

# MATH454 - Analysis 3

Measure spaces; Integration.

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# §1 SIGMA ALGEBRAS AND MEASURES

## §1.1 A Review of Riemann Integration

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $[a, b] \subset \mathbb{R}$ . Define a **partition** of  $[a, b]$  as the set

$$\text{part}([a, b]) := \{a =: x_0 < x_1 < \dots < x_N =: b\}.$$

We can then define the upper and lower Riemann integrals of  $f$  over the region  $[a, b]$  as

$$\begin{aligned} \text{upper:} \quad \int_a^b f(x) dx &:= \inf_{\text{part}([a, b])} \left\{ \sum_{i=1}^N \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \right\} \\ \text{lower:} \quad \int_a^b f(x) dx &:= \sup_{\text{part}([a, b])} \left\{ \sum_{i=1}^N \inf_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \right\}. \end{aligned}$$

We then say  $f$  **Riemann integrable** if these two quantities are equal, and denote this value by  $\int_a^b f(x) dx$ .

Many “nice-enough” (continuous, monotonic, etc.) functions are Riemann integrable, but many that we would like to be able to “integrate” are simply not, for instance Dirichlet’s function  $x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \cap [a, b] \\ 0 & x \in \mathbb{Q}^c \cap [a, b] \end{cases}$ . Hence, we need a more general notion of integration.

## §1.2 Sigma Algebras

↪ **Definition 1.1** (Sigma algebra): Let  $X$  be a *space* (a nonempty set) and  $\mathcal{F}$  a collection of subsets of  $X$ .  $\mathcal{F}$  a *sigma algebra* or simply  *$\sigma$ -algebra* of  $X$  if the following hold:

1.  $X \in \mathcal{F}$
2.  $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$  (closed under complement)
3.  $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$  (closed under countable unions)

↪ **Proposition 1.1:**

4.  $\emptyset \in \mathcal{F}$
5.  $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
6.  $A_1, \dots, A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
7.  $A, B \in \mathcal{F} \Rightarrow A \setminus B, B \setminus A \in \mathcal{F}$

⊗ **Example 1.1:** The “largest” sigma algebra of a set  $X$  is the power set  $2^X$ , the smallest the trivial  $\{\emptyset, X\}$ .

Given a set  $A \subset X$ , the set  $\mathcal{F}_A := \{\emptyset, X, A, A^c\}$  is a sigma algebra; given two disjoint sets  $A, B \subset X$ , then  $\mathcal{F}_{A,B} := \{\emptyset, X, A, A^c, B, B^c, A \cup B, A^c \cap B^c\}$  a sigma algebra.

↪ **Definition 1.2** (Generating a sigma algebra): Let  $X$  be a nonempty set, and  $\mathcal{C}$  a collection of subsets of  $X$ . Then, the  $\sigma$ -algebra *generated* by  $\mathcal{C}$ , denoted  $\sigma(\mathcal{C})$ , is such that

1.  $\sigma(\mathcal{C})$  a sigma algebra with  $\mathcal{C} \subseteq \sigma(\mathcal{C})$
2. if  $\mathcal{F}'$  a sigma algebra with  $\mathcal{C} \subseteq \mathcal{F}'$ , then  $\mathcal{F}' \supseteq \sigma(\mathcal{C})$

Namely,  $\sigma(\mathcal{C})$  is the smallest sigma algebra “containing” (as a subset)  $\mathcal{C}$ .

↪ **Proposition 1.2:**

1.  $\sigma(\mathcal{C}) = \bigcap \{ \mathcal{F} : \mathcal{F} \text{ a sigma algebra containing } \mathcal{C} \}$
2. if  $\mathcal{C}$  itself a sigma algebra, then  $\sigma(\mathcal{C}) = \mathcal{C}$
3. if  $\mathcal{C}_1, \mathcal{C}_2$  are two collections of subsets of  $X$  such that  $\mathcal{C}_1 \subseteq \mathcal{C}_2$ , then  $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_2)$

↪ **Definition 1.3** (The Borel sigma-algebra): The *Borel  $\sigma$ -algebra*, denoted  $\mathfrak{B}_{\mathbb{R}}$ , on the real line is given by

$$\mathfrak{B}_{\mathbb{R}} := \sigma(\{\text{open subsets of } \mathbb{R}\}).$$

We call sets in  $\mathfrak{B}_{\mathbb{R}}$  *Borel sets*.

↪ **Proposition 1.3:**  $\mathfrak{B}_{\mathbb{R}}$  is also generated by the sets

- $\{(a, b) : a < b \in \mathbb{R}\}$
- $\{(a, b] : a < b \in \mathbb{R}\}$
- $\{[a, b] : a < b \in \mathbb{R}\}$
- $\{[a, b) : a < b \in \mathbb{R}\} \oplus$
- $\{(-\infty, c) : c \in \mathbb{R}\}$
- $\{(-\infty, c] : c \in \mathbb{R}\}$
- etc.

PROOF. We prove just  $\oplus$ . It suffices to show that the generating sets of each  $\sigma$ -algebra is contained in the other  $\sigma$ -algebra. Let  $a < b \in \mathbb{R}$ . Then,

$$(a, b) = \bigcup_{n=1}^{\infty} \underbrace{\left[ a + \frac{1}{n}, b \right)}_{\in \oplus} \in \sigma(\{[a, b)\}) \Rightarrow \mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[a, b)\}).$$

Conversely,

$$[a, b) = \bigcap_{n=1}^{\infty} \left( a - \frac{1}{n}, b \right) \in \mathfrak{B}_{\mathbb{R}}.$$

■

↪ **Proposition 1.4:** All intervals (open, closed, half open, half closed, finite, etc) are Borel sets; any set obtained from countable set operations of intervals are Borel; all singletons are Borel; any finite and countable sets are Borel.

### §1.3 Measures

↪ **Definition 1.4** (Measurable Space): Let  $X$  be a space and  $\mathcal{F}$  a  $\sigma$ -algebra. We call the tuple  $(X, \mathcal{F})$  a *measurable space*.

↪ **Definition 1.5** (Measure): Let  $(X, \mathcal{F})$  be a measurable space. A *measure* is a function  $\mu : \mathcal{F} \rightarrow [0, \infty]$  satisfying

- (i)  $\mu(\emptyset) = 0$ ;
- (ii) if  $\{A_n\} \subseteq \mathcal{F}$  a sequence of (pairwise) disjoint sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

i.e.  $\mu$  is *countably additive*. We further call  $\mu$

- *finite* if  $\mu(X) < \infty$ ,
- a *probability measure* if  $\mu(X) = 1$ ,
- $\sigma$ -finite if  $\exists \{A_n\} \subseteq \mathcal{F}$  such that  $X = \bigcup_{n=1}^{\infty} A_n$  with  $\mu(A_n) < \infty \forall n \geq 1$ ,

and call the triple  $(X, \mathcal{F}, \mu)$  a *measure space*.

⊕ **Example 1.2:** The measure on  $\mathfrak{B}_{\mathbb{R}}$  given by

$$A \mapsto \begin{cases} |A| & \text{if } A \text{ finite} \\ \infty & \text{else} \end{cases}$$

is called the *counting measure*.

Fix  $x_0 \in \mathbb{R}$ , then the measure on  $\mathfrak{B}_{\mathbb{R}}$  given by

$$A \mapsto \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{else} \end{cases}$$

is called the *point mass at  $x_0$* .

↪ **Theorem 1.1** (Properties of Measures): Fix a measure space  $(X, \mathcal{F}, \mu)$ . The following properties hold:

1. (finite additivity) For any sequence  $\{A_n\}_{n=1}^N \subseteq \mathcal{F}$  of disjoint sets,

$$\mu\left(\bigcup_{n=1}^N A_n\right) = \sum_{n=1}^N \mu(A_n).$$

2. (monotonicity) For any  $A \subseteq B \in \mathcal{F}$ , then  $\mu(A) \leq \mu(B)$ .
3. (countable/finite subadditivity) For any sequence  $\{A_n\} \subseteq \mathcal{F}$  (**not** necessarily disjoint),

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

an analogous statement holding for a finite collection of sets  $A_1, \dots, A_N$ .

4. (continuity from below) For  $\{A_n\} \subseteq \mathcal{F}$  such that  $A_n \subseteq A_{n+1} \forall n \geq 1$  (in which case we say  $\{A_n\}$  “increasing” and write  $A_n \uparrow$ ) we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

5. (continuity from above) For  $\{A_n\} \subseteq \mathcal{F}, A_n \supseteq A_{n+1} \forall n \geq 1$  (we write  $A_n \downarrow$ ) we have that if  $\mu(A_1) < \infty$ ,

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

**Remark 1.1:** In 4., note that since  $A_n$  increasing, that the union  $\bigcup_{n=1}^{\infty} A_n \supseteq A_m$  for any arbitrarily large  $m$ ; indeed, one could logically right  $\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} A_n$ . In this notation, then, 4. simply states that we may interchange limit and measure. A similar argument can be viewed for 5. (how?).

**Remark 1.2:** The finiteness condition in 5. may be slightly modified such as to state that  $\mu(A_n) < \infty$  for some  $n$ ; remark why this would suffice to ensure the entire rest of the sequence has finite measure.

PROOF.

1. Extend  $A_1, \dots, A_N$  to an infinite sequence by  $A_n := \emptyset$  for  $n > N$ . Then this simply follows from countable additivity and  $\mu(\emptyset) = 0$ .
2. We may write  $B = A \cup (B \setminus A)$ ; this is a disjoint union of sets. By finite additivity, then,

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A),$$

since the measure is positive.

3. We prove only for a countable union; use the technique from 1. to extend to finite. We first “disjointify” the sequence such that we can use the countable additivity

axiom. Let  $B_1 = A_1, B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$  for  $n \geq 2$ . Remark then that  $\{B_n\} \subseteq \mathcal{F}$  is a disjoint sequence of sets, and that  $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ . By countable additivity and subadditivity,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

4. We again “disjointify” the sequence  $\{A_n\}$ . Put  $B_1 = A_1, B_n = A_n \setminus A_{n-1}$  for all  $n \geq 2$  (remark that this is equivalent to the construction from the previous proof because the sets are increasing). Then, again,  $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ , and in particular, for all  $N \geq 1, \bigcup_{n=1}^N B_n = A_N$ . Then

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \mu(B_n) \\ &= \lim_{N \rightarrow \infty} \mu\left(\bigcup_{n=1}^N B_n\right) = \lim_{N \rightarrow \infty} \mu(A_N). \end{aligned}$$

5. We yet again disjointify, backwards (in a way) from the previous case. Put  $B_n = A_1 \setminus A_n$  for all  $n \geq 1$ . Then,  $\{B_n\} \subseteq \mathcal{F}$ ,  $B_n$  increasing, and  $\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \bigcap_{n=1}^{\infty} A_n$ . Then, by continuity from below,

$$\mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n \rightarrow \infty} \mu(B_n) = \lim_{n \rightarrow \infty} \mu(A_1 \setminus A_n)$$

and also

$$\begin{aligned} \mu(A_1) &= \mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) + \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \\ &= \mu(A_1 \setminus A_n) + \mu(A_n), \end{aligned}$$

and combining these two equalities yields the desired result. ■

## §1.4 Constructing the Lebesgue Measure on $\mathbb{R}$

↪ **Definition 1.6** (Lebesgue outer measure): For all  $A \subseteq \mathbb{R}$ , define

$$m^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n, I_n \text{ open intervals} \right\},$$

called the *Lebesgue outer measure* of  $A$  (where  $\ell(I)$  is the length of interval  $I$ , i.e. the absolute value of the difference of its endpoints, if finite, or  $\infty$  if not).

↪ **Proposition 1.5:** The following properties of  $m^*$  hold:

1.  $m^*(A) \geq 0$  for all  $A \subseteq \mathbb{R}$ , and  $m^*(\emptyset) = 0$ .
2. (monotonicity) For  $A \subseteq B$ ,  $m^*(A) \leq m^*(B)$ .
3. (countable subadditivity) For  $\{A_n\}, A_n \subseteq \mathbb{R}$ ,  $m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n)$ .<sup>1</sup>
4. If  $I \subseteq \mathbb{R}$  an interval, then  $m^*(I) = \ell(I)$ .
5.  $m^*$  is translation invariant; for any  $A \subseteq \mathbb{R}, x \in \mathbb{R}$ ,  $m^*(A) = m^*(A + x)$  where  $A + x := \{a + x : a \in A\}$ .
6. For all  $A \subseteq \mathbb{R}$ ,  $m^*(A) = \inf\{m^*(B) : A \subseteq B \subseteq \mathbb{R}, B \text{ open}\}$ .
7. If  $A = A_1 \cup A_2 \subseteq \mathbb{R}$  with  $d(A_1, A_2) > 0$ ,<sup>2</sup> then  $m^*(A_1) + m^*(A_2) = m^*(A)$ .
8. If  $A = \bigcup_{k=1}^{\infty} J_k$  where  $J_k$ 's are "almost disjoint intervals" (i.e. share at most endpoints), then  $m^*(A) = \sum_{k=1}^{\infty} m^*(J_k) = \sum_{k=1}^{\infty} \ell(J_k)$ .

PROOF.

3. If  $m^*(A_n) = \infty$ , for any  $n$ , we are done, so assume wlog  $m^*(A_n) < \infty$  for all  $n$ . Then, for each  $n$  and  $\varepsilon > 0$ , one can choose open intervals  $\{I_{n,i}\}_{i \geq 1}$  such that  $A_n \subseteq \bigcup_{i=1}^{\infty} I_{n,i}$  and  $\sum_{i=1}^{\infty} \ell(I_{n,i}) \leq m^*(A_n) + \frac{\varepsilon}{2^n}$ . Hence

$$\begin{aligned} \bigcup_{n=1}^{\infty} A_n &\subseteq \bigcup_{n=1, i=1}^{\infty} I_{n,i} \\ \Rightarrow m^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \sum_{n,i=1}^{\infty} \ell(I_{n,i}) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{n,i}) \leq \sum_{n=1}^{\infty} \left(m^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon, \end{aligned}$$

and as  $\varepsilon$  arbitrary, the statement follows.

4. We prove first for  $I = [a, b]$ . For any  $\varepsilon > 0$ , set  $I_1 = (a - \varepsilon, b + \varepsilon)$ ; then  $I \subseteq I_1$  so  $m^*(I) \leq \ell(I_1) = (b - 1) + 2\varepsilon$  hence  $m^*(I) \leq b - a = \ell(I)$ . Conversely, let  $\{I_n\}$  be any open-interval converging of  $I$  (wlog, each of finite length; else the statement holds trivially). Since  $I$  compact, it can be covered by finitely many of the  $I_n$ 's, say  $\{I_n\}_{n=1}^N$ , denoting  $I_n = (a_n, b_n)$  (with relabelling, etc). Moreover, we can pick the  $a_n, b_n$ 's such that  $a_1 < a, b_N > b$ , and generally  $a_n < b_{n-1} \forall 2 \leq n \leq N$ . Then,

$$\begin{aligned} \sum_{n=1}^{\infty} \ell(I_n) &\geq \sum_{n=1}^N \ell(I_n) = b_1 - a_1 + \sum_{n=2}^N (b_n - a_n) \\ &\geq b_1 - a_1 + \sum_{n=2}^N (b_n - b_{n-1}) \\ &= b_N - a_1 \geq b - 1 = \ell(I), \end{aligned}$$

hence since the cover was arbitrary,  $m^*(A) \geq \ell(I)$ , and equality holds.

Now, suppose  $I$  finite, with endpoints  $a < b$ . Then for any  $\frac{b-a}{2} > \varepsilon > 0$ , then

$$[a + \varepsilon, b - \varepsilon] \subseteq I \subseteq [a - \varepsilon, b + \varepsilon],$$

<sup>1</sup>More generally, any set function on  $2^{\mathbb{R}}$  that satisfies 1., 2., and 3. is called an *outer measure*.

<sup>2</sup>Remark: this is a stronger requirement than disjointness!



hence by monotonicity and the previous part of this proof

$$m^*([a + \varepsilon, b - \varepsilon]) = b - a - 2\varepsilon \leq m^*(I) \leq b - a + 2\varepsilon = m^*([a - \varepsilon, b + \varepsilon]),$$

from which it follows that  $m^*(I) = b - a = \ell(I)$ .

Finally, suppose  $I$  infinite. Then,  $\forall M \geq 0, \exists$  closed, finite interval  $I_M$  with  $I_M \subseteq I$  and  $\ell(I_M) \geq M$ . Hence,  $m^*(I) \geq m^*(I_M) \geq M$  and thus as  $M$  arbitrary it must be that  $m^*(I) = \infty = \ell(I)$ .

6. Denote  $\tilde{m}(A) := \inf\{m^*(B) : A \subseteq B \subseteq \mathbb{R}, B \text{ open}\}$ . For any  $A \subseteq B \subseteq \mathbb{R}$  with  $B$  open, monotonicity gives that  $m^*(A) \leq m^*(B)$ , hence  $m^*(A) \leq \tilde{m}(A)$ . Conversely, assuming wlog  $m^*(A) < \infty$  (else holds trivially), then for all  $\varepsilon > 0$ , there exists  $\{I_n\}$  such that  $A \subseteq \bigcup_{n=1}^{\infty} I_n$  with  $\sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon$ . Setting  $B := \bigcup_{n=1}^{\infty} I_n$ , we have that  $A \subseteq B$  and  $m^*(B) = m^*(\bigcup I_n) \leq$  (by finite subadditivity)  $\sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon$  hence  $m^*(B) \leq m^*(A)$  for all  $B$ . Thus  $m^*(A) \geq \tilde{m}(A)$  and equality holds.

7. Put  $\delta := d(A_1, A_2) > 0$ . Clearly  $m^*(A) \leq m^*(A_1) + m^*(A_2)$  by finite subadditivity. wlog,  $m^*(A) < \infty$  (and hence  $m^*(A_i) < \infty, i = 1, 2$ ) (else holds trivially). Then  $\forall \varepsilon > 0, \exists \{I_n\} : A \subseteq \bigcup I_n$  and  $\sum \ell(I_n) \leq m^*(A) + \varepsilon$ . Then, for all  $n$ , we consider a “refinement” of  $I_n$ ; namely, let  $\{I_{n,i}\}_{i \geq 1}$  such that  $I_n \subseteq \bigcup_i I_{n,i}$  and  $\ell(I_{n,i}) < \delta$  and  $\sum_i \ell(I_{n,i}) \leq \ell(I_n) + \frac{\varepsilon}{2^n}$ . Relabel  $\{I_{n,i} : n, i \geq 1\} \rightsquigarrow \{J_m : m \geq 1\}$  (both are countable). Then,  $\{J_m\}$  defines an open-interval cover of  $A$ , and since  $\ell(J_m) < \delta$  for each  $m$ ,  $J_m$  intersects at most one  $A_i$ . For each  $m$  and  $p = 1, 2$ , put

$$M_p := \{m : J_m \cap A_p \neq \emptyset\},$$

noting that  $M_1 \cap M_2 = \emptyset$ . Then  $\{J_m : m \in M_p\}$  is an open covering of  $A_p$ , and so

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\leq \sum_{m \in M_1} \ell(J_m) + \sum_{m \in M_2} \ell(J_m) \\ &\leq \sum_{m=1}^{\infty} \ell(J_m) = \sum_{n,i=1}^{\infty} \ell(I_{n,i}) \\ &\leq \sum_n \left( \ell(I_n) + \frac{\varepsilon}{2^n} \right) \\ &= \sum_n \ell(I_n) + \varepsilon \\ &\leq m^*(A) + 2\varepsilon, \end{aligned}$$

and hence equality follows.

8. If  $\ell(J_k) = \infty$  for some  $k$ , then since  $J_k \subseteq A$ , subadditivity gives us that  $m^*(J_k) \leq m^*(A)$  and so  $m^*(A) = \infty = \sum_{k=1}^{\infty} \ell(J_k)$  (since if any  $J_k$  infinite, the sum of the lengths of all of them will also be infinite).

Suppose then  $\ell(J_k) < \infty$  for all  $k$ . Fix  $\varepsilon > 0$ . Then for all  $k \geq 1$ , choose  $I_k \subseteq J_k$  such that  $\ell(J_k) \leq \ell(I_k) + \frac{\varepsilon}{2^k}$ . For any  $N \geq 1$ , we can choose a subset  $\{I_1, \dots, I_N\}$  of intervals such that all are disjoint, with strictly positive distance between them, and so

$$\begin{aligned}
\bigcup_{k=1}^N I_k &\subseteq \bigcup_{k=1}^N I_k \subseteq A \\
\Rightarrow m^*(A) &\geq m^*\left(\bigcup_{k=1}^N I_k\right) \geq \sum_{k=1}^N \ell(I_k) \\
&\geq \sum_{k=1}^N \left(\ell(J_k) - \frac{\varepsilon}{2^k}\right) \\
&\geq \sum_{k=1}^N \ell(J_k) - \varepsilon \\
\Rightarrow m^*(A) &\geq \sum_{k=1}^{\infty} \ell(J_k),
\end{aligned}$$

the second inequality following from finite subadditivity. The converse of the final inequality holds trivially. ■

## §1.5 Lebesgue-Measurable Sets

↪ **Definition 1.7:**  $A \subseteq \mathbb{R}$  is  $m^*$ -measurable if  $\forall B \subseteq \mathbb{R}$ ,

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c).$$

**Remark 1.3:** By subadditivity,  $\leq$  always holds in the definition above.

↪ **Theorem 1.2** (Carathéodary's Theorem): Let

$$\mathcal{M} := \{A \subseteq \mathbb{R} : A \text{ } m^* \text{-measurable}\}.$$

Then,  $\mathcal{M}$  is a  $\sigma$ -algebra of subsets of  $\mathbb{R}$ .

Define  $m : \mathcal{M} \rightarrow [0, \infty]$ ,  $m(A) = m^*(A)$ . Then,  $m$  is a measure on  $\mathcal{M}$ , called the *Lebesgue measure* on  $\mathbb{R}$ . We call sets in  $\mathcal{M}$  *Lebesgue-measurable* or simply *measurable* (if clear from context) accordingly. We call  $(\mathbb{R}, \mathcal{M}, m)$  the *Lebesgue measure space*.

**PROOF.** The first two  $\sigma$ -algebra axioms are easy. We have for any  $B \subseteq \mathbb{R}$  that

$$m^*(B \cap \mathbb{R}) + m^*(B \cap \mathbb{R}^c) = m^*(B) + m^*(B \cap \emptyset) = m^*(B)$$

so  $\mathbb{R} \in \mathcal{M}$ . Further,  $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$  by the symmetry of the requirement for sets to be in  $\mathcal{M}$ .

The final axiom takes more work. We show first  $\mathcal{M}$  closed under finite unions; by induction it suffices to show for 2 sets. Let  $A_1, A_2 \in \mathcal{M}$ . Then, for all  $B \subseteq \mathbb{R}$ ,

$$\begin{aligned}
m^*(B) &= m^*(B \cap A_1) + m^*(B \cap A_1^c) \\
&= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*(B \cap A_1^c \cap A_2^c) \\
&= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*(B \cap (A_1 \cup A_2)^c)
\end{aligned}$$

Note that  $(B \cap A_1) \cup (B \cap A_1^c \cap A_2^c) = B \cap (A_1 \cup A_2)$ , hence by subadditivity,

$$m^*(B) \geq m^*(B \cap (A_1 \cup A_2)) + m^*(B \cap (A_1 \cup A_2)^c),$$

and since the other direction of the inequality comes for free, we conclude  $A_1 \cup A_2 \in \mathcal{M}$ .

Let now  $\{A_n\} \subseteq \mathcal{M}$ . We “disjointify”  $\{A_n\}$ ; put  $B_1 := A_1$ ,  $B_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i$ ,  $n \geq 2$ , noting  $\bigcup_n A_n = \bigcup_n B_n$ , and each  $B_n \in \mathcal{M}$ , as each is but a finite number of set operations applied to the  $A_n$ ’s, and thus in  $\mathcal{M}$  as demonstrated above. Put  $E_n := \bigcup_{i=1}^n B_i$ , noting again  $E_n \in \mathcal{M}$ . Then, for all  $B \subseteq \mathbb{R}$ ,

$$\begin{aligned} m^*(B) &= m^*\left(\underbrace{B \cap E_n}_{\text{chop up } B_n}\right) + m^*\left(\underbrace{B \cap E_n^c}_{E_n \subseteq \bigcup B_n \Rightarrow E_n^c \supseteq (\bigcup B_n)^c}\right) \\ &\geq m^*\left(B \cap \underbrace{E_n \cap B_n}_{=B_n}\right) + m^*\left(B \cap \underbrace{E_n \cap B_n^c}_{=E_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\ &\geq m^*(B \cap B_n) + m^*\left(\underbrace{B \cap E_{n-1}}_{\text{chop up } B_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\ &\geq m^*(B \cap B_n) + m^*(B \cap E_{n-1} \cap B_{n-1}) \\ &\quad + m^*(B \cap E_{n-1} \cap B_{n-1}^c) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right). \end{aligned}$$

Notice that the last line is essentially the second applied to  $B_{n-1}$ ; hence, we have a repeating (essentially, “descending”) pattern in this manner, which we repeat until  $n \rightarrow 1$ . We have, thus, that

$$m^*(B) \geq \sum_{i=1}^n [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right),$$

so taking  $n \rightarrow \infty$ ,

$$\begin{aligned} m^*(B) &\geq \sum_{i=1}^{\infty} [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\ &\geq m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right). \end{aligned}$$

As usual, the inverse inequality comes for free, and thus we can conclude  $\bigcup_{n=1}^{\infty} B_n$  also  $m^*$ -measurable, and thus so is  $\bigcup_{n=1}^{\infty} A_n$ . This proves  $\mathcal{M}$  a  $\sigma$ -algebra.

We show now  $m$  a measure. By previous propositions, we have that  $m \geq 0$  and  $m(\emptyset) = 0$  (since  $m = m^*|_{\mathcal{M}}$ ), so it remains to prove countable subadditivity.

Let  $\{A_n\} \subseteq \mathcal{M}$ -disjoint. Following precisely the same argument as above, used to prove that  $\mathcal{M}$  closed under countable unions, shows that for any  $n \geq 1$

$$m\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n m(A_i),$$

that is, finite additivity holds, and thus by subadditivity

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq m\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n m(A_i),$$

and so taking the limit of  $n \rightarrow \infty$ , we have

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} m(A_i),$$

with the converse inequality coming for free. Thus,  $m$  indeed a measure on  $\mathcal{M}$ . ■

↪ **Proposition 1.6:**  $\mathcal{M}, m$  translation invariant; for all  $A \in \mathcal{M}, x \in \mathbb{R}, x + A = \{x + a : a \in A\} \in \mathcal{M}$  and  $m(A) = m(A + x)$ .

**Remark 1.4:** We would like this to hold, heuristically, since if we shift sets on the real line, we should expect their length to remain constant.

PROOF. For all  $B \subseteq \mathbb{R}$ , we have (since  $m^*$  translation invariant)

$$\begin{aligned} m^*(B) &= m^*(B - x) = m^*\left(\underbrace{(B - x) \cap A}_{=B \cap (A+x)}\right) + m^*\left(\underbrace{(B - x) \cap A^c}_{=B \cap (A^c+x)=B \cap (A+x)^c}\right) \\ &= m^*(B \cap (A + x)) + m^*(B \cap (A + x)^c), \end{aligned}$$

thus  $A + x \in \mathcal{M}$ , and since  $m^*$  translation invariant, it follows that  $m$  is. ■

↪ **Theorem 1.3:**  $\forall a, b \in \mathbb{R}$  with  $a < b$ ,  $(a, b) \in \mathcal{M}$ , and  $m((a, b)) = b - a$ .

**Remark 1.5:** Again, we'd like this to hold, heuristically, since we would like the measure of an interval to simply be its length; we'd moreover like to be able to measure intervals, i.e. have intervals be contained in  $\mathcal{M}$ .

↪ **Corollary 1.1:**  $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M}$

PROOF.  $\mathfrak{B}_{\mathbb{R}}$  is generated by open intervals of the form  $(a, b)$ . All such intervals are in  $\mathcal{M}$  by the previous theorem, and hence the proof. ■

## §1.6 Properties of the Lebesgue Measure

↪ **Proposition 1.7** (Regularity Properties of  $m$ ): For all  $A \in \mathcal{M}$ , the following hold.

- For all  $\varepsilon > 0$ ,  $\exists G$  open such that  $A \subseteq G$  and  $m(G \setminus A) < \varepsilon$ .
- For all  $\varepsilon > 0$ ,  $\exists F$ -closed such that  $F \subseteq A$  and  $m(A \setminus F) \leq \varepsilon$ .
- $m(A) = \inf\{m(G) : G \text{ open}, G \supseteq A\}$ .
- $m(A) = \sup\{m(K) : K \text{ compact}, K \subseteq A\}$ .
- If  $m(A) < \infty$ , then for all  $\varepsilon > 0$ ,  $\exists K \subseteq A$  compact, such that  $m(A \setminus K) < \varepsilon$ .
- If  $m(A) < \infty$ , then for all  $\varepsilon \geq 0$ ,  $\exists$  finite collection of open intervals  $I_1, \dots, I_N$  such that  $m\left(A \Delta \left(\bigcup_{n=1}^N I_n\right)\right) \leq \varepsilon$ .

↪ **Proposition 1.8** (Completeness of  $m$ ):  $(\mathbb{R}, \mathcal{M}, m)$  is *complete*, in the sense that for all  $A \subseteq \mathbb{R}$ , if  $\exists B \in \mathcal{M}$  such that  $A \subseteq B$  and  $m(B) = 0$ , then  $A \in \mathcal{M}$  and  $m(A) = 0$ .

Equivalently, any subset of a null set is again a null set.

**Remark 1.6:** In general,  $A \in \mathcal{F}, B \subseteq A \not\Rightarrow B \in \mathcal{F}$ .

↪ **Proposition 1.9:** Up to rescaling,  $m$  is the unique, nontrivial measure on  $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$  that is finite on compact sets and is translation invariant, i.e. if  $\mu$  another such measure on  $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$  with  $\mu = c \cdot m$  for  $c > 0$ , then  $\mu = m$ .

**Remark 1.7:** Such a  $c$  is simply  $c = \mu((0, 1))$ .

To prove this proposition, we first introduce some helpful tooling:

↪ **Theorem 1.4** (Dynkin's  $\pi$ -d): Given a space  $X$ , let  $\mathcal{C}$  be a collection of subsets of  $X$ .  $\mathcal{C}$  is called a  $\pi$ -system if  $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$  (that is, it is closed under finite intersections).

Let  $\mathcal{F} = \sigma(\mathcal{C})$ , and suppose  $\mu_1, \mu_2$  are two finite measures on  $(X, \mathcal{F})$  such that  $\mu_1(X) = \mu_2(X)$  and  $\mu_1 = \mu_2$  when restricted to  $\mathcal{C}$ . Then,  $\mu_1 = \mu_2$  on all of  $\mathcal{F}$ .

↪ **Proposition 1.10:**  $\{\emptyset\} \cup \{(a, b) : a < b \in \mathbb{R}\}$  a  $\pi$ -system.

↪ **Proposition 1.11:** If  $\mu$  a measure on  $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$  such that for all intervals  $I$ ,  $\mu(I) = \ell(I)$ , then  $\mu = m$ .

