# **MATH457** - Algebra 4

Representation Theory; Galois Theory

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# §1 Representation Theory

## §1.1 Introduction

↪**Definition 1.1** (Linear Representation): A *linear representation* of a group $G$ is a vector space $V$ over a field $\mathbb{F}$ equipped with a map $G \times V \to V$ that makes $V$ a $G$-set in such a way that for each $g \in G$, the map $v \mapsto gv$ is a linear homomorphism of $V$.

This induces a homomorphism

$$\rho : G \to \operatorname{Aut}_{\mathbb{F}}(V),$$

or, in particular, when $n = \dim_{\mathbb{F}} V < \infty$, a homomorphism

$$\rho : G \to \mathrm{GL}_n(\mathbb{F}).$$

Alternatively, a linear representation $V$ can be viewed as a module over the group ring $\mathbb{F}[G] = \left\{\sum_{g \in G} : \lambda_g g : \lambda_g \in \mathbb{F}\right\}$ (where we require all but finitely many scalars $\lambda_g$ to be zero).

↪**Definition 1.2** (Irreducible Representation): A linear representation $V$ of a group $G$ is called *irreducible* if there exists no proper, nontrivial *subspace* $W \subsetneq V$ such that $W$ is $G$-stable.

⊛ **Example 1.1**:

1. Consider $G = \mathbb{Z}/2 = \{1, \tau\}$. If $V$ a linear representation of $G$ and $\rho : G \to \operatorname{Aut}(V)$. Then, $V$ uniquely determined by $\rho(\tau)$. Let $p(x)$ be the minimal polynomial of $\rho(\tau)$. Then, $p(x) \mid x^2 - 1$. Suppose $\mathbb{F}$ is a field in which $2 \neq 0$. Then, $p(x) \mid (x-1)(x+1)$ and so $p(x)$ has either $1, -1$, or both as eigenvalues and thus we may write
$$V = V_+ \oplus V_-,$$
where $V_\pm := \{v \mid \tau v = \pm v\}$. Hence, $V$ is irreducible only if one of $V_+, V_-$ all of $V$ and the other is trivial, or in other words $\tau$ acts only as multiplication by $1$ or $-1$.
2. Let $G = \{g_1, ..., g_N\}$ be a finite abelian group, and suppose $\mathbb{F}$ an algebraically closed field of characteristic 0 (such as $\mathbb{C}$). Let $\rho : G \to \operatorname{Aut}(V)$ and denote $T_j := \rho(g_j)$ for $j = 1, ..., N$. Then, $\{T_1, ..., T_N\}$ is a set of mutually commuting linear transformations. Then, there exists a simultaneous eigenvector, say $v$, for $\{T_1, ..., T_N\}$, and so $\operatorname{span}(v)$ a $G$-stable subspace of $V$. Thus, if $V$ irreducible, it must be that $\dim_{\mathbb{F}} V = 1$.