MATH251 - Algebra 2

Vector spaces, linear (in)dependence, span, bases; linear transformations, kernel, image, isomorphisms, nilpotent operators.

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Contents

1 Introduction

Remark 1.1. *This course is about vector spaces and linear transformations between them; a vector space involves multiplication by* scalars, *where the scalars come from some field. We recall first examples of fields, then vector spaces, as a motivation, before presenting a formal definition.*

1.1 Vector Spaces

Remark 1.2. *Much of this is recall from Algebra 1.*

⊛ **Example 1.1: Examples of Fields**

1. $\mathbb{Q}$; the field of rational numbers.
2. $\mathbb{R}$; the field of real numbers; $\mathbb{Q} \subseteq \mathbb{R}$.
3. $\mathbb{C}$; the field of complex numbers; $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.
4. $\mathbb{F}_p \equiv \mathbb{Z}/p\mathbb{Z} \equiv \{0, 1, \ldots, p-1\}; the (unique) field of \mathrm{p} elements, where \mathrm{p} prime.$[a]
    (a) $p = 2$; $\mathbb{F}_2 \equiv \{0, 1\}$.
    (b) $p = 3$; $\mathbb{F}_3 \equiv \{0, 1, 2\}$.
    (c) $\cdots$

[a] where $a +_p b :=$ remainder of $\frac{a+b}{p}$, $a \cdot_p b :=$ remainder of $\frac{a \cdot b}{p}$.

Remark 1.3. *Throughout the course, we will denote an abstract field as* $\mathbb{F}$.

⊛ **Example 1.2: Examples of Vector Spaces**

1. $\mathbb{R}^3 := \{(x, y, z) : x, y, z \in \mathbb{R}\}$. We can add elements in $\mathbb{R}^3$, and multiply them by real scalars.

2. $\mathbb{F}^n := \underbrace{\mathbb{F} \times \mathbb{F} \times \cdots \mathbb{F}}_{n \text{ times}} := \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{F}\}$, where $n \in \mathbb{N}$[1]; this is a generalization of the previous example, where we took $n = 3$, $\mathbb{F} = \mathbb{R}$. Operations follow identically; addition:

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

   and, taking a scalar $\lambda \in \mathbb{F}$, multiplication:

$$\lambda \cdot (a_1, a_2, \ldots, a_n) := (\lambda \cdot a_1, \lambda \cdot a_2, \ldots, \lambda \cdot a_n).$$

   We refer to these elements $(a_1, \cdots, a_n)$ as *vectors* in $\mathbb{F}^n$; the vector for which $a_i = 0 \, \forall \, i$ is the 0 *vector*, and is the additive identity, making $\mathbb{F}^n$ an abelian group under addition, that admits multiplication by scalars from $\mathbb{F}$.

3. $C(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : f \text{ continuous}\}$. Here, we have the constant zero function as our additive identity ($x \mapsto 0 \, \forall \, x$), and addition/scalar multiplication of two continuous real functions are continuous.

4. $\mathbb{F}[t] := \{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n : a_i \in \mathbb{F} \, \forall \, i, n \in \mathbb{N}\}$, ie, the set of all polynomials in t with coefficients from $\mathbb{F}$. Here, we can add two polynomials;

$$(a_0 + a_1 t + \cdots + a_n t^n) + (b_0 + b_1 t + \cdots + b_m t^m) := \sum_{i=0}^{\max\{n,m\}} (a_i + b_i) t^i,$$

   (where we "take" undefined a_i/b_i's as 0; that is, if $m > n$, then $a_{m-n}, a_{m-n+1}, \ldots, a_m$ are taken to be 0). Scalar multiplication is defined

$$\lambda \cdot (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) := \lambda a_0 + \lambda a_1 t + \lambda a_2 t^2 + \cdots + \lambda a_n t^n.$$

   Here, the zero polynomial is simply 0 (that is, $a_i = 0 \, \forall \, i$).

[1] Where we take $0 \in \mathbb{N}$, for sake of consistency. Moreover, by convention, we define $\mathbb{F}^0$ (that is, when $n = 0$) to be $\{0\}$; the trivial vector space.

↪ **Definition 1.1: Vector Space**

A *vector space* V over a field $\mathbb{F}$ is an *abelian group* with an operation denoted $+$ (or $+_V$) and identity element[2] denoted 0_V, equipped with *scalar multiplication* for each scalar $\lambda \in \mathbb{F}$ satisfying the following axioms:

1. $1 \cdot v = v$ for $1 \in \mathbb{F}$, $\forall v \in V$.
2. $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta)v$, $\forall \alpha, \beta \in \mathbb{F}, v \in V$.
3. $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$, $\forall \alpha, \beta \in \mathbb{F}, v \in V$.
4. $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$, $\forall \alpha \in \mathbb{F}, u, v \in V$.

We refer to elements $v \in V$ as *vectors*.

↪ **Proposition 1.1**

For a vector space V over a field $\mathbb{F}$, the following holds:

1. $0 \cdot v = 0_V$, $\forall v \in V$ (where $0 := 0_{\mathbb{F}}$)
2. $-1 \cdot v = -v$, $\forall v \in V$ (where $1 := 1_{\mathbb{F}}$)[3]
3. $\alpha \cdot 0_V = 0_V$, $\forall \alpha \in \mathbb{F}$

Proof.

1. $0 \cdot v = (0 + 0) \cdot v = 0 \cdot v + 0 \cdot v \implies 0 \cdot v = 0_V$ (by "cancelling" one of the $0 \cdot v$ terms on each side).
2. $v + (-1 \cdot v) = (1 \cdot v + (-1) \cdot v) = (1 - 1) \cdot v = 0 \cdot v = 0_V \implies (-1 \cdot v) = -v$.
3. $\alpha \cdot 0_V = \alpha \cdot (0_V + 0_V) = \alpha \cdot 0_V + \alpha \cdot 0_V \implies \alpha \cdot 0_V = 0_V$ (by, again, cancelling a term on each side).

■

↪ *Lecture 01; Last Updated: Wed Jan 10 14:16:29 EST 2024*

1.2 Creating Spaces from Other Spaces

[2] The "zero vector".

[3] NB: "additive inverse"

↪ **Definition 1.2: Product/Direct Sum of Vector Spaces**

For vector spaces U, V over the same field $\mathbb{F}$, we define their *product* (or *direct sum*) as the set

$$U \times V = \{(u, v) : u \in U, v \in V\},$$

with the operations:

$$(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$$
$$\lambda \cdot (u, v) := (\lambda \cdot u, \lambda \cdot v)$$

⊛ **Example 1.3: $\mathbb{F}$**

$\mathbb{F}^2 = \mathbb{F} \times \mathbb{F}$, where $\mathbb{F}$ is considered as the vector space over $\mathbb{F}$ (itself).

↪ **Definition 1.3: Subspace**

For a vector space V over a field $\mathbb{F}$, a *subspace* of V is a subset $W \subseteq V$ s.t.

1. $0_V \in W$[4]
2. $u + v \in W \, \forall \, u, v \in W$ (closed under addition)
3. $\alpha \cdot u \in W \, \forall \, u \in W, \alpha \in \mathbb{F}$[5]

Then, W is a vector space in its own right.

[4] This is equivalent to requiring that $W \neq \varnothing$; stated this way, axiom 3. would necessitate that $0 \cdot w = 0_V \in W$.

[5] Note that these axioms are equivalent to saying that W is a subgroup of V with respect to vector addition; 2. ensures closed under addition, and 3. ensures the existence of additive inverses (as per $-1 \cdot v = -v$).

⊛ **Example 1.4: Examples of Subspaces**

1. Let $V := \mathbb{F}^n$.

   - $W := \{(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n : x_1 = 0\} = \{(0, x_2, x_3, \ldots, x_n) : x_i \in \mathbb{F}\}$.
   - $W := \{(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n : x_1 + 2 \cdot x_2 = 0\}$

     Proof. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in W$. Then, $x + y = (x_1 + y_1, \ldots, x_n + y_n)$, and $x_1 + y_1 + 2 \cdot (x_2 + y_2) = x_1 + 2 \cdot x_2 + y_1 + 2 \cdot y_2 = 0 + 0 = 0 \implies x + y \in W$. Similar logic follows for axioms 2., 3. ■

   - (More generally)

     $$W := \{(x_1, \ldots, x_n) \in \mathbb{F}^n : \begin{matrix} a_{11}x_1 + \cdots a_{1n}x_n = 0 \\ a_{21}x_1 + \cdots + a_{2n}x_n = 0 \\ \ddots \\ a_{k1}x_1 + \cdots a_{kn}x_n = 0 \end{matrix}\},$$

     that is, a linear combination of homogenous "conditions" on each term.

   - $W^* := \{(x_1, \ldots, x_n) : x_1 + x_2 = 1\}$ is *not* a subspace; it is not closed under addition, nor under scalar multiplication.

2. Let $\mathbb{F}[t]_n := \{a_0 + a_1 t + \cdots + a_n t^n : a_i \in \mathbb{F}\}$. Then, $\mathbb{F}[t]_n$ is a subspace of $\mathbb{F}[t]$, the more general polynomial space. *However*, the set of all polynomials of degree *exactly* n (all axioms fail, in fact) is not a subspace of $\mathbb{F}[t]_n$.

   - $W := \{p(t) \in \mathbb{F}[t]_n : p(1) = 0\}$.
   - $W := \{p(t) \in \mathbb{F}[t]_n : p''(t) + p'(t) + 2p(t) = 0\}$.

3. Let $V := C(\mathbb{R})$ be the space of continuous function $\mathbb{R} \to \mathbb{R}$.

   - $W := \{f \in C(\mathbb{R}) : f(\pi) + 7f(\sqrt{2}) = 0\}$.
   - $W := C^1(\mathbb{R}) :=$ everywhere differentiable functions.
   - $W := \{f \in C(\mathbb{R}) : \int_0^1 f \, \mathrm{d}x = 0\}$.

↪ **Proposition 1.2**

Let W_1, W_2 be subspaces of a vector space V over $\mathbb{F}$. Then, define the following:

1. $W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$
2. $W_1 \cap W_2 := \{w \in V : w \in W_1 \land w \in W_2\}$

These are both subspaces of V.

Proof.

1. (a) $0_V \in W_1$ and $0_V \in W_2 \implies 0_V = 0_V + 0_V \in W_1 + W_2$.
   (b) $(u_1 + u_2) + (v_1 + v_2) = (u_1 + v_1) + (u_2 + v_2) \in W_1 + W_2$.
   (c) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v \in W_1 + W_2$
2. (a) $0_V \in W_1$ and $0_V \in W_2 \implies 0_V = 0_V + 0_V \in W_1 \cap W_2$.
   (b) $u, v \in W_1 \cap W_2 \implies u + v \in W_1 \land u + v \in W_2 \implies u + v \in W_1 \cap W_2$.
   (c) $\alpha \cdot u \in W_1 \land \alpha \cdot u \in W_2 \implies \alpha \cdot u \in W_1 \cap W_2$.

■

1.3 Linear Combinations and Span

↪ **Definition 1.4: Linear Combination**

Let V be a vector space over a field $\mathbb{F}$. For finitely many vectors $v_1, v_2, \ldots, v_n$, their *linear combination* is a sum of the form

$$\sum_{i=1}^{n} a_i v_i = a_1 \cdot v_1 + \cdots + a_n \cdot v_n,$$

where $a_i \in \mathbb{F} \, \forall \, i$.

A linear combination is called *trivial* if $a_i = 0 \, \forall \, i$, that is, all coefficients are 0.

If $n = 0$ (ie, we are "summing up" 0 vectors), we define the sum as the zero vector; $\sum_{i=1}^{0} a_i v_i := 0_V$.