PROOF. Consider for all  $n \geq 1$   $\mu|_{\mathfrak{B}_{[-n, n]}}$ . Clearly,  $\mu([-n, n]) = m([-n, n]) = 2n$ , and for all  $a, b \in \mathbb{R}$ ,  $\mu((a, b) \cap [-n, n]) = \ell((a, b) \cap [-n, n]) = m((a, b) \cap [-n, n])$ . Thus, by the previous theorem,  $\mu$  must match  $m$  on all of  $\mathfrak{B}_{[-n, n]}$ .

Let now  $A \in \mathfrak{B}_{\mathbb{R}}$ . Let  $A_n := A \cap [-n, n] \in \mathfrak{B}_{[-n, n]}$ . By continuity of  $m$  from below,

$$\begin{aligned}\mu(A) &= \lim_{n \rightarrow \infty} \mu(A_n) \\ &= \lim_{n \rightarrow \infty} m(A_n) \\ &= m(A),\end{aligned}$$

hence  $\mu = m$ . ■

↪ **Proposition 1.12:** If  $\mu$  a measure on  $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$  assigning finite values to compact sets and is translation invariant, then  $\mu = cm$  for some  $c > 0$ .

**Remark 1.8:** This proposition is also tacitly stating that  $\mathfrak{B}_{\mathbb{R}}$  translation invariant; this needs to be shown.

↪ **Lemma 1.1:**  $\mathfrak{B}_{\mathbb{R}}$  translation invariant; for any  $A \in \mathfrak{B}_{\mathbb{R}}, x \in \mathbb{R}, A + x \in \mathfrak{B}_{\mathbb{R}}$ .

PROOF. We employ the “good set strategy”; fix some  $x \in \mathbb{R}$  and let

$$\Sigma := \{B \in \mathfrak{B}_{\mathbb{R}} : B + x \in \mathfrak{B}_{\mathbb{R}}\}.$$

We have by construction  $\Sigma \subseteq \mathfrak{B}_{\mathbb{R}}$ . One can check too that  $\Sigma$  a  $\sigma$ -algebra. But in addition, its easy to see that  $\{(a, b) : a < b \in \mathbb{R}\} \subseteq \Sigma$ , since a translated interval is just another interval, and since these sets generate  $\mathfrak{B}_{\mathbb{R}}$ , it must be further that  $\mathfrak{B}_{\mathbb{R}} \subseteq \Sigma$ , completing the proof. ■

PROOF. (of the proposition) Let  $c = \mu((0, 1])$ , noting that  $c > 0$  (why? Consider what would happen if  $c = 0$ ).

This implies that  $\forall n \geq 1, \mu\left(\left(0, \frac{1}{n}\right]\right) = \frac{c}{n}$  (obtained by “chopping up”  $(0, 1]$  into  $n$  disjoint intervals); from here we can draw many further conclusions:

$$\begin{aligned}\forall m = 1, \dots, n-1, \mu\left(\left(0, \frac{m}{n}\right]\right) &= \frac{m}{n}c \\ \Rightarrow \forall q \in \mathbb{Q} \cap (0, 1], \mu((0, q]) &= qc \\ \Rightarrow \forall q \in \mathbb{Q}^+, \mu((0, q]) &= q \cdot c \text{ (translate)} \\ \Rightarrow \forall a \in \mathbb{R}, \mu((a, a+q]) &= q \cdot c \\ \Rightarrow \forall \text{intervals } I, \mu(I) &= c \cdot \ell(I) \text{ (continuity)} \\ \Rightarrow \forall n \geq 1, a, b \in \mathbb{R}, \mu((a, b) \cap [-n, n]) &= c \cdot \ell((a, b) \cap [-n, n]) = c \cdot m((a, b) \cap [-n, n]),\end{aligned}$$

but then,  $\mu = c \cdot m$  on  $\mathfrak{B}_{\mathbb{R}[-n, n]}$ , and by appealing again the Dynkin's,  $\mu = c \cdot m$  on all of  $\mathfrak{B}_{\mathbb{R}}$ . ■

↪ **Proposition 1.13** (Scaling):  $m$  has the *scaling property* that  $\forall A \in \mathcal{M}, c \in \mathbb{R}, c \cdot A = \{cx : x \in A\} \in \mathcal{M}$ , and  $m(c \cdot A) = |c| m(A)$ .

PROOF. Assume  $c \neq 0$ . Given  $A \subseteq \mathbb{R}$ , remark that  $\{I_n\}$  an open interval cover of  $A$  iff  $\{cI_n\}$  and open interval cover of  $cA$ , and  $\ell(cI_n) = |c| \ell(I_n)$ , and thus  $m^*(cA) = |c| m^*(A)$ .

Now, suppose  $A \in \mathcal{M}$ . Then, we have for any  $B \subseteq \mathbb{R}$ ,

$$\begin{aligned} m^*(B) &= |c| m^*\left(\frac{1}{c}B\right) = |c| m^*\left(\frac{1}{c}B \cap A\right) + |c| m^*\left(\frac{1}{c}B \cap A^c\right) \\ &= m^*(B \cap cA) + m^*(B \cap (cA)^c), \end{aligned}$$

so  $cA \in \mathcal{M}$ . ■

## §1.7 Relationship between $\mathfrak{B}_{\mathbb{R}}$ and $\mathcal{M}$

↪ **Definition 1.8**: Given  $(X, \mathcal{F}, \mu)$ , consider the following collection of subsets of  $X$ ,

$$\mathcal{N} := \{B \subseteq X : \exists A \in \mathcal{F} \text{ s.t. } \mu(A) = 0, B \subseteq A\}.$$

Put  $\overline{\mathcal{F}} := \sigma(\mathcal{F} \cup \mathcal{N})$ ; this is called the *completion* of  $\mathcal{F}$  with respect to  $\mu$ .

↪ **Proposition 1.14**:  $\overline{\mathcal{F}} = \{F \subseteq X : \exists E, G \in \mathcal{F} \text{ s.t. } \exists E \subseteq F \subseteq G \text{ and } m(G \setminus E) = 0\}$ .

PROOF. Put  $\mathcal{G}$  the set on the right; one can check  $\mathcal{G}$  a  $\sigma$ -algebra. Since  $\mathcal{F} \subseteq \mathcal{G}$  and  $\mathcal{N} \subseteq \mathcal{G}$ , we have  $\overline{\mathcal{F}} \subseteq \mathcal{G}$ .

Conversely, for any  $F \in \mathcal{G}$ , we have  $E, G \in \mathcal{F}$  such that  $E \subseteq F \subseteq G$  with  $m(G \setminus E) = 0$ . We can rewrite

$$F = \underbrace{E}_{\in \mathcal{F}} \cup \underbrace{(F \setminus E)}_{\substack{\subseteq G \setminus E \\ \Rightarrow \mu(G \setminus E) = 0 \\ \Rightarrow G \setminus E \in \mathcal{N}}},$$

hence  $F \in \mathcal{F} \cup \mathcal{N}$  and thus in  $\overline{\mathcal{F}}$ , and equality holds. ■

↪ **Definition 1.9**: Given  $(X, \mathcal{F}, \mu)$ ,  $\mu$  can be *extended* to  $\overline{\mathcal{F}}$  by, for each  $F \in \overline{\mathcal{F}}$  with  $E \subseteq F \subseteq G$  s.t.  $\mu(G \setminus E) = 0$ , put

$$\mu(F) = \mu(E) = \mu(G).$$

We call then  $(X, \overline{\mathcal{F}}, \mu)$  a *complete measure space*.

**Remark 1.9**: It isn't obvious that this is well defined a priori; in particular, the  $E, G$  sets are certainly not guaranteed to be unique in general, so one must check that this definition is valid regardless of choice of "sandwich sets".

↪ **Theorem 1.5:**  $(\mathbb{R}, \mathcal{M}, m)$  is the completion of  $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, m)$ .

PROOF. Given  $A \in \mathcal{M}$ , then  $\forall n \geq 1, \exists G_n$ -open with  $A \subseteq G_n$  s.t.  $m^*(G_n \setminus A) \leq \frac{1}{n}$  and  $\exists F_n$ -closed with  $F_n \subseteq A$  s.t.  $m^*(A \setminus F_n) \leq \frac{1}{n}$ .

Put  $C := \bigcap_{n=1}^{\infty} G_n$ ,  $B := \bigcap_{n=1}^{\infty} F_n$ , remarking that  $C, B \in \mathfrak{B}_{\mathbb{R}}$ ,  $B \subseteq A \subseteq C$ , and moreover

$$\begin{aligned} m(C \setminus A) &\leq \frac{1}{n}, m(A \setminus B) \leq \frac{1}{n} \\ \Rightarrow m(C \setminus B) &= m(C \setminus A) + m(A \setminus B) \leq \frac{2}{n}, \end{aligned}$$

but  $n$  can be arbitrarily large, hence  $m(C \setminus B) = 0$ ; in short, given a measurable set, we can “sandwich it” arbitrarily closely with Borel sets. Thus,  $A \in \overline{\mathfrak{B}_{\mathbb{R}}} \Rightarrow \mathcal{M} \subseteq \overline{\mathfrak{B}_{\mathbb{R}}}$ . But recall that  $\mathcal{M}$  complete, so  $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M} \Rightarrow \overline{\mathfrak{B}_{\mathbb{R}}} \subseteq \overline{\mathcal{M}} = \mathcal{M}$ , and thus  $\overline{\mathfrak{B}_{\mathbb{R}}} = \mathcal{M}$  indeed.

Heuristically, this means that any measurable set is “different” from a Borel set by at most a null set. ■

## §1.8 Some Special Sets

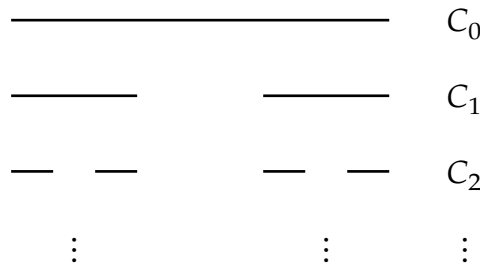
### 1.8.1 Uncountable Null Set?

Remark that for any countable set  $A \in \mathcal{M}$ ,  $m(A) = 0$ ; indeed, one may write  $A = \bigcup_{n=1}^{\infty} \{a_n\}$  for singleton sets  $\{a_n\}$ , and so

$$m(A) = \sum_{n=1}^{\infty} m(a_n) = 0.$$

One naturally asks the opposite question, does there exist a measurable, *uncountable* set with measure 0? We construct a particular one here, the Cantor set,  $C$ .

This requires an “inductive” construction. Define  $C_0 = [0, 1]$ , and define  $C_k$  to be  $C_{k-1}$  after removing the middle third from each of its disjoint components. For instance  $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ , then  $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ , and so on. This may be clearest graphically:



Remark that the  $C_n \downarrow$ . Put finally

$$C := \bigcap_{n=1}^{\infty} C_n.$$



↪ **Proposition 1.15:** The following hold for the Cantor set  $C$ :

1.  $C$  is closed (and thus  $C \in \mathfrak{B}_{\mathbb{R}}$ );
2.  $m(C) = 0$ ;
3.  $C$  is uncountable.

PROOF.

1. For each  $n$ ,  $C_n$  is the countable (indeed, finite) union of  $2^n$ -many disjoint, closed intervals, hence each  $C_n$  closed.  $C$  is thus a countable intersection of closed sets, and is thus itself closed.
2. For each  $n$ , each of the  $2^n$  disjoint closed intervals in  $C_n$  has length  $\frac{1}{3^n}$ , hence

$$m(C_n) = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$

Since  $\{C_n\} \downarrow$ , by continuity of  $m$  we have

$$m(C) = \lim_{n \rightarrow \infty} m(C_n) = \lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = 0.$$

3. This part is a little trickier. Notice that for any  $x \in [0, 1]$ , we can define a sequence  $(a_n)$  where each  $a_n \in \{0, 1, 2\}$ , and such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n};$$

in particular, this is just the base-3 representation of  $x$ , which we denote  $(x)_3 = (a_1 a_2 \dots)$ .

I claim now that

$$C = \{x \in [0, 1] : (x)_3 \text{ has no 1's}\}.$$

Indeed, at each stage  $n$  of the construction of the Cantor set, we get rid of the segment of the real line that would correspond to the  $a_n = 1$ . One should note that  $(x)_3$  not necessarily unique; for instance  $\left(\frac{1}{3}\right)_3 = (1, 0, 0, \dots) = (0, 2, 2, \dots)$ , but if we specifically consider all  $x$  such that there *exists* a base three representation with no 1's, i.e. like  $\frac{1}{3}$ , then  $C$  indeed captures all the desired numbers.

Thus, we have that

$$\text{card}(C) = \text{card}(\{a_n : a_n = 0, 2\}).$$

Define now the function

$$f : C \rightarrow [0, 1], \quad x \mapsto \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot \frac{1}{2^n}, \text{ where } (x)_3 = (a_n)$$

i.e., we “squish” the base-3 representation into a base-2 representation of a number.

This is surjective; for any  $y \in [0, 1]$ ,  $(b_n) := (y)_2$  contains only 0's and 1's, hence  $(2b_n)$

contains only 0's and 1's, so let  $x$  be the number such that  $(x)_3 = (2b_n)$ . This necessarily exists, indeed, we simply take our definitions backwards:

$$x := \sum_{n=1}^{\infty} \frac{2b_n}{3^n},$$

which maps to  $y$  under  $f$  and is contained in  $C$ . Hence,  $\text{card}(C) \geq \text{card}([0, 1])$ ; but  $[0, 1]$  uncountable, and thus so is  $C$ . ■

We can naturally extend the function  $f$  used here to map the entire interval  $[0, 1] \rightarrow [0, 1]$  as follows

$$f(x) := \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot \frac{1}{2^n} & \text{if } x \in C, (x)_3 = (a_n) \\ f(a) & \text{if } x \notin C \text{ then } x \in (a, b) \text{ s.t. } (a, b) \text{ removed from } [0, 1] \end{cases}.$$

This function is often called the *Devil's Staircase* or *Cantor-Lebesgue function*.

↪ **Proposition 1.16:**

1.  $f(0) = 0, f(1) = 1, f \equiv \frac{1}{2}$  on  $(\frac{1}{3}, \frac{2}{3}), f \equiv \frac{1}{4}$  on  $(\frac{1}{9}, \frac{2}{9})$
2.  $f : [0, 1] \rightarrow [0, 1]$  a surjection
3.  $f$  is nondecreasing
4.  $f$  is continuous

PROOF. 1., 2., clear from construction.

For 3., let  $x_1 < x_2 \in C$ , and suppose  $(x_1)_3 = (a_n), (x_2)_3 = (b_n)$ . Then, since  $x_1 < x_2$ , it must be that  $a_n, b_n$  can only be equal up to some finite  $N$ ; then the next  $0 = a_{N+1} < b_{N+1} = 2$ . Hence, it follows that the “modified binary expansion” that arises from  $f$  gives directly that  $f(x_1) \leq f(x_2)$ .

For 4.,  $f$  is clearly continuous on  $[0, 1] - C$ , since it is piecewise-constant here. Also,  $f$  is “one-sided continuous” at each of the “boundary points”  $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \dots$ . If  $x \in C$ , for any  $n \geq 1$ , there must be  $x_n, x_n'$  such that  $x_n < x < x_n'$  (if  $x = 0$ , only need  $x_n'$ , if  $x = 1$ , only need  $x_n$ ) and  $f(x_n') - f(x_n) \leq \frac{1}{2^n}$ . Then,  $f$  is continuous at  $x$  by monotonicity of  $f$ . ■

### 1.8.2 Non-Measurable Sets?

We've shown then that there is indeed an uncountable set of measure 0. Another question we may ask ourselves is, is there a  $A \subseteq \mathbb{R}$  that is non-measurable? The answer to this turns out to be yes, but the construction requires invoking the axiom of choice:

**Axiom 1** (Of Choice): If  $\Sigma$  a collection of nonempty sets, then  $\exists$  a function

$$S : \Sigma \rightarrow \bigcup_{A \in \Sigma} A,$$

such that  $A \in \sigma, S(A) \in A$ . Such a function is called a *selection function*, and  $S(A)$  a *representative* of  $A$ .

We construct now a non-measurable set, assuming the above. Consider  $[0, 1]$ , and define an equivalence relation  $\sim$  on  $[0, 1]$  by

$$a \sim b \Leftrightarrow a - b \in \mathbb{Q}.$$

Its easy to check that this is indeed an equivalence relation. Denote by  $E_a$  the equivalence class containing  $a$ , and set  $\Sigma = \{E_a : a \in [0, 1]\}$ . Note that for any  $E_a \in \Sigma, E_a \neq \emptyset$ .

Invoking the axiom of choice, we can select exactly one element  $S_a$  from  $E_a$  for each  $E_a \in \Sigma$ . Set

$$N := \{S_a : S_a \text{ is a representative of } E_a, E_a \in \Sigma\}.$$

**Proposition 1.17:**  $N$ , called a *Vitali set*, is non-measurable.

PROOF. Assume towards a contradiction that  $N$  indeed measurable,  $N \in \mathcal{M}$ . Consider  $[-1, 1] \cap \mathbb{Q}$ ; this is countable, so we can enumerate it  $\{q_k\}, k \geq 1$ . For each  $k$ , put

$$N_k := N + q_k.$$

By the assumption of measurability and translation invariance of  $m$ , it must be that each  $N_k$  measurable and has the same measure as  $N$ .

We claim each  $N_k$  disjoint. Assume not, then  $\exists k \neq \ell$  (i.e.  $q_k \neq q_\ell$ ) and  $S_a, S_b \in N$  such that  $S_a + q_k = S_b + q_\ell$ . But then  $S_a - S_b = q_\ell - q_k \in \mathbb{Q}$ , hence  $S_a \sim S_b$ . But we constructed  $N$  to have only one representative from each equivalence class, hence it must be that  $S_a = S_b$ , and so  $S_a + q_k = S_a + q_\ell \Rightarrow q_k = q_\ell$ , contradicting the assumed distinctness of the  $q$ 's; hence, the  $N_k$ 's indeed disjoint.

We claim next that  $[0, 1] \subseteq \bigcup_{k=1}^{\infty} N_k$ . Let  $x \in [0, 1]$ . Then,  $x \sim S_a$  for some unique  $S_a \in N$  and so  $x - S_a \in \mathbb{Q}$ . But also,  $x, S_a \in [0, 1]$ , hence  $x - S_a \in [-1, 1]$  (moreover,  $x - S_a \in [-1, 1] \cap \mathbb{Q}$ ) and there must exist a  $k$  such that  $x - S_a = q_k$ , since the  $q_k$ 's enumerate the entire  $[-1, 1] \cap \mathbb{Q}$ . Thus,  $x \in N_k$  by the construction of the  $N_k$ 's. Thus,  $[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k$  indeed.

On the other hand,  $\bigcup_{k=1}^{\infty} N_k \subseteq [-1, 2]$  and so we have the "bound"

$$[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k \subseteq [-1, 2].$$

Taking the measure of all sides then, we have the bound

$$1 \leq \mu\left(\bigcup_{n=1}^{\infty} N_k\right) \leq 3.$$

Invoking the disjointness of the  $N_k$ 's, we can also use countable additivity to write

$$\mu\left(\bigcup_{n=1}^{\infty} N_k\right) = \sum_{k=1}^{\infty} m(N_k) = \sum_{k=1}^{\infty} m(N),$$

but this final line is a sequence of positive, constant real numbers; hence, it is impossible for it to be within 1 and 3, and we have a contradiction. Hence,  $N$  indeed not measurable.

Remark that this proof also shows that  $m^*(N_k) > 0$  so  $m^*(N) > 0$  (given the interval bound on  $N$  we've found). ■

↪ **Proposition 1.18:** For every  $A \in \mathcal{M}$  such that  $m(A) > 0$ , there exists  $B \subseteq A$  such that  $B$  is non-measurable.

PROOF. Assume otherwise, that there is a  $A \in \mathcal{M}$  with  $m(A) > 0$  such that any subset  $B$  of  $A$  is also measurable.

Remark that  $A \subseteq \bigcup_{n \in \mathbb{Z}} A \cap [n, n+1]$ . Then, there exists an  $n$  such that  $m(A \cap [n, n+1]) > 0$  and thus, translating  $A' := A \cap [n, n+1] - n$ ,  $m(A') > 0$ , noting that  $A' \subseteq [0, 1]$ . Now, for any  $B' \subseteq A'$ ,  $B' + n \subseteq A$ . By assumption, then  $B' + n$  must be measurable so  $B'$  measurable.

In summary, then, we have  $A' \subseteq [0, 1]$  with  $m(A') > 0$  such that (by assumption)  $B'$  measurable for all  $B' \subseteq A'$ .

Let  $N, \{q_k\}, N_k$  be as in the previous proof. Set

$$A_k' := A' \cap N_k, k \geq 1.$$

Then,  $A_k'$  disjoint, and

$$A' = [0, 1] \cap A' \subseteq \bigcup_{k=1}^{\infty} (N_k \cap A') = \bigcup_{k=1}^{\infty} A_k'.$$

Since  $m(A') > 0$ , there exists a  $k$  such that  $m(A_k') > 0$ . Set, for this  $k$ ,

$$L := \{\ell \geq 1 : q_\ell + q_k \in [-1, 1]\}.$$

This set is again countably infinite. We translate, obtaining a disjoint sequence of sets  $\{q_\ell + A_k' : \ell \in L\}$ ; since  $q_\ell + q_k \in [-1, 1] \cap \mathbb{Q}$ , then  $q_\ell + q_k = q_m$  for some unique  $m$ , and so  $q_\ell + A_k' = q_\ell + A' \cap (N + q_k) \subseteq N_m$ . Hence, we have on the one hand that by countable additivity

$$\bigcup_{\ell \in L} (q_\ell + A_k') \subseteq [-1, 2] \Rightarrow \sum_{\ell \in L} m(q_\ell + A_k') \leq 3,$$

and so it must be that  $m(q_\ell + A_k') = m(A_k') = 0$  (else the series couldn't be finite), contradicting the finiteness assumption on  $m(A_k')$ . ■

### 1.8.3 Non-Borel Measurable Set?

We may ask, is there  $A \in \mathcal{M}$  such that  $A \notin \mathfrak{B}_{\mathbb{R}}$ ?

Let  $f : [0, 1] \rightarrow [0, 1]$  be the Cantor-Lebesgue function, and put  $g(x) = f(x) + x$ ; note that  $g$  is continuous and strictly increasing, and is defined  $g : [0, 1] \rightarrow [0, 2]$ . Remark that  $g$  bijective; the strictly increasing gives injective, and moreover  $g(0) = 0, g(1) = 2$  hence by intermediate value theorem it is surjective. Hence,  $g^{-1} : [0, 2] \rightarrow [0, 1]$  exists, and is also continuous, so in short  $g$  is a homeomorphism; it maps open to open, closed to closed. In particular, if  $A \in \mathfrak{B}_{\mathbb{R}}$ , then  $g(A) \in \mathfrak{B}_{\mathbb{R}}$ .

Recall that if  $(a, b)$  an open interval that gets removed from the construction of  $C$ , then  $f$  is constant and so  $g$  will map  $(a, b)$  to another open interval of the same length  $b - a$ . Thus,

$$m(g([0, 1] \setminus C)) = m([0, 1] \setminus C) = 1.$$

Hence,  $m(g(C)) = 2 - 1 = 1 > 0$ , since  $g(C \cup [0, 1] \setminus C) = [0, 2]$ . Hence, there exists a  $B \subseteq g(C)$  such that  $B \notin \mathcal{M}$ , as per the previous proposition.

Let  $A := g^{-1}(B)$ ; then  $A \subseteq g^{-1}(g(C)) = C$ . Since  $m(C) = 0$ ,  $A \in \mathcal{M}$  and  $m(A) = 0$ . But,  $A \notin \mathfrak{B}_{\mathbb{R}}$ ; if it were, then  $g(A) = B \in \mathfrak{B}_{\mathbb{R}}$ , since  $g$  “maintains” Borel sets, but  $B$  is not even Lebesgue measurable and so this is a contradiction).

## §2 INTEGRATION THEORY

### §2.1 Measurable Functions

We will be considering functions  $f$  defined on  $\mathbb{R}$  or some subset of  $\mathbb{R}$  that could take positive or negative infinity as its value i.e.

$$f : \mathbb{R} \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\},$$

where  $\overline{\mathbb{R}}$  the *extended real line*; we say  $f$  is  $\overline{\mathbb{R}}$ -valued. If  $f$  never takes  $\infty, -\infty$  for any  $x \in \mathbb{R}$ , we say  $f$  finite-valued, or just  $\mathbb{R}$ -valued.

For all  $a \in \mathbb{R}$ , we consider inverse images

$$f^{-1}([-\infty, a)) := \{x \in \mathbb{R} : f(x) \in [-\infty, a)\} = \{f < a\},$$

remarking the inclusion of  $-\infty$ ; similarly

$$f^{-1}((a, \infty]) := \{x \in \mathbb{R} : f(x) \in (a, \infty]\} = \{f > a\},$$

and so on, for any  $B \subseteq \mathbb{R}$ ,

$$f^{-1}(B) := \{x \in \mathbb{R} : f(x) \in B\} = \{f \in B\}.$$

Remark that

$$f^{-1}(B^c) = (f^{-1}(B))^c$$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B),$$

which extend naturally for countable unions/intersections.

↪ **Definition 2.1** (Measurable Function):  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  is *measurable* if  $\forall a \in \mathbb{R}$ ,

$$f^{-1}([-\infty, a)) \in \mathcal{M}.$$

↪ **Proposition 2.1** (Equivalent Definitions of Measurability):

$$\begin{aligned} f \text{ is measurable} &\Leftrightarrow \forall a \in \mathbb{R}, f^{-1}([a, \infty]) \in \mathcal{M} \\ &\Leftrightarrow \forall a \in \mathbb{R}, f^{-1}((a, \infty]) \in \mathcal{M} \\ &\Leftrightarrow \forall a \in \mathbb{R}, f^{-1}([-\infty, a]) \in \mathcal{M} \end{aligned}$$

PROOF. We prove just the last equivalence. Notice that  $\forall a \in \mathbb{R}$ , we can use the commuting of inverse images with countable unions, intersections, complement to write

$$f^{-1}([-\infty, a)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a - \frac{1}{n}\right)\right)$$

and

$$f^{-1}([-\infty, a]) = \bigcap_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a + \frac{1}{n}\right]\right).$$

■

↪ **Proposition 2.2**: If  $f$  finite-valued, Then

$$\begin{aligned} f \text{ is measurable} &\Leftrightarrow \forall a < b \in \mathbb{R}, f^{-1}((a, b)) \in \mathcal{M} \\ &\Leftrightarrow \dots \quad f^{-1}((a, b]) \in \mathcal{M} \\ &\Leftrightarrow \dots \quad f^{-1}([a, b)) \in \mathcal{M} \\ &\Leftrightarrow \dots \quad f^{-1}([a, b]) \in \mathcal{M}. \end{aligned}$$

↪ **Definition 2.2** (Extended Borel Sigma Algebra): Define the Borel “extended” algebra  $\mathfrak{B}_{\overline{\mathbb{R}}}$  of subsets of  $\overline{\mathbb{R}}$ , defined by

$$\mathfrak{B}_{\overline{\mathbb{R}}} := \sigma(\mathfrak{B}_{\mathbb{R}} \cup \{-\infty\}, \{\infty\}).$$

↪ **Proposition 2.3**:  $\mathfrak{B}_{\overline{\mathbb{R}}} = \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ .

PROOF. For every  $a \in \mathbb{R}$ , we may write

$$[-\infty, a) = \underbrace{(-\infty, a)}_{\in \mathfrak{B}_{\mathbb{R}}} \cup \{-\infty\} \in \mathfrak{B}_{\overline{\mathbb{R}}},$$

so  $\sigma(\{[-\infty, a) : a \in \mathbb{R}\}) \subseteq \mathfrak{B}_{\overline{\mathbb{R}}}$ .

Conversely, notice that

$$\{-\infty\} = \bigcap_{n=1}^{\infty} [-\infty, -n),$$

and

$$\{\infty\} = \overline{\mathbb{R}} - \left( \bigcup_{n=1}^{\infty} [-\infty, n) \right),$$

so  $\{-\infty\}, \{\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ . Hence, for any  $a \in \mathbb{R}$ ,

$$(-\infty, a) = [-\infty, a) - \{-\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\}),$$

and so  $\mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ .  $\{-\infty\}, \{\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$  already, and thus  $\mathfrak{B}_{\overline{\mathbb{R}}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ . ■

↪ **Proposition 2.4:**  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  measurable  $\Leftrightarrow$  for all  $B \in \mathfrak{B}_{\overline{\mathbb{R}}}$ ,  $f^{-1}(B) \in \mathcal{M}$ .

PROOF.  $\Leftarrow$  is immediate. For  $\Rightarrow$ , let  $\mathcal{C}$  be a collection of subsets of  $\overline{\mathbb{R}}$ , then put

$$f^{-1}(\mathcal{C}) := \{f^{-1}(B) : B \in \mathcal{C}\}.$$

By an assignment question (2.6),

$$f^{-1}(\sigma(\mathcal{C})) = \sigma(f^{-1}(\mathcal{C})).$$

Take  $\mathcal{C} = \{[-\infty, a) : a \in \mathbb{R}\}$ . Then,

$$f^{-1}(\sigma(\mathcal{C})) = f^{-1}(\mathfrak{B}_{\overline{\mathbb{R}}}) = \sigma(f^{-1}(\{[-\infty, a) : a \in \mathbb{R}\})).$$

But  $f$  measurable, so  $f^{-1}([- \infty, a)) \in \mathcal{M}$  for each  $a \in \mathbb{R}$ , hence  $\sigma(f^{-1}(\{[-\infty, a) : a \in \mathbb{R}\})) \subseteq \mathcal{M}$  and so  $f^{-1}(\sigma(\mathcal{C})) \subseteq \mathcal{M}$  completing the proof. ■

↪ **Corollary 2.1:** If  $f$  finite-valued, then  $f$  is measurable  $\Leftrightarrow$  for every  $B \in \mathfrak{B}_{\mathbb{R}}$ ,  $f^{-1}(B) \in \mathcal{M}$ .

↪ **Proposition 2.5:** Given  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ , define the *finite valued component* of  $f$  given by

$$f_{\mathbb{R}}(x) := \begin{cases} f(x) & : -\infty < f(x) < \infty \\ 0 & \text{otherwise} \end{cases}.$$

Then,  $f$  measurable  $\Leftrightarrow \forall B \in \mathfrak{B}_{\mathbb{R}}, f_{\mathbb{R}}^{-1}(B) \in \mathcal{M}$  AND  $\{f = \infty\}, \{f = -\infty\}$  both in  $\mathcal{M}$ .

PROOF. ( $\Leftarrow$ ) For any  $a \in \mathbb{R}$ ,

$$f^{-1}([- \infty, a)) = \{f = -\infty\} \cup f^{-1}((-\infty, a)) = \{f = -\infty\} \cup f_{\mathbb{R}}^{-1}((-\infty, a)),$$

a union of measurable sets and hence is itself measurable.

