

MATH249 - Complex Variables

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§1 COMPLEX NUMBERS

The complex numbers are the set

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\},$$

where $i^2 = -1$. This set is readily equipped with operations of addition, subtraction, multiplication and division; given two complex numbers $a + bi, c + di$, these operations are determined by the rules

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ (a + bi)(c + di) &= ac - bd + (ad + bc)i \\ \frac{1}{a + bi} &= \frac{a - bi}{a^2 + b^2},\end{aligned}$$

assuming in the final line that $a^2 + b^2 \neq 0$, i.e. that $a + bi \neq 0$ in \mathbb{C} . In particular, in the division line, we obtain the result by multiplying the top and bottom by the *conjugate* of $z := a + bi$; we denote

$$\bar{z} = a - bi,$$

noting that in particular,

$$z\bar{z} = a^2 + b^2 = |z|^2.$$

Any complex number $z = a + bi$ may be written in so-called *polar form*

$$z = r(\cos \theta + i \sin \theta), \quad r := \sqrt{a^2 + b^2} = |z|, \theta := \arg(z) = \arctan(b/a),$$

with the θ read modulo 2π . This is a useful representation for the sake of multiplication; given $z_i = r_i(\cos(\theta_i) + i \sin(\theta_i))$, $i = 1, 2$, we have

$$z_1 z_2 = \cdots = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].$$

In particular,

$$|z_1 z_2| = |z_1| |z_2|, \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

↪ **Theorem 1.1:** $\cos(\theta) + i \sin(\theta) = \exp(i\theta)$

PROOF. Taylor expand both sides. ■

In particular, this theorem gives a clear way to define the exponential of a complex number

$$e^{x+iy} = e^x e^{iy} = e^x (\cos(y) + i \sin(y)),$$

so that in particular, for any $z \in \mathbb{C}$,

$$|e^z| = e^{\operatorname{Re}(z)}, \quad \arg(e^z) = \operatorname{Im}(z).$$

§1.1 Fundamental Theorem of Algebra

↪ **Theorem 1.2** (Fundamental Theorem of Algebra): If $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ is a polynomial with complex coefficients $a_0, a_1, \dots, a_{n-1}, a_n$, then there exists a $z \in \mathbb{C}$ such that $f(z) = 0$.

PROOF. (A First Proof) Remark that if $|z| = R \gg 1$ (much larger than zero), then we have

$$\begin{aligned} |a_n z^n| &= |a_n| R^n, \\ |a_{n-1} z^{n-1} + \dots + a_1 z + a_0| &\leq |a_{n-1}| R^{n-1} + \dots + |a_1| R + |a_0| \\ &\leq (|a_{n-1}| + |a_{n-2}| + \dots + |a_1| + |a_0|) R^{n-1}. \end{aligned}$$

Let $z_0 \in \mathbb{C}$ be a point for which $|f(z_0)|$ is a minimum; this must exist for $|f|$ must be very large outside of the disc of radius R centered at the origin. Namely, $|z_0| < R$. We claim z_0 a root of f . We may assume without loss of generality that $z_0 = 0$, by replacing $f(z)$ with $f(z - z_0)$. We write

$$\begin{aligned} f(z) &= a_0 + \dots + a_k z^k + \dots + a_n z^n, \\ &= a_0 + a_k z^k \left(1 + \frac{a_{k+1}}{a_k} z + \dots + \frac{a_n}{a_k} z^{n-k} \right). \end{aligned}$$

where $a_k \neq 0$ the first nonzero coefficient with $k \geq 1$. If we can show $a_0 = 0$, we are done. Assume otherwise. Let

$$z := \left(-\frac{a_0}{a_k} \right)^{\frac{1}{k}} \varepsilon, \quad \varepsilon > 0.$$

With this value of z , we have

$$f(z) = a_0 - a_0 \varepsilon^k \left(1 + \underbrace{\dots}_{=o(\varepsilon)} \right) \approx a_0 (1 - \varepsilon^k).$$

By choosing ε sufficiently small, this implies

$$|f(z)| < |a_0| = |f(0)|,$$

which contradicts the assumed minimality of $z_0 = 0$, unless of course $a_0 = f(z_0) = 0$, providing the claim. ■

PROOF. (A Second Proof) We want to view $f(z)$ as a mapping $\mathbb{C} \rightarrow \mathbb{C}$. Assume $f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$. When $|z|$ large, we know

$$|f(z) - z^n| < C|z|^{n-1},$$

for some constant C independent of z . Remark that the map $\varphi : z \mapsto z^n$ maps a circle of radius R to a circle of radius R^n ; in particular, if we take a point $z = Re^{i\theta}$ on the circle of radius R of angle θ with the origin, and let θ vary from 0 to 2π , one “rotation” in the pre-image world will lead to n “rotations” in the image world. Similarly, for $z \mapsto f(z)$, the image of the R -radius circle may not be a circle, but a “fudged” circle; the curve of the image will still be some periodic curve. As we let $R \rightarrow 0$, though, the image will go

to the singular point a_0 . Thus, at some value of R , the image of the R -radius circle would have to pass through the origin, and thus this point must be a root of $f(z)$. ■

PROOF. (A Third Proof) We use a result that we will prove later in the class, Liouville's Theorem, which states that any bounded differentiable function $f : \mathbb{C} \rightarrow \mathbb{C}$ must be constant.

