

# MATH455 - Analysis 4

## Summary

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### 1 Linear Operators (Open Mapping; Closed Graph; Uniform Boundedness)

**Definition 1:** For  $X, Y$  normed vector spaces,  $\mathcal{L}(X, Y) := \left\{ T : X \rightarrow Y \mid \|T\| := \sup_{x \in X} \frac{\|Tx\|_Y}{\|x\|_X} < \infty \right\}$

**Theorem 1:**  $T : X \rightarrow Y$  bounded iff continuous; if  $X, Y$  Banach, so is  $\mathcal{L}(X, Y)$ .

**Theorem 2:**

- (i) Any two nvs of the same finite dimension are isomorphic;
- (ii) Any finite dimensional space complete, any finite dimensional subspace is closed;
- (iii)  $\overline{B(0, 1)}$  compact in  $X$  iff  $X$  finite dimensional.

**Theorem 3** (Open Mapping): Let  $T : X \rightarrow Y$  a bounded linear operator where  $X, Y$  Banach. Then, if  $T$  surjective,  $T$  open, that is,  $T(\mathcal{U})$  open in  $Y$  for any  $\mathcal{U}$  open in  $X$ .

*Remark 1:* By scaling & translating, openness of an operator is equivalent to proving  $T(B_X(0, 1))$  contains  $B_Y(0, r)$  for some  $r > 0$ .

**Corollary 1:** If  $T : X \rightarrow Y$  bounded, linear and bijective for  $X, Y$  Banach,  $T^{-1}$  continuous. In particular, if  $(X, \|\cdot\|_1), (X, \|\cdot\|_2)$  are two Banach spaces such that  $\|x\|_2 \leq C\|x\|_1$ , then  $\|\cdot\|_1, \|\cdot\|_2$  are equivalent.

**Theorem 4** (Closed Graph Theorem): Let  $T : X \rightarrow Y$  where  $X, Y$  Banach. Then  $T$  continuous iff  $T$  is closed, i.e. the graph  $G(T) := \{(x, Tx) : x \in X\} \subset X \times Y$  is closed in the product topology.

*Remark 2:* This theorem crucially uses the fact that the norm

$$\|(x, y)\|_* := \|x\|_X + \|y\|_Y$$

(among others) induces the product topology on  $X \times Y$ , hence in particular such a norm can be used to make  $X \times Y$  a nvs.

**Theorem 5** (Uniform Boundedness): Let  $X$  Banach and  $Y$  an nvs, and let  $\mathcal{F} \subset \mathcal{L}(X, Y)$  such that  $\forall x \in X, \exists M_x > 0$  s.t.  $\|Tx\|_Y \leq M_x \forall T \in \mathcal{F}$  (that is,  $\mathcal{F}$  pointwise bounded). Then,  $\mathcal{F}$  uniformly bounded, i.e. there is some  $M > 0$  such that  $\|T\|_Y \leq M$  for every  $T \in \mathcal{F}$ .

**Remark 3:** This is implied by the consequence of the Baire Category theorem that states that if  $\mathcal{F} \subset C(X)$  where  $X$  a complete metric space and  $\mathcal{F}$  pointwise bounded, then there is a nonempty open set  $\mathcal{O} \subset X$  such that  $\mathcal{F}$  uniformly bounded on  $\mathcal{O}$ . In the case of a nvs, by linearity, being uniformly bounded on an open set extends to being uniformly bounded on all of  $X$ .

**Theorem 6** (Banach-Saks-Steinhaus): Let  $X$  Banach and  $Y$  an nvs, and  $\{T_n\} \subset \mathcal{L}(X, Y)$  such that for every  $x \in X$ ,  $\lim_n T_n(x)$  exists in  $Y$ . Then

- (i)  $\{T_n\}$  uniformly bounded in  $\mathcal{L}(X, Y)$ ;
- (ii)  $T \in \mathcal{L}(X, Y)$  where  $T(x) := \lim_n T_n(x)$ ;
- (iii)  $\|T\| \leq \liminf_n \|T_n\|$ .

**Remark 4:** (i) follows from uniform boundedness, (ii) from just taking sums limits, (iii) from taking  $\lim(\inf)$ its.