( $\Rightarrow$ ) Remark that  $\{f = \infty\}, \{f = -\infty\} \in \mathcal{M}$  automatically. For any  $B \in \mathfrak{B}_{\mathbb{R}}$ , we have

$$f_{\mathbb{R}}^{-1}(B) = \{x \in \mathbb{R} : f_{\mathbb{R}}(x) \in B\} = \{x \in \mathbb{R} : f(x) \in B, -\infty < f < \infty\} \cup \{x \in \mathbb{R} : 0 \in B, f(x) = \pm\infty\} \in \mathcal{M}.$$

■

$\hookrightarrow$  **Definition 2.3:** If a statement is true for every  $x \in A$  where  $A \in \mathcal{M}$  s.t.  $m(A^c) = 0$ , then we say the statement is true a.e. (almost everywhere).

$\hookrightarrow$  **Proposition 2.6:** If  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  is measurable and  $f = g$  a.e. then  $g$  is measurable.

$\hookrightarrow$  **Corollary 2.2:** If  $f$  is finite-valued a.e., then  $f$  is measurable  $\Leftrightarrow f_{\mathbb{R}}$  is measurable  $\Leftrightarrow \forall a < b \in \mathbb{R}, f^{-1}((a, b)) \in \mathcal{M}$ .

$\hookrightarrow$  **Proposition 2.7:** If  $f \equiv c$  then  $f$  measurable.

If  $f = \mathbb{1}_A$  for some  $A \subseteq \mathbb{R}$ , then  $f$  is measurable  $\Leftrightarrow A \in \mathcal{M}$ .

PROOF. Assume  $f \equiv c$ . Then

$$f^{-1}([-\infty, a)) = \begin{cases} \mathbb{R} & \text{if } c < a \\ \emptyset & \text{if } c \geq a \end{cases} \in \mathcal{M}.$$

Assume now  $f = \mathbb{1}_A$ . For all  $a \in \mathbb{R}$ ,

$$f^{-1}([-\infty, a)) = \begin{cases} \mathbb{R} & \text{if } a > 1 \\ A^c & \text{if } 0 < a \leq 1 \in \mathcal{M} \Leftrightarrow A \in \mathcal{M}. \\ \emptyset & \text{if } a \leq 0 \end{cases}$$

■

$\hookrightarrow$  **Proposition 2.8:** If  $f$  is (finite-valued) continuous, then  $f$  is measurable.

PROOF.  $f : \mathbb{R} \rightarrow \mathbb{R}$  continuous  $\Leftrightarrow$  for all  $G \subseteq \mathbb{R}$  open,  $f^{-1}(G)$  open. For all  $a < b \in \mathbb{R}$ , then  $f^{-1}((a, b))$  open so  $f^{-1}((a, b)) \in \mathcal{M}$  so  $f$  measurable.

In fact, if  $f : \mathbb{R} \rightarrow \mathbb{R}$  continuous, then for all  $B \in \mathfrak{B}_{\mathbb{R}}, f^{-1}(B) \in \mathfrak{B}_{\mathbb{R}}$ ;

$$f^{-1}(\mathfrak{B}_{\mathbb{R}}) = f^{-1}(\sigma(\{\text{open sets}\})) = \sigma\left(\underbrace{f^{-1}(\{\text{open sets}\})}_{\text{all open}}\right) \subseteq \sigma(\{\text{open sets}\}) = \mathfrak{B}_{\mathbb{R}}.$$

Moreover, if  $f^{-1}$  (inverse) exists and is continuous, then for any  $B \in \mathfrak{B}_{\mathbb{R}}, f(B) \in \mathfrak{B}_{\mathbb{R}}$ . ■



↪ **Proposition 2.9:** If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is measurable and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, then  $g \circ f$  is measurable.

**Remark 2.1:** The order matters! The converse doesn't hold in general.

PROOF. For all  $a \in \mathbb{R}$ ,

$$\begin{aligned} (g \circ f)^{-1}((-\infty, a)) &= \{x \in \mathbb{R} : g(f(x)) < a\} \\ &= \{x \in \mathbb{R} : f(x) \in g^{-1}((-\infty, a))\} \\ &= f^{-1}(g^{-1}((-\infty, a))) \in \mathcal{M}. \end{aligned}$$

■

↪ **Proposition 2.10:** If  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  is measurable, then:

1. for every  $c \in \mathbb{R}$ ,  $cf$  is measurable (in particular  $-f$  measurable);
2.  $|f|$  is measurable;
3. for every  $k \in \mathbb{N}$ ,  $f^k$  is a measurable.

PROOF. We prove just 3. If  $k = 0$  this is trivial. For any  $a \in \mathbb{R}$ ,

$$(f^k)^{-1}([-\infty, a)) = \begin{cases} f^{-1}\left([-\infty, a^{\frac{1}{k}})\right) & \text{if } k \text{ is odd} \\ \emptyset & \text{if } k \text{ is even and } a \leq 0 \in \mathcal{M}. \\ f^{-1}\left([-a^{\frac{1}{k}}, a^{\frac{1}{k}})\right) & \text{if } k \text{ is even and } a > 0 \end{cases}$$

■

↪ **Proposition 2.11:** If  $f, g$  are two finite-valued measurable functions, then  $f + g, f \cdot g, f \vee g := \max\{f, g\}, f \wedge g := \min\{f, g\}$  are measurable functions, where

$$(f \vee g)(x) = \max\{f(x), g(x)\}.$$

PROOF. For all  $a \in \mathbb{R}$ ,

$$\begin{aligned} (f + g)^{-1}([-\infty, a)) &= \{x \in \mathbb{R} : f(x) + g(x) < a\} \\ &= \{x \in \mathbb{R} : f(x) < a - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \{x \in \mathbb{R} : f(x) < q < a - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \underbrace{\{x \in \mathbb{R} : f(x) < q\}}_{\in \mathcal{M}} \cap \underbrace{\{x \in \mathbb{R} : g(x) < a - q\}}_{\in \mathcal{M}} \in \mathcal{M}. \end{aligned}$$

This implies, then, that  $f - g$  measurable, as are  $(f + g)^2$  and  $(f - g)^2$ , and thus

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$$

is measurable.

We have too that

$$f \vee g = \frac{1}{2}(|f-g| + (f+g))$$

and so is measurable, and so

$$f \wedge g = -\max\{-f, -g\} = -(-f \vee -g)$$

is measurable. ■

↪ **Corollary 2.3:** If  $f$  is measurable, then  $f^+ := f \vee 0 = \max\{f, 0\}$  and  $f^- := -(f \wedge 0) = \max\{-f, 0\}$  are measurable, as is  $f \wedge k$  for any  $k \in \mathbb{R}$ .

**Remark 2.2:** Notice that  $f = f^+ - f^-$ , even with “infinities”, and  $|f| = f^+ + f^-$ .

↪ **Proposition 2.12:** Let  $\{f_n\}$  be a sequence of measurable functions. Then,  $\sup_n f_n$ ,  $\inf_n f_n$ ,  $\limsup_{n \rightarrow \infty} f_n$ , and  $\liminf_{n \rightarrow \infty} f_n$  are all measurable (where  $(\limsup_{n \rightarrow \infty} f_n)(x) := \limsup_{n \rightarrow \infty} f_n(x) = \inf_{m \geq 1} \sup_{n \geq m} f_n(x) = \lim_{m \rightarrow \infty} \sup_{n \geq m} f_n(x)$ ).

PROOF. To show  $\sup_n f_n$  measurable, we will show for all  $a \in \mathbb{R}$   $\{\sup_n f_n \leq a\} \in \mathcal{M}$ .

$$x \in \left\{ \sup_n f_n \leq a \right\} \Leftrightarrow \sup_n f_n(x) \leq a \Leftrightarrow f_n(x) \leq a \forall n \geq 1 \Leftrightarrow x \in \bigcap_{n=1}^{\infty} \{f_n \leq a\},$$

hence  $\{\sup_n f_n \leq a\} = \bigcap_{n=1}^{\infty} \underbrace{\{f_n \leq a\}}_{\in \mathcal{M}} \in \mathcal{M}$  and hence  $\sup_n f_n$  is measurable. Note that using  $\leq$  was important;  $\{\sup_n f_n < a\} \subsetneq \bigcap_{n=1}^{\infty} \{f_n < a\}$ , since the  $\sup_n f_n$  could equal  $a$ . We could say the following, however:

$$\left\{ \sup_n f_n < a \right\} = \bigcup_{k=1}^{\infty} \left\{ \sup_n f_n \leq a - \frac{1}{k} \right\} = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{ f_n \leq a - \frac{1}{k} \right\} \in \mathcal{M}.$$

Next, we have  $\inf_n f_n = -\sup_n (-f_n)$  so we are done.

For  $\limsup, \liminf$ , we have

$$\limsup_n f_n = \inf_{m \geq 1} \underbrace{\sup_{n \geq m} f_n}_{:= g_m}.$$

$g_m$  is measurable for each  $m \geq 1$ , hence  $\inf_m g_m$  is measurable, hence  $\limsup_n f_n$  is measurable. Similar logic follows for  $\liminf$ .

We could have show, more directly, that

$$\begin{aligned}
\left\{ \limsup_n f_n < a \right\} &= \left\{ \inf_{m \geq 1} \sup_{n \geq m} f_n < a \right\} \\
&= \bigcup_{m=1}^{\infty} \left\{ \sup_{n \geq m} f_n < a \right\} \\
&= \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \sup_{n \geq m} f_n \leq a - \frac{1}{k} \right\} \\
&= \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ f_n \leq a - \frac{1}{k} \right\}.
\end{aligned}$$

■

↪ **Proposition 2.13:** Let  $\{f_n\}$  be a sequence of measurable functions. Then, all of the following sets are also measurable:

$$\begin{aligned}
\{x \in \mathbb{R} : \lim_{n \rightarrow \infty} f_n(x) \text{ exists in } \mathbb{R}\} &=: \left\{ \lim_{n \rightarrow \infty} f_n \text{ exists in } \mathbb{R} \right\}, \\
\{\lim f_n = \infty\}, \{\lim f_n = -\infty\}, \{\lim f_n = c \in \mathbb{R}\}.
\end{aligned}$$

Moreover, if  $\lim_{n \rightarrow \infty} f_n$  exists (in  $\mathbb{R}$  or as  $\pm\infty$ ) a.e. with  $f = \lim_{n \rightarrow \infty} f_n$  a.e. then  $f$  is measurable.

PROOF. We have

$$\begin{aligned}
\{\lim f_n \text{ exists in } \mathbb{R}\} &= \{\limsup f_n = \liminf f_n \text{ and } -\infty < \limsup f_n < \infty\} \\
&= \{-\infty < \liminf f_n < \infty\} \cap \{-\infty < \limsup f_n < \infty\} \cap \{\limsup f_n - \liminf f_n = 0\} \in \mathcal{M}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\{\lim f_n = c\} &= \left\{ x \in \mathbb{R} : \forall k \geq 1, \exists n \geq 1 \text{ s.t. } \forall m \geq n, |f_n(x) - c| \leq \frac{1}{k} \right\} \\
&= \bigcap_{k=1}^{\infty} \bigcup_{\substack{n=1 \\ \forall \varepsilon = \frac{1}{k} > 0}}^{\infty} \bigcap_{\substack{m=n \\ \forall m \geq n}}^{\infty} \left\{ |f_n(x) - c| \leq \frac{1}{k} \right\}.
\end{aligned}$$

■

## §2.2 Approximation by Simple Functions

Given a function  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ , measurable, we may write

$$f = f^+ - f^-,$$

where  $f^+, f^-$  are non-negative measurable functions; so, it suffices to study non-negative measurable functions. For any  $n \geq 1$ , we have

$$f_n^+ := (f^+ \wedge n) \cdot \mathbb{1}_{[-n, n]},$$

i.e., we cap  $f^+$  at  $n$ , and disregard values of  $f^+$  outside of  $[-n, n]$ ; hence we limit our view to a  $2n \times n$  “box”. Then,  $f_n^+$  is non-negative, measurable, bounded (by  $n$ ), compactly supported (zero outside a bounded set), and in particular  $f_n^+ \uparrow$ , with limit

$$\lim_{n \rightarrow \infty} f_n^+ = f^+.$$

An identical construction follows for  $f^-$  with

$$f_n^- := (f^- \wedge n) \mathbb{1}_{[-n, n]},$$

with  $f_n^- \uparrow$  and

$$\lim_{n \rightarrow \infty} f_n^- = f^-.$$

Fix some  $n$  and consider  $f_n^+$ . For  $k = 0, 1, 2, \dots, 2^n n$ , define

$$A_{n,k} := \left\{ x \in [-n, n] : \frac{k}{2^n} \leq f_n^+(x) < \frac{k+1}{2^n} \right\} = \left\{ \frac{k}{2^n} \leq f_n^+ < \frac{k+1}{2^n} \right\} \cap [-n, n] \in \mathcal{M},$$

noting that  $A_{n,k} \cap A_{n,\ell} = \emptyset$  if  $k \neq \ell$ . Set now

$$\varphi_n := \sum_{k=0}^{n \cdot 2^n} \mathbb{1}_{A_{n,k}} \frac{k}{2^n} = \sum_{k=0}^{n \cdot 2^n} \begin{cases} \frac{k}{2^n} & \text{if } x \in A_{n,k} \\ 0 & \text{else} \end{cases}.$$

We call  $\varphi_n$  a “simple function”; more generally:

↪ **Definition 2.4:**  $\varphi$  is a *simple function* if  $\varphi = \sum_{k=1}^L \mathbb{1}_{E_k} \cdot a_k$  where  $L$  a positive integer,  $a_k$ 's are constant,  $E_k$ 's are measurable sets of finite measure.

Moreover, note that  $\varphi_n \uparrow$ ; at each new stage  $n \rightarrow n+1$ , the regions are cut in two,  $A_{n,k} = A_{n+1,2k} \cup A_{n+1,2k+1}$ . In addition, we have  $\varphi_n \leq f_n^+ \leq f^+$  for all  $n$ . Moreover, we have the following:

↪ **Proposition 2.14:**

$$\lim_{n \rightarrow \infty} \varphi_n(x) = f^+(x)$$

for all  $x \in \mathbb{R}$ .

PROOF. For all  $x \in \mathbb{R}$ , for sufficiently large  $n$  we have that  $x \in [-n, n]$  and so  $f^+(x) = f^+(x) \mathbb{1}_{[-n, n]}(x)$ . Assume for now  $f^+ < \infty$ . Then, for sufficiently large (r?)  $n$ , we can ensure  $f^+(x) < n$  and so  $f^+(x) = f_n^+(x)$  for such an  $x$ . Further, we have that  $x \in A_{n,k}$  for some  $k$  so  $\varphi_n(x) = \frac{k}{2^n}$  and  $f_n^+(x) < \frac{k+1}{2^n}$  and thus

$$0 \leq f_n^+(x) - \varphi_n(x) < \frac{k+1}{2^n} - \frac{k}{2^n} = 2^{-n}$$

by construction and so  $0 \leq f^+(x) - \varphi_n(x) \leq 2^{-n}$  and thus  $\lim_{n \rightarrow \infty} \varphi_n(x) = f^+(x)$ .

In the case that  $f^+(x) = \infty$ , then  $\varphi_n(x) = n$  for all sufficiently large  $n$  hence

$$\lim_{n \rightarrow \infty} \varphi_n(x) = \lim_{n \rightarrow \infty} n = \infty = f^+(x).$$

■

↪ **Theorem 2.1:** If  $g$  is measurable and non-negative, there exists a sequence of simple functions  $\{\varphi_n\}$  such that  $\varphi_n \uparrow$  and  $\lim_{n \rightarrow \infty} \varphi_n(x) = g(x)$  for every  $x \in \mathbb{R}$ .

We can repeat this same construction and proof for  $f^-$  with a sequence  $\tilde{\varphi}_n$ . Even better:

↪ **Theorem 2.2:** If  $f$  is measurable, then  $\exists$  a sequence of simple functions  $\{\psi_n\}$  such that  $|\psi_n| \uparrow$  and  $|\psi_n| \leq |f|$  for all  $n$  and for all  $x \in \mathbb{R}$ ,  $\lim_{n \rightarrow \infty} \psi_n(x) = f(x)$ .

PROOF. Take  $\psi_n = \varphi_n - \tilde{\varphi}_n$  as above; then for all  $x \in \mathbb{R}$ , at least one of  $\varphi_n(x), \tilde{\varphi}_n(x)$  equals zero. Then

$$|\psi_n| = \varphi_n + \tilde{\varphi}_n < f^+ + f^- = |f|,$$

and

$$\lim_{n \rightarrow \infty} \psi_n(x) = \lim_{n \rightarrow \infty} \varphi_n(x) - \lim_{n \rightarrow \infty} \tilde{\varphi}_n(x) = f^+ - f^- = f.$$

■

↪ **Definition 2.5 (Step Function):**  $\theta$  a *step function* if it takes the form

$$\theta(x) = \sum_{k=1}^L a_k \mathbb{1}_{I_k}(x),$$

where  $L \in \mathbb{N}$ ,  $a_k$ 's constant, and  $I_k$  finite, open intervals.

↪ **Theorem 2.3:** If  $f$  is measurable, then there exists a sequence of step functions  $\{\theta_n\}$  such that

$$\lim_{n \rightarrow \infty} \theta_n(x) = f(x) \text{ for **almost every** } x \in \mathbb{R}.$$

In particular, we do not have pointwise convergence as for general simple functions, but we have convergence outside a zero-measure set.

PROOF. Assume, wlog, that  $f$  non-negative (by the previous construction, we can “split”  $f$  if not and approximate its positive, negative parts). Given  $A \in \mathcal{M}$  with finite measure, recall that for every  $\varepsilon > 0$ , there exists finitely many finite open intervals  $I_1, \dots, I_N$  such that

$$m\left(A \triangle \left(\bigcup_{i=1}^N I_i\right)\right) < \varepsilon.$$

By renaming/rearranging  $I_i$ 's if necessary, we may assume that  $I_i$ 's are disjoint; hence

$$\mathbb{1}_{\bigcup_{i=1}^N I_i} = \sum_{i=1}^N \mathbb{1}_{I_i}.$$

Put

$$\theta_A := \sum_{i=1}^N \mathbb{1}_{I_i},$$

noting this is indeed a step function as the name suggests. Then, remark that

$$m(\underbrace{\{x \in \mathbb{R} : \mathbb{1}_A(x) \neq \theta_A(x)\}}_{=A \Delta (\bigcup_{i=1}^N I_i)}) < \varepsilon.$$

Since  $f$  measurable and non-negative,  $\exists \{\varphi_n\}$  sequence of simple functions with limit  $f$ . In particular,

$$\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}}.$$

Applying our above analysis to each  $A_{n,k}$ , then, we have that for any  $n \geq 1$  and  $k = 0, 1, \dots, n2^n$  we can find a step function  $\theta_{n,k}$  such that

$$m(\{x \in \mathbb{R} : \mathbb{1}_{A_{n,k}} \neq \theta_{n,k}(x)\}) < \frac{1}{2^n(n2^n + 1)} \quad (" = \varepsilon").$$

Put then

$$\theta_n := \sum_{k=0}^{n2^n} \frac{k}{2^n} \theta_{n,k},$$

which is itself a step function. Put

$$E_n := \{x \in \mathbb{R} : \theta_n(x) \neq \varphi_n(x)\}.$$

Then,

$$m(E_n) \leq m\left(\bigcup_{k=0}^{n2^n} \{\theta_{n,k} \neq \mathbb{1}_{A_{n,k}}\}\right) \leq \sum_{k=0}^{n2^n} m(\{\theta_{n,k} \neq \mathbb{1}_{A_{n,k}}\}) \leq 2^{-n}.$$

The  $\varphi_n$ 's are chosen such that  $\forall x \in \mathbb{R}, |\varphi_n(x) - f_n(x)| \leq \frac{1}{2^n}$ . Putting

$$F_n := \{x \in \mathbb{R} : |\theta_n(x) - f_n(x)| > 2^{-n}\},$$

then remark that  $F_n \subseteq E_n$  so  $m(F_n) \leq \frac{1}{2^n}$ .

We claim now that for a.e.  $x \in \mathbb{R}$ ,  $\exists m \geq 1$  such that  $\forall n \geq m, |\theta_n(x) - f_n(x)| \leq \frac{1}{2^n}$ , remarking that such an  $m$  is *dependent* on  $x$ . Consider the complement of this statement; if this set has measure 0, we are done. The logical negation would be "for every  $m \geq 1$ , exist  $n \geq m$  such that  $|\theta_n(x) - f_n(x)| > 2^{-n}$ ", which is equivalent to the set

$$\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in \mathbb{R} : |\theta_n(x) - f_n(x)| > 2^{-n}\} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n.$$

Let  $B_m := \bigcup_{n=m}^{\infty} F_n$ ; notice  $B_m \downarrow$ . Then, by continuity from above \*\*\*\*

$$m\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n\right) = \lim_{m \rightarrow \infty} m(B_m) \leq \lim_{m \rightarrow \infty} \sum_{n=m}^{\infty} m(F_n) \leq \lim_{m \rightarrow \infty} \sum_{n=m}^{\infty} \frac{1}{2^n} = 0,$$

since the tail of a convergent series must converge to zero. Hence, the set has measure 0 as desired so for almost every  $x \in \mathbb{R}$  there exists  $m \geq 1$  such that for all  $n \geq m$ ,  $|\theta_n - f_n| \leq \frac{1}{2^n}$ , hence almost everywhere  $\lim_{n \rightarrow \infty} (\theta_n - f_n) = 0$ . Therefore, almost everywhere,

$$\theta_n = (\theta_n - f_n) + f_n \xrightarrow{n \rightarrow \infty} f.$$

■

In this proof, we have proven (and then used) more generally:

↪ **Lemma 2.1** (Borel-Cantelli Lemma): If  $\{F_n\} \subseteq \mathcal{M}$  such that  $\sum_{n=1}^{\infty} m(F_n) < \infty$ , then

$$m\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n\right) = 0.$$

PROOF. Remark that  $\bigcup_{n=m}^{\infty} F_n$  a decreasing sequence of functions indexed by  $m$ . By continuity of the measure and subadditivity,

$$m\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n\right) = \lim_{m \rightarrow \infty} m\left(\bigcup_{n=m}^{\infty} F_n\right) \leq \lim_{m \rightarrow \infty} \sum_{n=m}^{\infty} m(F_n) = 0,$$

since the tail of a converging sequence must converge to zero. ■

### §2.3 Convergence Almost Everywhere vs Convergence in Measure

↪ **Definition 2.6** (Convergence Almost Everywhere): For measurable functions  $\{f_n\}, f$  we say  $f_n$  converges to  $f$  a.e. and write  $f_n \rightarrow f$  a.e. if for almost every  $x \in \mathbb{R}$ ,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

Similarly, we say  $f_n \rightarrow f$  a.e. on  $A$  if  $\exists B \subseteq A$  with  $m(B) = 0$  such that  $\forall x \in A - B$ ,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

↪ **Definition 2.7** (Convergence in Measure): For measurable, finite-valued functions  $\{f_n\}, f$  we say  $f_n$  converges to  $f$  in measure and write  $f_n \rightarrow f$  in measure if for every  $\delta > 0$ ,

$$\lim_{n \rightarrow \infty} m(\{x \in \mathbb{R} : |f_n(x) - f(x)| \geq \delta\}) = 0.$$

Similarly, we say  $f_n \rightarrow f$  in measure on  $A$  if  $\forall \delta > 0$ ,  $\lim_{n \rightarrow \infty} m(\{x \in A : |f_n(x) - f(x)| \geq \delta\}) = 0$ .

↪ **Proposition 2.15**: Given finite-valued measurable functions  $\{f_n\}, f$  and  $A \in \mathcal{M}$  with finite measure, then if  $f_n \rightarrow f$  a.e. on  $A$ , then  $f_n \rightarrow f$  in measure on  $A$ .

PROOF. For all  $\delta > 0$ ,

$$\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in A : |f_n(x) - f(x)| > \delta\} \subseteq \{x \in A : \lim_{n \rightarrow \infty} f_n(x) \neq f(x)\}.$$

The set on the RHS has measure zero and thus so does the left one. Then,

$$\lim_{m \rightarrow \infty} m\left(\bigcup_{n=m}^{\infty} \{x \in A : |f_n(x) - f(x)| > \delta\}\right) = 0$$

by continuity, and

$$\{|f_m - f| > \delta\} \subseteq \bigcup_{n=m}^{\infty} \{|f_n - f| > \delta\}$$

hence  $m(\{|f_m - f| > \delta\}) \leq m(\bigcup_{n=m}^{\infty} \{|f_n - f| > \delta\}) \xrightarrow{m \rightarrow \infty} 0$ . ■

⊗ **Example 2.1:** We give an example of why the assumption that  $m(A) < \infty$  is necessary. Let,  $f_n = \mathbb{1}_{[n, \infty)}$  and  $f \equiv 0$ . Then,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for every  $x \in \mathbb{R}$ . But  $m(\{x \in \mathbb{R} : |f_n(x) - f(x)| = 1\}) = m([n, \infty)) = \infty$ .

In general, the converse statement  $f_n \rightarrow f$  in measure does *not* imply that  $f_n \rightarrow f$  almost everywhere, even on finite measure sets. Put  $\varphi_{1,1} = \mathbb{1}_{[0,1)}$ ,  $\varphi_{2,1} = \mathbb{1}_{[0, \frac{1}{2})}$ ,  $\varphi_{2,2} = \mathbb{1}_{[\frac{1}{2}, 1)}$ ,  $\varphi_{3,1} = \mathbb{1}_{[0, \frac{1}{3})}$ ,  $\varphi_{3,2} = \mathbb{1}_{[\frac{1}{3}, \frac{2}{3})}$ ,  $\varphi_{3,3} = \mathbb{1}_{[\frac{2}{3}, 1)}$ , or in general  $\varphi_{k,j} = \mathbb{1}_{[\frac{j-1}{k}, \frac{j}{k})}$  for  $j = 1, \dots, k$ . Reorder  $\varphi_{k,j}$  “lexicographically” into  $\{f_n\}$ . Then, we claim  $f_n \rightarrow 0$  in measure on  $[0, 1)$ ; for any  $\delta \in (0, 1)$ ,

$$m(\{|f_n - 0| > \delta\}) = \frac{1}{k(n)} \rightarrow 0,$$

where  $k(n)$  the “row” that  $f_n$  comes from. Hence,  $f_n$  converges in measure. However,  $f_n$  does not converge almost everywhere on  $[0, 1)$ . Indeed, for each  $x \in \mathbb{R}$  and  $k \geq 1$ , there exists a *unique*  $j$  such that  $x \in [\frac{j-1}{k}, \frac{j}{k})$  hence  $\varphi_{k,j}(x) = 1$ , so in other notation there always exists an  $n$  such that  $f_n(x) = 1$ , and so precisely  $f_n(x) = 1$  for infinitely many  $n$ . Hence, we do not have convergence everywhere (in fact, anywhere).

↪ **Proposition 2.16:** Given  $\{f_n\}$ ,  $f$  measurable, finite-valued functions, if  $f_n \rightarrow f$  in measure, then there exists a subsequence  $\{f_{n_k}\}$  such that  $f_{n_k} \rightarrow f$  a.e. as  $k \rightarrow \infty$ .

PROOF. Assume  $f_n \rightarrow f$  in measure, that is for every  $\delta > 0$ ,  $m(\{|f_n - f| > \delta\}) \rightarrow 0$ .

Hence, for all  $k \geq 1$ , with  $\delta = \frac{1}{k}$ , we have that for some sufficiently large  $n_k$ , we have

that  $m\left(\underbrace{\{|f_{n_k} - f| > \frac{1}{k}\}}_{:=A_k}\right) \leq \frac{1}{k^2}$ , hence  $\sum_{k=1}^{\infty} m(A_k) < \infty$ . Hence,

$$m\left(\bigcap_{\ell=1}^{\infty} \bigcup_{k=\ell}^{\infty} A_k\right) = \lim_{\ell \rightarrow \infty} m\left(\bigcup_{k=\ell}^{\infty} A_k\right) \leq \lim_{\ell \rightarrow \infty} \sum_{k=\ell}^{\infty} m(A_k) = 0,$$



since  $\sum_{k=\ell}^{\infty} m(A_k)$  the tail of a converging series. Hence, complementing the above, a.e. there  $\exists \ell$  such that for every  $k \geq \ell$ ,  $|f_{n_k} - f| \leq \frac{1}{k}$  and so  $\lim_{k \rightarrow \infty} |f_{n_k} - f| = 0$  almost everywhere, and so  $f_{n_k} \rightarrow f$  a.e. (as  $k \rightarrow \infty$ ). ■

↪ **Proposition 2.17** (Subsequence Test): Given  $\{f_n\}$ ,  $f$  measurable, finite-valued functions,  $f_n \rightarrow f$  in measure  $\Leftrightarrow$  for every subsequence  $\{n_k\}$ , there exists a subsubsequence  $\{n_{k_\ell}\} \subset \{n_k\}$  such that  $f_{n_{k_\ell}} \rightarrow f$  in measure as  $\ell \rightarrow \infty$ .

PROOF.  $\Rightarrow$  is clear. For  $\Leftarrow$ , suppose towards a contradiction that  $f_n \not\rightarrow f$  in measure. Then,  $\exists \delta > 0$  and subsequence  $\{n_k\}$   $m(\{|f_{n_k} - f| > \delta\}) > \delta$  for every  $k$ . By the assumption of the RHS, there exists a further subsequence  $\{n_{k_\ell}\}$  such that  $f_{n_{k_\ell}} \rightarrow f$  in measure. This is a contradiction. ■

⊗ **Example 2.2** (Assignment Exercise): Prove that if  $f_n \rightarrow f$  in measure and  $g_n \rightarrow g$  in measure,  $f_n g_n \rightarrow fg$  in measure (everything finite valued, measurable).

## §2.4 Egorov's Theorem and Lusin's Theorem

Recall that if  $f$  is measurable, then  $\exists \{\theta_n\}$  sequence of step functions such that  $\theta_n \rightarrow f$  almost everywhere.

↪ **Theorem 2.4** (Egorov's): Given  $\{f_n\}$ ,  $f$  measurable functions and  $A \in \mathcal{M}$  with  $m(A) < \infty$ , if  $f_n \rightarrow f$  a.e. on  $A$ , then  $\forall \varepsilon > 0$ , there exists a closed subset  $A_\varepsilon \subseteq A$  with  $m(A \setminus A_\varepsilon) \leq \varepsilon$  such that  $f_n \rightarrow f$  uniformly on  $A_\varepsilon$ .

PROOF. We assume first  $f$  is finite-valued on  $A$  (otherwise, replace  $A$  with  $A \cap \{-\infty < f < \infty\}$ ; we'll deal with  $\{f = \pm\infty\}$  later). We want to show that  $\forall \varepsilon > 0, \exists$  closed  $A_\varepsilon \subseteq A$  s.t.  $m(A \setminus A_\varepsilon) < \varepsilon$  and  $\sup_{x \in A_\varepsilon} |f_n(x) - f(x)| \rightarrow 0$  as  $n \rightarrow \infty$ .

