

MATH458 - Differential Geometry

Based on lectures from Winter 2026 by Prof. Jean-Pierre Mutanguha.
Notes by Louis Meunier

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§1 SOME REVIEW

We will work in \mathbb{R}^n , usually with $n = 2, 3$. For vectors $v = (v_1, \dots, v_n), w = (w_1, \dots, w_n) \in \mathbb{R}^n$, we denote the dot product

$$v \cdot w = \sum_{i=1}^n v_i w_i.$$

More generally, an *inner product* on \mathbb{R}^n is any function $b : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ that is symmetric, bilinear and positive definite. For instance, if $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is linear and invertible $b_T(v, w) := T(v) \cdot T(w)$ a new inner product. In fact, it turns out every inner product on \mathbb{R}^n is of this form; this implies that every inner product is just a coordinate-change away from the dot product.

We will say a linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is *orthogonal* if it is inner product preserving, i.e. $T(v) \cdot T(w) = v \cdot w$ for every $v, w \in \mathbb{R}^n$.

Exercise 1.1: Show that T is inner product preserving iff it is norm preserving ($\|Tv\| = \|v\|$) iff it is distance preserving ($\|T(v - w)\| = \|v - w\|$).

Exercise 1.2: Show that if T orthogonal, it is a bijection with determinant ± 1 .

We say $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, linear, is *orientation preserving* if $\det(T) > 0$.

↪ **Definition 1.1** (Rigid Motion): A function $M : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a *rigid motion* if there exists an $a \in \mathbb{R}^n$ and $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ orthogonal and orientation preserving such that

$$M(v) = a + Tv, \quad \forall v \in \mathbb{R}^n.$$

We view the space \mathbb{E}^n as \mathbb{R}^n equipped with the Euclidean distance, which we'll denote $d_{\mathbb{E}}$ or d if no confusion arises, *up to rigid motions*. In practice, this means working in \mathbb{E}^n has no distinguished origin point or coordinate axes. However, also in practice, we will make the identification $\mathbb{E}^n \simeq \mathbb{R}^n$ by picking an origin and axes, as we will see.

However, working in \mathbb{E}^n , abstractly, still preserves orientation and distance, since these are both preserved under rigid motions.

For $r > 0$ and $\rho \in \mathbb{E}^n$, we write $\mathbb{D}_r(\rho)$ for the open unit disk, and $\mathbb{D}^n := \mathbb{D}_1(0) \subset \mathbb{R}^n$.

↪ **Theorem 1.1** (Heine-Borel): $C \subset \mathbb{E}^n$ compact iff closed and bounded.

Exercise 1.3: Let $r' > r > 0$ and $\rho \in \mathbb{E}^n$. Let $f : \mathbb{D}_{r'}(\rho) \rightarrow \mathbb{E}^n$ be continuous. Show that $f|_{\mathbb{D}_r(\rho)}$ uniformly continuous.

We'll denote the derivative of a function $f : \mathcal{U} \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ at a point a by $D_a f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, which is represented by the Jacobian $m \times n$ matrix $J(f)_a = \left(\frac{\partial f}{\partial x_1}|_a, \dots, \frac{\partial f}{\partial x_n}|_a \right)$.

↪ **Definition 1.2:** We will say $f : \mathcal{U} \rightarrow \mathbb{R}^m$ is C^k on \mathcal{U} if all the k th order partial derivatives of all of the component functions of f are continuous. We say f in C^∞ if it is in C^k for every $k \geq 1$. We write C^0 for the space of continuous functions.

Remark 1.1: $C^{k+1} \Rightarrow C^k$

§2 CURVES

↪ **Definition 2.1** (Parametrized curve/path): A *parametrized curve/path* in \mathbb{E}^n is a continuous function

$$\gamma : I \rightarrow \mathbb{E}^n,$$

where $I \subset \mathbb{R}$ an interval. We say γ *compact* if I is compact.

↪ **Definition 2.2** ((Regular) C^k parametrized curve): Fix coordinates in \mathbb{E}^n . Then, a (regular) C^k parametrized curve is a parametrized curve in which $\gamma \in C^k(I)$ (and for which $\frac{d\gamma}{dt}(t) \neq 0 \forall t \in I$).

Exercise 2.1: Regularity and differentiability is preserved under rigid motion, i.e. if γ a (regular) C^k parametrized curve and M a rigid motion on \mathbb{R}^n , then $\tilde{\gamma} := M \circ \gamma$ also (regular) C^k .

