

# MATH454 - Analysis 3

Measure spaces; Integration.

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# §1 SIGMA ALGEBRAS AND MEASURES

## §1.1 A Review of Riemann Integration

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $[a, b] \subset \mathbb{R}$ . Define a **partition** of  $[a, b]$  as the set

$$\text{part}([a, b]) := \{a =: x_0 < x_1 < \dots < x_N =: b\}.$$

We can then define the upper and lower Riemann integrals of  $f$  over the region  $[a, b]$  as

$$\begin{aligned} \text{upper:} \quad \int_a^b f(x) \, dx &:= \inf_{\text{part}([a, b])} \left\{ \sum_{i=1}^N \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \right\} \\ \text{lower:} \quad \int_a^b f(x) \, dx &:= \sup_{\text{part}([a, b])} \left\{ \sum_{i=1}^N \inf_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \right\}. \end{aligned}$$

We then say  $f$  **Riemann integrable** if these two quantities are equal, and denote this value by  $\int_a^b f(x) \, dx$ .

Many “nice-enough” (continuous, monotonic, etc.) functions are Riemann integrable, but many that we would like to be able to “integrate” are simply not, for instance Dirichlet’s function  $x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \cap [a, b] \\ 0 & x \in \mathbb{Q}^c \cap [a, b] \end{cases}$ . Hence, we need a more general notion of integration.

## §1.2 Sigma Algebras

↪ **Definition 1.1** (Sigma algebra): Let  $X$  be a *space* (a nonempty set) and  $\mathcal{F}$  a collection of subsets of  $X$ .  $\mathcal{F}$  a *sigma algebra* or simply  $\sigma$ -algebra of  $X$  if the following hold:

1.  $X \in \mathcal{F}$
2.  $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$  (closed under complement)
3.  $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$  (closed under countable unions)

↪ **Proposition 1.1:**

4.  $\emptyset \in \mathcal{F}$
5.  $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
6.  $A_1, \dots, A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
7.  $A, B \in \mathcal{F} \Rightarrow A \setminus B, B \setminus A \in \mathcal{F}$

⊗ **Example 1.1:** The “largest” sigma algebra of a set  $X$  is the power set  $2^X$ , the smallest the trivial  $\{\emptyset, X\}$ .

Given a set  $A \subset X$ , the set  $\mathcal{F}_A := \{\emptyset, X, A, A^c\}$  is a sigma algebra; given two disjoint sets  $A, B \subset X$ , then  $\mathcal{F}_{A,B} := \{\emptyset, X, A, A^c, B, B^c, A \cup B, A^c \cap B^c\}$  a sigma algebra.

↪ **Definition 1.2** (Generating a sigma algebra): Let  $X$  be a nonempty set, and  $\mathcal{C}$  a collection of subsets of  $X$ . Then, the  $\sigma$ -algebra *generated* by  $\mathcal{C}$ , denoted  $\sigma(\mathcal{C})$ , is such that

1.  $\sigma(\mathcal{C})$  a sigma algebra with  $\mathcal{C} \subseteq \sigma(\mathcal{C})$
2. if  $\mathcal{F}'$  a sigma algebra with  $\mathcal{C} \subseteq \mathcal{F}'$ , then  $\mathcal{F}' \supseteq \sigma(\mathcal{C})$

Namely,  $\sigma(\mathcal{C})$  is the smallest sigma algebra “containing” (as a subset)  $\mathcal{C}$ .

↪ **Proposition 1.2:**

1.  $\sigma(\mathcal{C}) = \bigcap \{ \mathcal{F} : \mathcal{F} \text{ a sigma algebra containing } \mathcal{C} \}$
2. if  $\mathcal{C}$  itself a sigma algebra, then  $\sigma(\mathcal{C}) = \mathcal{C}$
3. if  $\mathcal{C}_1, \mathcal{C}_2$  are two collections of subsets of  $X$  such that  $\mathcal{C}_1 \subseteq \mathcal{C}_2$ , then  $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_2)$

↪ **Definition 1.3** (The Borel sigma-algebra): The *Borel  $\sigma$ -algebra*, denoted  $\mathfrak{B}_{\mathbb{R}}$ , on the real line is given by

$$\mathfrak{B}_{\mathbb{R}} := \sigma(\{\text{open subsets of } \mathbb{R}\}).$$

We call sets in  $\mathfrak{B}_{\mathbb{R}}$  *Borel sets*.

