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ODEs MATH325

Course Outline:

Based on Lectures from Winter, 2024 by Prof. Antony Humphries.

Contents

1 Introduction

1.1 Definitions

↪ **Definition 1.1: Diffferential equation**

A *diffferential equation* (DE) is an equation with derivatives. *Ordinary* DE's (ODE) will be covered in this course; other types (PDE's, SDE's, DDE's, FDE's, etc.) exist as well but won't be discussed. ODE's only have one independent variable (typically, $y = f(x)$ or $y = f(t)$).

⊛ **Example 1.1: A Trivial Example**

$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x$. Integrating both sides:

$$\int \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int 6x\,\mathrm{d}x \implies y(x) = 3x^2 + C.$$

⊛ **Example 1.2: Another One**

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = 0 \implies y = at + b.$$

↪ **Definition 1.2: Order**

The order of a differential equation is defined as the order of the highest derivative in the equation.

1.2 Initival Values

Remark 1.1. *Note the existence of arbitrary constants in the previous examples, indicating infinite solutions. We often desire unique solutions by fixing these coefficients. For first order ODEs, we simply specify a single initial condition (say, some $y(x_0) = \alpha_0$). For higher order ODEs of degree n, we can either specify $n-1$ initial conditions for $n-1$ derivatives (say, $y(x_0) = \alpha_0$, $y'(x_0) = \beta_0$), or boundary conditions (say, $y(x_0) = \alpha_0, y(x_1) = \alpha_1$) where values for the solution itself are specified.*

⊛ **Example 1.3: A Less Trivial Example**

$\frac{\mathrm{d}y}{\mathrm{d}x} = y$. We cannot simply integrate both sides as before, as we have no way to know what $\int y\,\mathrm{d}x$ (the RHS) is equal to. We can fairly easily guess that $y = e^x$ is a solution; its derivative is equal to itself, hence it does indeed solve the equation. This is not the only solution; indeed, given $y = ce^x$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ce^x = y = ce^x.$$

Luckily, we were rather limited in how many places constants could appear; this doesn't always hold.

1.3 Physical Applications

⊛ **Example 1.4: Simple Pendulum**

Let θ be the angle of a pendulum of mass m from vertical and length l. Then, we have the equation of motion

$$ml\ddot{\theta} = -mg\sin\theta \implies \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \implies \ddot{\theta} + \omega^2\sin\theta = 0.$$

Take θ small, then, $\sin\theta \approx \theta$. Then, $\ddot{\theta} + \omega^2\theta = 0$. This is linear simple harmonic motion, and has periodic solutions; how do we know this is a valid solution to the non-linear model?

↪ *Lecture 01; Last Updated: Thu Jan 4 15:16:18 EST 2024*

⊛ **Example 1.5: Lorenz Equations**

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \sigma(y - x) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= rx - y - xz \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= xy - bz \end{aligned}$$

These are a famous set of equations originally derived from atmospheric modeling, known for its chaotic behavior for particular parameters. This is a nonlinear system of de's, and beyond the scope of this class (indeed, it is not solvable exactly).

1.4 Uniqueness

Given an ODE of the general form $y^{(n)} = f(t, y, y', \dots, y^{n-1})$, if we wish to determine $y^{(n)}(t_0)$ uniquely, we need to specify the initial conditions

$$y(t_0), y'(t_0), \dots, y^{(n-1)}(t_0).$$

Moreover, this not only determines uniqueness of $y^{(n)}(t_0)$, byt the uniqueness of solution y for $t \in I$ for some "interval of validity" I.

↪ **Definition 1.3: Autonomous/Nonautonomous**

An ODE of the form

$$y^{(n)} = f(y, y', \dots, y^{(n-1)})$$

is called *autonomous*; that is, if it has no explicit dependence on the independent variable. Otherwise, the system is called *nonautonomous*.

↪ Definition 1.4: Linear/Nonlinear

Linear ODEs of dimension n have a solution space which is a vector space of dimension n. As a result, solutions can be written as a linear combination of n basis solutions (or "fundamental set of solutions"). Solutions to nonlinear ODEs cannot be written this way (except locally).

Alternatively (but equivalently), if we can write an nth order ODE in the form

$$a_n(t)y^n(t) + \cdots a_1(t)y'(t) + a_0(t)y(t) = g(t),$$

or equivalently,

$$\sum_{i=0}^{n} a_i(t)y^i(t) = g(t), \quad ⊛$$

where each $a_i(t)$ and $g(t)$ are known functions of t, then we say that the ODE is linear. Otherwise, it is nonlinear.

⊛ Example 1.6

The pendulum

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

is autonomous and linear;

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

is autonomous and nonlinear, due to the $\sin\theta$ term (indeed, this is a nonlinear oscillator equation); a damped-forced oscillator

$$\ddot{\theta} + k^2\dot{\theta} + \omega^2\theta = A\sin(\mu t)$$

is nonautonomous and linear.

Remark 1.2. *Note that the following definitions apply only to linear ODEs.*

↪ Definition 1.5: Homogeneous/Nonhomogeneous

A linear ODE of the form ⊛ is *homogeneous* if $g(t) = 0$; otherwise it is *nonhomogeneous*.

