

# MATH455 - Analysis 4

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## §1 ABSTRACT METRIC AND TOPOLOGICAL SPACES

### §1.1 Review of Metric Spaces

Throughout fix  $X$  a nonempty set.

↪ **Definition 1.1** (Metric):  $\rho : X \times X \rightarrow \mathbb{R}$  is called a *metric*, and thus  $(X, \rho)$  a *metric space*, if for all  $x, y, z \in X$ ,

- $\rho(x, y) \geq 0$ ,
- $\rho(x, y) = 0 \Leftrightarrow x = y$ ,
- $\rho(x, y) = \rho(y, x)$ , and
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ .

↪ **Definition 1.2** (Norm): Let  $X$  a linear space. A function  $\| \cdot \| : X \rightarrow [0, \infty)$  is called a *norm* if for all  $u, v \in X$  and  $\alpha \in \mathbb{R}$ ,

- $\|u\| = 0 \Leftrightarrow u = 0$ ,
- $\|u + v\| \leq \|u\| + \|v\|$ , and
- $\|\alpha u\| = |\alpha| \|u\|$ .

**Remark 1.1:** A norm induces a metric by  $\rho(x, y) := \|x - y\|$ .

↪ **Definition 1.3:** Given two metrics  $\rho, \sigma$  on  $X$ , we say they are *equivalent* if  $\exists C > 0$  such that  $\frac{1}{C}\sigma(x, y) \leq \rho(x, y) \leq C\sigma(x, y)$  for every  $x, y \in X$ . A similar definition follows for equivalence of norms.

Given a metric space  $(X, \rho)$ , then, we have the notion of

- open balls  $B(x, r) = \{y \in X : \rho(x, y) < r\}$ ,
- open sets (subsets of  $X$  with the property that for every  $x \in X$ , there is a constant  $r > 0$  such that  $B(x, r) \subseteq X$ ), closed sets, closures, and
- *convergence*.

↪ **Definition 1.4** (Convergence):  $\{x_n\} \subseteq X$  converges to  $x \in X$  if  $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$ .

We have several (equivalent) notions, then, of continuity; via sequences,  $\varepsilon - \delta$  definition, and by pullbacks (inverse images of open sets are open).

↪ **Definition 1.5** (Uniform Continuity):  $f : (X, \rho) \rightarrow (Y, \sigma)$  uniformly continuous if  $f$  has a “modulus of continuity”, i.e. there is a continuous function  $\omega : [0, \infty) \rightarrow [0, \infty)$  such that

$$\sigma(f(x_1), f(x_2)) \leq \omega(\rho(x_1, x_2))$$

for every  $x_1, x_2 \in X$ .

**Remark 1.2:** For instance, we say  $f$  Lipschitz continuous if there is a constant  $C > 0$  such that  $\omega(\cdot) = C(\cdot)$ . Let  $\alpha \in (0, 1)$ . We say  $f$   $\alpha$ -Holder continuous if  $\omega(\cdot) = C(\cdot)^\alpha$  for some constant  $C$ .

↪ **Definition 1.6** (Completeness): We say  $(X, \rho)$  *complete* if every cauchy sequence in  $(X, \rho)$  converges to a point in  $X$ .

**Remark 1.3:** If  $(X, \rho)$  complete and  $E \subseteq X$ , then  $(E, \rho)$  is complete iff  $E$  closed in  $X$ .

## §1.2 Compactness, Separability

↪ **Definition 1.7** (Open Cover, Compactness):  $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$ , where  $X_\lambda$  open in  $X$  and  $\Lambda$  an arbitrary index set, an *open cover* of  $X$  if for every  $x \in X$ ,  $\exists \lambda \in \Lambda$  such that  $x \in X_\lambda$ .

$X$  is *compact* if every open cover of  $X$  admits a compact subcover. We say  $E \subseteq X$  compact if  $(E, \rho)$  compact.

