MATH358 - Advanced Calculus

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§1 Differentiation

For a function $f : (a,b) \to \mathbb{R}$, f differentiable at $x_0 \in (a,b)$ if $L := \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists, and write $f'(x_0) = L$. Equivalently, $f'(x_0)$ exists and is equal to L if

$$\frac{|f(x) - f(x_0) - L(x - x_0)|}{|x - x_0|} \to 0$$

as $x \to x_0$. This characterization motivates the generalization we'll follow in the general dimensional case.

Let $\Omega \subset \mathbb{R}^n$ a connected, open set. We call such a set a *domain* to follow. Let $f : \Omega \to \mathbb{R}^m$.

↪**Definition 1.1**: f differentiable at $x_0 \in \Omega$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ (which can be viewed as a matrix) such that

$$\lim_{\substack{x\to x_0 \\ x\neq x_0}} \frac{\|f(x) - f(x_0) - L(x - x_0)\|_{\mathbb{R}^m}}{\|x - x_0\|_{\mathbb{R}^n}} = 0.$$

Equivalently, $\forall\, \varepsilon > 0$, there is a $\delta > 0$ such that if $0 < \|x - x_0\| < \delta$, then

$$\|f(x) - f(x_0) - L(x - x_0)\| \leq \varepsilon \|x - x_0\|. \qquad \dagger$$

↪**Theorem 1.1**: If L as in the previous definition exists, then it is unique.

We write then $Df(x_0) := L$.

Proof. Suppose $L_1, L_2 : \mathbb{R}^n \to \mathbb{R}^m$ are two linear maps such that † holds. Fix $\varepsilon > 0$ and let $\delta > 0$ such that † holds for both L_1, L_2 with $\frac{\varepsilon}{2}$. Then, for x such that $0 < \|x - x_0\| < \delta$, then

$$\begin{aligned}\|(L_1 - L_2)(x - x_0)\| &\leq \|f(x) - f(x_0) - L_1(x - x_0)\| + \|f(x) - f(x_0) - L_2(x - x_0)\| \\ &\leq \varepsilon \|x - x_0\|.\end{aligned}$$

Put $h = \frac{x - x_0}{\|x - x_0\|}$ which is a unit vector in $\mathbb{R}^n$. Then, this gives

$$\|(L_1 - L_2)h\| \leq \varepsilon.$$

For any vector $y \in \mathbb{R}^n$, there is a constant $\rho = \|y\|$ and appropriate h such that $y = \rho h$, and so

$$\|(L_1 - L_2)(\rho h)\| = |\rho| \|(L_1 - L_2)h\| \leq |\rho| \varepsilon,$$

by linearity, and since ε arbitrary, it must be that $L_1 = L_2$. ■

↪**Proposition 1.1**: If $f : \Omega \to \mathbb{R}^m$ is differentiable at $x_0 \in \Omega$, then f is continuous at x_0.

Proof. Let $\varepsilon = 1$ and $\delta > 0$ such that † holds. Then, for x such that $0 < \| x - x_0\| < \delta$,

$$\begin{aligned}\|f(x) - f(x_0)\| &\leq \|L(x - x_0)\| + \|f(x) - f(x_0) - L(x - x_0)\| \\ &< (\|L\| + 1)\ \|x - x_0\| =: K\ \|x - x_0\|.\end{aligned}$$

where $\|L\|$ is the "maximal value" of $L(x - x_0)$, which is finite. Let, then, $\delta' < \min\left\{\delta, \frac{\varepsilon}{K}\right\}$. Then, if x is such that $\|x - x_0\| < \delta'$, then

$$\|f(x) - f(x_0)\| < K\ \|x - x_0\| < \varepsilon,$$

proving continuity. ■

↪**Definition 1.2**: Let $f = (f_1, ..., f_m) : \Omega \to R^m$. Define, for $i = 1, ..., n, j = 1, ..., m$,

$$\frac{\partial f_j}{\partial x_i}(x_1, ..., x_n) := \lim_{\substack{h \to 0 \\ h \neq 0}} \left[\frac{f_j(x_1, ..., x_i + h, ..., x_n) - f_{j(x_1, ..., x_n)}}{h}\right],$$

the partial derivative of the jth component of f with respect to x_i.

↪**Proposition 1.2**: If f differentiable at $x_0 \in \Omega$, then $\frac{\partial f_j}{\partial x_i}(x_0)$ exists for each $i = 1, ..., n, j = 1, ..., m$, and moreover, $L = Df(x_0) = \left(\frac{\partial f_j}{\partial x_i}\right)$.

Proof. Denote the entries of $L = \left(a_{ji}\right)$. Put for arbitrary h, $x = x_0 + he_i$. Then,

$$\frac{\|f(x) - f(x_0) - L(x - x_0)\|}{\|x - x_0\|} = \left(\sum_{j=1}^{m} \left[\frac{f_j(x) - f_j(x_0)}{h} - a_{ji}\right]^2\right)^{\frac{1}{2}}.$$

This term converges to zero by assumption as $h \to 0$, and by continuity of $(\cdot)^{\frac{1}{2}}$, and the fact that the summation is over nonnegative summands, it must be that

$$\lim_{h \to 0} \frac{f_j(x_0 + he_i) - f_j(x_0)}{h} = a_{ji}$$

for each i, j. The LHS limit is simply $\frac{\partial f_j}{\partial x_i}(x_0)$, completing the proof. ■