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Algebra 2 MATH251

Course Outline:

Based on Lectures from Winter, 2024 by Prof. Anush Tserunyan.

Contents

1 Introduction

Remark 1.1. *This course is about vector spaces and linear transformations between them; a vector space involves multiplication by* scalars, *where the scalars come from some field. We recall first examples of fields, then vector spaces, as a motivation, before presenting a formal definition.*

1.1 Vector Spaces

Remark 1.2. *Much of this is recall from Algebra 1.*

⊛ **Example 1.1: Examples of Fields**

1. $\mathbb{Q}$; the field of rational numbers.

2. $\mathbb{R}$; the field of real numbers; $\mathbb{Q} \subseteq \mathbb{R}$.

3. $\mathbb{C}$; the field of complex numbers; $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

4. $\mathbb{F}_p \equiv \mathbb{Z}/p\mathbb{Z} \equiv \{0, 1, \ldots, p-1\}; the(unique)field of \mathrm{p} elements, where \mathrm{p} prime.$[a]

   (a) $p = 2$; $\mathbb{F}_2 \equiv \{0, 1\}$.

   (b) $p = 3$; $\mathbb{F}_3 \equiv \{0, 1, 2\}$.

   (c) $\cdots$

[a] where $a +_p b :=$ remainder of $\frac{a+b}{p}$, $a \cdot_p b :=$ remainder of $\frac{a \cdot b}{p}$.

Remark 1.3. *Throughout the course, we will denote an abstract field as* $\mathbb{F}$.

⊛ **Example 1.2: Examples of Vector Spaces**

1. $\mathbb{R}^3 := \{(x, y, z) : x, y, z \in \mathbb{R}\}$. We can add elements in $\mathbb{R}^3$, and multiply them by real scalars.

2. $\mathbb{F}^n := \underbrace{\mathbb{F} \times \mathbb{F} \times \cdots \mathbb{F}}_{n \text{ times}} := \{(a_1, a_2, \ldots, a_n) : a_i \in \mathbb{F}\}$, where $n \in \mathbb{N}$[1]; this is a generalization of the previous example, where we took $n = 3, \mathbb{F} = \mathbb{R}$. Operations follow identically; addition:

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) := (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

   and, taking a scalar $\lambda \in \mathbb{F}$, multiplication:

$$\lambda \cdot (a_1, a_2, \ldots, a_n) := (\lambda \cdot a_1, \lambda \cdot a_2, \ldots, \lambda \cdot a_n).$$

   We refer to these elements $(a_1, \cdots, a_n)$ as *vectors* in $\mathbb{F}^n$; the vector for which

$a_i = 0 \,\forall\, i$ is the 0 *vector*, and is the additive identity, making $\mathbb{F}^n$ an abelian group under addition, that admits multiplication by scalars from $\mathbb{F}$.

3. $C(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} : f \text{ continuous}\}$. Here, we have the constant zero function as our additive identity ($x \mapsto 0 \,\forall\, x$), and addition/scalar multiplication of two continuous real functions are continuous.

4. $\mathbb{F}[t] := \{a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n : a_i \in \mathbb{F} \,\forall\, i, n \in \mathbb{N}\}$, ie, the set of all polynomials in t with coefficients from $\mathbb{F}$. Here, we can add two polynomials;

$$(a_0 + a_1 t + \cdots + a_n t^n) + (b_0 + b_1 t + \cdots + b_m t^m) := \sum_{i=0}^{\max\{n,m\}} (a_i + b_i) t^i,$$

(where we "take" undefined a_i/b_i's as 0; that is, if $m > n$, then $a_{m-n}, a_{m-n+1}, \ldots, a_m$ are taken to be 0). Scalar multiplication is defined

$$\lambda \cdot (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) := \lambda a_0 + \lambda a_1 t + \lambda a_2 t^2 + \cdots + \lambda a_n t^n.$$

Here, the zero polynomial is simply 0 (that is, $a_i = 0 \,\forall\, i$).

[1] Where we take $0 \in \mathbb{N}$, for sake of consistency. Moreover, by convention, we define $\mathbb{F}^0$ (that is, when $n = 0$) to be $\{0\}$; the trivial vector space.

