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Classical Mechanics PHYS251

Contents

1 Introduction & Notations

1.1 Vectors

This course deals mainly with *scalars* (magnitude) and *vectors* (magnitude and direction). We define algebra on vectors, briefly:

- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ *(commutativity of addition)*
- $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ *(associativity of addition)*
- $c(d\mathbf{A}) = (cd)\mathbf{A}$ *(associativity of scalar multiplication)*
- $(c + d)\mathbf{A} = c\mathbf{A} + d\mathbf{A}$ *(Distributivity of scalar multiplication)*
- $c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$ *(Distributivity of scalar multiplication)*

and the operators:

- $\mathbf{A} \times \mathbf{B} = \mathbf{C}$ s.t. $|\mathbf{C}| = |\mathbf{A}||\mathbf{B}| \sin\theta$, where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{C}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$. This is equivalent to computing $\det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$. This *cross product* is **anti-commutative**, meaning $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. Additionally, note that $\mathbf{A} \times \mathbf{A} = 0$.
- $\mathbf{A} \cdot \mathbf{B} = C = |\mathbf{A}||\mathbf{B}| \cos\theta$, where C is a scalar. Note that $C = 0$ when $\theta = \pi/2$, ie $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

1.2 Law of Cosines

Consider a (planar) triangle constructed of sides $\mathbf{C} = \mathbf{A} + \mathbf{B}$. We can write

$$\begin{aligned} \mathbf{C}^2 &= \mathbf{C}_x^2 + \mathbf{C}_y^2 \\ &= (|\mathbf{A}| - |\mathbf{B}| \cos\theta)^2 + (|\mathbf{B}| \sin\theta)^2 \\ &= |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}| \cos\theta \end{aligned}$$

1.3 Perspectives on the Cross Product

$$
\begin{aligned}
\vec{A} \times \vec{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\
&= A_x B_y (\mathbf{i} \times \mathbf{j}) + \cdots \\
&= (A_y B_z - A_z B_y)\mathbf{i} + \cdots \\
&\equiv (\mathbf{A} \times \mathbf{B})_k = \sum_{i=1}^{3} \sum j = 1^3 \mathcal{E}_{ijk} A_i B_j
\end{aligned}
$$

$$
\text{Where } \mathcal{E}_{ijk} = \begin{cases} 1; & \text{ijk even permutation of 123} \\ -1; & \text{ijk odd permutation of 123} \\ 0; & \text{otherwise} \end{cases}
$$

1.4 Describing a Particle in Space in Polar Coordinates

Consider a particle moving through space with a constant angular velocity $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega$. We can describe this movement in terms of planar coordinates as $\mathbf{r}(t) = r_0 \cos(\omega t)\mathbf{i} + r_0 \sin(\omega t)\mathbf{j}$. Differentiating with respect to time, we obtain $\mathbf{v}(t) = -r_0\omega(\sin(\omega t)\mathbf{i} - \cos(\omega t)\mathbf{j})$. Notice that $\mathbf{r} \cdot \mathbf{v} = 0 \forall t$; this should be familiar, as the velocity vector is always perpendicular to the position vector in purely circular motion. Differentiating again, we obtain $\mathbf{a}(t) = -r_0\omega^2(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}) = -\omega^2\mathbf{r}(t)$. In other words, the acceleration is always opposing the position vector (given the negative sign), and is proportional to the square of the angular velocity.

Assume now, instead, that the particle moves arbitrarily, described by a function $\mathbf{r}(t)$. In polar coordinates, this position vector is always travelling along the vector $\hat{\mathbf{r}}$, with magnitude r, and we can write $\mathbf{r}(t) = r \cdot \hat{\mathbf{r}}$. Differentiating:

$$
\begin{aligned}
\mathbf{v}(t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(r \cdot \hat{\mathbf{r}}) \\
&= \frac{\mathrm{d}r}{\mathrm{d}t}\hat{\mathbf{r}} + r\frac{\mathrm{d}\hat{\mathbf{r}}}{\mathrm{d}t} \\
&= \dot{r}\hat{\mathbf{r}} + r\frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) \\
&= \dot{r}\hat{\mathbf{r}} + r(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\dot{\theta} \\
&= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}
\end{aligned}
$$

Recalling that $\begin{cases} \hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \\ \hat{\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \end{cases}$. Differentiating again:

$$\begin{aligned}\mathbf{a}(t) = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}\right) \\ &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\hat{\mathbf{r}}) \\ &= \left(\underbrace{\ddot{r}}_{\text{radial}} - \underbrace{r\dot{\theta}^2}_{\text{centripetal}}\right)\hat{\mathbf{r}} + \left(\underbrace{2\dot{r}\dot{\theta}}_{\text{coriolis}} + \underbrace{r\ddot{\theta}}_{\text{tangential}}\right)\hat{\theta}\end{aligned}$$

we obtain a decomposition of the acceleration vector into radial ($\hat{\mathbf{r}}$) and tangential ($\hat{\theta}$) components, labelled accordingly. Note that the centripetal acceleration, labelled, is the same as the acceleration in circular motion mentioned previously.[1 2]

1.5 Newton's Laws of Motion

N1: *"in absence of external force, a body at rest remains at rest, and a body in motion remains in motion, with the same speed & same direction"*

This law defines *intertial frames*, ie ones in which the law holds.

