

# MATH255 - Analysis 2

Basic point-set topology; metric spaces; Hölder-Minkowski Inequalities; compactness.

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# 1 Introduction

## 1.1 Metric Spaces

### ↪ Definition 1.1: Metric Space

A set  $X$  is a *metric space* with distance  $d$  if

1. (symmetric)  $d(x, y) = d(y, x) \geq 0$
2.  $d(x, y) = 0 \iff x = y$
3. (triangle inequality)  $d(x, y) + d(y, z) \geq d(x, z)$

**Remark 1.1.** If 1., 3. are satisfied but not 2.,  $d$  can be called a “pseudo-distance”.

### ↪ Definition 1.2: Open Metric Space

Let  $(X, d)$  be a metric space. A subset  $A \subseteq X$  is open  $\iff \forall x \in A, \exists r = r(x) > 0$  s.t.  $B(x, r(x)) \subseteq A$ .

### ↪ Definition 1.3: Normed Space

Let  $X$  be a vector space over  $\mathbb{R}$ . The norm on  $X$ , denoted  $\|x\| \in \mathbb{R}$ , is a function that satisfies

1.  $\|x\| \geq 0$
2.  $\|x\| = 0 \iff x = 0$
3.  $\|c \cdot x\| = |c| \cdot \|x\|$
4.  $\|x + y\| \leq \|x\| + \|y\|$

If  $X$  is a normed vector space over  $\mathbb{R}$ , we can define a distance  $d$  on  $X$  by  $d(x, y) = \|x - y\|$ .

### ↪ Proposition 1.1

If  $X$  is a normed vector space over  $\mathbb{R}$ , a distance  $d$  on  $X$  by  $d(x, y) = \|x - y\|$  makes  $(X, d)$  a metric space.

Proof. 1.  $d(x, y) = \|x - y\| \geq 0$

2.  $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$

3.  $d(x, y) + d(y, z) = \|x - y\| + \|y - z\| \geq \|(x - y) + (y - z)\| = \|x - z\| := d(x, z)$



⊛ **Example 1.1:  $L^p$  distance in  $\mathbb{R}^n$**

Let  $\bar{x} \in \mathbb{R}^n$ ,  $x = (x_1, x_2, \dots, x_n)$ . The  $L^p$  norm is defined

$$\|x\|_p := (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{\frac{1}{p}}.$$

In the case  $p = 2, n = 2$ , we simply have the standard Euclidean distance over  $\mathbb{R}^2$ .

Unit Balls: consider when  $\|x\|_p \leq 1$ , over  $\mathbb{R}^2$ .

- $p = 1$  :  $|x_1| + |x_2| \leq 1$ ; this forms a “diamond ball” in the plane.
- $p = 2$  :  $\sqrt{|x_1|^2 + |x_2|^2} \leq 1$ ; this forms a circle of radius 1. Clearly, this surrounds a larger area than in  $p = 1$ .

A natural question that follows is what happens as  $p \rightarrow \infty$ ? Assuming  $|x_1| \geq |x_2|$ :

$$\begin{aligned} \|x\|_p &= (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \\ &= \left[ |x_1|^p \left( 1 + \left| \frac{x_2}{x_1} \right|^p \right) \right]^{\frac{1}{p}} \\ &= |x_1| \left( 1 + \left| \frac{x_2}{x_1} \right|^p \right)^{\frac{1}{p}} \end{aligned}$$

If  $|x_1| > |x_2|$ , this goes to  $|x_1|$ . If they are instead equal, then  $\|x\|_p = |x_1| \cdot 2^{\frac{1}{p}} \rightarrow |x_1| \cdot 1$  as well. Hence,  $\lim_{p \rightarrow \infty} \|x\|_p = \max\{|x_1|, |x_2|\}$ . Thus, the unit ball will approach  $\max\{|x_1|, |x_2|\} \leq 1$ , that is, the unit square.

↪ **Proposition 1.2**

Let  $x \in \mathbb{R}^n$ . Then,  $\|x\|_p \rightarrow \max\{|x_1|, \dots, |x_n|\}$  as  $p \rightarrow \infty$ .

**Remark 1.2.** This is an extension of the previous example to arbitrary real space; the proof follows nearly identically.

↪ **Definition 1.4: Convex Set**

Let  $X$  be a normed space, and take  $x, y \in X$ . The line segment from  $x$  to  $y$  is the set

$$\{t \cdot x + (1 - t) \cdot y : 0 \leq t \leq 1\}.$$

Let  $A \subseteq X$ .  $A$  is *convex*  $\iff \forall x, y \in A$ , we have that

$$(t \cdot x + (1 - t) \cdot y) \in A \forall 0 \leq t \leq 1.$$

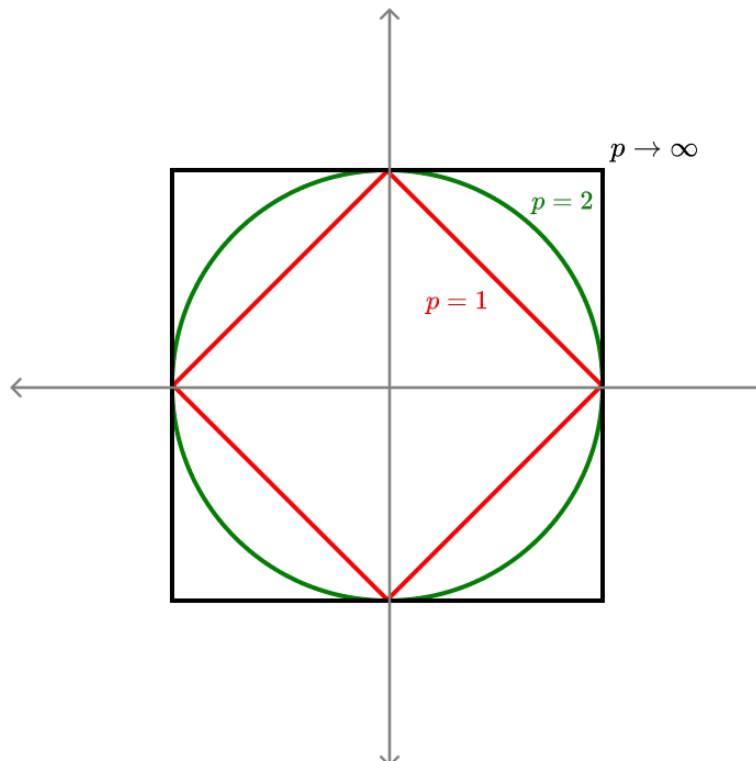


Figure 1: Regions of  $\mathbb{R}^2$  where  $\|x\|_p \leq 1$  for various values of  $p$ .

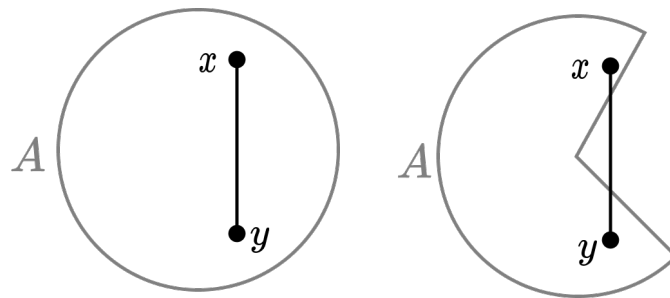


Figure 2: Convex (left) versus not convex (right) sets.

**Remark 1.3.** Think of this as saying “a set is convex iff every point on a line segment connected any two points is in the set”.

↪ **Definition 1.5:**  $\ell_p$

The space  $\ell_p$  of sequences is defined as

$$\{x = (x_1, x_2, \dots, x_n, \dots) : \sum_{n=1}^{\infty} |x_n|^p < +\infty\} \quad *.$$

Then,  $*$  defines the  $\ell^p$  norm on the space of sequences; that is,  $\|x\|_p := (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$ .

⊛ **Example 1.2:**  $\ell_p$ ,  $x_n = \frac{1}{n}$

. Let  $x_n = \frac{1}{n}$ . For which  $p$  is  $x \in \ell_p$ ? We have, raising the norm to the power of  $p$  for ease:

$$\begin{aligned} \|x\|_p^p &= |x_1|^p + |x_2|^p + \cdots + |x_n|^p + \cdots \\ &= 1^p + \left(\frac{1}{2}\right)^p + \cdots < \infty \iff p > 1. \end{aligned}$$

In the case that  $p = 1$ , this becomes a harmonic sum, which diverges.

⊛ **Example 1.3:**  $L^p$  space of functions

Let  $f(x)$  be a continuous function. We define the norm of  $f$  over an interval  $[a, b]$

$$\|f\|_p = \left[ \int_a^b |f(x)|^p dx \right]^{\frac{1}{p}}.$$

**Remark 1.4.** Triangle inequality for  $\|x\|_p$  or  $\|f\|_p$  is called *Minkowski inequality*;  $\|x\|_p + \|y\|_p \geq \|x + y\|_p$ . This will be discussed further.

⊛ **Example 1.4:** Distances between sets in  $\mathbb{R}^2$

Let  $A, B$  be bounded, closed, “nice” sets in  $\mathbb{R}^2$ . We define

$$d(A, B) := \text{Area}(A \triangle B),$$

where

$$A \triangle B : (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

It can be shown that this is a “valid” distance.

**Remark 1.5.**  $\triangle$  denotes the “symmetric difference” of two sets.

⊛ **Example 1.5:**  $p$ -adic distance

Let  $p$  be a prime number. Let  $x = \frac{a}{b} \in \mathbb{Q}$ , and write  $x = p^k \cdot \left(\frac{c}{d}\right)$ , where  $c, d$  are not divisible by  $p$ . Then, the  $p$ -adic norm is defined  $\|x\|_p := p^{-k}$ . It can be shown that this is a norm.

Suppose  $p = 2$ ,  $x = 28 = 4 \cdot 7 = 2^2 \cdot 7$ . Then,  $\|28\|_2 = 2^{-2} = \frac{1}{4}$ ; similarly,  $\|1024\|_2 = \|2^{10}\|_2 = 2^{-10}$ .

More generally, we have that  $\|2^k\|_2 = 2^{-k}$ ; conversely,  $\|2^{-k}\|_2 = 2^k$ . That is, the closer to 0, the larger the distance, and vice versa, contrary to our notion of Euclidean distance.