Suppose $p(z)$ a polynomial with no roots in \mathbb{C} . Let $f(z) = \frac{1}{p(z)}$ (this is well-defined, since by assumption p has no roots); this is bounded on \mathbb{C} , and has derivative $\frac{d}{dz}f(z) = -\frac{p'(z)}{p(z)}$. By Liouville's, f must be a constant and thus p must be a constant. ■

§1.2 Analytic, Holomorphic Functions

↪ **Definition 1.1** (Holomorphic/Analytic): A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is said to be *holomorphic* if it has a well-defined derivative, i.e. if the limit

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

exists and is well-defined (in the sense that it is independent of the "path" h takes to 0).

We may write $f : \mathbb{C} \rightarrow \mathbb{C}$ as

$$f(z) = f(x+iy) = u(x,y) + iv(x,y),$$

where $u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$. We can calculate $f'(z)$ in two different ways.

1. Restrict h to \mathbb{R} :

$$\begin{aligned} f'(z) &= \lim_{\substack{h \rightarrow 0, \\ h \in \mathbb{R}}} \frac{f(z+h) - f(z)}{h} = \lim_{\substack{h \rightarrow 0, \\ h \in \mathbb{R}}} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \\ &= \lim_{\substack{h \rightarrow 0, \\ h \in \mathbb{R}}} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{\substack{h \rightarrow 0, \\ h \in \mathbb{R}}} \frac{v(x+h,y) - v(x,y)}{h} \\ &= \frac{\partial u}{\partial x}(x,y) + i \frac{\partial v}{\partial x}(x,y). \end{aligned}$$

2. Restrict to h purely imaginary values:

$$\begin{aligned} f'(z) &= \lim_{\substack{h \rightarrow 0, \\ h \in \mathbb{R}}} \frac{f(z+ih) - f(z)}{ih} = \lim_{\substack{h \rightarrow 0, \\ h \in \mathbb{R}}} \frac{u(x,y+h) + iv(x,y+h) - u(x,y) - iv(x,y)}{ih} \\ &= \frac{1}{i} \frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) \\ &= \frac{\partial v}{\partial y}(x,y) - i \frac{\partial u}{\partial y}(x,y) \end{aligned}$$

These two computations must of course agree, which imply (equating real, imaginary parts)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are the *Cauchy-Riemann equations*. Viewing the pair $f = (u, v)$ as a mapping $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, the Cauchy-Riemann equations imply that the Jacobian of f is given in the form

$$J_f(x, y) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

↪ **Proposition 1.1:**

- If f, g are holomorphic and $a, b \in \mathbb{C}$, then $af + bg$ are also holomorphic, and moreover $(af + bg)' = af' + bg'$
- With $f(z) := z^n, f'(z) = nz^{n-1}$
- As a result, any polynomial on \mathbb{C} is holomorphic

↪ **Theorem 1.3:** If f satisfies the Cauchy-Riemann equations, then f is holomorphic.

PROOF. Write $f = u + iv$ as before. Let $h = h_1 + ih_2$. Then,

$$u(x + h_1, y + h_2) = u(x, y) + h_1 \partial_x u + h_2 \partial_y u + |h| \psi_1(h), \quad \psi_1(h) \rightarrow 0 \text{ as } h \rightarrow 0,$$

with similar for v with a remainder ψ_2 . Then, by Cauchy-Riemann,

$$f(z + h) = f(z) + (\partial_x v - i \partial_y u)(h_1 + ih_2) + \psi(h)|h|, \quad \psi(h) = o(|h|).$$

Dividing both sides by h and sending $h \rightarrow 0$ gives the result. ■

§1.3 Power Series

We say a series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n, z \in \mathbb{C}$, *converges* if $\lim_{N \rightarrow \infty} \sum_{n=0}^N a_n z^n$ exists as a complex number. We say it *converges absolutely* if $\lim_{N \rightarrow \infty} \sum_{n=0}^N |a_n| |z|^n$ exists.