↪ *Lecture 02; Last Updated: Mon Feb 12 08:36:21 EST 2024*

↪ **Definition 1.5: A More General Definition of Linear Combination**

For a (possibly infinite) set S of vectors from V, a *linear combination* of vectors in S is a linear combination of $a_1 v_1 + \cdots a_n v_n$ for some finite subset $\{v_1, \ldots, v_n\} \subseteq S$.[6]

[6] That is, we do not allow infinite sums.

↪ **Definition 1.6: Span**

For a subset $S \subseteq V$, we define its *span* as

$$\operatorname{Span}(S) := \text{ set of all linear combinations of } S := \{a_1 v_1 + \cdots a_n v_n : a_i \in \mathbb{F}, v_i \in S\}.$$

By convention, we set $\operatorname{Span}(\varnothing) = \{0_V\}$.

⊛ **Example 1.5**

Let $S := \{(1,0,-1),(0,1,-1),(1,1,-2)\} \subseteq \mathbb{R}^3$. Then,

$$0_{\mathbb{R}^3} = (0,0,0) = 1 \cdot (1,0,-1) + 1 \cdot (0,1,-1) + -1 \cdot (1,1,-2).$$

We claim, moreover, that $\operatorname{Span}(S) = U := \{(x,y,z) \in \mathbb{R}^3 : x+y+z=0\}$ (a plane through the origin).

Proof. Note that $S \subseteq U$, hence $S \subseteq \operatorname{Span} S \subseteq U$. OTOH, if $(x,y,z) \in U$, we have $z = -x-y$, and so

$$(x,y,z) = (x,y,-x-y) = x \cdot (1,0,-1) + y \cdot (0,1,-1) \in \operatorname{Span}(S)$$

hence $U \subseteq \operatorname{Span}(S)$ and thus $\operatorname{Span}(S) = U$. ■

Remark 1.4. *We implicitly used the following claim in the proof above; we prove it more generally.*

↪ **Proposition 1.3**

Let V be a vector space over $\mathbb{F}$ and let $S \subseteq V$. Then, $\operatorname{Span}(S)$ is always a subspace. Moreover, it is the smallest (minimal) subspace containing S (that is, for any subspace $U \supseteq S$, we have that $U \supseteq \operatorname{Span} S$).

Proof. Because adding/scalar multiplying linear combinations of elements of S again results in a linear combination of elements of S, and $0_V \in \operatorname{Span}(S)$ by definition, we have that $\operatorname{Span}(S)$ is indeed a subspace.

If $U \supset S$ is a subspace of V containing S, then by definition U is closed under addition, that is, taking linear combinations of its elements (in particular, of elements of S); hence, $U \supset \operatorname{Span}(S)$. ■

↪ **Lemma 1.1**

For $S \subseteq V$ and $v \in V$, $v \in \operatorname{Span}(S) \iff \operatorname{Span}(S \cup \{v\}) = \operatorname{Span}(S)$.

Proof. ($\implies$) Let $v \in \operatorname{Span}(S) \implies v = a_1 v_1 + \cdots a_n v_n, a_i \in \mathbb{F}, v_i \in V$. Then, for any linear combination

$$b_1 u_1 + \cdots b_m u_m + b \cdot v = b_1 u_1 + \cdots b_m u_m + b(a_1 v_1 + \cdots + a_n v_n)$$

is a linear combination of vectors in $S \cup \{v\}$ (first equality) or equivalently, a combination of vectors in S (second equality) and thus $\operatorname{Span}(S \cup \{v\}) \subseteq \operatorname{Span} S$. The reverse inclusion follows trivially.

$(\impliedby)$ $\mathrm{Span}(S \cup \{v\}) = \mathrm{Span}\, S \implies v \in \mathrm{Span}(S)$. ■

⊛ **Example 1.6**

(From the above example) We have

$$\mathrm{Span}(\{(1,0,-1),(0,1,-1)\} \cup \{(1,1,-2)\}) = \mathrm{Span}(\{(1,0,-1),(0,1,-1)\}),$$

since $(1,1,-2) \in \mathrm{Span}(\{(1,0,-1),(0,1,-1)\})$ (it was redundant, as it could be generated by the other two vectors).

↪ **Definition 1.7: Spanning Set**

Let V be a vector space over a field $\mathbb{F}$. We call $S \subseteq V$ a *spanning set* for V if $\mathrm{Span}(S) = V$. We call such a spanning set *minimal* if no proper subset of S is a spanning set ($\nexists v \in S$ s.t. $S \setminus \{v\}$ spanning).

Remark 1.5. *Note that any $S \subseteq V$ is a spanning for $\mathrm{Span}(S)$. But, S may not be minimal; indeed, consider the previous example. We were able to remove a vector from S while having the same span.*

⊛ **Example 1.7**

For $\mathbb{F}^n$ as a vector space over $\mathbb{F}$, the *standard spanning set*

$$\underset{n}{\mathrm{St}} := \{\underbrace{(1,\dots,0)}_{:=e_1}, \underbrace{(0,1,0,\dots,0)}_{:=e_2}, \dots, \underbrace{(0,\dots,1)}_{e_n}\}.$$

Given any $x := (x_1, \dots, x_n) \in \mathbb{F}^n$, we can write

$$x = x_1 \cdot e_1 + \cdots x_n \cdot e_n.$$

This is clearly minimal; removing any e_i would then result in a 0 in the ith "coordinate" of a vector, hence $\mathrm{St} \setminus \{e_i\}$ would span only vectors whose ith coordinate is 0.

↪ **Definition 1.8: Linear Dependence**

Let V be a vector space over a field $\mathbb{F}$. A set $S \subseteq V$ is said to be *linearly dependent* if there is a nontrivial linear combination of vectors in S that is equal to 0_V.

Conversely, S is called *linearly independent* if there is no nontrivial linear combination of vectors in S that is equal to 0_V; all linear combinations of vectors in S that equal 0_V are trivial.

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⊛ **Example 1.8**

1. The empty set $\varnothing$ is linearly independent; there are no non-trivial linear combinations that equal 0_V (there are no linear combinations at all).

2. For $v \in V$, the set $\{v\}$ is linearly dependent iff $v = 0_V$.

3. $S := \{(1,0,-1),(0,1,-1),(1,1,-2)\} := \{v_1,v_2,v_3\}$; S is linearly dependent ($v_1+v_2-v_3 = (0,0,0)$).

4. $V := \mathbb{F}^3$; $S := \{(1,0,-1),(0,1,-1),(0,0,1)\} = \{v_1,v_2,v_3\}$ is linearly independent.

   Proof. Suppose

$$
\begin{aligned}
a_1v_1 + a_2v_2 + a_3v_3 = 0_V \\
&\implies a_1 = 0 \wedge a_2 = 0 \wedge -a_1 - a_2 + a_3 = 0 \implies a_3 = 0 \\
&\implies a_1 = a_2 = a_3 = 0
\end{aligned}
$$

   Hence only a trivial linear combination is possible. ∎

5. St_n is linearly independent.

   Proof.

$$\sum_{i=1}^{n} a_i e_i = 0_{\mathbb{F}^n} \implies a_i = 0 \,\forall\, i$$

   ∎

↪ **Lemma 1.2**

Let V be a vector space over a field $\mathbb{F}$, and $S \subseteq V$ (possibly infinite).

1. S is linearly dependent $\iff$ there is a finite subset $S_0 \subseteq S$ that is linearly dependent.

2. S is linearly independent $\iff$ all finite subsets of S are linearly independent.

Proof. 2. follows from the negation of 1.

($\Longleftarrow$) Trivial.

($\implies$) Suppose S linearly dependent. Then, 0_V = some nontrivial linear combination of vectors $v_1, \ldots, v_n$ in S. Let $S_0 = \{v_1, \ldots, v_n\}$, then, S_0 is linearly dependent itself. ∎

1.4 Linear Dependence and Span

$\hookrightarrow$ **Proposition 1.4**

Let V be a vector space over a field $\mathbb{F}$ and $S \subseteq V$.

1. S linearly dependent $\iff$ $\exists v \in \operatorname{Span}(S \setminus \{v\})$.

2. S linearly independent $\iff$ there is no $v \in \operatorname{Span}(S \setminus \{v\})$.

Proof. 2. follows from the negation of 1.

($\implies$) Suppose S linearly dependent. Then, $0_V = \sum_{i=1}^n a_i v_i$ for some nontrivial linear combination of distinct vectors S. At least one of $a_i \neq 0$; we can assume wlog (reindexing) $a_1 \neq 0$. Then,

$$a_1 v_1 = -\sum_{i=2}^n a_i v_i \implies v_1 = (-a_1^{-1}) \sum_{i=2}^n a_i v_i = \sum_{i=2}^n (-a_1^{-1} a_i) v_i,$$

hence, $v_1 \in \operatorname{Span}(\{v_2, \ldots, v_n\}) \subseteq \operatorname{Span}(S \setminus \{v\})$

($\impliedby$) Suppose $v \in \operatorname{Span}(S \setminus \{v\})$, then $v = a_1 v_1 + \cdots + a_n v_n$, with $v_1, \ldots, v_n \in S \setminus \{v\}$, thus

$$0_V = a_1 v_1 + \cdots a_n v_n - v,$$

which is not a trivial combination (-1 on the v; v cannot "merge" with the other vectors), hence S is linearly dependent. ■

$\hookrightarrow$ **Corollary 1.1**

$S \subseteq V$ is linearly independent $\iff$ S a minimal spanning set of $\operatorname{Span} S$.

Proof. Follows from proposition 1.4, 2. ■

$\hookrightarrow$ **Definition 1.9: Maximally Independent**

Let V be a vector space over a field $\mathbb{F}$. A set $S \subseteq V$ is called *maximally independent* if S is linearly independent and $\nexists v \in V \setminus S$ s.t. $S \cup \{v\}$ is still linearly independent.

In other words, there is no proper supset $\tilde{S} \supsetneq S$ that is still independent.

$\hookrightarrow$ **Lemma 1.3**

If $S \subseteq V$ maximally independent, then S is spanning for V.

Proof. Let $S \subseteq V$ be maximally independent. Let $v \in V$; supposing $v \notin S$ (in the case that $v \in S$, then $v \in \operatorname{Span}(S)$ trivially). By maximality, $S \cup \{v\}$ is linearly dependent, hence there exists a nontrivial linear combination that equals

0_V. Since S independent, this combination must include v, with a nonzero coefficient. We can write

$$av + \sum_{i=1}^{n} a_i v_i = 0_V \quad a \neq 0, v_i \in S$$
$$\implies v = \sum_{i=1}^{n} (-a^{-1} a_i) v_i \in \operatorname{Span} S.$$

■

↪ **Theorem 1.1**

Let V be a vector space over a field $\mathbb{F}$ and let $S \subseteq V$. TFAE:

1. S is a minimal spanning set;
2. S is linearly independent and spanning;
3. S is a maximally linearly independent set;
4. Every vector in V is equal to *unique* linear combination of vectors in S.

↪ *Lecture 04; Last Updated: Fri Jan 26 13:32:10 EST 2024*

Proof. (1. $\implies$ 2.) Suppose S is spanning for V and is minimal. Then, by corollary 1.1, we have that S is linearly independent, and is thus both linearly independent and spanning.

(2. $\implies$ 3.) Suppose S is linearly independent and spanning. Let $v \in V \setminus S$; S is spanning, hence $v \in \operatorname{Span} S$, that is, there exists a linear combination of vectors in S that is equal to v:

$$v = a_1 v_1 + \cdots + a_n v_n, a_i \in \mathbb{F}, v_i \in S.$$

Thus, $0_V = a_1 v_1 + \cdots + a_n v_n - v$, thus $S \cup \{v\}$ is linearly dependent, and so S is maximally linearly independent.

(3. $\implies$ 1.) Suppose S is maximally linearly independent. By lemma 1.3, S is spanning, and since S is linearly independent, by corollary 1.1, S is minimally spanning for $\operatorname{Span} S$.

(2. $\implies$ 4.) Suppose S is linearly independent and spans V, and let $v \in V$. We have that $v \in \operatorname{Span} S$ and hence is equal to a linear combination of vectors in S. This gives existence; we now need to prove uniqueness.