↪ **Proposition 1.3**

$\|x\|_p$  as defined above is a well-defined norm over  $\mathbb{Q}$ .

## 2 Point-Set Topology

### 2.1 Definitions

#### ↪ **Definition 2.1: Topological space**

A set  $X$  is a topological space if we have a collection of subsets  $\tau$  of  $X$  called *open sets* s.t.

1.  $\emptyset \in \tau, X \in \tau$
2. Consider  $\{A_\alpha\}_{\alpha \in I}$  where  $A_\alpha$  an open set for any  $\alpha$ ; then,  $\bigcup_{\alpha \in I} A_\alpha \in \tau$ , that is, it is also an open set.
3. If  $J$  is a finite set, and  $A_\beta$  open for all  $\beta \in J$ , then  $\bigcap_{\beta \in J} A_\beta \in \tau$  is also open.

In other words, 2.: arbitrary unions of open sets are open, and 3.: finite intersections of open sets are open.

#### ↪ **Definition 2.2: Closed sets**

Closed sets are complements of open sets; hence, axioms for closed sets follow appropriately;

- 1.\*  $X, \emptyset$  closed;
- 2.\*  $B_\alpha$  closed  $\forall \alpha \in I \implies \bigcap_{\alpha \in I} B_\alpha$  closed.
- 3.\*  $B_\beta$  closed  $\forall \beta \in J, J$  finite, then  $\bigcup_{\beta \in J} B_\beta$  also closed.

↪ **Definition 2.3: Equivalence of Metrics**

Suppose we have a metric space  $X$  with two distances  $d_1, d_2$ ; will these necessarily admit the same topology?

A sufficient condition is that, if  $\forall x \neq y \in X, \exists 1 < C < +\infty$  s.t.

$$\frac{1}{C} < \frac{d_1(x, y)}{d_2(x, y)} < C.$$

That is, the distances are equivalent, up to multiplication by a constant.

Indeed, this condition gives that  $d_2 < C d_1$  and  $d_2 > \frac{d_1}{C}$ ; this gives

$$B_{d_1}(x, \frac{r}{C}) \subseteq B_{d_2}(x, r) \subseteq B_{d_1}(x, C \cdot r).$$

Hence,  $d_1, d_2$  define the same open/closed sets on  $X$  thus admitting the same topologies. We write  $d_1 \asymp d_2$ .

**Remark 2.1.** If  $d_1 \asymp d_2$  and  $d_2 \asymp d_3$ , then also  $d_1 \asymp d_3$ . Moreover, clearly,  $d_1 \asymp d_1$  and  $d_1 \asymp d_2 \implies d_2 \asymp d_1$ , hence this is a well-defined equivalence relation.

Hence, its enough to show that  $\forall 1 < p < +\infty$ , we have  $\|x\|_p \asymp \|x\|_\infty$  to show that any  $\|x\|_q$  norm are equivalent for all  $q$  on  $\mathbb{R}^n$ .

↪ **Definition 2.4: Interior, Boundary of a Topological Set**

Let  $X$  be a topological space,  $A \subseteq X$  and let  $x \in X$ . We have the following possibilities

1.  $\exists U$ -open :  $x \in U \subseteq A$ . In this case, we say  $x \in$  the *interior* of  $A$ , denoted

$$x \in \text{Int}(A).$$

2.  $\exists V$ -open :  $x \in V \subseteq X \setminus A = A^C$ . In this case, we write

$$x \in \text{Int}(X^C).$$

3.  $\forall U$ -open :  $x \in U, U \cap A \neq \emptyset$  AND  $U \cap A^C \neq \emptyset$ . In this case, we say  $x$  is in the *boundary* of  $A$ , and denote

$$x \in \partial A.$$

↪ **Definition 2.5: Closure**

$x \in \text{Int}(A)$  or  $x \in \partial A$  (that is,  $x \in \text{Int}(A) \cup \partial A$ )  $\iff$  every open set  $U$  that contains  $x$  intersects  $A$ .<sup>1</sup> Such points are called *limit points* of  $A$ . The set of all limits points of  $A$  is called the *closure* of  $A$ , denoted  $\bar{A}$ .

<sup>1</sup>“Requires” proof.

**Remark 2.2.** We have that

$$\text{Int}(A) \subseteq A \subseteq \overline{A} = \text{Int}(A) \cup \partial A.$$

↪ **Proposition 2.1: Properties of  $\text{Int}(A)$**

$\text{Int}(A)$  is open, and it is the largest open set contained in  $A$ . It is the union of all  $U$ -open s.t.  $U \subseteq A$ . Moreover, we have that

$$\text{Int}(\text{Int}(A)) = \text{Int}(A).$$

↪ **Proposition 2.2: Properties of  $\overline{A}$**

$\overline{A}$  is closed;  $\overline{A}$  is the smallest closed set that contains  $A$ , that is,  $\overline{A} = \bigcap B$  where  $B$  closed and  $A \subseteq B$ . We have too that

$$\overline{(\overline{A})} = \overline{A}.$$

↪ **Proposition 2.3**

1.  $A$  is open  $\iff A = \text{Int}(A)$
2.  $A$  is closed  $\iff A = \overline{A}$

## 2.2 Basis

↪ **Definition 2.6: Basis for a Topology**

Let  $\tau$  be a topology on  $X$ . Let  $\mathcal{B} \subseteq \tau$  be a collection of open sets in  $X$  such that every open set is a union of open sets in  $\mathcal{B}$ .

⊛ **Example 2.1: Example Basis**

$X = \mathbb{R}$ , and  $\mathcal{B} = \{\text{all open intervals } (a, b) : -\infty < a < b < +\infty\}$ .

↪ **Proposition 2.4**

Let  $\mathcal{B}$  be a collection of open sets in  $X$ . Then,  $\mathcal{B}$  is a basis  $\iff$

1.  $\forall x \in X, \exists U\text{-open} \in \mathcal{B} \text{ s.t. } x \in U$ .
2. If  $U_1 \in \mathcal{B}$  and  $U_2 \in \mathcal{B}$ , and  $x \in U_1 \cap U_2$ , then  $\exists U_3 \in \mathcal{B} \text{ s.t. } x \in U_3 \subseteq U_1 \cap U_2$ .

⊛ **Example 2.2**

Consider  $X = \mathbb{R}$ . Requirement 1. follows from taking  $U = (x - \varepsilon, x + \varepsilon)$  for any  $\varepsilon > 0$ . For 2., suppose  $x \in (a, b) \cap (c, d) =: U_1 \cap U_2$ . Let  $U_3 = (\max\{a, c\}, \min\{b, d\})$ ; then, we have that  $U_3 \subseteq U_1 \cap U_2$ , while clearly  $x \in U_3$ .



↪ **Proposition 2.5**

In a metric space, a basis for a topology is a collection of open balls,

$$\{B(x, r) : x \in X, r > 0\} = \{\{y \in X : d(x, y) < r\} : x \in X, r > 0\}.$$

*Proof.* We prove via proposition 2.4. Property 1. holds clearly;  $x \in B(x, \varepsilon)$ -open  $\subseteq \mathcal{B}$ .

For property 2., let  $x \in B(y_1, r_1) \cap B(y_2, r_2)$ , that is,  $d(x, y_1) < r_1$  and  $d(x, y_2) < r_2$ . Let

$$\delta := \min\{r_1 - d(x, y_1), r_2 - d(x, y_2)\}.$$

We claim that  $B(x, \delta) \subseteq U_1 \cap U_2$ .

Let  $z \in B(x, \delta)$ . Then,

$$d(z, y_1) \stackrel{\Delta \neq}{\leq} d(z, x) + d(x, y_1) < \delta + d(x, y_1) \leq r_1 - d(x, y_1) + d(x, y_1) = r_1,$$

hence, as  $d(z, y_1) < r_1 \implies z \in B(y_1, r_1) = U_1$ . Replacing each occurrence of  $y_1, r_1$  with  $y_2, r_2$  respectively gives identically that  $z \in B(y_2, r_2) = U_2$ . Hence, we have that  $B(x, \delta) \subseteq U_1 \cap U_2$  and 2. holds. ■

## 2.3 Subspaces

↪ **Definition 2.7**

Let  $X$  be a topological space and let  $Y \subseteq X$ . We define the subspace topology on  $Y$ :

1. Open sets in  $Y = \{Y \cap \text{open sets in } X\}$

↪ **Proposition 2.6: Consequences of Subspace Topologies**

Suppose  $\mathcal{B}$  is a basis for a topology in  $X$ . Then,  $\{U \cap Y : U \in \mathcal{B}\}$  forms a basis for the subspace  $Y \subseteq X$ .

Suppose  $X$  a metric space. Then,  $Y$  is also a metric space, with the same distance.

↪ **Proposition 2.7**

Let  $Y \subseteq X$  - a metric space. Then, the metric space topology for  $(Y, d)$  is the same as the subspace topology.

*Proof.* (Sketch) A basis for the open sets in  $X$  can be written  $\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)$ ; hence

$$Y \cap \left( \bigcup_{\alpha \in I} B(x_\alpha, r_\alpha) \right) = \bigcup_{\alpha \in I} (Y \cap B(x_\alpha, r_\alpha))$$

is an open set topology for  $Y$ . ■

↪ **Lemma 2.1**

Let  $A \subseteq X$ -open,  $B \subseteq A$ ;  $B$ -open in subspace topology for  $A \iff B$ -open in  $X$ .

↪ **Lemma 2.2**

Let  $Y \subseteq X$ ,  $A \subseteq Y$ . Then,  $\overline{A}$  in  $Y = Y \cap \overline{A}$  in  $X$ . We can denote this

$$\overline{A}_Y = \overline{A}_X \cap Y.$$

## 2.4 Continuous Functions

↪ **Definition 2.8: Continuous Function**

Let  $X, Y$  be topological spaces. Let  $f : X \rightarrow Y$ .  $f$  is *continuous*  $\iff \forall$  open  $V \in Y$ ,  $f^{-1}(V)$ -open in  $X$ .

↪ **Proposition 2.8**

This definition is consistent with the normal  $\varepsilon$ - $\delta$  definition on the real line.