↪ **Theorem 1.4:** Given $\sum_{n=0}^{\infty} a_n z^n$, there exists a number $0 \leq R \leq \infty$ for which

1. if $|z| < R$, then $\sum a_n z^n$ converges absolutely;
2. if $|z| > R$, then $\sum a_n z^n$ does not converge.

Furthermore,

$$\frac{1}{R} = \limsup_n |a_n|^{\frac{1}{n}}.$$

PROOF. Let $L = \frac{1}{R}$ and suppose $|z| < R$. There exists some $\varepsilon > 0$ such that

$$r := (L + \varepsilon)|z| < 1.$$

There exists some N such that $L + \varepsilon > |a_n|^{\frac{1}{n}}$ for all $n > N$ by definition of limsup's; thus

$$\begin{aligned} |z| |a_n|^{\frac{1}{n}} &< (L + \varepsilon)|z| = r < 1 \\ \Rightarrow |z|^n |a_n| &< r^n. \end{aligned}$$

But since $r < 1$, it follows that $\sum |a_n| |z|^n$ converges by comparing to the geometric series $\sum r^n$.

If $|z| > R$, there is an $\varepsilon > 0$ so that there are infinitely-many n 's for which $|a_n|^{\frac{1}{n}} > \frac{1}{R} - \varepsilon$, and so

$$|a_n|^{\frac{1}{n}}|z| > r > 1$$

hence $|a_n||z|^n > r^n$, so that $\sum |a_n||z|^n$ diverges by comparison. Moreover, we have shown that $|a_n||z|^n$ does not converge to zero, which implies the series does not even converge (“normally”). ■

⊗ **Example 1.1:**

1. $\sum_{n=0}^{\infty} n!z^n$ has $R = 0$
2. $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$ with $R = \infty$.
3. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ has $R = 1$.

↪ **Theorem 1.5:** A power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ admits a derivative on its disc of convergence, and $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$.

PROOF. Write $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ as the “potential” derivative we aim to show, remarking that this series converges and moreover has the same radius of convergence as f since $\lim n^{\frac{1}{n}} = 1$ and thus $\limsup a_n^{\frac{1}{n}} = \limsup (n a_n)^{\frac{1}{n}}$. Write

$$f(z) = S_N(z) + E_N(z), \quad S_N(z) := \sum_{n=0}^N a_n z^n, \quad E_N(z) := \sum_{n=N+1}^{\infty} a_n z^n.$$

Fix $z_0 \in D_R(0)$. We show $\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) \rightarrow 0$ as $h \rightarrow 0$. We can write

$$\begin{aligned} \frac{f(z_0+h)-f(z_0)}{h} - g(z_0) &= \frac{S_N(z_0+h) - S_N(z_0)}{h} - g(z_0) + \frac{E_N(z_0+h) - E_N(z_0)}{h} \\ &= \left\{ \frac{S_N(z_0+h) - S_N(z_0)}{h} - S'_N(z_0) \right\} + \{S'_N(z_0) - g(z_0)\} + \left\{ \frac{E_N(z_0+h) - E_N(z_0)}{h} \right\} \\ &= (A) + (B) + (C). \end{aligned}$$

For all $\varepsilon > 0$, there exists N_1 $|(B)| < \varepsilon$ for all $N > N_1$.

There exists N_2 such that $|(C)| < \varepsilon$ for all $N > N_2$, since we have

$$(C) = \sum_{n \geq N+1} a_n \frac{(z_0+h)^n - z_0^n}{h},$$

and

$$(z_0+h)^n - z_0^n = h \left((z_0+h)^{n-1} + (z_0+h)^{n-2} z_0 + \cdots + (z_0+h)^{n-j} z_0^j + \cdots + z_0^{n-1} \right).$$

Since $|z_0+h|, |z_0| < r < R$ for h sufficiently small, we know

$$|(z_0+h)^n - z_0^n| \leq |h| n r^{n-1},$$

so that

$$\left| \frac{(z_0+h)^n - z_0^n}{h} \right| \leq n r^{n-1}.$$

It follows that

$$|(C)| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}.$$

This is the tail of an absolutely converging series, hence as $N \rightarrow \infty$, $|(C)| \rightarrow 0$, so we have the claimed bound.

Finally, let $N := \max(N_1, N_2)$. We see that for any fixed N , $(A) \rightarrow 0$ as $h \rightarrow 0$ by the definition of the derivative, and thus we can take $h = h(N)$ sufficiently small so that $|(A)| < \varepsilon$. Combining all these bounds gives the proof. ■

↪ **Corollary 1.1:** $f(z) = \sum a_n z^n$ is infinitely differentiable in its radius of convergence.

↪ **Definition 1.2:** A function $f : \Omega \rightarrow \mathbb{C}$ is called *analytic* if it is equal to a power series on $D_\varepsilon(z_0)$ for all $z_0 \in \Omega$, for some $\varepsilon > 0$.