Suppose there exist two linear combinations that equal v,

$$v = a_1 v_1 + \cdots + a_n v_n = b_1 u_1 + \cdots + b_m u_m,$$

$a_i, b_j \in \mathbb{F}$, $v_i, u_j \in S$. With appropriate reindexing/relabelling and allowing certain scalars to equal 0, we can assume that the combinations use the same vectors (with potentially different coefficients), that is,

$$v = a_1 w_1 + \cdots + a_k w_k = b_1 w_1 + \cdots + a_k w_k.$$

This implies, then,

$$(a_1 - b_1) w_1 + \cdots + (a_k - b_k) w_k = 0_V,$$

and by the assumed linear independent of S, each coefficient $(a_i - b_i) = 0 \, \forall \, i \implies a_i = b_i \, \forall \, i$, hence, these are indeed the same representations, and thus this representation is unique.

(4. $\implies$ 2.) Suppose every vector in V admits a unique linear combination of vectors in S. Clearly, then, S is spanning. It remains to show S is linearly independent. Suppose

$$0_V = a_1 v_1 + \cdots + a_n v_n$$

for $v_i \in S$. But we have that every vector has a unique representation, and we know that $a_i = 0 \, \forall \, i$ is a (valid) linear combination that gives 0_V; hence, this must be the unique combination, $a_i = 0 \, \forall \, i$, and the linear combination above is trivial. Hence, S is linearly independent and spanning. ■

$\hookrightarrow$ **Definition 1.10: Basis**

If any (hence all) of the above statements hold, we call S a *basis* for V.

In the words of 4., we call the unique linear combination of vectors in S that is equal to v the *unique representation of v in S*. Its coefficients are called the *Fourier coefficients of v in S*.

⊛ **Example 1.9**

1. $\mathrm{St}_n = \{e_i : 1 \leqslant i \leqslant n\}$ is a basis for $\mathbb{F}^n$.
2. In $\mathbb{F}^3$, the set
$$\{(1, 0, -1), (0, 1, -1), (0, 0, 1)\}$$
is a basis; it is linearly independent and spanning.
3. For $\mathbb{F}[t]_n$, the standard basis is
$$\{1, t, t^2, \ldots, t^n\}.$$
4. For $\mathbb{F}[t]$, the standard basis is
$$S := \{1, t, t^2, \ldots\} = \{t^n : n \in \mathbb{N}\}.$$
5. Let $\mathbb{F}[\![t]\!]$ denote the space of all formal power series $\sum_{n \in \mathbb{N}} a_n t^n$; polynomials are an example, but with only finite nonzero coefficients. Note that, then, the set S defined above is not a basis for this "extended" set. We *can* in fact find a basis for this set; we need more tools first.

$\hookrightarrow$ **Theorem 1.2**

Every vector space has a basis.

Remark 1.6. *This theorem relies on assuming the Axiom of Choice.*

$\hookrightarrow$ Lecture 05; Last Updated: Wed Jan 17 13:37:26 EST 2024

Proof (Attempt). (Of theorem 1.2) We will try to "inductively" build a maximally independent set, as follows:

Begin with an empty set $S_0 := \varnothing$, and iteratively add more vectors to it. Let $v_0 \in V$ be a non-zero vector, and let $S_1 := \{v_0\}$.

If S_1 is maximal, then we are done. Otherwise, there exists a new vector $v_1 \in V \setminus S_1$ s.t. $S_2 := \{v_0, v_1\}$ is still independent.

If S_2 is maximal, then we are done. Otherwise, there exists a new vector $v_2 \in V \setminus S_2$ s.t. $S_3 := \{v_0, v_1, v_2\}$ is still independent.

Continue in this manner; this would take arbitrarily many finite, or even infinite, steps; we would need some "choice function" that would "allow" us to choose any particular ith vector v_i.

We can make this construction precise via the Axiom of Choice and transfinite induction (on ordinals); alternatively, we will prove a statement equivalent to the Axiom of Choice, Zorn's Lemma. ■

Remark 1.7. *Before stating Zorn's Lemma, we introduce the following terminology.*

↪ **Axiom 1.1: Axiom of Choice**

Let X be a set of nonempty sets. Then, there exists a choice function f defined on X that maps each set of X to an element of that set.

↪ **Definition 1.11: Inclusion-Maximal Element**

A *inclusion-maximal* element of I is a set $S \in I$ s.t. there is no strict super set $S' \supsetneq S$ s.t. $S' \in I$.

↪ **Definition 1.12: Chain**

Let X a set. Call a collection $\mathcal{C} \subseteq \mathcal{P}(X)$ a *chain* if any two $A, B \in \mathcal{C}$ are comparable, ie, $A \subseteq B$ or $B \subseteq A$.

↪ **Definition 1.13: Upper Bound**

An *upper bound* of a collection $\tau \subseteq \mathcal{P}(X)$ is a set $U \subseteq X$ s.t. $U \supseteq J \,\forall\, J \in \tau$; U contains the union of all sets in J.

⊛ **Example 1.10: Of The Previous Definitions**

Let $X := \mathbb{N}, I := \{\varnothing, \{0\}, \{1, 2\}, \{1, 2, 3\}\} \subseteq \mathcal{P}(\mathbb{N})$.

The maximal elements of I would be $\{0\}$ and $\{1, 2, 3\}$.

Chains would include $\mathcal{C}_0 := \{\varnothing, \{1, 2\}, \{1, 2, 3\}\}, \mathcal{C}_1 := \{\varnothing, \{0\}\}, \mathcal{C}_2 := \{\varnothing\}$ (or any set containing a single element).

The sets $\{0, 1, 2, 3\}$ and $\{0, 1, 2, 3, 4, 5\}$ are upper bounds for I, while neither is an element of I. The set $\{1, 2, 3\}$ is an upper bound for $\mathcal{C}_0$. A chain $\{\varnothing, \{0\}, \{0, 1\}, \{0, 1, 2\}, \dots\}$ has an upper bound of $\mathbb{N}$.

↪ **Lemma 1.4: Zorn's Lemma**

Let X be an ambient set and $I \subseteq \mathcal{P}(X)$ be a nonempty collection of subsets of X. If every chain $\mathcal{C} \subseteq I$ has an upper bound in I, then I has a maximal element.

"Proof". This is equivalent to the Axiom of Choice; proving it is beyond the scope of this course :(. ■

Proof of theorem 1.2, cnt'd. We obtain a maximal independent set using Zorn's Lemma.

Let I be the collection of all linearly independent subsets of V. I is nonempty; $\varnothing \in I$, as is $\{v\} \in I$ for any nonzero $v \in V$. To apply Zorn's, we need to show that every chain $\mathcal{C}$ if sets in I has an upper bound in I; that is, every linearly independent set has an upper bound that itself is linearly independent.

Let $\mathcal{C}$ be a chain in I. Let $S := \bigcup \mathcal{C}$ be the union of all sets in $\mathcal{C}$. To show S is linearly independent, it suffices to show that every finite subset $\{v_1, \ldots, v_n\} \subseteq S$ is linearly independent. Let $S_i \in \mathcal{C}$ be s.t. $v_i \in S_i$ for each i. Because $\mathcal{C}$ a chain, for each i, j we have either $S_i \subseteq S_j$ or $S_j \subseteq S_i$, and so we can order $S_1, \ldots, S_n$ in increasing order w.r.t $\subseteq$. This implies, then, there is a maximal S_{i_0} s.t. $S_{i_0} \supseteq S_i \,\forall\, i \in \{1, \ldots, n\}$. Moreover, we have that $\{v_1, \ldots, v_n\} \in S_{i_0}$, and that S_{i_0} is linearly independent and thus $\{v_1, v_2, \ldots, v_n\}$ is also linearly independent.

Thus, as we can apply Zorn's Lemma, we conclude that I has a maximal element, ie, there is a maximal independent set, and thus a V indeed has a basis. ■

↪ Lecture 06; Last Updated: Fri Jan 19 13:36:58 EST 2024

↪ **Theorem 1.3**

For every vector space V over a field $\mathbb{F}$, any two bases $\mathcal{B}_1, \mathcal{B}_2$ are equinumerous/of equal size/cardinality, ie, there is a bijection between $\mathcal{B}_1$ and $\mathcal{B}_2$.

Remark 1.8. *We will only prove this for vector spaces that admit a finite basis.*

↪ **Lemma 1.5: Steinitz Substitution**

Let V be a vector space over a field $\mathbb{F}$. Let $Y \subseteq V$ be a (possibly infinite) linearly independent set and let $Z \subseteq V$ be a finite spanning set. Then:

1. $k := |Y| \leqslant |Z| =: n$
2. There is $Z' \subseteq Z$ of size $n - k$ s.t. $Y \cup Z'$ is still spanning.

Proof. We prove by induction on k.

$k = 0$ gives that $Y = \varnothing$, and so $Z' = Z$ itself works ($Z' \cup Y = Z$) as a spanning set.

Suppose the statement holds for some $k \geqslant 0$. Let Y be an independent set such that $|Y| = k + 1$, ie

$$Y := \{y_1, y_2, \ldots, y_k, y_{k+1}\}, \quad y \in V.$$

By our inductive assumption, we can consider $Y' := \{y_1, \ldots, y_k\} \subseteq Y$ of size k, to obtain a set

$$Z' = \{z_1, z_2, \ldots, z_{n-k}\} \subseteq Z, \text{ s.t. } Y' \cup Z' = \{y_1, \ldots, y_k, z_1, \ldots, z_{n-k}\}$$

is spanning. As this is spanning, we can write y_{k+1} as a linear combination of vectors in $Y' \cup Z'$, ie

$$y_{k+1} = a_1 y_1 + \cdots + a_k y_k + b_1 z_1 + \cdots + b_{n-k} z_{n-k}, \quad a_i, b_j \in \mathbb{F}.$$

It must be that at least one of b_j's must be nonzero; if they were all zero, then y_{k+1} would simply be a linear combination of vector y_i giving that y_{k+1} linearly dependent, contradicting our construction of Y linearly independent.

Assume, wlog, $b_{n-k} \neq 0$. Then, we can write

$$z_{n-k} = b_{n-k}^{-1} y_{k+1} - b_{n-k}^{-1} a_1 y_1 - \cdots - b_{n-k}^{-1} a_k y_k - b_{n-k}^{-1} b_1 z_1 - \cdots - b_{n-k}^{-1} b_{n-k-1} z_{n-k-1},$$

and hence

$$z_{n-k} \in \operatorname{Span}\{y_1, \ldots, y_{k+1}, z_1, \ldots, z_{n-k-1}\} = \operatorname{Span}\left(\underbrace{\{y_1, \ldots, y_{k+1}\}}_{Y} \cup \underbrace{\{z_1, \ldots, z_{n-k-1}\}}_{:=Z''}\right).$$

We had that $Y' \cup Z'$ was spanning, and $(Y' \cup Z') \setminus (Y \cup Z'') = \{z_{n-k}\} \subseteq \operatorname{Span}(Y \cup Z'')$, and we thus have that $Y \cup Z''$ is also spanning. ■

↪ **Corollary 1.2: Finite Basis Case for theorem 1.3**

Let V be a vector space that admits a finite basis. Then, any two bases of V are equinumerous.

Proof. Let Y, Z be two finite bases for V. Then, Y is independent and Z is spanning, so by Steinitz Substitution, $|Y| \leqslant |Z|$. OTOH, Z is independent, and Y is spanning, so by Steinitz Substitution, $|Z| \leqslant |Y|$, and we conclude that $|Y| = |Z|$. Let $n := |Y|$.