For each  $k \geq 1$  and  $n \geq 1$ , put

$$E_n^{(k)} := \left\{ x \in A : |f_j(x) - f(x)| \leq \frac{1}{k} \forall j \geq n \right\}.$$

For fixed  $k$ , remark that  $E_n^{(k)} \subseteq E_{n+1}^{(k)}$ , i.e.  $E_n^{(k)}$  increasing (wrt  $n$ ), so we may consider

$$\bigcup_{n=1}^{\infty} E_n^{(k)} = \left\{ x \in A : \exists n \geq 1 \text{ s.t. } \forall j \geq n, |f_j(x) - f(x)| \leq \frac{1}{k} \right\} \supseteq \left\{ x \in A : \lim_{n \rightarrow \infty} f_n(x) = f(x) \right\} =: A'.$$

By assumption,  $m(A') = m(A)$ , so by continuity and the superset relation above,  $m(A) = m(A') \leq m\left(\bigcup_{n=1}^{\infty} E_n^{(k)}\right) = \lim_{n \rightarrow \infty} m(E_n^{(k)}) \leq m(A)$ , and thus  $\lim_{n \rightarrow \infty} m(E_n^{(k)}) = m(A)$  for every  $k \geq 1$ .

Given, then, any  $\varepsilon > 0$ , there exists a  $n_k$  such that  $m(A \setminus E_{n_k}^{(k)}) = m(A) - m(E_{n_k}^{(k)}) < \frac{1}{2^k} \frac{\varepsilon}{2}$ . Set

$$B := A \setminus \left( \bigcap_{k=1}^{\infty} E_{n_k}^{(k)} \right),$$

then

$$m(B) = m\left(\bigcup_{k=1}^{\infty} A \setminus E_{n_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} m(A \setminus E_{n_k}^{(k)}) \leq \frac{\varepsilon}{2}.$$

Put

$$\tilde{A} := A \setminus B = \bigcap_{k=1}^{\infty} E_{n_k}^{(k)}.$$

Then, if  $x \in \tilde{A}$ , then  $x \in E_{n_k}^{(k)}$  for every  $k$ , and hence for every  $k \geq 1$  and  $j \geq n_k$ ,  $|f_j(x) - f(x)| \leq \frac{1}{k}$ . This shows then that  $f_n \rightarrow f$  uniformly on  $\tilde{A}$ . By regularity of  $m$ , there exists a closed  $A_\varepsilon \subseteq \tilde{A}$  such that  $m(\tilde{A} \setminus A_\varepsilon) \leq \frac{\varepsilon}{2}$ . Then,  $f_n \rightarrow f$  uniformly on  $A_\varepsilon$ , and  $m(A \setminus A_\varepsilon) = m(A \setminus \tilde{A}) + m(\tilde{A} \setminus A_\varepsilon) < \varepsilon$ .

Now, if  $f = \infty / -\infty$  on  $A$ , then  $A = A^\infty \cup A^{-\infty} \cup A^\mathbb{R}$  (with  $A^\bullet := \{f = \bullet\} \cap A$ ). The last case is done. For  $A^\infty$  (similar construction for  $A^{-\infty}$ ), define for every  $k, n \geq 1$ ,

$$E_n^{(k)} := \{x \in A : f_j(x) > k \forall j \geq n\}.$$

Then, the remainder of the proof follows precisely the same for the sequence of sets  $E_n^{(k)}$ . ■

**Remark 2.3:**

1. The assumption  $m(A) < \infty$  is necessary. For instance  $f_n = \mathbb{1}_{[n, \infty)} \rightarrow 0$  pointwise, but for any  $a \in \mathbb{R}$ ,  $f_n$  does not converge to 0 uniformly on  $(a, \infty)$ .
2. In general, Egorov's  $\nRightarrow f_n \rightarrow f$  uniformly a.e.. For instance, on  $[0, 1]$ , let  $f_n(x) = x^n$  and  $f(x) \equiv 0$ . For every  $x \in [0, 1)$ ,  $f_n(x) \rightarrow f(x)$  as  $n \rightarrow \infty$ . Hence,  $f_n \rightarrow f$  a.e. on  $[0, 1]$  (the only point that doesn't converge, indeed, is at 1). If  $A \subseteq [0, 1]$  is closed such that  $1 \in A$ , then  $f_n \nrightarrow f$  uniformly on  $A$ . To see this, let  $\{x_m\} \subseteq A$  such that  $x_m \uparrow$  and  $\lim_{m \rightarrow \infty} x_m = 1$ . Then, for any fixed  $n$ ,

$$\sup_{x \in A} |f_n(x) - f(x)| \geq \sup_m |f_n(x_m) - f(x_m)| = \sup_m x_m^n = 1,$$

hence  $f_n$  does not converge uniformly on  $A$ .

**↪ Theorem 2.5** (Lusin's Theorem): Given  $f$  measurable and finite-valued and  $A \in \mathcal{M}$  with  $m(A) < \infty$ , for all  $\varepsilon > 0$ , there exists a closed  $A_\varepsilon \subseteq A$  with  $m(A \setminus A_\varepsilon) < \varepsilon$  such that  $f|_{A_\varepsilon}$  is continuous.

**Remark 2.4:** Lusin's Theorem states that  $f|_{A_\varepsilon}$  is continuous as a function on  $A_\varepsilon$ , which is *not* the same as saying  $f$  as a function on  $A$  is continuous at points in  $A_\varepsilon$ .

For instance,  $f = \mathbb{1}_{\mathbb{Q} \cap [0,1]}$  is not continuous anywhere on  $[0,1]$ . However,  $f|_{\mathbb{Q} \cap [0,1]}$  is constant and therefore continuous on  $\mathbb{Q} \cap [0,1]$ .

PROOF. Let  $\{\theta_n\}$  be a sequence of step functions such that  $\theta_n \rightarrow f$  a.e. on  $A$ . Note that  $\theta_n$  piecewise constant and hence piecewise continuous. Given  $\varepsilon > 0$  and  $n \geq 1$ , we can find an open set  $E_n$  such that  $\theta_n|_{E_n^c}$  is continuous and  $m(E_n) \leq \frac{\varepsilon}{2} \frac{1}{2^n}$ . Meanwhile, Egorov's implies that there exists a closed  $B \subseteq A$  such that  $m(A \setminus B) \leq \frac{\varepsilon}{2}$  such that  $\theta_n \rightarrow f$  uniformly on  $B$ . Set

$$A_\varepsilon = B \setminus \bigcup_{n=1}^{\infty} E_n,$$

noting that  $A_\varepsilon \subset A$  closed and

$$m(A \setminus A_\varepsilon) = m(A \setminus B) + m\left(\bigcup_{n=1}^{\infty} E_n\right) = \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} m(E_n) \leq \varepsilon.$$

Finally, on  $A_\varepsilon$ ,  $\theta_n \rightarrow f$  uniformly and  $\theta_n|_{A_\varepsilon}$  continuous, and hence  $f|_{A_\varepsilon}$  continuous (uniform limit of continuous functions is continuous). ■

**Remark 2.5:**

1. Lusin's Theorem  $\nRightarrow f$  is continuous almost everywhere in general. For instance, recall that fat Cantor set  $\tilde{C}$ , with  $m(\tilde{C}) = \frac{1}{2}$ . Let  $f = \mathbb{1}_{\tilde{C}}$ .  $f$  is NOT continuous a.e. on  $[0,1]$ , i.e.  $\forall B \subseteq [0,1]$  with  $m(B) = 1$ ,  $f|_B$  is NOT continuous. To see this, let  $\tilde{D} = [0,1] \setminus \tilde{C}$ . Since  $m(B) = 1$ , then  $m(\tilde{C} \cap B) = m(\tilde{D} \cap B) = \frac{1}{2}$ . Then for any  $x \in \tilde{C} \cap B$ ,  $f|_B$  is NOT continuous at  $x$ . If it were at say some  $x_0 \in \tilde{C} \cap B$ , then there must exist some  $\delta > 0$  such that for any  $x \in (x_0 - \delta, x_0 + \delta) \cap B$ ,  $|f(x) - f(x_0)| < \frac{1}{2}$ . Hence, for any  $x \in (x_0 - \delta, x_0 + \delta) \cap B$ ,  $\frac{1}{2} \leq f(x) \leq \frac{3}{2}$ . However,  $m((x_0 - \delta, x_0 + \delta) \cap B \cap \tilde{D}) > 0$  so it must be that  $\exists y \in (x_0 - \delta, x_0 + \delta) \cap B \cap \tilde{D} \Rightarrow f(y) = 0$ , a contradiction. How, then, does one apply Lusin's; that is,  $\forall \varepsilon > 0$ , there must exist some  $A_\varepsilon \subseteq [0,1]$  such that  $m([0,1] \setminus A_\varepsilon) < \varepsilon$  and  $f|_{A_\varepsilon} < \varepsilon$  (exercise)?
2. (Exercise) The  $\{\theta_n\}$ 's are not continuous on  $\mathbb{R}$ , but you can choose a sequence  $\{\tilde{\theta}_n\}$  to be continuous on  $\mathbb{R}$  such that  $\tilde{\theta}_n \rightarrow f$  a.e..
3. Lusin's Theorem  $\Rightarrow \forall k$  sufficiently large,  $\exists A_k \subseteq A$  closed such that  $m(A \setminus A_k) \leq \frac{1}{k}$  and  $f|_{A_k}$  continuous on  $A_k$ . In fact, we can construct them such that  $A_k \uparrow$  (otherwise replace  $A_k$  with  $\bigcup_{i=1}^k A_i$ ).

## §2.5 Construction of Integrals

### 2.5.1 Integral of Simple Functions

↪ **Definition 2.8:** Given a simple function  $\varphi = \sum_{k=1}^L a_k \mathbb{1}_{E_k}$ , the (Lebesgue) integral of  $\varphi$  is defined as

$$\int_{\mathbb{R}} \varphi(x) \, dx = \int_{\mathbb{R}} \varphi := \sum_{k=1}^L a_k \cdot m(E_k).$$

For any  $A \in \mathcal{M}$ ,  $\mathbb{1}_A \varphi$  is again a simple function and we define

$$\int_A \varphi := \int_{\mathbb{R}} \mathbb{1}_A \varphi.$$

↪ **Proposition 2.18** (Properties of  $\int_{\mathbb{R}} \varphi$ ):

1. (Well-definedness) The written representation of  $\varphi$  is not necessarily unique, but if  $\varphi = \sum_{k=1}^L a_k \mathbb{1}_{E_k} = \sum_{\ell=1}^M b_{\ell} \mathbb{1}_{F_{\ell}}$ , then

$$\sum_{k=1}^L a_k m(E_k) = \sum_{\ell=1}^M b_{\ell} m(F_{\ell}).$$

2. (Linearity) If  $\varphi, \psi$  two simple functions and  $a, b \in \mathbb{R}$ , then  $a\varphi + b\psi$  a simple function, and

$$\int_{\mathbb{R}} a\varphi + b\psi = a \cdot \int_{\mathbb{R}} \varphi + b \cdot \int_{\mathbb{R}} \psi.$$

3. (Finite Additivity) If  $\varphi$  a simple function,  $A, B \in \mathcal{M}$  with  $A \cap B = \emptyset$ , then

$$\int_{A \cup B} \varphi = \int_A \varphi + \int_B \varphi.$$

4. (Monotonicity) If  $\varphi, \psi$  are two simple functions with  $\varphi \leq \psi$ , then  $\int_{\mathbb{R}} \varphi \leq \int_{\mathbb{R}} \psi$ .
5. If  $\varphi$  a simple function then so is  $|\varphi|$  and  $|\int_{\mathbb{R}} \varphi| \leq \int_{\mathbb{R}} |\varphi|$ .

PROOF.

1. wlog, we may assume  $E_k$  and  $F_{\ell}$  are respectively disjoint. Set  $a_0 = b_0 = 0$ ,  $E_0 := \left(\bigcup_{k=1}^L E_k\right)^c$ ,  $F_0 := \left(\bigcup_{\ell=1}^M F_{\ell}\right)^c$  for convenience. Now,  $\{E_0, \dots, E_L\}, \{F_0, \dots, F_M\}$  are two partitions of  $\mathbb{R}$ . In particular, then, for each  $k$ ,  $\mathbb{1}_{E_k} = \sum_{\ell=0}^M \mathbb{1}_{E_k \cap F_{\ell}}$ , since  $E_k = \bigsqcup_{\ell=0}^M (E_k \cap F_{\ell})$ . Now, we have

$$\varphi = \sum_{k=0}^L a_k \mathbb{1}_{E_k} = \sum_{k=0}^L \sum_{\ell=0}^M a_k \mathbb{1}_{E_k \cap F_{\ell}}.$$

Similarly partitioning, we have

$$\varphi = \sum_{\ell=0}^M b_{\ell} \mathbb{1}_{F_{\ell}} = \sum_{\ell=0}^M \sum_{k=0}^L b_{\ell} \mathbb{1}_{E_k \cap F_{\ell}}.$$

If  $E_k \cap F_{\ell} \neq \emptyset$ , then  $a_k = b_{\ell}$ , and thus on the one hand

$$\int_{\mathbb{R}} \varphi = \sum_{k=0}^L \sum_{\ell=0}^M a_k m(E_k \cap F_{\ell})$$

and on the other

$$\int_{\mathbb{R}} \varphi = \sum_{\ell=0}^M \sum_{k=0}^L b_{\ell} m(E_k \cap F_{\ell}),$$

(with summation convention  $0 \cdot \infty = 0$ ). If  $m(E_k \cap F_{\ell}) > 0$ , then  $E_k \cap F_{\ell} \neq \emptyset$  and so  $a_k = b_{\ell}$  and so the two sums agree.

4. Assume  $\varphi = \sum_{k=1}^L a_k \mathbb{1}_{E_k}$ ,  $\psi = \sum_{\ell=1}^M b_{\ell} \mathbb{1}_{F_{\ell}}$ . Repeat the partitioning/rewriting steps from part 1, then note that since  $\varphi \leq \psi$ , if  $E_k \cap F_{\ell} \neq \emptyset$ , it must be that  $a_k \leq b_{\ell}$ , so if  $m(E_k \cap F_{\ell}) > 0$   $a_k \leq b_{\ell}$  and thus the monotonicity follows. ■

## 2.5.2 Integral of Non-Negative Functions

↪ **Definition 2.9:** If  $f$  a non-negative, measurable function then the integral of  $f$  is given by

$$\int_{\mathbb{R}} f(x) dx = \int_{\mathbb{R}} f := \sup \left\{ \int_{\mathbb{R}} \varphi : \varphi \text{ is simple and } \varphi \leq f \right\}.$$

↪ **Proposition 2.19:** The definition above agrees with that for simple functions that are also non-negative, namely this definition is consistent with the previous.

PROOF. Let  $\varphi$  be non-negative. Then  $\varphi \leq \varphi$  certainly so the first definition  $\int_{\mathbb{R}} \varphi \leq \sup \{\dots\}$ . Conversely, it suffices to show that for any non-negative simple  $\psi \leq \varphi$ ,  $\int_{\mathbb{R}} \psi \leq \int_{\mathbb{R}} \varphi$ , using the first definition. But this simply follows from monotonicity of  $\int$ , and we are done. ■

**Remark 2.6:** Given  $f \geq 0$  and measurable, this definition implies that there exists a sequence  $\{\varphi_n\}$  of simple functions such that  $\varphi_n \leq f$  and  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n = \int_{\mathbb{R}} f$ . We would like to show that, in some sense, the choice of  $\{\varphi_n\}$  is arbitrary.

↪ **Theorem 2.6:** Suppose  $f \geq 0$  and measurable. If  $\{\varphi_n\}$  a sequence of simple functions such that  $\varphi_n \uparrow$  and  $\lim_{n \rightarrow \infty} \varphi_n = f$  pointwise, then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n = \int_{\mathbb{R}} f.$$

PROOF. Since  $\varphi_n \leq f$  for all  $n \geq 1$ , then  $\int_{\mathbb{R}} \varphi_n \leq \int_{\mathbb{R}} f$  and so  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n \leq \int_{\mathbb{R}} f$  (nothing the limit on the LHS necessarily always exists by monotonicity). On the other hand, it suffices to show that  $\forall \psi \leq f$  simple, that  $\int_{\mathbb{R}} \psi \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n$ . Assume  $\psi = \sum_{k=1}^L a_k \mathbb{1}_{E_k} = \sum_{k=0}^L a_k \mathbb{1}_{E_k}$  where  $\{E_0, \dots, E_L\}$  forms a partition of  $\mathbb{R}$ . Since

$$\int_{\mathbb{R}} \psi = \sum_{k=0}^L a_k m(E_k)$$

and

$$\int_{\mathbb{R}} \varphi_n = \sum_{k=0}^L \int_{E_k} \varphi_n$$

by finite additivity. It suffices to show then that for each  $k = 0, \dots, L$ ,  $a_k m(E_k) \leq \lim_{n \rightarrow \infty} \int_{E_k} \varphi_n$ .

First, if  $a_k = 0$  or  $m(E_k) = 0$ , then we are done. Assume  $a_k, m(E_k) > 0$ . For each fixed  $k$ ,  $\lim_{n \rightarrow \infty} \varphi_n = f \geq \psi$  so for every  $x \in E_k$ ,  $\lim_{n \rightarrow \infty} \varphi_n(x) \geq \psi(x) = a_k$ . For any  $\varepsilon > 0$ , put

$$C_n^\varepsilon := \{x \in E_k : \varphi_n(x) \geq (1 - \varepsilon)a_k\}.$$

Since  $\varphi_n \leq \varphi_{n+1}$ ,  $C_n^\varepsilon \uparrow$  wrt  $n$ . Then note

$$\bigcup_{n=1}^{\infty} C_n^\varepsilon = E_k.$$

Then,

$$\lim_{n \rightarrow \infty} \int_{E_k} \varphi_n = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \mathbb{1}_{E_k} \varphi_n \geq \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \mathbb{1}_{C_n^\varepsilon} \varphi_n \geq \lim_{n \rightarrow \infty} (1 - \varepsilon)a_k m(C_n^\varepsilon) = (1 - \varepsilon)a_k m(E_k),$$

where we use the fact that  $\mathbb{1}_{E_k} \varphi_n \geq \mathbb{1}_{C_n^\varepsilon} \varphi_n \geq (1 - \varepsilon)a_k \mathbb{1}_{C_n^\varepsilon}$  and  $\lim_{n \rightarrow \infty} m(C_n^\varepsilon) = m(\bigcup_{n=1}^{\infty} C_n^\varepsilon) = m(E_k)$ . Since  $\varepsilon$  arbitrary, then

$$\lim_{n \rightarrow \infty} \int_{E_k} \varphi_n \geq a_k m(E_k),$$

and we are done. ■

↪ **Corollary 2.4:** For any  $f \geq 0$  measurable, if  $\forall n \geq 1, k = 0, 1, \dots, n2^n$  with  $A_{n,k} := \left\{ \frac{k}{2^n} \leq f < \frac{k+1}{2^n} \right\}$ , then

$$\int_{\mathbb{R}} f = \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} m(A_{n,k}).$$

PROOF. Let  $\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}}$ , then  $\varphi_n \uparrow$  and  $\varphi_n \rightarrow f$ . ■

↪ **Proposition 2.20** (Properties of Integral of Non-Negative Functions):

1. (Well-definedness) If  $f, g \geq 0$  measurable such that  $f = g$  a.e., then  $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$ .
2. (Linearity) For any  $f, g \geq 0$  measurable and  $a, b \geq 0$ , then  $\int_{\mathbb{R}} (af + bg) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g$ .
3. (Monotonicity) If  $f, g \geq 0$  measurable and  $f \leq g$  a.e., then  $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$ .
4. i. Let  $f \geq 0$  measurable, then  $\int_{\mathbb{R}} f = 0 \Leftrightarrow f \equiv 0$  a.e.  
 ii. Let  $f \geq 0$  measurable,  $A \in \mathcal{M}$ . Then  $\int_A f = 0 \Leftrightarrow$  either  $f \equiv 0$  a.e. on  $A$  or  $m(A) = 0$ .  
 iii. Let  $f \geq 0$  measurable, then if  $\int_{\mathbb{R}} f < \infty$  then  $f$  is finite valued a.e.
5. (Markov's Inequality) Let  $f \geq 0$  measurable and  $0 < a < \infty$ . Then,  $m(\{f > a\}) \leq \frac{1}{a} \int_{\mathbb{R}} f$ . In particular, if the RHS is finite,  $\lim_{a \rightarrow \infty} m(\{f > a\}) = 0$ , in fact in  $O\left(\frac{1}{a}\right)$ .

PROOF.

1. Let  $\{\varphi_n\}, \{\psi_n\}$  sequences of simple functions such that both are monotonically increasing with  $\varphi_n \rightarrow f, \psi_n \rightarrow g$ . Put  $h_n := \varphi_n \mathbb{1}_{\{f=g\}} + \psi_n \mathbb{1}_{\{f \neq g\}}$ ; then  $h_n$  again simple,  $h_n \uparrow$ , and  $h_n \rightarrow g$  everywhere. Then,

$$\int_{\mathbb{R}} g = \lim_n \int_{\mathbb{R}} h_n = \lim_n \left( \int_{\{f=g\}} \varphi_n + \int_{\{f \neq g\}} \psi_n \right) = \lim_n \int_{\{f=g\}} \varphi_n.$$

Meanwhile,

$$\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} \varphi_n = \lim_n \left( \int_{\{f=g\}} \varphi_n + \int_{\{f \neq g\}} \varphi_n \right) = \lim_n \int_{\{f=g\}} \varphi_n,$$

and so  $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$ .

2. Take  $\{\varphi_n\}, \{\psi_n\}$  as in the previous proof. Then  $\{h_n : a\varphi_n + b\psi_n\}$  again a sequence of monotonically increasing simple functions with limit  $af + bg$ . Then

$$\int_{\mathbb{R}} (af + bg) = \lim_n \int_{\mathbb{R}} h_n = \lim_n \int_{\mathbb{R}} (a\varphi_n + b\psi_n) = \lim_n \left( a \int_{\mathbb{R}} \varphi_n + b \int_{\mathbb{R}} \psi_n \right) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g.$$

3. wlog, assume that  $f \leq g$  everywhere by replacing  $f$  with  $f \mathbb{1}_{\{f \leq g\}}$ . Then,  $\{\varphi : \text{simple}, \varphi \leq f\} \subseteq \{\varphi : \text{simple}, \varphi \leq g\}$  and so  $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$ .
4. i.  $\Leftarrow$  clear. Conversely, we would like to prove that if  $A = \{f > 0\}, m(A) = 0$ . Put  $A_n := \{f \geq \frac{1}{n}\}$  for  $n \geq 1$ . Then,  $A_n \uparrow$  and  $\bigcup_{n=1}^{\infty} A_n = A$ . By continuity of  $m$ ,

$$m(A) = \lim_n m(A_n).$$

Suppose towards a contradiction that  $m(A) = \delta > 0$ . Then,  $\delta = \lim_n m(A_n)$ , and so must exist  $N \geq 1$  such that  $m(A_N) \geq \frac{\delta}{2}$ . Since  $f \geq f \mathbb{1}_{A_N} \geq \frac{1}{N} \mathbb{1}_{A_N}$ . By monotonicity,  $\int_{\mathbb{R}} f \geq \int_{\mathbb{R}} \frac{1}{N} \mathbb{1}_{A_N} = \frac{1}{N} m(A_N) \geq \frac{1}{N} \frac{\delta}{2} > 0$ , a contradiction.

ii. By i.,  $\int_A f = 0 \Leftrightarrow \mathbb{1}_A f \equiv 0$  a.e. on  $\mathbb{R}$ . If  $m(A) = 0$ , then  $\mathbb{1}_A \equiv 0$  a.e. so  $\mathbb{1}_A f \equiv 0$  a.e.. Else, if  $m(A) > 0$ , then  $f \equiv 0$  a.e. on  $A$ .

iii. Put  $A := \{f = \infty\}$ . Assume towards a contradiction that  $m(A) = \delta > 0$ . Then, for every  $n \geq 1, f \geq f \mathbb{1}_A \geq n \mathbb{1}_A$  and so  $\int_{\mathbb{R}} f \geq \int_{\mathbb{R}} n \mathbb{1}_A = nm(A) = n\delta$ . But this holds for any arbitrary  $n$ , so  $\int_{\mathbb{R}} f = \infty$ , a contradiction.

5. Put  $A_a := \{f > a\}$ . Then  $f \geq f \mathbb{1}_{A_a} > a \mathbb{1}_{A_a}$  so  $\int_{\mathbb{R}} f \geq am(A_a)$ .

■

### 2.5.3 Integral of General Measurable, Integrable Functions

↪ **Definition 2.10:** For  $f$  measurable,  $\int_{\mathbb{R}} f := \int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-$ , provided that at least one of  $\int_{\mathbb{R}} f^+, \int_{\mathbb{R}} f^-$  is finite; in particular,  $\int_{\mathbb{R}} f$  may be finite or infinite.

**Remark 2.7:** Only having  $\int_{\mathbb{R}} f$  being defined is not sufficient for the desirable properties (linearity, monotonicity) to hold.

↪ **Definition 2.11** (Integrable): A measurable function  $f$  is called *integrable*, denoted  $f \in L^1(\mathbb{R})$ , if both  $\int_{\mathbb{R}} f^+ < \infty$  and  $\int_{\mathbb{R}} f^- < \infty$ . Note that

$$\begin{aligned} f \in L^1(\mathbb{R}) &\Leftrightarrow \int_{\mathbb{R}} |f| < \infty \text{ (since } \int_{\mathbb{R}} |f| = \int_{\mathbb{R}} f^+ + \int_{\mathbb{R}} f^-) \\ &\Leftrightarrow \int_{\mathbb{R}} f \text{ finite valued.} \end{aligned}$$

↪ **Proposition 2.21** (Properties of Integrals of Integrable Functions):

1.  $|\int_{\mathbb{R}} f| \leq \int_{\mathbb{R}} |f|$
2.  $f \in L^1(\mathbb{R}) \Rightarrow f$  is finite valued a.e.
3. (Linearity) For  $f, g \in L^1(\mathbb{R})$  and  $a, b \in \mathbb{R}$ ,  $af + bg \in L^1(\mathbb{R})$  and  $\int_{\mathbb{R}} (af + bg) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g$
4. If  $f \in L^1(\mathbb{R})$  and  $A \in \mathcal{M}$  and  $m(A) = 0$  then  $\int_A f = 0$ ; in particular if  $f, g \in L^1(\mathbb{R})$  with  $f = g$  a.e. then  $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$
5. (Monotonicity) If  $f, g \in L^1(\mathbb{R})$  with  $f \leq g$  a.e., then  $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$

PROOF.

1.  $-\int_{\mathbb{R}} f^- \leq \int_{\mathbb{R}} f \leq \int_{\mathbb{R}} f^+$  and  $\int_{\mathbb{R}} f^{\pm} \leq \int_{\mathbb{R}} |f|$ .
2. We know  $\int_{\mathbb{R}} |f| < \infty$  so  $|f| < \infty$  a.e. by properties of integrals of non-negative functions so  $m(\{f = \pm\infty\}) = 0$
3.  $|af| \leq |a| |f|$  so by monotonicity of non-negative functions,  $\int_{\mathbb{R}} |af| \leq |a| \int_{\mathbb{R}} |f| < \infty$  so  $af$  in  $L^1(\mathbb{R})$ . Note then that

$$(af)^+ = \begin{cases} af^+ & \text{if } a \geq 0 \\ -af^- & \text{if } a < 0 \end{cases} \quad (af)^- = \begin{cases} af^- & \text{if } a \geq 0 \\ -af^+ & \text{if } a < 0 \end{cases}$$

so

$$\begin{aligned} \int_{\mathbb{R}} af &= \int_{\mathbb{R}} (af)^+ - \int_{\mathbb{R}} (af)^- \\ &= \begin{cases} \int_{\mathbb{R}} af^+ - \int_{\mathbb{R}} af^- & \text{if } a \geq 0 \\ \int_{\mathbb{R}} (-a)f^- - \int_{\mathbb{R}} (-a)f^+ & \text{if } a < 0 \end{cases} \\ &= \begin{cases} a(\int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-) & \text{if } a \geq 0 \\ (-a)(\int_{\mathbb{R}} f^- - \int_{\mathbb{R}} f^+) & \text{if } a < 0 \end{cases} = a \int_{\mathbb{R}} f. \end{aligned}$$

By the same argument  $bg \in L^1(\mathbb{R})$  and  $\int_{\mathbb{R}} (bg) = b \int_{\mathbb{R}} g$ . wlog,  $a = b = 1$ . We want to show  $f + g \in L^1(\mathbb{R})$ ; clearly  $|f + g| \leq |f| + |g| < \infty$  so it must be  $f + g \in L^1(\mathbb{R})$ . Set  $h := f + g$  then  $|h, f, g| < \infty$  a.e. and each of the integrals of  $|h, f, g| < \infty$ . Then,  $h^+ - h^- = f^+ - f^- + g^+ - g^-$ . Then  $h^+ + f^- + g^- = f^+ + g^+ + h^-$ , where now both sides are non-negative functions. By linearity of integrals of non-negative functions and since all terms finite a.e.,



$$\begin{aligned}
\int h^+ + \int f^- + \int g^- &= \int f^+ + \int g^+ + \int h^- \\
&\Rightarrow \int h^+ - \int h^- = \int f^+ - \int f^- + \int g^+ - \int g^- \\
&\Rightarrow \int (f + g) = \int h = \int f + \int g.
\end{aligned}$$

4.  $|\int_A f| \leq \int_A |f| = 0$ .

5. Put  $h = g - f$  (valid since  $f, g \in L^1(\mathbb{R})$ ) then  $h \geq 0$  a.e. Then  $\int_{\mathbb{R}} h \geq 0$  so by linearity  $\int_{\mathbb{R}} (g - f) = \int_{\mathbb{R}} g - \int_{\mathbb{R}} f \geq 0$ .

■

## §2.6 Convergence Theorems of Integral

↪ **Theorem 2.7** (Monotone Coverage Theorem (MON)): Assume  $\{f_n\}, f$  are non-negative, measurable functions. If  $f_n \uparrow$  and  $\lim_{n \rightarrow \infty} f_n = f$ , then

$$\int_{\mathbb{R}} f = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n.$$

**Remark 2.8:** When we write  $\lim_{n \rightarrow \infty} f_n = f$ , we mean pointwise convergence; however, one can replace these statements with convergence a.e. and obtain an equivalent, more general result wlog.

PROOF. By monotonicity of non-negative functions,  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n$  exists, forming an increasing sequence. Since  $f_n \leq f$ , then we know too that  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n \leq \int_{\mathbb{R}} f$ .