↪ **Definition 1.1: Vector Space**

A *vector space* V over a field $\mathbb{F}$ is an *abelian group* with an operation denoted $+$ (or $+_V$) and identity element[2] denoted 0_V, equipped with *scalar multiplication* for each scalar $\lambda \in \mathbb{F}$ satisfying the following axioms:

1. $1 \cdot v = v$ for $1 \in \mathbb{F}$, $\forall\, v \in V$.
2. $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta) v$, $\forall\, \alpha, \beta \in \mathbb{F}, v \in V$.
3. $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v$, $\forall\, \alpha, \beta \in \mathbb{F}, v \in V$.
4. $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v$, $\forall\, \alpha \in \mathbb{F}, u, v \in V$.

We refer to elements $v \in V$ as *vectors*.

[2] The "zero vector".

↪ **Proposition 1.1**

For a vector space V over a field $\mathbb{F}$, the following holds:

1. $0 \cdot v = 0_V$, $\forall\, v \in V$ (where $0 := 0_{\mathbb{F}}$)
2. $-1 \cdot v = -v$, $\forall\, v \in V$ (where $1 := 1_{\mathbb{F}}$)[3]

3. $\alpha \cdot 0_V = 0_V, \forall \alpha \in \mathbb{F}$

NB: "additive inverse"[3]

Proof. 1. $0 \cdot v = (0+0) \cdot v = 0 \cdot v + 0 \cdot v \implies 0 \cdot v = 0_V$ (by "cancelling" one of the $0 \cdot v$ terms on each side).

2. $v + (-1 \cdot v) = (1 \cdot v + (-1) \cdot v) = (1-1) \cdot v = 0 \cdot v = 0_V \implies (-1 \cdot v) = -v.$

3. $\alpha \cdot 0_V = \alpha \cdot (0_V + 0_V) = \alpha \cdot 0_V + \alpha \cdot 0_V \implies \alpha \cdot 0_V = 0_V$ (by, again, cancelling a term on each side).

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1.2 Creating Spaces from Other Spaces

↪ **Definition 1.2: Product/Direct Sum of Vector Spaces**

For vector spaces U, V over the same field $\mathbb{F}$, we define their *product* (or *direct sum*) as the set

$$U \times V = \{(u, v) : u \in U, v \in V\},$$

with the operations:

$$(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$$
$$\lambda \cdot (u, v) := (\lambda \cdot u, \lambda \cdot v)$$

⊛ **Example 1.3:** $\mathbb{F}$

$\mathbb{F}^2 = \mathbb{F} \times \mathbb{F}$, where $\mathbb{F}$ is considered as the vector space over $\mathbb{F}$ (itself).

↪ Definition 1.3: Subspace

For a vector space V over a field $\mathbb{F}$, a *subspace* of V is a subset $W \subseteq V$ s.t.

1. $0_V \in W$[4]
2. $u + v \in W \, \forall \, u, v \in W$ (closed under addition)
3. $\alpha \cdot u \in W \, \forall \, u \in W, \alpha \in \mathbb{F}$[5]

Then, W is a vector space in its own right.

[4] This is equivalent to requiring that $W \neq \varnothing$; stated this way, axiom 3. would necessitate that $0 \cdot w = 0_V \in W$.

[5] Note that these axioms are equivalent to saying that W is a subgroup of V with respect to vector addition; 2. ensures closed under addition, and 3. ensures the existence of additive inverses (as per $-1 \cdot v = -v$).

⊛ **Example 1.4: Examples of Subspaces**

1. Let $V := \mathbb{F}^n$.
   - $W := \{(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n : x_1 = 0\} = \{(0, x_2, x_3, \ldots, x_n) : x_i \in \mathbb{F}\}$.
   - $W := \{(x_1, x_2, \ldots, x_n) \in \mathbb{F}^n : x_1 + 2 \cdot x_2 = 0\}$

     Proof. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in W$. Then, $x + y = (x_1 + y_1, \ldots, x_n + y_n)$, and $x_1 + y_1 + 2 \cdot (x_2 + y_2) = x_1 + 2 \cdot x_2 + y_1 + 2 \cdot y_2 = 0 + 0 = 0 \implies x + y \in W$. Similar logic follows for axioms 2., 3. ■
   - (More generally)

     $$W := \{(x_1, \ldots, x_n) \in \mathbb{F}^n : \begin{matrix} a_{11}x_1 + \cdots a_{1n}x_n = 0 \\ a_{21}x_1 + \cdots + a_{2n}x_n = 0 \\ \ddots \\ a_{k1}x_1 + \cdots a_{kn}x_n = 0 \end{matrix}\},$$