It remains to show that there exist no infinite bases for V; it suffices to show that there is no independent set of size $n+1$. To this end, let $I \subseteq V$ such that $|I| = n+1$ be an independent set. Y is still spanning, hence, by the substitution lemma, $n+1 \leqslant n$, a contradiction. Hence, I as defined cannot exist and so any basis of V must be of size n. ■

↪ **Definition 1.14: Dimension**

Let V be a vector space over a field $\mathbb{F}$. The *dimension* of V, denote

$$\dim(V)$$

as the cardinality/size of any basis for V. We call V *finite dimensional* if $\dim(V)$ is a natural number, i.e. V admits a finite basis. Otherwise, we say V is infinite dimensional.

↪ **Corollary 1.3: of Steinitz Substitution**

Let V be a finite dimensional vector space over $\mathbb{F}$ and denote $n := \dim(V)$. Then:

1. Every linearly independent subset $I \subseteq V$ has size $\leqslant n$;
2. Every spanning set $S \subseteq V$ for V has size $\geqslant n$;
3. Every independent set I can be completed to a basis to V, ie, there exists a basis B for V s.t. $I \subseteq B$.

Proof. Fix a basis B for V, $|B| =: n$.

1. If I is a independent set, then because B spanning, Steinitz Substitution gives $|I| \leqslant |B|$.
2. If S spanning for V, then because B is linearly independent, Steinitz Substitution gives $|B| \leqslant |S|$.
3. Let I be an independent set. Then, because B is spanning, Steinitz Substitution gives $B' \subseteq B$ of size $n - |I|$ s.t. $I \cup B'$ is spanning. Moreover, $|I \cup B'| \leqslant n$, and by 2. it must have size $\geqslant n$, and thus has size precisely n and is thus a minimally spanning set and thus a basis.

■

↪ **Corollary 1.4: Monotonicity of Dimension**

Let V be a vector space over a field $\mathbb{F}$. For any subspace $W \subseteq$, $\dim W \leqslant \dim V$, and

$$\dim W = \dim V \iff W = V.$$

Proof. Let $B \subseteq W$ be a basis for W. Because B is independent, $|B| \leqslant \dim(V)$ by 1. of corollary 1.3, so $\dim(W) = |B| \leqslant \dim(V)$.

If $|B| = \dim(V)$, then B is a basis for V again by 1. of corollary 1.3, so $W = \mathrm{Span}(B) = V$. ■

↪ *Lecture 07; Last Updated: Mon Jan 22 13:43:44 EST 2024*

2 Linear Transformations

2.1 Definitions

↪ **Definition 2.1: Linear Transformation**

Let V, W be vector spaces over a field $\mathbb{F}$. A function $T : V \to W$ is called a *linear transformation* if it preserves the vector space structures, that is,

1. $T(v_0 + v_1) = T(v_0) + T(v_1)$, $\forall\, v_0, v_1 \in V$;
2. $T(\alpha \cdot v) = \alpha \cdot T(v)$, $\forall\, \alpha \in \mathbb{F}, v \in V$;
3. $T(0_V) = 0_W$.

Remark 2.1. *Note that 3. is redundant, implied by 2., but included for emphasis:*

$$T(0_V) = T(0_{\mathbb{F}} \cdot 0_V) = 0_{\mathbb{F}} \cdot T(0_V) = 0_W.$$

⊛ **Example 2.1: Linear Transformations**

1. $T : \mathbb{F}^2 \to \mathbb{F}^2$, $T(a_1, a_2) := (a_1 + 2a_2, a_1)$.
2. Let $\theta \in \mathbb{R}$, and let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation by θ. The linearity of this is perhaps most obvious in polar coordinates, ie $v \in \mathbb{R}^2$, $v = r(\cos\alpha, \sin\alpha)$ for appropriate r, α, and $T_\theta(v) = r(\cos(\alpha + \theta), \sin(\alpha + \theta))$.
3. $T : \mathbb{R}^2 \to \mathbb{R}^2$, a reflection about the x-axis, ie, $T(x, y) = (x, -y)$.
4. Projections, $T : \mathbb{F}^n \to \mathbb{F}^n$.
5. The transpose on $M_n(\mathbb{F})$, ie, $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$, where $A \mapsto A^t$.
6. The derivative on space of polynomials of degree leq n, $D : \mathbb{F}[t]_{n+1} \to \mathbb{F}[t]_n, p(t) \mapsto p'(t)$.

↪ **Theorem 2.1**

Linear transformations are completely determined by their values on a basis.

That is, let $\mathcal{B} := \{v_1, \dots, v_n\}$ be a basis for a vector space V over $\mathbb{F}$. Let W also be a vector space over $\mathbb{F}$ and let $w_1, \dots, w_n \in W$ be arbitrary vectors. Then, there is a unique linear transformation $T : V \to W$ s.t. $T(v_i) = w_i \,\forall\, i = 1, \dots, n$.

Proof. We aim to define $T(v)$ for arbitrary $v \in V$. We can write

$$v = a_1 v_1 + \cdots + a_n v_n$$

as the unique representation of v in terms of the basis $\mathcal{B}$. Then, we simply define

$$T(v) := a_1 w_1 + \cdots + a_n w_n,$$

for our given w_i's. Then, $T(v_i) = 1 \cdot w_i = w_i$, as desired, and T is linear;

1. Let $u, v \in V; u := \sum_n a_i v_i, v := \sum_n b_i v_i$. Then,

$$T(u+v) = T(\sum_n a_i v_i + \sum_n b_i v_i) = T(\sum_n (a_i + b_i) v_i) = \sum_n (a_i + b_i) w_i = \sum_n a_i w_i + \sum_n b_i w_i = T(u) + T(v).$$

2. Scalar multiplication follows similarly.

To show uniqueness, suppose T_0, T_1 are two linear transformations satisfying $T_0(v_i) = w_i = T_1(v_i)$. Let $v \in V$, and write $v = \sum_n a_i v_i$. By linearity,

$$T_k(v) = T_k(\sum_n a_i v_i) = \sum_n a_i T(v_i) = \sum_n a_i w_i,$$

for $k = 0, 1$, hence, $T_1(v) = T_0(v)$ for arbitrary v, hence the transformations are equivalent. ■

↪ **Definition 2.2: Some Important Transformations**

We denote $T_0 : V \to W$ by $T_0(v) := 0_W \,\forall\, v \in V$ the *zero transformation*. We denote $I_V : V \to V$, $I_V(v) := v \,\forall\, v \in V$, as the *identity transformation*.

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2.2 Isomorphisms, Kernel, Image

↪ **Definition 2.3: Isomorphism**

Let V, W be vector spaces over $\mathbb{F}$. An *isomorphism* from V to W is a linear transformation $T : V \to W$ (a homomorphism for vector spaces) which admits an inverse T^{-1} that is also linear.

If such an isomorphism exists, we say V and W are *isomorphic*.

↪ **Proposition 2.1**

$T : V \to W$ is an isomorphism $\iff$ T is linear and bijective.

Proof. The direction $\implies$ is trivial.

Suppose $T : V \to W$ is linear and bijective, ie T^{-1} exists. We need to show that T^{-1} is linear. Let $w_1, w_2 \in W, a_1, a_2 \in \mathbb{F}$. Then:

$$\begin{aligned} T^{-1}(a_1 w_1 + a_2 w_2) &= T^{-1}(a_1 T(T^{-1}(w_1)) + a_2 T(T^{-1}(w_2))) \\ \text{(by linearity of } T\text{)} \quad &= T^{-1}(T(a_1 T^{-1}(w_1) + a_2 T^{-1}(w_2))) \\ &= a_1 T^{-1}(w_1) + a_2 T^{-1}(w_2). \end{aligned}$$

■

Remark 2.2. *This proposition holds for all structures that only have operations; it does not for those with relations, such as graphs, orders, etc..*

↪ **Theorem 2.2**

For $n \in \mathbb{N}$, every n-dimensional vector space V over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$. In particular, all n-dim vector spaces over $\mathbb{F}$ are isomorphic.

Proof. Fix a basis $\mathcal{B} := \{v_1, \ldots, v_n\}$ for V, and let $T : V \to \mathbb{F}^n$ be the unique linear transformation determined by $\mathcal{B}$ with $T(v_i) = e_i$, where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{F}^n$. We show that T is a bijection.

(Injective) Suppose $T(x) = T(y), x, y \in V$. Write $x = a_1v_1 + \cdots + a_nv_n$, $y = b_1v_1 + \cdots + b_nv_n$, the unique representation of x, y in the basis $\mathcal{B}$. We have:

$$a_1e_1 + \cdots + a_ne_n = a_1T(v_1) + \cdots + a_nT(v_n) = T(a_1v_1 + \cdots + a_nv_n) = T(x) = T(y) = \cdots = b_1e_1 + \cdots + b_ne_n,$$

but by the uniqueness of representation in a basis, it follows that each $a_i = b_i$, hence, $x = y$.

(Surjective) Let $w \in \mathbb{F}^n$. Then, $w = a_1e_1 + \cdots + a_ne_n$ (uniquely). But then,

$$w = a_1T(v_1) + \cdots + a_nT(v_n) = T(a_1v_1 + \cdots + a_nv_n),$$

where $a_1v_1 + \cdots + a_nv_n \in V$, hence T indeed surjective. ∎

Remark 2.3. *Replacing $\mathbb{F}^n$ with an arbitrary n-dim vector space W over $\mathbb{F}$ yields the following.*

↪ **Theorem 2.3: Freeness of Vector Space**

Let W, V be vector spaces over $\mathbb{F}$ and let β, γ be bases for V, W respectively. Every bijection $T : \beta \to \gamma$ can be extended to an isomorphism $\hat{T} : V \to W$.

In particular, all vector spaces over $\mathbb{F}$ with equinumerous bases are isomorphic.

Remark 2.4. *The proof follows very similarly to the previous theorem, but extended to arbitrary, possible infinite, spaces.*

Proof. ∎

↪ **Definition 2.4: Image/Kernel**

For a linear transformation $T : V \to W$, where V, W are vector spaces over $\mathbb{F}$, we define the *image*

$$\mathrm{Im}(T) := T(V),$$

and its *kernel*

$$\mathrm{Ker}(T) = T^{-1}(\{0_W\}).$$

↪ **Proposition 2.2**

$\mathrm{Ker}(T)$ and $\mathrm{Im}\, T$ are subspaces of V, W resp.

Proof. ($\mathrm{Ker}(T)$) Let $v_0, v_1 \in \mathrm{Ker}\, T$ and $a_0, a_1 \in \mathbb{F}$, then

$$T(a_0 v_0 + a_1 v_1) = a_0 T(v_0) + a_1 T(v_1) = 0_W \implies a_0 v_0 + a_1 v_1 \in \mathrm{Ker}\, T.$$

($\mathrm{Im}(T)$) Let $w_0, w_1 \in \mathrm{Im}\, T$, $a_0, a_1 \in \mathbb{F}$. Then $w_i = T(v_i), v_i \in V$, and so

$$a_0 w_0 + a_1 w_1 = a_0 T(v_0) + a_1 T(v_1) = T(a_0 v_0 + a_1 v_1) \implies a_0 w_0 + a_1 w_1 \in \mathrm{Im}\, T.$$

∎

↪ **Proposition 2.3**

Let $T : V \to W$ be a linear transformation, where V, W vector spaces over $\mathbb{F}$. Let β be a (possibly infinite) basis for V. Then, $T(B)$ spans $\mathrm{Im}(T)$.

In particular, T is surjective iff $T(\beta)$ spans W.

Proof. Let $w \in \mathrm{Im}(T)$, so $w = T(v)$ for some $v \in V$, where we have $v := a_1 v_1 + \cdots + a_n v_n, v_i \in \beta$. Then,

$$w = T(v) = a_1 T(v_1) + \cdots + a_n T(v_n) \in \mathrm{Span}(\{T(v_1), \ldots, T(v_n)\}) \subseteq \mathrm{Span}(T(\beta)).$$

∎

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↪ **Proposition 2.4**

Let $T : V \to W$ be a linear transformation, where V, W vector spaces over $\mathbb{F}$. TFAE:

1. T is injective.
2. $\mathrm{Ker}(T)$ is the trivial subspace $\{0_V\}$.
3. $T(\beta)$ is independent for each basis β for V.