↪ **Definition 2.3:** Given a curve γ , we define

- the *velocity*, $v = \frac{d\gamma}{dt} : I \rightarrow \mathbb{R}^n$
- the *acceleration*, $a = \frac{d^2\gamma}{dt^2} : I \rightarrow \mathbb{R}^n$
- the *speed*, $\sigma = \|v\| = \left\| \frac{d\gamma}{dt} \right\| : I \rightarrow \mathbb{R}$,

whenever each of these quantities all exist.

Exercise 2.2: Speed is preserved by rigid motions.

↪ **Definition 2.4:** Let γ be a C^1 curve. The *arclength* of γ is defined by

$$\ell(\gamma) := \int_I \sigma(t) dt.$$

⊗ **Example 2.1:** Let $p, q \in \mathbb{E}^2$ with $d_{\mathbb{E}}(p, q) = 3$. Suppose $\gamma : [a, b] \rightarrow \mathbb{E}^2$ is a C^1 -path with $\gamma(a) = p, \gamma(b) = q$. Prove that $\ell(\gamma) \geq 3$, with equality holding iff $\gamma(I)$ is a line segment, with no change of direction.

(Hint: pick coordinates so that $p = 0$ and the x -axis passes through q to simplify computations.)

↪ **Definition 2.5** (Curve): A set $C \subset \mathbb{E}^n$ is a *curve* if it is connected, and for every $p \in C$, there exists a compact neighborhood N_p of p and a one-to-one, compact, parametrized curve $\gamma : I \rightarrow \mathbb{E}^n$ such that $\gamma(I) = C \cap N_p$.

A curve is called C^k if there exists γ as in the definition which are now required to be C^k .

I.e., a general curve is everywhere locally a compact parametrized curve.

Remark 2.1: One can relax the one-to-one/compact conditions to obtain either a global compact parametrization (which may not be one-to-one) or a parametrized curve with $I = \mathbb{R}$ with $\gamma(I) = \mathcal{C}$ and γ is periodic.

§2.1 Classification Theorem for Curves

↪ **Theorem 2.1** (Classification Theorem for Curves): Let $\mathcal{C} \subset \mathbb{E}^n$ a connected subset. Then, \mathcal{C} is a (regular) $[C^k]$ curve iff it is the image of a (regular) $[C^k]$ path $\gamma : I \rightarrow \mathbb{E}^n$ satisfying either

1. γ is one-to-one with $[C^k]$ continuous inverse
2. $I = \mathbb{R}$ and γ is periodic, and the restriction of γ to any interval I' shorter than the period is one-to-one.

If γ satisfies 1. or 2., we'll call it a *global parametrization* of \mathcal{C} .

Remark 2.2: This means we just need *one* path to describe a curve; but it may, in 2., loop back onto itself.

§2.2 Reparametrizations of Curves

↪ **Definition 2.6** (Reparametrization): Let $I, \tilde{I} \subset \mathbb{R}$ be intervals and $t : \tilde{I} \rightarrow I$ a continuous bijection (we'll call it a *change of parameters*). Then, the *reparametrization* of $\gamma : I \rightarrow \mathbb{E}^n$ using t is the composition $\tilde{\gamma} := \gamma \circ t : \tilde{I} \rightarrow \mathbb{E}^n$.

Suppose γ a regular C^k path and $t : \tilde{I} \rightarrow I$ a C^k bijection with a C^k inverse. Then $\tilde{\gamma}$ is a C^k -reparametrization of γ .

We say t is *orientation-preserving* (*-reversing*) if it is monotone increasing (decreasing).

Remark 2.3: γ also a reparametrization of $\tilde{\gamma}$ using the inverse $s := t^{-1}$.

↪ **Theorem 2.2:** Suppose $\gamma : I \rightarrow \mathbb{R}^n$ is C^1 and $\tilde{\gamma} : \tilde{I} \rightarrow \mathbb{R}^n$ a C^1 reparametrization of γ . Then $\ell(\gamma) = \ell(\tilde{\gamma})$, that is, arclength is invariant under change of parameters.

↪ **Theorem 2.3** (Arc-Length Parametrization): Let $\gamma : I \rightarrow \mathbb{E}^n$ be a regular C^k path. Then, there exists an orientation-preserving C^k reparametrization of γ , $\tilde{\gamma} : \tilde{I} \rightarrow \mathbb{E}^n$, with unit speed, i.e. $\|\dot{\tilde{\gamma}}\| \equiv 1$.