↪ Definition 1.6: Constant/Variable

A linear ODE of the form $*$ is *constant coefficient* if $a_j(t) =$ constant $\forall\, j$; if at least one a_j not constant, it is *non-constant* or *variable coefficient*.

Remark 1.3. *Note that while we define linearity of ODEs in terms of the form of $y^{(n)} = f(t, y, \ldots)$, this more "helpfully" relates to the form of the solution of such an ODE, which is indeed linear.*

1.5 Solutions

Given an n order ODE $y^{(n)} = f(t, y, \ldots)$, and assuming f continuous, then for $y(t)$ to be a solution, we need y to be n-times differentiable; hence, $y, \ldots, y^{(n-1)}$ must all exist and be continuous. Then, $y^{(n)}$, being a continuous function of continuous functions, is, itself, continuous.

↪ **Definition 1.7: Solution**

The function $y(t) : I \to \mathbb{R}$ is a solution to an ODE on an interval $I \subseteq \mathbb{R}$ if it is n-times differentiable on I, and satisfies the ODE on this interval.

Given an well-defined IVP with $n - 1$ initial values defined at t_0, then $y(t)$ is a solution if $t_0 \in I$, y satisfies the initial values, and $y(t)$ is a solution on the interval.

↪ **Definition 1.8: Interval of Validity**

The largest I on which $y(t) : I \to \mathbb{R}$ solves an ODE is called the *interval of validity* of the problem.

↪ *Lecture 02; Last Updated: Thu Jan 11 11:05:26 EST 2024*

2 First Order ODEs

2.1 Separable ODEs

↪ **Definition 2.1: Separable ODE**

An ODE of the form

$$y' = P(t)Q(y)$$

is called *separable*. We solve them:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= P(t)Q(y) \\ \implies \int \frac{1}{Q(y)} \,\mathrm{d}y &= \int P(t) \,\mathrm{d}t \,. \end{aligned}$$

Finish by evaluating both sides.

⊛ **Example 2.1**

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}t} = ty && (1) \\ &\implies \frac{1}{y} \,\mathrm{d}y = t \,\mathrm{d}t && (2) \\ &\implies \ln|y| = \frac{t^2}{2} + C && (3) \\ &\implies |y| = Ke^{\frac{t^2}{2}} \quad \text{where } K = e^C && (4) \\ &\implies y = Be^{\frac{t^2}{2}} \quad \text{where } B = \pm K = \pm e^C && (5) \end{aligned}$$

Note that we call line (3) an *implicit solution*. In this case, we could easily turn this into an explicit solution by solving for $y(t)$; this won't always be possible.

Note that it would appear, based on the definition, that $B \neq 0$ (as $e^{\cdots} \neq 0$); however, plugging $y = 0$ into (1) shows that this is indeed a solution. It is quite easy to verify that (5) is a valid solution;

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Be^{\frac{t^2}{2}}\right) = Bte^{\frac{t^2}{2}} = t \cdot y,$$

as desired; this holds $\forall\, B \in \mathbb{R}$.

Remark 2.1. *Is it valid to split the differentials like this?*

$$\begin{aligned} \frac{1}{Q(y)}\frac{\mathrm{d}y}{\mathrm{d}t} &= P(t) \\ \Longrightarrow \int \frac{1}{Q(t)}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t &= \int P(t)\,\mathrm{d}t \end{aligned}$$

Let $g(y) = \frac{1}{Q}(y)$ *and* $G(y) = \int g(y)\,\mathrm{d}y$. *By the chain rule,*

$$\frac{\mathrm{d}}{\mathrm{d}t}(G(y(t))) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}}{\mathrm{d}y}G(y(t)) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot g(y(t)) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{1}{Q(y(t))}.$$

Integrating both sides with respect to time, we have

$$\begin{aligned} G(y(t)) = \int \frac{1}{Q(y(t))}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t &= \int P(t)\,\mathrm{d}t + C \\ \Longrightarrow \int g(y)\,\mathrm{d}y &= \int P(t)\,\mathrm{d}t + C \\ \Longrightarrow \int \frac{1}{Q(y)}\,\mathrm{d}y &= \int P(t)\,\mathrm{d}t + C \end{aligned}$$

This was our original expression obtaining by "splitting", hence it is indeed "valid".

⊛ **Example 2.2**

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2}{1-y^2} \\ \Longrightarrow \int (1-y^2)\,\mathrm{d}y &= \int x^2\,\mathrm{d}x \\ \Longrightarrow y - \frac{y^3}{3} &= \frac{x^3}{x} + C \\ \Longrightarrow y - \frac{1}{3}(y^3 + x^3) &= C \end{aligned}$$

Suppose we have the same ODE but now with an IVP $y(0) = 4$. Then, plugging this into our implicit solution:

$$4 - \frac{1}{3}(64 + 0) = C \implies C = 4 - \frac{64}{3} = -\frac{52}{3},$$

so our IVP solution is

$$y - \frac{1}{3}(y^3 + x^3) = -\frac{52}{3}.$$

2.2 Linear First Order ODEs

↪ **Definition 2.2: Integrating Factor**

A linear first order ODE of the form

$$\begin{aligned} a_1(t)y'(t) + a_0(t)y(t) &= g(t) \\ \implies y' + \frac{a_0}{a_1}y &= \frac{g}{a_1} \\ \implies y' + p(t)y &= q(t). \end{aligned}$$