↪ **Proposition 1.3:**  $\mathfrak{B}_{\mathbb{R}}$  is also generated by the sets

- $\{(a, b) : a < b \in \mathbb{R}\}$
- $\{(a, b] : a < b \in \mathbb{R}\}$
- $\{[a, b] : a < b \in \mathbb{R}\}$
- $\{[a, b) : a < b \in \mathbb{R}\} \oplus$
- $\{(-\infty, c) : c \in \mathbb{R}\}$
- $\{(-\infty, c] : c \in \mathbb{R}\}$
- etc.

PROOF. We prove just  $\oplus$ . It suffices to show that the generating sets of each  $\sigma$ -algebra is contained in the other  $\sigma$ -algebra. Let  $a < b \in \mathbb{R}$ . Then,

$$(a, b) = \bigcup_{n=1}^{\infty} \underbrace{\left[ a + \frac{1}{n}, b \right)}_{\in \oplus} \in \sigma(\{[a, b)\}) \Rightarrow \mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[a, b)\}).$$

Conversely,

$$[a, b) = \bigcap_{n=1}^{\infty} \left( a - \frac{1}{n}, b \right) \in \mathfrak{B}_{\mathbb{R}}.$$

■

↪ **Proposition 1.4:** All intervals (open, closed, half open, half closed, finite, etc) are Borel sets; any set obtained from countable set operations of intervals are Borel; all singletons are Borel; any finite and countable sets are Borel.

### §1.3 Measures

↪ **Definition 1.4** (Measurable Space): Let  $X$  be a space and  $\mathcal{F}$  a  $\sigma$ -algebra. We call the tuple  $(X, \mathcal{F})$  a *measurable space*.

↪ **Definition 1.5** (Measure): Let  $(X, \mathcal{F})$  be a measurable space. A *measure* is a function  $\mu : \mathcal{F} \rightarrow [0, \infty]$  satisfying

- (i)  $\mu(\emptyset) = 0$ ;
- (ii) if  $\{A_n\} \subseteq \mathcal{F}$  a sequence of (pairwise) disjoint sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

i.e.  $\mu$  is *countably additive*. We further call  $\mu$

- *finite* if  $\mu(X) < \infty$ ,
- a *probability measure* if  $\mu(X) = 1$ ,
- $\sigma$ -finite if  $\exists \{A_n\} \subseteq \mathcal{F}$  such that  $X = \bigcup_{n=1}^{\infty} A_n$  with  $\mu(A_n) < \infty \forall n \geq 1$ ,

and call the triple  $(X, \mathcal{F}, \mu)$  a *measure space*.

⊕ **Example 1.2:** The measure on  $\mathfrak{B}_{\mathbb{R}}$  given by

$$A \mapsto \begin{cases} |A| & \text{if } A \text{ finite} \\ \infty & \text{else} \end{cases}$$

is called the *counting measure*.

Fix  $x_0 \in \mathbb{R}$ , then the measure on  $\mathfrak{B}_{\mathbb{R}}$  given by

$$A \mapsto \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{else} \end{cases}$$

is called the *point mass at  $x_0$* .

↪ **Theorem 1.1** (Properties of Measures): Fix a measure space  $(X, \mathcal{F}, \mu)$ . The following properties hold:

1. (finite additivity) For any sequence  $\{A_n\}_{n=1}^N \subseteq \mathcal{F}$  of disjoint sets,

$$\mu\left(\bigcup_{n=1}^N A_n\right) = \sum_{n=1}^N \mu(A_n).$$

2. (monotonicity) For any  $A \subseteq B \in \mathcal{F}$ , then  $\mu(A) \leq \mu(B)$ .
3. (countable/finite subadditivity) For any sequence  $\{A_n\} \subseteq \mathcal{F}$  (**not** necessarily disjoint),

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

an analogous statement holding for a finite collection of sets  $A_1, \dots, A_N$ .