PROOF. Pick $t_0 \in I$ and define

$$s : I \rightarrow \mathbb{R}, \quad s(t) := \int_{t_0}^t \|\dot{\gamma}(r)\| dr.$$

This integral exists and is bounded, and moreover,

$$\frac{ds}{dt} = \|\dot{\gamma}(t)\| > 0,$$

since γ regular. In particular, we see that s is invertible on its image $\tilde{I} := s(I)$, and increasing. Then, $s : I \rightarrow \tilde{I}$ an orientation-preserving, C^1 bijection with $s' > 0$. By the

inverse function theorem, $t := s^{-1} : \tilde{I} \rightarrow I$ exists and has the same desired properties. Moreover,

$$t'(s) = \frac{1}{s'(t(s))} = \frac{1}{\|\dot{\gamma}(t(s))\|}.$$

Letting $\tilde{\gamma} := \gamma \circ t$, then we see that

$$\|\dot{\tilde{\gamma}}(s)\| = \|\dot{\gamma} \circ t(s) \cdot t'(s)\| = \frac{1}{\|\dot{\gamma}(t(s))\|} \|\dot{\gamma}(t(s))\| \equiv 1.$$

■

Exercise 2.3: Any two arc-length parametrizations differ by some shifting in the domain, i.e. if $\gamma_i : I_i \rightarrow \mathbb{R}^n$ are two arc-length reparametrizations of a regular path $\gamma : I \rightarrow \mathbb{R}^n$ using a change of parameters $t_i : I_i \rightarrow I$ for $i = 1, 2$, then $h := t_2^{-1} \circ t_1 : I_2 \rightarrow I_1$ is a restriction of a rigid motion of \mathbb{R} ; specifically $h' \equiv 1$.

With this, we can try to define the length of a general curve C . Suppose $C \subset \mathbb{E}^n$ a compact curve with boundary $\{p, q\}$ (so satisfies the first point of the classification theorem).

1. If C a line segment, then we just define

$$\mathcal{L}_1(C) := d_{\mathbb{E}}(p, q).$$

2. If C regular, then we define

$$\mathcal{L}_2(C) := \ell(\gamma),$$

where γ is any parametrization of C .

Exercise 2.4: This definition of \mathcal{L}_2 is well-defined, i.e. independent of choice of parametrization.

↪ **Definition 2.7** (Rectifiable): Let C be a compact curve with boundary $\{p, q\}$. An *inscribed polygon* in C is a finite increasing sequence of points $\mathcal{D} = \{p_i\}_{i=0}^N$ of points in C with endpoints $p_0 = p, p_N = q$. We write

$$L(\mathcal{D}) := \sum_{i=0}^{N-1} d_{\mathbb{E}}(p_i, p_{i+1})$$

for the length of \mathcal{D} , and

$$|\mathcal{D}| := \max_{i=0}^{N-1} d_{\mathbb{E}}(p_i, p_{i+1})$$

for the size of \mathcal{D} .

A curve C is said to be *rectifiable* if there exists a real number $\mathcal{L}_3(C) \geq 0$ such that for all sequence $\{\mathcal{D}_m\}$ of inscribed polygons in C with $|\mathcal{D}_m| \xrightarrow{m \rightarrow \infty} 0$, we have

$$\lim_{m \rightarrow \infty} L(\mathcal{D}_m) = \mathcal{L}_3(C).$$

↪ **Proposition 2.1:** A unit-speed reparametrization is essentially unique, up to a shift in the domain I .

Exercise 2.5: Compute the arc-length parametrization of $\gamma(t) := (t, t^2)$.

↪ **Lemma 2.1:** Let $\tilde{\gamma} : \tilde{I} \rightarrow \mathbb{R}^n$ be a regular C^2 path with constant speed. Then, $\ddot{\tilde{\gamma}}$ will always be orthogonal to $\dot{\tilde{\gamma}}$.

PROOF. Suppose $\|\dot{\tilde{\gamma}}\| \equiv c$. We apply the product rule for dot products, to obtain

$$\begin{aligned} 0 &= \frac{d}{dt}(c^2) = \frac{d}{dt}\|\dot{\tilde{\gamma}}\|^2 \\ &= \frac{d}{dt}\dot{\tilde{\gamma}} \cdot \dot{\tilde{\gamma}} \\ &= 2\ddot{\tilde{\gamma}} \cdot (\dot{\tilde{\gamma}}), \end{aligned}$$

which gives the proof. ■

§2.3 Curvature

Let γ be a regular C^2 -path $\gamma : I \rightarrow \mathbb{R}^n$, there exists an orientation-preserving change of parameters $t : \tilde{I} \rightarrow I$ such that $\tilde{\gamma} := \gamma \circ t : \tilde{I} \rightarrow \mathbb{R}^n$ has unit speed. Let $s := t^{-1} : I \rightarrow \tilde{I}$.