Conversely, for every  $n$ , let  $\{\varphi_{n,k}\}_{k \in \mathbb{N}}$  be a sequence of simple functions such that  $\varphi_{n,k} \uparrow$  w.r.t  $k$  and  $\varphi_{n,k} \rightarrow f_n$  as  $k \rightarrow \infty$ ;

$f_1$	$f_2$	$\cdots$	$f_k$	$f_{k+1}$	$\cdots$	$\rightarrow f$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$		
$\varphi_{1,k}$	$\varphi_{2,k}$	$\ddots$	$\varphi_{k,k}$	$\varphi_{k+1,k}$	$\cdots$	
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\cdots$	
$\varphi_{1,2}$	$\varphi_{2,2}$	$\ddots$	$\varphi_{k,2}$	$\varphi_{k+1,2}$	$\cdots$	
$\varphi_{1,1}$	$\varphi_{2,1}$	$\cdots$	$\varphi_{k,1}$	$\varphi_{k+1,1}$	$\cdots$	

For each  $k \geq 1$ , let

$$g_k := \max\{\varphi_{1,k}, \varphi_{2,k}, \dots, \varphi_{k,k}\}.$$

Then,  $g_k$  simple for each  $k$ , and  $g_k \uparrow$  and  $g_k \leq f$ . So,  $\lim_{k \rightarrow \infty} g_k$  exists. Then, for all  $n \geq 1$ ,  $\lim_{k \rightarrow \infty} g_k \geq \lim_{k \rightarrow \infty} \varphi_{n,k} = f_n$  so  $\lim_{k \rightarrow \infty} g_k \geq \lim_{n \rightarrow \infty} f_n = f$ . Thus,  $\lim_{k \rightarrow \infty} \int_{\mathbb{R}} g_k = \int_{\mathbb{R}} f$  by a previous theorem. Since  $\forall k \geq 1, \varphi_{1,k}, \varphi_{2,k}, \dots, \varphi_{k,k} \leq f_k, g_k \leq f_k$  and thus by monotonicity  $\int_{\mathbb{R}} g_k \leq \int_{\mathbb{R}} f_k \Rightarrow \int_{\mathbb{R}} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}} g_k \leq \lim_{k \rightarrow \infty} \int_{\mathbb{R}} f_k$  as desired. ■

↪ **Corollary 2.5:** If  $\{f_n\}, f$  measurable functions such that  $f_n \uparrow$  and  $\lim_n f_n = f$  and  $\int_{\mathbb{R}} f_1^- < \infty$ , then  $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$ .

PROOF. Since  $f_n \uparrow, f_n \geq f_1$  so  $f \geq f_1$ . Then,  $f_n^- \leq f_1^-, f^- \leq f_1^-$ , all of these are finite valued a.e., and  $\int_{\mathbb{R}} f_n^- \leq \int_{\mathbb{R}} f_1^- < \infty$  and  $\int_{\mathbb{R}} f^- \leq \int_{\mathbb{R}} f_1^- < \infty$ . For each  $n \geq 1$ , set  $\tilde{f}_n := f_n + f_1^- = f_n^+ - f_n^- + f_1^- \geq 0$ , and  $\tilde{f}_n \uparrow$  with  $\lim_n \tilde{f}_n = f + f_1^- =: \tilde{f} \geq 0$ . By MON,  $\int_{\mathbb{R}} \tilde{f} = \lim_n \int_{\mathbb{R}} \tilde{f}_n$  so  $\int_{\mathbb{R}} (f + f_1^-) = \lim_n \int_{\mathbb{R}} (f_n + f_1^-)$ .

We have that  $\tilde{f}_n = f_n + f_1^- = f_n^+ - f_n^- + f_1^- \Rightarrow \tilde{f}_n + f_n^- = f_n^+ + f_1^-$ , which is valid since  $f_n^- < \infty$  a.e.. By linearity, then,

$$\begin{aligned} \int_{\mathbb{R}} \tilde{f}_n + \int_{\mathbb{R}} f_n^- &= \int_{\mathbb{R}} f_n^+ + \int_{\mathbb{R}} f_1^- \\ \Rightarrow \int_{\mathbb{R}} \tilde{f}_n &= \int_{\mathbb{R}} f_n^+ - \int_{\mathbb{R}} f_n^- + \int_{\mathbb{R}} f_1^- \quad \text{because } \int_{\mathbb{R}} f_n^- < \infty \\ \Rightarrow \int_{\mathbb{R}} \tilde{f}_n &= \int_{\mathbb{R}} f_n + \int_{\mathbb{R}} f_1^-. \end{aligned}$$

Similar work gives  $\int_{\mathbb{R}} \tilde{f} = \int_{\mathbb{R}} f + \int_{\mathbb{R}} f_1^-$ , and taking limits and using  $\lim_n \int_{\mathbb{R}} (f_n + f_1^-) = \int_{\mathbb{R}} (f + f_1^-)$  completes the proof. ■

↪ **Theorem 2.8 (Reverse MON):** Assume  $\{f_n\}$ , measurable such that  $f_n \downarrow$  and  $\lim_{n \rightarrow \infty} f_n = f$ . If  $\int_{\mathbb{R}} f_1^+ < \infty$ , then  $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$ .

PROOF. Consider  $\{-f_n\}$  and use the previous corollary. ■

↪ **Theorem 2.9 (Fatou's Lemma):** Assume  $\{f_n\}$  non-negative, measurable. Then

$$\int_{\mathbb{R}} \left( \liminf_{n \rightarrow \infty} f_n \right) \leq \liminf_{n \rightarrow \infty} \left( \int_{\mathbb{R}} f_n \right).$$

PROOF. For every  $m \geq 1$ , set  $g_m := \inf_{n \geq m} f_n$ . Then,  $g_m$  non-negative and  $g_m \uparrow$ , with  $\lim_m g_m = \liminf_n f_n$ . By MON,  $\int_{\mathbb{R}} \liminf_n f_n = \lim_{m \rightarrow \infty} \left( \int_{\mathbb{R}} g_m \right)$ . For every  $n \geq m$ ,  $g_m \leq f_n$ , so by monotonicity,  $\int_{\mathbb{R}} g_m \leq \int_{\mathbb{R}} f_n$  for every  $n \geq m$ , so  $\int_{\mathbb{R}} g_m \leq \inf_{n \geq m} \int_{\mathbb{R}} f_n$ , and hence  $\lim_{m \rightarrow \infty} \int_{\mathbb{R}} g_m \leq \lim_{m \rightarrow \infty} \inf_{n \geq m} \int_{\mathbb{R}} f_n = \liminf_n \left( \int_{\mathbb{R}} f_n \right)$ , and the proof follows. ■

↪ **Corollary 2.6:** Assume  $\{f_n\}$  measurable and there exists a measurable function  $g$  such that  $\int_{\mathbb{R}} g^- < \infty$  and  $f_n \geq g$  for every  $n$ . Then,

$$\int_{\mathbb{R}} \left( \liminf_n f_n \right) \leq \liminf_n \left( \int_{\mathbb{R}} f_n \right).$$

PROOF. Since  $f_n \geq g$  for all  $n \geq 1$ ,  $f_n^- \leq g^-$  so  $f_n^- < \infty$  a.e. and  $\int_{\mathbb{R}} f_n^- < \infty$ . Set  $\tilde{f}_n := f_n + g^- \geq 0$ . Then, apply Fatou to get  $\int_{\mathbb{R}} \liminf_n \tilde{f}_n \leq \liminf_n \int_{\mathbb{R}} \tilde{f}_n$ , then it suffices to check linearity. ■

↪ **Theorem 2.10** (Reverse Fatou): Assume  $\{f_n\}$  measurable and there exists a  $g$  measurable such that  $\int_{\mathbb{R}} g^+ < \infty$  and  $f_n \leq g$  for all  $n \geq 1$ . Then,

$$\int_{\mathbb{R}} \left( \limsup_n f_n \right) \geq \limsup_n \left( \int_{\mathbb{R}} f_n \right).$$

PROOF. Apply previous proof to  $\{-f_n\}$ . ■

**Remark 2.9:** The “floor”  $g$  is necessary. Let  $f_n(x) := \begin{cases} -1 & \text{if } x \geq n \\ 0 & \text{if } x < n \end{cases}$ . Then,  $f_n \uparrow$ , and  $\lim_n f_n = 0$  while  $\int_{\mathbb{R}} f_n = -\infty$  for every  $n$ , so MON doesn’t apply.

↪ **Theorem 2.11** (Dominated Convergence Theorem (DOM)): Assume  $\{f_n\}, f$  measurable with  $\lim_n f_n = f$ . If there exists a  $g \in L^1(\mathbb{R})$  such that  $|f_n| \leq |g|$  for all  $n$ , then  $f_n \rightarrow f$  in  $L^1(\mathbb{R})$  i.e.  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f_n - f| = 0$ . In particular,  $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$ .

PROOF. Since  $|f_n| \leq |g|$  and  $f = \lim_{n \rightarrow \infty} f_n$ , then  $|f| \leq |g|$ . So,  $\int_{\mathbb{R}} |f_n| \leq \int_{\mathbb{R}} |g| < \infty$  and similarly  $\int_{\mathbb{R}} |f| \leq \int_{\mathbb{R}} |g| < \infty$  so  $|f_n|, f \in L^1(\mathbb{R})$ .

Observe that  $|f_n - f| \leq 2|g|$ , and  $\int_{\mathbb{R}} (2|g|) < \infty$ . Applying Reverse Fatou to  $\{|f_n - f|\}_{n \in \mathbb{N}}$ , we find

$$\begin{aligned} \int_{\mathbb{R}} \left( \underbrace{\limsup_n (|f_n - f|)}_0 \right) &\geq \limsup_n \left( \int_{\mathbb{R}} |f_n - f| \right) \\ &\Rightarrow \lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f_n - f| = 0, \end{aligned}$$

so in particular

$$\left| \int_{\mathbb{R}} f_n - \int_{\mathbb{R}} f \right| = \left| \int_{\mathbb{R}} (f_n - f) \right| \leq \int_{\mathbb{R}} |f_n - f| \rightarrow 0$$

so  $\lim_n \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$ . ■

**Remark 2.10:** We must find  $g \in L^1(\mathbb{R})$  to dominate  $|g| \geq |f_n|$  irrespective of  $n$ . For instance, if  $f_n = \mathbb{1}_{[n, 2n]}$ , then  $\lim_n f_n = 0$ , but  $\int_{\mathbb{R}} f_n = n$  for all  $n \geq 1$ . DOM doesn’t apply, since we would need a constant 1 function to dominate all  $f_n$ , which is not integrable.

↪ **Proposition 2.22:** Assume  $f \in L^1(\mathbb{R})$ ,  $\{h_n\}$  a sequence of measurable functions that are uniformly bounded, i.e.  $\exists M > 0$  such that  $|h_n| \leq M$  a.e. for all  $n \geq 1$ . If  $h_n \rightarrow h$  a.e. for some measurable function  $h$ , then

$$\lim_n \int_{\mathbb{R}} (f h_n) = \int_{\mathbb{R}} (f h).$$

PROOF. For every  $n$ ,  $|f \cdot h_n| \leq M |f| \in L_1(\mathbb{R})$ . The conclusion follows from DOM. ■

↪ **Corollary 2.7:** If  $f \in L^1(\mathbb{R})$  then for all  $\varepsilon > 0$ , there exists a compact set  $K \subseteq \mathbb{R}$  such that  $\int_{K^c} |f| \leq \varepsilon$ .

PROOF. If  $h_n := \mathbb{1}_{[-n,n]}$ , the  $\lim_n \int_{\mathbb{R}} f h_n = \lim_n \int_{[-n,n]} f = \int_{\mathbb{R}} f$ , and also  $\lim_n \int_{\{\mathbb{R}-[-n,n]\}} f = 0$ . ■

↪ **Corollary 2.8:** If  $f \in L^1(\mathbb{R})$ , then for all  $\varepsilon > 0$ ,  $\exists N \geq 1$  such that  $\int_{\{|f| > N\}} |f| \leq \varepsilon$ .

PROOF. Let  $h_n = \mathbb{1}_{\{|f| > n\}}$  then  $\lim_{n \rightarrow \infty} \int_{\{|f| > n\}} f = 0$ . ■

↪ **Corollary 2.9:** If  $\{A_n\} \subseteq \mathcal{M}$  such that  $A_n \uparrow$ , then  $\int_{\cup_{n=1}^{\infty} A_n} f = \lim_{n \rightarrow \infty} \int_{A_n} f$  ( $\mathbb{1}_{A_n} f \rightarrow \mathbb{1}_{\cup_{n=1}^{\infty} A_n} f$ ).

↪ **Corollary 2.10** (Countable Additivity): If  $\{B_n\} \subseteq \mathcal{M}$  are disjoint, then  $\int_{\cup_{n=1}^{\infty} B_n} f = \sum_{n=1}^{\infty} \int_{B_n} f$ .

↪ **Corollary 2.11:** If  $\{A_n\} \subseteq \mathcal{M}$  such that  $A_n \downarrow$ , then  $\int_{\cap_{n=1}^{\infty} A_n} f = \lim_{n \rightarrow \infty} \int_{A_n} f$ .

↪ **Proposition 2.23:** Assume  $f$  is non-negative, measurable, and finite-valued a.e.. Then, for every  $k \in \mathbb{Z}$ , put  $A_k := \{x \in \mathbb{R} : 2^k \leq f(x) < 2^{k+1}\}$ . Then,

$$f \text{ integrable} \Leftrightarrow \int_{\mathbb{R}} f < \infty \Leftrightarrow \sum_{k \in \mathbb{Z}} 2^k m(A_k) < \infty.$$

PROOF. ( $\Rightarrow$ ) Note that the  $A_k$ 's disjoint and  $\cup_{k \in \mathbb{Z}} A_k = \{0 < f < \infty\}$ . So,

$$\int_{\mathbb{R}} f = \underbrace{\int_{\{f=0\}} f}_{=0 \text{ since } f=0} + \int_{\{0 < f < \infty\}} + \underbrace{\int_{\{f=\infty\}} f}_{=0 \text{ since } f < \infty \text{ a.e.}} = \sum_{k \in \mathbb{Z}} \int_{A_k} f.$$

For each  $k \in \mathbb{Z}$ , for every  $x \in A_k$ ,  $2^k \leq f(x) < 2^{k+1}$  so  $2^k m(A_k) \leq \int_{A_k} f(x) < 2^{k+1} m(A_k)$ . Hence,

$$\sum_{k \in \mathbb{Z}} 2^k m(A_k) \leq \sum_{k \in \mathbb{Z}} \int_{A_k} f = \int_{\mathbb{R}} f < \infty.$$

( $\Leftarrow$ ) Suppose  $\sum_{k \in \mathbb{Z}} 2^k m(A_k) < \infty$ . We know again

$$\int_{\mathbb{R}} f = \int_{\{0 < f < \infty\}} f \text{ By } \overline{\text{MON}} \sum_{k \in \mathbb{Z}} \int_{A_k} f < \sum_{k \in \mathbb{Z}} 2^{k+1} m(A_k) = 2 \sum_{k \in \mathbb{Z}} 2^k m(A_k) < \infty.$$

■

⊗ **Example 2.3:** Let  $f(x) = |x|^{-\alpha} \mathbb{1}_{[-1,1]}(x)$ , with  $f(0) = \infty$  and  $\alpha > 0$ ;  $f$  finite-valued a.e.. For every  $k \in \mathbb{Z}$ , put  $A_k := \{2^k \leq f < 2^{k+1}\} = \{x \in [-1, 1] : 2^k \leq |x|^{-\alpha} < 2^{k+1}\}$ . By definition,  $|f| \geq 1$ , so

$$A_k = \left[-2^{-\frac{k}{\alpha}}, -2^{-\frac{(k+1)}{\alpha}}\right) \cup \left(2^{-\frac{(k+1)}{\alpha}}, 2^{-\frac{k}{\alpha}}\right] \text{ for } k \geq 0, \quad A_k = \emptyset \text{ if } k < 0.$$

Hence,

$$\sum_{k \in \mathbb{Z}} 2^k m(A_k) = \sum_{k=0}^{\infty} 2^k \cdot 2 \cdot \left(1 - 2^{-\frac{1}{\alpha}}\right) 2^{-\frac{k}{\alpha}} = 2 \left(1 - 2^{-\frac{1}{\alpha}}\right) \sum_{k=0}^{\infty} 2^{k(1-\frac{1}{\alpha})}.$$

Hence, the series  $< \infty \Leftrightarrow \alpha < 1$ , and thus  $\int_{[-1,1]} |x|^{-\alpha} dx < \infty \Leftrightarrow \alpha < 1$ .

⊗ **Example 2.4:** Let  $g(x) = |x|^{-\beta} \mathbb{1}_{\mathbb{R}-[-1,1]}(x)$  with  $\beta > 0$ . We have  $|g| < 1$ ; we again put

$$A_k := \{2^k \leq g < 2^{k+1}\} = \begin{cases} \left[-2^{-\frac{k}{\beta}}, -2^{-\frac{(k+1)}{\beta}}\right) \cup \left(2^{-\frac{(k+1)}{\beta}}, 2^{-\frac{k}{\beta}}\right] & \text{if } k < 0 \\ \emptyset & \text{if } k \geq 0. \end{cases}$$

So,

$$\int_{\mathbb{R}-[-1,1]} |x|^{-\beta} dx < \infty \Leftrightarrow \sum_{k \in \mathbb{Z}} 2^k m(A_k) < \infty \Leftrightarrow \beta > 1.$$

⊗ **Example 2.5:** Let  $f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)$ . What is  $\lim_{n \rightarrow \infty} \int_{(0,\infty)} f_n(x) dx$ ? We have that for all  $x > 0$ ,  $\lim_{n \rightarrow \infty} f_n(x) = 0$ . We have that since  $|\sin(\frac{x}{n})| \leq 1$ , so

$$|f_n(x)| \leq \left(1 + \frac{x}{n}\right)^{-n} \leq \left(1 + \frac{x}{2}\right)^{-2} \quad \forall x > 0, \forall n \geq 2.$$

Let  $g(x) := \left(1 + \frac{x}{2}\right)^{-2}$ . We would like to apply DOM, so we need to check that  $g \in L^1((0, \infty))$ . We have that

$$\int_{(0,\infty)} g = \int_{(0,1]} g + \int_{(1,\infty)} g \leq \int_{(0,1]} 1 + \underbrace{\int_{(1,\infty)} \frac{4}{x^2} dx}_{\beta=2 \text{ of previous example}} < \infty,$$

so indeed  $g \in L^1((0, \infty))$ . Applying DOM, then, we have that

$$\lim_{n \rightarrow \infty} \int_{(0,\infty)} f_n = \int_{(0,\infty)} \lim_{n \rightarrow \infty} f_n = 0.$$

⊗ **Example 2.6:** Let  $c > 0$ ,  $f_n(x) = x^{-c}(\cosh x)^{-\frac{1}{n}}$ . What is  $\lim_n \int_{(1,\infty)} f_n$ ?

For every  $x > 1$ ,  $\cosh x > 1$ , so  $(\cosh x)^{-\frac{1}{n}} \uparrow$  with respect to  $n$ , with  $\lim_n (\cosh x)^{-\frac{1}{n}} = 1$ , so  $\lim_{n \rightarrow \infty} f_n(x) = x^{-c}$  for every  $x > 1$ . Let  $g(x) = x^{-c}$ , then. By previous examples, when  $c > 1$ ,  $g \in L^1((1, \infty))$  so DOM applies and thus

$$\lim_n \int_{(1,\infty)} f_n = \int_{(1,\infty)} \lim_n f_n = \int_{(1,\infty)} x^{-c} dx < \infty.$$

When  $0 < c \leq 1$ , by Fatou,

$$\lim_n \inf \int_{(1,\infty)} f_n \geq \int_{(1,\infty)} \lim_n \inf(f_n) = \int_{(1,\infty)} x^{-c} dx,$$

since  $f_n$  converges. When  $0 < c \leq 1$ , the RHS =  $\infty$ , and thus  $\lim_{n \rightarrow \infty} \int_{(1,\infty)} f_n = \infty$ .

⊗ **Example 2.7:** Let  $c \geq 0$ ,  $f_n(x) := \frac{n}{1+n^2x^2}$  for  $x \geq 0$ . What is  $\lim_n \int_{[c,\infty)} f_n$ ?

We have that

$$\lim_n f_n(x) = \begin{cases} 0 & \text{if } x > 0 \\ \infty & \text{if } x = 0 \end{cases}.$$

On  $x \in [1, \infty)$ ,  $f_n(x) \geq f_{n+1}(x)$  for all  $n \geq 1$ , namely  $f_n \downarrow$ , and so  $f_n(x) \leq f_1(x) = \frac{1}{1+x^2} \cdot f_1(x) \in L^1(\mathbb{R})$ , by comparison with  $\frac{1}{x^2}$  ( $\alpha = 2$ ).

If  $x \in (0, 1)$ ,  $f_n(x) = \frac{1}{x} \frac{nx}{1+(nx)^2} \leq A \frac{1}{x}$ , with  $A := \sup_{t>0} \frac{t}{1+t^2} < \infty$ . But  $\frac{A}{x} \notin L^1((0, 1))$ .

When  $c > 0$ , for all  $x \geq c$  and for all  $n \geq 1$ ,

$$f_n(x) \leq \mathbb{1}_{[1,\infty)}(x) \frac{1}{1+x^2} + \mathbb{1}_{[c,1)}(x) \frac{A}{x} \leq \mathbb{1}_{[1,\infty)}(x) \frac{1}{1+x^2} + \mathbb{1}_{[c,1)}(x) \frac{A}{c} \in L^1([c, \infty)).$$

Hence, we may apply DOM, so

$$\lim_n \int_{[c,\infty)} f_n = \int_{[c,\infty)} \lim_n f_n = 0,$$

when  $c > 0$ . However, when  $c = 0$ , we have no such dominating function; so what is  $\int_{[0,\infty)} f_n(x) dx$ ?

## §2.7 Riemann Integral vs Lebesgue Integral

Recall; let  $f$  be bounded on  $[a, b]$ . Then,  $f$  is Riemann integrable on  $[a, b]$  if

$$\begin{cases} f \text{ is continuous on } [a, b] \\ f \text{ is monotonic on } [a, b] \\ f \text{ is continuous except at possibly finitely many points in } [a, b] \end{cases}.$$

Recall the function  $f = \mathbb{1}_{\mathbb{Q} \cap [0,1]}$ .  $f$  is not Riemann integrable, but is Lebesgue integrable, because  $|f| \leq \mathbb{1}_{[0,1]} \in L^1(\mathbb{R})$ .

**Remark 2.11:**

1.  $\exists$  bounded functions on  $[a, b]$  that are not Riemann integrable.
2. In general,  $g$  being Riemann integrable and  $|f| \leq |g| \not\Rightarrow f$  is Riemann integrable ( $\mathbb{1}_{\mathbb{Q} \cap [0,1]} \leq \mathbb{1}_{[0,1]}$ ).
3. In general, DOM and MON do *not* apply to Riemann integrable. For instance, consider  $\{q_n\}$  an enumeration of  $\mathbb{Q} \cap [0, 1]$ . Define  $f_n(x) := \begin{cases} 1 & \text{if } x \in \{q_1, \dots, q_n\} \\ 0 & \text{else} \end{cases}$ .  $f_n \uparrow$ , with  $f_n \rightarrow \mathbb{1}_{\mathbb{Q} \cap [0,1]}$ . So, MON applies with the Lebesgue integral, but  $f_n$  is only discontinuous, for every  $n$ , at finitely many points, so  $f_n$  Riemann integrable with  $\int_0^1 f_n = 0$ , but the limit is not Riemann integrable.

**$\hookrightarrow$  Theorem 2.12:** Assume  $f$  is Riemann integrable on  $[a, b]$ . Then,  $f$  is Lebesgue integrable on  $[a, b]$ , i.e.  $f \in L^1([a, b])$ . Moreover,  $\int_a^{b^{(R)}} f = \int_{[a,b]} f$ .

PROOF.  $f$  is Riemann integrable on  $[a, b]$ , so there is some  $M > 0$  such that  $|f| \leq M$  on  $[a, b]$ . Further, there exist step functions  $\varphi_n, \psi_n$  with  $\varphi_n \leq f \leq \psi_n$  on  $[a, b]$  and  $|\varphi_n|, |\psi_n| \leq M$  for all  $n \geq 1$ , and

$$\lim_{n \rightarrow \infty} \int_a^{b^{(R)}} \varphi_n = \int_a^{b^{(R)}} f = \lim_{n \rightarrow \infty} \int_a^{b^{(R)}} \psi_n.$$

Denote  $\varphi := \lim_{n \rightarrow \infty} \varphi_n, \psi := \lim_{n \rightarrow \infty} \psi_n$ , which exist by Monotonicity. Since  $\varphi_n, \psi_n$  are step functions, they are measurable hence  $\varphi, \psi$  measurable with  $\varphi \leq f \leq \psi$ . Observe that the Lebesgue, Riemann integral coincide on step functions. Hence,  $\int_a^{b^{(R)}} \varphi_n = \int_{[a,b]} \varphi_n$ , same with  $\psi_n$ . By DOM, (with  $M$  as the dominator)

$$\int_{[a,b]} \varphi = \lim_n \int_{[a,b]} \varphi_n = \lim_n \int_a^{b^{(R)}} \varphi_n = \int_a^{b^{(R)}} (f) = \lim_n \int_a^{b^{(R)}} \psi_n = \lim_n \int_{[a,b]} \psi_n = \int_{[a,b]} \psi.$$

Since  $\varphi \leq \psi$  and  $\int_{[a,b]} (\psi - \varphi) = 0$ , we have that  $\psi = \varphi$  a.e. on  $[a, b]$  by properties of integrals of non-negative functions, and thus  $f = \varphi = \psi$  a.e. on  $[a, b]$ . In particular, then,  $f$  is measurable, being equal a.e. to measurable functions. Thus, since  $|f| \leq M$  on  $[a, b]$ ,  $f \in L^1([a, b])$ , and so since integrals agree on functions that are equal a.e.,  $\int_{[a,b]} f = \int_{[a,b]} \varphi = \int_a^{b^{(R)}} f$  as desired. ■

⊗ **Example 2.8:** We return to our example of computing  $\lim_{n \rightarrow \infty} \int_{[0, \infty)} \frac{n}{1+n^2x^2} dx$ . We may rewrite

$$\int_{[0, \infty)} \frac{n}{1+n^2x^2} dx = \int_{[0, T]} \frac{n}{1+n^2x^2} dx + \int_{[T, \infty)} \frac{n}{1+n^2x^2} dx$$

where  $T > 0$ . We know from the previous example that the RHS integral converges to 0 by application of DOM. Now,  $\frac{n}{1+n^2x^2}$  is continuous on  $[0, T]$  and thus Riemann integrable, and so by the previous theorem

$$\int_{[0, T]} \frac{n}{1+n^2x^2} = \int_{[0, T]}^{(R)} \frac{n}{1+n^2x^2} = \arctan(nT).$$

As  $n \rightarrow \infty$ ,  $\arctan(nT) \rightarrow \frac{\pi}{2}$ , and thus the limit of the whole integral indeed exists, and is in fact equal to  $\frac{\pi}{2}$ .

## §2.8 $L^p$ -space

↪ **Definition 2.12** ( $p$ -integrable): Let  $f$  measurable and  $1 \leq p < \infty$ . We say  $f$  is  $p$ -integrable and write  $f \in L^p(\mathbb{R})$  if  $\int_{\mathbb{R}} |f|^p < \infty$ , i.e.  $|f|^p \in L^1(\mathbb{R})$ .

For  $f \in L^p(\mathbb{R})$ , define the  $p$ -norm

$$\|f\|_p := \left( \int_{\mathbb{R}} |f|^p \right)^{\frac{1}{p}}.$$

**Remark 2.12:** When  $p = 1$ , we see that  $\|\cdot\|_1$  a norm fairly clearly from properties of the integral. We need to show this for more general  $p > 1$ .

**Remark 2.13:**  $\|\cdot\|_p$  also defined when  $p = \infty$ ; given  $f$  measurable, we define

$$\|f\|_{\infty} := \text{ess sup}_{x \in \mathbb{R}} |f(x)| := \inf \{a \in \overline{\mathbb{R}} : |f| \leq a \text{ a.e.}\}.$$

Then, we define

$$L^{\infty}(\mathbb{R}) := \{f \text{ measurable s.t. } \|f\|_{\infty} < \infty\}.$$

One can show that if  $f \in L^{\infty}(\mathbb{R})$ ,  $|f| \leq \|f\|_{\infty}$  a.e..

↪ **Theorem 2.13** (Hölder's Inequality): Let  $1 < p < \infty$  and let  $q := \frac{p}{p-1}$  (such a  $q$  is called the Hölder Conjugate of  $p$ ). If  $f \in L^p(\mathbb{R})$  and  $g \in L^q(\mathbb{R})$ , then  $fg \in L^1(\mathbb{R})$ , and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

In particular, if  $p = q = 2$ , then we have the *Cauchy-Schwarz Inequality*.



**Remark 2.14:**  $\frac{1}{p} + \frac{1}{q} = 1$ .

PROOF. We will employ “Young’s Inequality”, which states that for all  $a, b \geq 0$ ,  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$  where  $\frac{1}{p} + \frac{1}{q} = 1$ . Since  $f \in L^p, g \in L^q$ , set  $\tilde{f} := \frac{f}{\|f\|_p}$  and  $\tilde{g} := \frac{g}{\|g\|_q}$ . Then, a.e.

$$|\tilde{f}\tilde{g}| \leq \frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q}.$$

We have

$$\int_{\mathbb{R}} |\tilde{f}\tilde{g}| = \int_{\mathbb{R}} \frac{|fg|}{\|f\|_p \|g\|_q}$$

and

$$\int_{\mathbb{R}} \frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q} = \frac{1}{p} \frac{\int_{\mathbb{R}} |f|^p}{\|f\|_p^p} + \frac{1}{q} \frac{\int_{\mathbb{R}} |g|^q}{\|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1,$$

and thus

$$\int_{\mathbb{R}} |fg| = \|fg\|_1 \leq \|f\|_p \|g\|_q$$

as required. ■

**Remark 2.15:** This inequality also holds for  $p = 1, q = \infty$  (assignment question).

↪ **Lemma 2.2:** For all  $a, b \geq 0$ ,  $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$  where  $\frac{1}{p} + \frac{1}{q} = 1$ .

PROOF. ■

↪ **Theorem 2.14** (Minkowski's Inequality): Let  $1 \leq p < \infty$  and  $f, g \in L^p(\mathbb{R})$ . Then,  $f + g \in L^p(\mathbb{R})$ , and in particular

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

In particular, then,  $\|\cdot\|_p$  satisfies the triangle inequality and is indeed a norm on  $L^p(\mathbb{R})$ .

PROOF. We have  $|f + g|^p \leq 2^p(|f|^p + |g|^p)$  hence  $f + g \in L^p(\mathbb{R})$  since  $|f|^p, |g|^p \in L^1(\mathbb{R})$ . Further

$$\begin{aligned}
\int_{\mathbb{R}} |f + g|^p &= \int_{\mathbb{R}} |f + g| |f + g|^{p-1} \leq \int_{\mathbb{R}} |f| |f + g|^{p-1} + \int_{\mathbb{R}} |g| |f + g|^{p-1} \\
&\stackrel{\text{(Hölder's)}}{\leq} \left( \int_{\mathbb{R}} |f|^p \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}} |f + g|^{(p-1)q} \right)^{\frac{1}{q}} + \left( \int_{\mathbb{R}} |g|^p \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}} |f + g|^{(p-1)q} \right)^{\frac{1}{q}} \\
&\leq (\|f\|_p + \|g\|_p) \left( \int_{\mathbb{R}} |f + g|^p \right)^{\frac{1}{q}} \\
&\Rightarrow \|f + g\|_p = \left( \int_{\mathbb{R}} |f + g|^p \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}} |f + g|^p \right) \cdot \left( \int_{\mathbb{R}} |f + g|^p \right)^{-\frac{1}{q}} \\
&\leq (\|f\|_p + \|g\|_p) \left( \int_{\mathbb{R}} |f + g|^p \right)^{\frac{1}{q}} \cdot \left( \int_{\mathbb{R}} |f + g|^p \right)^{-\frac{1}{q}} = \|f\|_p + \|g\|_p \\
&\Rightarrow \|f + g\|_p \leq \|f\|_p + \|g\|_p
\end{aligned}$$

■

**Remark 2.16:** Minkowski's also holds for  $p = \infty$ .

↪ **Lemma 2.3:** Let  $1 \leq p < \infty$ . If  $\{g_k\} \in L^p(\mathbb{R})$  such that  $\sum_{k=1}^{\infty} \|g_k\|_p < \infty$ , then  $\exists G \in L^p(\mathbb{R})$  such that  $G_m := \sum_{k=1}^m g_k \rightarrow G$  as  $m \rightarrow \infty$  a.e. as well as in  $L^p(\mathbb{R})$ .

