

MATH457 - Algebra 4

Representation Theory; Galois Theory

Based on lectures from Winter 2025 by Prof. Henri Darmon.

Notes by Louis Meunier

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§1 REPRESENTATION THEORY

§1.1 Introduction

↪ **Definition 1.1** (Linear Representation): A *linear representation* of a group G is a vector space V over a field \mathbb{F} equipped with a map $G \times V \rightarrow V$ that makes V a G -set in such a way that for each $g \in G$, the map $v \mapsto gv$ is a linear homomorphism of V .

This induces a homomorphism

$$\rho : G \rightarrow \text{Aut}_{\mathbb{F}}(V),$$

or, in particular, when $n = \dim_{\mathbb{F}} V < \infty$, a homomorphism

$$\rho : G \rightarrow \text{GL}_n(\mathbb{F}).$$

Alternatively, a linear representation V can be viewed as a module over the group ring $\mathbb{F}[G] = \left\{ \sum_{g \in G} \lambda_g g : \lambda_g \in \mathbb{F} \right\}$ (where we require all but finitely many scalars λ_g to be zero).

↪ **Definition 1.2** (Irreducible Representation): A linear representation V of a group G is called *irreducible* if there exists no proper, nontrivial *subspace* $W \subsetneq V$ such that W is G -stable.

⊗ Example 1.1:

1. Consider $G = \mathbb{Z}/2 = \{1, \tau\}$. If V a linear representation of G and $\rho : G \rightarrow \text{Aut}(V)$. Then, V uniquely determined by $\rho(\tau)$. Let $p(x)$ be the minimal polynomial of $\rho(\tau)$. Then, $p(x) \mid x^2 - 1$. Suppose \mathbb{F} is a field in which $2 \neq 0$. Then, $p(x) \mid (x - 1)(x + 1)$ and so $p(x)$ has either $1, -1$, or both as eigenvalues and thus we may write

$$V = V_+ \oplus V_-,$$

where $V_{\pm} := \{v \mid \tau v = \pm v\}$. Hence, V is irreducible only if one of V_+, V_- all of V and the other is trivial, or in other words τ acts only as multiplication by 1 or -1 .

2. Let $G = \{g_1, \dots, g_N\}$ be a finite abelian group, and suppose \mathbb{F} an algebraically closed field of characteristic 0 (such as \mathbb{C}). Let $\rho : G \rightarrow \text{Aut}(V)$ and denote $T_j := \rho(g_j)$ for $j = 1, \dots, N$. Then, $\{T_1, \dots, T_N\}$ is a set of mutually commuting linear transformations. Then, there exists a simultaneous eigenvector, say v , for $\{T_1, \dots, T_N\}$, and so $\text{span}(v)$ a G -stable subspace of V . Thus, if V irreducible, it must be that $\dim_{\mathbb{F}} V = 1$.

↪ **Theorem 1.1:** If G a finite abelian group and V an irreducible finite dimensional representation over an algebraically closed field of characteristic 0 , then $\dim V = 1$.

PROOF. Let $\rho : G \rightarrow \text{Aut}(V)$, label $G = \{g_1, \dots, g_N\}$ and put $T_j := \rho(g_j)$ for $j = 1, \dots, N$. Then, $\{T_1, \dots, T_N\}$ a family of mutually commuting linear transformations on V . Then,

there is a simultaneous eigenvector v for $\{T_1, \dots, T_N\}$ and thus $\text{span}(v)$ is T_1, \dots, T_N -stable and so $V = \text{span}(v)$. ■

↪ **Lemma 1.1:** Let V be a finite dimensional vector space over \mathbb{C} and let $T_1, \dots, T_N : V \rightarrow V$ be a family of mutually commuting linear automorphisms on V . Then, there is a simultaneous eigenvector for T_1, \dots, T_N .

↪ **Proposition 1.1:** Let \mathbb{F} a field where $2 \neq 0$ and V an irreducible representation of S_3 . Then, there are three distinct (i.e., up to homomorphism) possibilities for V .

PROOF. Let $\rho : G \rightarrow \text{Aut}(V)$ and let $T = \rho((23))$. Then, notice that $p_T(x) \mid (x^2 - 1)$ so T has eigenvalues in $\{-1, 1\}$.

If the only eigenvalue of T is -1 , we claim that V one-dimensional.

If T has 1 as an eigenvalue. ■

↪ **Proposition 1.2:** D_8 has a unique faithful irreducible representation, of dimension 2 over a field F in which $0 \neq 2$.

PROOF. Write $G = D_8 = \{1, r, r^2, r^3, v, h, d_1, d_2\}$ as standard. Let ρ be our irreducible, faithful representation and let $T = \rho(r^2)$. Then, $p_T(x) \mid x^2 - 1 = (x - 1)(x + 1)$ and so $V = V_+ \oplus V_-$, the respective eigenspaces for $\lambda = +1, -1$ respectively for T . Then, notice that since r^2 in the center of G , both V_+ and V_- are preserved by the action of G , hence one must be trivial and the other the entirety of V . V can't equal V_+ , else $T = I$ on all of V hence ρ not faithful so $V = V_-$.