## 2 Hilbert Spaces; Weak Convergence

**Theorem 7** (Cauchy-Schwarz):  $|(u, v)| \leq \|u\|\|v\|$ .

**Theorem 8** (Orthogonality): If  $M \subset H$  a closed subspace, for every  $x \in H$ , there is a unique decomposition

$$x = u + v, \quad u \in M, v \in M^\perp := \{v \in H \mid (v, y) = 0 \forall y \in M\},$$

and

$$\|x - u\| = \inf_{y \in M} \|x - y\|, \quad \|x - v\| = \inf_{y \in M^\perp} \|x - y\|.$$

**Theorem 9** (Riesz): For  $f \in H^* := \mathcal{L}(H, \mathbb{R})$ , there is a unique  $y \in H$  such that  $f(y) = (y, x), \forall x \in H$ .

**Theorem 10** (Bessel's Inequality): If  $\{e_n\} \subset H$  orthonormal, then  $\sum_{i=1}^\infty |(x, e_i)|^2 \leq \|x\|^2$ .

**Theorem 11** (Equivalent Notions of Orthonormal Basis): If  $\{e_n\} \subset H$  orthonormal, TFAE:

- (i) if  $(x, e_i) = 0$  for every  $i$ ,  $x = 0$ ;
- (ii) Parseval's identity holds,  $\|x\|^2 = \sum_{i=1}^\infty (x, e_i)^2$ , for every  $x \in H$ ;
- (iii)  $\{e_i\}$  a basis for  $H$ , that is  $x = \sum_{i=1}^\infty (x, e_i)e_i$  for every  $x \in H$ .

**Theorem 12:**  $H$  is separable (has a countable dense subset) iff  $H$  has a countable basis.

**Theorem 13** (Properties of the Adjoint): For  $T : H \rightarrow H$ , the *adjoint*  $T^* : H \rightarrow H$  is defined as the operator with the property  $(Tx, y) = (x, T^*y)$  for every  $x, y \in H$ . Then:

- if  $T \in \mathcal{L}(H)$  then  $T^* \in \mathcal{L}(H)$  and  $\|T^*\| = \|T\|$ ;
- $(T^*)^* = T$ ;
- $(T + S)^* = T^* + S^*$ ;
- $(T \circ S)^* = S^* \circ T^*$ ;
- if  $T \in \mathcal{L}(H)$ , then
  - $N(T^*) = R(T)^\perp$ , and similarly,
  - $N(T) = R(T^*)^\perp$ .

Note that then  $R(T)^\perp$  closed, so one finds  $(R(T)^\perp)^\perp = \overline{R(T)}$ .

**Definition 2** (Weak Convergence): We say  $\{x_n\} \subset X$  converges weakly to  $x \in X$  and write  $x_n \rightharpoonup x$  if for every  $T \in X^*$ ,  $Tx_n \rightarrow Tx$ . By Riesz, this is equivalent to saying  $(x_n, y) \rightarrow (x, y)$  for every  $y \in X$ .

We define, then,  $\sigma(X, X^*)$  to be the weak topology (on  $X$ ) generated by the collection of families  $X^*$ ; i.e., the coarsest topology for which every functional  $T \in X^*$  is continuous.

**Theorem 14** (Properties of Weak Convergence):

- (i) If  $x_n \rightharpoonup x$ , then  $\{x_n\}$  bounded in  $H$  and  $\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$ .
- (ii) If  $y_n \rightarrow y$  (strongly) and  $x_n \rightharpoonup x$  (weakly) then  $(x_n, y_n) \rightarrow (x, y)$ .

**Theorem 15** (Helley's Theorem): Let  $X$  a separable normed vector space and  $\{f_n\} \subset X^*$  such that there is a  $C > 0$  such that  $|f_n(x)| \leq C\|x\|$  for every  $x \in X$  and  $n \geq 1$ . Then, there is a subsequence  $\{f_{n_k}\}$  and  $f \in X^*$  such that  $f_{n_k}(x) \rightarrow f(x)$  for every  $x \in X$ .

*Remark 5:* This is just the Arzelà-Ascoli Lemma; by linearity, the uniform boundedness implies uniform Lipschitz continuity and thus equicontinuity.

