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Classical Mechanics PHYS251

Course Summary:

Covers classical mechanic topics including applying Newton's Laws, the Work-Energy Theorem, conservation of momentum, oscillations, frames of reference, centres of mass, orbits, Kepler's Laws, Lagrangian mechanics. Large Calculus focus, representing situations functionally with differential equations, graphical analysis of trends.

Contents

1 Introduction & Notations

Content in this section provides a brief overview of the course, as well as basic problem solving, vector algebra, etc. It is safe to skip without loss of continuity.

1.1 Vectors

This course deals mainly with *scalars* (magnitude) and *vectors* (magnitude and direction). We define algebra on vectors, briefly:

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ *(commutativity of addition)*
- $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ *(associativity of addition)*
- $c(d\mathbf{A}) = (cd)\mathbf{A}$ *(associativity of scalar multiplication)*
- $(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$ *(Distributivity of scalar multiplication)*
- $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$ *(Distributivity of scalar multiplication)*

and the operators:

- $\mathbf{A} \times \mathbf{B} = \mathbf{C}$ s.t. $|\mathbf{C}| = |\mathbf{A}||\mathbf{B}| \sin\theta$, where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{C}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. This is equivalent to computing $\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$. This *cross product* is **anti-commutative**, meaning $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. Additionally, note that $\mathbf{A} \times \mathbf{A} = 0$.
- $\mathbf{A} \cdot \mathbf{B} = C = |\mathbf{A}||\mathbf{B}| \cos\theta$, where C is a scalar. Note that $C = 0$ when $\theta = \pi/2$, ie $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

1.2 Law of Cosines

Consider a (planar) triangle constructed of sides $\mathbf{C} = \mathbf{A} + \mathbf{B}$. We can write

$$\begin{aligned} \mathbf{C}^2 &= \mathbf{C}_x^2 + \mathbf{C}_y^2 \\ &= (|\mathbf{A}| - |\mathbf{B}| \cos\theta)^2 + (|\mathbf{B}| \sin\theta)^2 \\ &= |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos\theta \end{aligned}$$

1.3 Perspectives on the Cross Product

$$
\begin{aligned}
\vec{A} \times \vec{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\
&= A_x B_y(\mathbf{i} \times \mathbf{j}) + \cdots \\
&= (A_y B_z - A_z B_y)\mathbf{i} + \cdots \\
&\equiv (\mathbf{A} \times \mathbf{B})_k = \sum_{i=1}^{3} \sum j = 1^3 \mathcal{E}_{ijk} A_i B_j
\end{aligned}
$$

Where $\mathcal{E}_{ijk} = \begin{cases} 1; & \text{ijk even permutation of } 123 \\ -1; & \text{ijk odd permutation of } 123 \\ 0; & \text{otherwise} \end{cases}$

1.4 Describing a Particle in Space in Polar Coordinates

Consider a particle moving through space with a constant angular velocity $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega$. We can describe this movement in terms of planar coordinates as $\mathbf{r}(t) = r_0 \cos(\omega t)\mathbf{i} + r_0 \sin(\omega t)\mathbf{j}$. Differentiating with respect to time, we obtain $\mathbf{v}(t) = -r_0\omega(\sin(\omega t)\mathbf{i} - \cos(\omega t)\mathbf{j})$. Notice that $\mathbf{r} \cdot \mathbf{v} = 0 \forall t$; this should be familiar, as the velocity vector is always perpendicular to the position vector in purely circular motion. Differentiating again, we obtain $\mathbf{a}(t) = -r_0\omega^2(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}) = -\omega^2\mathbf{r}(t)$. In other words, the acceleration is always opposing the position vector (given the negative sign), and is proportional to the square of the angular velocity.

Assume now, instead, that the particle moves arbitrarily, described by a function $\mathbf{r}(t)$. In polar coordinates, this position vector is always travelling along the vector $\hat{\mathbf{r}}$, with magnitude r, and we can write $\mathbf{r}(t) = r \cdot \hat{\mathbf{r}}$. Differentiating:

$$
\begin{aligned}
\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(r \cdot \hat{\mathbf{r}}\right) \\
&= \frac{\mathrm{d}r}{\mathrm{d}t}\hat{\mathbf{r}} + r\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} \\
&= \dot{r}\hat{\mathbf{r}} + r\frac{\mathrm{d}}{\mathrm{d}t}\left(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}\right) \\
&= \dot{r}\hat{\mathbf{r}} + r\left(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}\right)\dot{\theta} \\
&= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}
\end{aligned}
$$

Recalling that $\begin{cases} \hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \\ \hat{\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \end{cases}$. Differentiating again:

$$\begin{aligned} \mathbf{a}(t) = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}\right) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\hat{\mathbf{r}}) \\ &= \left(\underbrace{\ddot{r}}_{\text{radial}} - \underbrace{r\dot{\theta}^2}_{\text{centripetal}}\right)\hat{\mathbf{r}} + \left(\underbrace{2\dot{r}\dot{\theta}}_{\text{coriolis}} + \underbrace{r\ddot{\theta}}_{\text{tangential}}\right)\hat{\theta} \end{aligned}$$

we obtain a decomposition of the acceleration vector into radial ($\hat{\mathbf{r}}$) and tangential ($\hat{\theta}$) components, labelled accordingly. Note that the centripetal acceleration, labelled, is the same as the acceleration in circular motion mentioned previously.[1][2]

1.5 Newton's Laws of Motion

N1: *"in absence of external force, a body at rest remains at rest, and a body in motion remains in motion, with the same speed & same direction"*

This law defines *intertial frames*, ie ones in which the law holds.