↪ **Corollary 1.2:** f analytic $\Rightarrow f$ holomorphic

Remark 1.1: We'll see later that these are actually equivalent notions.

§1.4 Integration Along Curves

↪ **Definition 1.3:** A parametrized curve is a function $\gamma : [0, 1] \rightarrow \mathbb{C}$ where γ is differentiable with continuous derivative, with $\gamma'(t) \neq 0$ for all $t \in [0, 1]$.

↪ **Definition 1.4:** We'll say two parametrized curves $\gamma, \tilde{\gamma}$ are equivalent if there exists a smooth function $s : [0, 1] \rightarrow [0, 1]$ smooth with $s'(t) > 0$ and such that $\tilde{\gamma} = \gamma \circ s$.

We will consider curves as defined up to equivalency in this way.

↪ **Definition 1.5:** If γ is a parametrized curve, define

$$\int_{\gamma} f(z) dz := \int_0^1 f(\gamma(t)) \gamma'(t) dt.$$

If γ a piecewise smooth curve, i.e. γ can locally be written as $t \mapsto z(t) \in \mathbb{C}$ for $t \in [a_k, a_{k+1})$ for $k = 0, \dots, n-1$ for some sequence $a_k < a_{k+1}$, then

$$\int_{\gamma} f(z) dz := \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t)) z'(t) dt.$$

An obvious generalization holds for integration along more general intervals.

↪ **Proposition 1.2:** Path integrals are independent of choice of parametrization.

↪ **Definition 1.6** (Length of a curve): Define, for γ given by $z : I \rightarrow \mathbb{C}$,

$$\text{length}(\gamma) := \int_{\gamma} |dz| = \int_I |z'(t)| dt.$$

↪ **Proposition 1.3:** Let f, g continuous and $\alpha, \beta \in \mathbb{C}$. Then we have

1. Linearity:

$$\int_{\gamma} (\alpha f + \beta g) dz = \alpha \int_{\gamma} f dz + \beta \int_{\gamma} g dz.$$

2.
$$\int_{\gamma} f(z) dz = - \int_{\gamma^{-}} f(z) dz,$$

where γ^{-} is the *reverse path* of γ .

3.
$$\left| \int_{\gamma} f(z) dz \right| \leq \sup_{z \in \gamma} |f(z)| \text{length}(\gamma).$$

↪ **Definition 1.7 (Primitive):** A *primitive* of a continuous function f on a domain Ω is a function F such that $F' = f$ on Ω .

↪ **Proposition 1.4:** If f , continuous, has a primitive F on Ω and γ is a curve in Ω beginning at w_1 and ending at w_2 , then

$$\int_{\gamma} f dz = F(w_2) - F(w_1).$$

§1.5 Cauchy's Theorem

↪ **Theorem 1.6 (Cauchy):** If γ is a closed path contained in a region $\Omega \subset \mathbb{C}$ and its interior, and f is holomorphic in Ω , then $\int_{\gamma} f(z) dz = 0$.

It will take us some building to get here. In a simple case, though, we have a positive result:

↪ **Corollary 1.3:** If f has a primitive F on Ω , then Cauchy's theorem holds for f for any γ a closed path in $\text{int}(\Omega)$

PROOF. Apply the last proposition; now, $F(w_2) = F(w_1)$, so we have the result. ■

With some more work, we can also establish the proof for γ some simple contour.

↪ **Proposition 1.5 (Goursat's Lemma):** Let γ be a closed triangle in Ω and f a holomorphic function on Ω . Then $\int_{\gamma} f(z) dz = 0$.

PROOF. I'll add it later. Basically, follows from subsequent subdivision of the triangles and approximation of the total integral of f over these triangles. ■

↪ **Corollary 1.4:** If R a closed rectangle and Ω and f holomorphic on Ω , then $\int_R f(z) dz = 0$.

PROOF. A rectangle can be written as two triangles, with the "inner region" cancelling. ■

1.5.1 Primitives

↪ **Theorem 1.7:** Let f be holomorphic on an open disc Ω . Then, f has a primitive on that disc.

