MATH574 - Dynamical Systems

Based on lectures from Winter 2025 by Prof. Antony Humphries.
Notes by Louis Meunier

Contents

§1 Examples of Dynamical Systems

Roughly speaking, a dynamical system is a system that evolves in time, with common examples being a differential equation, in the continuous case, or a map, in the discrete case.

⊛ **Example 1.1** (The Logistic Map):

§2 Existence-Uniqueness Theory

↪**Definition 2.1** (Lipschitz): We say a function $f : \mathbb{R}^p \to \mathbb{R}^p$ is Lipschitz on $B \subseteq \mathbb{R}^p$ if there is a constant $L > 0$ such that $\|f(x) - f(y)\| \leq L \|x - y\|$ for every $x, y \in B$. We call L a "Lipschitz" constant. It is certainly not unique in general.

We say f *globally Lipschitz* if it is Lipschitz on $B = \mathbb{R}^p$, and f *locally Lipschitz* if f Lipschitz on every bounded domain $B \subseteq \mathbb{R}^p$ (note: the L will in general depend on the domain).

↪**Theorem 2.1**: Let $f : \mathbb{R}^p \to \mathbb{R}^p$ be a locally Lipschitz function. Then, there exists a unique solution to the initial value problem $\dot{u} = f(u), u(0) = u_0$ on some interval $t \in (-T_1(u_0), T_2(0))$, where $-T_1(u_0) < 0 < T_2(u_0)$ and

- either $T_2(u_0) = +\infty$ or $\|u(t)\| \to \infty$ as $t \to T_2(u_0)$, and
- either $T_1(u_0) = \infty$ or $\|u(t)\| \to -\infty$ as $t \to -T_1(u_0)$.

Heuristically, this first condition states that either our solution exists for all (forward) time after $-T_1(u_0)$, or it blows up in finite time, with a similar interpretation for the second, going backwards.

↪**Proposition 2.1**: Let $\dot{u} = f(u)$ where f locally Lipschitz. Let $B \subseteq \mathbb{R}^p$ be a bounded subset such that initial conditions $u_0, v_0 \in B$ define solutions $u(t), v(t)$ with $u(t), v(t) \in B$ for all $t \in [0, T]$. Let L be a Lipschitz constant for f on B. Then,

$$e^{-Lt} \|u_0 - v_0\| \leq \|u(t) - v(t)\| \leq e^{Lt} \|u_0 - v_0\| \qquad \forall\, t \in [0, T].$$

This provides a bound on how quickly solutions grow, decay in B.

↪**Corollary 2.1**: Let f locally Lipschitz and $u_0 \neq v_0$. Then, $u(t) \neq v(t)$ for all time such that the solutions both exist.

§3 Limit Sets and the Evolution Operator

We state definitions in this section first for ODEs, but they generalize.

↪**Definition 3.1** (Evolution Operator): Given $\dot{u} = f(u)$, the *evolution operator* is the map

$$S(t) : \mathbb{R}^p \to \mathbb{R}^p, \qquad t \geq 0$$

such that $u(t) = S(t)u_0$.

Such a map also defines a *semi-group* $\{S(t) : t \geq 0\}$ under composition, namely it is closed under repeated composition and this operator is associative.

For $B \subseteq \mathbb{R}^p$ define

$$S(t)B := \bigcup_{u \in B} S(t)u = \{u(t) = S(t)u_0 : u_0 \in B\}.$$

↪**Definition 3.2** (Forward/Positive Orbit): We define the *forward orbit* of a point u_0 as

$$\Gamma^+(u_0) := \bigcup_{t \geq 0} S(t)u_0,$$

i.e. the set of all points u_0 may "visit" as time increases.

↪**Definition 3.3** (Backwards/Negative Orbit): Similarly, define a *backwards orbit* (if one exists)

$$\Gamma^-(u_0) := \{u(t) : t \leq 0\},$$

s.t. $\forall\, t \leq s \leq 0, S(-t)u(t) = u_0$ and $S(s-t)u(t) = u(s)$.

Note that a negative orbit won't be unique in general, eg in maps, periodic points may multiple preimages.

↪**Definition 3.4** (Complete Orbit): If a negative orbit for u_0 exists, define the *complete orbit* through u_0 as

$$\Gamma(u_0) := \Gamma^+(u_0) \cup \Gamma^-(u_0).$$

Notice that if $v \in \Gamma(u_0)$, then $\Gamma(v) = \Gamma(u_0)$; namely a complete orbit through v exists.

↪**Definition 3.5** (Invariance): The set B is said to be *positively invariant* if $S(t)B \subseteq B$ for all $t \geq 0$. Similarly, B is said to be *negatively invariant* if $B \subseteq S(t)B$ for all $t \geq 0$.

↪**Definition 3.6** (ω-limit sets): A point $x \in \mathbb{R}^p$ is called an *ω-limit point* of u_0 if there exists a sequence $\{t_n\}$ with $t_n \to \infty$ such that $S(t_n)u_0 \to x$ as $n \to \infty$. The set of all such points for an initial condition u_0 is denoted $\omega(u_0)$, and called the ω-limit set of u_0.