N2: $\vec{F} = m\vec{a}$

N3: *"if a body **b** applies a force on body **a**, then **a** applies a force on **b** such that $\vec{F}_a = -\vec{F}_b$ (equal and opposite)"*[3]

Example 1.1. *Consider two blocks **A**, **B** where **A** lies atop **B** which all lie upon "the earth".*

***A** experiences the force W_A due to gravity and the force F_1 back from **B**.*

***B** experiences the normal force N from the table, the force of gravity W_B, and finally the force F_2 due to **A**.*

We can write (from N2)

$$\vec{F}_1 + \vec{W}_A = m_A\vec{a}_A$$

and similarly

$$\vec{N} + \vec{W}_B + \vec{F}_2 = m_B\vec{a}_B$$

Static situation $\implies$ $\vec{a}_A = \vec{a}_B = 0$. *We can further simplify writing* $F_1 = m_A g$ *(N2), and finally*

$$N = W_A + W_B = (m_A + m_B)g.$$

Example 1.2. *Consider a mass m_1 laying on a table, connected via a string to a mass m_2 hanging off the table, where the string is of small mass and is under a tension T. (Looking at a functionally massless stirng, we can consider there*

[1] None of the other "components" of acceleration are present in the constant angular velocity case because (1) $\ddot{r} = 0$, ie no radial acceleration, so the radial and coriolis components are zero, and (2) $\dot{\theta} = \omega \implies \ddot{\theta} = 0$, so the tangential acceleration is zero, leaving only the centripetal acceleration.

[2] See https://notes.louismeunier.net/Calculus%20A%2C%20B/calculus.pdf#page=85for a different perspective on this topic.

[3] $\implies$ conservation of momentum …

to be a constant tension as we can say $T - T' = \delta m a \implies T - T' = 0 \implies T = T'$, where T, T' is the tension on a particular segment of the string in opposing directions).

On m_1, we have the normal force N, tension T, and weight $m_1 \cdot g$.

On m_2, we have the weight $m_2 \cdot g$, the tension T (equal throughout string as explained above).

Together, we can write (N2; taking x to represent a motion "left" and z to represent a motion "down")

$$-T = m_1 \ddot{x}$$

and

$$m_2 g - T = m_2 \ddot{z}.$$

Noting that the string must "retain" its length we can write

$$l(t) = x(t) + z(t),$$

where x and z represent the length of the string in the x/z axes. However, as l must stay constant, we can differentiate twice wrt time to obtain

$$0 = \ddot{x} + \ddot{z}.$$

All together, then we have

$$\ddot{x} = \frac{-m_2 g}{m_1 + m_2}.$$

Integrating twice, we have

$$x(t) = \frac{-m_2 g}{m_1 + m_2} \cdot \frac{t^2}{2}.$$

Example 1.3. *Consider a mass m_1 connected to a string which lies along a pulley P_1, which then attaches to the center of pulley P_2, which has a string whicch, on one end, is attached to a mass m_2, and is grounded in the other.*

On m_1, we have tension T and the weight $m_1 \cdot g$.

On m_2, we have the weight $m_2 \cdot g$ and the tension T' (no reason for $T = T'$; strings aren't connected).

Consider P_1 - it is nailed to the wall at its center, and experiences some F ("up") from the nail, as well as the tension T_1 down (twice).

On P_2, we have the tension T_1 ("up") and the tension T_2 ("down", "twice").

We can write

$$T_1 - W_1 = m_1 a_1$$

$$T_2 - W_2 = m_2 a_2$$

$$T_1 - 2T_2 = m_{P_2} \cancel{a_{P_2}} \implies T_1 = 2T_2$$

As last time, we can use the fact that the strings cannot stretch. Consider m_1 to be at height y_1, P_1 to be at height y_{P_1}, P_2 to be at height y_{P_2}, and m_2 to be at height y_2.

We can then take the length l_1 of the string about P_1 as

$$l_1 = (y_{P_1} - y_1) + (y_{P_2} - y_2) + \pi R_{P_1}$$

Differentiating twice wrt time, we have

$$0 = a_1 + a_{P_2} \implies a_1 = -a_{P_2}.$$

Analyzing P_2 similarly, we will obtain

$$a_2 = 2a_{P_2}.$$

All together, we have

$$a_1 = \left(\frac{2m_2 - m_1}{4m_2 + m_1}\right) g.$$

Consider a collection of particles. The force on any particular particle, say 1, we can denote

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \cdots + \vec{F}_{1n} + \vec{F}_1^{\text{ext}}.$$

Notice that any force $\vec{F}_{nm} = -\vec{F}_{mn}$, by N3. Thus, if we add all the forces on all the particles in the bag, we will always have a "pairing" of forces such that each $\vec{F}_{mn}$ is "canceled" by another, leaving behind only the external forces, ie

$$\vec{F}^{\text{ext}} = \sum_i \vec{F}_i^{\text{ext}}.$$

Say there are no external forces; then, we have

$$\sum_i m_i \vec{a}_i = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i \vec{v}_i = 0.$$

And thus, we have conservation of momentum.