Proof. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ , continuous; that is,  $\forall \varepsilon > 0, \forall x \in \mathbb{R} \exists \delta > 0$  s.t.  $|x_1 - x| < \delta$ , then  $|f(x_1) - f(x)| < \varepsilon$ .

Let  $V \subseteq \mathbb{R}$  open. Let  $y \in V$ . Then,  $\exists \varepsilon : (y - \varepsilon, y + \varepsilon) \subseteq V$ . Let  $y = f(x)$ , hence  $y \in f^{-1}(V)$ . Now, if  $d(x, x_1) < \delta$ , we have that  $d(f(x_1), f(x)) < \varepsilon$  (by continuity of  $f$ ), hence  $f(x_1) \in (y - \varepsilon, y + \varepsilon) \subseteq V$ ; moreover,  $(x - \delta, x + \delta) \subseteq f^{-1}(V)$ , thus  $f^{-1}(V)$  is open as required.

The inverse of this proof follows identically. ■

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↪ **Proposition 2.9**

Suppose  $\mathcal{B}$  forms a basis of topology for  $Y$ . Then,  $f : X \rightarrow Y$  is continuous if  $f^{-1}(U)$  open  $\forall U \in \mathcal{B}$ .

Proof. If  $U$ -open set in  $Y$ , then  $\exists I$ -index set and a collection of open sets  $\{A_\alpha\}_{\alpha \in I}$ ,  $A_\alpha \in \mathcal{B}$ , s.t.  $U = \cup_{\alpha \in I} A_\alpha$ . Then, we have

$$f^{-1}(U) = f^{-1}(\cup_{\alpha \in I} (A_\alpha)) = \cup_{\alpha \in I} \underbrace{f^{-1}(A_\alpha)}$$

Hence, if each  $f^{-1}(A_\alpha)$  open, then  $\cup_{\alpha \in I} f^{-1}(A_\alpha)$  open; hence it suffices to check if  $f^{-1}(U) \forall U$ -open in  $V$  is open to see if  $f$  continuous. ■

↪ **Theorem 2.1: Continuity of Composition**

If  $f : X \rightarrow Y$  continuous and  $g : Y \rightarrow Z$  continuous, then  $g \circ f$  continuous as well.

Proof. Let  $U$ -open in  $Z$ . Then

$$(g \circ f)^{-1}(U) = f^{-1}(\underbrace{g^{-1}(U)}_{\substack{\text{open in } Y \\ \text{open in } X}})$$

■

↪ **Proposition 2.10**

If  $f : X \rightarrow Y$  continuous and  $A \subseteq X$ ,  $A$  has subspace topology, then  $f|_A : A \rightarrow Y$  is also continuous.<sup>2</sup>

Proof. Let  $U$ -open in  $Y$ . Then

$$(f|_A)^{-1}(U) = \underbrace{f^{-1}(U)}_{\text{open}} \cap \underbrace{A}_{\text{open}}$$

By the definition of subspace topology, this is an open set and hence  $f|_A$  is continuous. ■

## 2.5 Product Spaces

↪ **Definition 2.9: Finite Product Spaces**

Let  $X_1, \dots, X_n$  be topological spaces. We define

$$(X_1 \times X_2 \times \dots \times X_n),$$

and aim to define a *product topology*; a basis of which consists of cylinder sets.

↪ **Definition 2.10: Cylinder Set**

A *cylinder set* has the form

$$A_1 \times A_2 \times \dots \times A_n$$

where each  $A_j$ -open in  $X_j$ .

⊗ **Example 2.3**

Given an open interval  $(a_1, b_1), (a_2, b_2) \subset \mathbb{R}$ , the set  $(a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$  is a basis for the topology on  $\mathbb{R}^2$ .

<sup>2</sup>We denote  $f|_A$  as the restriction of the domain of  $f$  to  $A$ .

↪ **Definition 2.11: Projection**

Let  $X_1 \times X_2 \times \cdots \times X_n =: X$ . The *projection*  $\pi_j : X \rightarrow X_j$  maps  $(x_1, \dots, x_n) \rightarrow x_j \in X_j$ .

**Remark 2.3.** One can show  $\pi_j$  continuous.

↪ **Definition 2.12: Coordinate Function**

Given a function  $f : Y \rightarrow X_1 \times \cdots \times X_n = (x_1(y), x_2(y), \dots, x_n(y))$ . The *coordinate function* is

$$f_j = \pi_j \circ f; \quad f_j = x_j(y).$$

↪ **Proposition 2.11**

$f : Y \rightarrow X = X_1 \times \cdots \times X_n$  continuous  $\iff f_j : Y \rightarrow X_j$  continuous.

*Proof.* Its enough to show that  $\forall U \in \mathcal{B}$ -basis for  $X$ -product space,  $f^{-1}(U)$ -open in  $Y$ . Take  $U = A_1 \times \cdots \times A_n$ -open. Then, we claim that

$$f^{-1}(U) = f^{-1}(A_1 \times \cdots \times A_n) = f_1^{-1}(A_1) \cap f_2^{-1}(A_2) \cap \cdots \cap f_n^{-1}(A_n). \quad \star$$

If this holds, then as each  $f_i$  continuous (being a composition of continuous functions) and each  $A_i$  open in  $X_i$ , then each  $f_i^{-1}(A_i)$  open in  $Y$  and hence  $\star$ , being the finite intersection of open sets in  $Y$ , is itself open in  $Y$ . ■

⊛ **Example 2.4: Fourier Transform: Motivation for Infinite Product Topologies**

Let  $f \in C([0, 2\pi])$  is real-valued. We write the  $n$ th Fourier coefficients

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos(nx) dx - i \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin(nx) dx.\end{aligned}$$

And the Fourier transform of  $f$  as the infinite product

$$f(x) \mapsto (\dots, \hat{f}(-n), \hat{f}(-n+1), \dots, \hat{f}(-1), \hat{f}(0), \hat{f}(1), \dots, \hat{f}(n), \dots) \in \prod_{n \in \mathbb{Z}} (\mathbb{C})_n.$$

Hence, this is an (countably, as indexed by integers) infinite product space.

Now, let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Suppose  $f(x) \rightarrow 0$  “fast enough” as  $|x| \rightarrow \infty$  and  $f$  continuous. Then, we can define the Fourier coefficients

$$\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-itx} dx,$$

where  $t \in \mathbb{R}$ . We then have the transform

$$f \mapsto \{\hat{f}(t)\}_{t \in \mathbb{R}}.$$

In this case, our index set is  $\mathbb{R}$  is (uncountably) infinite.

↪ **Definition 2.13: Product Topology/Cylinder Sets for  $\infty$  Products**

Let  $X = \prod_{\alpha \in I} X_\alpha$ . Then, a basis for  $X$  is given by cylinder sets of the form  $A = \prod_{\alpha \in I} A_\alpha$  where  $A_\alpha$ -open in  $X_\alpha$ , AND  $A_\alpha = X_\alpha$  except for finitely many indices  $\alpha$ .

That is, there exists a finite set  $J = (\alpha_1, \dots, \alpha_k) \subseteq I$ , such that we can write  $A = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$  (where  $A_\alpha$  open in  $X_\alpha$ ).

↪ **Proposition 2.12**

Given  $f : Y \rightarrow \prod_{\alpha \in I} X_\alpha = X$ , then (taking  $f_\alpha = \pi_\alpha \circ f$  as before) we have that  $f$  is continuous in  $X \iff f_\alpha : Y \rightarrow X_\alpha$  continuous in  $X_\alpha \forall \alpha \in I$ .

**Remark 2.4.** Extension of proposition 2.11 to infinite product space.

Proof. Write  $U = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$ . Then,

$$f^{-1}(U) = \bigcap_{\alpha \in J} f_\alpha^{-1}(A_\alpha)$$

which is open in  $Y$ , hence  $f$  continuous. ■

**Remark 2.5.** The intersection of the entire spaces give no restriction.

↪ Lecture 03; Last Updated: Fri Jan 19 11:49:27 EST 2024

## 2.6 Metrizable

### ↪ Proposition 2.13

Different metrics can define the same topology.

#### ⊛ Example 2.5

1. Different  $\ell_p$  metrics in  $\mathbb{R}^n$  (PSET 1)
2. Let  $(X, d)$  be a metric space. Then,

$$\tilde{d}(x, y) := \frac{d(x, y)}{d(x, y) + 1}$$

is also a metric (the first two axioms are trivial), and defines the same topology. Note, moreover, that  $\tilde{d}(x, y) \leq 1 \forall x, y$ ; this distance is bounded, and can often be more convenient to work with in particular contexts.

### ↪ Question 2.1

Suppose  $(X_k, d_k)$  are metric spaces  $\forall k \geq 1$ . Then, we can define the product topology  $\tau$  on

$$X := \prod_{k=1}^{\infty} X_k.$$

Does the product topology  $\tau$  come from a metric? That is, is  $\tau$  metrizable?

**Remark 2.6.** There do indeed exist examples of non-metrizable topological spaces; this question is indeed well-founded.

*Answer.* Let  $\underline{x} = (x_1, x_2, \dots, x_n, \dots), \underline{y} = (y_1, y_2, \dots, y_n, \dots) \in \prod_{k=1}^{\infty} X_k$  (where  $x_i, y_i \in X_i$ ) be infinite sequences of elements. Then, for each metric space  $X_k$  take the metric

$$\tilde{d}_k(x_k, y_k) = \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)}$$

(as in the example above). Then, we define

$$D(\underline{x}, \underline{y}) = \sum_{k=1}^{\infty} \frac{\tilde{d}_k(x_k, y_k)}{2^k},$$

noting that  $D(\underline{x}, \underline{y}) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$  (by our construction, “normalizing” each metric), hence this is a valid, *converging* metric (which wouldn’t otherwise be guaranteed if we didn’t normalize the metrics). It remains to show whether this metric omits the same topology as  $\tau$ . ■

## 2.7 Compactness, Connectedness

### ↪ Definition 2.14: Compact

A set  $A$  in a topological space is said to be *compact* if every cover has a finite subcover. That is, if

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha - \text{open},$$

then  $\exists \{\alpha_1, \dots, \alpha_n \in I\}$  such that  $A \subseteq \bigcup_{i=1}^n U_{\alpha_i}$ .

### ↪ Proposition 2.14

A closed interval  $[a, b]$  is compact.