     that is, a linear combination of homogenous "conditions" on each term.
   - $W^* := \{(x_1, \ldots, x_n) : x_1 + x_2 = 1\}$ is *not* a subspace; it is not closed under addition, nor under scalar multiplication.
2. Let $\mathbb{F}[t]_n := \{a_0 + a_1 t + \cdots + a_n t^n : a_i \in \mathbb{F}\}$. Then, $\mathbb{F}[t]_n$ is a subspace of $\mathbb{F}[t]$, the more general polynomial space. *However*, the set of all polynomials of degree *exactly* n (all axioms fail, in fact) is not a subspace of $\mathbb{F}[t]_n$.
   - $W := \{p(t) \in \mathbb{F}[t]_n : p(1) = 0\}$.
   - $W := \{p(t) \in \mathbb{F}[t]_n : p''(t) + p'(t) + 2p(t) = 0\}$.
3. Let $V := C(\mathbb{R})$ be the space of continuous function $\mathbb{R} \to \mathbb{R}$.

- $W := \{f \in C(\mathbb{R}) : f(\pi) + 7f(\sqrt{2}) = 0\}$.
- $W := C^1(\mathbb{R}) :=$ everywhere differentiable functions.
- $W := \{f \in C(\mathbb{R}) : \int_0^1 f \,\mathrm{d}x = 0\}$.

↪ **Proposition 1.2**

Let W_1, W_2 be subspaces of a vector space V over $\mathbb{F}$. Then, define the following:

1. $W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$
2. $W_1 \cap W_2 := \{w \in V : w \in W_1 \land w \in W_2\}$

These are both subspaces of V.

Proof.

1. (a) $0_V \in W_1$ and $0_V \in W_2 \implies 0_V = 0_V + 0_V \in W_1 + W_2$.
   (b) $(u_1 + u_2) + (v_1 + v_2) = (u_1 + v_1) + (u_2 + v_2) \in W_1 + W_2$.
   (c) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v \in W_1 + W_2$
2. (a) $0_V \in W_1$ and $0_V \in W_2 \implies 0_V = 0_V + 0_V \in W_1 \cap W_2$.
   (b) $u, v \in W_1 \cap W_2 \implies u + v \in W_1 \land u + v \in W_2 \implies u + v \in W_1 \cap W_2$.
   (c) $\alpha \cdot u \in W_1 \land \alpha \cdot u \in W_2 \implies \alpha \cdot u \in W_1 \cap W_2$.

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1.3 Linear Combinations and Space

↪ **Definition 1.4: Linear Combination**

Let V be a vector space over a field $\mathbb{F}$. For finitely many vectors $v_1, v_2, \ldots, v_n$, their *linear combination* is a sum of the form

$$\sum_{i=1}^{n} a_i v_i = a_1 \cdot v_1 + \cdots + a_n \cdot v_n,$$

where $a_i \in \mathbb{F} \,\forall\, i$.

A linear combination is called *trivial* if $a_i = 0 \,\forall\, i$, that is, all coefficients are 0.

If $n = 0$ (ie, we are "summing up" 0 vectors), we define the sum as the zero vector; $\sum_{i=1}^{0} a_i v_i := 0_V$.

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↪ **Definition 1.5: A More General Definition of Linear Combination**

For a a (possible infinite) set S of vectors from V, a *linear combination* of vectors in S is a linear combination of $a_1 v_1 + \cdots a_n v_n$ for some finite subset $\{v_1, \ldots, v_n\} \subseteq S$.[6]

[6]That is, we do not allow infinite sums.

↪ **Definition 1.6: Span**

For a subset $S \subseteq V$, we define its *span* as

$$\operatorname{Span}(S) := \text{ set of all linear combinations of } S := \{a_1 v_1 + \cdots a_n v_n : a_i \in \mathbb{F}, v_i \in S\}.$$

By convention, we set $\operatorname{Span}(\varnothing) = \{0_V\}$.

⊛ **Example 1.5**

Let $S := \{(1, 0, -1), (0, 1, -1), (1, 1, -2)\} \subseteq \mathbb{R}^3$. Then,

$$0_{\mathbb{R}^3} = (0, 0, 0) = 1 \cdot (1, 0, -1) + 1 \cdot (0, 1, -1) + -1 \cdot (1, 1, -2).$$

We claim, moreover, that $\operatorname{Span}(S) = U := \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ (a plane through the origin).