1.6 Projectile Motion

Say we have a particle with an initial velocity v_0 launched at an angle θ. This particle experiences just one force, mg, and by N2 we have

$$\cancel{m}\ddot{y} = -\cancel{m}g.$$

Integrating, we have

$$y(t) = y(t=0) + \underbrace{v_0 \sin\theta}_{\text{vertical component of } v_0} t - \frac{gt^2}{2}.$$

In the x, we have $m\ddot{x} = 0$, $x(t) = v_0 \cos\theta t$. From here, we can rewrite $x(t)$ as t as a function of x and substitute into y for a function $y = f(x)$. This gives

$$y = \frac{v_0 \sin\theta x}{v_0 \cos\theta} - \frac{g}{2}\frac{x^2}{v_0^2 \cos^2\theta} (= ax - bx^2)$$

1.7 Forces

We (generally) have:

- 2 "protons" @ 1 fermi (fm = $10^{-15}m$)
- Nuclear strong, $2 \times 10^3 N$
- E-M, $2 \times 10^2 N$
- Weak force, $2 \times 10^{-11} N$
- Gravity, $2 \times 10^{-34} N$

We have the force between two point masses, m_a, m_b, at a distance r_{ab}:

$$\vec{F} = \frac{-Gm_a m_b}{r_{ab}^2},$$

noting that the negative sign $\implies$ gravity *attractive*, and where the gravitational constant $G = 6.67 \times 10^{-11} \frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}$.

We consider some "extended object", a spherical shell of mass M a distance r (center) from a "smaller" object m. We can subdivide this shell into infinitesimal pieces, then sum (integrate) their respective forces using the formula above to find the total force.

Say the distance between a particular "subsection" of the sphere is s from m, and the change in angle in the shell section we have $\frac{\mathrm{d}}{\mathrm{d}\theta}$, with θ the angle from the shell section to r, and φ is the angle between s and r.

We have

$$F = \frac{-Gm\mathrm{d}M}{s^2} \cdot \cos\varphi.$$

...

2 Statics

We have **static** motion when the net external force is 0. Clearly, then,

$$\sum F_{\text{ext}} = m\cancel{\vec{a}} = 0 \iff \vec{a} = 0.$$

When analyzing a system we thus need to balance:

- **Forces,** include
  - Tension;
  - Normal ($\perp$ to surface, N3);
  - Friction (opposing direction of motion; $F_k = \mu_k N$, $F_s \leq \mu_s N$, kinetic vs static frictions resp.);

– Gravity ($F_g = \frac{GMm}{R^2} \implies F_g = mg$, on earth);

– Spring (*Hooke's Law*, $\vec{F_s} = -k\vec{r}$)

- **Torques,** $\vec{\tau} = \vec{r} \times \vec{F}$. When dealing with rotational motion, sums of torques must be 0 as well to conserve stasis.

Example 2.1. *Consider a ladder of length l and mass m leaning against a frictionless wall with another end on a floor with coefficient of friction μ. Assuming the center of gravity is at the geometric center of the ladder ($\frac{l}{2}$), we can analyze the system as follows.*
Take N_1, N_2 to represent the normal forces on the ladder from the floor and the wall resp, and F_f as the friction force of the floor. We summarize the forces in the $[x]$ and $[y]$, and the torques:

$$[y] \quad 0 = N_1 - mg \implies N_1 = mg \quad \textbf{\textit{(1)}}$$

$$[x] \quad 0 = N_2 - F_f \implies N_2 = F_f \quad \textbf{\textit{(2)}}$$

$$[\tau] \quad 0 = \vec{F} \times \vec{r} \overset{\mathit{1D}}{=} F \cdot r = \underbrace{mg \cdot \frac{l}{2} \cdot \cos\theta}_{\text{rotation about c.o.m.}} - \underbrace{N_2 \cdot l \sin\theta}_{\text{about "floor point"}} \implies N_2 = \frac{mg}{2\tan\theta} \quad \textbf{\textit{(3)}}$$

*From **(1),(2),(3)**, we can find that the angle required for the ladder to not slip (ie, net forces and net torques sum to zero; will not slip nor rotate and fall), we require that*

$$\tan\theta \geq \frac{1}{2\mu}.$$

3 Non-Zero Forces

3.1 Trends of Common Forces

3.2 Generalizing Forces as Functions

Consider the general form of Newton's Second Law:

$$\vec{F} = m\vec{a}.$$

This $\vec{F}$ can take a number of forms, depending on its dependence on different variables. Consider;

- **Time**: $F(t) = m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = m\frac{\mathrm{d}v}{\mathrm{d}t}$. This is separable ODE:

$$\int \mathrm{d}v = v(t) = \int \mathrm{d}t \, \frac{F(t)}{m}$$
$$\implies x(t) = \int \mathrm{d}t \, v = \iint \mathrm{d}t \, \frac{F(t)}{m}$$