Next, it must be that $\rho(h)$ has both eigenvalues 1 and -1 . Let $v_1 \in V$ be such that $hv_1 = v_1$ and $v_2 = rv_1$. We claim that $W := \text{span}\{v_1, v_2\}$, namely $V = W$ 2-dimensional.

We simply check each element. $rv_1 = v_2$ and $rv_2 = r^2v_1 = -v_1$ which are both in W hence r and thus $\langle r \rangle$ fixes W . Next, $hv_1 = v_1$ and $vv_2 = vrv_1 = rhv_1 = rv_1 = v_2$ (since $rhr^{-1} = v$) and so $hv_2 = -v_2$ and $vv_1 = -v_1$ and so W G -stable. Finally, d_1 and d_2 are just products of these elements and so W G -stable. ■

↪ **Definition 1.3** (Isomorphism of Representations): Given a group G and two representations $\rho_i : G \rightarrow \text{Aut}_{\mathbb{F}}(V_i)$, $i = 1, 2$ an isomorphism of representations is a vector space isomorphism $\varphi : V_1 \rightarrow V_2$ that respects the group action, namely

$$\varphi(gv) = g\varphi(v)$$

for every $g \in G, v \in V_1$.

§1.2 Maschke's Theorem

↪ **Theorem 1.2** (Maschke's): Any representation of a finite group G over \mathbb{C} can be written as a direct sum of irreducible representations, i.e.

$$V = V_1 \oplus \cdots \oplus V_t,$$

where V_j irreducible.

Remark 1.1: $|G| < \infty$ essential. For instance, consider $G = (\mathbb{Z}, +)$ and 2-dimensional representation given by $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Then, $n \cdot e_1 = e_1$ and $n \cdot e_2 = ne_1 + e_2$. We have that $\mathbb{C}e_1$ irreducible then. But if $v = ae_1 + e_2 \in W := V \setminus \mathbb{C}e_1$, then $Gv = (a+1)e_1 + e_2$ so $Gv - v = e_1 \in W$, contradiction.

Remark 1.2: $|\mathbb{C}|$ essential. Suppose $F = \mathbb{Z}/3\mathbb{Z}$ and $V = Fe_1 \oplus Fe_2 \oplus Fe_3$, and $G = S_3$ acts on V by permuting the basis vectors e_i . Then notice that $F(e_1 + e_2 + e_3)$ an irreducible subspace in V . Let $W = F(w)$ with $w := ae_1 + be_2 + ce_3$ be any other G -stable subspace. Then, by applying (123) repeatedly to w and adding the result, we find that $(a+b+c)(e_1 + e_2 + e_3) \in W$. Similarly, by applying (12), (23), (13) to w , we find $(a-b)(e_1 - e_2)$, $(b-c)(e_2 - e_3)$, $(a-c)(e_1 - e_3)$ all in W . It must be that at least one of $a-b, a-c, b-c$ nonzero, else we'd have $w \in F(e_1 + e_2 + e_3)$. Assume wlog $a-b \neq 0$. Then, we may apply $(a-b)^{-1}$ and find $e_1 - e_2 \in W$. By applying (23), (13) to this vector and scaling, we find further $e_2 - e_3$ and $e_1 - e_3 \in W$. But then,

$$2(e_1 - e_2) + 2(e_1 - e_3) = e_1 + e_2 + e_3 \in W,$$

so $F(e_1 + e_2 + e_3)$ a subspace of W , a contradiction.

↪ **Proposition 1.3:** Let V be a representation of $|G| < \infty$ over \mathbb{C} and let $W \subseteq V$ a sub-representation. Then, W has a G -stable complement W' , such that $V = W \oplus W'$.

PROOF. Denote by ρ the homomorphism induced by the representation. Let W_0 be any complementary subspace of W and let

$$\pi : V \rightarrow W$$

be a projection onto W along W_0 , i.e. $\pi^2 = \pi$, $\pi(V) = W$, and $\ker(\pi) = W_0$. Let us "replace" π by the "average"

$$\tilde{\pi} := \frac{1}{\#G} \sum_{g \in G} \rho(g) \pi \rho(g)^{-1}.$$

Then the following hold:

- (1) $\tilde{\pi}$ G -equivariant, that is $\tilde{\pi}(gv) = g\tilde{\pi}(v)$ for every $g \in G, v \in V$.
- (2) $\tilde{\pi}$ a projection onto W .

Let $W' = \ker(\tilde{\pi})$. Then, W' G -stable, and $V = W \oplus W'$. ■

We present an alternative proof to the previous proposition by appealing to the existence of a certain inner product on complex representations of finite groups.