PROOF. Put  $\widetilde{G}_m := \sum_{k=1}^m |g_k|$  and  $\widetilde{G} := \sum_{k=1}^{\infty} |g_k|$ . Then,  $\widetilde{G}_m \uparrow$  with  $\lim_{m \rightarrow \infty} \widetilde{G}_m = \widetilde{G}$ . By MON,

$$\int_{\mathbb{R}} \widetilde{G}^p = \lim_{m \rightarrow \infty} \int_{\mathbb{R}} \widetilde{G}_m^p = \lim_{m \rightarrow \infty} \|\widetilde{G}_m\|_p^p \leq \lim_{m \rightarrow \infty} \left( \sum_{k=1}^m \|g_k\|_p \right)^p$$

where the final inequality is by Minkowski's. Then,

$$\leq \left( \lim_{m \rightarrow \infty} \sum_{k=1}^m \|g_k\|_p \right)^p = \left( \sum_{k=1}^{\infty} \|g_k\|_p \right)^p < \infty, \text{ by assumption}$$

Hence,  $\widetilde{G} \in L^p(\mathbb{R})$  and  $\|\widetilde{G}\|_p \leq \sum_{k=1}^{\infty} \|g_k\|_p$  and thus  $\widetilde{G}$  finite-valued a.e. and hence  $\sum_{k=1}^{\infty} g_k$  absolutely convergent a.e.. Set  $G = \lim_{m \rightarrow \infty} G_m = \sum_{k=1}^{\infty} g_k$  a.e.. Moreover, we know

$$|G| = \left| \sum_{k=1}^{\infty} g_k \right| \leq \sum_{k=1}^{\infty} |g_k| = \widetilde{G} \Rightarrow G \in L^p(\mathbb{R})$$

and

$$|G - G_m| \leq \sum_{k=m+1}^{\infty} |g_k|.$$

Fix  $\varepsilon > 0$ . Since  $\sum_{k=1}^{\infty} \|g_k\|_p < \infty$ , exists some  $M \geq 1$  such that  $\sum_{k=M+1}^{\infty} \|g_k\|_p < \varepsilon$ . Then,

$$\begin{aligned}
\int_{\mathbb{R}} |G - G_M|^p &\leq \int_{\mathbb{R}} \left( \sum_{k=M+1}^{\infty} |g_k| \right)^p = \lim_{L \rightarrow \infty} \int_{\mathbb{R}} \left( \sum_{k=M+1}^L |g_k| \right)^p \\
&\stackrel{\text{(Minkowski)}}{\leq} \lim_{L \rightarrow \infty} \left( \sum_{k=M+1}^L \|g_k\|_p \right)^p \\
&= \left( \sum_{k=M+1}^{\infty} \|g_k\|_p \right)^p \leq \varepsilon
\end{aligned}$$

hence  $G_m \rightarrow G$  in  $L^p(\mathbb{R})$ . ■

↪ **Theorem 2.15:** Let  $1 \leq p < \infty$ . Then  $L^p(\mathbb{R})$  is a complete normed space under the  $p$ -norm.

PROOF. Let  $f_n \in L^p(\mathbb{R})$  be a Cauchy sequence under  $\|\cdot\|_p$ . We can choose a subsequence  $\{n_k\}$  such that for every  $k \geq 1$ ,  $\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k}$ . Set  $g_k := f_{n_{k+1}} - f_{n_k}$ . By the lemma, if  $G_m := \sum_{k=1}^m g_k$ , there exists some  $G \in L^p(\mathbb{R})$  such that  $G_m \rightarrow G$  a.e. and in  $L^p(\mathbb{R})$ . In fact, we have

$$G_m = \sum_{k=1}^m g_k = \sum_{k=1}^m (f_{n_{k+1}} - f_{n_k}) = f_{n_{m+1}} - f_{n_1},$$

hence

$$G = \lim_{m \rightarrow \infty} G_m = \left( \lim_{m \rightarrow \infty} f_{n_{m+1}} \right) - f_{n_1}.$$

Let  $f := G + f_{n_1}$ . Then,  $f = \lim_{m \rightarrow \infty} f_{n_m}$  a.e. and since  $G_m \rightarrow G$  in  $L^p$ , we have that  $f_{n_m} \rightarrow f$  in  $L^p$  as  $m \rightarrow \infty$ . It remains to show convergence in  $L^p$  along the whole subsequence.

Fix  $\varepsilon > 0$ . Let  $N \geq 1$  such that  $\sup_{k, \ell \geq N} \|f_k - f_\ell\|_p < \varepsilon$  and  $m$  sufficiently large such that  $n_m > N$  and  $\|f_{n_m} - f\|_p \leq \varepsilon$ . Then,

$$\|f_n - f\|_p \leq \underbrace{\|f_n - f_{n_m}\|_p}_{< \varepsilon} + \underbrace{\|f_{n_m} - f\|_p}_{< \varepsilon} < 2\varepsilon,$$

completing the proof. ■

**Remark 2.17:**  $L^\infty$  also complete.

### 2.8.1 Dense Subspaces of $L^p(\mathbb{R})$

↪ **Lemma 2.4:** Bounded and compactly supported functions are dense in  $L^p(\mathbb{R})$ .

PROOF. Given  $f \in L^p(\mathbb{R})$ , set

$$f_n(x) = \mathbb{1}_{[-n, n]}(x) \cdot f(x) \cdot \mathbb{1}_{\{|f| \leq n\}}(x)$$

which are bounded and compactly supported on  $[-n, n]$ . We claim  $f_n \rightarrow f$  in  $L^p(\mathbb{R})$ .

We have  $\int_{\mathbb{R}} |f_n - f|^p$  nonzero only if  $x \notin [-n, n]$  or  $|f(x)| > n$ . Hence

$$\int_{\mathbb{R}} |f_n - f|^p \leq \int_{\mathbb{R} \setminus [-n, n]} |f|^p + \int_{\{|f| > n\}} |f|^p \rightarrow 0 \text{ as } n \rightarrow \infty.$$

■

↪ **Lemma 2.5:** Simple functions are dense in  $L^p(\mathbb{R})$ .

PROOF. For  $f \in L^p(\mathbb{R})$ , let  $f_n$  be as in the previous proof. For each  $n \geq 1, k = 0, 1, \dots, n2^n - 1$ , set

$$A_{n,k} := \left\{ x \in [-n, n] : \frac{k}{2^n} \leq f_n^+ < \frac{k+1}{2^n} \right\}, \quad \varphi_n^+ := \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{A_{n,k}},$$

and

$$B_{n,k} := \left\{ x \in [-n, n] : \frac{k}{2^n} \leq f_n^- < \frac{k+1}{2^n} \right\}, \quad \varphi_n^- := \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{B_{n,k}}.$$

Put  $\varphi_n := \varphi_n^+ - \varphi_n^-$ . This is a simple function, and  $|\varphi_n| \leq n$  and supported on  $[-n, n]$  for every  $n$  hence  $\varphi_n \in L^p(\mathbb{R})$ . In addition,  $\lim_n \varphi_n(x) = f(x)$ . In particular, for any  $n \geq 1$ ,

$$|f_n(x) - \varphi_n(x)| \leq |f_n^+(x) - \varphi_n^+(x)| + |f_n^-(x) - \varphi_n^-(x)| \leq 2 \cdot 2^{-n}.$$

Then, in particular

$$\begin{aligned} \|f - \varphi_n\|_p &\leq \underbrace{\|f - f_n\|_p}_{\rightarrow 0} + \underbrace{\|f_n - \varphi_n\|_p}_{= \left( \int_{[-n, n]} |f_n - \varphi_n|^p \right)^{\frac{1}{p}}}, \\ &\leq \left( (2 \cdot 2^{-n})^p m([-n, n]) \right)^{\frac{1}{p}} \rightarrow 0 \end{aligned}$$

and so indeed  $\varphi_n \rightarrow f$  in  $L^p(\mathbb{R})$ .

■

↪ **Theorem 2.16:** Let  $C_c(\mathbb{R})$  denote the space of continuous and compactly supported functions. Then,  $C_c(\mathbb{R})$  is dense in  $L^p(\mathbb{R})$  for  $1 \leq p < \infty$ .

PROOF. Give  $f \in L^p(\mathbb{R})$ , let  $\{\varphi_n\}$  simple functions as in the previous proof. Recall that, for every  $n \geq 1$ , there exists a step function  $\theta_n$  such that  $\theta_n \leq \sup_x |\varphi_n(x)| \leq n$ , is supported on  $[-n-1, n+1]$ , and  $\{\theta_n \neq \varphi_n\}$  has arbitrarily small measure. In particular, we choose  $\theta_n$  such that  $m(\{\theta_n \neq \varphi_n\}) \leq 2^{-n-1}$  for every  $n \geq 1$ .

Recall that given a step function  $\theta_n$ , there exists a function  $\tilde{\theta}_n$  continuous on  $\mathbb{R}$ ,  $\tilde{\theta}_n$  is supported on  $[-n-2, n+2]$ , and  $m(\{\tilde{\theta}_n - \theta_n\}) \leq 2^{-n-1}$ . Thus,  $\{\tilde{\theta}_n\} \subseteq C_c(\mathbb{R})$ , and

$$m(\{\tilde{\theta}_n - \varphi_n\}) \leq m(\{\tilde{\theta}_n - \theta_n\}) + m(\{\theta_n \neq \varphi_n\}) \leq 2^{-n}.$$

So, we have

$$\begin{aligned}
\|f - \tilde{\theta}_n\|_p &\leq \underbrace{\|f - \varphi_n\|_p}_{\rightarrow 0 \text{ by lemma}} + \underbrace{\|\varphi_n - \tilde{\theta}_n\|_p}_{= (\int_{\mathbb{R}} |\varphi_n - \tilde{\theta}_n|^p)^{\frac{1}{p}}} \\
&= (\int_{\{\tilde{\theta}_n \neq \varphi_n\}} |\varphi_n - \tilde{\theta}_n|^p)^{\frac{1}{p}} \\
&\leq ((2n)^p 2^{-n})^{\frac{1}{p}} \rightarrow 0
\end{aligned}$$

and thus  $\tilde{\theta}_n \rightarrow f$  in  $L^p(\mathbb{R})$ . ■

**Remark 2.18:** The density of  $C_c(\mathbb{R})$  in  $L^p(\mathbb{R})$  is useful in the study of properties of generic  $L^p$  functions. For instance, show that if  $f \in L^p(\mathbb{R})$ , then  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f(x + \frac{1}{n}) - f(x)|^p dx = 0$ , that is  $f(\cdot + \frac{1}{n}) \rightarrow f$  in  $L^p(\mathbb{R})$  using this density.

**Remark 2.19:**  $C_c(\mathbb{R})$  is NOT dense in  $L^\infty(\mathbb{R})$ .

## §2.9 Convergence Modes and Uniform Integrability

Recall that, given  $\{f_n\}, f$  measurable and finite-valued a.e., we have the following notions of convergence

1.  $f_n \rightarrow f$  in measure  $\Rightarrow \exists \{n_k\}$  such that  $f_{n_k} \rightarrow f$  a.e. as  $k \rightarrow \infty$
2.  $f_n \rightarrow f$  a.e. on  $A \in \mathcal{M}$  with  $m(A) < \infty \Rightarrow f_n \rightarrow f$  in measure on  $A$
3.  $f_n \rightarrow f$  in  $L^p(\mathbb{R})$ .

↪ **Proposition 2.24:** If  $\{f_n\}, f$  in  $L^p(\mathbb{R})$  for  $1 \leq p < \infty$  and  $f_n \rightarrow f$  in  $L^p(\mathbb{R})$ , then  $f_n \rightarrow f$  in measure.

PROOF. For  $\delta > 0$ , we have

$$m(\{|f_n - f| > \delta\}) = \int_{\{|f_n - f| > \delta\}} 1 dx.$$

Remark that  $1 \leq \frac{|f_n - f|}{\delta}$  over  $\{|f_n - f| > \delta\}$ ; further  $1^p = 1 \leq \left(\frac{|f_n - f|}{\delta}\right)^p$ . Hence,

$$\leq \int_{\{|f_n - f| > \delta\}} \frac{|f_n - f|^p}{\delta^p} dx \leq \frac{1}{\delta^p} \int_{\mathbb{R}} |f_n - f|^p \leq \frac{1}{\delta^p} \|f_n - f\|_p^p.$$

But by assumption  $\|f_n - f\|_p^p \rightarrow 0$  for any  $\delta > 0$ , hence  $m(\{|f_n - f| > \delta\}) \rightarrow 0$  i.e.  $f_n \rightarrow f$  in measure. ■

**Remark 2.20:** In general, convergence in  $L^p \not\Rightarrow$  convergence a.e., with the same counter example from convergence in measure  $\not\Rightarrow$  convergence a.e..

**Remark 2.21:** When do we have convergence a.e.  $\Rightarrow$  convergence in  $L^p$ ? This doesn't hold in general, unless some integral convergence theorem from before holds.

**Remark 2.22:** When do we have convergence in measure  $\Rightarrow$  convergence in  $L^p$ ? No in general, unless one of the integral convergence theorem holds; with some slight adaptation.

**Proposition 2.25** (MON, Measure Version (mMON)): Let  $f_n$  non-negative with  $f_n \uparrow$  and  $f_n \rightarrow f$  in measure. Then,

$$\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n.$$

PROOF.  $f_n \rightarrow f$  in measure implies  $f_{n_k} \rightarrow f$  almost everywhere along some subsequence  $n_k$ , so it must be that  $f$  non-negative. Suppose the claim fails. Then, there exists some subsequence  $\{n_\ell\}$  such that  $\int_{\mathbb{R}} f_{n_\ell} \not\rightarrow \int_{\mathbb{R}} f$ . However, along this subsequence we also have  $f_{n_\ell} \rightarrow f$  in measure, and hence exists a subsubsequence  $n_{\ell_p}$  such that  $f_{n_{\ell_p}} \rightarrow f$  a.e.. Then, by MON applied to this subsubsequence, we know that

$$\lim_p \int_{\mathbb{R}} f_{n_{\ell_p}} = \int_{\mathbb{R}} f,$$

a contradiction. ■

**Proposition 2.26** (mDOM): If  $f_n \in L^1(\mathbb{R})$  with  $f_n \rightarrow f$  in measure and there exists some  $g \in L^1(\mathbb{R})$  such that  $|f_n| \leq |g|$ , then  $f_n \rightarrow f$  in  $L^1(\mathbb{R})$ .

Recall that if  $f \in L^1(\mathbb{R})$ , then  $\int_{\{|f|>n\}} |f| \rightarrow 0$  as  $n \rightarrow \infty$ . The converse does not hold in general; consider  $f \equiv 1$ . However, we can achieve a partial converse.

For  $A \in \mathcal{M}$ , we say  $f \in L^1(A)$  if  $\int_A |f| < \infty$ .

**Proposition 2.27:** Given  $A \in \mathcal{M}$  with  $m(A) < \infty$ , then

$$f \in L^1(A) \Leftrightarrow \lim_n \int_{A \cap \{|f|>n\}} |f| = 0.$$

PROOF. ( $\Rightarrow$ ) We've proven before, c.f. properties of integral of non-negative functions.

( $\Leftarrow$ ) Choose  $N$  such that  $\int_{A \cap \{|f|>N\}} |f| \leq 1$ . Then,

$$\begin{aligned} \int_A |f| &= \int_{A \cap \{|f| \leq N\}} |f| + \int_{A \cap \{|f| > N\}} |f| \\ &\leq N \cdot m(A) + 1 < \infty. \end{aligned}$$

■

↪ **Definition 2.13** (Uniform Integrability): Given  $\{f_n\}$  measurable and  $A \in \mathcal{M}$ , we say  $\{f_n\}$  is uniformly integrable on  $A$  if

$$\lim_{M \rightarrow \infty} \left( \sup_{n \geq 1} \left( \int_{A \cap \{|f_n| > M\}} |f_n| \right) \right) = 0.$$

↪ **Proposition 2.28**: Let  $\{f_n\}$  measurable,  $A \in \mathcal{M}$ .

1. If  $m(A) < \infty$  and  $\{f_n\}$  uniformly integrable on  $A$ , then  $\{f_n\}$  is bounded in  $L^1(A)$ , that is  $\sup_{n \geq 1} \int_A |f_n| < \infty$ .
2. If  $\{f_n\}$  is bounded in  $L^p(A)$  for any  $1 < p < \infty$ , then  $\{f_n\}$  is uniformly integrable on  $A$ .

PROOF.

1. Let  $M$  such that  $\sup_{n \geq 1} \int_{A \cap \{|f_n| > M\}} |f_n| \leq 1$ . Then, we have that

$$\begin{aligned} \sup_{n \geq 1} \int_A |f_n| &= \sup_{n \geq 1} \left( \int_{A \cap \{|f_n| \leq M\}} |f_n| + \int_{A \cap \{|f_n| > M\}} |f_n| \right) \\ &\leq M \cdot m(A) + 1 < \infty. \end{aligned}$$

2. For any  $M > 0$ , note that  $1 \leq \left( \frac{|f_n|}{M} \right)^{p-1}$  over  $A \cap \{|f_n| > M\}$ . So,

$$\begin{aligned} \sup_n \int_{A \cap \{|f_n| > M\}} |f_n| &\leq \sup_n \int_{A \cap \{|f_n| > M\}} |f_n| \left( \frac{|f_n|}{M} \right)^{p-1} \\ &\leq \underbrace{\frac{1}{M^{p-1}}}_{>0} \underbrace{\sup_n \int_A |f_n|^p}_{<\infty} \rightarrow 0 \text{ as } M \rightarrow \infty. \end{aligned}$$

■

**Remark 2.23**: Notice that 2. does *not* require finiteness of the measure of  $A$ , in particular one can take  $A = \mathbb{R}$ .

↪ **Proposition 2.29**: Given  $\{f_n\}$  measurable and  $A \in \mathcal{M}$  with  $m(A) < \infty$ , TFAE:

- (i)  $f_n \in L^1(A) \forall n \geq 1, f \in L^1(A)$  and  $f_n \rightarrow f$  in  $L^1(A)$ ,
- (ii)  $\{f_n\}$  is uniformly integrable on  $A$  and  $f_n \rightarrow f$  in measure on  $A$ .

PROOF. (i)  $\Rightarrow$  (ii) Assume  $f_n \rightarrow f$  in  $L^1(A)$ , hence  $\int_A |f_n| \rightarrow \int_A |f|$  so  $\{f_n\}$  bounded in  $L^1(A)$ . Note we've already proven that  $f_n \rightarrow f$  in measure. For  $M > 0$ ,

$$\begin{aligned}
\int_{A \cap \{|f_n| > M\}} |f_n| &\leq \int_{A \cap \{|f_n| > M\}} |f_n - f| + \int_{A \cap \{|f_n| > M\}} |f| \\
&\leq \underbrace{\int_A |f_n - f|}_{\rightarrow 0} + \underbrace{\int_{A \cap \{|f_n| > M\} \cap \{|f| \leq \sqrt{M}\}} |f|}_{\leq \sqrt{M} \cdot m(A \cap \{|f_n| > M\})} + \underbrace{\int_{A \cap \{|f_n| > M\} \cap \{|f| > \sqrt{M}\}} |f|}_{\leq \int_{A \cap \{|f| > \sqrt{M}\}} |f| \rightarrow 0 \text{ since } f \in L^1} \\
&\leq \sqrt{M} \frac{\sup_n \int_A |f_n|}{M} \rightarrow 0 \text{ as } M \rightarrow \infty \quad (\text{Markov's})
\end{aligned}$$

Fix  $\varepsilon > 0$ . Choose  $N$  such that for all  $n \geq N$ ,  $\int_A |f_n - f| \leq \frac{\varepsilon}{3}$ , choose  $M$  such that  $\int_{A \cap \{|f| > \sqrt{M}\}} |f| < \frac{\varepsilon}{3}$  and  $\frac{\sup_n \int_A |f_n|}{\sqrt{M}} < \frac{\varepsilon}{3}$ . Thus,

$$\sup_{n \geq N} \int_{A \cap \{|f_n| > M\}} |f_n| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

We want this to hold for  $N = 1$  for uniformity, i.e. we need to deal with the first  $N - 1$  terms. We achieve this by making  $M$  larger if necessary such that

$$\int_{A \cap \{|f_k| > M\}} |f_k| \leq \varepsilon$$

for every  $k = 1, 2, \dots, N - 1$ .

(ii)  $\Rightarrow$  (i) assignment question. ■

### §3 PRODUCT SPACE

#### §3.1 Preparations

Given a measure space  $(X, \mathcal{F}, \mu)$  with  $\mu$  a  $\sigma$ -finite measure (i.e. there exists a sequence  $\{X_n\} \subseteq \mathcal{F}$  such that  $X_n \uparrow$  and  $\bigcup_n X_n = X$ , and  $\mu(X_n) < \infty$  for each  $n$ ).

$\hookrightarrow$  **Definition 3.1** (Measurable):  $f : X \rightarrow \overline{\mathbb{R}}$  is  $\mathcal{F}$ -measurable if  $\forall a \in \mathbb{R}, f^{-1}([-\infty, a)) \in \mathcal{F}$ .

We have similar properties for  $f$  in general as in the Lebesgue setting. -For  $f$   $\mathcal{F}$ -measurable,  $cf, f^k, |f|, f \wedge a, f \vee b, f^+, f^-$  are all  $\mathcal{F}$ -measurable for  $a, b, c \in \mathbb{R}$ .

- For  $f, g$   $\mathcal{F}$ -measurable,  $f + g, f - g, f \cdot g, f \wedge g, f \vee g$  are all  $\mathcal{F}$ -measurable.
- If  $\{f_n\}$   $\mathcal{F}$ -measurable,  $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n$  are  $\mathcal{F}$ -measurable.

We may “dissect” functions as before. For  $f$   $\mathcal{F}$ -measurable, write  $f = f^+ - f^-$ , and put for  $n \geq 1$  and  $\bullet = +, -$ ,

$$f_n^\bullet := \mathbb{1}_{X_n}(f^\bullet \wedge n).$$

Then,  $f_n^\bullet \uparrow f^\bullet$ . Put

$$\varphi_n^\bullet := \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}^\bullet},$$

where, for  $k = 0, 1, \dots, n2^n$  for  $n \geq 1$ ,

$$A_{n,k}^\bullet = \left\{ x \in X_n : \frac{k}{2^n} \leq f_n^\bullet < \frac{k+1}{2^n} \right\} \in \mathcal{F}.$$



Then, we may define the integral of the simple function

$$\int_X \varphi_n^\bullet d\mu := \sum_{k=0}^{n2^n} \frac{k}{2^n} \mu(A_{n,k}^\bullet).$$

Define then

$$\int_X f^\bullet d\mu := \lim_n \int_X \varphi_n^\bullet d\mu,$$

and

$$\int_X f d\mu := \int_X f^+ d\mu - \int_X f^- d\mu.$$

We say, then,  $f \in L^1(\mu)$  if  $\int_X |f| d\mu < \infty$ . This generalizes the notion of integration to a (slightly more) general  $\sigma$ -algebra.

### §3.2 Product Lebesgue $\sigma$ -Algebra

We will restrict our constructions to the product of 2 spaces, i.e.  $\mathbb{R}^2$ , but generalizes for general  $\mathbb{R}^d$ .

↪ **Definition 3.2** (Product  $\sigma$ -algebra): The *product  $\sigma$ -algebra* of subsets of  $\mathbb{R}^2$ , denoted by  $\mathcal{M} \otimes \mathcal{M}$  or simply  $\mathcal{M}^2$ , is defined as

$$\mathcal{M}^2 := \sigma(\{A \times B : A, B \in \mathcal{M}\}),$$

where

$$A \times B := \{(x, y) : x \in A, y \in B\}$$

as is standard.

Notice  $\mathcal{M}^2$  contains

- rectangles  $I_1 \times I_2$ ,  $I_1, I_2$  intervals;
- singletons  $\{(x, y)\}$ ;
- open sets, closed sets, and so  $\mathfrak{B}(\mathbb{R}^2) := \sigma(\{\text{open sets in } \mathbb{R}^2\}) \subseteq \mathcal{M}^2$ .

Given  $G$  open, then for every  $x \in G$ , there exists some disc centered at  $x$  contained entirely in  $G$ . Moreover, there exist  $(a_1, a_2), (b_1, b_2)$  with  $a_i, b_i \in \mathbb{Q}$  such that  $x \in (a_1, a_2) \times (b_1, b_2) \subset G$ . Then,  $G = \bigcup_{x \in G} (a_1, a_2) \times (b_1, b_2)$ .

↪ **Definition 3.3** (Slice): Given  $E \subseteq \mathbb{R}^2$ , then for every  $x \in \mathbb{R}$ , define

$$E_x := \{y \in \mathbb{R} : (x, y) \in E\} \subseteq \mathbb{R},$$

called the *slice* of  $E$  at  $x$ . Similarly, define for  $y \in \mathbb{R}$ ,

$$E^y := \{x \in \mathbb{R} : (x, y) \in E\} \subseteq \mathbb{R}.$$

↪ **Proposition 3.1:** If  $E \in \mathcal{M}^2$ , then for every  $x \in \mathbb{R}$ ,  $E_x \in \mathcal{M}$ , and for every  $y \in \mathbb{R}$   $E^y \in \mathcal{M}$ ; that is, product measurability  $\Rightarrow$  marginal measurability.

PROOF. Define

$$\mathcal{A} := \{E \subseteq \mathbb{R}^2 : \forall x \in \mathbb{R}, E_x \in \mathcal{M}\}.$$

We claim  $\mathcal{A}$  a  $\sigma$ -algebra of subsets of  $\mathbb{R}^2$ .

- $\mathbb{R}^2 \in \mathcal{A}$ ? Yes, since for every  $x \in \mathbb{R}$ ,  $\mathbb{R}_x^2 = \mathbb{R} \in \mathcal{M}$ .
- Let  $E \in \mathcal{A}$ . Then,  $E_x \in \mathcal{M}$  for every  $x \in \mathbb{R}$ . But we have too

$$(E^c)_x = (E_x)^c,$$

and since  $E_x \in \mathcal{M} \Rightarrow (E_x)^c \in \mathcal{M}$ , it follows that  $E^c \in \mathcal{A}$ .

- If  $\{E_n\} \subseteq \mathcal{A}$ , then for every  $x \in \mathbb{R}$ ,

$$\left(\bigcup_n E_n\right)_x = \left(\bigcup_n (E_n)_x\right) \in \mathcal{M}$$

so  $\bigcup_n E_n \in \mathcal{A}$ .

Hence,  $\mathcal{A}$  indeed a  $\sigma$ -algebra of subsets of  $\mathbb{R}^2$ . For every  $A, B \in \mathcal{M}$ , we claim  $A \times B \in \mathcal{A}$ . We have that for every  $x \in \mathbb{R}$ ,

$$(A \times B)_x = \begin{cases} \emptyset & \text{if } x \notin A \\ B & \text{if } x \in A \end{cases} \in \mathcal{M},$$

hence  $A \times B \in \mathcal{A}$ . Thus, since such sets generate  $\mathcal{M}^2$ , it follows that  $\mathcal{M}^2 \subseteq \mathcal{A}$ , and so every set in  $\mathcal{M}^2$  has the desired property.

An identical proof follows for  $E^y$ -type slices. ■

**Remark 3.1:** Notice we didn't prove  $\mathcal{A} = \mathcal{M}^2$ , indeed, because its not true.

For instance, let  $E = N \times A$  with  $N$  the Vitali set and  $A \in \mathcal{M}$ . Then, for every  $x \in A$ ,  $E_x = \begin{cases} A & \text{if } x \in N \\ \emptyset & \text{if } x \notin N \end{cases} \in \mathcal{M}$ , but  $E \notin \mathcal{M}^2$ , because for every  $y \in \mathbb{R}$ ,  $E^y = \begin{cases} N & \text{if } y \in A \\ \emptyset & \text{else} \end{cases}$ .

In fact, there exists sets such that  $E_x$  and  $E^y \in \mathcal{M}$  for every  $x, y \in \mathbb{R}$ , but  $E \notin \mathcal{M}^2$  (the *Sierpinski set*).

However, if  $E \subseteq \mathbb{R}^2$  a product set, i.e.  $E = A \times B$  for some  $A, B \subseteq \mathbb{R}$ , then  $A, B \in \mathcal{M} \Rightarrow E \in \mathcal{M}^2$ .

↪ **Definition 3.4** (Slice of sets): Let  $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$  a function. For every  $x \in \mathbb{R}$ , define

$$f_x : \mathbb{R} \rightarrow \overline{\mathbb{R}}, \quad f_x(y) := f(x, y),$$

called the *slice* of  $f$  at  $x$ . Similarly define  $f^y$ .

⊗ **Example 3.1:** If  $f = \mathbb{1}_E$  for some  $E \subseteq \mathbb{R}^2$ , then  $f_x = \mathbb{1}_{E_x}$ .

↪ **Proposition 3.2:** If  $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$  is  $\mathcal{M}^2$ -measurable, then for every  $x \in \mathbb{R}$ ,  $f_x$  is  $\mathcal{M}$ -measurable, and for every  $y \in \mathbb{R}$   $f^y$  is  $\mathcal{M}$ -measurable.