- **Position**: $F(x) = m\frac{\mathrm{d}v}{\mathrm{d}t}$. We can rewrite our acceleration by the chain rule as $\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}v$ (NB: this "trick" is very helpful in evaluating physical ODE problems such as this, where the reliance on a particular function of time is unknown, but can be obtained from physical properties...). We can then write:

$$\begin{aligned} m\frac{\mathrm{d}v}{\mathrm{d}x}v &= F(x) \\ m\int \mathrm{d}v\, v &= \int F(x)\,\mathrm{d}x \\ \frac{1}{2}mv^2\bigg|_D &= \int_D \mathrm{d}x\, F(x) \qquad \textit{NB: Work-Energy Thm} \end{aligned}$$

- **Velocity**: $m\frac{\mathrm{d}v}{\mathrm{d}t} = F(v) \implies m\int \mathrm{d}v\, \frac{1}{F(v)} = \int \mathrm{d}t$; we will see this more later.

3.3 Viscous Resistance

Consider some force that is dependent on velocity, $F(v)$ (friction, air resistance for instance, is often approximately modeled this way). Let's approximate this force function at small velocities by writing its Taylor Series expansion about $v = 0$ (note that higher order terms $O(v^n)$ can be consider negligible assuming $0 \leq v < 1$, a typical approach to this type of small-perturbation analysis):

$$F(v) = F(0) + v\,\frac{\partial F}{\partial v}\bigg|_{v=0} + \frac{v^2}{2!}\cdot\frac{\partial^2 F}{\partial v^2}\bigg|_{v=0} + \cdots + O(v^n)$$

Note that $v = 0 \implies a = 0 \implies F(0) = 0$, and thus the first term can be disregarded. Under a further assumption that $v \ll 1$ and remains so, we can eliminate $O(n^2)$ terms, and would be left with

$$F(v) = -v\,\frac{\partial F}{\partial v}\bigg|_{v=0} = -cv,$$

with $c := \frac{\partial F}{\partial v}\big|_{v=0}$, a constant. This can be rewritten and solved as a simple ODE;

$$\begin{aligned} F(v) = m\frac{\mathrm{d}v}{\mathrm{d}t} &= -cv \\ \frac{m}{v}\,\mathrm{d}v &= -c\,\mathrm{d}t \\ m\ln v &= -ct + k \\ v(0) = v0 &\implies k = m\ln v_0 \\ &\implies v(t) = v_0 e^{-\frac{ct}{m}} \\ \tau := \frac{m}{c}, &\quad v(t) = v_0 e^{-\frac{t}{\tau}} \\ &\implies x(t) = x_0 + v_0\tau(1 - e^{-\frac{t}{\tau}}) \end{aligned}$$

Example 3.1 (Air resistance $\propto v$)**.** *Consider a sky diver falling, with some initial velocity v_0, under the influence of gravity F_g downwards (positive) and air resistance F_v upwards (negative). Assuming that air resistance is proportional* *and opposing to velocity, we can write $F_v = -cv$ where c a constant. From Newton's Second:*

$$\begin{aligned}
\sum F = ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = F_g + F_v = mg - cv \\
\int_{v_0}^{v} \mathrm{d}v^* \, \frac{m}{mg - ct} = \int_0^t \mathrm{d}t^* \\
-\frac{m}{c} \ln\left(\frac{mg - cv}{mg - cv_0}\right) = t \\
v(t) = g\tau(1 - e^{-\frac{t}{\tau}}) + v_0 e^{-\frac{t}{\tau}}, \quad \tau := \frac{m}{c}
\end{aligned}$$

A common follow-up to this analysis is to compute the terminal velocity v_t. We will consider two methods to compute it in this situation.
First, consider the equation we found for $v(t)$. Taking the limit of $t \to \infty$, we have

$$\begin{aligned}
v_t = \lim_{t\to\infty} v(t) &= \lim_{t\to\infty} \left[g\tau(1 - \cancelto{0}{e^{-\frac{t}{\tau}}}) + \cancelto{0}{v_0 e^{-\frac{t}{\tau}}} \right] \\
&= g\tau = \frac{gm}{c}
\end{aligned}$$

Alteratively, consider our original formula,

$$ma = mg - cv.$$

Terminal velocity is reached when $a = 0$; more physically, when the forces in opposing directions completely cancel each other. Thus, we can solve for v directly;

$$m \cdot 0 = mg - cv \implies mg = cv_t \implies v_t = \frac{gm}{c},$$

equivalent to our previous computation.

Example 3.2 (Air resistance $\propto v^2$)**.** *Consider the previous example with $F_v = -cv^2$. We can write*

$$\sum F = m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - cv^2 \implies v_t = \sqrt{\frac{mg}{c}}.$$

A closed form of the position of the trajectory is possible, but difficult.

4 Oscillations

Many natural systems experience *oscillations*, characterized by some type of force that leads to repetitive motion.