4. (continuity from below) For  $\{A_n\} \subseteq \mathcal{F}$  such that  $A_n \subseteq A_{n+1} \forall n \geq 1$  (in which case we say  $\{A_n\}$  “increasing” and write  $A_n \uparrow$ ) we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

5. (continuity from above) For  $\{A_n\} \subseteq \mathcal{F}, A_n \supseteq A_{n+1} \forall n \geq 1$  (we write  $A_n \downarrow$ ) we have that if  $\mu(A_1) < \infty$ ,

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

**Remark 1.1:** In 4., note that since  $A_n$  increasing, that the union  $\bigcup_{n=1}^{\infty} A_n \supseteq A_m$  for any arbitrarily large  $m$ ; indeed, one could logically right  $\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} A_n$ . This this notation, then, 4. simply states that we may interchange limit and measure. A similar argument can be viewed for 5. (how?).

**Remark 1.2:** The finiteness condition in 5. may be slightly modified such as to state that  $\mu(A_n) < \infty$  for some  $n$ ; remark why this would suffice to ensure the entire rest of the sequence has finite measure.

PROOF.

1. Extend  $A_1, \dots, A_N$  to an infinite sequence by  $A_n := \emptyset$  for  $n > N$ . Then this simply follows from countable additivity and  $\mu(\emptyset) = 0$ .
2. We may write  $B = A \cup (B \setminus A)$ ; this is a disjoint union of sets. By finite additivity, then,

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A),$$

since the measure is positive.

3. We prove only for a countable union; use the technique from 1. to extend to finite. We first “disjointify” the sequence such that we can use the countable additivity

axiom. Let  $B_1 = A_1, B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$  for  $n \geq 2$ . Remark then that  $\{B_n\} \subseteq \mathcal{F}$  is a disjoint sequence of sets, and that  $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ . By countable additivity and subadditivity,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

4. We again “disjointify” the sequence  $\{A_n\}$ . Put  $B_1 = A_1, B_n = A_n \setminus A_{n-1}$  for all  $n \geq 2$  (remark that this is equivalent to the construction from the previous proof because the sets are increasing). Then, again,  $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$ , and in particular, for all  $N \geq 1, \bigcup_{n=1}^N B_n = A_N$ . Then

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \mu(B_n) \\ &= \lim_{N \rightarrow \infty} \mu\left(\bigcup_{n=1}^N B_n\right) = \lim_{N \rightarrow \infty} \mu(A_N). \end{aligned}$$

5. We yet again disjointify, backwards (in a way) from the previous case. Put  $B_n = A_1 \setminus A_n$  for all  $n \geq 1$ . Then,  $\{B_n\} \subseteq \mathcal{F}$ ,  $B_n$  increasing, and  $\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \bigcap_{n=1}^{\infty} A_n$ . Then, by continuity from below,

$$\mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n \rightarrow \infty} \mu(B_n) = \lim_{n \rightarrow \infty} \mu(A_1 \setminus A_n)$$

and also

$$\begin{aligned} \mu(A_1) &= \mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) + \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \\ &= \mu(A_1 \setminus A_n) + \mu(A_n), \end{aligned}$$

and combining these two equalities yields the desired result. ■

## §1.4 Constructing the Lebesgue Measure on $\mathbb{R}$

↪ **Definition 1.6** (Lebesgue outer measure): For all  $A \subseteq \mathbb{R}$ , define

$$m^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n, I_n \text{ open intervals} \right\},$$

called the *Lebesgue outer measure* of  $A$  (where  $\ell(I)$  is the length of interval  $I$ , i.e. the absolute value of the difference of its endpoints, if finite, or  $\infty$  if not).