To solve, we multiply by some integrating factor $\mu(t)$;

$$\mu(t)y'(t) + p(t)\mu(t)y(t) = \mu(t)q(t)$$

It would be quite convenient if $p(t)\mu(t) = \mu'(t)$; in this case, we'd have

$$\begin{aligned} \mu(t)y' + \mu'(t)y &= \mu(t)q(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)y(t)) &= \mu(t)q(t) \\ \implies \mu(t)y(t) &= \int \mu(t)q(t)\,\mathrm{d}t + C \\ \implies y(t) &= \frac{1}{\mu(t)}\int \mu(t)q(t)\,\mathrm{d}t + \frac{C}{\mu(t)} \end{aligned}$$

Now, what is $\mu(t)$? We required that

$$\begin{aligned} \mu'(t) &= p(t)\mu \\ \frac{\mathrm{d}\mu}{\mathrm{d}t} &= p(t)\mu \\ \implies \int \frac{\mathrm{d}\mu}{\mu} = \int p(t)\,\mathrm{d}t \implies \ln|\mu| &= \int p(t)\,\mathrm{d}t \\ \implies \mu(t) &= Ke^{\int p(t)\mathrm{d}t} \end{aligned}$$

However, note in our whole process earlier, we need only one μ; hence, for convenience, we can disregard any constants of integration and simply take

$$\boxed{\text{Integrating Factor:} \quad \mu(t) := e^{\int p(t)\mathrm{d}t}}$$

Then, our original linear ODE has general solution

$$y(t) = Ce^{-\int p(t)\mathrm{d}t} + e^{-\int p(t)\mathrm{d}t} \int e^{\int p(t)\mathrm{d}t} q(t) \,\mathrm{d}t\,.$$

⊛ **Example 2.3**

$$\begin{aligned}
ty' + 3y - t^2 &= 0 \\
y' + \frac{3}{t}y &= t \\
&\implies \mu(t) = e^{\int \frac{3}{t}\mathrm{d}t} = e^{3\ln|t|} = t^3 \\
&\implies t^3y' + 3t^2y = t^4 \\
&\implies \frac{\mathrm{d}}{\mathrm{d}t}(yt^3) = t^4 \\
&\implies yt^3 = \int t^4 \,\mathrm{d}t \\
&\implies y = \frac{1}{t^3} \cdot \frac{t^5}{5} + \frac{C}{t^3} = \frac{t^2}{5} + \frac{C}{t^3}
\end{aligned}$$

Note the division by zero issue when $t = 0$; this is not an issue with the solution method, but indeed with the ODE itself. The ODE breaks down when $t = 0$ for the same reason.

Thus, this solution is valid for $t \in (-\infty, 0) \cup (0, \infty) =: I_1 \cup I_2$; if we are given an IVP $y(t_0) = y_0$, if $t_0 < 0$, then the interval of validity is I_1, and if $t_0 > 0$, the interval of validity is I_2.

2.3 Exact Equations

↪ **Definition 2.3: Exact Equations**

A first order ODE of the form

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \iff \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{M(x, y)}{N(x, y)}$$

is said to be *exact* if

$$\frac{\partial}{\partial y}M(x, y) = \frac{\partial}{\partial x}N(x, y) \iff M_y(x, y) = N_x(x, y).$$

Suppose we have a solution $f(x, y(x)) = C$. Then,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(f(x, y(x))) &= 0 \\ \implies \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \implies \frac{f_x}{f_y} &= -\frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}$$

Now, with $f_x(x, y) = M(x, y)$ and $f_y = N(x, y)$, then $M_y(x, y) = f_{xy}(x, y)$ and $N_x = f_{yx}(x, y)$. Assuming f continuous with existing, continuous partial derivatives, then $f_{xy} = f_{yx}$ and hence $M_y(x, y) = N_x(x, y)$. Thus, a function f such that $f_x = M$ and $f_y = N$ yields a solution to the ODE.

⊛ **Example** 2.4

$$\begin{aligned} 2xy^2 \,\mathrm{d}x + 2x^2 y \,\mathrm{d}y = 0 &\equiv M \,\mathrm{d}x + N \,\mathrm{d}y = 0 \\ \implies M_y = 4xy, \qquad &\implies N_x = 4xy \\ f_x = M = 2xy^2 \implies f(x, y) &= x^2y^2 + C + F(y) \\ f_y = N = 2x^2y \implies f(x, y) &= x^2y^2 + C + F(x) \\ \implies f(x, y) &= x^2y^2 + C = K \end{aligned}$$

We can rearrange this as an explicit solution

$$y = \frac{k}{x}$$

for some constant k.

↪ Lecture 03; Last Updated: Tue Jan 16 10:10:00 EST 2024

↪ **Theorem** 2.1

This technique works generally.