3'. $T(\beta)$ is independent for some basis β for V.

Proof. (1. $\implies$ 2.) Trivial; only 0_V can be mapped to 0_W.

(2. $\implies$ 1.) Suppose $\mathrm{Ker}(T) = \{0_V\}$ and let $T(x) = T(y), x, y \in V$. By linearity,

$$T(x - y) = T(x) - T(y) = 0_W \implies x - y \in \mathrm{Ker}(T) \implies x - y = 0_V \implies x = y.$$

(2. $\implies$ 3.) Fix a basis β for V. To show that $T(\beta)$ linearly independent, take an arbitrary linear combination $a_1w_1 + \cdots + a_nw_n \in T(\beta)$. Suppose $\sum_i a_iw_i = 0_W$. Since $w_i \in T(\beta)$, $w_i = T(v_i), v_i \in \beta$, hence

$$\begin{aligned} 0_W = a_1w_1 + \cdots + a_nw_n = a_1T(v_1) + \cdots + a_nT(v_n) = T(a_1v_1 + \cdots + a_nv_n) \\ \implies a_1v_1 + \cdots + a_nv_n \in \mathrm{Ker}(T) \\ \implies a_1v_1 + \cdots + a_nv_n = 0_V, \end{aligned}$$

but each v_i is linearly independent, hence this must be a trivial linear combination, and thus $a_i = 0 \,\forall\, i$.

(3) $\implies$ (3') Trivial; stronger statement implies weaker statement.

(3') $\implies$ (2) Suppose $T(\beta)$ linearly independent for some basis β for V. Suppose $T(v) = 0_W, v \in V$. We write

$$v = a_1v_1 + \cdots a_nv_n, v_i \in \beta.$$

Then,

$$0_W = T(v) = T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n),$$

but $\{T(v_i)\} \subseteq T(\beta)$ is linearly independent, hence, this combination must be trivial and each $a_i = 0$, and thus $v = 0_V$ and so $\mathrm{Ker}(T) = \{0_V\}$ is trivial. ■

↪ **Definition 2.5: Rank, nullity**

Let V, W be vector spaces over $\mathbb{F}$ and $T : V \to W$ be linear. Define *rank* of T as

$$\mathrm{rank}(T) := \dim(\mathrm{Im}(T)),$$

and *nullity* of T as

$$\mathrm{nullity}(T) := \dim(\mathrm{Ker}(T)).$$

↪ **Theorem 2.4: Rank-Nullity Theorem**

Let V, W be vector spaces over $\mathbb{F}$, $\dim(V) < \infty$. Let $T : V \to W$ be a linear transformation. Then,

$$\mathrm{nullity}(T) + \mathrm{rank}(T) = \dim(V).$$

Remark 2.5. *Intuitively: the nullity is the number of vectors we "collapse"; the rank is what is left. Together, we have the entire space.*

Remark 2.6. *This follows directly from the first isomorphism theorem for vector spaces, and the fact that* $\dim(V/\mathrm{Ker}(T)) = \dim(V) - \dim(\mathrm{Ker}(T))$*; however, we will prove it without this result below.*

Proof. Let $\{v_1, \ldots, v_k\}$ be a basis for $\mathrm{Ker}(T)$, and complete it to a basis $\beta := \{v_1, \ldots, v_k, u_1, \ldots, u_{n-k}\}$ for V, where $n := \dim(V)$. We need to show that $\dim(\mathrm{Im}(T)) = n - k$.

Recall that $\{T(v_1), \ldots, T(v_k), T(u_1), \ldots, T(u_{n-k})\}$ spans $\mathrm{Im}(T)$. But $v_1, \ldots, v_k \in \mathrm{Ker}(T)$, so $T(v_i) = 0_W \,\forall\, i = 1, \ldots, k$. Hence, letting $\gamma := \{T(u_1), \ldots, T(u_{n-k})\}$ spans $\mathrm{Im}(T)$. It remains to show that γ is independent.

Let $a_1 T(u_1) + \cdots a_{n-k} T(u_{n-k}) = 0_W$; by linearity,

$$\begin{aligned} T(a_1 u_1 + \cdots + a_{n-k} u_{n-k}) = 0_W \\ \implies a_1 u_1 + \cdots + a_{n-k} u_{n-k} \in \operatorname{Ker}(T) \\ \implies a_1 u_1 + \cdots + a_{n-k} u_{n-k} = b_1 v_1 + \cdots + b_k v_k, \end{aligned}$$

but each of these $u_i, v_j \in \beta$, hence, each coefficient must be identically zero as β linearly independent, and thus $\dim(\operatorname{Im}(T)) = n - k$. This completes the proof. ■

↪ Corollary 2.1: Pigeonhole Principle for Dimension

Let $T : V \to W$ be a linear transformation. If T injective, then $\dim(W) \geqslant \dim(V)$.

Proof. If $\dim(V) < \infty$, then $\dim(\operatorname{Im}(T)) = \dim(V)$, and we have that $\dim(\operatorname{Im}(T)) \leqslant \dim(W)$ and conclude $\dim(V) \leqslant \dim(W)$.

If $\dim(V) = \infty$, then $\dim(\operatorname{Im}(T)) = \infty$ and $\dim(W) \geqslant \dim(\operatorname{Im}(T)) = \infty$. ■

↪ Corollary 2.2

Let $n \in \mathbb{N}$ and V, W be n-dimensional vector spaces over $\mathbb{F}$. For a linear transformation $T : V \to W$, TFAE:

1. T injective;
2. T surjective;
3. $\operatorname{rank}(T) = n$.

Proof. (2. $\iff$ 3.) Follows from $\operatorname{rank}(T) = \dim(\operatorname{Im}(T)) = n \iff \operatorname{Im}(T) = W$.

(1. $\implies$ 3.) We have $\operatorname{nullity}(T) = 0$ so $\operatorname{rank}(T) = \dim(V) = n$.

(3. $\implies$ 1.) If $\operatorname{rank}(T) = n$, then $\operatorname{nullity}(T) = 0$. ■

↪ Lecture 10; Last Updated: Mon Feb 5 14:03:23 EST 2024

↪ Theorem 2.5: First Isomorphism Theorem for Vector Spaces

Let V, W be vector spaces over $\mathbb{F}$. Let $T : V \to W$ be a linear transformation. Then,

$$V / \operatorname{Ker}(T) \cong \operatorname{Im}(T),$$

by the isomorphism given by $v + \operatorname{Ker}(T) \mapsto T(v)$.

Proof. From group theory, we know that $\hat{T} : V/\operatorname{Ker}(T) \to \operatorname{Im}(T)$, where $\hat{T}(v + \operatorname{Ker}(T)) := T(v)$ is well-defined, and is an isomorphism of abelian groups. We need only to check that $\hat{T}$ is linear, namely, that is respects scalar multiplication.

We have

$$\begin{aligned}\hat{T}(a \cdot (v + \mathrm{Ker}(T))) &= \hat{T}((a \cdot v) + \mathrm{Ker}(T)) \\ &= T(av) = a \cdot T(v) \\ &= a\hat{T}(v + \mathrm{Ker}(T)),\end{aligned}$$

as desired. ∎

2.3 The Space $\mathrm{Hom}(V, W)$

↪ **Definition 2.6: Homomorphism Space**

For vector spaces V, W over $\mathbb{F}$, let $\mathrm{Hom}(V, W)$ (also denoted $\ell(V, W)$) denote the set of all linear transformations from V to W. We can turn this into a vector space over $\mathbb{F}$ as follows:

1. *Addition of linear transformations:* for $T_0, T_1 \in \mathrm{Hom}(V, W)$, define

$$(T_0 + T_1) : V \to W, \quad v \mapsto T_0(v) + T_1(v).$$

   $(T_0 + T_1)$ is clearly a linear transformation, as the linear combination of linear transformations T_0, T_1.

2. *Scalar multiplication of linear transformations:* for $T \in \mathrm{Hom}(V, W)$, $a \in \mathbb{F}$, define

$$(a \cdot T) : V \to W, \quad v \mapsto a \cdot T(v),$$

   which is again clearly linear in its own right.

↪ **Proposition 2.5**

Endowed with the operations described above, $\mathrm{Hom}(V, W)$ is a vector space over $\mathbb{F}$.

Proof. Follows easily from the definitions. ∎

↪ **Theorem 2.6: Basis for** $\mathrm{Hom}(V, W)$

For vector spaces V, W over $\mathbb{F}$ and bases β, γ for V, W resp., the following set

$$\{T_{v,w} = v \in \beta, w \in \gamma\},$$

is a basis for $\mathrm{Hom}(V, W)$, where for each $v \in \beta$ and $w \in \gamma$, $T_{v,w} \in \mathrm{Hom}(V, W)$ defined as the unique linear transformation such that

$$T_{v,w}(v') = \begin{cases} w & v' = v \\ 0_W & v' \neq v \iff \in \beta \setminus \{v\} \end{cases}.$$

Proof. Left as a (homework) exercise. ■

↪ **Corollary 2.3**

If V, W finite dimensional, then $\dim(\mathrm{Hom}(V, W)) = \dim(V) \cdot \dim(W)$.

↪ **Proposition 2.6**

Let $\beta = \{v_1, \ldots, v_n\}$, $\gamma = \{w_1, \ldots, w_m\}$ be bases for V, W resp. Then, by theorem 2.6,

$$\{T_{v_i,w_j} : i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}\}$$

is a basis for $\mathrm{Hom}(V, W)$, and it has $n \cdot m$ vectors by construction.

2.4 Matrix Representation of Linear Transformations, Finite Fields

Consider a linear transformation $T : \mathbb{F}^n \to \mathbb{F}^m$ between finite fields. We know that T is uniquely determined by its value of basis vectors, so fix the standard bases

$$\beta = \{e_1^{(n)}, \ldots, e_n^{(n)}\} = \{v_1, \ldots, v_n\},$$

and note that T is determined by $\{T(v_1), \ldots, T(v_n)\} \subseteq \mathbb{F}^m$.

Remark 2.7. *We denote vectors in* $\mathbb{F}^n$ *as column vectors, ie* $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$.

Each $T(v_i)$ is a column vector in $\mathbb{F}^m$, and we an put these into a $m \times n$ matrix, namely:[7]

[7]Where $[T]$ denotes a matrix named "T".

$$[T] := \begin{pmatrix} | & & | \\ T(v_1) & \cdots & T(v_n) \\ | & & | \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{n}$$

We call this the *matrix representation* of T in the standard bases. The operation of multiplying an $m \times n$ matrix and a $n \times 1$ vector is precisely defined so that

↪ **Proposition 2.7**

$T(v) = [T] \cdot v$ for all $v \in \mathbb{F}^n$.

Proof. Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, where $v = x_1 v_1 + \cdots + x_n v_n$. Then

$$T(v) = x_1 T(v_1) + \cdots + x_n T(v_n)$$
$$T(v_i) = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix}$$

so

$$T(v) = \begin{pmatrix} a_{11} \cdot x_1 + \cdots + a_{1n} \cdot x_n \\ \ddots \\ a_{m1} \cdot x_1 + \cdots + a_{mn} \cdot x_n \end{pmatrix} = [T] \cdot v$$

■

↪ **Definition 2.7**

For a given $m \times n$ matrix A over $\mathbb{F}$, define $L_A : \mathbb{F}^n \to \mathbb{F}^m$ by $L_A(v) := A \cdot v$, where v is viewed as an $n \times 1$ column. It follows from definition that the L_A is linear.

In other words, every $T \in \operatorname{Hom}(\mathbb{F}^n, \mathbb{F}^m)$ is equal to L_A for some A.