4.1 Simple Springs

Hooke's Law states that $F_s = -kx$, where k is a constant spring coefficient (typically $0 < k < 1$), and x is the displacement from equilibrium. In general, we can write

$$F_s = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}x}v = -kx$$
$$m\int \mathrm{d}v\, v = -k\int \mathrm{d}x\, x$$
$$m\frac{v^2}{2} = -k\frac{x^2}{2}$$
$$mv = -kx \implies v = -\frac{k}{m}x$$

Often, we write $\omega^2 := \frac{k}{m}$, the phase of the oscillator.

4.1.1 Series vs Parallel

Consider a spring of coefficient k_1 connected end-to-end to a spring of k_2, and let x_1, x_2 represent the equilibrium points of the spring system resp (where $x_2 = x_1 +$ equilibrium of spring k_2). Consider the springs as massless (or of being of infinitesimal, negligible mass), so the only forces in question are due to the spring force(s).
At the endpoint of spring 1, we have

$$F_1 : k_1x_1 = (x_2 - x_1)k_2 \implies x_2 = \left(\frac{k_1 + k_2}{k_2}\right)x_1.$$

If we consider a mass on the end point of the system, we can write

$$F = -k_2\underbrace{(x_2 - x_1)}_{\text{displacement}} = -\left(\frac{k_1k_2}{k_1 + k_2}\right)x_2$$
$$= -k_{\text{eff}}x_2$$

Note that $\frac{1}{k_{\text{eff}}} = \frac{1}{k_1} + \frac{1}{k_2}$; this is the relationship between any number of springs connected in **series**.

Next, consider two springs of k_1, k_2 each connecting a mass to a wall. We can write the force on the mass, taking x to represent the displacement of the mass from equilibrium, as

$$F = -k_1x - k_2x = -(k_1 + k_2)x = -k_{\text{eff}}x,$$

where $k_{\text{eff}} = k_1 + k_2$; these springs are in **parallel**.

(Naturally, all of these analyses are under the assumption of no torsion, no gravity, etc (no other external forces))

4.2 Simple Harmonic Motion

Consider again a spring (k). From N2, we can write:

$$\begin{aligned} m\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} = -kx &\implies m\ddot{x} + kx = 0 \\ &\implies \ddot{x} + \omega_0^2 x = 0 \end{aligned}$$

This is a (fairly straightforward) second order linear homogenous equation (we will see non-homogenous versions of this later, when we introduce external forcing, etc.). The typical means of solving these is to assume $x(t)$ is of the form $Ae^{\alpha t}$, plug in, and solve for the relevant constants (noting that $\alpha \in \mathbb{C}$):

$$\begin{aligned} &(Ae^{\alpha t})'' + \omega^2 Ae^{\alpha t} = 0 \\ &\cancel{A}\alpha^2 \cancel{e^{\alpha t}} + \omega^2 \cancel{Ae^{\alpha t}} = 0 \\ &\qquad\qquad\qquad \implies \alpha^2 = -\omega^2 \implies \alpha = \pm i\omega, i = \sqrt{-1} \end{aligned}$$

Note that all dependence on time cancelled, as desired, as well as A; we are working with a second-order ODE, so we must have two "free" constants dependent on initial conditions, represented by this A.
Now, we have solutions of the form $x(t) = Ae^{\pm i\omega t}$. This form doesn't help us much physically, but we can simplify using Euler's formula, or, since we are dealing with purely a sign change in our parameters, the definition of $\cos(ix) = \cosh x = \frac{e^x + e^{-x}}{2} \implies \frac{e^{i\omega t} + e^{-i\omega t}}{2} = \cos(\omega t)$. Generally, then, we can write

$$x(t) = A\cos(\omega t + \varphi),$$

where A, φ are constants, typically defined by IVs.
Note that, equivalently, we could have written

$$x(t) = B\sin\omega t + C\cos\omega t,$$

but chose to rewrite in terms of φ for a more intuitive, physical meaning. Specifically, we have

$$A\cos\varphi = C, A\sin\varphi = -B, \tan\varphi = -\frac{B}{C}.$$

Graphically, this means that spring motion is a constant-amplitude sinusoidal wave that will oscillate forever. Realistically, forces such as friction, gravity, etc would cause a damping force over time and eventually slow the mass to a halt.

4.3 Damping Forces

Consider a mass m connected to a spring k, but now add some form of resistance (typically described as a piston), where force opposes the motion of the mass and is proportional to velocity. We write $F_r = -bv$ where b a constant.

Writing the equation of motion:

$$
\begin{aligned}
m\ddot{x} &= -kx - bv \\
\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x &= 0 \\
\equiv \ddot{x} + \gamma\dot{x} + \omega^2 x &= 0
\end{aligned}
$$

We now have a second order linear homogenous ode, but with an added $\dot{x}$. We can try the same approach, with a trial solution $x(t) = Ae^{\alpha t}$:

$$
\begin{aligned}
(Ae^{\alpha t})'' + \gamma(Ae^{\alpha t})' + \omega^2 Ae^{\alpha t} &= 0 \\
\underbrace{\alpha^2 + \alpha\gamma + \omega^2}_{\text{characteristic polynomial}} &= 0 \\
&\implies \alpha = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega^2}
\end{aligned}
$$

We thus have a number of cases based on the discriminant. Let $\tilde{\gamma} = \frac{\gamma}{2}$.

