MATH475 - PDEs

Summary

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1 First-Order Equations

Definition 1 (Method of Characteristics): A *characteristic* of a PDE

$$\begin{cases} F[u] = 0, \boldsymbol{x} \in \mathbb{R}^N \\ u(\boldsymbol{x}) = \varphi(\boldsymbol{x}), \boldsymbol{x} \in \Gamma \subset \mathbb{R}^{N-1} \end{cases}$$

is a curve upon which a solution to the PDE is constant. With appropriate assumptions on the PDE and its given initial data, one can find the value of a solution $u(\boldsymbol{x})$ to F anywhere by

- Given $\boldsymbol{x}$, find the characteristic curve γ that passes through $\boldsymbol{x}$; one should take care to parametrize γ (for convenience) such that $\gamma(0)$ lies on Γ.
- "Trace back" along γ to where it hits the initial data. We have then that $u(\boldsymbol{x}) = u(\gamma(0))$.

Theorem 1 (Linear Equations): Given a linear PDE of the form

$$\begin{cases} a(x,y)u_x + b(x,y)u_y = c_1(x,y)u + c_2(x,y) \\ u(x,y) = \varphi(x,y) \text{ on } \Gamma \subset \mathbb{R} \end{cases},$$

the characteristics $\gamma(s) = (x,y,z)(s)$ of $u(x,y)$ is given by the solution to the system of ODEs

$$\begin{cases} \dot{x}(s) = a(x(s),y(s)) \\ \dot{y}(s) = b(x(s),y(s)) \\ \dot{z}(s) = c_1(x(s),y(s))z(s) + c_2(x(s),y(s)), \\ x(0) := x_0, y(0) := y_0 \\ z(0) := z_0 = u(x_0,y_0) = \varphi(x_0,y_0) \end{cases}$$

where x_0, y_0 such that $(x_0,y_0) \in \Gamma$.

Remark 1: Notice that the x, y and z equations are decoupled. Hence, one can begin by solving for $x(s)$, $y(s)$ then plugging into the ODE for $z(s)$ to finish.

Remark 2: One can pick x_0, y_0 (with caveats) for convenience, as long as the point (x_0, y_0) lies on Γ, ensuring we can find u here. For simple data like $u(x,0) = \varphi(x)$ for $x \in \mathbb{R}$, it is easiest to pick $y_0 := 0$, then letting x_0 be free; this serves as a "parametrization" of the curves; not in the sense that s is a parameter, rather a parametrization of the family of characteristics, i.e. one should end up with a family $\{\gamma\}_{x_0 \in \mathbb{R}}$.

Remark 3: In temporal equations, i.e. where y (for instance) equals t, we will often have $b(x,t) \equiv 1$; in this case, one can often reparametrize with t rather than s, since the ODE for $\dot{t}(s)$ will just result in $t(s) = s + t_0$, effectively reducing from a system of 3 to 2 equations.

Remark 4: This method extends naturally to higher-dimensions equations; a PDE on $\mathbb{R}^N$ will result in $N+1$ ODEs to solve. Note that characteristics are *still* curves in this case, *not* $N-1$ dimensional manifolds as one mihgt expect!!

Theorem 2 (Semiilinear Equations): Given a semiilinear PDE of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u),$$

where c may be nonlinear, we have characteristics given by

$$\begin{cases} \dot{x}(s) = a(\cdots) \\ \dot{y}(s) = b(\cdots) \\ \dot{z}(s) = c(\cdots) \end{cases}.$$

Theorem 3 (Quasilinear Equations): Given a quasilinear equation of the form

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u),$$

characteristics are given as in previous cases, though are ODEs are now all coupled.

Remark 5: "Unique"/classical solutions may not exist for all initial data in quasilinear equations; in particular, if the initial data $u(x,0) = g(x)$ is nondecreasing, then our characteristic curves will intersect $g(x)$ precisely once and we are all good; in general, this may not hold.