Proof. If<sup>3</sup>  $a = b$ , this is clear. Suppose  $a < b$ , and let  $[a, b] \subseteq \bigcup_{i \in I} U_i =: \mathcal{U}$  be an arbitrary cover. Then, we proceed in the following steps:

1. **Claim:** Given  $x \in [a, b]$ ,  $x \neq b$ ,  $\exists y \in [a, b]$  s.t.  $[x, y]$  has a finite subcover.

Let  $x \in [a, b]$ ,  $x \neq b$ . Then,  $\exists U_\alpha \in \mathcal{U} : x \in U_\alpha$ . Since  $U_\alpha$  open, and  $x \neq b$ , we further have that  $\exists c \in [a, b]$  s.t.  $[x, c] \subseteq U_\alpha$ .

Now, let  $y \in (x, c)$ ; then, the interval  $[x, y] \subseteq [x, c] \subseteq U_\alpha$ , that is,  $[x, y]$  has a finite subcover.

2. Define  $C := \{y \in [a, b] : [a, y] \text{ has a finite subcover}\}$ . We note that

- $C \neq \emptyset$ ; taking  $x = a$  in Step 1. above, we have that  $\exists y \in [a, b]$  such that  $[a, y]$  has a finite step cover, so this  $y \in C$ .
- $C$  bounded; by construction,  $\forall y \in C, a < y \leq c$ .

Thus, we can validly define  $c := \sup C$ , noting that  $a < c \leq b$ . Ultimately, we wish to prove that  $c = b$ , completing the proof that  $[a, b]$  has a finite subcover.

3. **Claim:**  $c \in C$ .

Let  $U_\beta \in \mathcal{U} : c \in U_\beta$ . Then, by the openness of  $U_\beta$ ,  $\exists d \in [a, b]$  s.t.  $(d, c] \subseteq U_\beta$ .

Supposing  $c \notin C$ , then  $\exists z \in C$  such that  $z \in (d, c)$ ; if one did not exist, then this would imply that  $d$  was a smaller upper bound than  $c$ , a contradiction. Thus,  $[z, c] \subseteq (d, c] \subseteq U_\beta$ .

Moreover, we have that, given  $z \in C$ ,  $[a, z]$  has a finite subcover; call it  $U_z \subseteq \mathcal{U}$ . This gives, then:

$$[a, c] = [a, z] \cup [z, c] \subseteq U_z \cup U_\beta.$$

But this is a finite subcover of  $[a, c]$ , contradicting the fact that  $c \notin C$ . We conclude, then, that  $c \in C$  after all.

4. **Claim:**  $c = b$ .

Suppose not; then, since we have  $c \leq b$ , then assume  $c < b$ . Then, applying Step 1. with  $x = c$  (which we can do, by our assumption of  $c \neq b$ !), then we have that  $\exists y > c$  s.t.  $[c, y]$  has a finite subcover, call this  $U_y \subseteq \mathcal{U}$ .

Moreover, we had  $c \in C$ , hence  $[a, c]$  has a finite subcover, call this  $U_c \subseteq \mathcal{U}$ .

<sup>3</sup>This proof is adapted from that of Theorem 27.1 in Munkre's Topology, an identical theorem but applied to more general ordered topologies.

Then, this gives us that

$$[a, y] = [a, c] \cup [c, y] \subseteq U_c \cup U_y,$$

that is,  $[a, y]$  has a finite subcover, and so  $y \in C$ . But recall that  $y > c$ ; hence, this a contradiction to  $c$  being the least upper bound of  $C$ . We conclude that  $c = b$ , and thus  $[a, b]$  has a finite subcover, and is thus compact. ■

**Remark 2.7.** A similar proof shows that  $[a, b]$  is connected; we cannot cover it by two disjoint open sets.

### ↪ **Theorem 2.2: On Compactness**

Let  $A \subseteq \mathbb{R}^n$ . Then,  $A$  compact  $\iff A$  closed and bounded.

### ↪ **Proposition 2.15**

If  $X, Y$  are compact topological spaces, then  $X \times Y$  is compact.

**Remark 2.8.** By induction, if  $X_1, \dots, X_n$  compact, so is  $\prod_{i=1}^n X_i$ .

### ↪ **Proposition 2.16**

A closed subset of a compact topological space is compact in the subspace topology.

*Proof.* (Of theorem 2.2)

( $\Leftarrow$ ) If  $A \subseteq \mathbb{R}^n$  closed and bounded, then  $A \subseteq [-R, +R]^n$  for some  $R > 0$  (it is contained in some “ $n$ -cube”). Then, we have that  $[-R, R]$  is compact, by proposition 2.14, proposition 2.15, and proposition 2.16,  $A$  itself compact.

( $\Rightarrow$ ) Suppose  $A \subseteq \mathbb{R}^n$  is compact. Then,  $\bigcup_{x \in A} B(x, \varepsilon)$  for some  $\varepsilon > 0$  is an open cover of  $A$ . As  $A$  compact, there must exist a finite subcover of this cover,  $A \subseteq \bigcup_{i=1}^N B(x_i, r_i)$ . Let  $R := \max_{i=1}^N (\|x_i\| + r_i)$ . Then,  $A \subseteq \overline{B(0, R)}$ , that is,  $A$  is bounded.

Now, suppose  $x$  is a limit point of  $A$ . Then, any neighborhood of  $x$  contains a point in  $A$ , so  $\forall r > 0, B(x, r) \cap A \neq \emptyset$ , and so  $\overline{B(x, r)}$  also contains a point of  $A$  for any  $r > 0$ .

Now, suppose  $x \notin A$  (looking for a contradiction). Then,

$$U := \bigcup_{r>0} U_r := \bigcup_{r>0} (\mathbb{R}^n \setminus \overline{B(x, r)}) = \mathbb{R}^n \setminus \{x\}$$

is an open cover for the set  $A$ .  $A$  being compact implies that  $U$  has an finite subcover such that  $A \subset U_{r_1} \cup U_{r_2} \cup \dots \cup U_{r_N}$ . Let  $r_0 = \min_{i=1}^N r_i$ . Then,  $A \subseteq U_{r_0}$ , and  $A \cap \overline{B(x, r_0)} = \emptyset$ ; but this is a contradiction to the definition of a limit point, hence any limit point  $x$  is contained in  $A$  and  $A$  is thus closed by definition. ■

### ↪ **Proposition 2.17**

Compact  $\implies$  sequentially compact; that is, every sequence in a compact set has a convergent subsequence.



↪ **Definition 2.15: Connected**

A topological space  $X$  is *not connected* if  $X = U \cup V$  for two open, nonempty, disjoint sets  $U, V$ .

If this does not hold,  $X$  is said to be *connected*.

A set  $A \subseteq X$  is not connected if  $A$  is not connected in the subspace topology  $\iff A = \subseteq U \cup V$ , for  $U, V$ -open in  $X$ ,  $(U \cap A) \neq \emptyset$ ,  $(V \cap A) \neq \emptyset$  and  $U \cap V = \emptyset$ .

↪ **Theorem 2.3**

Let  $X$  be a connected topological space. Let  $f : X \rightarrow Y$  be a continuous function. Then,  $f(X)$  is also connected.

Proof. Suppose, seeking a contradiction, that  $X$  is connected, but  $f(X)$  is not. Then, we can write  $f(X) \subseteq Y$  as  $f(X) \subseteq U \cup V$ , such that  $U, V$  open in  $Y$  and  $U \cap V = \emptyset$ . Then,

$$(U \cap f(X)) \cap (V \cap f(X)) = \emptyset.$$

We also have that

$$X \subseteq \underbrace{f^{-1}(U)}_{\text{open in } X, \neq \emptyset} \cup \underbrace{f^{-1}(V)}_{\text{open in } X, \neq \emptyset}.$$

$f^{-1}(U) \cap f^{-1}(V) = \emptyset$  (that is, they are disjoint) by our assumption; this is a contradiction to the connectedness of  $X$ , as we are able to write it as a subset of two disjoint open sets. Hence,  $f(X)$  is indeed connected. ■

↪ **Lemma 2.3**

Any interval  $(a, b), [a, b], [a, b), \dots, \subseteq \mathbb{R}$  is connected.

Proof. ■

↪ **Theorem 2.4: “Intermediate Value Theorem”**

Suppose  $X$  is connected and  $f : X \rightarrow \mathbb{R}$  is a continuous function. Then,  $f$  takes intermediate values.

More precisely, let  $a = f(x), b = f(y)$  for  $x, y \in X$ . Assume  $a < b$ . Then,  $\forall a < c < b, \exists z \in X$  s.t.  $f(z) = c$ .

Proof. Suppose, seeking a contradiction, that  $\exists c : a < c < b$  s.t.  $c \notin f(X)$  (that is, there exists an intermediate value that is “not reached” by the function).

Let  $U = (-\infty, c)$  and  $V = (c, +\infty)$ ; note that these are disjoint open sets. Then, we have that

$$X = f^{-1}(U) \cup f^{-1}(V),$$

by our assumption of  $c \notin f(X)$ . But this gives that  $X$  is not connected, as the union of two open (by continuity), disjoint, nonempty ( $f(x) = a \in U \implies x \in f^{-1}(U)$ , and  $f(y) = b \in V \implies y \in f^{-1}(V)$ ) sets, a contradiction. ■

↪ **Theorem 2.5**

Suppose  $X$  is compact,  $Y$ -topological space,  $f : X \rightarrow Y$  is a continuous function. Then,  $f(X)$  is also compact.

Proof. Let  $\{U_\alpha\}_{\alpha \in I}$  be an open cover of  $f(X) \subseteq Y$ , that is,

$$f(X) \subseteq \bigcup_{\alpha \in I} U_\alpha \implies X \subseteq f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(U_\alpha) =: \bigcup_{\alpha \in I} V_\alpha - \text{open.}$$

Then, this is an open cover of  $X$ ;  $X$  is compact, thus there exists a finite subcover, that is, indices  $\{\alpha_1, \dots, \alpha_n\} \subseteq I$  such that  $X = \bigcup_{i=1}^n V_{\alpha_i}$ . Thus,

$$f(X) \subseteq \bigcup_{i=1}^n U_{\alpha_i},$$

which is a finite subcover of  $f(X)$ . Thus,  $f(X)$  is compact. ■

**Remark 2.9.** Recall the “extreme value theorem”: let  $f : [a, b] \rightarrow \mathbb{R}$  a continuous function; then, a minimum and maximum is obtained for  $f(x)$  on this interval for values in this interval.