- **underdamped;** $\tilde{\gamma}^2 < \omega^2 \implies$ discriminant $\in \mathbb{C}$, thus $\alpha = -\frac{\gamma}{2} \pm i\sqrt{\omega^2 - \frac{\gamma^2}{4}}$. We then have a solution

$$
\begin{aligned}
x(t) = e^{-\tilde{\gamma}t} &= A_1 e^{i\tilde{\omega}t} + A_2 e^{-i\tilde{\omega}t} \\
\implies &\boxed{x(t) = \underbrace{Ae^{-\tilde{\gamma}t}}_{\text{damping, }\to 0} \underbrace{\cos(\tilde{\omega}t + \varphi)}_{\text{oscillating}}}
\end{aligned}
$$

  As $t \to \infty$, $x(t) \to 0$ because of the damping term. The inverse exponential goes to zero, functionally diminishing the amplitude of the cosine with time.

- **overdamped;** $\tilde{\gamma}^2 > \omega^2 \implies$ discriminant $\in \mathbb{R}$, thus $\alpha = -\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 - \omega^2} \implies \alpha \le 0$. This yields a purely exponential solution,

$$
\boxed{x(t) = Ae^{-\alpha_- t} + Be^{-\alpha_+ t}}
$$

  since the oscillating term comes from an imaginary number in the exponential. Thus, there is no oscillation, and the solution tends to zero.

- **critically damped;** $\tilde{\gamma}^2 = \omega^2 \implies$ discriminant $= 0$, we have only one root, and we thus are missing a solution case. One can use a new trial solution $x(t) = Bte^{-\tilde{\gamma}t}$, and after some computation, we have

$$
\boxed{x(t) = Ae^{-\tilde{\gamma}t} + Bte^{-\tilde{\gamma}t}}
$$

  Note the linear t term in one of the exponentials. This will persist when looking at $x'(t)$, and will thus dominate the inverse exponential as $t \to 0$, hence the term "critically damped", as this type of damping will bring $x(t)$ to zero the fastest given the same initial conditions and spring constant.

4.4 Driven Oscillators and Resonance

Consider a spring (k) connected to a mass m on one end and to a wall which is moving as a function of time, where $x_w(t)$ represents the location of the wall from equilibrium. We write

$$x_w(t) = x_0 \cos \omega t,$$

ie the wall oscillates. From N2:

$$m\ddot{x} = -kx + kx_w = -kx + \underbrace{kx_0}_{:=F_0} \cos \omega t$$
$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

Solving for $x(t)$:

$$x(t) = B \cos(\omega_0 t + \varphi) + \frac{F_0}{m} \left(\frac{1}{w_0^2 - \omega^2} \right) \cos(\omega t).$$

This is the general equation for a **driven oscillator**.
Note too that

$$\lim_{\omega \to \omega_0^-} = \infty; \quad \lim_{\omega \to \omega_0^+} = -\infty,$$

an effect we call **resonance**. In reality, of course, some sort of damping (eg friction) makes these not go to infinity, but grow large/small regardless.

4.4.1 With Damping

Consider

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t,$$

an equation for a damped, driven oscillator. This can be solved similarly to the previous, and also gives different damped cases. Consider the underdamped case:

$$Ae^{-\tilde{\gamma} t} \cos(\tilde{\omega} t + \varphi_1) + \frac{F_0}{m} \cdot \frac{\cos(\omega t + \varphi_2)}{[(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2]^{\frac{1}{2}}}.$$

When we are near resonance, $\omega \approx \omega_0 \implies \omega_0^2 - \omega^2 = (\omega_0 - \omega)(\omega_0 + \omega) \approx 2\omega_0$, so we can approximate the denominator of the driven part as

$$[\cdots]^{\frac{1}{2}} \approx \left[(\omega_0 - \omega)^2 + \left(\frac{\gamma}{2} \right)^2 \right].$$

This $\left(\frac{\gamma}{2}\right)^2$ term serves as a "regulator" to the resonance effect.

4.5 Coupled Oscillators

Consider a series of three springs and two equal masses of m between a wall. Let x_1 be the displacement of the leftmost mass, and x_2 the displacement of the rightmost. We can write

$$\textbf{(i)} \quad m\ddot{x}_1 = -kx_1 + k(x_2 - x_1); \qquad \textbf{(ii)} \quad m\ddot{x}_2 = -k(x_2 - x_1) - kx_2$$

Adding and subtracting **(i),(ii)**:

$$\ddot{x}_1 + \ddot{x}_2 + \omega_0^2(x_1 + x_2) = 0; \qquad (\ddot{x}_1 - \ddot{x}_2) + 3\omega_0^2(x_1 - x_2) = 0.$$

We can let $y_1(t) = x_1 + x_2$ and $y_2(t) = x_1 - x_2$, and solve these separately as two homogenous second order ODEs, then add them back together. These y_1, y_2 are called the **normal modes**. This will yields

$$x_1(t) = B_+ \cos(\omega_0 t + \varphi_+) + B_- \cos\left(\sqrt{3}\omega_0 t + \varphi_-\right)$$
$$x_2(t) = B_+ \cos(\omega_0 t + \varphi_+) - B_- \cos\left(\sqrt{3}\omega_0 t + \varphi_-\right)$$

We dub $\omega_0, \sqrt{3}\omega_0$ the **normal frequencies**. If both masses are given the same initial displacements, they will both oscillate at ω_0; if they are given the same initial displacement but with opposite signs, they will oscillate at $\sqrt{3} \cdot \omega_0$.