PROOF. Assume wlog that Ω centered at the origin. Fix $z \in \Omega$ and let γ_z be the path that first travels horizontally from 0 to $\operatorname{Re}(z)$ along the real axis, then vertical to z . Define

$$F(z) := \int_{\gamma_z} f(w) \, dw.$$

We claim $F'(z) = f(z)$. Let $h \in \mathbb{C}$ be small so that $z + h \in \Omega$, and consider the difference

$$F(z + h) - F(z) = \int_{\gamma_{z+h}} f(w) \, dw - \int_{\gamma_z} f(w) \, dw.$$

These integrals have f being integrated from 0 horizontally to $\operatorname{Re}(z + h)$ then vertically to $z + h$, then, in the *opposite* orientation, from z to $\operatorname{Re}(z)$, then $\operatorname{Re}(z)$ to 0. In particular, the two components $z \rightarrow \operatorname{Re}(z)$ cancel in these two integrals, being oppositely oriented, so we are left with the contour from z vertically to $\operatorname{Re}(z)$, horizontally to $\operatorname{Re}(z + h)$, the vertically to $z + h$. Connect z to $z + \operatorname{Re}(h)$ via a horizontal line, and z to $z + h$ via a diagonal. This forms an (oriented) triangle and a rectangle, plus an extra diagonal, which by Goursat's lemma must all integrate out to zero (draw it). Thus,

$$F(z + h) - F(z) = \int_{\eta} f(w) \, dw,$$

where η the diagonal from z to $z + h$. Since f continuous, $f(w) = f(z) + \psi(w)$ where $\psi(w) \rightarrow 0$ as $w \rightarrow z$; thus,

$$\begin{aligned} F(z + h) - F(z) &= f(z) \int_{\eta} dw + \int_{\eta} \psi(w) \, dw \\ &= f(z)h + \int_{\eta} \psi(w) \, dw \\ \Rightarrow f(z) &= \lim_{h \rightarrow 0} \frac{F(z + h) - F(z)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \int_{\eta} \psi(w) \, dw. \end{aligned}$$

But since

$$\frac{1}{h} \left| \int_{\eta} \psi(w) \, dw \right| \leq \frac{1}{h} \sup_{\eta} |\psi| |\eta| = \sup_{\eta} |\psi| \xrightarrow{h \rightarrow 0} 0,$$

we have proven the claim. ■

↪ **Theorem 1.8** (Cauchy's Integral Formula): Let f holomorphic on Ω containing the closure of a disc D . Let C be the boundary of this disc, then for any $z \in D$,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} \, d\xi.$$

Remark 1.2: The same result holds for more general curves C as long as $z \in \operatorname{int}(C)$; how/when the results extend should be clear from the proof.

↪ **Corollary 1.5:** $f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2} dz$, and more generally,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

So in general, f holomorphic $\Rightarrow f$ is infinitely differentiable.

↪ **Corollary 1.6:** $|f^{(n)}(z_0)| \leq \frac{n! \|f\|_{C_R(z_0)}}{R^n}$, where $C_R(z_0)$ the circle of radius R centered at z_0 .

↪ **Theorem 1.9:** f is analytic centered at $z = z_0$.

PROOF. We can write

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw,$$

for some circle C containing z . We can expand

$$\begin{aligned} \frac{1}{w-z} &= \frac{1}{(w-z_0) - (z-z_0)} \\ &= \frac{1}{w-z_0} \cdot \frac{1}{1 - \frac{z-z_0}{w-z_0}} \\ &= \frac{1}{w-z_0} \sum_{n=0}^{\infty} \left[\frac{z-z_0}{w-z_0} \right]^n \\ &= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}} \end{aligned}$$

so that

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_C f(w) \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(w-z_0)^{n+1}} dw \\ &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left[\int_C \frac{f(w)}{(w-z_0)^{n+1}} dw \right] (z-z_0)^n \\ &= \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad a_n := \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}} dw. \end{aligned}$$

But we also realize that

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

from our previous result, hence we conclude

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n,$$

as we expect from the real-valued analog. ■

Remark 1.3: In particular, this implies, from our previous result, that $|a_n| \leq \frac{C}{R^n}$, where C a constant uniform in n and R the radius of the circle upon which we're integrating. In particular, this means

$$|a_n|^{1/n} \leq \frac{C^{1/n}}{R},$$

which we see converges to $\frac{1}{R}$ as $n \rightarrow \infty$, hence our series above has radius of convergence at least R ; i.e., the power series for f converges on any $D_R(z_0) \subset \Omega$.

Thus, we've shown that holomorphic \Rightarrow analytic, and thus the two are equivalent (with appropriate assumptions on the space upon which they are defined, etc) since we showed the converse earlier.

\hookrightarrow Theorem 1.10 (Liouville's Theorem): If f is holomorphic on \mathbb{C} and bounded, then f is constant.