↪ *Lecture 11; Last Updated: Fri Feb 9 14:12:09 EST 2024*

↪ **Proposition 2.8**

The map

$$\begin{aligned} \operatorname{Hom}(\mathbb{F}^n, \mathbb{F}^m) &\to M_{m\times n}(\mathbb{F}) \\ T &\mapsto [T] \end{aligned}$$

is an isomorphism of vector spaces, with inverse

$$\begin{aligned} M_{m\times n}(\mathbb{F}) &\to \operatorname{Hom}(\mathbb{F}^n, \mathbb{F}^m) \\ A &\mapsto L_A. \end{aligned}$$

Proof. Linearity: Let $\beta = \{v_1, \dots, v_n\}$ be the standard basis for $\mathbb{F}^n$. Fix $T_1, T_2 \in \operatorname{Hom}(\mathbb{F}^n, \mathbb{F}^m)$ and $\alpha \in \mathbb{F}$.

1.

$$\begin{aligned} [T_1 + T_2] &= \begin{pmatrix} & | & \\ \cdots & (T_1+T_2)(v_i) & \cdots \\ & | & \end{pmatrix} = \begin{pmatrix} & | & \\ \cdots & T_1(v_i) + T_2(v_i) & \cdots \\ & | & \end{pmatrix} \\ &= \begin{pmatrix} & | & \\ \cdots & T_1(v_i) & \cdots \\ & | & \end{pmatrix} + \begin{pmatrix} & | & \\ \cdots & T_2(v_i) & \cdots \\ & | & \end{pmatrix} \\ &= [T_1] + [T_2] \end{aligned}$$

2. It remains to show that $\alpha \cdot [T] = [\alpha \cdot T]$; the proof follows similarly to 1.

Inverse: We need to show that 1. $A \mapsto L_A \mapsto [L_A]$ is the identity on $M_{m\times n}(\mathbb{F})$, and conversely, that 2. $T \mapsto [T] \mapsto L_{[T]}$ is the identity on $\operatorname{Hom}(\mathbb{F}^n, \mathbb{F}^m)$.

1. We need to show that $[L_A] = A$. The jth column of $[L_A]$ is $L_A(v_j) = A \cdot v_j = j$th column of $A =: A^{(j)}$. Hence, the jth column of $[L_A]$ is equal to the jth column of A, and thus they are equal.

2. We showed this in proposition 2.7.

■

↪ **Corollary 2.4**

$\dim(\operatorname{Hom}(\mathbb{F}^n, \mathbb{F}^m)) = \dim(M_{m\times n}(\mathbb{F})) = m \cdot n.$

Remark 2.8. *This was stated previously in proposition 2.6 by constructing an explicit basis. Indeed, this basis is precisely the image of the standard basis for $M_{m\times n}(\mathbb{F})$ under the map $A \mapsto L_A$.*

2.5 Matrix Representation of Linear Transformations, General Spaces

Remark 2.9. *The previous section was concerned with representing transformations between finite fields $\mathbb{F}^n, \mathbb{F}^m$; this section aims to make the same construction for any finite dimensional V, W.*

↪ **Definition 2.8: Coordinate Vector**

Let V be a finite dimensional space over $\mathbb{F}$ and let $\beta := \{v_1, \ldots, v_n\}$ be a basis for V. Let $v \in V$, with (unique) representation $v = a_1 v_1 + \cdots + a_n v_n$. We denote

$$[v]_\beta := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$$

the *coordinate vector* of v in base β.

Remark 2.10. *Recall that $V \cong \mathbb{F}^n$ where $\dim(V) = n$, by the unique linear transformation $v_i \mapsto e_i$, where $\{e_1, \ldots, e_n\}$ the standard basis for $\mathbb{F}^n$. We denote this transformation*

$$I_\beta : V \to \mathbb{F}.$$

For an arbitrary $v \in V$, $I_\beta(v)$ maps v to its coordinate vector:

$$\begin{aligned} I_\beta(v) = I_\beta(a_1 v_1 + \cdots + a_n v_n) &= a_1 I_\beta(v_1) + \cdots a_n I_\beta(v_n) \\ &= a_1 e_1 + \cdots + a_n e_n = [v]_\beta. \end{aligned}$$

↪ **Proposition 2.9**

The map

$$I_\beta : V \to \mathbb{F}^n, \quad v \mapsto [v]_\beta$$

is an isomorphism.

Suppose we are given a linear transformation $T : V \to W$, where V, W finite dimensional spaces over $\mathbb{F}$. Fix $\beta := \{v_1, \ldots, v_n\}$ and $\gamma := \{w_1, \ldots, w_m\}$ as bases for V, W resp. We can denote $[T(v_i)]_\gamma$ as $T(v_i)$ in base γ (in the field m), and construct a matrix for T:[8]

$$[T]_\beta^\gamma := \begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix}$$

We call this the *matrix representation* of T from β to γ.

[8] Where we denote $[T]_\beta^\gamma$ as the matrix representation of the transform $T : V \to W$, with basis β, γ for V, W respectively.

↪ **Theorem 2.7**

Let $T : V \to W, \beta, \gamma$ as above.

1. The following diagram commutes:

$$\begin{tikzcd} \bullet V \arrow[r, "T"] \arrow[d, "I_\beta"'] & \bullet W \arrow[d, "I_\gamma"] \\ \bullet \mathbb{F}^n \arrow[r, dashed, "L_{[T]_\beta^\gamma}"'] & \bullet \mathbb{F}^m \end{tikzcd}$$

   Namely, $I_\gamma \circ T = L_{[T]_\beta^\gamma} \circ I_\beta$, or equivalently, given $v \in V$, $[T(v)]_\gamma = [T]_\beta^\gamma \cdot [v]_\beta$.

2. The map $\mathrm{Hom}(V, W) \to M_{m\times n}(\mathbb{F}), T \mapsto [T]_\beta^\gamma$ is a vector space isomorphism with inverse begin the map $M_{m\times n}(\mathbb{F}) \to \mathrm{Hom}(V, W), A \mapsto I_\gamma^{-1} \circ L_A \circ I_\beta$

Proof. 2. is left as a (homework) exercise; it follows directly from 1.

Fix $v \in V$. We need to show that $I_\gamma \circ T(v) = L_{[T]_\beta^\gamma} \circ I_\beta(v)$. We have

$$I_\gamma \circ T(v) = [T(v)]_\gamma.$$

OTOH,

$$L_{[T]_\beta^\gamma} \circ I_\beta(v) = L_{[T]_\beta^\gamma}([v]_\beta) = [T]_\beta^\gamma \cdot [v]_\beta.$$

We need to show, then, that $[T(v)]_\gamma = [T]_\beta^\gamma \cdot [v]_\beta$. Let $v = a_1 v_1 + \cdots + a_n v_n$, so $[v]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. Recall that $[T]_\beta^\gamma = \begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix}$. Thus, we have

$$\begin{aligned} [T]_\beta^\gamma \cdot [v]_\beta = a_1[T(v_1)]_\gamma + \cdots + a_n[T(v_n)]_\gamma &= [a_1 T(v_1) + \cdots + a_n T(v_n)]_\gamma \quad \textit{(by linearly of } I_\gamma\textit{)} \\ &= [T(a_1 v_1 + \cdots + a_n v_n)]_\gamma \quad \textit{(by linearity of } T\textit{)} \\ &= [T(v)]_\gamma, \end{aligned}$$

which is precisely what we wanted to show. ■

Remark 2.11. *For $A \in M_{m\times n}(\mathbb{F})$ and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$, we have*

$$A \cdot x = x_1 \cdot A^{(1)} + x_2 \cdot A^{(2)} + \cdots + x_n \cdot A^{(n)},$$

where $A^{(j)}$ is the jth column of A; thus $A \cdot x$ is a linear combination of A, with coefficients given by the vector x; this *interpretation can make it easier to make sense of computations.*

↪ *Lecture 12; Last Updated: Fri Feb 9 11:12:11 EST 2024*

2.6 Composition of Linear Transformations, Matrix Multiplication

↪ **Proposition 2.10**

Composition is associative; given $T : V \to W, S : W \to U$, and $R : U \to X$, then

$$(R \circ S) \circ T = R \circ (S \circ T).$$

Proof. Fix $v \in V$. Then

$$(R \circ S) \circ T(v) = (R \circ S)(T(v)) = R(S(T(v)))$$

OTOH:

$$R \circ (S \circ T)(v) = R((S \circ T)(v)) = R(S(T(v))).$$

■

Let $A \in M_{m \times n}(\mathbb{F})$ and $B \in M_{l \times m}(\mathbb{F})$. Then, $L_A : \mathbb{F}^n \to \mathbb{F}^m$ and $L_B : \mathbb{F}^m \to \mathbb{F}^l$, and have composition $L_B \circ L_A : \mathbb{F}^n \to \mathbb{F}^l$. We know that $L_B \circ L_A$ is a linear transformation, and thus must be equal to L_C for some matrix $C \in M_{l \times n}(\mathbb{F})$. Indeed, C is the matrix representation of the transformation $[L_B \circ L_A]$, as proven previously.

Let $\beta = \{e_1, \ldots, e_n\}$ for $\mathbb{F}^n$, then

$$[L_B \circ L_A] = \begin{pmatrix} | & & | \\ L_B \circ L_A(e_1) & \cdots & L_B \circ L_A(e_n) \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ B \cdot (A \cdot e_1) & \cdots & B \cdot (A \cdot e_n) \\ | & & | \end{pmatrix}$$

↪ **Definition 2.9: Matrix Multiplication**

For matrices $A \in M_{m \times n}(\mathbb{F})$ and $B \in M_{l \times m}(\mathbb{F})$, define their product $B \cdot A$ to be the matrix

$$[L_B \circ L_A] = \begin{pmatrix} | & & | \\ B \cdot (A \cdot e_1) & \cdots & B \cdot (A \cdot e_n) \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ B \cdot A^{(1)} & \cdots & B \cdot A^{(2)} \\ | & & | \end{pmatrix} = (c_{ij})_{1 \leqslant i \leqslant l}^{1 \leqslant j \leqslant n}$$

where $A^{(j)}$ is the jth column of A, $c_{ij} := \begin{pmatrix} - & B_{(i)} & - \end{pmatrix} \cdot \begin{pmatrix} | \\ A^{(j)} \\ | \end{pmatrix}$.

> ↪ **Proposition 2.11**
>
> $[L_B \circ L_A] = B \cdot A$, ie $L_B \circ L_A = L_{B \cdot A}$.

Proof. Follows from our definition. ■

> ↪ **Corollary 2.5**
>
> Matrix multiplication is association; $C \cdot (B \cdot A) = (C \cdot B) \cdot A$ for $A \in M_{m \times n}(\mathbb{F}), B \in M_{l \times m}(\mathbb{F}), C \in M_{k \times l}(\mathbb{F})$.