Note that, while this is being applied in the context of springs, similar ideas can be used in other forces proportional to displacement.

Example 4.1. *Take a "coupled" system of pendulums attached to a ceiling, with point masses at their ends and a spring connecting the two masses. Find their equations of motion and their normal frequencies.*

5 Momentum

5.1 Some Derivations

We can write

$$\vec{F}^{\text{ext}} = \sum_i m_i \ddot{r}_i = M\ddot{\vec{R}},$$

where $\vec{R}$ is the "center of mass coordinate"., ie

$$\vec{R} - \frac{1}{M}\sum_i m_i \vec{r}_i.$$

5.2 Center of Mass

Example 5.1 (Linear density). *Consider a rod with linear density (the density at a given point is proportional to the distance along the rod), λ;*

$$\lambda(s) = \lambda_0 \left(\frac{S}{L}\right) = \frac{\mathrm{d}m}{\mathrm{d}x},$$

where S is the point along the length. The total mass:

$$M = \int_{rod} \mathrm{d}m = \int_0^L \mathrm{d}m = \int_0^L \lambda \, \mathrm{d}x = \frac{x^2}{2} \cdot \frac{\lambda_0}{L}\bigg|_0^L = \frac{L\lambda_0}{2}.$$

We can now compute the center of mass X*:*

$$\mathsf{X} = \frac{1}{M}\sum_i x_i m_i \overset{continuous}{=} \frac{1}{M}\int_0^L \mathrm{d}m \, x = \frac{2}{L\lambda_0}\int_0^L \lambda \, \mathrm{d}x = \frac{2}{L\lambda_0}\int_0^L \lambda_0 \frac{x^2}{L} \, \mathrm{d}x = \frac{2}{L\lambda_0} \cdot \frac{\lambda_0 L^3}{3L} = \frac{2}{3}L.$$

Note that there is no dependence on λ!

Example 5.2 (Triangular sheet). *Consider a triangular sheet of height h and base b, of uniform density ρ. We write*

$$\vec{R} = \frac{1}{M} \cdot \int \underbrace{\rho \, \mathrm{d}V}_{\mathrm{d}M} \vec{r}.$$

We have that $\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$. We will assume that it is negligibly thin, so let $\mathrm{d}z = 0$. This means we can write mass as $M = \rho \, \mathrm{d}x \, \mathrm{d}y := \rho A$, where A is the area of the sheet. We can now compute its center of mass:

$$X_c = \frac{1}{A}\iint \mathrm{d}x\mathrm{d}y x = \frac{hb}{2}\int \mathrm{d}x \cdot x \cdot \int_0^{\frac{h}{b}x} = \frac{2}{3}b$$

$$Y_c = \cdots = \frac{h}{3}$$

Thus, the center of mass is $(X_c, Y_c) = (\frac{2}{3}b, \frac{h}{3})$.

5.3 Variable Mass Problems

We define **momentum** as $P(t) = M \cdot v$, and $\frac{\mathrm{d}}{\mathrm{d}t}P(t) = F$. A common technique to approaching problems involving variable mass is to approach it first via infinitesimal changes in different variables over some Δt, then bring $\Delta t \to 0$ to find F. Generally, though, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}P = \frac{\mathrm{d}}{\mathrm{d}t}(Mv) = \frac{\mathrm{d}M}{\mathrm{d}t}v + M\frac{\mathrm{d}v}{\mathrm{d}t}M = \dot{M}v + M\dot{v} = F.$$

Example 5.3. *Consider a cart of mass M, collecting rain at a rate $\frac{\mathrm{d}m}{\mathrm{d}t} = \sigma$, and traveling at a constant velocity v. What force must we apply for this constant velocity to persist, if any? Consider its momentum at some time t, and* *swiftly later at $t + \Delta t$:*

$$\begin{aligned} P(t) &= Mv \\ P(t+\Delta t) &= (M+\Delta)v \\ \lim_{\Delta t} \frac{\Delta P}{\Delta t} &= \lim \frac{\Delta m}{\Delta t} v \implies \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t} v = \boxed{\sigma v = F} \end{aligned}$$

Say, now, we let $F = 0$. We would then have, setting M_c as the mass of the cart alone,

$$\begin{aligned} P(t) &= (M_c + \sigma t)v \\ P(t+\Delta t) &= (M_c + \sigma t + \sigma \Delta t)(v + \Delta v) \\ &\implies \Delta P = \sigma v \Delta t + (M_c + \sigma t)\Delta v = 0 \\ &\implies \int \mathrm{d}t \, \frac{\sigma}{M_c + \sigma t} = -\int \mathrm{d}v \, \frac{1}{v} \\ &\implies v = v_0 \frac{M_c}{M_c + \sigma t} \end{aligned}$$