↪ **Definition 2.8** (Curvature of a parametrized curve): Define the curvature of γ as above at some time $t \in I$ to be

$$\kappa_\gamma : I \rightarrow \mathbb{R}_+, \quad \kappa_\gamma(t) := \|(\ddot{\tilde{\gamma}} \circ s)(t)\|.$$

Exercise 2.6: Show that this definition is well-defined, i.e. independent of choice of unit-speed parametrization.

↪ **Definition 2.9** (Curvature of a curve): Given a regular C^2 curve $\mathcal{C} \subset \mathbb{R}^n$, there exists (by the classification theorem) a global, regular, C^2 parametrization of \mathcal{C} , $\gamma : I \rightarrow \mathbb{R}^n$. For a point $p \in \mathcal{C}$, then, there exists some $t \in I$ such that $\gamma(t) = p$. Define, then, the curvature of \mathcal{C} at p , then, to be the curvature of γ at time t .

Exercise 2.7: Show that this definition is well-defined, i.e., independent of choice of regular global parametrization. One will need to appeal to the inverse function theorem, to show that any two such parametrizations differ by an orientation-preserving change of parameters.

Exercise 2.8: Show that curvature is preserved by rigid motions of \mathbb{R}^n , i.e. given M a rigid motion of \mathbb{R}^n and a regular C^2 curve γ , then

$$\kappa_{M \circ \gamma} = \kappa_\gamma.$$

Remark 2.4: In particular, this exercise gives the curvature is an *inherit property* of curves in \mathbb{E}^n , not just in \mathbb{R}^n .

Remark 2.5: The definition of κ_γ is a little bothersome in the sense that it requires computing an arc-length parametrization. The follow result shows how we can compute it regardless.

↪ **Proposition 2.2:**

$$\kappa_\gamma = \frac{1}{\|\dot{\gamma}\|^2} \left\| \ddot{\gamma} - \frac{\ddot{\gamma} \cdot \dot{\gamma}}{\dot{\gamma} \cdot \dot{\gamma}} \dot{\gamma} \right\| = \frac{\|\ddot{\gamma}^\perp\|}{\|\dot{\gamma}\|^2},$$

where we use the “ \perp ” notation to indicate the orthogonal complement of $\ddot{\gamma}$ with respect to $\dot{\gamma}$.

PROOF. I’ll add it later. It’s just repeated application of the chain rule and product rule. ■

Exercise 2.9: Compute the curvature of parabola $C := \{(x, y) \mid y = x^2\} \subset \mathbb{R}^2$ at any point.

↪ **Theorem 2.4:** The quantity $\frac{\|\ddot{\gamma}^\perp\|}{\|\dot{\gamma}\|^2}$ is preserved under reparametrization.

Remark 2.6: This is really more of a corollary of the previous proposition. Moreover, this implies that our definition of curvature is “correct” as a property of curves in \mathbb{E}^n rather than just \mathbb{R}^n .

↪ **Definition 2.10:** Let $\gamma : I \rightarrow \mathbb{R}^n$ a regular path. We define

- $T(t) := \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$, then *unit tangent* at time t

If $\gamma \in C^2$,

- $N(t) := \frac{\ddot{\gamma}(t)^\perp}{\|\ddot{\gamma}(t)^\perp\|}$, the *unit normal* at time t
- the *osculating plane* at time is the plane in \mathbb{R}^n that contains the point $\gamma(t)$ and is spanned by $\{\dot{\gamma}(t), \ddot{\gamma}(t)\}$ (supposing $\kappa_\gamma \neq 0$)
- the *osculating circle* at time t as the circle laying in the osculating plane of radius $\frac{1}{\kappa(t)}$ and centered at $\gamma(t) + \frac{N(t)}{\kappa(t)}$
- the *evolute* of γ is the map

$$t \in I \mapsto \gamma(t) + \frac{N(t)}{\kappa(t)} = \text{center of oscualting circle at } t$$

Remark 2.7: $\ddot{\gamma}^\perp \neq 0 \Leftrightarrow \kappa_\gamma \neq 0 \Leftrightarrow \{\dot{\gamma}, \ddot{\gamma}\}$ a linearly independent set.

Exercise 2.10: A circle of radius r , i.e. the curve defined implicitly by $\{x^2 + y^2 = r^2\}$, has curvature $\frac{1}{r}$.

This exercise shows that the osculating circle at a point on a curve has the same curvature as the curve at that point.

Exercise 2.11: Suppose $n = 2$ and a curve is given *explicitly* by $y = f(x)$ where $f : \mathbb{R} \rightarrow \mathbb{R}$ sufficiently differentiable. Compute the curvature in terms of f and its derivatives. Do the same if the curve is given *implicitly* as the set of $(x, y) \in \mathbb{R}^2$ such that $g(x, y) = 0$ where $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ sufficiently differentiable.