Proof. Given an exact ODE of the form $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$, we need to show that $\exists f(x, y)$ s.t. $f(x, y) = c$ solves the ODE. Let

$$f(x, y) = \int_{x_0}^{x} M(s, y)\,\mathrm{d}s + g(y)$$

for some function $g(y)$ to be chosen such that $f_y = N$. But we have

$$\begin{aligned} N(x, y) = f_y(x, y) &= \frac{\partial}{\partial y}\left[\int_{x_0}^{x} M(s, y)\,\mathrm{d}s + g(y)\right] \\ &= g'(y) + \frac{\partial}{\partial y}\int_{x_0}^{x} M(s, y)\,\mathrm{d}s \\ \implies g'(y) &= N(x, y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(s, y)\,\mathrm{d}s\,. \end{aligned}$$

But the LHS is a function of y only, while the RHS depends explicitly on x; hence, this technique will only work if the entire expression is actually independent of x. To show this, we take the partial of the RHS with respect to x:

$$\begin{aligned}\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right] &= N_x(x,y) - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s \\ &= N_x(x,y) - \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right] \\ &= N_x(x,y) - \frac{\partial}{\partial y}\left[M(x,y)\right] \\ &= N_x - M_y = 0,\end{aligned}$$

as the ODE is exact. Hence, the RHS is indeed a function of y alone. So, integrating both sides with respect to y:

$$g(y) = \int \left[N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right]\mathrm{d}y\,,$$

which gives us a $f(x,y)$ of

$$\begin{aligned} f(x,y) &= \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + \int\left[N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right]\mathrm{d}y\,, \\ \implies f(x,y) &= \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + \int_{y_0}^{y} N(x,t)\,\mathrm{d}t - \int_{y_0}^{y}\int_{x_0}^{x} M_y(s,t)\,\mathrm{d}s\,\mathrm{d}t \qquad \star\end{aligned}$$

which satisfies $f_x = M$ and $f_y = N$. Then, for $f(x,y) = C$, we have

$$\frac{\partial f}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial f}{\partial y} = M + \frac{\mathrm{d}y}{\mathrm{d}x}N = 0 \implies M\,\mathrm{d}x + N\,\mathrm{d}y = 0,$$

as desired.

Note that $\star$ is evaluated over a rectangle $[x_0, x] \times [y_0, y]$, but holds for any connected domain containing (x_0, y_0) and (x, y).

Also note that, as described, $g(y)$ is not a function of x; hence, we can pick x arbitrarily. Suppose we take $x = x_0$, then

$$f(x,y) = \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + \int_{y_0}^{y} N(x_0,t)\,\mathrm{d}t\,.$$

■

Remark 2.2. *We could have taken $g(x)$ and started from $f_y = N$. Then, we would have had the formula*

$$f(x,y) = \int_{y_0}^{y} N(x,t)\,\mathrm{d}t + \int_{x_0}^{x} M(s,y_0)\,\mathrm{d}y\,.$$

⊛ **Example 2.5**

$$2xy\,\mathrm{d}x + (x^2 - 1)\,\mathrm{d}y = 0.$$

We have $M(x,y) = 2xy$ and $N(x,y) = x^2 - 1$, so $M_y = 2x = N_y$ and the ODE is exact; hence, a solution exists of the form $f(x,y) = c$ where $f_x = M, f_y = N$.

$$f(x,y) = \int M(x,y)\,\mathrm{d}x = \int 2xy\,\mathrm{d}x = x^2y + k_1(y)$$
$$f(x,y) = \int N(x,y)\,\mathrm{d}y = \int (x^2-1)\,\mathrm{d}y = x^2y - y + k_2(x)$$

Hence $k_1(y) = -y$ and $k_2(x) = 0$, so

$$f(x,y) = x^2y - y = y(x^2-1),$$

so solutions to the original ODE are

$$y(x^2-1) = C \implies y = \frac{C}{x^2-1}.$$

2.4 Exact ODEs Via Integrating Factors

Suppose

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$

but $M_y \neq N_x$, that is, the ODE is not exact. Can we find an integrating factor $\mu(x,y)$ s.t.

$$[\mu(x,y)M(x,y)]\,\mathrm{d}x + [\mu(x,y)N(x,y)]\,\mathrm{d}y = 0$$

is exact? If so, such a μ must satisfy

$$\begin{aligned}
&\frac{\partial}{\partial y}[\mu(x,y)M(x,y)] = \frac{\partial}{\partial x}[\mu(x,y)N(x,y)] \\
&\implies \mu_y M + \mu M_y = \mu_x N + \mu N_x \\
&\implies N\mu_x - M\mu_y = (M_y - N_x)\,\mu \quad \circledast
\end{aligned}$$

This is not a generally easily soluble PDE; we will consider cases where μ is a function of only one independent variable, which greatly simplifies the expression; this could be simply $\mu(x)$, $\mu(y)$, or even $\mu(x \cdot y)$.

Suppose $\mu = \mu(x) \implies \mu_y = 0$. Then, $\circledast$ becomes

$$N\mu' = (M_y - N_x)\mu \implies \mu' = \left(\frac{M_y - N_x}{N}\right)\mu.$$

This is valid, provided the expression $\left(\frac{M_y - N_x}{N}\right)$ is a function solely of x. In this case, this becomes a linear first order ODE, with solution

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,\mathrm{d}x}.$$

OTOH, if $\mu = \mu(y)$, we can similarly derive

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\,\mathrm{d}y},$$

with a similar stipulation on the expression $\left(\frac{N_x - M_y}{M}\right)$ being a function of y solely.