Theorem 4 (Fully Nonlinear Equations):

2 The Wave Equation

Definition 2: The (general) wave equation in $\mathbb{R}^N$ is given by

$$\{u_{tt} = c^2 \Delta u, \boldsymbol{x} \in \mathbb{R}^N$$

where $\Delta u = \sum_{i=1}^N u_{x_i x_i}$ the *Laplacian* of u and $c > 0$.

Theorem 5 (1D): In $N = 1$, the general solution to the wave equation for $x \in \mathbb{R}$ with initial data $u(x,0) = \varphi(x), u_x(x,0) = \psi(x)$ is given by *D'Alembert's formula*

$$u(x,t) = \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s.$$

Remark 6: We prove/derive this formula by

(i) Factor the wave equation $(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$

(ii) Make a change of variables $\xi = x + ct, \eta = x - ct$ in which we see $u = f(x+ct) + g(x-ct)$ for any sufficiently smooth functions f, g

(iii) Solve for f, g in terms of φ, ψ

Theorem 6 (1D, semi-infinite): In $N = 1$, the "semi-infinite equation", namely th wave equation restricted to $x \geq 0$ with boundary condition $u(0,t) = 0$ for all $t \geq 0$, has solution given by

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\varphi_{\text{odd}}(x+ct) + \varphi_{\text{odd}}(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(s)\,\mathrm{d}s \\ &= \begin{cases} \frac{1}{2}(\varphi(x+ct) + \varphi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,\mathrm{d}s & \text{if } x \geq ct \\ \frac{1}{2}(\varphi(x+ct) - \varphi(ct-x)) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)\,\mathrm{d}s & \text{if } 0 \leq x \leq ct \end{cases}, \end{aligned}$$

where $\varphi_{\text{odd}}(x) := \begin{cases} \varphi(x) & \text{if } x \geq 0 \\ -\varphi(-x) & \text{if } x < 0 \end{cases}$, etc.

Remark 7: Domain of dependence, influence are quite different in the semi-infinite case:

Theorem 7 (3D Wave Equation): The solution to the 3D wave equation on all of $\mathbb{R}^3$ is given by

$$u(\boldsymbol{x},t) = \frac{1}{4\pi c^2 t^2}\iint_{\partial B(\boldsymbol{x},ct)} \varphi(\boldsymbol{y}) + \nabla\varphi(\boldsymbol{y}) \cdot (\boldsymbol{y} - \boldsymbol{x}) + t\psi(\boldsymbol{y})\,\mathrm{d}S_{\boldsymbol{y}}.$$

3 Distributions

Definition 3: Let $C_c^\infty(\mathbb{R})$ denote the space of *test functions*, smooth (infinitely differentiable) functions with compact support. Then, a *distribution* F is an element of the dual of $C_c^\infty(\mathbb{R})$, that is, a linear functional acting on smooth functions to return real numbers.

If f a (sufficiently nice) function, we have a natural way of associating f to a functional F_f; for any test function φ, we define

$$\langle F_f, \varphi \rangle := \int_{-\infty}^{\infty} f(x)\varphi(x)\,\mathrm{d}x.$$

Definition 4 (Derivative): The *derivative* of a functional F is defined such that for any $\varphi \in C_c^\infty(\mathbb{R})$,

$$\langle F', \varphi \rangle = -\langle F, \varphi' \rangle.$$

Definition 5 (Delta Function): δ_0 is defined as the distribution such that for any test function φ,

$$\langle \delta_0, \varphi \rangle = \varphi(0).$$

Definition 6: Let f_n be a sequence of functions and F a distribution. We say $f_n \to F$ *in the sense of distributions* (itsod) if for every test function φ,

$$\langle f_n, \varphi \rangle \to \langle F, \varphi \rangle$$

as a sequence of real numbers.

Theorem 8: Let $f_n(x) := (n - n^2\ |x|)\mathbb{1}_{[-\frac{1}{n}, \frac{1}{n}]}(x)$ for $n \geq 1$. Then, $f_n \to \delta_0$ itsod.

4 Fourier Transform

Definition 7: Let $f \in L^1(\mathbb{R})$. We define for every $k \in \mathbb{R}$

$$\hat{f}(k) := \int_{-\infty}^{\infty} f(x) e^{-ikx} \,\mathrm{d}x =: \mathcal{F}\{f\}(k),$$

the *Fourier transform* of f.