5.3.1 Rocket Motion

Consider a rocket of mass m moving at velocity $\vec{v}$ and expelling fuel of mass Δm, which leaves the rocket at some velocity Δu. We can write:

$$\vec{P}(t) = m\vec{v}; \qquad \vec{P}(t+\Delta t) = (m - \Delta m)(\vec{v} + \Delta v) + \Delta m(\vec{u} + \vec{v} + \Delta v)$$

Working out the differentials, this yields the famous **rocket equation**

$$\boxed{\vec{F} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} - \frac{\mathrm{d}m}{\mathrm{d}t}\vec{u}}$$

Example 5.4 (Rocket Eqn: Free space)**.** $\vec{F} = 0 \implies m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t}\vec{u} \implies \mathrm{d}\vec{v} = \vec{u}\frac{\mathrm{d}m}{m} \implies \vec{v_f} = -\vec{u}\ln\left(\frac{m_0}{m_f}\right)$.

Example 5.5 (Rocket Eqn: Force due to gravity)**.** $F = mg = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} - \vec{u}\frac{\mathrm{d}m}{\mathrm{d}t} \implies \vec{v_f} = gt - \vec{u}\ln\left(\frac{m_0}{m_f}\right)$

Example 5.6 (Rope falling on a scale)**.** *Consider a rope of mass m that is held above a scale such that part of it is resting on the scale. What does the scale read when the rope is left to fall?*

6 Work & Energy

6.1 Introduction: 1 Dimension

Consider force as a function of position, $\vec{F}(\vec{r}) = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$. We can simplify, in one direction,

$$
\begin{aligned}
m\frac{\mathrm{d}v}{\mathrm{d}t} &= m\frac{\mathrm{d}v}{\mathrm{d}x}\cdot \overset{v}{\cancel{\frac{\mathrm{d}x}{\mathrm{d}t}}} = F(x) \\
\implies \int_{v_0}^{v} \mathrm{d}v^*\, v^* &= \int_{x_0}^{x} \mathrm{d}x^*\, F(x^*) \\
\underbrace{\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2}_{\Delta\text{KE}} &= \underbrace{\int_{x_0}^{x} \mathrm{d}x^*\, F(x^*)}_{:=\text{Work}}
\end{aligned}
$$

This derives the **Work-Energy Theorem**, ie

$$\Delta\text{KE} = W.$$

Example 6.1 (Work due to gravity)**.** *Consider $F = \vec{g}$. We can write*

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = \int_{z_0}^{z_1} -mg\,\mathrm{d}z = -mg(z_1 - z_2).$$

Assume we end at zero velocity $v_1 = 0$ and define our "start" as zero displacement $z_0 = 0$, then we have

$$z_1 = \frac{v_0^2}{2g},$$

ie, the final position has no dependence on mass.

6.2 Extension to Higher Dimensions

We wrote previously $\vec{F}(\vec{r}) = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$, a vector valued function with vector valued arguments. We can manipulate this into a nicer form akin to the 1-dim case as follows;

$$
\begin{aligned}
\vec{F}\cdot\mathrm{d}\vec{r} &= m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\cdot\mathrm{d}\vec{r} \\
\star\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} &= \vec{v} \implies \mathrm{d}\vec{r} = \mathrm{d}t\,\vec{v}\star \\
\oint_R \vec{F}\cdot\mathrm{d}\vec{r} &= m\int_R \frac{\mathrm{d}\vec{v}}{\cancel{\mathrm{d}t}}\cdot\vec{v}\,\cancel{\mathrm{d}t}
\end{aligned}
$$

Note that, we are "canceling" the differentials rather informally, but with an equivalent result as properly taking a double integral over time as well as distance. Note that, as $\vec{r}$ is some arbitrary curve/trajectory in space, we are "formally" taking a line integral, hence the $\oint$. We call the trajectory we are integrating over R; say it has endpoints a, b. We can then simplify

$$\oint_R \vec{F} \cdot \mathrm{d}\vec{r} = m \int_a^b \mathrm{d}\vec{v}\,\vec{v} = \frac{1}{2} m \left(v_a^2 - v_b^2 \right),$$

where our RHS is the familiar KE. Thus,

$$\boxed{\oint_R \vec{F} \cdot \mathrm{d}\vec{r} = \Delta \mathrm{KE}} = W.$$

Generally, the LHS can't necessarily be integrated but we can make some simplifying observations. First $\mathrm{d}\vec{r}$ is always tangent to the trajectory $\vec{R}$, as should be familiar from the definition of a derivative. We can break down the force components along a given trajectory as the parallel and perpendicular components to the trajectory at a given point;

$$\vec{F} = F_\parallel + F_\perp \implies \vec{F} \cdot \mathrm{d}r = (F_\parallel + F_\perp) \cdot \mathrm{d}r\,.$$

However, by definition, the dot product of a vector with a perpendicular is equal to zero, and thus $\mathrm{d}r \cdot F_\perp = 0$, and we thus need only be concerned with the parallel components of force.