elements in the sets), hence $B_Y(0, 2c) \subseteq \overline{T(B_X(0, 1))}$.

We claim next that $T(B_X(0, 1)) \supseteq B_Y(0, c)$. Choose $y \in Y$ with $\|y\|_Y < c$. By the first claim, $B_Y(0, c) \subseteq \overline{T\left(B_X\left(0, \frac{1}{2}\right)\right)}$, so for every $\varepsilon > 0$ there is some $z \in X$ with $\|z\|_X < \frac{1}{2}$ and $\|y - Tz\|_Y < \varepsilon$. Let $\varepsilon = \frac{c}{2}$ and $z_1 \in X$ such that $\|z_1\|_X < \frac{1}{2}$ and $\|y - Tz_1\|_Y < \frac{c}{2}$. But the first claim can also be written as $B_Y\left(0, \frac{c}{2}\right) \subseteq \overline{T\left(B_X\left(0, \frac{1}{4}\right)\right)}$ so if $\varepsilon = \frac{c}{4}$, let $z_2 \in X$

such that $\|z_2\|_X < \frac{1}{4}$ and $\|(y - Tz_1) - Tz_2\|_Y < \frac{c}{4}$. Continuing in this manner we find that

$$B_Y\left(0, \frac{c}{2^k}\right) \subseteq \overline{T\left(B_X\left(0, \frac{1}{2^{k+1}}\right)\right)},$$

so exists $z_k \in X$ such that $\|z_k\|_X < \frac{1}{2^k}$ and $\|y - T(z_1 + \cdots + z_k)\|_Y < \frac{c}{2^k}$. Let $x_n = z_1 + \cdots + z_n \in X$. Then $\{x_n\}$ is Cauchy in X, since

$$\|x_n - x_m\|_X \leq \sum_{k=m}^{n} \|z_k\|_X < \sum_{k=m}^{n} \frac{1}{2^k} \to 0.$$

Since X a Banach space, $x_n \to \overline{x}$ and in particular $\|\overline{x}\| \leq \sum_{k=1}^{\infty} \|z_k\|_X < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$, so $\overline{x} \in B_X(0,1)$. Since T bounded it is continuous, so $Tx_n \to T\overline{x}$, so $y = T\overline{x}$ and thus $B_Y(0,c) \subseteq T(B(0,1))$. ■

↪**Corollary 2.2**: Let X, Y Banach and $T : X \to Y$ be bounded, linear and bijective. Then, T^{-1} continuous.

↪**Corollary 2.3**: Let $(X, \|\cdot\|_1), (X, \|\cdot\|_2)$ be Banach spaces. Suppose there exists $c > 0$ such that $\|x\|_2 \leq C\|x\|_1$ for every $x \in X$. Then, $\|\cdot\|_1, \|\cdot\|_2$ are equivalent.

Proof. Let T be the identity linear operator and use the previous corollary. ■

↪**Definition 2.5** (T closed): If X, Y are nvs and T is linear, the *graph* of T is the set

$$G(T) = \{(x, Tx) \mid x \in X\} \subseteq X \times Y.$$

We then say T is *closed* if $G(T)$ closed in $X \times Y$.

Remark 2.1: Since X, Y are nvs, they are metric spaces so first countable, hence closed $\leftrightarrow$ contains all limit points.

In the product topology, a countable base for $X \times Y$ at (x, y) is given by

$$\left\{B_X\left(x, \frac{1}{n}\right) \times B\left(y, \frac{1}{m}\right)\right\}_{n,m\in\mathbb{N}}.$$

Then, $G(T)$ closed iff $G(T)$ contains all limit points. How can we put a norm on $X \times Y$ that generates this product topology? Let

$$\|(x, y)\|_1 := \|x\|_X + \|y\|_Y.$$

If $(x_n, y_n) \to (x, y)$ in the product topology, then since Π_1, Π_2 continuous maps, $(x_n, y_n) \to (x, y)$ in the $\|\cdot\|_1$ topology. On the other hand if $(x_n, y_n) \to (x, y)$ in the $\|\cdot\|_1$ norm, then

$$\|x_n - x\|_X \leq \|(x_n, y_n) - (x, y)\|_1,$$

hence since the RHS $\to 0$ so does the LHS and so $x_n \to x$ in $\|\cdot\|_X$; similar gives $y_n \to y$ in $\|\cdot\|_Y$. From here it follows that $(x_n, y_n) \to (x, y)$ in the product topology.

So, to prove $G(T)$ closed, we just need to prove that if $x_n \to x$ in X and $Tx_n \to y$, then $y = Tx_n$.

↪**Theorem 2.4** (Closed Graph Theorem): Let X, Y be Banach spaces and $T : X \to Y$ linear. Then, T is continuous iff T is closed.

Proof. ($\Rightarrow$) Immediate from the above remark.