Theorem 9 (Derivative of a Fourier Transform): Assume $f \in L^1(\mathbb{R})$ n-times differentiable, then for any positive integer $1 \leq \ell \leq n$,

$$\widehat{\frac{\mathrm{d}^{(\ell)} f}{\mathrm{d}x^{(\ell)}}}(k) = i^\ell k^\ell \hat{f}(k).$$

Theorem 10: Let $f, \hat{f} \in L^1$ be continuous. Then, for every $x \in \mathbb{R}$,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \,\mathrm{d}x.$$

More generally, given $g(k)$, we define the *Inverse Fourier Transform* (IFT) as

$$\check{g}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(k) e^{ikx} \,\mathrm{d}k.$$

Definition 8 (Convolution): Let f, g be integrable, then we define the *convolution*

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x - y) g(y) \,\mathrm{d}y.$$

Theorem 11 (Properties of Convolution):

- $(f * g)' = (f' * g) = (f * g')$ (supposing f or g differentiable).
- $\left(\widehat{f * g}\right)(k) = \hat{f}(k)\hat{g}(k)$

5 Diffusion Equation

Definition 9: For $\alpha > 0$, the *diffusion equation* in 1 space dimension is

$$u_t = \alpha u_{xx}, \qquad u(x, 0) = g(x), \qquad x \in \mathbb{R}, t > 0.$$

In $\mathbb{R}^N$, we have similarly

$$u_t = \alpha \Delta u_{xx}.$$

Theorem 12: The following solves the heat equation, under assumptions of integrability:

$$u(x, t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\alpha t}} g(y) \, \mathrm{d}y.$$

In particular,

$$\lim_{t \to 0^+} u(x, t) = g(x)$$

for every $x \in \mathbb{R}$.

Let $\Phi(x, t) := \frac{1}{\sqrt{4\pi\alpha}} e^{-\frac{x^2}{4\alpha t}}$, this is called the *heat kernel*. Then, notice that

$$u(x, t) = (\Phi(\cdot, t) * g)(x).$$

Theorem 13: Φ as the following properties:

(i) $\forall\, t > 0, \int_{-\infty}^{\infty} \Phi(x, t) \, \mathrm{d}x = 1$.
(ii) Φ is just the normal probability density function with mean 0 and variance $2\alpha t$.
(iii) Φ solves the heat equation itself.
(iv) As $t \to 0^+$, $\Phi \to \delta_0$ in the sense of distributions.

6 Laplace's Equation

Definition 10: We call a function *harmonic* if $\Delta u = 0$.

Given a bounded domain Ω and a function g, we call

$$\begin{cases} \Delta u = 0 \text{ on } \Omega \\ u = g \text{ on } \partial\Omega \end{cases} \qquad \text{[D]}$$

the Dirichlet problem of the Laplacian

Theorem 14 (Properties of Harmonic Functions):

(Mean Value Property) Let $\Omega \subset \mathbb{R}^3$ a domain and $u \in C^2(\Omega)$ harmonic. Let $x_0 \in \Omega$ and $r > 0$ such that $B(x_0, r) \subset \Omega$. Then,

$$u(x_0) = \frac{1}{4\pi r^2} \iint_{\partial B(x_0,r)} u(x)\, \mathrm{d}S_x.$$

This actually holds if and only if.

(Maximum Principle) Let Ω bounded and connected, $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$. If u attains its maximum in Ω, then u must be identically constant on $\overline{\Omega}$.

(Stability of the Dirichlet Problem) Let g_1, g_2 continuous on $\partial\Omega$ and let u_i solve

$$\begin{cases} \Delta u_i = 0 \text{ on } \Omega \\ u_i = g_i \text{ on } \partial\Omega \end{cases}$$

for $i = 1, 2$. Then,

$$\max_{x \in \Omega} |u_1 - u_2| \leq \max_{x \in \partial\Omega} |g_1 - g_2|.$$

(Dirichlet's Principle) Let $\Omega \subset \mathbb{R}^N$ be a bounded domain $\mathcal{A}_h := \left\{\omega \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right) : \omega = h \text{ on } \partial\Omega\right\}$ for some function h. Let

$$E[\omega] := \frac{1}{2} \int_\Omega |\nabla\omega|^2 \, \mathrm{d}x.$$

Then, $u \in \mathcal{A}_h$ minimizes E if and only if u solves the Dirichlet problem with $u = h$ on $\partial\Omega$.