↪ **Theorem 2.6**

Let  $X$  compact, and  $f : X \rightarrow \mathbb{R}$  a continuous function. Then,

$$\max_{x \in X} f(x) \text{ and } \min_{x \in X} f(x)$$

are both attained.

Proof.  $f(X) \subseteq \mathbb{R}$  is compact by theorem 2.5, and so by theorem 2.2,  $f(X)$  is closed and bounded. Let, then,  $m = \inf f(X)$  and  $M = \sup f(X)$ ; these necessarily exist, since  $f(X)$  is bounded. Both  $m$  and  $M$  are limit points of  $f(X)$ . But  $f(X)$  is closed, and hence contains all of its limit points, and thus  $m \in f(X)$  and  $M \in f(X)$ , and thus  $\exists y_m : f(y_m) = m$  and  $y_M : f(y_M) = M$ . ■

↪ **Definition 2.16: Path Connected**

A set  $A \subseteq X$  is called *path connected* if  $\forall x, y \in A, \exists f : [a, b] \rightarrow X$ , continuous, s.t.  $f(a) = x, f(b) = y$  and  $f([a, b]) \subseteq A$ .

The set  $\{f(t) : a \leq t \leq b\}$  is called a *path* from  $x$  to  $y$ .

↪ **Theorem 2.7: Path connected  $\implies$  connected**

If  $A \subseteq X$  is path connected, then  $A$  is connected.

Proof. Suppose, seeking a contradiction, that  $A$  is path connected, but not connected. Then, we can write  $A \subseteq U \cup V$ , for open, disjoint, nonempty subsets  $U, V \subseteq X$ .

Let  $x \in U \cap A$  and  $y \in V \cap A$ . Then,  $\exists f : [a, b] \rightarrow A$  s.t.  $f(a) = x, f(b) = y$ , and  $f([a, b]) \subseteq A$ , by the path connectedness of  $A$ . Then,

$$[a, b] \subseteq f^{-1}(A) \subseteq \underbrace{f^{-1}(U \cap A)}_{\text{open}} \cup \underbrace{f^{-1}(V \cap A)}_{\text{open}} =: \underbrace{U_1}_{a \in} \cup \underbrace{U_2}_{b \in},$$

that is,  $[a, b]$  is contained in a union of open, nonempty, disjoint sets, contradicting  $[a, b]$  the connectedness of  $[a, b]$  by lemma 2.3. Thus,  $A$  is connected. ■

**Remark 2.10.** A counterexample to the opposite side of the implication is the Topologist's sine curve, the set

$$\{(x, \sin(\frac{1}{x})) : x \in (0, 1]\} \cup \{0\} \times [-1, 1].$$

This set is connected in  $\mathbb{R}^2$ , but is not path connected.

### ↪ **Proposition 2.18**

For open sets in  $\mathbb{R}^n$ , path connected  $\iff$  connected.

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## 2.8 Path Components, Connected Components

**Remark 2.11.** Remark that if a metric space  $X$  is not connected, then we can write  $X = U \cup V$  where  $U, V$  are open, nonempty and disjoint. It follows, then, that  $U = V^C$  (and vice versa) and hence  $U, V$  are both open and closed.

### ↪ **Definition 2.17: Connected Component**

A connected component of  $x \in X$  is the largest connected subset of  $X$  that contains  $x$ .

#### ⊛ **Example 2.6**

Let  $X = (0, 1) \cup (1, 2)$ . Here, we have two connected components,  $(0, 1)$  and  $(1, 2)$

#### ⊛ **Example 2.7: Middle Thirds Cantor Set**

Let  $C_0 := [0, 1]$ , and given  $C_n$ , define  $C_{n+1} := \frac{1}{3}(C_n \cup (2 + C_n))$  for  $n \geq 0$ .  $C_\infty$  is totally disconnected.

### ↪ **Definition 2.18: Path Component**

A path component  $P(x)$  of  $x \in X$  is the largest path connected subset of  $X$  that contains  $x$ .

↪ **Proposition 2.19**

$P(x) = \{x \in X : \exists \text{ continuous path } \gamma : [0, 1] \rightarrow X : \gamma(0) = x, \gamma(1) = y\}.$

**Remark 2.12.** Where we “start” a path does not matter. We write  $x \sim y$  if  $\exists \gamma$  from  $x$  to  $y$ ; this is an equivalence relation on the elements of  $X$ .

**Remark 2.13.** The choice of  $[0, 1]$  here is arbitrary; any closed interval is homeomorphic.

↪ **Lemma 2.4**

If  $P(x) \cap P(y) \neq \emptyset$ , then  $P(x) = P(y)$ .

Proof.  $P(x) \cap P(y) \neq \emptyset \implies \exists z : x \sim z \wedge y \sim z \implies x \sim y.$  ■

↪ **Lemma 2.5**

If  $A \subseteq X$  is connected, then  $\overline{A}$  is also connected.

↪ **Lemma 2.6**

Suppose  $A \subseteq X$  is both open and closed. Then, if  $C \subseteq X$  is connected and  $C \cap A \neq \emptyset$ , then  $C \subseteq A$ .

Proof. If  $A$  is both open and closed, then  $C \cap A$  is both open and closed in  $C$ . If  $C \cap A^c \neq \emptyset$ , then this is also open and closed in  $C$ . Hence, we can write  $C = (C \cap A) \cup (C \cap A^c)$ , that is, a disjoint union of two nonempty open sets, contradicting the connectedness of  $C$ . Hence,  $C \cap A^c = \emptyset$ , and so  $C \subseteq A$ . ■

↪ **Proposition 2.20**

Let  $\{C_\alpha\}_{\alpha \in I}$  be a collection of nonempty connected subspaces of  $X$  s.t.  $\forall \alpha, \beta \in I, C_\alpha \cap C_\beta \neq \emptyset$ . Then,  $\cup_{\alpha \in I} C_\alpha$  is connected.

↪ **Proposition 2.21**

Suppose each  $x \in X$  has a path-connected neighborhood. Then, the path components in  $X$  are the same as the connected components in  $X$ .

## 2.8.1 Cantor Staircase Function

↪ **Definition 2.19: An Explicit Definition**

Let  $x \in C : x = 0.a_1a_2a_3 \dots$  (base 3), ie  $a_j = \begin{cases} 0 \\ 2 \end{cases}$ . Define

$$f(x) = \begin{cases} \sum \frac{a_j/2}{2^j} & x \in C \\ \text{extend by continuity} & x \notin C. \end{cases}$$

That is, if  $x \notin C$ , set  $f(y) = \sup_{x \in C, x < y} f(x) = \inf_{x \in C, x > y} f(x)$ .

↪ **Definition 2.20: Complement Definition**

To construct the complement of the Cantor set, begin with  $[0, 1]$  and at a step  $n$ , we remove  $2^n$  open intervals from this interval.  $f(x)$  will be constant on each of these intervals with values  $\frac{k}{2^n}$  where  $k$  odd and  $0 < k < 2^n$ . Extend by continuity to all  $x \in C$ .

**Remark 2.14.** *Wikipedia's explanation of this is far better than whatever this definition is trying to say.*

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## 3 $L^p$ Spaces

### 3.1 Review of $\ell^p$ Norms

**Remark 3.1.** Recall that for  $1 \leq p \leq +\infty$ , we define for  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  the norm

$$\|x\|_p = (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}, \quad \|x\|_\infty = \max_{i=1}^n |x_i|.$$

Similarly, for infinite vector spaces, we had, for  $x = (x_1, \dots, x_n, \dots)$ , the norm

$$\|x\|_p = \left( \sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}, \quad \|x\|_\infty = \sup_{i \geq 1} |x_i|.$$

Here, we define

$$\ell_p := \{x = (x_1, \dots, x_n) : \|x\|_p < +\infty\}.$$

### 3.2 $\ell^p$ Norms, Hölder-Minkowski Inequalities

#### ↪ **Definition 3.1: Hölder Conjugates**

For  $1 \leq p, q \leq +\infty$ , we say that  $p, q$  are said to be *Hölder conjugates* if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

**Remark 3.2.** We refer to these simply as “conjugates” throughout as no other conception of conjugate numbers will be discussed.

Further, we take by convention  $\frac{1}{\infty} = 0$ .

#### ↪ **Proposition 3.1: Hölder’s Inequality**

Let  $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$ . Suppose  $p, q : 1 \leq p, q \leq +\infty$  are conjugate. Then,

$$\langle x, y \rangle_{\mathbb{R}^n} := \left| \sum_{i=1}^n x_i y_i \right| \leq \|x\|_p \cdot \|y\|_q$$

#### ⊛ **Example 3.1**

For the case  $p = 1$  or  $\infty$  (functionally, the same case):

#### ↪ **Lemma 3.1**

Let  $p, q$  be conjugates, and  $x, y \geq 0$ . Then,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

**Remark 3.3.** If the inequality holds, then, for some  $t > 0$ , let  $\tilde{x} = t^{\frac{1}{p}} \cdot x$ ,  $\tilde{y} = t^{\frac{1}{q}} y$ . Substituting  $x$  for  $\tilde{x}$  and  $y$  for  $\tilde{y}$ , we have

$$\text{LHS: } \tilde{x}\tilde{y} = t^{\frac{1}{p}}x \cdot t^{\frac{1}{q}}y = t^{\frac{1}{p}+\frac{1}{q}} \cdot xy = xy$$

$$\text{RHS: } \dots = t\left(\frac{x^p}{p} + \frac{y^q}{q}\right)$$

That is, we have

$$t \cdot xy \leq t \left( \frac{x^p}{p} + \frac{y^q}{q} \right),$$

hence, the inequality is preserved under multiplication by a positive scalar; moreover, the original inequality holds iff this “scaled” version holds. Hence, choosing  $t$  such that  $\tilde{y} = 1$  (let  $t = \left(\frac{1}{y}\right)^q$ ), it suffices to prove the lemma for  $y = 1$ .