Proof. $C \cdot (B \cdot A) = [L_C \circ (L_B \circ L_A)] = [(L_C \circ L_B) \circ L_A] = (C \cdot B) \cdot A$. ■

Remark 2.12. *This is proven by the linear transformation representation of matrices; try proving this* directly *from our definition.*

> ↪ **Corollary 2.6**
>
> Let V, W, U be finite-dimensional vector spaces over $\mathbb{F}$, $T : V \to W, S : W \to U$ be linear transformations and α, β, γ be bases for V, W, U resp. Then,
>
> $$[S \circ T]_\alpha^\gamma = [S]_\beta^\gamma \cdot [T]_\alpha^\beta.$$

Proof. Follows from the commutativity of the diagrams:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\wr"'] & W \arrow[r, "S"] \arrow[d, "\wr"'] & U \arrow[d, "\wr"'] \\ \mathbb{F}^n \arrow[r, "{[T]_\alpha^\beta}"'] & \mathbb{F}^m \arrow[r, "{[S]_\beta^\gamma}"'] & \mathbb{F}^l \end{tikzcd} \quad \iff \quad \begin{tikzcd} V \arrow[r, "T \circ S"] \arrow[d, "\wr"'] & U \arrow[d, "\wr"'] \\ \mathbb{F}^n \arrow[r, "{[S \circ T]_\alpha^\gamma}"'] & \mathbb{F}^l \end{tikzcd}$$

In "words", for $v \in V$,

$$[S \circ T]_\alpha^\gamma \cdot [v]_\alpha = [(S \circ T)(v)]_\alpha^\gamma = [S(T(v))]_\alpha = [S]_\beta^\gamma \cdot [T(v)]_\beta = [S]_\beta^\gamma \cdot [T]_\alpha^\beta \cdot [v]_\alpha,$$

ie we have shown that $L_{[S \circ T]_\alpha^\gamma} = L_{[S]_\beta^\gamma \cdot [T]_\alpha^\beta}$. Because $A \mapsto L_A$ is an isomorphism, it follows that $[S \circ T]_\alpha^\gamma = [S]_\beta^\gamma \cdot [T]_\alpha^\beta$. ■

↪ Lecture 13; Last Updated: Sat Feb 3 22:20:36 EST 2024

2.7 Inverses of Transformations and Matrices

Remark 2.13. *Recall that, given a function $f : X \to Y$, a function $g : Y \to X$ is called*

1. *a* left inverse *of f if $g \circ f = \mathrm{Id}_X$;*
2. *a* right inverse *of f if $f \circ g = \mathrm{Id}_X$;*
3. *a (two-sided)* inverse *of f if g both a left and right inverse of f.*

If an inverse exists, it is unique; let g_0, g_1 be inverse of f,then, $g_0 = g_0 \circ (f \circ g_1) = (g_0 \circ f) \circ g_1 = g_1$.

↪ **Proposition 2.12**

Let $f : X \to Y$. Then,

1. f has a left-inverse $\iff$ f injective;
2. f has a right-inverse $\iff$ f surjective;
3. f has an inverse $\iff$ f bijective.

Proof. ((a), $\implies$) Suppose $g : Y \to X$ is a left-inverse of f and $f(x_1) = f(x_2)$. Then, $g \circ f(x_1) = g \circ f(x_2) \implies x_1 = x_2$ and so f injective.

((b), $\implies$) Suppose $g : Y \to X$ is a right-inverse of f and let $y \in Y$. Then, $f(g(y)) = y \implies y \in f(X)$.

The remainder of the cases and directions are left as an exercise. ■

Remark 2.14. *Proof of* (b), $\impliedby$ *uses Axiom of Choice.*

⊛ **Example 2.2**

1. The differentiation transform $\delta : \mathbb{F}[t]_{n+1} \to \mathbb{F}[t]_n, p(t) \mapsto p'(t)$ has a right inverse, the integration transform, $\iota : \mathbb{F}[t]_n \to \mathbb{F}[t]_{n+1}$, $p(t) \mapsto$ antiderivative of $p(t)$; conversely, ι has left inverse δ; they do not admit inverses.
2. Let $f : \mathbb{F}[\![t]\!] \to \mathbb{F}[\![t]\!]$ be the left-shift map, where $\sum_{n=0}^{\infty} a_n t^n \mapsto \sum_{n=1}^{\infty} a_n t^{n-1}$. Then, $g : \mathbb{F}[\![t]\!] \to \mathbb{F}[\![t]\!]$ with $\sum_{n=0}^{\infty} a_n t^n \mapsto \sum_{n=0} a_n t^{n+1}$, the right-shift map, is a right inverse of f, but f has no left inverse (it is not injective).

Remark 2.15. *The existence of only one-sided inverses existing happens only when in infinite-dimensional vectors spaces, or when the dimension of the domain is not the same as the dimension of the codomain.*

↪ **Corollary 2.7: Of Rank-Nullity Theorem**

Let $T : V \to W$ s.t. $\dim(V) = \dim(W) < \infty$. TFAE:

1. T has a left-inverse;
2. T has a right-inverse;
3. T is invertible (has an inverse).

Proof. We have already that T injective $\iff$ T surjective $\iff$ T bijective. ■

↪ **Definition 2.10: Matrix Inverse**

We call a $n \times n$ matrix B over $\mathbb{F}$ the *inverse* of an $n \times n$ matrix A over $\mathbb{F}$ if $A \cdot B = B \cdot A = I_n$. We denote $B = A^{-1}$.

↪ **Proposition 2.13**

Let $A \in M_n(\mathbb{F})$. Then,

1. L_A is invertible $\iff$ A is invertible, in which case $L_A^{-1} = L_{A^{-1}}$;
2. A is invertible $\iff$ it has a left-inverse, ie $B \cdot A = I_n$ $\iff$ it has a right-inverse, ie $A \cdot B = I_n$.

Proof. 1. L_A invertible $\iff$ $\exists T : \mathbb{F}^n \to \mathbb{F}^n$-linear s.t. $L_A \circ T = T \circ L_A = I_{\mathbb{F}^n}$ $\iff$ $\exists$ a matrix $B \in M_n(\mathbb{F})$ such that $L_A \circ L_B = L_B \circ L_A = I_{\mathbb{F}^n}$ $\iff$ there is a matrix $B \in M_n(\mathbb{F})$ s.t. $L_{AB} = L_{BA} = I_{\mathbb{F}^n}$ $\iff$ there is a $B \in M_n(\mathbb{F})$ s.t. $A \cdot B = B \cdot A = I_n$.

2. Follows directly from corollary 2.7 and part 1.

■

2.8 Invariant Subspaces and Nilpotent Transformations

↪ **Definition 2.11: T-Invariant**

Let $T : V \to V$ be a linear transformation.[9] We call a subspace $W \subseteq V$ *T-invariant* if $T(W) \subseteq W$.

⊛ **Example 2.3: Examples of Invariant Subspaces**

1. For any $T : V \to V$, $\mathrm{Im}(T)$ is T-invariant.
2. For any $T : V \to V$, $\mathrm{Ker}(T)$ is T-invariant, since $T(v) = 0_V \in \mathrm{Ker}(T) \,\forall\, v \in \mathrm{Ker}(T)$. Moreover, for any $n \in \mathbb{N}$, the space $\mathrm{Ker}(T^n)$ is T-invariant.[10]

↪ Lecture 14; Last Updated: Mon Feb 12 08:34:27 EST 2024

↪ **Proposition 2.14**

For a linear operator $T : V \to V$, the following hold:

1. $V \supseteq \mathrm{Im}(T) \supseteq \mathrm{Im}(T^2) \supseteq \cdots \supseteq \mathrm{Im}(T^n) \supseteq \cdots$. Moreover, $\mathrm{Im}(T^n)$ is T-invariant for any $n \in \mathbb{N}$.
2. $\{0_V\} \subseteq \mathrm{Ker}(T) \subseteq \mathrm{Ker}(T^2) \subseteq \cdots \subseteq \mathrm{Ker}(T^n) \subseteq \cdots$. Moreover, $\mathrm{Ker}(T^n)$ is T-invariant for any $n \in \mathbb{N}$.

Proof. 1. If $x \in \mathrm{Im}(T^{n+1})$, then $x = T^{n+1}(y) = T^n(T(y)) \in \mathrm{Im}(T^n)$ for some $y \in V$, hence $\mathrm{Im}(T^{n+1}) \subseteq \mathrm{Im}(T^n)$. If $x \in \mathrm{Im}(T^n)$, then $x = T^n(y)$ so $T(x) = T(T^n(y)) = T^n(T(y)) \in \mathrm{Im}(T^n)$, so $T(\mathrm{Im}(T^n)) \subseteq \mathrm{Im}(T^n)$.

[9] Because the domain and codomain are the same, we often call T a "linear operator".
[10] $T^n := T \circ T \circ \cdots \circ T$, n times; $T^0 := I_V$.

2. If $x \in \mathrm{Ker}(T^n)$, then $T^{n+1}(x) = T(T^n(x)) = T(0_V) = 0_V$ hence $x \in \mathrm{Ker}(T^{n+1})$ so $\mathrm{Ker}(T^n) \subseteq \mathrm{Ker}(T^{n+1})$. Moreover, $T(x) \in \mathrm{Ker}(T^n)$ since $T(x) \in \mathrm{Ker}(T^{n-1}) \subseteq \mathrm{Ker}(T^n)$, since $T^{n-1}(T(x)) = T^n(x) = 0_V$ so $T(\mathrm{Ker}(T^n)) \subseteq \mathrm{Ker}(T^n)$.

■

⊛ Example 2.4: More Examples of Invariant Subspaces

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x, y, z) := (2x + y, 3x - y, 7z)$. Then, the $x - y$ plane, $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ is T-invariant, as is the z axis, $\{(x, y, z) \in \mathbb{R}^3 : x = y = 0\}$. Hence, we can decompose $\mathbb{R}^3$ into two T-invariant subspaces, namely $x - y$ plane $\oplus$ z-axis.

↪ Definition 2.12: Nilpotent

In a ring R, an element $r \in R$ is called *nilpotent* if $r^n = 0$ for some $n \in \mathbb{N}^+$.

A linear transformation $T : V \to V$ is called nilpotent if $T^n = 0$ for some $n \in \mathbb{N}^+$.[11]

For a matrix $A \in M_n(\mathbb{F})$, A is called nilpotent if $A^n = 0_n$ for some $n \in \mathbb{N}^+$.

[11] One can verify that all linear transformations $T : V \to V$ from a vector space to itself form a ring with $(\circ, +)$, ie composition and ("standard") addition of transformations. The same holds for linear operators defined over an abelian group (where the same $+$ operation is endowed by the ring).

⊛ **Example 2.5: Examples of Nilpotent Transformations**

1. Let V, n-dimensional vector space over $\mathbb{F}$ with basis $\beta := \{v_1, \dots, v_n\}$. Let $T : V \to V$ be the unique linear transformation that "shifts" β: ie, $T(v_1) := 0_V$, $T(v_2) := v_1, \dots, T(v_n) = v_{n-1}$.

2. The differentiation operation, $\delta : \mathbb{F}[t]_n \to \mathbb{F}[t]_n$ is nilpotent, since $\delta^{n+1} = 0$ for any polynomial.

3. For any matrix $A \in M_n(\mathbb{F})$, A is nilpotent iff $L_A : \mathbb{F}^n \to \mathbb{F}^n$ is nilpotent.

   Proof. $L_{A^k} = L_A^k \implies A^k = 0 \iff L_{A^k} = 0 \iff L_A^k = 0$ ∎

4. $n \times n$ matrices that are strictly upper triangular[12] are nilpotent. For instance, for 3×3, we need to show[13]

$$\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}^3 = 0 \iff \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}^3 \cdot \begin{pmatrix} \star \\ \star \\ \star \end{pmatrix} = 0$$

   We have:

$$\begin{aligned}
\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \star \\ \star \\ \star \end{pmatrix} &= \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} \star \\ \star \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \star \\ \star \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \star \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\end{aligned}$$

↪ **Proposition 2.15**

If V is n-dimensional and $T : V \to V$ is a linear nilpotent transformation, then $T^n = 0$.

Proof. Left as a (homework) exercise. ∎

[13] ie zeros everywhere except cells strictly above diagonal.

[13] Where we denote arbitrary elements $\star$; different $\star$s are not necessarily equal.

↪ Definition 2.13: Domain Restriction

For a function $f : X \to Y$ and $A \subseteq X$, we define the *restriction* of f to A as the function $f|_A : A \to Y$ given by $a \mapsto f(a)$.

↪ Definition 2.14: Direct Sum

Let V be a vector space over $\mathbb{F}$, and let $W_0, W_1 \subseteq V$ be subspaces of V. If

1. $W_0 \cap W_1 = \{0_V\}$ (the subspaces are *linearly independent*), and
2. $W_0 + W_1 = \{w_0 + w_1 : w_0 \in W_0, w_1 \in W_1\} = V$,

we write $V = W_0 \oplus W_1$, and say V is the *direct sum* if W_0, W_1.

↪ Theorem 2.8: Fitting's Lemma

For finite dimensional vector space V over $\mathbb{F}$ and a linear transformation $T : V \to V$, there is a decomposition

$$V = U \oplus W$$

as a direct sum of T-invariant subspaces U, W such that $T|_U : U \to U$ is nilpotent and $T|_W : W \to W$ is an isomorphism.