↪ **Proposition 1.5:** The following properties of  $m^*$  hold:

1.  $m^*(A) \geq 0$  for all  $A \subseteq \mathbb{R}$ , and  $m^*(\emptyset) = 0$ .
2. (monotonicity) For  $A \subseteq B$ ,  $m^*(A) \leq m^*(B)$ .
3. (countable subadditivity) For  $\{A_n\}, A_n \subseteq \mathbb{R}$ ,  $m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n)$ .<sup>1</sup>
4. If  $I \subseteq \mathbb{R}$  an interval, then  $m^*(I) = \ell(I)$ .
5.  $m^*$  is translation invariant; for any  $A \subseteq \mathbb{R}, x \in \mathbb{R}$ ,  $m^*(A) = m^*(A + x)$  where  $A + x := \{a + x : a \in A\}$ .
6. For all  $A \subseteq \mathbb{R}$ ,  $m^*(A) = \inf\{m^*(B) : A \subseteq B \subseteq \mathbb{R}, B \text{ open}\}$ .
7. If  $A = A_1 \cup A_2 \subseteq \mathbb{R}$  with  $d(A_1, A_2) > 0$ ,<sup>2</sup> then  $m^*(A_1) + m^*(A_2) = m^*(A)$ .
8. If  $A = \bigcup_{k=1}^{\infty} J_k$  where  $J_k$ 's are "almost disjoint intervals" (i.e. share at most endpoints), then  $m^*(A) = \sum_{k=1}^{\infty} m^*(J_k) = \sum_{k=1}^{\infty} \ell(J_k)$ .

PROOF.

3. If  $m^*(A_n) = \infty$ , for any  $n$ , we are done, so assume wlog  $m^*(A_n) < \infty$  for all  $n$ . Then, for each  $n$  and  $\varepsilon > 0$ , one can choose open intervals  $\{I_{n,i}\}_{i \geq 1}$  such that  $A_n \subseteq \bigcup_{i=1}^{\infty} I_{n,i}$  and  $\sum_{i=1}^{\infty} \ell(I_{n,i}) \leq m^*(A_n) + \frac{\varepsilon}{2^n}$ . Hence

$$\begin{aligned} \bigcup_{n=1}^{\infty} A_n &\subseteq \bigcup_{n=1, i=1}^{\infty} I_{n,i} \\ \Rightarrow m^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \sum_{n,i=1}^{\infty} \ell(I_{n,i}) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{n,i}) \leq \sum_{n=1}^{\infty} \left(m^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon, \end{aligned}$$

and as  $\varepsilon$  arbitrary, the statement follows.

4. We prove first for  $I = [a, b]$ . For any  $\varepsilon > 0$ , set  $I_1 = (a - \varepsilon, b + \varepsilon)$ ; then  $I \subseteq I_1$  so  $m^*(I) \leq \ell(I_1) = (b - 1) + 2\varepsilon$  hence  $m^*(I) \leq b - a = \ell(I)$ . Conversely, let  $\{I_n\}$  be any open-interval converging of  $I$  (wlog, each of finite length; else the statement holds trivially). Since  $I$  compact, it can be covered by finitely many of the  $I_n$ 's, say  $\{I_n\}_{n=1}^N$ , denoting  $I_n = (a_n, b_n)$  (with relabelling, etc). Moreover, we can pick the  $a_n, b_n$ 's such that  $a_1 < a, b_N > b$ , and generally  $a_n < b_{n-1} \forall 2 \leq n \leq N$ . Then,

$$\begin{aligned} \sum_{n=1}^{\infty} \ell(I_n) &\geq \sum_{n=1}^N \ell(I_n) = b_1 - a_1 + \sum_{n=2}^N (b_n - a_n) \\ &\geq b_1 - a_1 + \sum_{n=2}^N (b_n - b_{n-1}) \\ &= b_N - a_1 \geq b - 1 = \ell(I), \end{aligned}$$

hence since the cover was arbitrary,  $m^*(A) \geq \ell(I)$ , and equality holds.