Fix now $n = 2$. Let $\gamma : I \rightarrow \mathbb{R}^2$ be a regular C^2 curve and fix $t \in I$. Let us assume (by changing coordinates if necessary) that $\gamma(t) = 0$ and the x -axis is parallel to $T(t)$, i.e. $T(t) = (1, 0)$. Then, we see that we may write

$$\frac{\ddot{\gamma}(t)^\perp}{\|\dot{\gamma}(t)\|^2} = \text{constant} \times (0, 1).$$

Specifically, the “constant” here is what we call the *signed curvature* of γ at time t , and is computed as :

↪ **Definition 2.11** (Signed curvature): Let γ as in the above, then the *signed curvature* is the quantity

$$\kappa_\gamma^\pm(t) = \frac{1}{\|\dot{\gamma}(t)\|^2} \frac{\ddot{\gamma}(t) \cdot \dot{\gamma}(t)^*}{\|\dot{\gamma}(t)\|},$$

where we use the notation v^* as a rotation of $v = (v_1, v_2)$ by an angle of $\frac{\pi}{2}$, counter-clockwise, i.e. $v^* = (-v_2, v_1)$.

Exercise 2.12: $\kappa_\gamma^\pm(-t) = -\kappa_\gamma^\pm(t)$.

Exercise 2.13: Suppose $\gamma(t) = (x(t), y(t))$, then show

$$\kappa_\gamma = \frac{\dot{x}\ddot{y} - \ddot{x}\dot{y}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}.$$

↪ **Definition 2.12** (Angle function): Let $\gamma : I \rightarrow \mathbb{R}^2$ be a regular C^2 curve parametrized by arc length with basepoint $s_0 \in I$. We assume wlog $s_0 = 0$ (by translating if necessary) and that $\dot{\gamma}(0) = (1, 0)$ (by changing coordinates). We define the *angle function* of γ by

$$\theta : I \rightarrow \mathbb{R}, \quad \theta(0) = 0, \theta(s) := \int_0^s \kappa_\gamma^\pm(u) \, du.$$

In particular, $\frac{d\theta}{ds} = \kappa_\gamma^\pm(s)$.

Remark 2.8: We can view $s \mapsto \dot{\gamma}(s)$ as a new C^1 -parametrized curve, in which case its arc length is given by

$$\int_0^s \|\ddot{\gamma}(u)\| du = \int_0^s \kappa_{\dot{\gamma}}(u) du.$$

So, in a sense, the θ angle function is the “signed arc-length” of $\dot{\gamma}$, i.e. it accounts for backtracking.

Moreover, since we have an arc length parametrization, we know $\dot{\gamma}$ a unit vector, hence we can view the map $t \mapsto \dot{\gamma}(t)$ as a map from I to the unit circle in \mathbb{R}^2 . Hence, θ is meant to capture the angle of this unit vector for any t , i.e. $\dot{\gamma} = (\cos, \sin) \circ \theta$.

↪ **Theorem 2.5** (Fundamental Theorem of Plane Paths): Let $s_0 \in I$ be a given base point and let $\kappa : I \rightarrow \mathbb{R}$ be a C^{k-2} function ($2 \leq k \leq \infty$). Then, for each $p \in \mathbb{R}^2$ and $\theta_0 \in \mathbb{R}$, there is a unique regular C^k path $\gamma : I \rightarrow \mathbb{R}^2$, parametrized by arc-length, such that

1. $\kappa_{\dot{\gamma}} = \kappa$,
2. $\dot{\gamma}(s_0) = (\cos(\theta_0), \sin(\theta_0))$,
3. $\gamma(s_0) = p$.

Remark 2.9: The choice of p, θ_0 just correspond to a translation, rotation (resp.) of \mathbb{R}^2 of our curve, i.e. this means our curve is uniquely determined up to rigid motion.

Remark 2.10: This essentially says that, given the curvature of a curve in the plane, we can reconstruct the curve.

PROOF. We seek to find $\gamma : I \rightarrow \mathbb{R}^2$ and $\theta : I \rightarrow \mathbb{R}$ such that

$$\frac{d\gamma}{ds} = (\cos \theta, \sin \theta), \gamma(s_0) = p$$

and

$$\frac{d\theta}{ds} = \kappa, \theta(s_0) = \theta_0.$$

By the fundamental theorem of calculus, we know

$$\theta(s) = \int_{s_0}^s \kappa(u) du + \theta_0$$

is the unique solution for $\theta(s)$ with the given properties, which in turn implies

$$\gamma(s) = \left(\int_{s_0}^s \cos(\theta(u)) du, \int_{s_0}^s \sin(\theta(u)) du \right) + p,$$

which is again unique by FTC. ■

Remark 2.11: This theorem essentially says that a curve is uniquely determined by its signed curvature. However, the same is not true if we just take the curvature. For instance, the curves given explicitly by $y = x^3$, $y = |x|^3$ have the same curvature everywhere but clearly do not described the same curves.