**Theorem 16** (Weak Compactness): Every bounded sequence in  $H$  has a weakly converging subsequence.

*Remark 6:* This is a consequence of Helley's.

### 3 $L^p$ Spaces

**Theorem 17** (Basic Properties of  $L^p(\Omega)$ ):

- (i) (Holder's Inequality)  $\|fg\|_1 \leq \|f\|_p \|g\|_q$  for  $f \in L^p(\Omega)$ ,  $g \in L^q(\Omega)$  and  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $1 \leq p \leq q \leq \infty$ ;
- (ii) (Riesz-Fischer Theorem)  $L^p(\Omega)$  is a Banach space for every  $1 \leq p \leq \infty$ ;
- (iii)  $C_c(\mathbb{R}^d)$ , simple functions, and step functions are all dense in  $L^p(\mathbb{R}^d)$  for every finite  $p$ ;
- (iv)  $L^p(\Omega)$  is separable for every finite  $p$ ;
- (v) If  $\Omega \subset \mathbb{R}^d$  has finite measure, then  $L^p(\Omega) \subset L^{p'}(\Omega)$  for every  $p \leq p'$ ;
- (vi) If  $f \in L^p(\Omega) \cap L^q(\Omega)$  for  $1 \leq p \leq q \leq \infty$ , then  $f \in L^r(\Omega)$  for every  $r \in [p, q]$ .

**Theorem 18** (Riesz Representation for  $L^p(\Omega)$ ): Let  $1 \leq p < \infty$  and  $q$  the Holder conjugate of  $p$ . Then, if  $T \in (L^p(\Omega))^*$ , there is a unique  $g \in L^q(\Omega)$  such that

$$Tf = \int_{\Omega} fg, \quad \forall f \in L^p(\Omega),$$

and  $\|T\| = \|g\|_q$ .

*Remark 7:* When  $p = 2 = q$ , then  $L^p(\Omega)$  is a Hilbert space so this reduces to the typical Hilbert space theory.

**Theorem 19** (Weak Convergence in  $L^p(\Omega)$ ):

- Let  $p \in (1, \infty)$  and  $\{f_n\} \subset L^p(\Omega)$ , then by Riesz,  $f_n \rightharpoonup f$  iff  $\int_{\Omega} f_n g \rightarrow \int_{\Omega} fg$  for every  $g \in L^q(\Omega)$ .
- Suppose  $f_n$  are bounded and  $f \in L^p(\Omega)$ , then  $f_n \rightharpoonup f$  if and only if  $f_n \rightarrow f$  pointwise a.e..
- (Radon-Riesz) For  $p \in (1, \infty)$ , let  $\{f_n\} \subset L^p(\Omega)$  such that  $f_n \rightharpoonup f$ . Then,  $f_n \rightarrow f$  strongly if and only if  $\|f_n\|_p \rightarrow \|f\|$ .

**Theorem 20** (Weak Compactness in  $L^p(\Omega)$ ): Let  $p \in (1, \infty)$ . Then, every bounded sequence in  $L^p(\Omega)$  has a weakly converging subsequence in  $L^p(\Omega)$ .

*Remark 8:* This is essentially the same as the Hilbert space proof.

**Theorem 21** (Properties of Convolutions):

- (i)  $(f * g) * h = f * (g * h)$
- (ii) With  $\tau_z f(x) := f(x - z)$ ,  $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$
- (iii)  $\text{supp}(f * g) \subseteq \text{supp}(f) + \text{supp}(g) = \{x + y \mid x \in \text{supp}(f), y \in \text{supp}(g)\}$

**Theorem 22** (Young's Inequality): Let  $f \in L^1(\mathbb{R}^d)$  and  $g \in L^p(\mathbb{R}^d)$  for any  $p \in [1, \infty]$ , then

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p,$$

so in particular  $f * g \in L^p(\Omega)$ .

