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Analysis 2 MATH255

Course Outline:

Based on Lectures from Winter, 2024 by Prof. Dmitry Jackobson.

Contents

1 Introduction

1.1 Metric Spaces

↪ **Definition 1.1: Metric Space**

A set X is a *metric space* with distance d if

1. (symmetric) $d(x, y) = d(y, x) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. (triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$

Remark 1.1. *If 1., 3. are satisfied but not 2., d can be called a "pseudo-distance".*

↪ **Definition 1.2: Normed Space**

Let X be a vector space over $\mathbb{R}$. The norm on X, denoted $||x|| \in \mathbb{R}$, is a function that satisfies

1. $||x|| \geq 0$
2. $||x|| = 0 \iff x = 0$
3. $||c \cdot x|| = |c| \cdot ||x||$
4. $||x + y|| \leq ||x|| + ||y||$

If X is a normed vector space over $\mathbb{R}$, we can define a distance d on X by $d(x, y) = ||x - y||$.

↪ **Proposition 1.1**

If X is a normed vector space over $\mathbb{R}$, a distance d on X by $d(x, y) = ||x - y||$ makes (X, d) a metric space.

Proof.

1. $d(x, y) = ||x - y|| \geq 0$
2. $d(x, y) = 0 \iff ||x - y|| = 0 \iff x - y = 0 \iff x = y$
3. $d(x, y) + d(y, z) = ||x - y|| + ||y - z|| \geq ||(x - y) + (y - z)|| = ||x - z|| := d(x, z)$

■

⊛ **Example 1.1:** L^p **distance in** $\mathbb{R}^n$

Let $\overline{x} \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n)$. The L^p norm is defined

$$||x||_p := (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}.$$

In the case $p = 2, n = 2$, we simply have the standard Euclidean distance over $\mathbb{R}^2$.

Unit Balls: consider when $||x||_p \leq 1$, over $\mathbb{R}^2$.

- $p = 1 : |x_1| + |x_2| \leq 1$; this forms a "diamond ball" in the plane.
- $p = 2 : \sqrt{|x_1|^2 + |x_2|^2} \leq 1$; this forms a circle of radius 1. Clearly, this surrounds a larger area than in $p = 2$.

A natural question that follows is what happens as $p \to \infty$? Assuming $|x_1| \geq |x_2|$:

$$\begin{aligned} ||x||_p &= (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \\ &= \left[|x_1|^p \left(1 + \left|\frac{x_2}{x_1}\right|^p\right)\right]^{\frac{1}{p}} \\ &= |x_1| \left(1 + \left|\frac{x_2}{x_1}\right|^p\right)^{\frac{1}{p}} \end{aligned}$$

If $|x_1| > |x_2|$, this goes to $|x_1|$. If they are instead equal, then $||x||_p = |x_1| \cdot 2^{\frac{1}{p}} \to |x_1| \cdot 1$ as well. Hence, $\lim_{p\to\infty} ||x||_p = \max\{|x_1|, |x_2|\}$. Thus, the unit ball will approach $\max\{|x_1|, |x_2|\} \leq 1$, that is, the unit square.

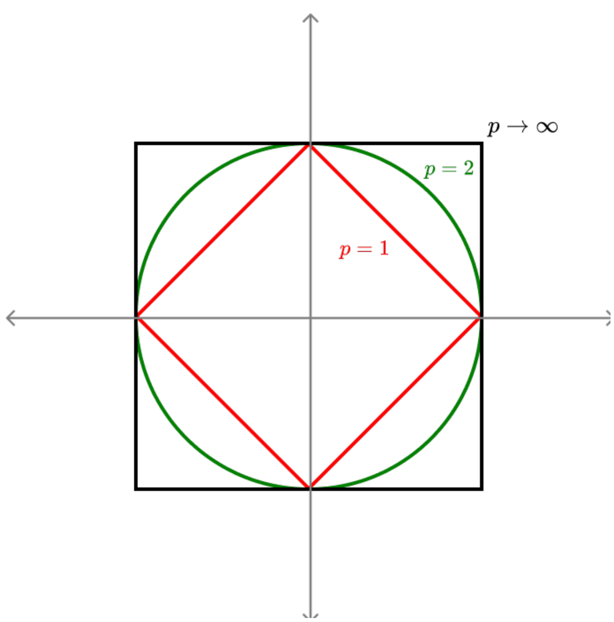

Figure 1: Regions of $\mathbb{R}^2$ where $||x||_p \leq 1$ for various values of p.

↪ **Proposition** **1.2**

Let $x \in \mathbb{R}^n$. Then, $||x||_p \to \max\{|x_1|, \dots, |x_n|\}$ as $p \to \infty$.