A more abstract manner of characterizing the angle function for a more general curve is as follows. If $\gamma : I \rightarrow \mathbb{R}^2$ a regular, C^2 curve, then the angle function $\theta : I \rightarrow I'$ where I' some other interval of \mathbb{R} , is such that

$$T = \rho \circ \theta,$$

where $\rho : I' \rightarrow \mathbb{R}^2$ is the standard parametrization of the circle given by $\rho(\theta) := (\cos(\theta), \sin(\theta))$ and T the unit tangent vector viewed as a map $I \rightarrow \mathbb{R}^2$.

Exercise 2.14: Show that the signed curvature of γ is preserved under rigid motion, hence is well-defined as a property of a curve in \mathbb{E}^n . (Note that the signed curvature is the derivative of the θ function, hence it suffices to prove this property for θ)

§2.4 3-Dimensional Space Paths

We wish to derive an analogous “fundamental” result for curves in \mathbb{R}^3 . However, we have no notion of “signed curvature” in this case. Moreover, as we’ll see, we actually need a second “intrinsic” (called *torsion*) of the curve to uniquely identify it.

Fix $\gamma : I \rightarrow \mathbb{R}^3$ regular and C^2 and with strictly positive curvature (turns out, there’s not much we can say when the curvature is 0).

Define as before

$$T := \frac{\dot{\gamma}}{\|\dot{\gamma}\|}, \quad N := \frac{\ddot{\gamma}^\perp}{\|\ddot{\gamma}^\perp\|}$$

the unit tangent and normal vectors. Remark that $T \cdot N = 0$. Since we are in \mathbb{R}^3 , there exists a unique third vector, which we denote B and call it the *binormal* such that $\{T, N, B\}$ is an orthonormal, positively oriented basis (in the sense that the matrix consisting of columns T, N, B in that order is orthogonal with determinant 1) of \mathbb{R}^3 , i.e.

$$B := T \times N.$$

The basis $\{T, N, B\} \subset \mathbb{R}^3$ is called the *Frenet frame* associated to γ .

We’ll be interested in the dynamics of this frame, i.e. how T, N, B resp. change in time. We need to additionally assume $\gamma \in C^3$ for this, so that we may take derivatives of N . We’ll also assume γ is parametrized by arc-length for convenience. We find that with these assumptions,

$$\begin{aligned} T &= \dot{\gamma} \\ \Rightarrow \dot{T} &= \ddot{\gamma} = \|\ddot{\gamma}\|N = \kappa N. \end{aligned}$$

In addition,

$$\|B\| = 1 \Rightarrow \dot{B} \cdot B = 0$$

and

$$B = T \times N \Rightarrow \dot{B} = \dot{T} \times N + T \times \dot{N} = \kappa \underbrace{N \times N}_{=0} + T \times \dot{N} \Rightarrow \dot{B} \cdot T = 0,$$

hence \dot{B} is simultaneously orthogonal to B and T , hence

$$\dot{B} = \text{const}(-N).$$

We call this constant the *torsion* τ of γ at time s , which is given by

$$\tau := -\dot{B} \cdot N.$$

Finally, to compute \dot{N} , we have that

$$\begin{aligned} \|N\| = 1 &\Rightarrow \dot{N} \cdot N = 0 \\ T \cdot N = 0 &\Rightarrow 0 = \dot{T} \cdot N + T \cdot \dot{N} = \underbrace{\kappa \|N\|^2}_{=1} + T \cdot \dot{N} \Rightarrow T \cdot \dot{N} = -\kappa \\ B \cdot N = 0 &\Rightarrow 0 = \dot{B} \cdot N + B \cdot \dot{N} = -\tau + B \cdot \dot{N} \Rightarrow B \cdot \dot{N} = \tau. \end{aligned}$$

This implies

$$\dot{N} = -\kappa T + \tau B.$$

In summary, we can succinctly write

$$\begin{pmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \quad (\text{The Frenet equations}).$$