⊛ **Example 2.6**

$$xy\,\mathrm{d}x + (2x^2 + 3y^2 - 20)\,\mathrm{d}y = 0,$$

with $M(x,y) = xy \implies M_y = x$ and $N(x,y) = 2x^2 + 3y^2 - 20 \implies N_x = 4x$. We have $M_y - N_x = x - 4x = -3x$ (so the ODE is not exact). We write

$$\frac{M_y - N_x}{M} = \frac{-3x}{xy} = \frac{-3}{y},$$

which is a function solely of y; hence, can find a $\mu(y)$:

$$\mu(y) = e^{-\int \frac{M_y - N_x}{M}\mathrm{d}y} = e^{-\int -\frac{3}{y}\mathrm{d}y} = e^{3\ln y} = y^3,$$

noting that we, as before, do not care about any integrating factors; we are seeking a single function that works. Multiplying this into our original ODE:

$$\underbrace{xy^4}_{:=\tilde{M}}\,\mathrm{d}x + \underbrace{(2x^2 + 3y^2 - 20)y^3}_{:=\tilde{N}}\,\mathrm{d}y = 0.$$

And indeed, we have

$$\tilde{M}_y = 4xy^3; \quad \tilde{N}_x = 4xy^3 \implies \tilde{M}_y = \tilde{N}_x,$$

as desired.

↪ *Lecture 04; Last Updated: Tue Jan 23 10:02:55 EST 2024*

↪ *Lecture 05; Last Updated: Tue Jan 23 10:23:37 EST 2024*

2.5 Qualitative Methods and Theory

Remark 2.3. *Read the first few chapters of Strogatz's Nonlinear Dynamics and Chaos book and you should be all good.*

⊛ **Example 2.7**

Show that $y' = y^{\frac{1}{3}}$ with $y(0) = 0$ has infinite solutions.

↪ *Lecture 06; Last Updated: Tue Jan 23 11:21:04 EST 2024*

2.6 Existence and Uniqueness

↪ **Definition 2.4: Lipschitz Continuity**

A function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is said to be *Lipschitz continuous* in y on the rectangle $R = \{(x, y) : x \in [a, b], y \in [c, d]\} = [a, b] \times [c, d]$ if there exists a constant $L > 0$ s.t.

$$|f(x, y_1) - f(x, y_2)| \leq L |y_1 - y_2|, \quad \forall (x, y_1), (x, y_2) \in R.$$

L is called the *Lipschitz constant*.

Remark 2.4. *Note that we define in terms on continuity in y; the x variable in each coordinate is kept constant.*

↪ **Lemma 2.1**

If $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous in x, y in the rectangle R, then f is Lipschitz in y on R.

Remark 2.5. *This result gives Differentiable $\implies$ Lipschitz Continuous $\implies$ Continuous.*

Proof. Using FTC, we have

$$\begin{aligned} f(x, y_2) &= f(x, y_1) + \int_{y_1}^{y_2} f_y(x, y) \, \mathrm{d}y \\ \implies |f(x, y_2) - f(x, y_1)| &= \left| \int_{y_1}^{y_2} f_y(x, y) \right| \leq \int_{y_1}^{y_2} |f_y(x, y)| \, \mathrm{d}y \\ &\leq |y_2 - y_1| \cdot \max_{(x,y) \in R} |f_y(x, y)|, \end{aligned}$$

noting that this maximum exists, and is attained, because f_y is continuous on a compact set. This gives, then, that f is Lipschitz in y with $L = \max_{(x,y) \in \mathbb{R}} |f_y(x, y)|$. ■

$\hookrightarrow$ Theorem 2.2: Existence and Uniqueness for Scalar First Order IVPs

If $f(t,y)$, $f_y(t,y)$ are continuous in t and y on a rectangle $R = \{(t,y) : t \in [t_0 - a, t_0 + a], y \in [y_0 - b, y_0 + b]\} = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$, then $\exists h \in (0, a]$ s.t. the IVP

$$y' = f(t,y), y(t_0) = y_0$$

has a unique solution, defined for $t \in [t_0 - h, t_0 + h]$. Moreover, this solution satisfies $y(t) \in [y_0 - b, y_0 + b] \, \forall \, t \in [t_0 - h, t_0 + h]$.

Remark 2.6. *A stronger theorem also holds with a weakened condition on f that requires only f Lipschitz. Clearly, f_y continuous $\implies$ f Lipschitz, so we will use this fact to prove the statement, but won't prove it for the only Lipschitz case for sake of conciseness.*

Proof. Rewrite the IVP as

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s)) \, \mathrm{d}s \, .$$

We will show this form has a unique solution, using an iteration method (namely, Picard Iteration).

We will begin by guessing a solution of the IVP, $y_0(t) = y_0$, $\forall \, t \in [t_0 - a, t_0 + a]$. This clearly satisfies the initial condition, but not the ODE itself.