Proof. If  $x = 0$  or  $y = 0$ , then the entire LHS becomes 0 and we are done; assume  $x, y > 0$ ; by the previous remark, assume wlog  $y = 1$ . Then, we have

$$\begin{aligned} x \cdot y \leq \frac{x^p}{p} + \frac{y^q}{q} &\iff x \cdot 1 \leq \frac{x^p}{p} + \frac{1}{q} \\ &\iff \frac{x^p}{p} - x + \frac{1}{q} =: f(x) \geq 0. \end{aligned}$$

Taking the derivative, we have

$$\begin{aligned} f'(x) &= \frac{px^{p-1}}{p} - 1 = x^{p-1} - 1 \\ p > 1 &\implies p - 1 > 0 \implies \begin{cases} f'(x) > 0 & \forall x > 1 \\ f'(x) = 0 & x = 1 \\ f'(x) < 0 & \forall 0 < x < 1 \end{cases} \end{aligned}$$

Hence,  $x = 1$  is a local minimum of the function, and thus  $f(x) \geq f(1) \forall 0 < x \leq 1$ . But  $f(1) = \frac{1^p}{p} - 1 + \frac{1}{q} = 1 - 1 = 0$ , hence  $f(x) \geq 0 \forall x \geq 0$ , as desired, and the inequality holds. ■

Proof. Assume  $\|x\|_p = \|y\|_q = 1$ . Then,

$$\begin{aligned}
\left| \sum_{i=1}^n x_i y_i \right| &\leq \sum_{i=1}^n |x_i y_i| && \text{(by triangle inequality)} \\
&\leq \sum_{i=1}^n \left| \frac{x_i^p}{p} + \frac{y_i^q}{q} \right| && \text{(by lemma 3.1)} \\
&= \frac{1}{p} \left( \sum_{i=1}^n |x_i|^p \right) + \frac{1}{q} \left( \sum_{i=1}^n |y_i|^q \right) \\
&= \frac{1}{p} \|x\|_p^p + \frac{1}{q} \|y\|_q^q && \text{(by staring)} \\
&= \frac{1}{p} \cdot 1^p + \frac{1}{q} \cdot 1^q = \frac{1}{p} + \frac{1}{q} = 1 && \text{(by assumption)} \\
&= \|x\|_p \cdot \|y\|_q,
\end{aligned}$$

and the proposition holds, in the special case  $\|x\|_p = \|y\|_q = 1$ .

If  $\|x\|_p = 0$  or  $\|y\|_q = 0$ , then  $x_1 = \dots = x_n = 0$  or  $y_1 = \dots = y_n = 0$ , resp, then we'd have ( $\|x\|_p = 0$  case)

$$0 \cdot y_1 + \dots + 0 \cdot y_n \leq 0,$$

which clearly holds.

Assume, then,  $\|x\|_p > 0, \|y\|_q > 0$ . Let  $\tilde{x} := \frac{x}{\|x\|_p}, \tilde{y} := \frac{y}{\|y\|_q}$ . Then,

$$\|\tilde{x}\|_p^p = \frac{(\sum_{i=1}^n |x_i|^p)}{\|x\|_p^p} = \frac{\|x\|_p^p}{\|x\|_p^p} = 1 \implies \|\tilde{x}\|_p = 1.$$

The same case holds for  $\tilde{y}$ , hence  $\|\tilde{y}\|_q = 1$ ; that is, we have “rescaled” both vectors. Hence, we can use the case we proved above for when the norms were identically 1 on  $\tilde{x}, \tilde{y}$ . We have:

$$\left| \sum_{i=1}^n \tilde{x}_i \tilde{y}_i \right| \leq 1$$

But by definition of  $\tilde{x}, \tilde{y}$ , we have

$$\left| \sum_{i=1}^n \tilde{x}_i \tilde{y}_i \right| = \left| \frac{1}{\|x\|_p \|y\|_q} \sum_{i=1}^n x_i y_i \right| \leq 1 \implies \left| \sum_{i=1}^n x_i y_i \right| \leq \|x\|_p \cdot \|y\|_q,$$

and the proof is complete. ■

### ↪ **Proposition 3.2: Minkowski Inequality**

Let  $1 \leq p \leq \infty, x, y \in \mathbb{R}^n$ . Then,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

**Remark 3.4.** This is just the triangle inequality for  $\ell_p$  norms.



Proof. The cases  $p = 1, \infty$  are left as an exercise.

Assume  $1 < p < \infty$ . Then,

$$\begin{aligned}
 \|x + y\|_p^p &= \sum_{j=1}^n |x_j + y_j|^p = \sum_{j=1}^n |x_j + y_j| |x_j + y_j|^{p-1} \\
 &\leq \sum_{j=1}^{\infty} (|x_j| + |y_j|) \cdot |x_j + y_j|^{p-1} \\
 &= \underbrace{\sum_{j=1}^n |x_j| \cdot |x_j + y_j|^{p-1}}_{:=A} + \overbrace{\sum_{j=1}^n |y_j| \cdot |x_j + y_j|^{p-1}}^{:=B} \quad \circledast
 \end{aligned}$$

Let  $\vec{u} = (|x_1|, \dots, |x_n|)$  and  $\vec{v} = (|x_1 + y_1|^{p-1}, \dots, |x_n + y_n|^{p-1})$ , then,  $A = \vec{u} \cdot \vec{v} = \langle \vec{u}, \vec{v} \rangle_{\mathbb{R}^n}$ . We have

$$\begin{aligned}
 \|\vec{u}\|_p &= \left( \sum_{i=1}^n (|x_i|^p) \right)^{\frac{1}{p}} = \|x\|_p \\
 \|\vec{v}\|_q &= \left( \sum_{i=1}^n (|x_i + y_i|^{p-1})^q \right)^{\frac{1}{q}} \\
 &= \left[ \sum_{i=1}^n (|x_i + y_i|^{p-1})^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}} \\
 &= \left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\
 &= \|x + y\|_p^{p-1}
 \end{aligned}$$

where the second-to-last line follows from  $p, q$  being conjugate, hence  $q = \frac{p}{p-1}$ . Thus, by Hölder's Inequality, we have that

$$A = \langle \vec{u}, \vec{v} \rangle \leq \|\vec{u}\|_p \cdot \|\vec{v}\|_q = \|x\|_p \cdot \|x + y\|_p^{p-1}.$$

By a similar construction, we can show that

$$B \leq \|y\|_p \cdot \|x + y\|_p^{p-1}.$$

Thus, returning to our original inequality in  $\circledast$ , we have

$$\begin{aligned}
 \|x + y\|_p^p &\leq A + B \\
 &\leq \|x\|_p \cdot \|x + y\|_p^{p-1} + \|y\|_p \cdot \|x + y\|_p^{p-1} \\
 \implies \|x + y\|_p &\leq \|x\|_p + \|y\|_p,
 \end{aligned}$$

and the proof is complete. ■

### 3.3 An Aside on Complete Metric Spaces

#### ↪ **Theorem 3.1**

The sequence of centers of balls with monotonically decreasing radii is a Cauchy sequence in  $X$ .

Proof. Let  $\varepsilon > 0$  and let  $N : \forall j > N, r_j < \varepsilon$ . Then,

$$d(x_j, x_k) < r_{\min(j,k)} = r_j$$

■

#### ↪ **Definition 3.2: Complete Metric Space**

A metric space is complete if every Cauchy sequence converges to a limit in that space.

#### ⊗ **Example 3.2: Examples of Complete Metric Spaces**

1.  $\mathbb{R}$ ,  $p$ -adic integers ( $\mathbb{Z}_p$ )/rationals ( $\mathbb{Q}_p$ ).
2.  $\ell_p = \{x = (x_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |x_i|^p < +\infty\}, 1 \leq p \leq +\infty$
3.  $\ell_{\infty} = \{x = (x_i) : \sup_{i=1}^{\infty} |x_i| < +\infty\}$ .

#### ↪ **Proposition 3.3**

Hölder's Inequality and Minkowski Inequality inequalities hold for infinite sequences. that is,

1. if  $x = (x_i) \in \ell_p$  and  $y = (y_i) \in \ell_q$  with  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$\left| \sum_{i=1}^{\infty} x_i y_i \right| \leq \|x\|_{\ell_p} \|y\|_{\ell_q}.$$

2. if  $x, y \in \ell_p$ , then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

**Remark 3.5.** 2. gives the triangle inequality for the  $\|x\|_p$  norm on  $\ell_p$ .

Moreover,

$$\begin{aligned} \|c \cdot x\|_p &= \|(c_1 x_1, \dots, c_n x_n, \dots)\|_p \\ &= \left( \sum_{i=1}^{\infty} |c x_i|^p \right)^{\frac{1}{p}} = \left( \sum_{i=1}^{\infty} |c|^p |x_i|^p \right)^{\frac{1}{p}} \\ &= (|c|^p)^{\frac{1}{p}} \|x\|_p = c \cdot \|x\|_p \end{aligned}$$

Proof. (of 2.) If  $x, y \in \ell_p$ , we have that  $\sum_{i=1}^{\infty} |x_i|^p < +\infty$ ,  $\sum_{i=1}^{\infty} |y_i|^p < +\infty$ , so  $\exists N > 0 : \sum_{i=N+1}^{\infty} |x_i|^p < \varepsilon$ ,  $\sum_{i=N+1}^{\infty} |y_i|^p < \varepsilon$ . Let  $x_i^{(n)} = (x_1, \dots, x_n, 0, 0, \dots)$  be  $(x)$  truncated after  $n$  (finite) coordinates. This gives

$$\|(x_i + y_i)^{(n)}\|_p \leq \|x_i^{(n)}\|_p + \|y_i^{(n)}\|_p \leq \|x\|_p + \|y\|_p$$

by Minkowski on finite spaces. Taking  $n \rightarrow \infty$  (ie, “detruncating”), we have  $(x + y) \in \ell_p$ , and thus  $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ .

1. left as an exercise. ■

### ↪ Proposition 3.4

Let  $1 \leq p \leq +\infty$ , and  $\|x\|_{\infty} = \sup_{i=1}^{\infty} |x_i| = A < +\infty$ ,  $\|y\|_{\infty} = \sup_{i=1}^{\infty} |y_i| = B < +\infty$ . Then, the triangle inequality  $\|x + y\|_{\infty} \leq \|x\|_{\infty} + \|y\|_{\infty}$  holds.