PROOF. Observe that for every  $B \subseteq \mathbb{R}$ ,

$$(f^{-1}(B))_x = f_x^{-1}(B)$$

for every  $x \in \mathbb{R}$ , with similar for  $f^y$ . In particular, then, if  $f$   $\mathcal{M}^2$ -measurable, then for every  $a \in \mathbb{R}$ ,  $f^{-1}([-\infty, a)) \in \mathcal{M}^2$  hence  $f_x^{-1}([-\infty, a)) = (f^{-1}([-\infty, a)))_x \in \mathcal{M}$ , with the same idea following for  $f^y$ . ■

**Remark 3.2:**

- If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is continuous, then  $f$  is measurable. For every  $a \in \mathbb{R}$ ,  $f^{-1}((-\infty, a))$  open by virtue (indeed, definition) of continuity, hence in  $\mathcal{M}^2$ .
- If  $f = \mathbb{1}_E$  for some  $E \subseteq \mathbb{R}^2$ ,  $f$   $\mathcal{M}^2$ -measurable  $\Leftrightarrow E \in \mathcal{M}^2$ .
- In general, there exists  $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$  such that  $f_x$   $\mathcal{M}$ -measurable but  $f$  is not  $\mathcal{M}^2$ -measurable.
- If  $f(x, y) = h(x)g(y)$  for some non-trivial  $h, g : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ , then  $f$  is  $\mathcal{M}^2$ -measurable  $\Leftrightarrow$  both  $h$  and  $g$  are  $\mathcal{M}$ -measurable. We show  $\Leftarrow$ ;

$$\begin{aligned} f^{-1}([-\infty, a)) &= \{(x, y) : h(x)g(y) < a\} \\ &= \{(x, y) : h(x) = 0, 0 < a\} \\ &\quad \cup \left\{ (x, y) : h(x) > 0, g(y) < \frac{a}{h(x)} \right\} \\ &\quad \cup \left\{ (x, y) : h(x) < 0, g(y) > \frac{a}{h(x)} \right\} \\ &= \{x : h(x) = 0\} \times \mathbb{R} \cap \{0 < a\} \in \mathcal{M}^2 \\ &\quad \cup \left( \bigcup_{q \in \mathbb{Q}} \underbrace{\left\{ x : 0 < h(x), q < \frac{a}{h(x)} \right\}}_{\in \mathcal{M}} \times \underbrace{\{y : g(y) < q\}}_{\in \mathcal{M}} \right) \\ &\quad \cup \left( \bigcup_{q \in \mathbb{Q}} \underbrace{\left\{ x : 0 > h(x), q > \frac{a}{h(x)} \right\}}_{\in \mathcal{M}} \times \underbrace{\{y : g(y) > q\}}_{\in \mathcal{M}} \right) \in \mathcal{M}^2 \end{aligned}$$

### §3.3 Product Measure

↪ **Definition 3.5:** Given  $E \in \mathcal{M}^2$ , define functions

$$I_E^{(1)} : \mathbb{R} \rightarrow \overline{\mathbb{R}}, \quad I_E^{(1)}(x) := m(E_x)$$

and

$$I_E^{(2)} : \mathbb{R} \rightarrow \overline{\mathbb{R}}, \quad I_E^{(2)}(y) := m(E^y).$$

↪ **Theorem 3.1:** Given  $E \in \mathcal{M}^2$ ,  $I_E^{(1)}, I_E^{(2)}$  are  $\mathcal{M}$ -measurable functions, and in particular

$$\int_{\mathbb{R}} I_E^{(1)}(x) dx = \int_{\mathbb{R}} I_E^{(2)}(y) dy. \quad \circledast$$

PROOF. If indeed  $I_E^{(1)}, I_E^{(2)}$   $\mathcal{M}$ -measurable, then the integrals of the functions are well-defined, being non-negative functions.

Set

$$\Sigma := \left\{ E \in \mathcal{M}^2 : I_{E_N}^{(1)}, I_{E_N}^{(2)} \text{ are } \mathcal{M}\text{-measurable and } \circledast \text{ holds, for } E_N := E \cap [-N, N]^2 \text{ for all } N > 0 \right\}.$$

Note that for every  $E \in \mathcal{M}^2$ , for all  $N > 0$ ,

$$I_{E_N}^{(1)}(x) = \begin{cases} m((E_N)_x) & \text{if } x \in [-N, N] \\ 0 & \text{o.w.} \end{cases} = \mathbb{1}_{[-N, N]}(x) I_{E_N}^{(1)}(x).$$

similarly for  $I_{E_N}^{(2)}$ .

Let  $\mathcal{C} := \{A \times B : A, B \in \mathcal{M}\}$  (recall  $\mathcal{M}^2 = \sigma(\mathcal{C})$ ).

• *Claim 1:*  $\mathcal{C} \subseteq \Sigma$

For every  $N > 0$ ,  $E_N = (A \times B) \cap [-N, N]^2 = A_N \times B_N$  ( $A_N := A \cap [-N, N]$ ). Then,

$$I_{E_N}^{(1)}(x) = I_{A_N \times B_N}^{(1)}(x) = \begin{cases} m(B_N) & \text{if } x \in A_N \\ 0 & \text{if } x \notin A_N \end{cases}, \quad I_{E_N}^{(2)}(y) = I_{A_N \times B_N}^{(2)}(y) = \begin{cases} m(A_N) & \text{if } y \in B_N \\ 0 & \text{if } y \notin B_N \end{cases},$$

and so  $I_{E_N}^{(1)}, I_{E_N}^{(2)}$  are measurable seeing as they are both just indicator functions of measurable sets times a constant. In particular,

$$\int_{\mathbb{R}} I_{E_N}^{(1)} = m(B_N)m(A_N) = \int_{\mathbb{R}} I_{E_N}^{(2)},$$

as required. Hence, indeed  $E_N \in \Sigma$  and so  $\mathcal{C} \subseteq \Sigma$ .

• *Claim 2:*  $\mathbb{R}^2 \in \Sigma$

For every  $N > 0$ ,

$$I_{[-N, N]^2}^{(1)}(x) = \begin{cases} 2N & \text{if } x \in [-N, N] \\ 0 & \text{o.w.} \end{cases},$$

similar for  $I_{[-N, N]^2}^{(2)}$ .  $I_{[-N, N]^2}^{(1)}, I_{[-N, N]^2}^{(2)}$  are both  $\mathcal{M}$ -measurable, and their integrals agree, and so it follows that  $\mathbb{R}^2 \in \Sigma$ .

- *Claim 3:*  $E \in \Sigma \Rightarrow E^c \in \Sigma$

For each  $N > 0$ , denote

$$F_N := E^c \cap [-N, N]^2.$$

$I_{F_N}^{(1)} = 0$  outside of  $[-N, N]$ , and for  $x \in [-N, N]$ ,

$$(F_N)_x = \{y : (x, y) \in E^c \cap [-N, N]^2\} = [-N, N] \setminus E_x = [-N, N] \setminus (E_N)_x$$

so

$$I_{F_N}^{(1)}(x) = 2N - I_{E_N}^{(1)}(x)$$

for  $x \in [-N, N]$ . Similarly,

$$I_{F_N}^{(2)}(y) = \begin{cases} 2N - I_{E_N}^{(2)}(y) & \text{if } y \in [-N, N] \\ 0 & \text{o.w.} \end{cases}.$$

In particular, then,  $I_{F_N}^{(1)}, I_{F_N}^{(2)}$  measurable, and

$$\begin{aligned} \int_{\mathbb{R}} I_{F_N}^{(1)} &= \int_{[-N, N]} 2N - I_{E_N}^{(1)} = 4N^2 - \int_{\mathbb{R}} I_{E_N}^{(1)} \\ \int_{\mathbb{R}} I_{F_N}^{(2)} &= \int_{[-N, N]} 2N - I_{E_N}^{(2)} = 4N^2 - \int_{\mathbb{R}} I_{E_N}^{(2)}, \end{aligned}$$

but we know  $\int_{\mathbb{R}} I_{E_N}^{(1)} = \int_{\mathbb{R}} I_{E_N}^{(2)}$  since  $E_N \in \Sigma$ , hence it follows that  $\int_{\mathbb{R}} I_{F_N}^{(1)} = \int_{\mathbb{R}} I_{F_N}^{(2)}$  and so it follows that  $E^c \in \Sigma$ .

- *Claim 4:*  $\{E_k\} \subseteq \Sigma \Rightarrow E := \bigcup_{k=1}^{\infty} E_k \in \Sigma$ .

Wlog,  $E_n$ 's disjoint. For  $N > 0$ ,  $E_N = \bigcup_{k=1}^{\infty} E_{k,N}$ .

$$I_{E_N}^{(1)}(x) = \mathbb{1}_{[-N, N]}(x) m\left(\bigcup_{k=1}^{\infty} (E_{k,N})_x\right) = \mathbb{1}_{[-N, N]} \sum_{k=1}^{\infty} m((E_{k,N})_x) = \sum_{k=1}^{\infty} I_{E_{k,N}}^{(1)}(x),$$

with similarly  $I_{E_N}^{(2)}(y) = \sum_{k=1}^{\infty} I_{E_{k,N}}^{(2)}(y)$ . This implies  $I_{E_N}^{(1)}, I_{E_N}^{(2)}$  are  $\mathcal{M}$ -measurable, and in particular

$$\int_{\mathbb{R}} I_{E_N}^{(1)} = \sum_{k=1}^{\infty} \int_{\mathbb{R}} I_{E_{k,N}}^{(1)}, \quad \int_{\mathbb{R}} I_{E_N}^{(2)} = \sum_{k=1}^{\infty} \int_{\mathbb{R}} I_{E_{k,N}}^{(2)},$$

which are equal since by assumption  $E_k \in \Sigma$ . Hence,  $E \in \Sigma$ , and thus by Claims 2-4,  $\Sigma$  a  $\sigma$ -algebra of subsets of  $\mathbb{R}^2$ , and thus by Claim 1  $\Sigma = \mathcal{M}^2$ .

Hence, for every  $E \in \mathcal{M}^2$ ,  $E \in \Sigma$  and so all the statements hold for  $E_N$  for every  $N > 0$ . Then,

$$I_E^{(1)}(x) = \lim_{N \rightarrow \infty} \mathbb{1}_{[-N, N]}(x) m((E_N)_x) = \lim_{N \rightarrow \infty} m((E_N)_x) = m(E_x) = \lim_{N \rightarrow \infty} I_{E_N}^{(1)}(x),$$

and in particular  $\{I_{E_N}^{(1)}\} \uparrow$ , hence  $I_E^{(1)}$   $\mathcal{M}$ -measurable, and

$$\int_{\mathbb{R}} I_E^{(1)} = \lim_{N \rightarrow \infty} \int_{\mathbb{R}} I_{E_N}^{(1)},$$

with similar for  $I_E^{(2)}$ , by monotonicity. Thus, since  $\int_{\mathbb{R}} I_{E_N}^{(1)} = \int_{\mathbb{R}} I_{E_N}^{(2)}$  for every  $N$ , the proof follows. ■

↪ **Definition 3.6:** Define a non-negative set function on  $(\mathbb{R}^2, \mathcal{M}^2)$  by

$$m(E) := \int_{\mathbb{R}} I_E^{(1)}(x) dx = \int_{\mathbb{R}} I_E^{(2)}(x) dx, \quad E \in \mathcal{M}^2.$$

$m$  is called the *Lebesgue measure* on  $\mathbb{R}^2$ .

↪ **Proposition 3.3:**  $m$  is indeed a measure on  $(\mathbb{R}^2, \mathcal{M}^2)$ .

PROOF.

- $m(\emptyset) = \int_{\mathbb{R}} 0 = 0$
- If  $\{E_k\} \subseteq \mathcal{M}^2$  disjoint, let  $E = \bigcup_{k=1}^{\infty} E_k$ . Then

$$m(E) = \sum_{k=1}^{\infty} m(E_k),$$

since for every  $x \in \mathbb{R}$ ,  $E_x = \bigcup_{k=1}^{\infty} (E_k)_x$  disjoint, so

$$\int_{\mathbb{R}} I_E^{(1)} = \sum_{k=1}^{\infty} \int_{\mathbb{R}} I_{E_k}^{(1)},$$

and the proof follows. ■

**Remark 3.3:**

1. For any  $E = I_1 \times I_2$ ,  $m(E) = \ell(I_1) \cdot \ell(I_2)$ . It follows that any singleton, and countable set, and any line on  $\mathbb{R}^2$  is a null set.
2. If  $A \subseteq \mathbb{R}$  is a null set in  $\mathcal{M}$ , then  $A \times \mathbb{R}, \mathbb{R} \times A$  are null sets, in  $\mathcal{M}^2$ .
3.  $\mathcal{M}^2$  is *not* complete under  $m$ , since for instance if  $N \subset \mathbb{R}$  the Vitali set,  $a \in \mathbb{R}$ , then  $\{a\} \times N \subseteq \{a\} \times \mathbb{R}$  is a subset of a null set, but  $\{a\} \times N$  is not measurable.
4. It is possible to construct  $m$  on  $\mathbb{R}^2$  through the “outer measure” approach. We take  $E \subseteq \mathbb{R}^2$ , and define

$$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \text{Area}(R_n) : R_n \text{'s closed, finite rectangles s.t. } E \subseteq \bigcup_{n=1}^{\infty} R_n \right\}.$$

Then,  $m^*$  satisfies similar properties as the 1-dimensional analog. We then say a set  $E$  is measurable if for every  $F \subseteq \mathbb{R}^2$ ,  $m^*(F) = m^*(F \cap E) + m^*(F \cap E^c)$ . Collect all such sets,  $\overline{\mathcal{M}^2} := \{E \subseteq \mathbb{R}^2 : E \text{ measurable}\}$ . This is a  $\sigma$ -algebra of subsets of  $\mathbb{R}^2$ , with  $m := m^*|_{\overline{\mathcal{M}^2}}$  a measure when restricted to it. Indeed,  $m$  matches the Lebesgue measure defined above, and  $\overline{\mathcal{M}^2}$ , as suggestively notated, the completion of  $\mathcal{M}^2$  under the Lebesgue measure. In addition,  $\overline{\mathcal{M}^2} = \mathfrak{B}_{\mathbb{R}^2}$ .

5. The Lebesgue measure  $m$  on  $\mathbb{R}^2$  is the unique measure on  $\mathcal{M}^2/\mathfrak{B}_{\mathbb{R}^2}/\overline{\mathcal{M}^2}$  such that for all  $I_1 \times I_2$  rectangles,  $m(I_1 \times I_2) = \ell(I_1)\ell(I_2)$ . This is because  $\mathcal{I} := \{I_1 \times I_2 : I_1, I_2 \text{ finite intervals}\}$  is a  $\pi$ -system and generates  $\mathfrak{B}_{\mathbb{R}^2}$ .
6. The Lebesgue measure on  $\mathbb{R}^2$  is translation invariant (rectangle area is invariant under translation). Namely, show that  $m_z : \mathcal{M}^2 \rightarrow [0, \infty]$ ,  $m_z(E) := m(E + z)$  is a measure and  $m_z = m$  on  $\mathcal{I}$ .
7. The Lebesgue measure  $m$  on  $\mathbb{R}^2$  is the only measure on  $\mathcal{M}^2/\mathfrak{B}_{\mathbb{R}^2}/\overline{\mathcal{M}^2}$  that is translation invariant and assigns 1 to  $[0, 1] \times [0, 1]$ .

### §3.4 Fubini's Theorem

↪ **Definition 3.7:** Let  $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$  be  $\mathcal{M}^2$ -measurable and non-negative. Define the functions

$$I_f^{(1)}(x) := \int_{\mathbb{R}} f(x, y) dy = \int_{\mathbb{R}} f_x(y) dy, \quad I_f^{(2)}(y) := \int_{\mathbb{R}} f(x, y) dx = \int_{\mathbb{R}} f^y(x) dx.$$

**Remark 3.4:** Given  $f : \mathbb{R}^2 \rightarrow [0, \infty]$ ,  $\mathcal{M}^2$ -measurable and non-negative, the integral of  $f$  wrt the Lebesgue measure on  $\mathbb{R}^2$  is denoted by  $\int_{\mathbb{R}^2} f(x, y) dx dy$  or  $\int_{\mathbb{R}^2} f$  if there is no ambiguity.

↪ **Theorem 3.2** (Tonelli's): Let  $f : \mathbb{R}^2 \rightarrow [0, \infty]$  be  $\mathcal{M}^2$ -measurable and non-negative. Then,

$$\int_{\mathbb{R}^2} f(x, y) \, dx \, dy = \int_{\mathbb{R}} I_f^{(1)}(x) \, dx = \int_{\mathbb{R}} I_f^{(2)}(y) \, dy,$$

or more explicitly,

$$\int_{\mathbb{R}^2} f(x, y) \, dx \, dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x, y) \, dy \right) dx = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x, y) \, dx \right) dy.$$

PROOF. Since  $f$   $\mathcal{M}^2$ -measurable, non-negative, there exists  $\{\varphi_n\}$ -sequence of simple functions with  $\varphi_n \uparrow f$ , and  $\int_{\mathbb{R}^2} f = \lim_n \int_{\mathbb{R}^2} \varphi_n$ , where, eg

$$\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}}, \quad A_{n,k} := \left\{ (x, y) \in [-n, n]^2 : \frac{k}{2^n} \leq f(x, y) < \frac{k+1}{2^n} \right\}, \quad k = 0, 1, \dots, n2^n.$$

So,

$$\int_{\mathbb{R}^2} f(x, y) \, dx \, dy = \lim_n \sum_{k=0}^{n2^n} \frac{k}{2^n} m(A_{n,k}).$$

On the other hand,  $\forall x \in \mathbb{R}$ , by MON

$$\begin{aligned} I_f^{(1)}(x) &= \int_{\mathbb{R}} f(x, y) \, dy = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n(x, y) \, dy \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} I_{A_{n,k}}^{(1)}(x) \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} m((A_{n,k})_x). \end{aligned}$$

We have then, again by MON, that

$$\int_{\mathbb{R}} I_f^{(1)}(x) \, dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \sum_{k=0}^{n2^n} \frac{k}{2^n} I_{A_{n,k}}^{(1)}(x) \, dx = \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} \int_{\mathbb{R}} I_{A_{n,k}}^{(1)}(x) \, dx.$$

Similarly, we find

$$\int_{\mathbb{R}} I_f^{(2)}(y) \, dy = \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} \int_{\mathbb{R}} I_{A_{n,k}}^{(2)}(y) \, dy.$$

By definition,

$$m(A_{n,k}) = \int_{\mathbb{R}} I_{A_{n,k}}^{(1)}(x) \, dx = \int_{\mathbb{R}} I_{A_{n,k}}^{(2)}(y) \, dy,$$

hence all of our terms actually agree, and bringing them together gives the proof. ■



↪ **Definition 3.8:** Given  $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$   $\mathcal{M}^2$ -measurable, we write  $f \in L^1(\mathbb{R}^2)$  if

$$\int_{\mathbb{R}^2} |f(x, y)| dx dy < \infty,$$

or equivalently if

$$\int_{\mathbb{R}^2} f^+ \text{ and } \int_{\mathbb{R}^2} f^- < \infty.$$

**Remark 3.5:** Suppose  $f \in L^1(\mathbb{R}^2)$ . Then by Tonelli's,

$$\int_{\mathbb{R}^2} |f(x, y)| dx dy = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x, y)| dy \right) dx = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x, y)| dx \right) dy,$$

and in particular all integrals are finite; namely,  $I_{|f|}^{(1)}, I_{|f|}^{(2)} \in L^1(\mathbb{R})$ .

↪ **Theorem 3.3** (Fubini's): If  $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$  is  $\mathcal{M}^2$ -measurable and  $f \in L^1(\mathbb{R}^2)$ , then

1.  $I_f^{(1)}, I_f^{(2)} \in L^1(\mathbb{R})$  (product integrability  $\Rightarrow$  marginal integrability)
2.  $I_f^{(1)}(x)$  finite-valued for a.e.  $x \in \mathbb{R} \Rightarrow f_x \in L^1(\mathbb{R})$  for a.e.  $x \in \mathbb{R}$ , similar for  $I_f^{(2)}$ , i.e.  $f_y \in L^1(\mathbb{R})$  for a.e.  $y \in \mathbb{R}$ .
3.  $\int_{\mathbb{R}^2} f(x, y) dx dy = \int_{\mathbb{R}} I_f^{(1)}(x) dx = \int_{\mathbb{R}} I_f^{(2)}(y) dy$ .

PROOF. Assume  $f \in L^1(\mathbb{R}^2)$ . Then by Tonelli's

$$\int_{\mathbb{R}} |I_f^{(1)}| \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x, y)| dy \right) dx = \int_{\mathbb{R}^2} |f(x, y)| dx dy < \infty \Rightarrow I_f^{(1)} \in L^1(\mathbb{R}).$$

We have 1.  $\Rightarrow$  2..

Now, write  $f = f^+ - f^-$ .  $f \in L^1(\mathbb{R}^2)$  gives that  $f^+, f^- \in L^1(\mathbb{R}^2)$  so  $f^+, f^-$  each finite valued a.e.. By Tonelli's, then,

$$\int_{\mathbb{R}} I_{f^+}^{(1)}(x) dx = \int_{\mathbb{R}} I_{f^+}^{(2)}(y) dy = \int_{\mathbb{R}^2} f^+(x, y) dx dy < \infty,$$

same with  $f^-$ . Then,  $I_{f^+}^{(1)}, I_{f^+}^{(2)}, I_{f^-}^{(1)}, I_{f^-}^{(2)} \in L^1(\mathbb{R})$ , hence are finite-valued a.e.. By linearity on  $L^1$  functions, then

$$\int_{\mathbb{R}} I_{f^+}^{(1)}(x) dx - \int_{\mathbb{R}} I_{f^-}^{(1)}(x) dx = \int_{\mathbb{R}} I_{f^+}^{(1)} - I_{f^-}^{(1)}.$$

For a.e.  $x \in \mathbb{R}$ ,  $f_x^+, f_x^- \in L^1(\mathbb{R})$ , so by linearity

$$I_{f^+}^{(1)}(x) - I_{f^-}^{(1)}(x) = \int_{\mathbb{R}} f_x^+(y) dy - \int_{\mathbb{R}} f_x^-(y) dy = \int_{\mathbb{R}} (f_x^+ - f_x^-) = \int_{\mathbb{R}} f_x,$$

so

$$\int_{\mathbb{R}} I_{f^+}^{(1)} - \int_{\mathbb{R}} I_{f^-}^{(1)} = \int_{\mathbb{R}} I_f^{(1)},$$

with similarly for

$$\int_{\mathbb{R}} I_{f^+}^{(2)} - \int_{\mathbb{R}} I_{f^-}^{(2)} = \int_{\mathbb{R}} I_f^{(2)}.$$

All together, then,

$$\int_{\mathbb{R}} I_f^{(1)} = \int_{\mathbb{R}} I_f^{(2)} = \int_{\mathbb{R}^2} (f^+ - f^-) = \int_{\mathbb{R}^2} f.$$

■

**Remark 3.6:** In general,  $I_f^{(1)}, I_f^{(2)} \in L^1(\mathbb{R}) \not\Rightarrow f \in L^1(\mathbb{R}^2)$ . For instance, let

$$f(x, y) = \begin{cases} 1 & \text{if } x < y < x + 1 \\ -1 & \text{if } x - 1 < y < x \\ 0 & \text{else} \end{cases}$$

Then, for all  $x \in \mathbb{R}$ ,

$$I_f^{(1)}(x) = \int_{\mathbb{R}} f(x, y) dy = 0,$$

and similarly

$$I_f^{(2)}(y) = \int_{\mathbb{R}} f(x, y) dx = 0,$$

so  $I_f^{(1)}, I_f^{(2)} \in L^1(\mathbb{R})$ , but  $f \notin L^1(\mathbb{R}^2)$ .

**Remark 3.7:** If  $f : \mathbb{R}^2 \rightarrow \bar{\mathbb{R}}$  is  $\overline{\mathcal{M}^2}$ -measurable, Tonelli's, Fubini's still hold. We'll use the same notations in this case.

In fact, there exists a  $\tilde{f} : \mathbb{R}^2 \rightarrow \bar{\mathbb{R}}$  that is  $\mathcal{M}^2$ -measurable such that  $\tilde{f} = f$  a.e. (exercise).

**Remark 3.8:** The constructions above extend to  $\mathbb{R}^d$ ,  $d \geq 3$ . In particular, we have

- $\mathcal{M}^d := \sigma(\{A_1 \times \cdots \times A_d : A_i \in \mathcal{M}\})$ .
- The product measure  $m$  is the Lebesgue measure on  $(\mathbb{R}^d, \mathcal{M}^d)$ .
- $\overline{\mathcal{M}^d}$  is the completion of  $\mathcal{M}^d$  under  $m$ .
- Tonelli's, Fubini's hold, with “ $d$ -embedded” integrals.
- $m$  shares similar properties on  $\mathbb{R}^d$  as on  $\mathbb{R}$ ;
  - translation invariance,
  - scaling property,
  - regularity, (outer: for every  $E \in \mathcal{M}^d$ ,  $m(E) = \inf\{m(G) : G \text{ open s.t. } E \subseteq G\}$ , inner: for every  $E \in \mathcal{M}^d$ ,  $m(E) = \sup\{m(K) : K \text{ compact s.t. } E \supseteq K\}$ ).

## §4 DIFFERENTIATION

In the Riemann setting, differentiation and integration are closely related. For instance, if  $F(x) := \int_a^x f(t) dt$  for some Riemann-integrable  $f$  on  $[a, b]$  and  $x \in [a, b]$ , then  $F$  is differentiable and  $F' = f$  on  $(a, b)$ . Or, if  $F$  differentiable, and  $F'$  is Riemann integrable on some  $[a, b]$ , then  $F(b) - F(a) = \int_a^b F'(t) dt$ . How much does this extend to the Lebesgue setting?

## §4.1 Hardy-Littlewood Maximal Function

↪ **Definition 4.1** (Hardy-Littlewood Maximal Function): Suppose  $f \in L^1(\mathbb{R})$ . The *Hardy-Littlewood Maximal Function* (H-L max.), denoted  $f^*$ , is defined as

$$f^*(x) := \sup_{I \in \mathcal{I}(x)} \frac{1}{m(I)} \int_I |f|,$$

where  $\mathcal{I}(x) := \{I : I \text{ an open interval containing } x\}$ .

↪ **Proposition 4.1**: Given  $f \in L^1(\mathbb{R})$ ,  $f^*$  is measurable.

PROOF.  $f^* \geq 0$ , so it suffices to show that for every  $a \geq 0$ ,  $\{f^* > a\}$  is measurable. Let  $x \in \{f^* > a\}$ . Then,  $a < f^*(x)$ , hence there must exist some  $I \in \mathcal{I}(x)$  such that  $\frac{1}{m(I)} \int_I |f| > a$ .  $I$  is open, and  $x \in I$ , so there exists some  $\delta > 0$  such that  $(x - \delta, x + \delta) \subseteq I$ . For every  $y \in (x - \delta, x + \delta)$ ,  $y \in I$ , hence  $I \in \mathcal{I}(y)$ . So,  $f^*(y) \geq \frac{1}{m(I)} \int_I |f| > a$ . Thus,  $y \in \{f^* > a\}$  as well. It follows, then, that  $(x - \delta, x + \delta) \subseteq \{f^* > a\}$ , hence  $\{f^* > a\}$  is open, and so in particular is measurable. ■

↪ **Lemma 4.1** (Vitali's Covering Lemma): Assume that  $\mathcal{I} := \{I_1, \dots, I_N\}$  a finite collection of open intervals. Then, there exists a sub-collection  $\{I_{k_1}, \dots, I_{k_M}\} \subset \mathcal{I}$  such that  $I_{k_i} \cap I_{k_j} = \emptyset$  for all  $i \neq j$  and

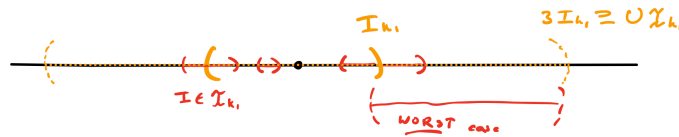
$$m\left(\bigcup_{i=1}^N I_i\right) \leq 3 \sum_{j=1}^M m(I_{k_j}).$$

PROOF. Assume wlog that  $m(I_i) < \infty$  for all  $1 \leq i \leq N$ ; if otherwise exists an  $i$  such that  $m(I_i) = \infty$ , then simply take your subcollection  $I_{k_1} := I_i$ , and the claim holds trivially.

Begin with the largest interval in  $\mathcal{I}$ , call it  $I_{k_1}$ . Let

$$\mathcal{I}_{k_1} := \{I \in \mathcal{I} : I \cap I_{k_1} \neq \emptyset\}.$$

For any  $I \in \mathcal{I}_{k_1}$ ,  $I \cap I_{k_1} \neq \emptyset$  and  $m(I) \leq m(I_{k_1})$ , so in particular  $I \subseteq 3I_{k_1}$  (if  $I_{k_1} = (a, b)$ ,  $3I_{k_1} := (a - 3(b - a), a + 3(b - a))$ ).



Then, in particular

$$\bigcup_{I \in \mathcal{I}_{k_1}} I \subseteq 3I_{k_1}.$$

Consider now  $\mathcal{I} \setminus \mathcal{I}_{k_1}$ , and choose the largest interval in the remaining part of the collection. Call it  $I_{k_2}$ . Set

$$\mathcal{I}_{k_2} := \{I \in \mathcal{I} \setminus \mathcal{I}_{k_1} : I \cap I_{k_2} \neq \emptyset\}.$$

Similarly to before,  $\bigcup_{I \in \mathcal{I}_{k_2}} I \subseteq 3I_{k_2}$ . By choice, too,  $I_{k_1} \cap I_{k_2} = \emptyset$ .

Repeat this process, until  $\mathcal{I} \setminus (\mathcal{I}_{k_1} \cup \dots \cup \mathcal{I}_{k_M}) = \emptyset$ , i.e. we have no intervals left in the original collection. Then, we obtain  $I_{k_1}, \dots, I_{k_M}$  disjoint intervals, with corresponding subcollections  $\mathcal{I}_{k_1}, \dots, \mathcal{I}_{k_M}$  forming a partition of  $\mathcal{I}$ . Then,

$$m\left(\bigcup_{n=1}^N I_n\right) = \sum_{j=1}^M m\left(\bigcup_{I \in \mathcal{I}_{k_j}} I\right) \leq 3 \sum_{j=1}^M m(I_{k_j}).$$

■

↪ **Proposition 4.2:** Suppose  $f \in L^1(\mathbb{R})$  and let  $f^*$  be the H-L max function of  $f$ . Then, for every  $\varepsilon > 0$ ,

$$m(\{x \in \mathbb{R} : f^*(x) > \varepsilon\}) \leq \frac{3}{\varepsilon} \|f\|_1 = \frac{3}{\varepsilon} \int_{\mathbb{R}} |f|.$$

PROOF. Fix  $\varepsilon > 0$  and put  $B := \{x \in \mathbb{R} : f^*(x) > \varepsilon\}$ . By inner regularity,

$$m(B) = \sup\{m(K) : B \supseteq K \text{ compact}\}.$$

It suffices to show then that  $m(K) \leq \frac{3}{\varepsilon} \|f\|_1$  for every compact  $K \subseteq B$ . For every  $x \in K$ ,  $f^*(x) > \varepsilon$  so there exists some open interval  $I_x$  such that  $x \in I_x$  and  $\frac{1}{m(I_x)} \int_{I_x} |f| > \varepsilon$ .

Hence, we may cover  $K \subseteq \bigcup_{x \in K} I_x$ . Since  $K$  compact it admits a finite subinterval, call it  $\mathcal{I} = \{I_1, \dots, I_N\}$ , such that  $K \subseteq \bigcup_{n=1}^N I_n$ . By the Covering Lemma,

$$m(K) \leq m\left(\bigcup_{n=1}^N I_n\right) \leq 3 \sum_{j=1}^M m(I_{k_j}),$$

for some disjoint subcollection  $I_{k_1}, \dots, I_{k_M}$ . Meanwhile, for every  $1 \leq j \leq M$ ,

$$m(I_{k_j}) < \frac{1}{\varepsilon} \int_{I_{k_j}} |f|,$$

hence, we find

$$m(K) \leq 3 \sum_{j=1}^M \frac{1}{\varepsilon} \int_{I_{k_j}} |f| = \frac{3}{\varepsilon} \int_{\bigcup_{j=1}^M I_{k_j}} |f| \leq \frac{3}{\varepsilon} \int_{\mathbb{R}} |f| = \frac{3}{\varepsilon} \|f\|_1.$$

■

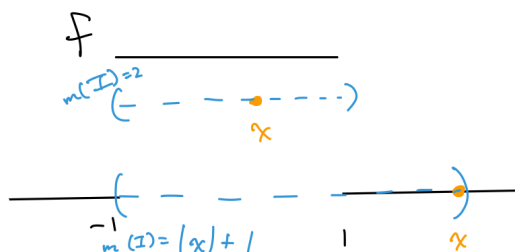
↪ **Corollary 4.1:** Given  $f \in L^1(\mathbb{R})$ ,  $f^*$  is finite-valued a.e..