**Theorem 23** (Derivatives of Convolutions): Let  $f \in L^1(\mathbb{R}^d)$  and  $g \in C^1(\mathbb{R}^d)$  with  $|\partial_i g| \in L^\infty(\mathbb{R}^d)$  for  $i = 1, \dots, d$ . Then,  $f * g \in C^1(\mathbb{R}^d)$ , and in particular

$$\partial_i(f * g) = f * (\partial_i g).$$

*Remark 9:* This holds more generally for many different assumptions on  $f, g$  but you basically need to be able to apply dominated convergence theorem to pass the limit involved in taking the derivative under the integral sign.

This extends for  $g \in C^k(\mathbb{R}^d)$ ; in particular, if  $g \in C^\infty(\mathbb{R}^d)$ , then  $f * g \in C^\infty(\mathbb{R}^d)$ . It also holds for the gradient, i.e.  $\nabla(f * g) = f * (\nabla g)$  (where the convolution is component-wise in the gradient vector).

**Theorem 24** (Good Kernels): A *good kernel* is a parametrized family of functions  $\{\rho_\varepsilon : \varepsilon \in \mathbb{R}\}$  with the properties

- (i)  $\int_{\mathbb{R}^d} \rho_\varepsilon(y) dy = 1$ ,
- (ii)  $\int_{\mathbb{R}^d} |\rho_\varepsilon(y)| dy \leq M$ ,
- (iii) for every  $\delta > 0$ ,  $\int_{|y| > \delta} |\rho_\varepsilon(y)| dy \rightarrow 0$  as  $\varepsilon \rightarrow 0^+$ .

The canonical, and in particular both smooth and compactly supported, example is

$$\rho(x) := \begin{cases} C \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| \leq 1, \\ 0 & \text{o.w.} \end{cases}$$

where  $C = C(d)$  a scaling constant such that  $\rho$  integrates to 1. Then  $\rho_\varepsilon(x) := \left(\frac{1}{\varepsilon^d}\right) \rho\left(\frac{x}{\varepsilon}\right)$  is a good kernel, supported on  $B(0, \varepsilon)$ . Then:

- (i) if  $f \in L^\infty(\mathbb{R}^d)$ ,  $f_\varepsilon := \rho_\varepsilon * f$  and  $f$  continuous at  $x$ , then  $f_\varepsilon(x) \rightarrow f(x)$  as  $\varepsilon \rightarrow 0$ ;
- (ii) if  $f \in C(\mathbb{R}^d)$  then  $f_\varepsilon \rightarrow f$  uniformly on compact sets;
- (iii) if  $f \in L^p(\mathbb{R}^d)$  with  $p$  finite, then  $f_\varepsilon \rightarrow f$  in  $L^p(\mathbb{R}^d)$ .

*Remark 10:* Part 3. follows immediately from 2. by density of  $C_c(\mathbb{R}^d)$  in  $L^p(\mathbb{R}^d)$ .

**Corollary 2:**  $C_c^\infty(\mathbb{R}^d)$  dense in  $L^p(\mathbb{R}^d)$  for any finite  $p$ .

**Theorem 25** (Weierstrass Approximation Theorem): Polynomials are dense in  $C([a, b])$ , i.e. for any  $f \in C([a, b])$  and  $\eta > 0$ , there is a polynomial  $p(x)$  such that  $\|p - f\|_{L^\infty([a, b])} < \eta$ .

**Theorem 26** (Strong Compactness): Let  $\{f_n\} \subseteq L^p(\mathbb{R}^d)$  for  $p$  finite, such that

- $\{f_n\}$  uniformly bounded in  $L^p(\mathbb{R}^d)$ , and
- $\lim_{|h| \rightarrow 0} \|f_n - \tau_h f_n\|_p = 0$  uniformly in  $n$ , i.e. for every  $\eta > 0$  there is a  $\delta > 0$  such that  $|h| < \delta$  implies  $\|f_n - \tau_h f_n\|_p < \eta$  for every  $n \geq 1$ .

Then, for every  $\Omega \subset \mathbb{R}^d$  of finite measure, there exists a subsequence  $\{f_{n_k}\}$  such that  $f_{n_k} \rightarrow f$  in  $L^p(\Omega)$ .

*Remark 11:* This is Arzelà-Ascoli in disguise!