Now, given $y_n(t)$, we define

$$y_{n+1}(t) = y(t_0) + \int_{t_0}^{t} f(s, y_n(s)) \, \mathrm{d}s \, .$$

If this terminates, that is, $y_{n+1}(t) = y_n(t) \, \forall \, t \in [t_0 - a, t_0 + a]$, then $y_n(t)$ solves the IVP.

We now show that this iteration is both well-defined, and converges to unique solution.

By construction, $y_0 : [t_0 - a, t_0 + a] \to [y_0 - b, y_0 + b]$, and is continuous. As a bounded function on a bounded interval, it is integrable, and the first step of our step is well-defined.

Now suppose $y_n(t) : [t_0 - a, t_0 + a] \to [y_0 - b, y_0 + b]$ is continuous and integrable. Then,

$$y_{n+1}(t) = y(t_0) + \int_{t_0}^{t} f(s, y_n(s)) \, \mathrm{d}s$$

is continuous as well, as f is continuous and $y_n(s)$ is as well. It is not guaranteed to be restricted to $[y_0 - b, y_0 + b]$, however.

Since f continuous and attains its maximum on R, let

$$M := \max_{(t,y) \in R} |f(t,y)| < \infty.$$

We have, then, that

$$
\begin{aligned}
y_{n+1}(t) - y(t_0) &= \int_{t_0}^{t} f(s, y_n(s))\,\mathrm{d}s \\
\implies |y_{n+1}(t) - y(t_0)| &\le |t - t_0|\, M
\end{aligned}
$$

Hence, if we choose $h : Mh \le b$, and then $y_{n+1}(t) : [t_0 - h, t_0 + h] \to [y_0 - b, y_0 + b]$ and we can iterative inductively, $y_n(t) : [t_0 - h, t_0 + h] \to [y_0 - b, y_0 + b]\,\forall\, n$. Here, we take $h = \min\{\frac{b}{M}, a\}$.

Now, let $I = [t_0 - h, t_0 + h]$, then $y_n(t) : I \to [y_0 - b, y_0 + b]$ for all n. Each iterate satisfies $y_n(t_0) = y(t_0) = y_0$; it remains to show that the iteration converges.

Let $C(I, [y_0 - b, y_0 + b])$ be the space of continuous functions $f : I \to [y_0 - b, y_0 + b]$, noting that $y_n \in C\,\forall\, n$. We define a mapping on C, $T : C \to C$ by

$$
v = Tu, v(t) = y_0(t_0) + \int_{t_0}^{t} f(s, u(s))\,\mathrm{d}s\,.
$$

Then, $y_{n+1} = Ty_n$. We aim to show that this iteration converges uniquely; we will do this by showing T is a contraction mapping.

For $y \in C$ define the norm $||y||_\infty$ by $||y||_\infty := \max_{t\in I} |y(t)|$. This is a norm;

1. $\forall\, k \in \mathbb{R}, ||ky||_\infty = |k|\, ||y||_\infty$.
2. $||y||_\infty = 0 \iff \max_{t\in I} |y(t)| = 0 \iff y(t) = 0\,\forall\, t \in I$.
3. $||y_1 + y_2||_\infty = \max_{t\in I} |y_1 + y_2| \le \max_{t\in I}(|y_1| + |y_2|) \le \max_{t\in I} |y_1| + \max_{t\in I} |y_2| = ||y_1||_\infty + ||y_2||_\infty$.

Now let $u, v \in C$. Then,

$$
\begin{aligned}
||Tu - Tv||_\infty &= \max_{t\in I} |Tu(t) - Tv(t)| \\
&= \max_{t\in I} \left| y(t_0) + \int_{t_0}^{t} f(s, u(s)\,\mathrm{d}s) - y_0 + \int_{t_0}^{t} f(s, v(s))\,\mathrm{d}s \right| \\
&= \max_{t\in I} \left| \int_{t_0}^{t} f(s, u(s)) - f(s, v(s))\,\mathrm{d}s \right| \\
&\le \max_{t\in I} \int_{t_0}^{t} |f(s, u(s)) - f(s, v(s))|\,\mathrm{d}s \\
&\le \max_{t\in I} |t - t_0| \cdot \max_{s\in I} |f(s, u(s)) - f(s, v(s))| \\
&\le hL \cdot \max_{s\in I} |u(s) - v(s)| \\
&= hL \cdot ||u - v||_\infty,
\end{aligned}
$$

hence, we have a contraction mapping if $hL < 1$; if $hL \ge 1$, let $h < \min\{a, \frac{b}{m}, \frac{1}{L}\} > 0$. With such an h, $\exists \mu \in (0, 1) : hL \le \mu < 1$, and $||Tu - Tv||_\infty \le \mu||u - v||_\infty$, hence, a contraction mapping.

The contractive mapping theorem, which will not be proven, states that any contraction mapping has a unique fixed point $y = Ty$; moreover, for any $y_0 \in C$, the iteration $y_{n+1} = Ty_n$ converges to y.

To see this, suppose $u = Tn, v = Tv$ are two solutions of our IVP. Then, by the contraction quality,

$$||u - v||_\infty = ||Tu - Tv||_\infty \leq \mu ||u - v||_\infty,$$

a contradiction unless $||u - v||_\infty = 0 \iff u = v$, hence, we have uniqueness of our solution; that is, our IVP has at most one solution. It remains to show that this solution exists.

Consider a sequence y_n, with $y_{n+1} = Ty_n$. Then,

$$\sum_{i=0}^{N} ||y_{i+1} - y_i||_\infty \leq \mu^N ||y_1 - y_0||_\infty,$$

by the contractive property, thus,

$$\sum_{i=0}^{\infty} ||y_{i+1} - y_i|| \leq (\sum_{i=0}^{\infty} \mu^j) ||y_1 - y_0||_\infty = \frac{1}{1-\mu} ||y_1 - y_0||_\infty = R_0,$$

for some radius (real number) R_0. Similarly, looking only at the tail of the series,

$$\sum_{j=n}^{\infty} ||y_{j+1} - y_j||_\infty \leq \frac{\mu^n}{1-\mu} ||y_1 - y_0||_\infty = \mu^n R_0,$$

that is, a "smaller" radius. We could, but won't, show that this sequence is Cauchy, and space C we are working in is complete and hence this sequence converges to some limit in the space; moreover, the limit of this sequence satisfies the IVP by construction. This is beyond the scope of this course. ■

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⊛ Example 2.8: Using Picard Iteration

$$y' = 2t(1+y) =: f(t, y), \quad y(0) = 0.$$

This ODE is linear and separable, and has solution $y(t) = e^{t^2} - 1$ (solving whichever way you like). We can alternatively solve this using Picard Iteration.

Let $y_0(t) = 0 \, \forall \, t$, noting that the IC is satisfied. We define

$$y_{n+1}(t) = \cancelto{0}{y(0)} + \int_{\cancelto{0}{t_0}}^{t} f(s, y_n(s)) \, \mathrm{d}s \,,$$

where $f(s, y_n(s)) = 2s(1 + y(s))$. This gives

$$\begin{aligned} y_{n+1}(t) = \int_0^t 2s(1+y_n(s))\,\mathrm{d}s\,. \\ \implies y_1(t) = \int_0^t 2s(1+y_0(s))\,\mathrm{d}s = \int_0^t 2s\,\mathrm{d}s = t^2 \\ \implies y_2(t) = \int_0^t 2s(1+s^2)\,\mathrm{d}s = t^2 + \frac{1}{2}t^4 \\ \implies y_3(t) = \cdots = t^2 + \frac{1}{2!}t^4 + \frac{1}{3!}t^6 \\ \cdots \implies y_n(t) = \sum_{k=1}^{n} \frac{t^{2k}}{k!} \\ \implies \lim_{n\to\infty} y_n(t) = \sum_{k=1}^{\infty} \frac{(t^2)^k}{k!} = e^{t^2} - 1, \end{aligned}$$

the same solution as previously shown.

Remark 2.7. *The previous example worked nicely due to $y_n(t)$ always being a simple polynomial with a familiar convergence. This is not always (nor often) the case.*

Remark 2.8. *Recall the example $y' = y^{\frac{1}{3}}$ with multiple solutions. In the language of the theorem, $f(t, y) = y^{\frac{1}{3}}$ is continuous, but $f_1(t, y) = \frac{1}{3}y^{-\frac{2}{3}}$ becomes unbounded as $y \to 0$, and the function is thus not Lipschitz in a neighborhood of $y = 0$.*

Remark 2.9. *Recall that this theorem guarantees solutions in a closed rectangular region; it is possible, under certain conditions, to extend the solution beyond the bounds. But how far?*

⊛ **Example 2.9**

$$y' = y^2, \quad y(0) = 1.$$

This has a solution $y(t) = \frac{1}{c-t} = \frac{1}{1-t}$ (with IC). Notice that $y(t) \to +\infty$ as $t \to 1$. By this observation, we have that, if we were to repeat Picard iteration for increasing time t, the rectangular domains of our validity of each piecewise solution would be bounded by 1.

↪ **Corollary 2.1**

If $f(t, y)$ and $f_y(t, y)$ are continuous for all $t, y \in \mathbb{R}$, then $\exists t_- < t_0 < t_+$ such that the IVP

$$y' = f(t, y), \quad y(t_0) = y_0$$

has a unique solution $y(t)\,\forall\, t \in (t_-, t_+)$, and moreover, either $t_+ = +\infty$ or $\lim_{t\to t_+} |y(t)| = \infty$, and either $t_- = -\infty$ or $\lim_{t\to t_-} |y(t)| = \infty$.

Remark 2.10. *Finding t_-, t_+ requires the solution. In example 2.9, $t_- = -\infty$, $t_+ = 1$. Changing the IC will naturally change these values.*

↪ **Theorem 2.3**

If $p(t), g(t)$ continuous on an open interval $I = (\alpha, \beta)$ and $t_0 \in I$, then the IVP

$$y'(t) + p(t)y = g(t), \quad y(t_0) = y_0$$

has a unique solution $y(t) : I \to \mathbb{R}$.