Remark 1.2. *This is an extension of the previous example to arbitrary real space.*

Proof. ■

↪ **Definition** **1.3: Convex Set**

Let X be a normed space, and take $x, y \in X$. The line segment from x to y is the set

$$\{t \cdot x + (1-t) \cdot y : 0 \leq t \leq 1\}.$$

Let $A \subseteq X$. A is *convex* $\iff$ $\forall x, y \in A$, we have that

$$(t \cdot x + (1-t) \cdot y) \in A \, \forall \, 0 \leq t \leq 1.$$

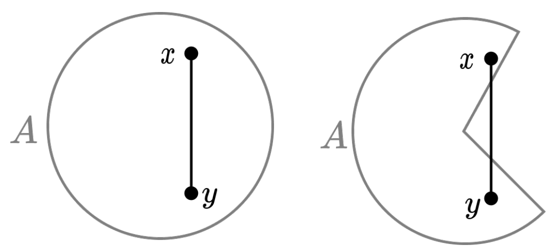

Figure 2: Convex (left) versus not convex (right) sets.

Remark 1.3. *Think of this as saying "a set is convex iff every point on a line segment connected any two points is in the set".*

↪ **Definition** **1.4:** ℓ_p

The space ℓ_p of sequences is defined as

$$\{x = (x_1, x_2, \dots, x_n, \dots) : \sum_{n=1}^{\infty} |x_n|^p < +\infty\} \quad * .$$

Then, $*$ defines the ℓ^p norm on the space of sequences; that is, $||x||_p := (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$.

⊛ **Example** **1.2:** ℓ_p, $x_n = \frac{1}{n}$

. Let $x_n = \frac{1}{n}$. For which p is $x \in \ell_p$? We have, raising the norm to the power of p for ease:

$$\begin{aligned} ||x||_p^p &= |x_1|^p + |x_2|^p + \cdots + |x_n|^p + \cdots \\ &= 1^p + \left(\frac{1}{2}\right)^p + \cdots < \infty \iff p > 1. \end{aligned}$$

In the case that $p = 1$, this becomes a harmonic sum, which diverges.

⊛ **Example 1.3: L^p space of functions**

Let $f(x)$ be a continuous function. We define the norm of f over an interval $[a, b]$

$$||f||_p = \left[\int_a^b |f(x)|^p \, dx\right]^{\frac{1}{p}}.$$

Remark 1.4. *Triangle inequality for $||x||_p$ or $||f||_p$ is called Minkowski inequality; $||x||_p + ||y||_p \geq ||x + y||_p$. This will be discussed further.*

⊛ **Example 1.4: Distances between sets in $\mathbb{R}^2$**

Let A, B be bounded, closed, "nice" sets in $\mathbb{R}^2$. We define

$$d(A, B) := \text{Area}(A \triangle B),$$

where

$$A \triangle B : (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

It can be shown that this is a "valid" distance.

Remark 1.5. *$\triangle$ denotes the "symmetric difference" of two sets.*

⊛ **Example 1.5: p-adic distance**

Let p be a prime number. Let $x = \frac{a}{b} \in \mathbb{Q}$, and write $x = p^k \cdot \left(\frac{c}{d}\right)$, where c, d are not divisible by p. Then, the p-adic norm is defined $||x||_p := p^{-k}$. It can be shown that this is a norm.

Suppose $p = 2$, $x = 28 = 4 \cdot 7 = 2^2 \cdot 7$. Then, $||28||_2 = 2^{-2} = \frac{1}{4}$; similarly, $||1024||_2 = ||2^{10}||_2 = 2^{-10}$.

More generally, we have that $||2^k||_2 = 2^{-k}$; coversely, $||2^{-k}|| = 2^k$. That is, the closer to 0, the larger the distance, and vice versa, contrary to our notion of Euclidean distance.

↪ **Proposition 1.3**

$||x||_p$ as defined above is a well-defined norm over $\mathbb{Q}$.

Proof. ■

1.2 Topological Spaces

↪ **Definition 1.5: Topological space**

A set X is a topological space if we have a collection of subsets τ of X called *open sets* s.t.

1. $\varnothing \in \tau, X \in \tau$
2. Consider $\{A_\alpha\}_{\alpha \in I}$ where A_α an open set for any α; then, $\bigcup_{\alpha \in I} A_\alpha \in \tau$, that is, it is also an open set.
3. If J is a finite set, and A_β open for all $\beta \in J$, then $\bigcap_{\beta \in J} A_\beta \in \tau$ is also open.

In other words, 2.: arbitrary unions of open sets are open, and 3.: finite intersections of open sets are open.

↪ **Definition 1.6: Closed sets**

Closed sets are complements of open sets; hence, axioms for closed sets follow appropriately;

1.* $X, \varnothing$ closed;

2.* B_α closed $\forall\, \alpha \in I \implies \bigcap_{\alpha \in I} B_\alpha$ closed.

3.* B_β closed $\forall\, \beta \in J$, J finite, then $\bigcup_{\beta \in J} B_\beta$ also closed.

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