Proof. We have

$$\sup_{i=1}^{\infty} |x_i + y_i| \leq \sup_{i=1}^{\infty} (|x_i| + |y_i|) \leq \sup_{i=1}^{\infty} |x_i| + \sup_{i=1}^{\infty} |y_i| = \|x\|_{\infty} + \|y\|_{\infty}.$$

### ↪ Proposition 3.5

$\|x\|_{\infty} := \sup_{i=1}^{\infty} |x_i|$  is a well-defined norm on  $\ell_{\infty}$ .

Proof. The triangle inequality is prove in proposition 3.4. The remainder of the requirements are left as an exercise. ■

### ↪ Proposition 3.6

$\ell_p \subseteq \ell_q$  if  $p < q$ .

Proof. Let  $x \in \ell_p$ . If  $\sum_{i=1}^{\infty} |x_i|^p < +\infty$ , then  $\exists N : \forall i \geq N, |x_i| \leq 1$ . Then,

$$\begin{aligned} \sum_{i \geq N} |x_i|^q &\leq \sum_{i \geq N} |x_i|^p < \infty \\ \implies \sum_{i=1}^{\infty} |x_i|^q &< +\infty \implies x \in \ell_q \\ &\implies \ell_p \subseteq \ell_q \end{aligned}$$

### 3.4 Contraction Mapping Theorem

#### ↪ Definition 3.3: Contraction Mapping

Let  $(X, d)$  be a metric space. A *contraction mapping* on  $X$  is a function  $f : X \rightarrow X$  for which  $\exists$  a constant  $0 < c < 1$  such that

$$d(f(x), f(y)) \leq c \cdot d(x, y) \quad \forall x, y \in X.$$

#### ↪ Theorem 3.2: Contraction Mapping Theorem

Let  $(X, d)$  be a complete metric space, and let  $f : X \rightarrow X$  be a contraction. Then, there exists a unique fixed point  $z$  of  $f$  such that  $f(z) = z$ .

Moreover,  $f^{[n]}(x) := f \circ f \circ \cdots \circ f(x) \rightarrow z$  as  $n \rightarrow \infty$  for any  $x \in X$ .

**Remark 3.6.** The “functional construction” of the Cantor set is an example of a contraction mapping, with  $f_1(x) = \frac{x}{3}$ ,  $f_2(x) = \frac{x+2}{3}$ . The first has a fixed point of 0, and the second a fixed point of 1.

**Remark 3.7.** This is a generalization of *this proof* done in Analysis I, an equivalent claim over the reals.

*Proof.* Fix  $x \in X$ . Consider the sequence  $\{x_0, x_1, x_2, \dots, x_n, \dots\} := \{x, f(x), f \circ f(x), \dots, f^{[n]}(x), \dots\}$  (we call  $f^{[n]}$  the *orbit* of  $x$  under iterations of  $f$ ). We claim that this is a Cauchy sequence. Let  $n \in \mathbb{N}$  arbitrary, then we have, by the property of the contraction mapping,

$$d(f^{[n+1]}(x) - f^{[n]}(x)) \leq c \cdot d(f^{[n]}(x) - f^{[n-1]}(x)) \leq c^2 d(f^{[n-1]}(x) - f^{[n-2]}(x)).$$

Arguing inductively, it follows that

$$d(f^{[n+1]}(x) - f^{[n]}(x)) \leq c^n d(f(x), x). \quad \star$$

Let now  $m, k \in \mathbb{N}$ ,  $m, k > 0$ . It follows that

$$\begin{aligned} d(f^{[m]}, f^{[m+k]}(x)) &\leq d(f^{[m]}(x), f^{[m+1]}(x)) + d(f^{[m+1]}(x), f^{[m+2]}(x)) + \cdots + d(f^{[m+k-1]}(x), f^{[m+k]}(x)) \\ &\stackrel{\star}{\leq} d(x, f(x)) [c^m + c^{m+1} + \cdots + c^{m+k-1}] \\ &\leq c^m d(x, f(x)) [1 + c + \cdots + c^k + c^{k+1} + \cdots] = \frac{c^m d(x, f(x))}{1 - c} \end{aligned}$$

Now, given  $\varepsilon > 0$ , choose  $N$  such that  $\frac{c^N d(x, f(x))}{1 - c} < \varepsilon$ . It follows, then, that  $\{f^{[n]}(x)\}_{n \in \mathbb{N}}$  a Cauchy sequence, and thus converges,  $f^{[n]}(x) \rightarrow z$  as  $n \rightarrow \infty$  for some  $z$ .

We further have to show that  $f(z) = z$ . It is easy to show that  $f$  continuous due to the contraction mapping (it is clearly Lipschitz with constant  $c$ ), and it thus follows that

$$\lim_{n \rightarrow \infty} f(f^{[n]}(x)) = \lim_{n \rightarrow \infty} f^{[n]}(x) \implies f(z) = z,$$

by sequential characterization of continuous functions.

Finally, we need to show that this limit is unique. Suppose  $\exists y_1 \neq y_2$ , ie two fixed points with  $f(y_1) = y_1$  and  $f(y_2) = y_2$ . Then, by the property of the contraction mapping,

$$d(f(y_1), f(y_2)) \leq c \cdot d(y_1, y_2),$$

but by assumption of being fixed points,

$$d(f(y_1), f(y_2)) = d(y_1, y_2),$$

implying  $d(y_1, y_2) \leq c \cdot d(y_1, y_2)$ . This is only possible if  $d(y_1, y_2) = 0$ , and thus  $y_1 = y_2$  and the fixed point is indeed unique. ■

### ↪ **Theorem 3.3: $\ell_p$ complete**

The space  $\ell_p$  is complete for all  $1 \leq p \leq +\infty$ .

Equivalently, if  $(x^1), (x^2), \dots, (x^n)$  is a Cauchy sequence in  $\ell^p$ ,  $\exists y \in \ell^p$  s.t.  $x^n \rightarrow y$  as  $n \rightarrow \infty$ .

Proof. (Sketch) We suppose first  $p < +\infty$ . Consider an arbitrary number of Cauchy sequences in  $\ell_p$ :

$$\begin{aligned} x^{(1)} &= (x_1^{(1)}, \dots, x_n^{(1)}, \dots) \\ x^{(2)} &= (x_1^{(2)}, \dots, x_n^{(2)}, \dots) \\ &\vdots \quad \quad \quad \vdots \\ x^{(k)} &= (x_1^{(k)}, \dots, x_n^{(k)}, \dots) \in \ell_p \end{aligned}$$

We claim that, for any  $k \in \mathbb{N}$ , the  $(x_k^{(n)})_{n \in \mathbb{N}}$  is a Cauchy sequence; note that in this definition we are taking a *fixed-index* (namely, the  $k$ th) element from different sequences (namely, the  $n$ th sequence).

Since  $x^{(1)}, x^{(2)}, \dots, x^{(n)}, \dots$  are Cauchy sequences in  $\ell^p$ , we have for a fixed  $\varepsilon > 0$ ,  $\exists N \in \mathbb{N} : \forall m, n > N$ ,  $d_p(x^{(m)}, x^{(n)}) < \varepsilon$ :

$$\begin{aligned} d_p(x^{(m)}, x^{(n)})^p &= \|x^{(m)} - x^{(n)}\|_p^p = \sum_{i=1}^{\infty} |x_i^{(m)} - x_i^{(n)}|^p < \varepsilon^p \\ |x_k^{(m)} - x_k^{(n)}|^p &\leq \sum_{i=1}^{\infty} |x_i^{(m)} - x_i^{(n)}|^p \implies |x_k^{(m)} - x_k^{(n)}|^p < \varepsilon^p \\ &\implies |x_k^{(m)} - x_k^{(n)}| < \varepsilon, \end{aligned}$$

since we are taking “less of the summands in the second line”. It follows, then, that for each  $k$ ,  $\exists z_k : x_k^{(n)} \rightarrow z_k$  as  $n \rightarrow \infty$ . Let  $z = (z_1, \dots, z_n, \dots)$ . We claim that  $x^{(n)} \rightarrow z \in \ell_p$  as  $n \rightarrow \infty$ .

First, we show that  $d_p(x^{(n)}, z) \rightarrow 0$  as  $n \rightarrow \infty$  (that is,  $x^{(n)} \rightarrow z$  as  $n \rightarrow \infty$ ). Fix  $\varepsilon > 0$ , and choose  $N \in \mathbb{N}$  for

which  $d_p(x^{(m)}, x^{(n)}) < \varepsilon \ \forall m, n \geq N$  (by Cauchy). Fix  $K \in \mathbb{N}, K > 0$ .

$$d_p^p(x^{(n)}, z) = \|x^{(n)} - z\|_p^p = \sum_{i=1}^{\infty} |x_i^{(n)} - z_i|^p$$

$$\|x^{(m)} - x^{(n)}\|_p^p < \varepsilon^p \implies \sum_{i=1}^K |x_i^{(m)} - x_i^{(n)}|^p \leq \varepsilon^p$$

Let  $m \rightarrow \infty$ ; then  $x_i^{(m)} \rightarrow z_i$  (note that  $i$  fixed!), and we have

$$\sum_{i=1}^K |z_i - x_i^{(n)}|^p \leq \varepsilon^p.$$

Let  $K \rightarrow \infty$ ; then,

$$\sum_{i=1}^{\infty} |z_i - x_i^{(n)}|^p \leq \varepsilon^p \implies \|z - x^{(n)}\|_p \leq \varepsilon \implies d_p(z, x^{(n)}) \leq \varepsilon,$$

and thus  $x^{(n)} \rightarrow z$  as  $n \rightarrow \infty$ .

It remains to show that  $z \in \ell_p$ , ie  $\|z\|_p < +\infty$ . We have:

$$\|z\|_p \leq \underbrace{\|z - x^{(n)}\|_p}_{\rightarrow 0} + \|x^{(n)}\|_p.$$

For sufficiently large  $n$ ,  $\|z - x^{(n)}\| \leq 1$  (for instance);  $x^{(n)} \in \ell_p$ , hence  $\|x^{(n)}\|_p < +\infty$  (say,  $\|x^{(n)}\|_p \leq M$ ). Thus:

$$\|z\|_p \leq 1 + M < +\infty \implies z \in \ell_p,$$

and the proof is complete. ■

### 3.5 Compactness in Metric Spaces

#### ↪ **Definition 3.4: Totally Bounded**

Let  $(X, d)$  be a metric space. If for every  $\varepsilon > 0$ ,  $\exists x_1, \dots, x_n \in X, n = n(\varepsilon) : \bigcup_{i=1}^n B(x_i, \varepsilon) = X$ , we say  $X$  is *totally bounded*.