PROOF. For every  $N > 0$ ,  $m(\{f^* > N\}) \leq \frac{3}{N} \|f\|_1$ . Taking  $N \rightarrow \infty$ , we find then  $m(\{f^* > N\}) \rightarrow 0$ , and since  $m(\{f^* = \infty\}) \leq m(\{f^* > N\}) \forall N > 0$  it follows that  $m(\{f^* = \infty\}) = 0$ . ■

**Remark 4.1:** While a Markov-like inequality,  $f^*$  need not be integrable in general. For instance, let  $f = \mathbb{1}_{[-1,1]} \in L^1(\mathbb{R})$ . Then, consider  $f^*$ , and in particular consider the average of  $f$  over intervals  $I = (a, b)$ ;

$$\frac{1}{b-a} \int_a^b f(t) dt = \begin{cases} 0 & \text{if } (a, b) \cap [-1, 1] = \emptyset \\ \frac{\min\{b, 1\} - \max\{a, -1\}}{b-a} & \text{if } (a, b) \cap [-1, 1] \neq \emptyset. \end{cases}$$

So, we find that  $f^*(x) = 1$  if  $x \in (-1, 1)$  (take your  $I = [-1, 1]$ , this achieves max), and  $f^*(x) = \frac{2}{|x|+1}$  if  $x \notin (-1, 1)$  (you want as much of the  $[-1, 1]$  support as possible, and with your other endpoint as close to  $x$  as possible).  $f^*$  not integrable.



## §4.2 Lebesgue Differentiation Theorem

↪ **Theorem 4.1** (Lebesgue Differentiation Theorem): Given  $f \in L^1(\mathbb{R})$ , for a.e.  $x \in \mathbb{R}$ , if  $\{I_n\}$  a sequence of open intervals such that  $x \in I_n \forall n \geq 1$  and  $\lim_{n \rightarrow \infty} m(I_n) = 0$  (we say  $\{I_n\}$  a sequence of intervals *shrinking to*  $x$ ), then

$$\lim_{n \rightarrow \infty} \frac{1}{m(I_n)} \int_{I_n} |f(t) - f(x)| dt = 0.$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{1}{m(I_n)} \int_{I_n} f(t) dt = f(x).$$

PROOF. The “In particular” comes from the fact that, for  $x$  such that  $f(x) < \infty$ ,

$$\left| \frac{1}{m(I_n)} \int_{I_n} f(t) dt - f(x) \right| = \left| \frac{1}{m(I_n)} \int_{I_n} f(t) - f(x) dt \right| \leq \frac{1}{m(I_n)} \int_{I_n} |f(t) - f(x)| dt,$$

so if the RHS  $\rightarrow 0$ , so does the left.

Without loss of generality, assume  $f$  finite valued everywhere, and only use finite-valued intervals  $I_n$ . For every  $k \geq 1$ , define

$$B_k := \left\{ x \in \mathbb{R} : \exists \{I_n\} \subseteq \mathcal{I}(x) \text{ with } \lim_{n \rightarrow \infty} m(I_n) = 0 \text{ s.t. } \limsup_{n \rightarrow \infty} \frac{1}{m(I_n)} \int_{I_n} |f(t) - f(x)| dt \geq \frac{1}{k} \right\}.$$

Notice  $B_k \uparrow$  and  $\bigcup_{k=1}^{\infty} B_k = \{x \in \mathbb{R} : \text{theorem fails}\}$ . So, it suffices to show that  $m(B_k) = 0$  for every  $k \geq 1$ .

Fix an arbitrary  $\varepsilon > 0$ . Continuous, compactly supported functions are dense in  $L^1(\mathbb{R})$  so we may find  $g \in C_c(\mathbb{R})$  such that  $\|f - g\|_1 \leq \varepsilon$ . Since  $g$  continuous and compactly supported, for every  $x \in \mathbb{R}$  and  $k \geq 1$ , there exists some  $\alpha > 0$  such that if  $|t - x| \leq \alpha$ ,  $|g(t) - g(x)| \leq \frac{1}{3k}$ .

Given any  $x \in \mathbb{R}$  and any sequence  $\{I_n\} \subseteq \mathcal{I}(x)$  with  $\lim_n m(I_n) = 0$ , we have

$$\frac{1}{m(I_n)} \int_{I_n} |f(t) - f(x)| dt \leq \frac{1}{m(I_n)} \int_{I_n} |f(t) - g(t)| dt \quad (1)$$

$$+ \frac{1}{m(I_n)} \int_{I_n} |g(t) - g(x)| dt \quad (2)$$

$$+ |g(x) - f(x)| \quad (3)$$

by triangle inequality, adding/subtracting  $g(t), g(x)$ . We know that when  $n$  sufficiently large such that  $m(I_n) < \alpha$ ,  $|g(t) - g(x)| \leq \frac{1}{3k} \forall t \in I_n$ , hence (2)  $\leq \frac{1}{3k}$  for sufficiently large  $n$ . For  $x$  to be in  $B_k$ , we need too that  $\limsup_n ((1) + (2) + (3)) > \frac{1}{k}$ . But we know that (2)  $\leq \frac{1}{3k}$  for all sufficiently large  $n$ , we must have that  $\limsup_n ((1) + (3)) > \frac{2}{3k}$ . Let

$$C_k := \left\{ x \in \mathbb{R} : \limsup_n (1) > \frac{1}{3k} \right\}, \quad D_k := \left\{ x \in \mathbb{R} : \limsup_n (3) > \frac{1}{3k} \right\},$$

then remark  $m(B_k) \leq m(C_k) + m(D_k)$  since  $B_k \subseteq C_k \cup D_k$ . Then,

$$m(D_k) = m\left(\left\{|f - g| > \frac{1}{3k}\right\}\right) \leq 3k \|f - g\|_1 \leq 3k\varepsilon,$$

by Markov's, and

$$\begin{aligned} m(C_k) &= m\left(\left\{\limsup_n \frac{1}{m(I_n)} \int_{I_n} |f - g| > \frac{1}{3k}\right\}\right) \\ &\leq m\left(\left\{(f - g)^* > \frac{1}{3k}\right\}\right) \leq 3 \cdot 3k \|f - g\|_1 = 9k\varepsilon, \end{aligned}$$

by using the previous H-L inequality. Hence, we find

$$m(B_k) \leq 12k\varepsilon,$$

and, sending  $\varepsilon \rightarrow 0$  we find  $m(B_k) = 0$ , completing the proof. ■

↪ **Definition 4.2** (Lebesgue Point): We call  $x$  a *Lebesgue point* of  $f$  if the Lebesgue Differentiation Theorem holds for  $f$  at  $x$ .

**Remark 4.2:** In the statement, the “a.e.  $x \in \mathbb{R}$ ” cannot be replaced with “pointwise”. For example, consider  $f = \mathbb{1}_{[0,1]}$  and  $I_n = \left(-\frac{1}{n}, \frac{1}{n}\right) \in \mathcal{I}(0)$ . Then

$$\frac{1}{m(I_n)} \int_{I_n} f = \frac{2}{n} \int_{-\frac{1}{n}}^{\frac{1}{n}} \mathbb{1}_{[0,1]} = \frac{1}{2} \neq f(0),$$

hence 0 not a Lebesgue point of  $f$ .

↪ **Corollary 4.2:** If  $f \in L^1(\mathbb{R})$ , then for a.e.  $x \in \mathbb{R}$ ,

$$0 = \lim_{h \rightarrow 0} \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - f(x)| dt \quad (\text{i})$$

$$= \lim_{h \rightarrow 0} \frac{1}{2h} \int_{-h}^h |f(x+y) - f(x)| dy \quad (\text{ii}).$$

PROOF. (i)  $\Rightarrow$  (ii) by translation of the integral.

If  $x \in \mathbb{R}$  is such that (i) fails, then there is a sequence  $\{h_n\} \in \mathbb{R}^+$  such that  $\lim h_n = 0$  and  $\lim_n \frac{1}{2h_n} \int_{x-h_n}^{x+h_n} |f(t) - f(x)| dt \neq 0$ . Then, the Lebesgue Diff Thm fails at  $x$  for  $I_n = (x - h_n, x + h_n)$  so  $x$  not a Lebesgue point. Hence,  $\{x : \text{(i) fails}\}$  a null set.

In particular, this implies that  $\lim_{h \rightarrow 0} \frac{1}{2h} \int_{x-h}^{x+h} f(t) dt = f(x)$  for a.e.  $x \in \mathbb{R}$ , so as a function of  $x$ ,  $\lim_{h \rightarrow 0} \frac{1}{2h} \int_{x-h}^{x+h} f(t) dt$  measurable. ■

↪ **Corollary 4.3:** Given  $f \in L^1(\mathbb{R})$ , then  $|f| \leq f^*$  a.e..

PROOF. Apply LDT to  $|f|$ . This implies that for a.e.  $x \in \mathbb{R}$ ,  $|f(x)| = \lim_{n \rightarrow \infty} \frac{1}{m(I_n)} \int_{I_n} |f(t)| dt$ ,  $I_n := \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ . By definition,  $\lim_{n \rightarrow \infty} \frac{1}{m(I_n)} \int_{I_n} |f(t)| dt \leq f^*(x)$ , being the supremum of such quantities, and the proof follows. ■

↪ **Theorem 4.2:** Given  $f \in L^1(\mathbb{R})$  and  $a \in \mathbb{R}$ , define  $F(x) := \int_a^x f(t) dt$  for every  $x \geq a$ . Then,  $F$  is uniformly continuous,  $F'(x)$  exists and is equal to  $f(x)$  for a.e.  $x \in \mathbb{R}$ .

↪ **Lemma 4.2:** If  $f \in L^1(\mathbb{R})$ , then for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\forall I$ -interval with  $m(I) \leq \delta$ ,  $\int_I |f| \leq \varepsilon$ .

PROOF. Recall  $f^* < \infty$  a.e.. Let  $A_N := \{f^* > N\}$ , then  $\mathbb{1}_{A_N} \rightarrow 0$  almost everywhere as  $N \rightarrow \infty$ . Hence,

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}} \mathbb{1}_{A_N} |f| = 0$$

by DOM, and so

$$\int_{\{f^* > N\}} |f| \rightarrow 0$$

as  $N \rightarrow \infty$ . Given  $\varepsilon$ , then,  $\exists N$  such that  $\int_{\{f^* > N\}} |f| \leq \frac{\varepsilon}{2}$ . Let  $\delta = \frac{\varepsilon}{2N}$ . Then, for every  $I$  (wlog open) with  $m(I) < \delta$ ,

$$\begin{aligned} \int_I |f| &= \int_{I \cap \{f^* > N\}} |f| + \int_{I \cap \{f^* \leq N\}} |f| \\ &\leq \int_{\{f^* > N\}} |f| + \int_{I \cap \{f^* \leq N\}} |f| \\ &\leq \frac{\varepsilon}{2} + N \cdot m(I) \\ &\leq \frac{\varepsilon}{2} + N \frac{\varepsilon}{2N} = \varepsilon. \end{aligned}$$

■

PROOF. (of  $\hookrightarrow$  [Theorem 4.2](#)) For every  $\varepsilon > 0$ , let  $\delta > 0$  as in the lemma. Then for every  $x > y \geq a$  such that  $|x - y| \leq \delta$ ,

$$|F(x) - F(y)| = \left| \int_y^x f(t) dt \right| \leq \int_{(y,x)} |f| \leq \varepsilon,$$

so  $F$  uniformly continuous.

Let  $x \in \mathbb{R}$  be a Lebesgue point of  $f$  and such that  $f(x)$  is finite valued. Then,

$$\begin{aligned} \left| \frac{1}{h} (F(x+h) - F(x)) - f(x) \right| &= \left| \frac{1}{h} \int_x^{x+h} f(t) dt - f(x) \right| \\ &= \left| \frac{1}{h} \int_x^{x+h} f(t) - f(x) dt \right| \\ &\leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt \\ &\leq 2 \frac{1}{2h} \int_{x-h}^{x+h} |f(t) - f(x)| dx \rightarrow 0 \text{ as } h \rightarrow 0^+. \end{aligned}$$

We can similarly show that  $\lim_{h \rightarrow 0} \left| \frac{1}{h} (F(x) - F(x-h)) - f(x) \right| = 0$ .

■

**Remark 4.3:** In general, the a.e. statement cannot be dropped. For instance, if  $f = \mathbb{1}_{\{0\}}$ ,  $F \equiv 0$  and  $F' \equiv 0$  but  $F'(0) \neq f(0)$ .

### §4.3 Monotonic (Increasing) Functions

Let  $F$  an increasing function on  $[a, b]$  (we restrict to finite-valued functions). If needed, we can extend  $F$  to beyond this interval by setting  $F(x) = F(a)$  everywhere to the left of  $a$ , and  $F(x) = F(b)$  everywhere to the right of  $b$ ; then  $F$  still increasing on  $\mathbb{R}$ .

$\hookrightarrow$  **Proposition 4.3:** If  $F$  increasing on  $[a, b]$ , then  $F$  is continuous except at most countably many points in  $[a, b]$ . In particular,  $F$  is measurable.



**Remark 4.4:** For general functions and  $x \in \mathbb{R}$ , we define

$$\begin{aligned}\overline{D}_r F(x) &:= \limsup_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}, & \underline{D}_r F(x) &:= \liminf_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} \\ \overline{D}_\ell F(x) &:= \limsup_{h \rightarrow 0} \frac{F(x) - F(x-h)}{h}, & \underline{D}_\ell F(x) &:= \liminf_{h \rightarrow 0} \frac{F(x) - F(x-h)}{h}.\end{aligned}$$

If  $\overline{D}_r F(x) = \underline{D}_r F(x) = \overline{D}_\ell F(x) = \underline{D}_\ell F(x)$ ,  $F'(x)$  exists and equals this equal limit.

**Proposition 4.4:** Assume  $F$  increasing on  $[a, b]$ . Then,  $F'$  exists a.e. on  $[a, b]$ .

**PROOF.** For every  $x \in [a, b]$ ,  $\underline{D}_r F(x) \leq \overline{D}_r F(x)$ ,  $\underline{D}_\ell F(x) \leq \overline{D}_\ell F(x)$ . If we can show  $\overline{D}_r F(x) \leq \underline{D}_\ell F(x)$  and  $\overline{D}_\ell F(x) \leq \underline{D}_r F(x)$  a.e., we'd be done.

Set  $E := \{x \in [a, b] : \overline{D}_\ell F(x) > \underline{D}_r F(x)\}$ . We want to show  $m(E) = 0$ . For  $p < q \in \mathbb{Q}$ , set

$$E_{pq} := \{x \in [a, b] : \overline{D}_\ell F(x) > q \text{ and } \underline{D}_r F(x) < p\}.$$

Notice that  $E = \bigcup_{p < q \in \mathbb{Q}} E_{pq}$ , which is a countable union, so it suffices to show that  $m(E_{pq}) = 0$ .

Suppose otherwise, that there is some  $\delta > 0$  such that  $m(E_{pq}) = \delta$ . fix any  $0 < \varepsilon < \frac{\delta}{2}$ . Choose an open  $G \supseteq E_{pq}$  with  $m(G) \leq m(E_{pq}) + \varepsilon = \delta + \varepsilon$ . Consider

$$\mathcal{J} := \left\{ I = [x, x+h] \subseteq G : \frac{F(x+h) - F(x)}{h} < p \right\}.$$

For every  $x \in E_{pq}$ ,  $x \in G$ -open, and  $\underline{D}_r F(x) < p$ . So, for every  $x \in E_{pq}$ , there exists  $I = [x, x+h] \in \mathcal{J}$  for arbitrarily small  $h$ . In particular,  $\mathcal{J}$  a Vitali covering of  $E_{pq}$  (see following lemma). Hence, there exists a disjoint subcollection  $I_1, \dots, I_N \in \mathcal{J}$  such that  $m(E_{pq} \setminus \bigcup_{i=1}^N I_i) \leq \varepsilon$ . Write  $I_i = [x_i, x_i + h_i]$ ,  $i = 1, \dots, N$ . Define

$$\tilde{G} := \bigcup_{i=1}^N (x_i, x_i + h_i), \quad \widetilde{E}_{p,q} = E_{p,q} \cap \tilde{G}.$$

Then,

$$m(\widetilde{E}_{p,q}) = m(E_{p,q} \cap \tilde{G}) = m\left(E_{p,q} \cap \bigcup_{i=1}^N I_i\right) = m(E_{p,q}) - m\left(E_{p,q} \setminus \bigcup_{i=1}^N I_i\right) \geq \delta - \varepsilon.$$

Since  $\widetilde{E}_{p,q} \subseteq \tilde{G}$  and  $\tilde{G}$  is open, define  $\tilde{\mathcal{J}} := \left\{ \tilde{I} = [y-r, y] \subseteq \tilde{G} : \frac{F(y) - F(y-r)}{r} > q \right\}$ . Then,  $\tilde{\mathcal{J}}$  is a Vitali covering of  $\widetilde{E}_{p,q}$ , and we can extract disjoint  $\tilde{I}_1, \dots, \tilde{I}_M \in \tilde{\mathcal{J}}$  such that  $m(\widetilde{E}_{p,q} \setminus \bigcup_{j=1}^M \tilde{I}_j) \leq \varepsilon$ . Hence,

$$\begin{aligned}m\left(\bigcup_{j=1}^M \tilde{I}_j\right) &= m(\widetilde{E}_{p,q}) - m\left(\widetilde{E}_{p,q} \setminus \bigcup_{j=1}^M \tilde{I}_j\right) \\ &\geq \delta - \varepsilon - \varepsilon = \delta - 2\varepsilon.\end{aligned}$$

Write  $\tilde{I}_j := [y_j - r_j, y_j]$ ,  $j = 1, \dots, M$ . Then,  $\tilde{I}_1, \dots, \tilde{I}_M$  form a disjoint subintervals of  $I_1, \dots, I_N$ . Since  $F$  is increasing,

$$\sum_{j=1}^M (F(y_j) - F(y_j - r_j)) \leq \sum_{i=1}^N (F(x_i + h) - F(x_i)).$$

We have that

$$\text{RHS} \leq p \sum_{i=1}^N h_i = pm \left( \bigcup_{i=1}^N I_i \right) \leq pm(G) \leq p(\delta + \varepsilon),$$

and similarly,

$$\text{LHS} \geq q \sum_{j=1}^M r_j = qm \left( \bigcup_{j=1}^M \tilde{I}_j \right) \geq q(\delta - 2\varepsilon),$$

hence we find

$$q(\delta - 2\varepsilon) \leq p(\delta + \varepsilon),$$

but  $\varepsilon$  arbitrary, so we find  $q\delta \leq p\delta$ , contradicting  $p < q$ , hence  $\delta = 0$ . ■

**Remark 4.5:** We tacitly used that  $\overline{D_r}F$ , etc, are measurable to say that  $E$  measurable. This needs to be proven.

↪ **Lemma 4.3** (Vitali's Covering Theorem): Given a set  $E \subseteq \mathbb{R}$ , a collection  $\mathcal{J}$  of intervals is called a *Vitali covering* of  $E$  if for every  $x \in E$  and  $\varepsilon > 0$ , there is an  $I \in \mathcal{J}$  such that  $x \in I$  and  $m(I) < \varepsilon$ .

If  $E \in \mathcal{M}$  with  $m(E) < \infty$  and  $\mathcal{J}$  a Vitali covering of  $E$ , then for every  $\varepsilon > 0$ , there is a finite subcollection  $I_1, I_2, \dots, I_N \in \mathcal{J}$  such that  $I_i \cap I_j = \emptyset$  for  $i \neq j$ , and

$$m \left( E \setminus \bigcup_{i=1}^N I_i \right) \leq \varepsilon.$$

PROOF. Assume, wlog,  $m \left( \bigcup_{I \in \mathcal{J}} I \right) < \infty$ . Else, let  $G$  open such that  $G \supseteq E$  and  $m(G) < \infty$ , and redefine  $\mathcal{J}' := \{I \in \mathcal{J} : I \subseteq G\}$ , then, if  $\mathcal{J}$  a Vitali covering of  $E$ , so is  $\mathcal{J}'$ .

Defining then

$$\alpha_1 := \sup \{m(I) : I \in \mathcal{J}\},$$

we know  $\alpha_1 < \infty$ . Then,  $\exists I_1 \in \mathcal{J}$  such that  $m(I_1) > \frac{\alpha_1}{2}$ . Then, consider

$$\mathcal{J}_1 := \{I \in \mathcal{J} : I \cap I_1 = \emptyset\}.$$

Define

$$\alpha_2 := \sup \{m(I) : I \in \mathcal{J}_1\} < \infty,$$

so there exists  $I_2 \in \mathcal{J}_1$  such that  $m(I_2) > \frac{\alpha_2}{2}$ , and put  $\mathcal{J}_2 := \{I \in \mathcal{J} : I \cap I_1 = \emptyset \text{ and } I \cap I_2 = \emptyset\}$ . Repeat this procedure; this will generate a sequence  $\{\alpha_k\}$  and  $\{I_k\}$  such that the  $I_k$ 's are disjoint, and  $\alpha_k = \sup\{m(I) : I \in \mathcal{J}, I \cap I_j = \emptyset \forall j = 1, \dots, k-1\}$ . Since  $\bigcup_{k=1}^{\infty} I_k \subseteq \bigcup_{I \in \mathcal{J}} I$  and disjointness,

$$\sum_{k=1}^{\infty} m(I_k) = m\left(\bigcup_{k=1}^{\infty} I_k\right) < \infty.$$

In addition,  $m(I_k) > \frac{\alpha_k}{2}$  hence  $\sum_{k=1}^{\infty} m(I_k) \geq \sum_{k=1}^{\infty} \frac{\alpha_k}{2}$ , noticing that  $\alpha_k \rightarrow 0$  as  $k \rightarrow \infty$  and in particular this means  $\sum_{k=1}^{\infty} \frac{\alpha_k}{2}$  a converging series. Fix  $\varepsilon > 0$ , then there exists some  $N$  sufficiently large such that

$$\sum_{k=N+1}^{\infty} m(I_k) < \frac{\varepsilon}{5}.$$

We claim that  $\{I_i\}_{i=1}^N$  satisfies our desired properties, namely that  $m\left(E \setminus \bigcup_{i=1}^N I_i\right) < \varepsilon$ . It suffices to show  $m\left(E \setminus \bigcup_{i=1}^N \bar{I}_i\right) < \varepsilon$ , since this adds/removes only points so doesn't change the measure. Since  $\bigcup_{i=1}^N \bar{I}_i$  closed, then for every  $x \in E \setminus \bigcup_{i=1}^N \bar{I}_i$ ,  $\text{dist}\left(x, \bigcup_{i=1}^N \bar{I}_i\right) = \lambda > 0$ . Since  $\mathcal{J}$  a Vitali covering, there is some  $I^* \in \mathcal{J}$  such that  $x \in I^*$  and  $m(I^*) < \lambda$ . Hence, it must be that  $I^* \cap I_i = \emptyset$  for every  $i = 1, \dots, N$ . Hence,  $m(I^*) \leq \alpha_{N+1}$ . Let  $N^* \geq N + 1$  be such that  $\alpha_{N^*+1} < m(I^*) \leq \alpha_{N^*}$ . So, there is a  $j = N + 1, \dots, N^*$  such that  $I^* \cap I_j \neq \emptyset$  (we "start seeing"  $I^*$  at the  $j$ th step). Now,

$$m(I^*) \leq \alpha_{N^*} \leq \alpha_j \leq 2m(I_j).$$

In particular,  $I^* \subseteq "5I_j"$ , where  $5I_j$  is  $I_j$  "expanded" 5 times. So,

$$E \setminus \bigcup_{i=1}^N \bar{I}_i \subseteq \bigcup_{k=N+1}^{\infty} "5I_k",$$

so

$$m\left(E \setminus \bigcup_{i=1}^N \bar{I}_i\right) \leq m\left(\bigcup_{k=N+1}^{\infty} "5I_k"\right) \leq 5 \sum_{k=N+1}^{\infty} m(I_k) = 5 \cdot \frac{\varepsilon}{5} = \varepsilon,$$

as we aimed to show. ■

↪ **Proposition 4.5:** Assume  $F : [a, b] \rightarrow \mathbb{R}$  is increasing. Then,  $F' \in L^1([a, b])$ ,  $F' \geq 0$  a.e. on  $[a, b]$ , and  $\int_a^b F'(t) dt \leq F(b) - F(a)$ .

PROOF.  $F' \geq 0$  clear. For a.e.  $x \in [a, b]$ ,  $F'(x) = \lim_{n \rightarrow \infty} G_n(x)$  where  $G_n(x) := \frac{F\left(x + \frac{1}{n}\right) - F(x)}{\frac{1}{n}}$ , expanding  $F$  to be constant its last value outside of  $[a, b]$  if necessary. Then, by Fatou,

$$\begin{aligned}
\int_a^b F'(x) \, dx &\leq \liminf_n \int_a^b G_n(x) \, dx \\
&= \liminf_n \left( n \left( \int_a^b F\left(x + \frac{1}{n}\right) \, dx - \int_a^b F(x) \, dx \right) \right) \\
&= \liminf_n \left( n \left( \int_{a+\frac{1}{n}}^{b+\frac{1}{n}} F(t) \, dt - \int_a^b F(t) \, dt \right) \right) \\
&= \liminf_n \left( n \left( \int_b^{b+\frac{1}{n}} F(t) \, dt - \int_a^{a+\frac{1}{n}} F(t) \, dt \right) \right) \\
&\leq \liminf_n \left( n \left( F(b) \frac{1}{n} - F(a) \frac{1}{n} \right) \right) = F(b) - F(a).
\end{aligned}$$

This proves in turn  $F' \in L^1([a, b])$ . ■

**Remark 4.6:** The inequality established may be strict. Recall  $f$  the Cantor-Lebesgue function. It is continuous and increasing, so  $f'$  exists almost everywhere on  $[0, 1]$ , indeed, for every  $x \in [0, 1] \setminus C$ ,  $f'(x) = 0$ . Then,  $f' \equiv 0$  a.e. on  $[0, 1]$  so  $\int_0^1 f'(x) \, dx = 0$ , while  $f(1) - f(0) = 1$ .

#### §4.4 Functions of Bounded Variation

↪ **Definition 4.3** (Bounded Variation): A function  $F : [a, b] \rightarrow \mathbb{R}$  is of *bounded variation* on  $[a, b]$ , denoted by  $f \in \text{BV}([a, b])$  if

$$T_F(a, b) := \sup \left\{ \sum_{k=1}^N |F(x_k) - F(x_{k-1})| : N \geq 1, a = x_0 < x_1 < x_2 < \dots < x_N = b \right\} < \infty.$$

We call  $x_0, \dots, x_N$  a partition of  $[a, b]$ , and  $T_F$  the *total variation* of  $F$  over  $[a, b]$ .

↪ **Proposition 4.6:**

1. For  $x \in [a, b]$ , set  $V_F(x) := T_F(a, x)$ . Then,  $V_F : [a, b] \rightarrow [0, \infty]$  is increasing and for every  $a \leq x \leq y \leq b$ ,  $V_F(y) - V_F(x) = T_F((x, y))$ .
2. If  $F, G \in \text{BV}([a, b])$ , then both  $F + G, F - G \in \text{BV}([a, b])$  and  $T_{F+G}(a, b) \leq T_F(a, b) + T_G(a, b)$ .
3. If  $F$  is monotonic on  $[a, b]$ ,  $F \in \text{BV}([a, b])$ .
4. If  $f \in L^1(\mathbb{R})$  and  $F(x) := \int_a^x f(t) \, dt$  for  $x \in [a, b]$ , then  $F \in \text{BV}([a, b])$  and  $T_F(a, b) \leq \int_a^b |f(t)| \, dt$ .
5. If  $F \in \text{BV}([a, b])$ ,  $F$  is bounded on  $[a, b]$ .
6. If  $F$  is continuous on  $[a, b]$  and differentiable everywhere on  $(a, b)$ , and there is some  $M > 0$  such that  $|F'(x)| \leq M$  for every  $x \in (a, b)$ , then  $F \in \text{BV}([a, b])$ .
7. In 6., the boundedness of  $F'$  cannot be dropped.

PROOF.

3. For any partition  $a = x_0 < \dots < x_N = b$ , we have

$$\sum_{k=1}^N |F(x_k) - F(x_{k-1})| = \sum_{k=1}^N F(x_k) - F(x_{k-1}) = F(b) - F(a),$$

which was true of any partition so in particular  $T_F(a, b) = |F(b) - F(a)|$ .

4. For any partition  $a = x_0 < \dots < x_N = b$ ,

$$\begin{aligned} \sum_{k=1}^N |F(x_k) - F(x_{k-1})| &= \sum_{k=1}^N \left| \int_{x_{k-1}}^{x_k} f(t) dt \right| \\ &\leq \sum_{k=1}^N \int_{x_{k-1}}^{x_k} |f(t)| dt = \int_a^b |f(t)| dt < \infty. \end{aligned}$$

5. For every  $x \in [a, b]$ ,

$$\begin{aligned} |F(x)| &\leq |F(x) - F(a)| + |F(a)| \leq T_F(a, x) + |F(a)| \\ &\leq T_F(a, b) + |F(a)| < \infty. \end{aligned}$$

6. By the mean value theorem, for every  $x < y \in (a, b)$ , there is some  $z \in (x, y)$  such that  $\frac{F(y) - F(x)}{y - x} = F'(z)$ . Hence, for every such  $x, y$ ,

$$|F(y) - F(x)| \leq M(y - x).$$

For any partition  $a = x_0 < \dots < x_N = b$ , then,  $\sum_{k=1}^N |F(x_k) - F(x_{k-1})| \leq M(b - a)$ .

7. For instance, consider  $F(x) = \begin{cases} x \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ . Then  $F$  continuous on  $[0, 1]$  and differentiable on  $(0, 1)$ , but  $F \notin \text{BV}([0, 1])$ .

■

↪ **Theorem 4.3:** A function  $F : [a, b] \rightarrow \mathbb{R}$  is in  $\text{BV}([a, b])$  if and only if there exist  $H, G : [a, b] \rightarrow \mathbb{R}$  increasing such that  $F(x) = H(x) - G(x)$  for every  $x \in [a, b]$ .

PROOF. ( $\Leftarrow$ ) Increasing functions are in  $\text{BV}([a, b])$ , so  $H, G$  and thus  $H - G \in \text{BV}([a, b])$ .

( $\Rightarrow$ ) Assume  $F \in \text{BV}([a, b])$ . Let  $H(x) = V_F(x)$  for  $x \in [a, b]$ , which is increasing on  $[a, b]$ . Let  $G = H - F$ , which we claim is also increasing. For every  $x < y \in [a, b]$ ,

$$\begin{aligned} G(y) - G(x) &= H(y) - H(x) - (F(y) - F(x)) \\ &= T_F(x, y) - (F(y) - F(x)) \\ &\geq T_F(x, y) - |F(y) - F(x)| \geq 0. \end{aligned}$$

■

↪ **Theorem 4.4:** If  $F : [a, b] \rightarrow \mathbb{R}$  is of bounded variation, then  $F$  is continuous on  $[a, b]$  except at at most countably many points,  $F'$  exists almost everywhere  $[a, b]$ , and  $F' \in L^1([a, b])$ .