Remark 2.11. *In other words, this is a special case of the corollary above for linear ODEs; any "misbehavior" of the solutions would be solely due to discontinuities in the defining ODE.*

3 Second Order ODEs

3.1 Introduction

Second Order ODEs are of the form

$$y'' = f(t, y, y').$$

There is no general technique to solving these; we will be looking at special classes throughout.

Specifically in the case of nonlinear odes, there are two special cases we can solve,

1. f does not depend on y; ie $y'' = f(t, y')$. A substitution $u = y'$ yields $u' = f(t, u)$, hence this is just a first order ODE, with corresponding $y(t) = k_1 + \int u(t) \, \mathrm{d}t$.
2. f does not depend on t; ie $y'' = f(y, y')$. Let $u = y'$, so $u' = y'' = f(y, u)$. Consider $u = u(y(t))$, then,

   $$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = u\frac{\mathrm{d}u}{\mathrm{d}y},$$

   and so

   $$u\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{\mathrm{d}u}{\mathrm{d}t} = f(y, u) \implies \frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{y}f(y, u),$$

   which again yields a first order ODE, in $u = u(y)$.

⊛ **Example 3.1: Of Case 2.**

$$y'' + \omega^2 y = 0^a$$

Rewrite this as $y'' = -\omega^2 y = f(y, y')$, and let $u = y'$, then $\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{u}f(y, u) = \frac{1}{u}[-\omega^2 y]$. This is a separable equation:

$$\begin{aligned} u\,\mathrm{d}u &= -\omega^2 y\,\mathrm{d}y \\ \frac{1}{2}u^2 &= -\frac{1}{2}\omega^2 y^2 + c \\ \implies u^2 &= -\omega^2 y^2 + c' \\ \implies u = \pm\sqrt{k^2-\omega^2y^2} \implies \frac{\mathrm{d}y}{\mathrm{d}t} &= \pm\sqrt{k^2-\omega^2y^2} \end{aligned}$$

Which is just another separable equation[b]:

$$\begin{aligned} \pm\int \mathrm{d}t &= \frac{1}{\omega}\int \frac{\mathrm{d}y}{\sqrt{\frac{k^2}{\omega^2}-y^2}} \\ \implies \frac{1}{\omega}\arcsin\left(\frac{\omega y}{k}\right) &= \pm t + C \\ \implies \frac{\omega y}{k} = \sin\left(\pm\omega t \pm \omega\tilde{C}\right) &= \pm\sin\left(\omega t + \omega\tilde{C}\right) \\ \implies y(t) &= \pm\frac{k}{\omega}\sin\left(\omega t + \omega\tilde{C}\right) \\ \implies y(t) &= K\sin(\omega t + C), \end{aligned}$$

which can be rewritten $y(t) = k_1\sin(\omega t) + k_2\cos(\omega t)$ with the appropriate substitutions.

[a]This is the equation for a simple harmonic oscillator.
[b]Please excuse the sloppy use of constants, it doesn't really matter.

Remark 3.1. *This is not the easiest way to solve this equation. More generally, this technique can lead to intractable integrals.*

⊛ **Example 3.2: Nonlinear Pendulum**

$$y'' + \omega^2\sin y = 0.$$

Making the same substitution as before, $u = y'$, we have

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}y} &= -\frac{1}{u}\omega^2\sin y \\ \int u\,\mathrm{d}u &= \int -\omega^2\sin y\,\mathrm{d}y \\ \frac{1}{2}u^2 &= \omega^2\cos y + c_1 \\ \frac{1}{2}(y')^2 &= \omega^2\cos y + c_1 \\ y' &= \pm\sqrt{2c_1 + 2\omega^2\cos y} \\ \pm\int\mathrm{d}t &= \int\frac{\mathrm{d}y}{\sqrt{2c+2\omega^2\cos y}}, \end{aligned}$$

where the integral on the RHS is some type of elliptic integral.

3.2 Linear, Homogeneous

We will solve a general form

$$a(t)y'' + b(t)y' + c(t)y = 0 \quad \circledast\,.$$

3.2.1 Principle of Superposition

↪ **Theorem 3.1: Superposition of Solutions to Linear Second Order ODEs**

If $y_1(t), y_2(t)$ solve $\circledast$ for $t \in I$-interval, then $y(t) = k_1y_1(t) + k_2y_2(t)$, for constants k_1, k_2 solves $\circledast$ on I as well. In other words, linear combinations of solutions are themselves solutions.

Remark 3.2. *This can be extended quite naturally to any linear order of ODE.*

Proof. This is clear by just plugging into the problem; let $y(t) = k_1y_1(t) + k_2y_2(t)$. Then:

$$\begin{aligned} a(t)y''(t) + b(t)y'(t) + c(t)y(t) &= a(t)(k_1y_1'' + k_2y_2'') + b(t)(k_1y_1' + k_2y_2') + c(t)(k_1y_1 + k_2y_2) \\ &= k_1(ay_1'' + by_1' + cy_1) + k_2(ay_2'' + by_2' + cy_2) \\ &= k_1 \cdot 0 + k_2 \cdot 0 = 0, \end{aligned}$$

as desired. ■

↪ *Lecture 08; Last Updated: Tue Jan 30 11:24:41 EST 2024*