Definition 11 (Fundamental Solution to the Laplacian): The *fundamental solution* to the Laplacian over $\mathbb{R}^N$ is given by

$$\Theta(x) = \begin{cases} \frac{1}{2\pi} \ln|x| & N = 2 \\ -\frac{1}{(4\pi)\ |x|} & N = 3, \\ \frac{1}{N(N-2)\omega_N\ |x|^{N-2}} & N \geq 4 \end{cases}$$

where ω_N the volume of the unit sphere in $\mathbb{R}^N$.

Theorem 15: In the sense of distributions, $\Delta\Phi = \delta_0$.

Theorem 16 (Representation Formula): Let Ω bounded and $u \in C^2\left(\overline{\Omega}\right)$ and harmonic on Ω. Then, for $x_0 \in \Omega$,

$$u(x_0) = \int_{\partial\Omega} u(x) \frac{\partial\Phi(x - x_0)}{\partial n} - \Phi(x - x_0) \frac{\partial u(x)}{\partial n} \, \mathrm{d}S_x.$$

Theorem 17: For [D], define the *Green's function* of Ω as the function $G(x, x_0)$, for $x \in \overline{\Omega}, x_0 \in \Omega$, such that, for $x \neq x_0$,

$$G(x, x_0) = \Phi(x - x_0) + H_{x_0}(x),$$

where H_{x_0} harmonic and $H_{x_0}(x) = -\Phi(x - x_0)$ for $x \in \partial\Omega$. Suppose such a G exists, then $u(x_0)$ solves [D], where for $x_0 \in \partial\Omega$,

$$u(x_0) = \int_{\partial\Omega} g(x) \frac{\partial}{\partial n} G(x, x_0).$$

Remark 8: Assuming existence, the proof follows by applying the representation formula and Green's Second identity.

Theorem 18 (Properties of Green's Function): Let G be the Green's function for some Ω. Then,

(i) G is unique
(ii) $G(x, x_0) = G(x, x_0)$ for every $x \neq x_0 \in \Omega$.

7 Fourier Series

8 Helpful Identities

Theorem 19 (Averaging Lemma): Let φ continuous on $\mathbb{R}^3$. Then, for any $x_0 \in \mathbb{R}^3$,

$$\varphi(x_0) = \lim_{r \to 0^+} \frac{3}{4\pi r^3} \int_{B(x_0, r)} \varphi(x) \, \mathrm{d}x.$$

Similar statements hold in arbitrary dimensions.

Remark 9: This is just a special case of the Lebesgue Differentiation Theorem.

Theorem 20 (Vector Field Integration by Parts): Let $\boldsymbol{u}$ be a C^1 vector field and v a C^1 function defined on some $\Omega \subseteq \mathbb{R}^3$. Then,

$$\int_\Omega \boldsymbol{u} \cdot \nabla v \, \mathrm{d}x = -\int_\Omega (\operatorname{div} \boldsymbol{u}) v \, \mathrm{d}x + \int_{\partial\Omega} (v\boldsymbol{u}) \cdot \boldsymbol{n} \, \mathrm{d}S_x.$$

Remark 10: Computed $\left(u^i v\right)_{x_i}$ for $i = 1, 2, 3$, sum over the indices, integrate, apply the divergence theorem.

Theorem 21 (Green's Identities): Let $u, v \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ for some bounded domain Ω. Then

$$1. \quad \int_\Omega v \Delta u \, \mathrm{d}x = \int_{\partial\Omega} v \frac{\partial u}{\partial n} - \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$$

$$2. \quad \int_\Omega v \Delta u - u \Delta v \, \mathrm{d}x = \int_{\partial\Omega} v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \, \mathrm{d}S_x.$$