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Analysis I, II

MATH254

Contents

1 Logic, Sets, and Functions

1.1 Mathematical Induction & The Naturals

The **natural numbers**, $\mathbb{N} = \{1, 2, 3, \ldots\}$, are specified by the 5 **Peano Axioms**:

(1) $1 \in \mathbb{N}$ [1]

(2) every natural number has a successor in $\mathbb{N}$

(3) 1 is not the successor of any natural number

(4) if the successor of x is equal to the successor of y, then x is equal to y [2]

(5) **the axiom of induction**

The **Axiom of Induction** (AI), can be stated in a number of ways.

Axiom 1.1 (*AI.i*)

Let $S \subseteq \mathbb{N}$ with the properties:

(a) $1 \in S$

(b) if $n \in S$, then $n + 1 \in S$[3]

then $S = \mathbb{N}$.

Example 1.1. *Prove that, for every $n \in \mathbb{N}$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2} (\equiv (1))$*

Proof (via AI.i): *Let S be the subset of $\mathbb{N}$ for which (1) holds; thus, our goal is to show $S = \mathbb{N}$, and we must prove (a) and (b) of AI.i.*

- *by inspection, $1 \in S$ since $1 = \frac{1(1+1)}{2} = 1$, proving (a)*
- *assume $n \in S$; then, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ by definition of S. Adding $n + 1$ to both sides yields:*

$$
\begin{aligned}
1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) && (1)\\
&= (n+1)(\frac{n}{2} + 1) && (2)\\
&= \frac{(n+1)(n+2)}{2} && (3)\\
&= \frac{(n+1)((n+1)+1)}{2} && (4)
\end{aligned}
$$

[1] using 0 instead of 1 is also valid, but we will use 1 here.

[2] axioms (2)-(4) can be equivalently stated in terms of a successor function $s(n)$ more rigorously, but won't here

[3] *(a)* is called the **inductive base**; *(b)* the **inductive step**. All AI restatements are equivalent in having both of these, and only differentiate on their specific values.

Line (4) is equivalent to statement (1) (substituting n for $n+1$), and thus if $n \in S$, then $n+1 \in S$ and (b) holds. Thus, by AI.i, $S = \mathbb{N}$ and $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ holds $\forall n \in \mathbb{N}$. ■

Exercise 1.1. *Prove (by induction), that for every $n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.*

(Solution)

Example 1.1 can also be proven directly (Gauss' method).

Proof: Let $A(n) = 1+2+3+\cdots+n$. We can write $2 \cdot A(n) = 1+2+3+\cdots+n+1+2+3+\cdots+n$. Rearranging terms (1 with n, 2 with $n-1$, etc.), we can say $2 \cdot A(n) = (n+1) + (n+1) + \cdots$, where $(n+1)$ is repeated n times; thus, $2 \cdot A(n) = n(n+1)$, and $A(n) = \frac{n(n+1)}{2}$. ■

Axiom 1.2 (*AI.ii*)

Let $S \subseteq \mathbb{N}$ s.t.
(a) $m \in S$
(b) $n \in S \implies n+1 \in S$
then $\{m, m+1, m+2, \dots\} \subseteq S$.

Exercise 1.2. *Using AI.ii, prove that for $n \geq 2$, $n^2 > n+1$*

(Solution)

Axiom 1.3 (*Principle of Complete Induction, AI.iii*)

Let $S \subseteq \mathbb{N}$ s.t.
(a) $1 \in S$
(b) if $1, 2, \dots, n-1 \in S$, *then* $n \in S$
then $S = \mathbb{N}$.

Finally, combing AI.ii and AI.iii;

Axiom 1.4 (*AI.iv*)

Let $S \subseteq \mathbb{N}$ s.t.:
(a) $m \in S$
(b) if $m, m+1, \dots, m+n \in S$, *then* $m+n+1 \in S$
then $\{m, m+1, m+2, \dots\} \subseteq S$.

Theorem 1.1 (*Fundamental Theorem of Arithmetic*)

Every natural number n can be written as a product of one or more primes. [4]

Proof: Let S be the set of all natural numbers that can be written as a product of one or more primes. We will use AI.iv to show $S = \{2, 3, \dots\}$.