($\Leftarrow$) Consider the function

$$x \mapsto \|x\|_* := \|x\|_X + \|Tx\|_Y.$$

So by the above, T closed implies $(X, \|\cdot\|_*)$ is complete, i.e. if $x_n \to x$ in $\|\cdot\|_*$ in X iff $x_n \to x$ in $\|\cdot\|_X$ and $Tx_n \to Tx$ in $\|\cdot\|_Y$. However, $\|\cdot\|_X \leq \|\cdot\|_*$, hence since $\left(X, \|\cdot\|_X\right)$ and $\left(X, \|\cdot\|_*\right)$ are Banach spaces, by the corollary, there is some $C > 0$ such that $\|\cdot\|_* \leq C\|\cdot\|_X$. So,

$$\|x\|_X + \|Tx\|_Y \leq C\|x\|_X,$$

so

$$\|Tx\|_Y \leq \|x\|_X + \|Tx\|_Y \leq C\|x\|_X,$$

so T bounded and thus continuous. ■

Remark 2.2: The Closed Graph Theorem simplifies proving continuity of T. It tells us we can assume if $x_n \to x$, $\{Tx_n\}$ Cauchy so $\exists\, y$ such that $Tx_n \to y$ since Y is Banach. So, it suffices to check that $y = Tx$ to check continuity; we don't need to check convergence of Tx_n.

§2.4 Uniform Boundedness Principle

Recall the following consequence of the Baire Category Theorem:

↪**Theorem 2.5**: Let $\mathcal{F} \subseteq C(X)$ where (X, ρ) a complete metric space. Suppose $\mathcal{F}$ pointwise bounded. Then, there exists a nonempty open set $\mathcal{O} \subseteq X$ such that there is some $M > 0$ such that $|f(x)| \leq M$ for every $x \in \mathcal{O}, f \in \mathcal{F}$.

This leads to the following result:

↪**Theorem 2.6** (Uniform Boundedness Principle): Let X a Banach space and Y a nvs. Consider $\mathcal{F} \subseteq \mathcal{L}(X, Y)$. Suppose $\mathcal{F}$ is pointwise bounded, i.e. for every $x \in X$, there is some $M_x > 0$ such that

$$\|Tx\|_Y \leq M_x, \forall T \in \mathcal{F}.$$

Then, $\mathcal{F}$ is uniformly bounded, i.e. $\exists M > 0$ such that

$$\|T\|_Y \leq M, \forall T \in \mathcal{F}.$$

Proof. For every $T \in \mathcal{F}$, let $f_T : X \to \mathbb{R}$ be given by

$$f_T(x) = \|Tx\|_Y.$$

Since $T \in \mathcal{L}(X, Y)$, T is continuous, so $x_n \underset{X}{\to} x \Rightarrow Tx_n \underset{Y}{\to} Tx$, hence $\|Tx_n\|_Y \to \|Tx\|_Y$ so f_T continuous for each T i.e. $f_T \in C(X)$, so $\{f_T\} \subseteq C(X)$ pointwise bounded. So by the previous theorem, there is some ball $B(x_0, r) \subseteq X$ and some $K > 0$ such that $\|Tx\| \leq K$ for every $x \in B(x_0, r)$ and $T \in \mathcal{F}$. Thus, for every $x \in B(0, r)$,

$$\begin{aligned}
\|Tx\| &= \|T(x - x_0 + x_0)\| \\
&\leq \left\| T \underbrace{(x - x_0)}_{\in B(x_0, r)} \right\| + \|Tx_0\| \\
&\leq K + M_{x_0}, \qquad \forall x \in B(0, r), T \in \mathcal{F}.
\end{aligned}$$

Thus, for every $x \in B(0, 1)$,

$$\|Tx\| = \frac{1}{r} \left\| T \underbrace{(rx)}_{\in B(0,r)} \right\| \leq \frac{1}{r}\left(K + M_{x_0}\right) =: M,$$

so its clear $\|T\| \leq M$ for every $T \in \mathcal{F}$. ■

↪**Theorem 2.7** (Banach-Saks-Steinhaus): Let X a Banach space and Y a nvs. Let $\{T_n\} \subseteq \mathcal{L}(X, Y)$. Suppose for every $x \in X$, $\lim_{n\to\infty} T_n(x)$ exists in Y. Then,

a. $\{T_n\}$ are uniformly bounded in $\mathcal{L}(X, Y)$;

b. For $T : X \to Y$ defined by

$$T(x) := \lim_{n\to\infty} T_n(x),$$

we have $T \in \mathcal{L}(X, Y)$;

c. $\|T\| \leq \liminf_{n\to\infty} \|T_n\|$ (*lower semicontinuity result*).

Proof. (a) For every $x \in X$, $T_n(x) \to T(x)$ so $\|Tx\| < \infty$ hence $\sup_n \|T_n x\| < \infty$. By uniform boundedness, then, we find $\sup_n \|T\| =: C < \infty$.

(b) T is linear (by linearity of T_n). By (a),

$$\|T_n x\| \leq C\|x\|,$$

for every n, x, so

$$\|Tx\| \leq C\|x\| \; \forall \, x \in X,$$

so T bounded.

(c) We know

$$\|T_n x\| \leq \|T_n\| \|x\| \; \forall \, x \in X,$$

so

$$\frac{\|T_n x\|}{\|x\|} \leq \|T_n\|,$$

so

$$\liminf_n \frac{\|T_n x\|}{\|x\|} = \frac{\|Tx\|}{\|x\|} \leq \liminf_n \|T_n\|,$$

so by "suping" both sides,

$$\|T\| \leq \liminf_n \|T_n\|.$$

■

Remark 2.3:

- We do note have $T_n \to T$ in $\mathcal{L}(X, Y)$ i.e. with respect to the operator norm.
- If Y is a Banach space, then $\lim_{n\to\infty} T_n(x)$ exists in $Y \Leftrightarrow \{T_n x\}$ Cauchy in Y for every $x \in X$.