PROOF. We know that for any $z_0 \in \mathbb{C}$,

$$|f'(z_0)| \leq \frac{\|f\|_C}{R},$$

for any circle C with z_0 center and of radius R . Since f bounded, this means

$$|f'(z_0)| \leq \frac{1}{R} \sup_{\mathbb{C}} |f| \rightarrow 0, R \rightarrow \infty.$$

This means $f'(z_0) = 0$ everywhere and thus f is constant. We could take $R \rightarrow \infty$ since f holomorphic everywhere hence on every disc $D_R(z_0)$ for $R > 0$. ■

§1.6 Rigidity of Holomorphic Functions

\hookrightarrow Theorem 1.11: Suppose that f holomorphic in Ω and vanishes on a sequence of distinct points $z_1, \dots, z_n \in \Omega$ with a limit point $z_\infty \in \Omega$. Then, $f \equiv 0$ on an open disc about z_∞ .

PROOF. Let D be a disc centered at z_∞ and contained in Ω . We write

$$f(z) = \sum_{n \geq N} \frac{f^{(n)}(z_\infty)}{n!} (z - z_\infty)^n = a_N (z - z_\infty)^N \sum_{n=0}^{\infty} \frac{a_{N+n+1}}{a_N} (z - z_\infty)^n$$

where $N \geq 1$ the minimal integer such that $f^{(N)}(z_0) \neq 0$ and $a_n := \frac{f^{(n)}(z_\infty)}{n!}$. We see that if D sufficiently small, both

$$(z - z_\infty)^N, \quad \left(1 + \frac{a_{N+1}}{a_N} (z - z_\infty) + \dots \right)$$

has no additional zeros in a sufficiently small disc centered at z_∞ ; but this contradicts the fact that $z_n \rightarrow z_\infty$, i.e. there should be infinitely many zeros when $n \rightarrow \infty$. This is a contradiction, and hence there is no minimal N for which $f^{(n)}(z_\infty)$ doesn't vanish.

Hence, it must be that f identically zero on this small disc. ■

↪ **Proposition 1.6:** If f holomorphic and $f(z) = 0$ on a small disc $D \subset \Omega$ then $f \equiv 0$ on Ω .

PROOF. Let

$$U = \text{int}(\{z \in \Omega : f(z) = 0\}).$$

This set is open and nonempty ($D \subset U$). It is also closed; to see this, let $\{z_n\} \subset U$ with limit z . Then by the previous theorem, $f(z) = 0$, and thus $z \in U$ so U closed. But Ω connected, so $\Omega = U$. ■

This basically says that local behavior of holomorphic functions gives us information about the global behaviour.

↪ **Corollary 1.7** (Principle of Analytic Continuation): If f, g are holomorphic on Ω and $f(z) = g(z)$ for either

- (a) z in a nonempty open subset of Ω , or
- (b) a sequence $\{z_n\}$ and its limit point Then $f = g$ on Ω .

PROOF. Consider $f - g$ and apply the previous. ■

1.6.1 Special Cases

1. Let $f(z) = e^z$ and let $g(z)$ be any other holomorphic extension of e^x . Then, $f = g$ on \mathbb{R} , and thus agree everywhere; this is the unique extension of the exponential, i.e. $e^{x+iy} = e^x(\cos y + i \sin y)$.
2. Consider the Riemann zeta function,

$$\zeta(k) = \sum_{n \geq 1} \frac{1}{n^k},$$

converges for $k = 2, 3, \dots$. If we allow $k = u + iv \in \mathbb{C}$, we can write

$$\frac{1}{n^k} = \exp\left(\log\left(\frac{1}{n}\right)(u + iv)\right)$$

thus

$$\left|\frac{1}{n^k}\right| = \exp\left(\log\left(\frac{1}{n}\right)u\right) = \frac{1}{n^u},$$

so that

$$|\zeta(u + iv)| < \sum_{n=1}^{\infty} \left|\frac{1}{n^{u+iv}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^u},$$

which converges when $u > 1$. Thus, $\zeta(s)$ for $s \in \mathbb{C}$ converges (absolutely) whenever $\text{Re}(s) > 1$. Riemann showed that $\zeta(s)$ admits a holomorphic extension to $\mathbb{C} - \{1\}$.

§1.7 Singularities of $f(z)$

↪ **Definition 1.8:** If $f(z)$ is holomorphic on $D_r(z_0) - \{z_0\}$ for some $r > 0$, then z_0 is called a *singularity* of $f(z)$.

↪ **Definition 1.9:**

1. z_0 is called a *removable singularity* if $f(z)$ extends to a holomorphic function on $D_r(z_0)$
2. If $\frac{1}{f(z)}$ has a removable singularity at z_0 , then z_0 is called a *pole* of $f(z)$
3. Otherwise, z_0 is called an *essential singularity* of f .

⊗ **Example 1.2:**

1. $f(z) = \frac{\sin(z)}{z}$ has a removable singularity at 0 (taking $f(0) = 1$ extends f to a holomorphic function everywhere).
2. $f(z) = \frac{1}{z}$ has a pole at 0.
3. $f(z) = e^{\frac{1}{z}}$ at 0 has an essential singularity.