Proof. Recall that $\mathrm{Im}(T) \supseteq \cdots \supseteq \mathrm{Im}(T^n)$ and $\mathrm{Ker}(T) \subseteq \cdots \subseteq \mathrm{Ker}(T^n)$. Both of these become constant eventually, ie the inequalities become strict equalities, hence $\exists N \in \mathbb{N}^+$ such that $\forall\, k \in \mathbb{N}$, $\mathrm{Im}(T^{N+k}) = \mathrm{Im}(T^N)$ and $\mathrm{Ker}(T^{N+k}) = \mathrm{Ker}(T^N)$.

Let $U := \mathrm{Ker}(T^N)$ and $W := \mathrm{Im}(T^N)$. These are clearly T-invariant.

$T^N(\mathrm{Ker}(T^N)) = \{0_V\}$, and $T(\mathrm{Im}(T^N)) = \mathrm{Im}(T^{N+1}) = \mathrm{Im}(T^N) = W$ and thus $T|_W : W \to W$ is surjective and hence $T|_W$ must be injective and thus an isomorphism.

It remains to show that $V = U \oplus W$. If $v \in U \cap W, T^N(v) = 0_V$ but $T|_W$ an isomorphism so $T^N(v) = 0 \iff v = 0_V$, hence $U \cap W = \{0_V\}$.

Thus, we have $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W) = \dim(U) + \dim(W) = \dim(V)$; moreover, it must be that $U + W = V$.[14] ■

↪ *Lecture 15; Last Updated: Fri Feb 9 13:40:20 EST 2024*

2.9 Dual Spaces

[14]It is precisely here that we use finiteness of V.

↪ **Definition 2.15: Dual Space**

For a vector space V over a field $\mathbb{F}$, linear transformations from $V \to \mathbb{F}$ (where we view $\mathbb{F}$ as a one-dimensional vector space over $\mathbb{F}$) are called *linear functionals*. The space of linear functionals (namely, $\mathrm{Hom}(V, \mathbb{F})$) is denoted V^*, and called the *dual space* of V.

↪ **Proposition 2.16**

If V is finite dimensional, $\dim(V^*) = \dim(V)$.[15]

Proof. For finite dimensional V, we know that $\dim(\mathrm{Hom}(V, \mathbb{F})) = \dim(V) \cdot \dim(\mathbb{F}) = \dim(V)$, hence $\dim(V^*) = \dim(V)$. In the same notation with which we proved this originally in proposition 2.6; fix a basis $\beta := \{v_1, \dots, v_n\}$ for V and the standard basis $\gamma := \{1\}$ for $\mathbb{F}$, and defined $\beta^* := \{f_1, \dots, f_n\}$, where $f_i := T_{v_i,1} : V \to \mathbb{F}$ maps $v_i \mapsto 1$ and every other basis vector to $0_{\mathbb{F}}$. ■

Remark 2.16. *The basis β^* for V^* is called the* dual basis. *Explicitly, we have:*

↪ **Corollary 2.8**

Let V be a finite dimensional vector space over $\mathbb{F}$ and let $\beta := \{v_1, \dots, v_n\}$ be a basis for V. Then,

$$\beta^* := \{f_1, \dots, f_n\}$$

is a basis for V^*. Moreover, for each linear functional $f \in V^*$,

$$f = \sum_{i=1}^{n} f(v_i) \cdot f_i.$$

Proof. Linear indepedence: let $a_1 f_1 + \cdots + a_n f_n = 0_{V^*} =: 0$. Then,

$$(a_1 f_1 + \cdots + a_n f_n)(v_i) = a_i f_i(v_i) = a_i \cdot 1 = a_i \implies a_i = 0,$$

hence β^* indeed linearly independent.

Spanning: let $f \in V^*$. We claim that $f = \sum_{i=1}^{n} f(v_i) f_i$. It suffices to show these two sides are equal on the basis vectors, as linear transformations are determined by their effect on basis vectors. We have:

$$\left(\sum_{i=1}^{n} f(v_i) f_i\right)(v_j) = \sum_{i=1}^{n} f(v_i) f_i(v_j) = \sum_{i=1}^{n} f(v_i) \cdot \delta_{ij} = f(v_j),$$

as desired.[16] ■

[15] This does *not* hold for infinite dimensional spaces.

[16] Where $\delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ is the Kronecker delta.

⊛ **Example 2.6**

1. Let $V := \mathbb{F}^n$ and $\beta := \{v_1, \dots, v_n\}$ be a basis for $\mathbb{F}^n$, viewed as column vectors, and let $\beta^* := \{f_1, \dots, f_n\}$ be the dual basis for V^*. Recall that $f_i : \mathbb{F}^n \to \mathbb{F}$, hence $f_i := L_{A_i}$ for some matrix $A_i \in M_{1\times n}(\mathbb{F}) :=$ space of $1 \times n$ row vectors. Hence, $A_i = e_i^t$.

2. Consider V^{**}, the dual of the dual. If V is finite-dimensional, then as $\dim(V) = \dim(V^*)$, we have $\dim(V) = \dim(V^*) = \dim(V^{**})$, ie, they are (abstractly) isomorphic.

   We have that $T : V \to V^*, v_i \mapsto f_i$ is an isomorphism; we define an explicit isomorphism to V^{**} below.

↪ **Definition 2.16**

Let V be an arbitrary vector space over $\mathbb{F}$. For each $x \in V$, define $\hat{x} \in V^{**}$ by $\hat{x} : V^* \to \mathbb{F}$, where $\hat{x}(f) := f(x)$.

Remark 2.17. *Note that $\hat{x}$ is linear.*

↪ **Theorem 2.9**

The map $x \mapsto \hat{x} : V \to V^{**}$ is a linear injection. In particular, if V is finite dimensional, it is an isomorphism.

Proof. Let $x \in V$ and suppose $\hat{x} = 0_{V^{**}}$. Let β be a basis for V and β^* its dual basis. Let $x = a_1v_1 + \cdots + a_nv_n$ for $v_i \in \beta, a_i \in \mathbb{F}$. Let f_i such that $f_i(v_j) = \delta_{ij}v_j$. Then,

$$\hat{x}f_i = f_i(x) = f_i(a_1v_1 + \cdots a_nv_n) = a_i = 0,$$

hence, $a_i = 0 \,\forall\, i$. Hence, $x = 0$, and thus $\hat{x}$ has a trivial kernel and is thus injective. ■

↪ Lecture 16; Last Updated: Mon Feb 12 13:34:17 EST 2024

Remark 2.18. *Notice that to get an isomorphism $V \cong V^*$, we fixed a basis for V to define it. However, for $V \cong V^{**}$, we had a canonical isomorphism independent of choice of basis. Writing $S \subseteq V$, $\hat{S} := \{\hat{x} : x \in S\} \subseteq V^{**}$, our theorem says that $\hat{V} = V^{**}$ for finite-dimensional V.*

↪ **Definition 2.17: Annihilator**

Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$. We call

$$S^\perp := \{f \in V^* : f|_S = 0\} = \{f \in V^* : f(u) = 0 \,\forall\, u \in S\}$$

the *annihilator* of S.

↪ **Proposition 2.17**

Let V be a vector space over $\mathbb{F}$ and $S \subseteq V$.

1. $S^\perp$ is a subspace of V^*[17]
2. $S_1 \subseteq S_2 \subseteq V \implies S_1^\perp \supseteq S_2^\perp$
3. $S^\perp = (\operatorname{Span}(S))^\perp$

Proof. 1. If $f_1, f_2 \in S^\perp, a \in \mathbb{F}$, then $\forall\, u \in S$,

$$(af_1 + f_2)(u) = af_1(u) + f_2(u) = a \cdot 0 + 0,$$

so $af_1 + f_2 \in S^\perp$.

2. Clear.
3. If $f \in V^*$ takes all vectors in S to 0, then it does the same for zeros.

■

↪ **Proposition 2.18**

If V is finite dimensional and $U \subseteq V$ a subspace, then $(U^\perp)^\perp = \hat{U}$.

Proof. We know that $V^{**} = \hat{V}$, so we fix $\hat{x} \in \hat{V}$ and show that

$$\hat{x} \in (U^\perp)^\perp \iff \hat{x} \in \hat{U} \iff x \in U.$$

We have

$$\hat{x} \in (U^\perp)^\perp :\iff \forall\, f \in U^\perp, \hat{x}(f) = f(x) = 0$$

hence if $x \in U$, then $\hat{x} \in (U^\perp)^\perp$, so $\hat{U} \subseteq (U^\perp)^\perp$.

Conversely, let $\hat{x} \in (U^\perp)^\perp$. Then, $\forall\, f \in U^\perp$, $f(x) = 0$.

Suppose towards a contradiction that $x \notin U$. We aim to define $f \in U^\perp$ s.t. $f(x) = 1$, obtaining a contradiction. Let $\{u_1, \ldots, u_k\}$ be a basis for U, noting that $\{u_1, \ldots, u_k, x\}$ still linearly independent by assumption of $x \notin U = \operatorname{Span}(\{u_1, \ldots, u_k\})$. Thus, we can extend this to a basis $\beta = \{u_1, \ldots, u_k, x, v_1, \ldots, v_m\}$ for V. Define $f : V \to \mathbb{F} \in V^*$ as the unique linear transformation such that $f(u_i) = f(v_j) = 0$ and $f(x) = 1$. Then, $f \in U^\perp$ by definition, and $f(x) = 1$ by definition. This is a contradiction that $x \notin U$. ■

[17] Even if S is not a subspace itself.

↪ **Corollary 2.9**

For a finite dimensional V and subspace $U \subseteq V$,

$$U = \{x \in V : \forall f \in U^{\perp}, f(x) = 0\}.$$

↪ **Definition 2.18: Dual/Transpose of T**

Let V, W be vector spaces over a field $\mathbb{F}$ and $T : V \to W$. We define the *dual/transpose* of T is the map $T^* : W^* \to V^*$, given by $g \mapsto g \circ T$. Ie, $T^*(g)(v) := g \circ T^*(v) = g(T(v))$.

↪ **Proposition 2.19**

Let V, W be vector spaces over a field $\mathbb{F}$ and $T : V \to W$.

1. T^* is linear.
2. $\mathrm{Ker}(T^*) = (\mathrm{Im}(T))^{\perp}$.
3. $\mathrm{Im}(T^*) \subseteq (\mathrm{Ker}(T))^{\perp}$ and is equal if V, W are finite dimensional.
4. If V, W are finite dimensional and β, γ are bases resp., then

$$[T^*]_{\gamma^*}^{\beta^*} = ([T]_{\beta}^{\gamma})^t.$$

Proof. 1. $T^*(ag_1 + g_2) = (ag_1 + g_2) \circ T = a \cdot g_1 \circ T + g_2 \circ T = a \cdot T^*(g_1) + T^*(g_2)$, $\forall g_1, g_2 \in W^*, a \in \mathbb{F}$.

2. For $g \in W^*$,

$$\begin{aligned} g \in \mathrm{Ker}(T^*) :\iff T^*(g) = 0_{V^*} &\iff T^*(g)(v) = 0 \,\forall v \in V \\ &\iff g(T(v)) = 0 \,\forall v \in V \\ &\iff g(w) = 0 \,\forall w \in \mathrm{Im}(T) \\ &\iff g \in (\mathrm{Im}(T))^{\perp} \end{aligned}$$

3. Fix $f = T^*(g) \in \mathrm{Im}(T^*)$, $g \in W^*$, and $u \in \mathrm{Ker}(T)$, noting that $f(u) = T^*(g)(u) = g(T(u)) = g(0_W) = 0$ so $f \in (\mathrm{Ker}(T))^{\perp}$.

$\blacksquare$

↪ *Lecture 17; Last Updated: Mon Feb 12 14:26:13 EST 2024*