↪ **Theorem 2.6** (Fundamental Theorem of Space Paths): Let $I \subset \mathbb{R}$ be an interval with basepoint $s_0 \in I$. Suppose $\tau : I \rightarrow \mathbb{R}$ is a C^{k-3} function and $\kappa : I \rightarrow \mathbb{R}_{>0}$ is a C^{k-2} function (where $3 \leq k \leq \infty$). Then, for any initial point $p_0 \in \mathbb{R}^3$, initial velocity $v_0 \in \mathbb{R}^3$, and initial normal direction $n_0 \in \mathbb{R}^3$ such that $\|v_0\| = \|n_0\| = 1$ and $v_0 \cdot n_0 = 1$, there is a *unique* regular C^k path $\gamma : I \rightarrow \mathbb{R}^3$ parametrized by arc length and satisfying:

1. $\kappa_\gamma = \kappa$,
2. $\tau_\gamma = \tau$,
3. $\gamma(s_0) = p_0$,
4. $\dot{\gamma}(s_0) = v_0$,
5. $\ddot{\gamma} \frac{s_0}{\|\dot{\gamma}(s_0)\|} = n_0$.

Remark 2.12: The last three requirements say that this curve is uniquely defined up to rigid motion, hence unique in \mathbb{E}^3 ; translations will simply change the initial point p_0 , and rotations will change the angles of v_0, n_0 .

PROOF. Remark that the Frenet equations are a system of (9) first order ODEs with given initial condition. The Picard-Lindelhoff theorem from ODEs says that there exist unique function $T, N, B : I \rightarrow \mathbb{R}^3$ satisfying the equations with $T(s_0) = v_0, N(s_0) = n_0, B(s_0) = v_0 \times n_0$. We need to show that these are the Frenet frame of some curve.

First, we show they are a positively oriented orthogonal basis. Indeed, remark that, using the Frenet equations,

$$\frac{d}{ds}(T \cdot N) = \kappa(N \cdot N) - \kappa(T \cdot T) + \tau(T \cdot B)$$

$$\frac{d}{ds}(T \cdot B) = \kappa(N \cdot B) - \tau(T \cdot N)$$

$$\frac{d}{ds}(N \cdot B) = -\kappa(T \cdot B) + \tau(B \cdot B) - \tau(N \cdot N)$$

$$\frac{d}{ds}(T \cdot T) = 2\kappa(T \cdot N)$$

$$\frac{d}{ds}(N \cdot N) = -2\kappa(T \cdot N) + 2\tau(N \cdot B)$$

$$\frac{d}{ds}(B \cdot B) = -2\tau(N \cdot B).$$

These are a system of ODEs for the quantities $T \cdot N, T \cdot B$, etc with initial conditions $0, 0, 0, 1, 1, 1$. However, the system can also be solved by $T \cdot N \equiv 0, T \cdot B \equiv 0$, etc, and so by uniqueness of solutions to linear ODEs, it follows that $T \cdot N = 0$, etc, which proves the orthonormality. To show positive orientation, it suffices to show that $(T \times N) \cdot B \equiv 1$. This is true at the basepoint of time by choice of initial conditions, and if we take the derivative, we find

$$\frac{d}{ds}((T \times N) \cdot B) = \kappa(N \times N) \cdot B + [(T \times (-\kappa T)) + T \times (\tau B)] \cdot B + (T \times N) \cdot (-\tau N),$$

which we see to be equal to zero by our orthonormality proof from above. Thus, $\{T, N, B\}$ is indeed a positively-oriented orthonormal basis.

Finally, we need to show that there exists a unique curve with T as its unit tangent (from which the remainder of the quantities N , etc will follow); indeed, we have

$$\gamma : I \rightarrow \mathbb{R}^3, \quad \gamma(s) = p_0 + \int_{s_0}^s T(u) \, du$$

is the unique curve with $\dot{\gamma} = T$; the fact that $\gamma \in C^k$ follows from $T \in C^{k-1}$. ■

Exercise 2.15: With the same assumptions as above, also assume $\sigma : I \rightarrow \mathbb{R}_{>0} \in C^{k-1}$. Then, there exists a unique C^k regular path $\gamma \in \mathbb{E}^3$ such that

$$\|\dot{\gamma}\| = \sigma, \quad \kappa_\gamma = \kappa, \quad \tau_\gamma = \tau.$$

We're interested in defining the torsion for more general paths in a consistent way. Let γ a regular C^3 curve in \mathbb{R}^3 with $\kappa > 0$. Let $\tilde{\gamma} : \tilde{I} \rightarrow \mathbb{R}^3$ be a arc-length reparametrization using $t : \tilde{I} \rightarrow I$, and let $s = t^{-1}$, and define

$$\tau := \tilde{\tau} \circ s,$$

where $\tilde{\tau}$ the torsion of $\tilde{\gamma}$, as defined above.