↪ **Definition 1.8** (Totally Bounded,  $\varepsilon$ -nets):  $(X, \rho)$  *totally bounded* if  $\forall \varepsilon > 0$ , there is a finite cover of  $X$  of balls of radius  $\varepsilon$ . If  $E \subseteq X$ , an  $\varepsilon$ -*net* of  $E$  is a collection  $\{B(x_i, \varepsilon)\}_{i=1}^N$  such that  $E \subseteq \bigcup_{i=1}^N B(x_i, \varepsilon)$  and  $x_i \in X$  (note that  $x_i$  need not be in  $E$ ).

↪ **Definition 1.9** (Sequentially Compact):  $(X, \rho)$  *sequentially compact* if every sequence in  $X$  has a convergence subsequence whose limit is in  $X$ .

↪ **Definition 1.10** (Relatively/Pre- Compact):  $E \subseteq X$  *relatively compact* if  $\bar{E}$  compact.

↪ **Theorem 1.1:** TFAE:

- $X$  complete and totally bounded;
- $X$  compact;
- $X$  sequentially compact.

**Remark 1.4:**  $E \subseteq X$  relatively compact if every sequence in  $E$  has a convergent subsequence.

Let  $f : (X, \rho) \rightarrow (Y, \sigma)$  continuous with  $(X, \rho)$  compact. Then,

- $f(X)$  compact in  $Y$ ;
- if  $Y = \mathbb{R}$ , the max and min of  $f$  over  $X$  are achieved;
- $f$  is uniformly continuous.

Let  $C(X) := \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous}\}$  and  $\|f\|_\infty := \max_{x \in X} |f(x)|$  the sup (max, in this case) norm. Then,

**→ Theorem 1.2:** Let  $(X, \rho)$  compact. Then,  $(C(X), \|\cdot\|_\infty)$  is complete.

PROOF. Let  $\{f_n\} \subseteq C(X)$  Cauchy with respect to  $\|\cdot\|_\infty$ . Then, there exists a subsequence  $\{f_{n_k}\}$  such that for each  $k \geq 1$ ,  $\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq 2^{-k}$ .

Let  $j \in \mathbb{N}$ . Then, for any  $k \geq 1$ ,

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell=k}^{\infty} 2^{-\ell}$$

and hence for each  $x \in X$ , with  $c_k := f_{n_k}(x)$ ,

$$|c_{k+j} - c_k| \leq \sum_{\ell=k}^{\infty} 2^{-\ell}.$$

The RHS is the tail of a converging series, and thus  $|c_{k+j} - c_k| \rightarrow 0$  as  $k \rightarrow \infty$  i.e.  $\{c_k\}$  a Cauchy sequence, in  $\mathbb{R}$ .  $(\mathbb{R}, |\cdot|)$  complete, so  $\lim_{k \rightarrow \infty} c_k =: f(x)$  exists for each  $x \in X$ . So, for each  $x \in X$ , we find

$$|f_{n_k}(x) - f(x)| \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

and since the RHS is independent of  $x$ , we may pass to the sup norm, and find

$$\|f_{n_k} - f\|_\infty \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

with the RHS  $\rightarrow 0$  as  $k \rightarrow \infty$ . Hence,  $f_{n_k} \rightarrow f$  in  $C(X)$  as  $k \rightarrow \infty$ . In other words, we have uniform convergence of  $\{f_{n_k}\}$ . Each  $\{f_{n_k}\}$  continuous, and thus  $f$  also continuous, and thus  $f \in C(X)$ .

It remains to show convergence along the whole sequence. Suppose otherwise. Then, there is some  $\alpha > 0$  and a subsequence  $\{f_{n_j}\} \subseteq \{f_n\}$  such that  $\|f_{n_j} - f\|_\infty > \alpha > 0$  for every  $j \geq 1$ . Then, let  $k$  be sufficiently large such that  $\|f - f_{n_k}\|_\infty \leq \frac{\alpha}{2}$ . Then, for every  $j \geq 1$  and  $k$  sufficiently large,

$$\begin{aligned} \|f_{n_j} - f_{n_k}\|_\infty &\geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty \\ &> \alpha - \frac{\alpha}{2} > 0, \end{aligned}$$

which contradicts the Cauchy-ness of  $\{f_n\}$ , completing the proof. ■