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ODEs MATH325

Course Outline:

Based on Lectures from Winter, 2024 by Prof. Antony Humphries.

Contents

1 Introduction

1.1 Definitions

↪ **Definition 1.1: Diffferential equation**

A *diffferential equation* (DE) is an equation with derivatives. *Ordinary* DE's (ODE) will be covered in this course; other types (PDE's, SDE's, DDE's, FDE's, etc.) exist as well but won't be discussed. ODE's only have one independent variable (typically, $y = f(x)$ or $y = f(t)$).

⊛ **Example 1.1: A Trivial Example**

$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x$. Integrating both sides:

$$\int \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int 6x\,\mathrm{d}x \implies y(x) = 3x^2 + C.$$

⊛ **Example 1.2: Another One**

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = 0 \implies y = at + b.$$

↪ **Definition 1.2: Order**

The order of a differential equation is defined as the order of the highest derivative in the equation.

1.2 Initival Values

Remark 1.1. *Note the existence of arbitrary constants in the previous examples, indicating infinite solutions. We often desire unique solutions by fixing these coefficients. For first order ODEs, we simply specify a single initial condition (say, some $y(x_0) = \alpha_0$). For higher order ODEs of degree n, we can either specify $n-1$ initial conditions for $n-1$ derivatives (say, $y(x_0) = \alpha_0$, $y'(x_0) = \beta_0$), or boundary conditions (say, $y(x_0) = \alpha_0, y(x_1) = \alpha_1$) where values for the solution itself are specified.*

⊛ **Example 1.3: A Less Trivial Example**

$\frac{\mathrm{d}y}{\mathrm{d}x} = y$. We cannot simply integrate both sides as before, as we have no way to know what $\int y\,\mathrm{d}x$ (the RHS) is equal to. We can fairly easily guess that $y = e^x$ is a solution; its derivative is equal to itself, hence it does indeed solve the equation. This is not the only solution; indeed, given $y = ce^x$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ce^x = y = ce^x.$$

Luckily, we were rather limited in how many places constants could appear; this doesn't always hold.

1.3 Physical Applications

⊛ **Example 1.4: Simple Pendulum**

Let θ be the angle of a pendulum of mass m from vertical and length l. Then, we have the equation of motion

$$ml\ddot{\theta} = -mg\sin\theta \implies \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \implies \ddot{\theta} + \omega^2 \sin\theta = 0.$$

Take θ small, then, $\sin\theta \approx \theta$. Then, $\ddot{\theta} + \omega^2\theta = 0$. This is linear simple harmonic motion, and has periodic solutions; how do we know this is a valid solution to the non-linear model?

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⊛ **Example 1.5: Lorenz Equations**

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \sigma(y - x) \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= rx - y - xz \\ \frac{\mathrm{d}z}{\mathrm{d}t} &= xy - bz \end{aligned}$$

These are a famous set of equations originally derived from atmospheric modeling, known for its chaotic behavior for particular parameters. This is a nonlinear system of de's, and beyond the scope of this class (indeed, it is not solvable exactly).

1.4 Uniqueness

Given an ODE of the general form $y^{(n)} = f(t, y, y', \dots, y^{n-1})$, if we wish to determine $y^{(n)}(t_0)$ uniquely, we need to specify the initial conditions

$$y(t_0), y'(t_0), \dots, y^{(n-1)}(t_0).$$

Moreover, this not only determines uniqueness of $y^{(n)}(t_0)$, byt the uniqueness of solution y for $t \in I$ for some "interval of validity" I.

↪ **Definition 1.3: Autonomous/Nonautonomous**

An ODE of the form
$$y^{(n)} = f(y, y', \dots, y^{(n-1)})$$
is called *autonomous*; that is, if it has no explicit dependence on the independent variable. Otherwise, the system is called *nonautonomous*.

↪ **Definition 1.4: Linear/Nonlinear**

Linear ODEs of dimension n have a solution space which is a vector space of dimension n. As a result, solutions can be written as a linear combination of n basis solutions (or "fundamental set of solutions"). Solutions to nonlinear ODEs cannot be written this way (except locally).

Alternatively (but equivalently), if we can write an nth order ODE in the form
$$a_n(t)y^n(t) + \cdots a_1(t)y'(t) + a_0(t)y(t) = g(t),$$
or equivalently,
$$\sum_{i=0}^{n} a_i(t)y^i(t) = g(t), \quad \circledast$$
where each $a_i(t)$ and $g(t)$ are known functions of t, then we say that the ODE is linear. Otherwise, it is nonlinear.

⊛ **Example 1.6**

The pendulum
$$\ddot{\theta} + \omega^2 \sin\theta = 0$$
is autonomous and linear;
$$\ddot{\theta} + \omega^2 \sin\theta = 0$$
is autonomous and nonlinear, due to the $\sin\theta$ term (indeed, this is a nonlinear oscillator equation); a damped-forced oscillator
$$\ddot{\theta} + k^2\dot{\theta} + \omega^2\theta = A\sin(\mu t)$$
is nonautonomous and linear.

Remark 1.2. *Note that the following definitions apply only to linear ODEs.*

↪ **Definition 1.5: Homogeneous/Nonhomogeneous**

A linear ODE of the form $\circledast$ is *homogeneous* if $g(t) = 0$; otherwise it is *nonhomogeneous*.

↪ **Definition 1.6: Constant/Variable**

A linear ODE of the form $*$ is *constant coefficient* if $a_j(t) = \text{constant } \forall j$; if at least one a_j not constant, it is *non-constant* or *variable coefficient*.

Remark 1.3. *Note that while we define linearity of ODEs in terms of the form of* $y^{(n)} = f(t, y, \ldots)$, *this more "helpfully" relates to the form of the solution of such an ODE, which is indeed linear.*

1.5 Solutions

Given an n order ODE $y^{(n)} = f(t, y, \ldots)$, and assuming f continuous, then for $y(t)$ to be a solution, we need y to be n-times differentiable; hence, $y, \ldots, y^{(n-1)}$ must all exist and be continuous. Then, $y^{(n)}$, being a continuous function of continuous functions, is, itself, continuous.

↪ **Definition 1.7: Solution**

The function $y(t) : I \to \mathbb{R}$ is a solution to an ODE on an interval $I \subseteq \mathbb{R}$ if it is n-times differentiable on I, and satisfies the ODE on this interval.

Given an well-defined IVP with $n-1$ initial values defined at t_0, then $y(t)$ is a solution if $t_0 \in I$, y satisfies the initial values, and $y(t)$ is a solution on the interval.

↪ **Definition 1.8: Interval of Validity**

The largest I on which $y(t) : I \to \mathbb{R}$ solves an ODE is called the *interval of validity* of the problem.

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2 First Order ODEs

2.1 Separable ODEs

↪ **Definition 2.1: Separable ODE**

An ODE of the form

$$y' = P(t)Q(y)$$

is called *separable*. We solve them:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= P(t)Q(y) \\ \implies \int \frac{1}{Q(y)}\,\mathrm{d}y &= \int P(t)\,\mathrm{d}t\,. \end{aligned}$$

Finish by evaluating both sides.

⊛ **Example 2.1**

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= ty && (1) \\ &\implies \frac{1}{y}\,\mathrm{d}y = t\,\mathrm{d}t && (2) \\ &\implies \ln|y| = \frac{t^2}{2} + C && (3) \\ &\implies |y| = Ke^{\frac{t^2}{2}} \quad \text{where } K = e^C && (4) \\ &\implies y = Be^{\frac{t^2}{2}} \quad \text{where } B = \pm K = \pm e^C && (5) \end{aligned}$$

Note that we call line (3) an *implicit solution*. In this case, we could easily turn this into an explicit solution by solving for $y(t)$; this won't always be possible.

Note that it would appear, based on the definition, that $B \neq 0$ (as $e^{\cdots} \neq 0$); however, plugging $y = 0$ into (1) shows that this is indeed a solution. It is quite easy to verify that (5) is a valid solution;

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Be^{\frac{t^2}{2}}\right) = Bte^{\frac{t^2}{2}} = t \cdot y,$$

as desired; this holds $\forall\, B \in \mathbb{R}$.

Remark 2.1. *Is it valid to split the differentials like this?*

$$\begin{aligned} \frac{1}{Q(y)}\frac{\mathrm{d}y}{\mathrm{d}t} &= P(t) \\ \implies \int \frac{1}{Q(t)}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t &= \int P(t)\,\mathrm{d}t \end{aligned}$$

Let $g(y) = \frac{1}{Q}(y)$ *and* $G(y) = \int g(y)\,\mathrm{d}y$. *By the chain rule,*

$$\frac{\mathrm{d}}{\mathrm{d}t}(G(y(t))) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}}{\mathrm{d}y}G(y(t)) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot g(y(t)) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{1}{Q(y(t))}.$$

Integrating both sides with respect to time, we have

$$\begin{aligned} G(y(t)) = \int \frac{1}{Q(y(t))} \frac{\mathrm{d}y}{\mathrm{d}t} \, \mathrm{d}t &= \int P(t) \, \mathrm{d}t + C \\ \implies \int g(y) \, \mathrm{d}y &= \int P(t) \, \mathrm{d}t + C \\ \implies \int \frac{1}{Q(y)} \, \mathrm{d}y &= \int P(t) \, \mathrm{d}t + C \end{aligned}$$

This was our original expression obtaining by "splitting", hence it is indeed "valid".

⊛ **Example 2.2**

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2}{1 - y^2} \\ \implies \int (1 - y^2) \, \mathrm{d}y &= \int x^2 \, \mathrm{d}x \\ \implies y - \frac{y^3}{3} &= \frac{x^3}{x} + C \\ \implies y - \frac{1}{3}(y^3 + x^3) &= C \end{aligned}$$

Suppose we have the same ODE but now with an IVP $y(0) = 4$. Then, plugging this into our implicit solution:

$$4 - \frac{1}{3}(64 + 0) = C \implies C = 4 - \frac{64}{3} = -\frac{52}{3},$$

so our IVP solution is

$$y - \frac{1}{3}(y^3 + x^3) = -\frac{52}{3}.$$

2.2 Linear First Order ODEs

↪ **Definition 2.2: Integrating Factor**

A linear first order ODE of the form

$$\begin{aligned} a_1(t) y'(t) + a_0(t) y(t) &= g(t) \\ \implies y' + \frac{a_0}{a_1} y &= \frac{g}{a_1} \\ \implies y' + p(t) y &= q(t). \end{aligned}$$

To solve, we multiply by some integrating factor $\mu(t)$;

$$\mu(t)y'(t) + p(t)\mu(t)y(t) = \mu(t)q(t)$$

It would be quite convenient if $p(t)\mu(t) = \mu'(t)$; in this case, we'd have

$$\begin{aligned}
\mu(t)y' + \mu'(t)y &= \mu(t)q(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)y(t)) &= \mu(t)q(t) \\
\implies \mu(t)y(t) &= \int \mu(t)q(t)\,\mathrm{d}t + C \\
\implies y(t) &= \frac{1}{\mu(t)}\int \mu(t)q(t)\,\mathrm{d}t + \frac{C}{\mu(t)}
\end{aligned}$$

Now, what is $\mu(t)$? We required that

$$\begin{aligned}
\mu'(t) &= p(t)\mu \\
\frac{\mathrm{d}\mu}{\mathrm{d}t} &= p(t)\mu \\
\implies \int \frac{\mathrm{d}\mu}{\mu} = \int p(t)\,\mathrm{d}t \implies \ln|\mu| &= \int p(t)\,\mathrm{d}t \\
\implies \mu(t) &= Ke^{\int p(t)\mathrm{d}t}
\end{aligned}$$

However, note in our whole process earlier, we need only one μ; hence, for convenience, we can disregard any constants of integration and simply take

$$\boxed{\text{Integrating Factor:}\quad \mu(t) := e^{\int p(t)\mathrm{d}t}}$$

Then, our original linear ODE has general solution

$$y(t) = Ce^{-\int p(t)\mathrm{d}t} + e^{-\int p(t)\mathrm{d}t}\int e^{\int p(t)\mathrm{d}t}q(t)\,\mathrm{d}t\,.$$

⊛ **Example 2.3**

$$
\begin{aligned}
ty' + 3y - t^2 &= 0 \\
y' + \frac{3}{t}y &= t \\
&\implies \mu(t) = e^{\int \frac{3}{t}\,\mathrm{d}t} = e^{3\ln|t|} = t^3 \\
&\implies t^3y' + 3t^2y = t^4 \\
&\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(yt^3\right) = t^4 \\
&\implies yt^3 = \int t^4\,\mathrm{d}t \\
&\implies y = \frac{1}{t^3}\cdot\frac{t^5}{5} + \frac{C}{t^3} = \frac{t^2}{5} + \frac{C}{t^3}
\end{aligned}
$$

Note the division by zero issue when $t = 0$; this is not an issue with the solution method, but indeed with the ODE itself. The ODE breaks down when $t = 0$ for the same reason.

Thus, this solution is valid for $t \in (-\infty, 0) \cup (0, \infty) =: I_1 \cup I_2$; if we are given an IVP $y(t_0) = y_0$, if $t_0 < 0$, then the interval of validity is I_1, and if $t_0 > 0$, the interval of validity is I_2.

2.3 Exact Equations

↪ **Definition 2.3: Exact Equations**

A first order ODE of the form

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \iff \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{M(x,y)}{N(x,y)}
$$

is said to be *exact* if

$$
\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y) \iff M_y(x,y) = N_x(x,y).
$$

Suppose we have a solution $f(x, y(x)) = C$. Then,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(f(x,y(x))) &= 0 \\
\implies \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\implies \frac{f_x}{f_y} &= -\frac{\mathrm{d}y}{\mathrm{d}x}
\end{aligned}
$$

Now, with $f_x(x,y) = M(x,y)$ and $f_y = N(x,y)$, then $M_y(x,y) = f_{xy}(x,y)$ and $N_x = f_{yx}(x,y)$. Assuming f continuous with existing, continuous partial derivatives, then

$f_{xy} = f_{yx}$ and hence $M_y(x,y) = N_x(x,y)$. Thus, a function f such that $f_x = M$ and $f_y = N$ yields a solution to the ODE.

⊛ **Example 2.4**

$$\begin{aligned} 2xy^2\,\mathrm{d}x + 2x^2y\,\mathrm{d}y = 0 \equiv M\,\mathrm{d}x + N\,\mathrm{d}y = 0 \\ \implies M_y = 4xy, \quad \implies N_x = 4xy \\ f_x = M = 2xy^2 \implies f(x,y) = x^2y^2 + C + F(y) \\ f_y = N = 2x^2y \implies f(x,y) = x^2y^2 + C + F(x) \\ \implies f(x,y) = x^2y^2 + C = K \end{aligned}$$

We can rearrange this as an explicit solution

$$y = \frac{k}{x}$$

for some constant k.

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↪ **Theorem 2.1**

This technique works generally.

Proof. Given an exact ODE of the form $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$, we need to show that $\exists f(x,y)$ s.t. $f(x,y) = c$ solves the ODE. Let

$$f(x,y) = \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + g(y)$$

for some function $g(y)$ to be chosen such that $f_y = N$. But we have

$$\begin{aligned} N(x,y) = f_y(x,y) &= \frac{\partial}{\partial y}\left[\int_{x_0}^{x} M(s,y)\,\mathrm{d}s + g(y)\right] \\ &= g'(y) + \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s \\ &\implies g'(y) = N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\,. \end{aligned}$$

But the LHS is a function of y only, while the RHS depends explicitly on x; hence, this technique will only work if the entire expression is actually independent of x. To show this, we take the partial of the RHS with respect to x:

$$\begin{aligned}\frac{\partial}{\partial x}\left[N(x,y)-\frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right] &= N_x(x,y)-\frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s \\ &= N_x(x,y)-\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right] \\ &= N_x(x,y)-\frac{\partial}{\partial y}\left[M(x,y)\right] \\ &= N_x - M_y = 0,\end{aligned}$$

as the ODE is exact. Hence, the RHS is indeed a function of y alone. So, integrating both sides with respect to y:

$$g(y)=\int\left[N(x,y)-\frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right]\mathrm{d}y\,,$$

which gives us a $f(x,y)$ of

$$\begin{aligned} f(x,y) &= \int_{x_0}^{x} M(s,y)\,\mathrm{d}s+\int\left[N(x,y)-\frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right]\mathrm{d}y\,, \\ \implies f(x,y) &= \int_{x_0}^{x} M(s,y)\,\mathrm{d}s+\int_{y_0}^{y} N(x,t)\,\mathrm{d}t-\int_{y_0}^{y}\int_{x_0}^{x} M_y(s,t)\,\mathrm{d}s\,\mathrm{d}t \qquad \star\end{aligned}$$

which satisfies $f_x = M$ and $f_y = N$. Then, for $f(x,y)=C$, we have

$$\frac{\partial f}{\partial x}+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial f}{\partial y}=M+\frac{\mathrm{d}y}{\mathrm{d}x}N=0 \implies M\,\mathrm{d}x+N\,\mathrm{d}y=0,$$

as desired.

Note that $\star$ is evaluated over a rectangle $[x_0,x]\times[y_0,y]$, but holds for any connected domain containing (x_0,y_0) and (x,y).

Also note that, as described, $g(y)$ is not a function of x; hence, we can pick x arbitrarily. Suppose we take $x=x_0$, then

$$f(x,y)=\int_{x_0}^{x} M(s,y)\,\mathrm{d}s+\int_{y_0}^{y} N(x_0,t)\,\mathrm{d}t\,.$$

■

Remark 2.2. *We could have taken $g(x)$ and started from $f_y = N$. Then, we would have had the formula*

$$f(x,y)=\int_{y_0}^{y} N(x,t)\,\mathrm{d}t+\int_{x_0}^{x} M(s,y_0)\,\mathrm{d}y\,.$$

⊛ **Example 2.5**

$$2xy\,\mathrm{d}x+(x^2-1)\,\mathrm{d}y=0.$$

We have $M(x,y) = 2xy$ and $N(x,y) = x^2 - 1$, so $M_y = 2x = N_y$ and the ODE is exact; hence, a solution exists of the form $f(x,y) = c$ where $f_x = M$, $f_y = N$.

$$f(x,y) = \int M(x,y)\,\mathrm{d}x = \int 2xy\,\mathrm{d}x = x^2y + k_1(y)$$
$$f(x,y) = \int N(x,y)\,\mathrm{d}y = \int (x^2-1)\,\mathrm{d}y = x^2y - y + k_2(x)$$

Hence $k_1(y) = -y$ and $k_2(x) = 0$, so

$$f(x,y) = x^2y - y = y(x^2-1),$$

so solutions to the original ODE are

$$y(x^2-1) = C \implies y = \frac{C}{x^2-1}.$$

2.4 Exact ODEs Via Integrating Factors

Suppose
$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$
but $M_y \neq N_x$, that is, the ODE is not exact. Can we find an integrating factor $\mu(x,y)$ s.t.

$$[\mu(x,y)M(x,y)]\,\mathrm{d}x + [\mu(x,y)N(x,y)]\,\mathrm{d}y = 0$$

is exact? If so, such a μ must satisfy

$$\begin{aligned}\frac{\partial}{\partial y}[\mu(x,y)M(x,y)] &= \frac{\partial}{\partial x}[\mu(x,y)N(x,y)]\\ \implies \mu_y M + \mu M_y &= \mu_x N + \mu N_x\\ \implies N\mu_x - M\mu_y &= (M_y - N_x)\,\mu \quad \circledast\end{aligned}$$

This is not a generally easily soluble PDE; we will consider cases where μ is a function of only one independent variable, which greatly simplifies the expression; this could be simply $\mu(x), \mu(y)$, or even $\mu(x \cdot y)$.

Suppose $\mu = \mu(x) \implies \mu_y = 0$. Then, $\circledast$ becomes

$$N\mu' = (M_y - N_x)\mu \implies \mu' = \left(\frac{M_y - N_x}{N}\right)\mu.$$

This is valid, provided the expression $\left(\frac{M_y - N_x}{N}\right)$ is a function solely of x. In this case, this becomes a linear first order ODE, with solution

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,\mathrm{d}x}.$$

OTOH, if $\mu = \mu(y)$, we can similarly derive

$$\mu(x) = e^{\int \frac{N_x - M_y}{M} \,\mathrm{d}y},$$

with a similar stipulation on the expression $\left(\frac{N_x - M_y}{M}\right)$ being a function of y solely.

⊛ **Example 2.6**

$$xy \,\mathrm{d}x + (2x^2 + 3y^2 - 20) \,\mathrm{d}y = 0,$$

with $M(x,y) = xy \implies M_y = x$ and $N(x,y) = 2x^2 + 3y^2 - 20 \implies N_x = 4x$. We have $M_y - N_x = x - 4x = -3x$ (so the ODE is not exact). We write

$$\frac{M_y - N_x}{M} = \frac{-3x}{xy} = \frac{-3}{y},$$

which is a function solely of y; hence, can find a $\mu(y)$:

$$\mu(y) = e^{-\int \frac{M_y - N_x}{M} \mathrm{d}y} = e^{-\int -\frac{3}{y} \mathrm{d}y} = e^{3 \ln y} = y^3,$$

noting that we, as before, do not care about any integrating factors; we are seeking a single function that works. Multiplying this into our original ODE:

$$\underbrace{xy^4}_{:= \tilde{M}} \,\mathrm{d}x + \underbrace{(2x^2 + 3y^2 - 20)y^3}_{:= \tilde{N}} \,\mathrm{d}y = 0.$$

And indeed, we have

$$\tilde{M}_y = 4xy^3; \quad \tilde{N}_x = 4xy^3 \implies \tilde{M}_y = \tilde{N}_x,$$

as desired.

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