↪ **Proposition 2.3:** Let γ be as above. Then,

$$\kappa = \frac{\|\dot{\gamma} \times \ddot{\gamma}\|}{\|\dot{\gamma}\|^3}, \quad \tau = \left(\frac{\dot{\gamma} \times \ddot{\gamma}}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} \right) \cdot \ddot{\gamma}$$

PROOF. We know $\kappa = \frac{\|\ddot{\gamma}^\perp\|}{\|\dot{\gamma}\|^2}$. In \mathbb{R}^3 , $\|\dot{\gamma} \times \ddot{\gamma}\|$ is the area of the parallelogram with sides $\dot{\gamma}, \ddot{\gamma}$, or equivalently, twice the area of the triangle with base $\dot{\gamma}$ and height $\ddot{\gamma}^\perp$ (the perpendicular to the base $\dot{\gamma}$), i.e.

$$\|\dot{\gamma} \times \ddot{\gamma}\| = \|\dot{\gamma}\| \|\ddot{\gamma}^\perp\|,$$

which proves the first claim. The second claim follows from lots of careful chain rules.

■

Exercise 2.16: Is torsion preserved by reversals? i.e., if $\bar{\gamma} := \gamma \circ \bar{t}$ where $\bar{t}(t) = -t$, is $\tau_{\bar{\gamma}}(\bar{t}) = \tau_{\gamma}(t)$?

Exercise 2.17: (Twisted Cubic) Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ be given by $\gamma(t) = (t, t^2, t^3)$. Show that $\kappa(0) = 2, \tau(0) = 3$.

Exercise 2.18: (Helix) Let $\gamma(t) = (\cos(t), \sin(t), t)$. Show that $\kappa \equiv \frac{1}{2}, \tau \equiv \frac{1}{2}$.

Exercise 2.19: Find an example where $\kappa \equiv \frac{1}{2}, \tau \equiv -\frac{1}{2}$.

§2.5 Global Theorems/Properties of Plane Curves

Let \mathbb{S}^1 denote the unit circle in \mathbb{R}^2 centered at the origin, with global periodic parametrization $\rho(t) = (\cos(t), \sin(t))$. Given a C^0 curve in \mathbb{S}^1 by $g : I \rightarrow \mathbb{S}^1$, a function $\theta : I \rightarrow \mathbb{R}$ is called a *lift* of g via ρ if

1. it is C^0
2. $g = \rho \circ \theta$

↪ **Theorem 2.7:** Fix $t_0 \in \mathbb{R}, \theta_0 \in \mathbb{R}$ such that $g(t_0) = (\cos \theta_0, \sin \theta_0)$. Then, there exists a unique lift θ of g such that $\theta(t_0) = \theta_0$.

If $g : \mathbb{R} \rightarrow \mathbb{S}^1$ a periodic path with period $[a, b]$, then for any lift θ of g ,

$$|\theta(b) - \theta(a)| = 2\pi n, \quad n \in \mathbb{Z}_+,$$

where n the number of times the curve “goes around” \mathbb{S}^1 .

↪ **Theorem 2.8** (Hopf's Umlaufsatz): If $C \subset \mathbb{R}^2$ a regular closed curve periodic (with period $[a, b]$) parametrization $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$, then for any lift θ of its tangent vector T (i.e., θ is an angle function), $|\theta(b) - \theta(a)| = 2\pi$.

We say γ is *positively/ccw oriented* if $\theta(b) - \theta(a) = 2\pi$, and *negatively/cw oriented* if $\theta(b) - \theta(a) = -2\pi$.

↪ **Theorem 2.9** (Jordan Curve Theorem): Let $C \subset \mathbb{R}^2$ a regular closed curve. Then, $\mathbb{R}^2 \setminus C$ has two connected components; one bounded (“inside” of C) and one unbounded (“outside” of C).

We can then say γ is *positively oriented* if T^* points inside C , and *negatively oriented* if T^* points outside C . It turns out these different notions of orientation are equivalent.

↪ **Theorem 2.10** (Isoperimetric Inequality): Let $C \subset \mathbb{R}^2$ a regular closed curve. Let ℓ = length of C and A = area of inside of C . Then,

$$4\pi A \leq \ell^2,$$

with equality iff C is a circle.

PROOF. (Sketch of Hopf’s) ■

§3 APPENDIX

↪ **Proposition 3.1**: For $u, v \in \mathbb{R}^3$, $\|u \times v\|$ is the area of the parallelogram with side u, v .

↪ **Proposition 3.2**:

↪ **Proposition 3.3**: