MATH249 - Complex Variables

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§1 Complex Numbers

The complex numbers are the set

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\},$$

where $i^2 = -1$. This set is readily equipped with operations of addition, subtraction, multiplication and division; given two complex numbers $a + bi, c + di$, these operations are determined by the rules

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$
$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2},$$

assuming in the final line that $a^2 + b^2 \neq 0$, i.e. that $a + bi \neq 0$ in $\mathbb{C}$. In particular, in the division line, we obtain the result by multiplying the top and bottom by the *conjugate* of $z := a + bi$; we denote

$$\bar{z} = a - bi,$$

noting that in particular,

$$z\bar{z} = a^2 + b^2 = |z|^2.$$

Any complex number $z = a + bi$ may be written in so-called *polar form*

$$z = r(\cos\theta + i\sin\theta), \quad r := \sqrt{a^2 + b^2} = |z|, \theta := \arg(z) = \arctan(b/a),$$

with the θ read modulo 2π. This is a useful representation for the sake of multiplication; given $z_i = r_i(\cos(\theta_i) + i\sin(\theta_i)), i = 1, 2$, we have

$$z_1 z_2 = \cdots = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

In particular,

$$|z_1 z_2| = |z_1||z_2|, \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

↪**Theorem 1.1**: $\cos(\theta) + i\sin(\theta) = \exp(i\theta)$

Proof. Taylor expand both sides. ■

In particular, this theorem gives a clear way to define the exponential of a complex number

$$e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)),$$

so that in particular, for any $z \in \mathbb{C}$,

$$|e^z| = e^{\mathrm{Re}(z)}, \quad \arg(e^z) = \mathrm{Im}(z).$$

§1.1 Fundamental Theorem of Algebra

↪**Theorem 1.2** (Fundamental Theorem of Algebra): If $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial with complex coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$, then there exists a $z \in \mathbb{C}$ such that $f(z) = 0$.

Proof. *(A First Proof)* Remark that if $|z| = R \gg 1$ (much larger than zero), then we have

$$\begin{aligned} |a_n z^n| &= |a_n| R^n, \\ |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| &\leq |a_{n-1}| R^{n-1} + \cdots + |a_1| R + |a_0| \\ &\leq (|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|) R^{n-1}. \end{aligned}$$

Let $z_0 \in \mathbb{C}$ be a point for which $|f(z_0)|$ is a minimum; this must exist for $|f|$ must be very large outside of the disc of radius R centered at the origin. Namely, $|z_0| < R$. We claim z_0 a root of f. We may assume without loss of generality that $z_0 = 0$, by replacing $f(z)$ with $f(z - z_0)$. We write

$$\begin{aligned} f(z) &= a_0 + \cdots + a_k z^k + \cdots + a_n z^n, \\ &= a_0 + a_k z^k \left(1 + \frac{a_{k+1}}{a_k} z + \cdots + \frac{a_n}{a_k} z^{n-k} \right). \end{aligned}$$

where $a_k \neq 0$ the first nonzero coefficient with $k \geq 1$. If we can show $a_0 = 0$, we are done. Assume otherwise. Let

$$z := \left(-\frac{a_0}{a_k} \right)^{\frac{1}{k}} \varepsilon, \quad \varepsilon > 0.$$

With this value of z, we have

$$f(z) = a_0 - a_0 \varepsilon^k \left(1 + \underbrace{\cdots}_{= o(\varepsilon)} \right) \approx a_0 \left(1 - \varepsilon^k \right).$$

By choosing ε sufficiently small, this implies

$$|f(z)| < |a_0| = |f(0)|,$$

which contradicts the assumed minimality of $z_0 = 0$, unless of course $a_0 = f(z_0) = 0$, providing the claim. ■

Proof. (A Second Proof) We want to view $f(z)$ as a mapping $\mathbb{C} \to \mathbb{C}$. Assume $f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$. When $|z|$ large, we know

$$|f(z) - z^n| < C|z|^{n-1},$$

for some constant C independent of z. Remark that the map $\varphi : z \mapsto z^n$ maps a circle of radius R to a circle of radius R^n; in particular, if we take a point $z = Re^{i\theta}$ on the circle of radius R of angle θ with the origin, and let θ vary from 0 to 2π, one "rotation" in the pre-image world will lead to n "rotations" in the image world. Similarly, for $z \mapsto f(z)$, the image of the R-radius circle may not be a circle, but a "fudged" circle; the curve of the image will still be some periodic curve. As we let $R \to 0$, though, the image will go to the singular point a_0. Thus, at some value of R, the image of the R-radius circle would have to pass through the origin, and thus this point must be a root of $f(z)$. ■

Proof. (A Third Proof) We use a result that we will prove later in the class, Liouville's Theorem, which states that any bounded differentiable function $f : \mathbb{C} \to \mathbb{C}$ must be constant.

Suppose $p(z)$ a polynomial with no roots in $\mathbb{C}$. Let $f(z) = \frac{1}{p(z)}$ (this is well-defined, since by assumption p has no roots); this is bounded on $\mathbb{C}$, and has derivative $\frac{\mathrm{d}}{\mathrm{d}z} f(z) = -\frac{p'(z)}{p(z)}$. By Liouville's, f must be a constant and thus p must be a constant. ■

§1.2 Analytic, Holomorphic Functions

↪**Definition 1.1** (Holomorphic/Analytic): A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *holomorphic* is it has a well-defined derivative, i.e. if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is well-defined (in the sense that it is independent of the "path" h takes to 0).

We may write $f : \mathbb{C} \to \mathbb{C}$ as

$$f(z) = f(x+iy) = u(x,y) + iv(x,y),$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$. We can calculate $f'(z)$ in two different ways.

1. Restrict h to $\mathbb{R}$:

$$\begin{aligned} f'(z) = \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{f(z+h) - f(z)}{h} &= \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \\ &= \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{v(x+h,y) - v(x,y)}{h} \\ &= \frac{\partial u}{\partial x}(x,y) + i \frac{\partial v}{\partial x}(x,y). \end{aligned}$$

2. Rescrict to h purely imaginary values:

$$\begin{aligned} f'(z) = \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{f(z+ih) - f(z)}{ih} &= \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{u(x,y+h) + iv(x,y+h) - u(x,y) - iv(x,y)}{ih} \\ &= \frac{1}{i} \frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) \\ &= \frac{\partial v}{\partial y}(x,y) - i \frac{\partial u}{\partial y}(x,y) \end{aligned}$$

These two computations must of course agree, which imply (equating real, imaginary parts)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are the *Cauchy-Riemann equations*. Viewing the pair $f = (u, v)$ as a mapping $\mathbb{R}^2 \to \mathbb{R}^2$, the Cauchy-Riemann equations imply that the Jacobian of f is given in the form

$$J_f(x, y) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

↪**Proposition 1.1**:
- If f, g are holomorphic and $a, b \in \mathbb{C}$, then $af + bg$ are also holomorphic, and moreover $(af + bg)' = af' + bg'$
- With $f(z) := z^n, f'(z) = nz^{n-1}$
- As a result, any polynomial on $\mathbb{C}$ is holomorphic

↪**Theorem 1.3**: If f satisfies the Cauchy-Riemann equations, then f is holomorphic.

PROOF. Write $f = u + iv$ as before. Let $h = h_1 + ih_2$. Then,

$$u(x + h_1, y + h_2) = u(x, y) + h_1 \partial_x u + h_2 \partial_y u + |h| \psi_1(h), \quad \psi_1(h) \to 0 \text{ as } h \to 0,$$

with similar for v with a remainder ψ_2. Then, by Cauchy-Riemann,

$$f(z + h) = f(z) + \left(\partial_x v - i\partial_y u\right)(h_1 + ih_2) + \psi(h)|h|, \quad \psi(h) = o(|h|).$$

Dividng both sides by h and sending $h \to 0$ gives the result. ∎

§1.3 Power Series

We say a series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n, z \in \mathbb{C}$, *converges* if $\lim_{N \to \infty} \sum_{n=0}^{N} a_n z^n$ exists as a complex number. We say it a converges *absolutely* if $\lim_{N \to \infty} \sum_{n=0}^{N} |a_n||z|^n$ exists.

↪**Theorem 1.4**: Given $\sum_{n=0}^{\infty} a_n z^n$, there exists a number $0 \leq R \leq \infty$ for which
1. if $|z| < R$, then $\sum a_n z^n$ converges absolutely;
2. if $|z| > R$, then $\sum a_n z^n$ does not converge.

Furthermore,

$$\frac{1}{R} = \limsup_n |a_n|^{\frac{1}{n}}.$$

PROOF. Let $L = \frac{1}{R}$ and suppose $|z| < R$. There exists some $\varepsilon > 0$ such that

$$r := (L + \varepsilon)|z| < 1.$$

There exists some N such that $L + \varepsilon > |a_n|^{\frac{1}{n}}$ for all $n > N$ by definition of limsup's; thus

$$\begin{aligned} |z||a_n|^{\frac{1}{n}} &< (L + \varepsilon)|z| = r < 1 \\ \Rightarrow |z|^n |a_n| &< r^n. \end{aligned}$$

But since $r < 1$, it follows that $\sum |a_n||z|^n$ converges by comparing to the geometric series $\sum r^n$.

If $|z| > R$, there is an $\varepsilon > 0$ so that there are infinitely-many n's for which $|a_n|^{\frac{1}{n}} > \frac{1}{R} - \varepsilon$, and so

$$|a_n|^{\frac{1}{n}}|z| > r > 1$$

hence $|a_n||z|^n > r^n$, so that $\sum |a_n||z|^n$ diverges by comparison. Moreover, we have shown that $|a_n||z|^n$ does not converge to zero, which implies the series does not even converge ("normally"). ■

⊛ **Example 1.1**:
1. $\sum_{n=0}^{\infty} n! z^n$ has $R = 0$
2. $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$ with $R = \infty$.
3. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ has $R = 1$.

↪**Theorem 1.5**: A power series $f(z) = \sum_{n=0}^{n} a_n z^n$ admits a derivative on its disc of convergence, and $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$.

Proof. Write $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ as the "potential" derivative we aim to show, remarking that this seires converges and moreover has the same radius of convergence as f since $\lim n^{\frac{1}{n}} = 1$ and thus $\limsup a_n^{\frac{1}{n}} = \limsup (n a_n)^{\frac{1}{n}}$. Write

$$f(z) = S_N(z) + E_N(z), \quad S_N(z) := \sum_{n=0}^{N} a_n z^n, E_N(z) := \sum_{n=N+1}^{\infty} a_n z^n.$$

Fix $z_0 \in D_R(0)$. We show $\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) \to 0$ as $h \to 0$. We can write

$$\begin{aligned} \frac{f(z_0+h) - f(z_0)}{h} - g(z_0) &= \frac{S_N(z_0+h) - S_N(z_0)}{h} - g(z_0) + \frac{E_N(z_0+h) - E_N(z_0)}{h} \\ &= \left\{\frac{S_N(z_0+h) - S_N(z_0)}{h} - S'_{N(z_0)}\right\} + \{S'_N(z_0) - g(z_0)\} + \left\{\frac{E_N(z_0+h) - E_N(z_0)}{h}\right\} \\ &= (A) + (B) + (C). \end{aligned}$$

For all $\varepsilon > 0$, there exists N_1 $|(B)| < \varepsilon$ for all $N > N_1$.

There exists N_2 such that $|(C)| < \varepsilon$ for all $N > N_2$, since we have

$$(C) = \sum_{n \geq N+1} a_n \frac{(z_0+h)^n - z_0^n}{h},$$

and

$$(z_0+h)^n - z_0^n = h\Big((z_0+h)^{n-1} + (z_0+h)^{n-2} z_0 + \cdots + (z_0+h)^{n-j} z_0^j + \cdots + z_0^{n-1}\Big).$$

Since $|z_0 + h|, |z_0| < r < R$ for h sufficiently small, we know

$$\left|(z_0+h)^n - z_0^n\right| \leq |h| n r^{n-1},$$

so that

$$\left|\frac{(z_0+h)^n - z_0^n}{h}\right| \leq n r^{n-1}.$$

It follows that

$$|(C)| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}.$$

This is the tail of an absolutely converging series, hence as $N \to \infty$, $|(C)| \to 0$, so we have the claimed bound.

Finally, let $N := \max(N_1, N_2)$. We see that for any fixed N, $(A) \to 0$ as $h \to 0$ by the definition of the derivative, and thus we can take $h = h(N)$ sufficiently small so that $|(A)| < \varepsilon$. Combining all these bounds gives the proof. ■

↪**Corollary 1.1**: $f(z) = \sum a_n z_n$ is infinitely differentiable in its radius of convergence.

↪**Definition 1.2**: A function $f : \Omega \to \mathbb{C}$ is called *analytic* if it is equal to a power series on $D_\varepsilon(z_0)$ for all $z_0 \in \Omega$, for some $\varepsilon > 0$.

↪**Corollary 1.2**: f analytic $\Rightarrow f$ holomorphic

Remark 1.1: We'll see later that these are actually equivalent notions.

§1.4 Integration Along Curves

↪**Definition 1.3**: A parametrized curve is a function $\gamma : [0,1] \to \mathbb{C}$ where γ is differentiable with continuous derivative, with $\gamma'(t) \neq 0$ for all $t \in [0,1]$.

↪**Definition 1.4**: We'll say two parametrized curves $\gamma, \tilde{\gamma}$ are equivalent if there exists a smooth function $s : [0,1] \to [0,1]$ smooth with $s'(t) > 0$ and such that $\tilde{\gamma} = \gamma \circ s$.

We will consider curves as defined up to equivalency in this way.

↪**Definition 1.5**: If γ is a parametrized curve, define

$$\int_\gamma f(z)\,\mathrm{d}z := \int_0^1 f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

If γ a piecewise smooth curve, i.e. γ can locally be written as $t \mapsto z(t) \in \mathbb{C}$ for $t \in [a_k, a_{k+1})$ for $k = 0, \ldots, n-1$ for some sequence $a_k < a_{k+1}$, then

$$\int_\gamma f(z)\,\mathrm{d}z := \sum_{k=0}^{n+1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\,\mathrm{d}t.$$

An obvious generalization holds for integration along more general intervals.

↪**Proposition 1.2**: Path integrals are independent of choice of parametrization.

↪**Definition 1.6** (Length of a curve): Define, for γ given by $z : I \to \mathbb{C}$,

$$\operatorname{length}(\gamma) := \int_\gamma |\,\mathrm{d}z| = \int_I |z'(t)|\,\mathrm{d}t.$$

↪**Proposition 1.3**: Let f, g continuous and $\alpha, \beta \in \mathbb{C}$. Then we have

1. Linearity:
$$\int_\gamma (\alpha f + \beta g)\,\mathrm{d}z = \alpha \int_\gamma f\,\mathrm{d}z + \beta \int_\gamma g\,\mathrm{d}z.$$
2. $$\int_\gamma f(z)\,\mathrm{d}z = -\int_{\gamma^-} f(z)\,\mathrm{d}z,$$
where γ^- is the *reverse path* of γ.
3. $\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \sup_{z\in\gamma}|f(z)|\,\mathrm{length}(\gamma)$.

↪**Definition 1.7** (Primitive): A *primitive* of a continuous function f on a domain Ω is a function F such that $F' = f$ on Ω.

↪**Proposition 1.4**: If f, continuous, has a primitive F on Ω and γ is a curve in Ω beginning at w_1 and ending at w_2, then
$$\int_\gamma f\,\mathrm{d}z = F(w_2) - F(w_1).$$

§1.5 Cauchy's Theorem

↪**Theorem 1.6** (Cauchy): If γ is a closed path contained in a region $\Omega \subset \mathbb{C}$ and its interior, and f is holomorphic in Ω, then $\int_\gamma f(z)\,\mathrm{d}z = 0$.

It will take us some building to get here. In a simple case, though, we have a positive result:

↪**Corollary 1.3**: If f has a primitive F on Ω, then Cauchy's theorem holds for f for any γ a closed path in $\mathrm{int}(\Omega)$

Proof. Apply the last proposition; now, $F(w_2) = F(w_1)$, so we have the result. ■

With some more work, we can also establish the proof for γ some simple contour.

↪**Proposition 1.5** (Gorsut's Lemma): Let γ be a closed triangle in Ω and f a holomorphic function on Ω. Then $\int_\gamma f(z)\,\mathrm{d}z = 0$.

Proof. I'll add it later. Basically, follows from subsequent subdivision of the triangles and approximation of the total integral of f over these triangles. ■

↪**Corollary 1.4**: If R a closed rectangle and Ω and f holomorphic on Ω, then $\int_R f(z)\,\mathrm{d}z = 0$.

Proof. A rectangle can be written as two triangles, with the "inner region" cancelling. ■

1.5.1 Primitives

↪**Theorem 1.7**: Let f be holomorphic on an open disc Ω. Then, f has a primitive on that disc.

Proof. Assume wlog that Ω centered at the origin. Fix $z \in \Omega$ and let γ_z be the path that first travels horizontally from 0 to $\mathrm{Re}(z)$ along the real axis, then vertical to z. Define

$$F(z) := \int_{\gamma_z} f(w)\,\mathrm{d}w.$$

We claim $F'(z) = f(z)$. Let $h \in \mathbb{C}$ be small so that $z + h \in \Omega$, and consider the difference

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\,\mathrm{d}w - \int_{\gamma_z} f(w)\,\mathrm{d}w.$$

These integrals have f being integrated from 0 horizontally to $\mathrm{Re}(z+h)$ then vertically to $z + h$, then, in the *opposite* orientation, from z to $\mathrm{Re}(z)$, then $\mathrm{Re}(z)$ to 0. In particular, the two components $z \to \mathrm{Re}(z)$ cancel in these two integrals, being oppositely oriented, so we are left with the contour from z vertically to $\mathrm{Re}(z)$, horizontally to $\mathrm{Re}(z+h)$, the vertically to $z + h$. Connect z to $z + \mathrm{Re}(h)$ via a horizontal line, and z to $z + h$ via a diagonal. This forms an (oriented) triangle and a rectangle, plus an extra diagonal, which by Gorsut's lemma must all integrate out to zero (draw it). Thus,

$$F(z+h) - F(z) = \int_\eta f(w)\,\mathrm{d}w,$$

where η the diagonal from z to $z + h$. Since f continuous, $f(w) = f(z) + \psi(w)$ where $\psi(w) \to 0$ as $w \to z$; thus,

$$\begin{aligned}
F(z+h) - F(z) &= f(z) \int_\eta \mathrm{d}w + \int_\eta \psi(w)\,\mathrm{d}w \\
&= f(z)h + \int_\eta \psi(w)\,\mathrm{d}w \\
\Rightarrow f(z) &= \lim_{h\to 0} \frac{F(z+h) - F(z)}{h} - \lim_{h\to 0} \frac{1}{h} \int_\eta \psi(w)\,\mathrm{d}w.
\end{aligned}$$

But since

$$\frac{1}{h}\left|\int_\eta \psi(w)\right| \leq \frac{1}{h} \sup_\eta |\psi|\,|\eta| = \sup_\eta |\psi| \underset{h\to 0}{\to} 0,$$

we have proven the claim. ■

↪**Theorem 1.8** (Cauchy's Integral Formula): Let f holomorphic on Ω containing the closure of a disc D. Let C be the boundary of this disc, then for any $z \in D$,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi.$$

Remark 1.2: The same result holds for more general curves C as long as $z \in \mathrm{int}(C)$; how/when the results extend should be clear from the proof.

↪**Corollary 1.5**: $f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z$, and more generally,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z.$$

So in general, f holomorphic $\Rightarrow f$ is infinitely differentiable.

↪**Corollary 1.6**: $\left|f^{(n)}(z_0)\right| \leq \frac{n!\|f\|_{C_R(z_0)}}{R^n}$, where $C_r(z_0)$ the circle of radius R centered at z_0.

↪**Theorem 1.9**: f is analytic centered at $z = z_0$.

Proof. We can write

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,\mathrm{d}w,$$

for some circle C containing z. We can expand

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{(w-z_0)-(z-z_0)} \\
&= \frac{1}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}} \\
&= \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left[\frac{z-z_0}{w-z_0}\right]^n \\
&= \sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}
\end{aligned}$$

so that

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C f(w)\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(w-z_0)^{n+1}}\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left[\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w\right](z-z_0)^n \\
&= \sum_{n=0}^{\infty} a_n(z-z_0)^n, \qquad a_n := \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,\mathrm{d}w.
\end{aligned}$$

But we also realize that

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

from our previous result, hence we conclude

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n,$$

as we expect from the real-valued analog. ■

Remark 1.3: In particular, this implies, from our previous result, that $|a_n| \leq \frac{C}{R^n}$, where C a constant uniform in n and R the radius of the circle upon which we're integrating. In particular, this means

$$|a_n|^{1/n} \leq \frac{C^{1/n}}{R},$$

which we see converges to $\frac{1}{R}$ as $n \to \infty$, hence our series above has radius of convergence at least R; i.e., the power series for f converges on any $D_R(z_0) \subset \Omega$.

Thus, we've shown that holomorphic $\Rightarrow$ analytic, and thus the two are equivalent (with appropriate assumptions on the space upon which they are defined, etc) since we showed the converse earlier.

↪**Theorem 1.10** (Lioville's Theorem): If f is holomorphic on $\mathbb{C}$ and bounded, then f is constant.

PROOF. We know that for any $z_0 \in \mathbb{C}$,

$$|f'(z_0)| \leq \frac{\|f\|_C}{R},$$

for any circle C with z_0 center and of radius R. Since f bounded, this means

$$|f'(z_0)| \leq \frac{1}{R} \sup_{\mathbb{C}} |f| \to 0, R \to \infty.$$

This means $f'(z_0) = 0$ everywhere and thus f is constant. We could take $R \to \infty$ since f holomorphic everywhere hence on every disc $D_R(z_0)$ for $R > 0$. ■

§1.6 Rigidity of Holomorphic Functions

↪**Theorem 1.11**: Suppose that f holomorphic in Ω and vanishes on a sequence of distinct points $z_1, \ldots, z_n \in \Omega$ with a limit point $z_\infty \in \Omega$. Then, $f \equiv 0$ on an open disc about z_∞.

PROOF. Let D be a disc centered at z_∞ and contained in Ω. We write

$$f(z) = \sum_{n \geq N} \frac{f^{(n)}(z_\infty)}{n!}(z - z_\infty)^n = a_N (z - z_\infty)^N \sum_{n=0}^{\infty} \frac{a_{N+n+1}}{a_N}(z - z_\infty)^n$$

where $N \geq 1$ the minimal integer such that $f^{(N)}(z_0) \neq 0$ and $a_n := \frac{f^{(n)}(z_\infty)}{n!}$. We see that if D sufficiently small, both

$$(z - z_\infty)^n, \quad \left(1 + \frac{a_{N+1}}{a_N}(z - z_\infty) + \cdots\right)$$

has no additional zeros in a sufficiently small disc centered at z_∞; but this contradicts the fact that $z_n \to z_\infty$, i.e. there should be infinitely many zeros when $n \to \infty$. This is a contradiction, and hence there is no minimal N for which $f^{(n)}(z_\infty)$ doesn't vanish. Hence, it must be that f identically zero on this small disc. ■

↪**Proposition 1.6**: If f holomorphic and $f(z) = 0$ on a small disc $D \subset \Omega$ then $f \equiv 0$ on Ω.

PROOF. Let

$$U = \text{int}(\{z \in \Omega : f(z) = 0\}).$$

This set is open and nonempty $(D \subset U)$. It is also closed; to see this, let $\{z_n\} \subset U$ with limit z. Then by the previous theorem, $f(z) = 0$, and thus $z \in U$ so U closed. But Ω connected, so $\Omega = U$. ■

This basically says that local behavior of holomorphic functions gives us information about the global behaviour.

↪**Corollary 1.7** (Principle of Analytic Continuation): If f, g are holomorphic on Ω and $f(z) = g(z)$ for either
(a) z in a nonempty open subset of Ω, or
(b) a sequence $\{z_n\}$ and its limit point Then $f = g$ on Ω.

PROOF. Consider $f - g$ and apply the previous. ■

1.6.1 Special Cases

1. Let $f(z) = e^z$ and let $g(z)$ be any other holomorphic extension of e^x. Then, $f = g$ on $\mathbb{R}$, and thus agree everywhere; this is the unique extension of the exponential, i.e. $e^{x+iy} = e^x(\cos y + i \sin y)$.
2. Consider the Riemann zeta function,

$$\zeta(k) = \sum_{n \geq 1} \frac{1}{n^k},$$

converges for $k = 2, 3, \ldots$. If we allow $k = u + iv \in \mathbb{C}$, we can write

$$\frac{1}{n^k} = \exp\left(\log\left(\frac{1}{n}\right)(u + iv)\right)$$

thus

$$|\frac{1}{n^k}| = \exp\left(\log\left(\frac{1}{n}\right)u\right) = \frac{1}{n^u},$$

so that

$$|\zeta(u + iv)| < \sum_{n=1}^{\infty} |\frac{1}{n^{u+iv}}| = \sum_{n=1}^{\infty} \frac{1}{n^u},$$

which converges when $u > 1$. Thus, $\zeta(s)$ for $s \in \mathbb{C}$ converges (absolutely) whenever $\text{Re}(s) > 1$. Riemann showed that $\zeta(s)$ admits a holomorphic extension to $\mathbb{C} - \{1\}$.