↪ **Theorem 3.4**

Let  $(X, d)$  be a metric space. TFAE:

1.  $X$  is complete and totally bounded;
2.  $X$  is compact;
3.  $X$  is sequentially compact (every sequence has a convergent subsequence).

*Proof.* (1.  $\implies$  2.) Suppose  $X$  complete and totally bounded. Assume towards a contradiction that  $X$  not compact, ie there exists an open cover  $\{U_\alpha\}_{\alpha \in I}$  of  $X$  with no finite subcover.

$X$  being totally bounded gives that it can be covered by finitely many open balls of radius  $\frac{1}{2}$ . It must be that at least one of these open balls cannot be finitely covered, otherwise we would have a finite subcover. Let  $F_1$  be the closure of this ball.  $F_1$  closed, with diameter  $\text{diam}(F_1) \leq 1$ .  $X$ .

We also have that  $X$  can be covered by finitely many balls of radius  $\frac{1}{4}$ ; again, there must be at least one ball  $B_1$  such that  $B_1 \cap F_1$  cannot be covered by finitely many open sets from the cover. Let  $F_2 = \overline{B_1} \cap F_1$ -closed, with  $\text{diam}(F_2) \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ .<sup>4</sup>

Arguing inductively, at some step  $n$ ,  $X$  can be covered by finitely many balls of radius  $\frac{1}{2^n}$ ; at least one of these balls  $B$  cannot be covered by a finite subcover hence  $B \cap F_{n-1}$  cannot be covered by finitely many  $U_\alpha$ 's. Let  $F_n = \overline{B} \cap F_{n-1}$ -closed, with  $\text{diam}(F_n) \leq \frac{1}{2^{n-1}}$ .

As such, we have a nested sequence  $F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$  of closed sets, where  $\text{diam}(F_k) \leq \frac{1}{2^{k-1}} \rightarrow 0$  as  $k \rightarrow \infty$ .

↪ **Lemma 3.1** (Cantor Intersection Theorem).  $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$ .

*Proof.* (Of Lemma) Let  $x_k \in F_k$ . Then,  $\{x_k\}_{k \in \mathbb{N}}$  is a Cauchy sequence, since

$$d(x_n, x_{n+k}) \leq \text{diam}(F_n) + \dots + \text{diam}(F_{n+k}) \leq \frac{1}{2^{n-1}},$$

by the nested property, which can be made arbitrarily small for sufficiently large  $n, k$ . Hence,  $x_n \rightarrow y \in X$  for some  $y$ , as  $X$  complete. The tail of  $x_n$  lies in  $F_n$  for all sufficiently large  $n$ , and as each  $F_n$  closed, the limit must lie in  $F_n$  for all sufficiently large  $n$ . We conclude the intersection nonempty. ■

This  $y$  from the lemma is covered by some  $U_{\alpha_0}$ -open for some  $\alpha_0 \in I$ . Being open,  $\exists \varepsilon > 0 : B(y, \varepsilon) \subseteq U_{\alpha_0}$ . Let  $n : \frac{1}{2^{n-1}} < \varepsilon$ . Then,  $y \in F_n$ , and as  $\text{diam}(F_n) \leq \frac{1}{2^{n-1}}$ , we have that  $F_n \subseteq B(y, \frac{1}{2^{n-1}}) \subseteq B(y, \varepsilon) \subseteq U_{\alpha_0}$ . But then, we have that  $F_n$  covered by a single open set  $U_{\alpha_0}$ , a contradiction to our inductive construction of  $F_n$ . We conclude  $X$  compact.

(2.  $\implies$  3.) Suppose  $X$  compact. Let  $\{x_n\}_{n \in \mathbb{N}} \in X$ . Let  $F_n = \overline{\bigcup_{k \geq n} \{x_k\}}$ -closed; we have too that  $F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$ .

↪ **Definition 3.5: Finite Intersection Property**

$\mathcal{F}$  has finite intersection property provided any finite subcollection of sets in  $\mathcal{F}$  has a non-empty intersection.

<sup>4</sup> $B_1$  has radius  $\frac{1}{4}$  and hence diameter  $\frac{1}{2}$ . The intersection of  $B_1$  with a set with a larger diameter must have diameter  $\leq \frac{1}{2}$ .

↪ **Lemma 3.2** (Finite Intersection Formulation of Compactness).  $X$ -compact  $\iff$  every collection  $\mathcal{F}$  of closed subsets of  $X$  with finite intersection property has non-empty intersection.

Proof. ■

This lemma directly gives that  $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$ ,  $\{F_n\}_{n \in \mathbb{N}}$  being a collection of closed subsets with any subset having nonempty intersection (by the nestedness). Let  $y \in \bigcap_{n=1}^{\infty} F_n$ . Take  $B(y, \frac{1}{k})$ , which thus has nonempty intersection with  $\{x_k\}_{k \geq n} \forall n$ , ie  $\exists n_1 : d(y, x_{n_1}) < 1$  and  $\exists n_2 > n_1 : d(y, x_{n_2}) < \frac{1}{2}$ . Arguing inductively,  $\exists n_j > n_{j-1} : d(y, x_{n_j}) < \frac{1}{j}$  for any given  $n_{j-1}$ . It follows that  $\lim_{j \rightarrow \infty} x_{n_j} = y$ , and thus  $\{x_{n_j}\}$  is a convergent subsequence of  $\{x_n\}$  that converges within  $X$ , and thus  $X$  is sequentially compact.

(3.  $\implies$  1.) Suppose  $X$  sequentially compact. Let  $\{x_n\} \in X$  be a Cauchy sequence in  $X$ , which thus have a convergent subsequence  $\{x_{n_k}\} \rightarrow y$ .

↪ **Lemma 3.3.** Let  $\{x_n\}$  be a Cauchy sequence in  $X$  where  $X$  sequentially compact. Then, if  $\{x_{n_k}\} \rightarrow y$ , so does  $\{x_n\} \rightarrow y$

Proof. ■

Then,  $\{x_n\}_n \rightarrow y$  and so  $X$  complete.

Suppose  $X$  not totally bounded, ie  $\exists \varepsilon > 0 : X$  cannot be covered by a finite union of balls of  $B(x_j, \varepsilon)$ . Let  $x_1 \in X$  s.t.  $B(x_1, \varepsilon) \not\supseteq X$ ;  $\exists x_2 \in X \setminus B(x_1, \varepsilon)$ , and so  $X \not\subseteq B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$  by assumption. Then, choose  $x_3 \in X \setminus (B(x_1, \varepsilon) \cup B(x_2, \varepsilon))$ . Arguing inductively, we have that  $\exists x_n \in X \setminus (\bigcup_{i=1}^n B(x_i, \varepsilon))$ , noting that  $d(x_n, x_j) \geq \varepsilon \forall 1 \leq j \leq n$ .

Consider the sequence  $\{x_j\}_{j \in \mathbb{N}}$ :

↪ **Lemma 3.4.**  $\{x_j\}$  cannot have a convergent subsequence.

Proof. Follows by  $d(x_m, x_n) \geq \varepsilon \forall m, n$ . ■

This contradicts our assumption that  $X$  sequentially compact, and we conclude  $X$  must be totally bounded. ■



⊛ **Example 3.3: Complete Metric Space Example:  $L^p$  norm**

Let  $f \in C([a, b])$ . We define the norm

$$\|f\|_p := \left( \int_a^b |f(x)|^p dx \right)^{\frac{1}{p}}.$$

As desired,  $\|f\|_p \geq 0$ ;  $\|f\|_p = 0 \iff f \equiv 0$ ;  $\|c \cdot f\|_p = c \cdot \|f\|_p$ .

Hölder's and Minkowski's inequalities for functions also hold; for  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 \leq p, q \leq \infty$ ,

$$\int |fg| \leq \|f\|_p \cdot \|g\|_q; \quad \|f + g\|_p \leq \|f\|_p + \|g\|_p,$$

respectively.

We similarly have the  $L^\infty$  norm, namely, for a function  $f : [a, b] \rightarrow \mathbb{R}$ ,

$$\|f\|_\infty = \sup_{x \in [a, b]} |f(x)|,$$

which obeys all the necessary properties as well.

Let  $f_n \rightarrow f$  in  $C([a, b])$ , wrt  $\|\cdot\|_\infty$ , where  $\{f_n\}_{n \in \mathbb{N}}$  a sequence of functions. Namely, we say that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \sup_{x \in [a, b]} |f_n(x) - f(x)| < \varepsilon.$$

If this holds, we say that  $f_n$  *uniformly converges*.

We say that  $f_n(x) \rightarrow f(x)$  *pointwise* on  $[a, b]$  if  $\forall x \in [a, b], f_n(x) \rightarrow f(x)$ . Note that uniform convergence implies pointwise convergence, but not the converse.

↪ **Theorem 3.5**

Suppose  $f_n(x)$  continuous, and  $f_n(x) \rightarrow f(x)$  uniformly on  $[a, b]$ . Then,  $f(x)$  also continuous on  $[a, b]$ .

**Proof.** Fix  $\varepsilon > 0$ ,  $x_0 \in [a, b]$ . We have that  $\exists N : n \geq N, |f_n(x) - f(x)| < \frac{\varepsilon}{3}, \forall x \in [a, b]$ .

Let  $n \geq N$ .  $f_n(x)$  continuous at  $x_0$ , hence  $\exists \delta(x_0) > 0 : |y - x_0| \implies |f_n(y) - f_n(x_0)| < \frac{\varepsilon}{3}$ . We have

$$\begin{aligned} |f(x_0) - f(y)| &\leq |f(x_0) - f_n(x_0)| + |f_n(x_0) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

completing the proof. ■

**Remark 3.8.** This does not hold with pointwise convergence.

**Remark 3.9.** We will prove later that  $C([a, b])$  is complete for  $\|f\|_\infty$ , but not for arbitrary  $\|f\|_p$ ,  $1 \leq p < +\infty$ . To “complete”  $C([a, b])$  for  $p \neq \infty$ , we will need to consider measurable functions and redefine our notion of integration.