Given a bounded set B, the ω-limit set of B is defined as

$$\omega(B) := \{x \in \mathbb{R}^p : \exists\, t_n \to \infty, y_n \in B \text{ s.t. } S(t_n)y_n \to x\}.$$

Remark 3.1: In general, $\omega(B)$ is *not* the union of ω-limit sets of points in B.

↪**Theorem 3.1**: For any $u_0 \in \mathbb{R}^p$,

$$\omega(u_0) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \{S(t)u_0\}},$$

and similarly for any bounded $B \subseteq \mathbb{R}^p$,

$$\omega(B) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B}.$$

↪**Definition 3.7** (α-limit set): A point $x \in \mathbb{R}^p$ is called an *α-limit point* for $u_0 \in \mathbb{R}^p$ if there exists a negative orbit through u_0 and a sequence $\{t_n\}$ with $t_n \to -\infty$ such that $u(t_n) \to x$. The set of all such points for u_0 is denoted $\alpha(u_0)$.

↪**Theorem 3.2**: If $\Gamma^+(u_0)$ bounded, then $\omega(u_0)$ is a non-empty, compact, invariant, connected set.

↪**Definition 3.8** (Attraction): We say a set A *attracts* B if for every $\varepsilon > 0$, there is a $t^* = t^*(\varepsilon, A, B)$ such that $S(t)B \subseteq N(A, \varepsilon)$ for every $t \geq t^*$, where $N(A, \varepsilon)$ denotes the ε-neighborhood of A.

A compact, invariant set A is called an *attractor* if it attracts an open neighborhood of itself, i.e. $\exists\, \varepsilon > 0$ such that A attracts $N(A, \varepsilon)$.

A *global attractor* is an attractor that attracts every bounded subset of $\mathbb{R}^p$.

↪**Theorem 3.3** (Continuous Gronwall Lemma): Let $z(t)$ be such that $\dot{z} \leq az + b$ for some $a \neq 0, g \in \mathbb{R}$ and $z(t) \in \mathbb{R}$. Then, $\forall\, t \geq 0$,

$$z(t) \leq e^{at}z(0) + \frac{b}{a}(e^{at} - 1).$$

↪**Theorem 3.4** (ω-limit sets as attractors): Assume $B \subseteq \mathbb{R}^p$ is a bounded, open set such that $S(t)B \subseteq \overline{B}\, \forall\, t > 0$. Then, $\omega(B) \subseteq B$, and $\omega(B)$ is an attractor, which attracts B. Furthermore,

$$\omega(B) = \bigcap_{t \geq 0} S(t)B.$$

↪**Definition 3.9** (Dissipative): A dynamical system is called *dissipative* if there exists a bounded set B such $\forall\, A$ bounded, there exists a $t^* = t^*(A) > 0$ such that $S(t)A \subseteq B \;\; \forall\, t \geq t^*$. We then call such a B an *absorbing set*.

Remark 3.2: B absorbing $\Rightarrow \omega(A) \subseteq \omega(B)$. Moreover, $\omega(B)$ attracts A for every bounded set A. I.e., $\omega(B)$ is a global attractor.

§4 Stability Theory

↪**Definition 4.1** (Stable/Unstable Manifolds): If u^* a steady state of a dynamical system, the *stable manifold* of u^* is defined as the set

$$\{u \in \mathbb{R}^p : \omega(u) = u^*\},$$

and similarly, the *unstable manifold* is defined

$$\{u \in \mathbb{R}^p : \Gamma^-(u) \,\exists \text{ and } \alpha(u) = u^*\}.$$

↪**Definition 4.2** (Lyapunov Stability): A steady state u^* is called *Lyapunov stable* if $\forall\, \varepsilon > 0$, there exists a $\delta > 0$ such that if $\|u^* - v\| < \delta$, then $\|S(t)v - u^*\| < \varepsilon$ for all time $t \geq 0$.

↪**Definition 4.3** (Quasi-Asymptotically Stable): A steady state u^* is called *Quasi-asymptotically stable* (qas) if there exists a $\delta > 0$ such that if $\|u - u^*\| < \delta$, $\lim_{t \to \infty} \|S(t)u - u^*\| = 0$.

↪**Definition 4.4** (Asymptotically Stable): A steady state u^* is called asymptotically stable if it is both Lyapunov stable and qas.

↪**Definition 4.5** (Linearization): Consider a dynamical system $\dot{u} = f(u)$, where $f(u^*) = 0$. Let $v(t) = u(t) - u^*$, then, $\dot{v} = f(u^* + v)$, and $v^* = 0$ corresponds to a fixed point. Taylor expanding $\dot{v}$, we find

$$\begin{aligned}\dot{v} &= f(u^* + v) \\ &= f(u^*) + J_f(u^*)v + \mathcal{O}\left(\|v\|^2\right) \\ &= J_f(u^*) \cdot v + \mathcal{O}\left(\|v\|^2\right),\end{aligned}$$

where $J_f(u^*)$ the Jacobian matrix of f evaluated at u^*. The linear system

$$\dot{v} = J_f(u^*)v$$

is called the *linearization* of $\dot{u} = f(u)$ at u^*.