§1.7 Singularities of $f(z)$

↪**Definition 1.8**: If $f(z)$ is holomorphic on $D_r(z_0) - \{z_0\}$ for some $r > 0$, then z_0 is called a *singularity* of $f(z)$.

↪**Definition 1.9**:

1. z_0 is called a *removable* singularity if $f(z)$ extends to a holomorphic function on $D_r(z_0)$
2. If $\frac{1}{f(z)}$ has a removable singularity at z_0, then z_0 is called a *pole* of $f(z)$
3. Otherwise, z_0 is called an *essential singularity* of f.

⊛ **Example 1.2**:

1. $f(z) = \frac{\sin(z)}{z}$ has a removable singularity at 0 (taking $f(0) = 1$ extends f to a holomorphic function everywhere).
2. $f(z) = \frac{1}{z}$ has a pole at 0.
3. $f(z) = e^{\frac{1}{z}}$ at 0 has an essential singularity.

↪**Proposition 1.7** (Local expansions at z_0): If f is a nonzero holomorphic at z_0, then there exists a unique $m \geq 0$ such that

$$f(z) = (z - z_0)^m g(z),$$

where $g(z_0) \neq 0$.

We call m the *order of vanishing* of f at z_0.

↪**Proposition 1.8**: If $f(z)$ has a pole at $z = z_0$, then there exists a unique integer $m < 0$ such that

$$f(z) = (z - z_0)^m g(z)$$

where $g(z)$ holomorphic near z_0 and non-vanishing at z_0.

Proof. We know $\frac{1}{f(z)}$ holomorphic near z_0 so by the previous $\frac{1}{f(z)} = (z - z_0)^m g(z)$ so $f(z) = (z - z_0)^{-m} g^{-1}(z)$. Since $g(z)$ holomorphic near z_0, so is $g(z)^{-1}$. ■

↪**Definition 1.10**: A function f which is holomorphic on $\Omega - \{z_1, \ldots, z_k\}$ and has poles at $z_1, \ldots, z_k$ is called *meromorphic* on Ω.

↪**Definition 1.11**: For $f(z)$ meromorphic, put $\operatorname{ord}_{z_0}(f) =$ unique $m \in \mathbb{Z}$ such that $f(z) = (z - z_0)^m h(z)$.

Remark 1.4: $\operatorname{ord}_{z_0}(f) = 0 \Rightarrow f(z_0) \neq 0$, $\operatorname{ord}_{z_0}(f) > 0 \Rightarrow f(z_0) = 0$, $\operatorname{ord}_{z_0}(f) < 0 \Rightarrow f$ has a pole at z_0.

↪**Corollary 1.8**: If f has a pole of order m then f admits a *Laurent series expansion*

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots.$$

Proof. For $m \geq 1$, write

$$f(z) = (z - z_0)^{-m} h(z),$$

where we can expand

$$h(z) = a_{-m} + a_{-m+1}(z - z_0) + \cdots$$

since h holomorphic. ■

↪**Definition 1.12**: The quantity

$$\frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z - z_0}$$

is called the *principal part* of $f(z)$ at $z = z_0$, and is denoted $\mathrm{PP}(f)$. Thus, we may write $f(z) = \mathbb{P}(f) + g(z)$ where g is holomorphic at z_0.

Thus, if f is meromorphic at z_0, we have two representations of f:

1. $f(z) = (z -_0)^m h(z)$ where h is holomorphic with $h(z_0) \neq 0$ and $m = \mathrm{ord}_{z_0} f < 0$
2. $f(z) = \mathrm{PP}(f) + g(z)$ where $\mathrm{PP}(z)$ is a polynomial in $(z - z_0)^{-1}$ with finite degree $-\mathrm{ord}_{z_0} f$ and $g(z)$ is holomorphic.

↪**Definition 1.13**: If $\mathrm{PP}(f) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{a_{-1}}{z-z_0}$, then we call a_{-1} the *residue* of f.

Recall that

$$\int_C \frac{\mathrm{d}z}{z^n} = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

where C any circle about the origin.

↪**Theorem 1.12** (Residue Formula): Suppose $f(z)$ is meromorphic at z_0, and let C be a sufficiently small circle around z_0 contained inside the region of holomorphicity of $f(z)$. Then,

$$\mathrm{res}_{z_0} f(z) = \frac{1}{2\pi i} \int_C f(z)\, \mathrm{d}z.$$

Proof. Clear consequence of the second representation of f from above and the previous remarks. ■