#### 4 Fourier Analysis

**Definition 3** (Fourier Series): Let  $L^2(\mathbb{T}) = \{f : \mathbb{T} \rightarrow \mathbb{R} \mid \int_{\mathbb{T}} f^2 < \infty\}$  equipped with the inner product  $(f, g) = \int_{\mathbb{T}} f \bar{g}$ . Then,  $e_n(x) := e^{2\pi i n x}$ , for  $n \in \mathbb{Z}$ , is an orthonormal basis for  $L^2(\mathbb{T})$ . The *Fourier coefficients* of a function  $f$  are defined then, for  $n \in \mathbb{Z}$ ,

$$\hat{f}(n) = (f, e_n) = \int_{\mathbb{T}} f(x) e^{-2\pi i n x} dx,$$

and so the *complex Fourier series* is defined

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}.$$

**Theorem 27** (Riemann-Lebesgue Lemma): If  $f \in L^2(\mathbb{T})$ ,  $\lim_{n \rightarrow \infty} |\hat{f}(n)| = 0$ .

*Remark 12:* By expanding the real and complex parts of the coefficients, this also implies

$$\lim_{n \rightarrow \infty} \int_{\mathbb{T}} f(x) \sin(2n\pi x) dx = \lim_{n \rightarrow \infty} \int_{\mathbb{T}} f(x) \cos(2n\pi x) dx = 0.$$

**Definition 4** (Dirichlet Kernel): The *Dirichlet Kernel* is the sequence of functions defined

$$D_N(x) := \sum_{n=-N}^N e^{2\pi i n x} = \frac{\sin(2\pi(N + \frac{1}{2})x)}{\sin(2\pi \frac{x}{2})}.$$

Then, the partial sum  $S_N f(x) := \sum_{n=-N}^N \hat{f}(n) e^{2\pi i n x} = (f * D_N)(x)$ .

**Theorem 28** (Convergence Results):

- If  $f \in L^2(\mathbb{T})$  and Lipschitz at  $x_0$ , then  $S_N f(x_0) \rightarrow f(x_0)$
- If  $f \in L^2(\mathbb{T}) \cap C^2(\mathbb{T})$ , then  $S_N f \rightarrow f$  uniformly on  $\mathbb{T}$ .

**Definition 5** (Fourier Transform): The *Fourier Transform* of  $f : \mathbb{R} \rightarrow \mathbb{C}$  is defined

$$\hat{f}(\zeta) := \int_{\mathbb{R}} f(x) e^{-2\pi i \zeta x} dx.$$

The *Inverse Fourier Transform* of  $f \in L^1(\mathbb{R})$  is defined

$$\check{f}(x) := \int_{\mathbb{R}} f(\zeta) e^{2\pi i \zeta x} d\zeta = \widehat{f(-\cdot)}(x).$$

**Theorem 29** (Properties of the Fourier Transform): Let  $f, g \in L^1(\mathbb{R})$ .

- (i)  $\hat{f}, \check{f} \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$
- (ii)  $\widehat{\tau_y f}(\zeta) = e^{-2\pi i \zeta y} \hat{f}(\zeta)$ , and  $\tau_\eta \hat{f}(\zeta) = \widehat{e^{2\pi i \eta(\cdot)} f(\cdot)}(\zeta)$
- (iii)  $\widehat{f * g} = \hat{f} \cdot \hat{g}$
- (iv)  $\int_{\mathbb{R}} f(x) \hat{g}(x) dx = \int_{\mathbb{R}} \hat{f}(x) g(x) dx$
- (v) Let  $h(x) := e^{\pi a x^2}$  for  $a > 0$ , then  $\hat{f}(\zeta) = \frac{1}{\sqrt{a}} e^{-\pi \frac{\zeta^2}{a}}$

**Theorem 30** (Fourier Inversion): If  $f \in L^1(\mathbb{R})$  and  $\hat{f} \in L^1(\mathbb{R})$ , then  $f$  agrees almost everywhere with some  $f_0 \in C(\mathbb{R})$  and  $\hat{\hat{f}} = \check{\check{f}} = f_0$ .

**Theorem 31** (Plancherel's Theorem): If  $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ , then  $\hat{f} \in L^2(\mathbb{R})$  and  $\|f\|_2 = \|\hat{f}\|_2$ .