Now, suppose  $I$  finite, with endpoints  $a < b$ . Then for any  $\frac{b-a}{2} > \varepsilon > 0$ , then

$$[a + \varepsilon, b - \varepsilon] \subseteq I \subseteq [a - \varepsilon, b + \varepsilon],$$

<sup>1</sup>More generally, any set function on  $2^{\mathbb{R}}$  that satisfies 1., 2., and 3. is called an *outer measure*.

<sup>2</sup>Remark: this is a stronger requirement than disjointness!

hence by monotonicity and the previous part of this proof

$$m^*([a + \varepsilon, b - \varepsilon]) = b - a - 2\varepsilon \leq m^*(I) \leq b - a + 2\varepsilon = m^*([a - \varepsilon, b + \varepsilon]),$$

from which it follows that  $m^*(I) = b - a = \ell(I)$ .

Finally, suppose  $I$  infinite. Then,  $\forall M \geq 0, \exists$  closed, finite interval  $I_M$  with  $I_M \subseteq I$  and  $\ell(I_M) \geq M$ . Hence,  $m^*(I) \geq m^*(I_M) \geq M$  and thus as  $M$  arbitrary it must be that  $m^*(I) = \infty = \ell(I)$ .

6. Denote  $\tilde{m}(A) := \inf\{m^*(B) : A \subseteq B \subseteq \mathbb{R}, B \text{ open}\}$ . For any  $A \subseteq B \subseteq \mathbb{R}$  with  $B$  open, monotonicity gives that  $m^*(A) \leq m^*(B)$ , hence  $m^*(A) \leq \tilde{m}(A)$ . Conversely, assuming wlog  $m^*(A) < \infty$  (else holds trivially), then for all  $\varepsilon > 0$ , there exists  $\{I_n\}$  such that  $A \subseteq \bigcup_{n=1}^{\infty} I_n$  with  $\sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon$ . Setting  $B := \bigcup_{n=1}^{\infty} I_n$ , we have that  $A \subseteq B$  and  $m^*(B) = m^*(\bigcup I_n) \leq$  (by finite subadditivity)  $\sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon$  hence  $m^*(B) \leq m^*(A)$  for all  $B$ . Thus  $m^*(A) \geq \tilde{m}(A)$  and equality holds.

7. Put  $\delta := d(A_1, A_2) > 0$ . Clearly  $m^*(A) \leq m^*(A_1) + m^*(A_2)$  by finite subadditivity. wlog,  $m^*(A) < \infty$  (and hence  $m^*(A_i) < \infty, i = 1, 2$ ) (else holds trivially). Then  $\forall \varepsilon > 0, \exists \{I_n\} : A \subseteq \bigcup I_n$  and  $\sum \ell(I_n) \leq m^*(A) + \varepsilon$ . Then, for all  $n$ , we consider a “refinement” of  $I_n$ ; namely, let  $\{I_{n,i}\}_{i \geq 1}$  such that  $I_n \subseteq \bigcup_i I_{n,i}$  and  $\ell(I_{n,i}) < \delta$  and  $\sum_i \ell(I_{n,i}) \leq \ell(I_n) + \frac{\varepsilon}{2^n}$ . Relabel  $\{I_{n,i} : n, i \geq 1\} \rightsquigarrow \{J_m : m \geq 1\}$  (both are countable). Then,  $\{J_m\}$  defines an open-interval cover of  $A$ , and since  $\ell(J_m) < \delta$  for each  $m$ ,  $J_m$  intersects at most one  $A_i$ . For each  $m$  and  $p = 1, 2$ , put

$$M_p := \{m : J_m \cap A_p \neq \emptyset\},$$

noting that  $M_1 \cap M_2 = \emptyset$ . Then  $\{J_m : m \in M_p\}$  is an open covering of  $A_p$ , and so

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\leq \sum_{m \in M_1} \ell(J_m) + \sum_{m \in M_2} \ell(J_m) \\ &\leq \sum_{m=1}^{\infty} \ell(J_m) = \sum_{n,i=1}^{\infty} \ell(I_{n,i}) \\ &\leq \sum_n \left( \ell(I_n) + \frac{\varepsilon}{2^n} \right) \\ &= \sum_n \ell(I_n) + \varepsilon \\ &\leq m^*(A) + 2\varepsilon, \end{aligned}$$

and hence equality follows.