Proof. Note that $S \subseteq U$, hence $S \subseteq \operatorname{Span} S \subseteq U$. OTOH, if $(x, y, z) \in U$, we have $z = -x - y$, and so

$$(x, y, z) = (x, y, -x - y) = x \cdot (1, 0, -1) + y \cdot (0, 1, -1) \in \operatorname{Span}(S)$$

hence $U \subseteq \operatorname{Span}(S)$ and thus $\operatorname{Span}(S) = U$. ■

Remark 1.4. *We implicitly used the following claim in the proof above; we prove it more generally.*

↪ **Proposition 1.3**

Let V be a vector space over $\mathbb{F}$ and let $S \subseteq V$. Then, $\operatorname{Span}(S)$ is always a subspace. Moreover, it is the smallest (minimal) subspace containing S (that is, for any subspace $U \supseteq S$, we have that $U \supseteq \operatorname{Span} S$).

Proof. Because adding/scalar multiplying linear combinations of elements of S again results in a linear combination of elements of S, and $0_V \in \operatorname{Span}(S)$ by definition, we have that $\operatorname{Span}(S)$ is indeed a subspace.

If $U \supset S$ is a subspace of V containing S, then by definition U is closed under addition, that is, taking linear combinations of its elements (in particular, of elements of S); hence, $U \supset \operatorname{Span}(S)$. ■

↪ **Lemma 1.1**

For $S \subseteq V$ and $v \in V$, $v \in \mathrm{Span}(S) \iff \mathrm{Span}(S \cup \{v\}) = \mathrm{Span}(S)$.

Proof. ($\implies$) Let $v \in \mathrm{Span}(S) \implies v = a_1 v_1 + \cdots a_n v_n, a_i \in \mathbb{F}, v_i \in V$. Then, for any linear combination

$$b_1 u_1 + \cdots b_m u_m + b \cdot v = b_1 u_1 + \cdots b_m u_m + b(a_1 v_1 + \cdots + a_n v_n)$$

is a linear combination of vectors in $S \cup \{v\}$ (first equality) or equivalently, a combination of vectors in S (second equality) and thus $\mathrm{Span}(S \cup \{v\}) \subseteq \mathrm{Span}\, S$. The reverse inclusion follows trivially.

($\impliedby$) $\mathrm{Span}(S \cup \{v\}) = \mathrm{Span}\, S \implies v \in \mathrm{Span}(S)$. ■

⊛ **Example 1.6**

(From the above example) We have

$$\mathrm{Span}(\{(1,0,-1),(0,1,-1)\} \cup \{(1,1,-2)\}) = \mathrm{Span}(\{(1,0,-1),(0,1,-1)\}),$$

since $(1,1,-2) \in \mathrm{Span}(\{(1,0,-1),(0,1,-1)\})$ (it was redundant, as it could be generated by the other two vectors).

↪ **Definition 1.7: Spanning Set**

Let V be a vector space over a field $\mathbb{F}$. We call $S \subseteq V$ a *spanning set* for V if $\mathrm{Span}(S) = V$. We call such a spanning set *minimal* if no proper subset of S is a spanning set ($\nexists v \in S$ s.t. $S \setminus \{v\}$ spanning).

Remark 1.5. *Note that any $S \subseteq V$ is a spanning for $\mathrm{Span}(S)$. But, S may not be minimal; indeed, consider the previous example. We were able to remove a vector from S while having the same span.*

⊛ **Example 1.7**

For $\mathbb{F}^n$ as a vector space over $\mathbb{F}$, the *standard spanning set*

$$\underset{n}{\mathrm{St}} := \{\underbrace{(1,\ldots,0)}_{:=e_1}, \underbrace{(0,1,0,\ldots,0)}_{:=e_2}, \ldots, \underbrace{(0,\ldots,1)}_{e_n}\}.$$

Given any $x := (x_1, \ldots, x_n) \in \mathbb{F}^n$, we can write

$$x = x_1 \cdot e_1 + \cdots x_n \cdot e_n.$$

This is clearly minimal; removing any e_i would then result in a 0 in the ith "coordinate" of a vector, hence $\mathrm{St} \setminus \{e_i\}$ would span only vectors whose ith coordinate is 0.

↪ **Definition 1.8: Linear Dependence**

Let V be a vector space over a field $\mathbb{F}$. A set $S \subseteq V$ is said to be *linearly dependent* if there is a nontrivial linear combination of vectors in S that is equal to 0_V.

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