↪ **Definition 1.4:** Given a vector space V over \mathbb{C} , a *Hermitian pairing/inner product* is a hermitian-bilinear map $V \times V \rightarrow \mathbb{C}$, $(v, w) \mapsto \langle v, w \rangle$ such that

- linear in the first coordinate;
- conjugate-linear in the second coordinate;
- $\langle v, v \rangle \in \mathbb{R}^{\geq 0}$ and equal to zero iff $v = 0$.

↪ **Theorem 1.3:** Let V be a finite dimensional complex representation of a finite group G . Then, there is a hermitian inner product $\langle \cdot, \cdot \rangle$ such that $\langle gv, gw \rangle = \langle v, w \rangle$ for every $g \in G$ and $v, w \in V$.

PROOF. Let $\langle \cdot, \cdot \rangle_0$ be any inner product on V (which exists by defining $\langle e_i, e_j \rangle_0 = \delta_i^j$ and extending by conjugate linearity). We apply “averaging”:

$$\langle v, w \rangle := \frac{1}{\#G} \sum_{g \in G} \langle gv, gw \rangle.$$

Then, one can check that $\langle \cdot, \cdot \rangle$ is hermitian linear, positive, and in particular G -equivariant. ■

From this, the previous proposition follows quickly by taking $W' = W^\perp$, the orthogonal complement to W with respect to the G -invariant inner product that the previous theorem provides.

From this proposition, Maschke’s follows by repeatedly applying this logic. Since at each stage V is split in two, eventually the dimension of the resulting dimensions will become zero since V finite dimensional. Hence, the remaining vector spaces V_1, \dots, V_t left will necessarily be irreducible, since if they weren’t, we could apply the proposition further.

↪ **Theorem 1.4 (Schur’s Lemma):** Let V, W be irreducible representations of a group G . Then,

$$\text{Hom}_G(V, W) = \begin{cases} 0 & \text{if } V \not\cong_G W \\ \mathbb{C} & \text{if } V \cong_G W \end{cases}$$

where $\text{Hom}_G(V, W) = \{T : V \rightarrow W \mid T \text{ linear and } G\text{-equivariant}\}$.

PROOF. Suppose $V \not\cong_G W$ and let $T \in \text{Hom}_G(V, W)$. Then, notice that $\ker(T)$ a subrepresentation of V (a subspace that is a representation in its own right), but by assumption V irreducible hence either $\ker(T) = V$ or $\{0\}$.

If $\ker(T) = V$, then T trivial, and if $\ker(T) = \{0\}$, then this implies $T : V \rightarrow \text{im}(T) \subset W$ a representation isomorphism, namely $\text{im}(T)$ a irreducible subrepresentation of W . This implies that, since W irreducible, $\text{im}(T) = W$, contradicting the original assumption.

Suppose now $V \cong W$. Let $T \in \text{Hom}_G(V, W) = \text{End}_G(V)$. Since \mathbb{C} algebraically closed, T has an eigenvalue, λ . Then, notice that $T - \lambda I \in \text{End}_G(V)$ and so $\ker(T - \lambda I) \subset V$ a, necessarily trivial because V irreducible, subrepresentation of V . Hence, $T - \lambda I = 0 \Rightarrow T = \lambda I$ on V . It follows that $\text{Hom}_G(V, W)$ a one-dimensional vector space over \mathbb{C} , so namely \mathbb{C} itself. ■

↪ **Corollary 1.1:** Given a general representation $V = \bigoplus_{j=1}^t V_j^{m_j}$,

$$m_j = \dim_{\mathbb{C}} \text{Hom}_G(V_j, V).$$

↪ **Definition 1.5 (Trace):** The trace of an endomorphism $T : V \rightarrow V$ is the trace of any matrix defining T . Since the trace is conjugation-invariant, this is well-defined regardless of basis.

↪ **Proposition 1.4:** Let $W \subseteq V$ a subspace and $\pi : V \rightarrow W$ a projection. Then, $\text{tr}(\pi) = \dim(W)$.

↪ **Theorem 1.5:** If $\rho : G \rightarrow \text{Aut}_{\mathbb{F}}(V)$ a complex representation of G , then

$$\dim(V^G) = \frac{1}{\#G} \sum_{g \in G} \text{tr}(\rho(g)),$$

where $V^G = \{v \in V : gv = v \ \forall g \in G\}$.

PROOF. Let $\pi = \frac{1}{\#G} \sum_{g \in G} \rho(g)$. Then, notice that $\text{im}(\pi) = V^G$ and $\pi^2 = \pi$ hence a projection from V onto V^G . Using the previous proposition and linearity of the trace completes the proof. ■

§1.3 Characters

↪ **Definition 1.6:** Let $\dim(V) < \infty$ and G a group. The *character* of V is the function

$$\chi_V : G \rightarrow \mathbb{C}, \quad \chi_V(g) := \text{tr}(\rho(g)).$$