↪ **Proposition 1.7** (Local expansions at z_0): If f is a nonzero holomorphic at z_0 , then there exists a unique $m \geq 0$ such that

$$f(z) = (z - z_0)^m g(z),$$

where $g(z_0) \neq 0$.

We call m the *order of vanishing* of f at z_0 .

↪ **Proposition 1.8:** If $f(z)$ has a pole at $z = z_0$, then there exists a unique integer $m < 0$ such that

$$f(z) = (z - z_0)^m g(z)$$

where $g(z)$ holomorphic near z_0 and non-vanishing at z_0 .

PROOF. We know $\frac{1}{f(z)}$ holomorphic near z_0 so by the previous $\frac{1}{f(z)} = (z - z_0)^m g(z)$ so $f(z) = (z - z_0)^{-m} g^{-1}(z)$. Since $g(z)$ holomorphic near z_0 , so is $g(z)^{-1}$. ■

↪ **Definition 1.10:** A function f which is holomorphic on $\Omega - \{z_1, \dots, z_k\}$ and has poles at z_1, \dots, z_k is called *meromorphic* on Ω .

↪ **Definition 1.11:** For $f(z)$ meromorphic, put $\text{ord}_{z_0}(f) =$ unique $m \in \mathbb{Z}$ such that $f(z) = (z - z_0)^m h(z)$.

Remark 1.4: $\text{ord}_{z_0}(f) = 0 \Rightarrow f(z_0) \neq 0$, $\text{ord}_{z_0}(f) > 0 \Rightarrow f(z_0) = 0$, $\text{ord}_{z_0}(f) < 0 \Rightarrow f$ has a pole at z_0 .

↪ **Corollary 1.8:** If f has a pole of order m then f admits a *Laurent series expansion*

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$

PROOF. For $m \geq 1$, write

$$f(z) = (z - z_0)^{-m} h(z),$$

where we can expand

$$h(z) = a_{-m} + a_{-m+1}(z - z_0) + \dots$$

since h holomorphic. ■

↪ **Definition 1.12:** The quantity

$$\frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots + \frac{a_{-1}}{z - z_0}$$

is called the *principal part* of $f(z)$ at $z = z_0$, and is denoted $\text{PP}(f)$. Thus, we may write $f(z) = \text{PP}(f) + g(z)$ where g is holomorphic at z_0 .

Thus, if f is meromorphic at z_0 , we have two representations of f :

1. $f(z) = (z - z_0)^m h(z)$ where h is holomorphic with $h(z_0) \neq 0$ and $m = \text{ord}_{z_0} f < 0$
2. $f(z) = \text{PP}(f) + g(z)$ where $\text{PP}(z)$ is a polynomial in $(z - z_0)^{-1}$ with finite degree $-\text{ord}_{z_0} f$ and $g(z)$ is holomorphic.

↪ **Definition 1.13:** If $\text{PP}(f) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \dots + \frac{a_{-1}}{z - z_0}$, then we call a_{-1} the *residue* of f .

Recall that

$$\int_C \frac{dz}{z^n} = \begin{cases} 0 & n \neq 1 \\ 2\pi i & n = 1 \end{cases}$$

where C any circle about the origin.

↪ **Theorem 1.12 (Residue Formula):** Suppose $f(z)$ is meromorphic at z_0 , and let C be a sufficiently small circle around z_0 contained inside the region of holomorphicity of $f(z)$. Then,

$$\text{res}_{z_0} f(z) = \frac{1}{2\pi i} \int_C f(z) dz.$$

PROOF. Clear consequence of the second representation of f from above and the previous remarks. ■

↪ **Corollary 1.9:** Suppose f is holomorphic in an open set containing some toy contour γ and its interior, except at points z_1, \dots, z_N , then

$$\int_{\gamma} f(z) dz = 2\pi i \sum_{k=1}^N \text{res}_{z_k} f.$$

↪ **Definition 1.14:** For a holomorphic function f , we define the logarithmic derivative of f by

$$D_{\log} f = \frac{f'}{f},$$

which is holomorphic except possibly where f vanishes; hence it is meromorphic wherever f holomorphic.

↪ **Proposition 1.9:**

- $D_{\log}(fg) = D_{\log}(f) + D_{\log}(g)$
- $\text{res}_{z_0} D_{\log}(f) = \text{ord}_{z_0}(f)$
- If C a contour such that f never vanishes on C , then $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \sum_{z_0 \in \text{Int}(C)} \text{ord}_{z_0}(f)$ (argument principle)

↪ **Theorem 1.13 (Rouche):** Suppose f, g are holomorphic on Ω , and $|f(z)| > |g(z)|$ for all $z \in C$ and both f, g never vanish on C , then $f(z)$ and $f(z) + g(z)$ have the same number of zeros in $\text{int}(C)$.