N2: $\vec{F} = m\vec{a}$

N3: *"if a body **b** applies a force on body **a**, then **a** applies a force on **b** such that $\vec{F}_a = -\vec{F}_b$ (equal and opposite)"*[3]

Example 1.1. *Consider two blocks **A**, **B** where **A** lies atop **B** which all lie upon "the earth".*

***A** experiences the force W_A due to gravity and the force F_1 back from **B**.*

***B** experiences the normal force N from the table, the force of gravity W_B, and finally the force F_2 due to **A**.*

We can write (from N2)

$$\vec{F}_1 + \vec{W}_A = m_A\vec{a}_A$$

and similarly

$$\vec{N} + \vec{W}_B + \vec{F}_2 = m_B\vec{a}_B$$

Static situation $\implies \vec{a}_A = \vec{a}_B = 0$. *We can further simplify writing $F_1 = m_A g$ (N2), and finally*

$$N = W_A + W_B = (m_A + m_B)g.$$

[1]None of the other "components" of acceleration are present in the constant angular velocity case because (1) $\ddot{r} = 0$, ie no radial acceleration, so the radial and coriolis components are zero, and (2) $\dot{\theta} = \omega \implies \ddot{\theta} = 0$, so the tangential acceleration is zero, leaving only the centripetal acceleration.

[2]See https://notes.louismeunier.net/Calculus%20A%2C%20B/calculus.pdf#page=85for a different perspective on this topic.

[3] $\implies$ conservation of momentum …

Example 1.2. *Consider a mass m_1 laying on a table, connected via a string to a mass m_2 hanging off the table, where the string is of small mass and is under a tension T. (Looking at a functionally massless stirng, we can consider there to be a constant tension as we can say $T - T' = \delta m a \implies T - T' = 0 \implies T = T'$, where T, T' is the tension on a particular segment of the string in opposing directions).*

On m_1, we have the normal force N, tension T, and weight $m_1 \cdot g$.

On m_2, we have the weight $m_2 \cdot g$, the tension T (equal throughout string as explained above).

Together, we can write (N2; taking x to represent a motion "left" and z to represent a motion "down")

$$-T = m_1 \ddot{x}$$

and

$$m_2 g - T = m_2 \ddot{z}.$$

Noting that the string must "retain" its length we can write

$$l(t) = x(t) + z(t),$$

where x and z represent the length of the string in the x/z axes. However, as l must stay constant, we can differentiate twice wrt time to obtain

$$0 = \ddot{x} + \ddot{z}.$$

All together, then we have

$$\ddot{x} = \frac{-m_2 g}{m_1 + m_2}.$$

Integrating twice, we have

$$x(t) = \frac{-m_2 g}{m_1 + m_2} \cdot \frac{t^2}{2}.$$

Example 1.3. *Consider a mass m_1 connected to a string which lies along a pulley P_1, which then attaches to the center of pulley P_2, which has a string whicch, on one end, is attached to a mass m_2, and is grounded in the other.*

On m_1, we have tension T and the weight $m_1 \cdot g$.

On m_2, we have the weight $m_2 \cdot g$ and the tension T' (no reason for $T = T'$; strings aren't connected).

Consider P_1 - it is nailed to the wall at its center, and experiences some F ("up") from the nail, as well as the tension T_1 down (twice).

On P_2, we have the tension T_1 ("up") and the tension T_2 ("down", "twice").

We can write

$$\begin{aligned} T_1 - W_1 &= m_1 a_1 \\ T_2 - W_2 &= m_2 a_2 \\ T_1 - 2T_2 &= m_{P_2} \cancel{a_{P_2}} \implies T_1 = 2T_2 \end{aligned}$$

As last time, we can use the fact that the strings cannot stretch. Consider m_1 to be at height y_1, P_1 to be at height y_{P_1}, P_2 to be at height y_{P_2}, and m_2 to be at height y_2.

We can then take the length l_1 of the string about P_1 as

$$l_1 = (y_{P_1} - y_1) + (y_{P_2} - y_2) + \pi R_{P_1}$$

Differentiating twice wrt time, we have

$$0 = a_1 + a_{P_2} \implies a_1 = -a_{P_2}.$$

Analyzing P_2 similarly, we will obtain

$$a_2 = 2a_{P_2}.$$

All together, we have

$$a_1 = \left(\frac{2m_2 - m_1}{4m_2 + m_1} \right) g.$$

Consider a collection of particles. The force on any particular particle, say 1, we can denote

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \cdots + \vec{F}_{1n} + \vec{F}_1^{\text{ext}}.$$

Notice that any force $\vec{F}_{nm} = -\vec{F}_{mn}$, by N3. Thus, if we add all the forces on all the particles in the bag, we will always have a "pairing" of forces such that each $\vec{F}_{mn}$ is "canceled" by another, leaving behind only the external forces, ie

$$\vec{F}^{\text{ext}} = \sum_i \vec{F}_i^{\text{ext}}.$$

Say there are no external forces; then, we have

$$\begin{aligned}\sum_i m_i \vec{a}_i &= 0\\ \frac{\mathrm{d}}{\mathrm{d}t}\sum_i m_i \vec{v}_i &= 0.\end{aligned}$$

And thus, we have conservation of momentum.

1.6 Projectile Motion

Say we have a particle with an initial velocity v_0 launched at an angle θ. This particle experiences just one force, mg, and by N2 we have

$$\cancel{m}\ddot{y} = -\cancel{m}g.$$

Integrating, we have

$$y(t) = y(t=0) + \underbrace{v_0 \sin\theta}_{\text{vertical component of } v_0} t - \frac{gt^2}{2}.$$

In the x, we have $m\ddot{x} = 0$, $x(t) = v_0 \cos\theta t$. From here, we can rewrite $x(t)$ as t as a function of x and substitute into y for a function $y = f(x)$. This gives

$$y = \frac{v_0 \sin\theta x}{v_0 \cos\theta} - \frac{g}{2}\frac{x^2}{v_0^2 \cos^2\theta} (= ax - bx^2)$$