↪**Proposition 4.1**: The general solution to the linearized system

$$\dot{v} = Jv, \qquad v(0) = v_0,$$

is

$$v(t) = e^{tJ} \cdot v_0,$$

where $e^{\cdot}$ the matrix exponential defined by the (always convergent) series

$$e^M = \sum_{j=0}^{\infty} \frac{M^j}{j!}.$$

Suppose $\dot{v} = Jv$ and J complex diagonlizable with eigenvalues $\lambda_1, ..., \lambda_n$. Then, J conjugate to the diagonal matrix Λ with diagonal entries equal to the eigenvalues, namely

$$J = P\Lambda P^{-1}.$$

It follows that

$$v(t) = Pe^{t\Lambda}P^{-1}v_0.$$

Equivalently (changing coordinates), letting $w(t) = P^{-1}v(t)$, we find

$$w(t) = e^{t\Lambda}w(0),$$

noting that now, since Λ diagonal,

$$e^{t\Lambda} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}.$$

↪**Definition 4.6** (Linear Stable, Unstable, Centre Manifolds): Supposing 0 a steady state and $J_f(0)$ complex diagonalizable, define respectively the *linear* stable, unstable, and centre manifolds:

$$E^s(0) := \{u \mid u \text{ spanned by eigenvectors with } \Re(\lambda) < 0\}$$
$$E^u(0) := \{u \mid u \text{ spanned by eigenvectors with } \Re(\lambda) > 0\}$$
$$E^c(0) := \{u \mid u \text{ spanned by eigenvectors with } \Re(\lambda) = 0\}.$$

Notice that if $u_0 \in E^s(0)$, then the corresponding solution with initial condition u_0, $u(t)$, converges to 0 as $t \to \infty$, with similar conditions for $u_0 \in E^u(0)$.

↪**Definition 4.7** (Hyperbolic): A steady state u^* is called *hyperbolic* if $J_f(u^*)$ has no eigenvalues with $\Re(\lambda) = 0$, i.e. $\dim(E^c(u^*)) = 0$.

↪**Theorem 4.1**: If u^* a hyperbolic steady state of $\dot{u} = f(u)$, and all of the eigenvalues of $J_f(u^*)$ have strictly negative real part, then u^* is asymptotically stable.

↪**Theorem 4.2**: If u^* a steady state and $J_f(u^*)$ has a steady state with eigenvalue having real part strictly positive real part, then u^* unstable (namely not Lyapunov stable).

Remark 4.1: These theorems describe cases in which the linearization is correct in predicting the nonlinear behaviour.

Remark 4.2: The second theorem can only guarantee non-Lyapunov stability because linearization is a local process - quasi-asymptotic stability is "more global", and not picked up by the linearization necessarily.

↪**Theorem 4.3** (Hartman-Grobman Theorem): If f continuously differentiable and $\dot{u} = f(u)$ has a hyperbolic steady state u^*, then there exists an open ball $B(u^*, \delta) \subseteq \mathbb{R}^p$, an open set $0 \in N$ and a homeomorphism

$$H : B(u^*, \delta) \to N$$

such that while $u(t) \in B(u^*, \delta)$ a solution to $\dot{u} = f(u)$, then $v(t) = H(u(t))$ a solution of $\dot{v} = J_f(u^*)v$.

↪**Definition 4.8** (Stable, Unstable Manifold): The *stable, unstable* manifolds of a steady state u^* are defined

$$W^s(u^*) := \{u \in \mathbb{R}^p \mid S(t)u \to u^* \text{ as } t \to \infty\}$$
$$W^u(u^*) := \{u \in \mathbb{R}^p \mid \Gamma^-(u) \,\exists \text{ and } S(t)u \to u^* \text{ as } t \to -\infty\}.$$

§5 Delay Differential Equations

A delay differential equation (DDE) is generally speaking an ODE that depends on the state of the system in the past. We'll focus on DDEs of the form

$$\dot{u}(t) = f(u(t), u(t-\tau)),$$

where $u \in \mathbb{R}^p, f : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$, and $\tau > 0$ a fixed time delay.

The "canonical" first example of a DDE is $\dot{u}(t) = u(t-\tau)$ for $t \geq 0$. Notice that for any time $t \in [0, \tau]$, then, $\dot{u}(t)$ depends on u for times that are not given by the DDE directly. In short, then, we need to supply not just an initial value to the DDE, but a whole initial data, namely $u(t) = \varphi(t)$ for $t \in [-\tau, 0]$.

Suppose for now we take $\varphi \equiv 1$, so we wish to solve the DDE with initial data

$$\begin{cases} \dot{u}(t) = u(t-\tau) & t > 0 \\ u(t) = 1 & -\tau \leq t \leq 0 \end{cases}.$$