PROOF. Consider $f_t(z) := f(z) + tg(z)$ for $t \in [0, 1]$; this is holomorphic. Remark that f_t has no zeros on C , since $|f_t(z)| \geq \|f(z)\| - t\|g(z)\| > 0$ by assumption. Define

$$N(t) = \#\{\text{zeros of } f_t(z) \text{ in } C, \text{ with multiplicity}\} = \frac{1}{2\pi i} \int_C \frac{f'_t(z)}{f_t(z)} dz,$$

using the argument principle. Since $N(t) : [0, 1] \rightarrow \mathbb{N}_{\geq 0}$ clearly continuous in t from this representation, it thus must be constant i.e. number of zeros of $f + g = N(1) = N(0) = \text{number of zeros of } f$, as claimed. ■

↪ **Corollary 1.10:** Suppose $f(z)$ never vanishes on a contour C and let $\lambda \in \mathbb{C}$ such that $|\lambda| < \sup_{z \in C} |f(z)|$. Then,

$$\# \text{ of zeros of } f(z) \text{ in } \text{int}(C) = \# \text{ of zeros of } f(z) - \lambda \text{ in } \text{int}(C)$$

↪ **Theorem 1.14 (Open Mapping Theorem):** If $f : \Omega \rightarrow \mathbb{C}$ is holomorphic and nonconstant, and U open in Ω , then $f(U)$ is also open.

PROOF. Fix $w_0 = f(z_0)$ for $z_0 \in \Omega$. There is a circle C centered at z_0 such that $f(z) - w_0 \neq 0$ on C . Let $0 < \varepsilon < \inf_{z \in C} |f(z) - w_0|$. Consider $D_\varepsilon(w_0)$; we need to show that it is contained in the image of f . Let $w \in D_\varepsilon(w_0)$, then $|w - w_0| < \varepsilon$. Then

$$f(z) - w = (f(z) - w_0) + (w_0 - w),$$

so by the previous result, the number of zeros of $f(z) - w_0$ in $\text{int}(C)$ is the same as the number of zeros as $f(z) - w$ in $\text{int}(C)$. $f(z) - w_0$ has at least one zero here, since $f(z_0) = w_0$, and thus $f(z) - w$ has at least one zero in $\text{int}(C)$ i.e. $w \in \text{image}(f)$. This concludes the proof. ■

↪ **Theorem 1.15 (Maximum Modulus Principle):** If f is a nonconstant holomorphic function on Ω , then f cannot attain a maximum inside Ω .

PROOF. Suppose otherwise, that f attains a max at $z_0 \in \Omega$. Take a small disc D centered at z_0 , then since f open, $f(D)$ is open and so there exists some neighborhood

of $f(z_0)$ in the image of f . This implies there exists some $z \in D$ such that $|f(z)| > |f(z_0)|$, which is a contradiction. ■

↪ **Corollary 1.11:** If f holomorphic and nonconstant on Ω bounded, then $|f(z)|$ obtains its maximum on $\partial\Omega$.

§1.8 Primitives and the Logarithm

We aim to extend the definition of the logarithm to complex numbers. We know $\log(x) = \int_1^x \frac{dt}{t}$ for $x \in \mathbb{R}$, so heuristically, we'd like

$$\log(z) := \int_{\gamma} \frac{dz}{z},$$

where γ any “nice” path joining 1 to z . However, there are a lot of choices of such paths - is this well-defined, i.e. independent of choice of path? Suppose we have two paths γ_1, γ_2 from 1 to z , with the region bounded by the difference of these curves containing zero, then

$$\int_{\gamma_1} \frac{dz}{z} - \int_{\gamma_2} \frac{dz}{z} = 2\pi i,$$

assuming things are correctly oriented. Thus in general, this doesn't make sense... so for “general” γ , $\int_{\gamma} \frac{dz}{z}$ are only well-defined up to integer multiples of $2\pi i$; we say that the complex logarithm is multi-valued.

↪ **Definition 1.15:** Let α, β be two points in a region Ω . Let γ_0, γ_1 be two paths joining α to β . We say γ_0, γ_1 are *homotopic* if there exists a family of paths γ_t for $t \in [0, 1]$ satisfying:

1. $\gamma_{t=0} = \gamma_0, \gamma_{t=1} = \gamma_1$
2. $\gamma_t(0) = \alpha, \gamma_t(1) = \beta$ for all t
3. $(t, s) \mapsto \gamma_t(s)$ is jointly continuous in (t, s) .

↪ **Proposition 1.10:**

- Being homotopic is an equivalence relation on the set of paths between α and β