- *(a)* holds; 2 is prime and thus $2 \in S$
- suppose that $2, 3, \dots, 2+n \in S$. Consider $2 + (n+1)$:

[4] 1 is not a prime number

- if $2+(n+1)$ is *prime*, then $2+(n+1) \in S$, as all primes are products of 1 and themselves and are thus in S by definition.
- if $2+(n+1)$ is *not prime*, then it can be written as $2+(n+1) = a \cdot b$ where $a, b \in \mathbb{N}$, and $1 < a < 2+(n+1)$ and $1 < b < 2+(n+1)$. By the definition of S, $a, b \in S$, and can thus be written as the product of primes. Let $a = p_1 \cdot \dots \cdot p_l$ and $b = q_1 \cdot \dots \cdot q_j$, where the p's and q's are prime and $l, j \geq 1$. Then, $a \cdot b$ is a product of primes, and thus so is $2+(n+1)$. Thus, $2+(n+1) \in S$, and by AI.iv, $S = \{2, 3, 4, \dots\}$ ■

1.2 Extensions: Integers, Rationals, Reals

Consider the set of naturals $\mathbb{N} = \{1, 2, 3, \dots\}$. Adding 0 to $\mathbb{N}$ defines $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We define the **integers** as the set $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, or the set of all positive and negative whole numbers.

Within $\mathbb{Z}$, we can define multiplication, addition and subtraction, with the neturals of 1 and 0, respectively. However, we cannnot define division, as we are not guaranteed a quotient in $\mathbb{Z}$. This necessitates the **rationals**, $\mathbb{Q}$. We define

$$\mathbb{Q} = \{\frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}.$$

On $\mathbb{Q}$, we have the familiar operations of multiplication, addition, subtraction and properties of associativity, distributivity, etc. We can also define division, as $\frac{\frac{p}{q}}{\frac{p'}{q'}} = \frac{pq'}{qp'}$.

We can also define a relation $<$ between fractions, such that

- $x < y$ and $y < z \implies x < z$
- $x < y \implies x + z < y + z$

$\mathbb{Q}$, together with its operations and relations above, is called an **ordered field**.

2 Appendix

Axiom 1.1 (*AI.i*)

Let $S \subseteq \mathbb{N}$ *with the properties:*

(a) $1 \in S$

(b) if $n \in S$*, then* $n+1 \in S$[5]

then $S = \mathbb{N}$.

Exercise 1.1. *Prove (by induction), that for every* $n \in \mathbb{N}$, $1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Proof of Exercise 1.1 : Follows a similar structure to the previous example. Let S be the subse4t of $\mathbb{N}$ for which the statement holds. $1 \in S$ by inspection (*(a)* holds), and we prove *(b)* by assuming $n \in S$ and showing $n+1 \in S$ (algebraically). Thus, by AI.i, $S = \mathbb{N}$ and the statement holds $\forall n \in \mathbb{N}$. ■

Axiom 1.2 (*AI.ii*)

Let $S \subseteq \mathbb{N}$ *s.t.*

(a) $m \in S$

(b) $n \in S \implies n+1 \in S$

then $\{m, m+1, m+2, \dots\} \subseteq S$.

Exercise 1.2. *Using AI.ii, prove that for* $n \geq 2$, $n^2 > n+1$

Proof of Exercise 1.2 : Again, very similar to the previous induction examples. Take S to be the subset of $\mathbb{N}$ for which the statement holds. *(a)* of AI.ii holds by inspection (where $m = 2$), and *(b)* holds by assuming $n \in S$ and showing that $n+1 \in S$. Thus, $S = \{2, 3, 4, \dots\}$, and the statement holds $\forall n \geq 2$.

Axiom 1.3 (*Principle of Complete Induction, AI.iii*)

Let $S \subseteq \mathbb{N}$ *s.t.*

(a) $1 \in S$

(b) if $1, 2, \dots, n-1 \in S$*, then* $n \in S$

then $S = \mathbb{N}$.

Axiom 1.4 (*AI.iv*)

Let $S \subseteq \mathbb{N}$ *s.t.:*

(a) $m \in S$

(b) if $m, m+1, \dots, m+n \in S$*, then* $m+n+1 \in S$

then $\{m, m+1, m+2, \dots\} \subseteq S$.

Theorem 1.1 (*Fundamental Theorem of Arithmetic*)

Every natural number n can be written as a product of one or more primes. [6]