8. If  $\ell(J_k) = \infty$  for some  $k$ , then since  $J_k \subseteq A$ , subadditivity gives us that  $m^*(J_k) \leq m^*(A)$  and so  $m^*(A) = \infty = \sum_{k=1}^{\infty} \ell(J_k)$  (since if any  $J_k$  infinite, the sum of the lengths of all of them will also be infinite).

Suppose then  $\ell(J_k) < \infty$  for all  $k$ . Fix  $\varepsilon > 0$ . Then for all  $k \geq 1$ , choose  $I_k \subseteq J_k$  such that  $\ell(J_k) \leq \ell(I_k) + \frac{\varepsilon}{2^k}$ . For any  $N \geq 1$ , we can choose a subset  $\{I_1, \dots, I_N\}$  of intervals such that all are disjoint, with strictly positive distance between them, and so



$$\begin{aligned}
\bigcup_{k=1}^N I_k &\subseteq \bigcup_{k=1}^N I_k \subseteq A \\
\Rightarrow m^*(A) &\geq m^*\left(\bigcup_{k=1}^N I_k\right) \geq \sum_{k=1}^N \ell(I_k) \\
&\geq \sum_{k=1}^N \left(\ell(J_k) - \frac{\varepsilon}{2^k}\right) \\
&\geq \sum_{k=1}^N \ell(J_k) - \varepsilon \\
\Rightarrow m^*(A) &\geq \sum_{k=1}^{\infty} \ell(J_k),
\end{aligned}$$

the second inequality following from finite subadditivity. The converse of the final inequality holds trivially. ■

## §1.5 Lebesgue-Measurable Sets

↪ **Definition 1.7:**  $A \subseteq \mathbb{R}$  is  $m^*$ -measurable if  $\forall B \subseteq \mathbb{R}$ ,

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c).$$

**Remark 1.3:** By subadditivity,  $\leq$  always holds in the definition above.

↪ **Theorem 1.2** (Carathéodary's Theorem): Let

$$\mathcal{M} := \{A \subseteq \mathbb{R} : A \text{ } m^* \text{-measurable}\}.$$

Then,  $\mathcal{M}$  is a  $\sigma$ -algebra of subsets of  $\mathbb{R}$ .

Define  $m : \mathcal{M} \rightarrow [0, \infty]$ ,  $m(A) = m^*(A)$ . Then,  $m$  is a measure on  $\mathcal{M}$ , called the *Lebesgue measure* on  $\mathbb{R}$ . We call sets in  $\mathcal{M}$  *Lebesgue-measurable* or simply *measurable* (if clear from context) accordingly. We call  $(\mathbb{R}, \mathcal{M}, m)$  the *Lebesgue measure space*.

**PROOF.** The first two  $\sigma$ -algebra axioms are easy. We have for any  $B \subseteq \mathbb{R}$  that

$$m^*(B \cap \mathbb{R}) + m^*(B \cap \mathbb{R}^c) = m^*(B) + m^*(B \cap \emptyset) = m^*(B)$$

so  $\mathbb{R} \in \mathcal{M}$ . Further,  $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$  by the symmetry of the requirement for sets to be in  $\mathcal{M}$ .

The final axiom takes more work. We show first  $\mathcal{M}$  closed under finite unions; by induction it suffices to show for 2 sets. Let  $A_1, A_2 \in \mathcal{M}$ . Then, for all  $B \subseteq \mathbb{R}$ ,

$$\begin{aligned}
m^*(B) &= m^*(B \cap A_1) + m^*(B \cap A_1^c) \\
&= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*(B \cap A_1^c \cap A_2^c) \\
&= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*(B \cap (A_1 \cup A_2)^c)
\end{aligned}$$

Note that  $(B \cap A_1) \cup (B \cap A_1^c \cap A_2) = B \cap (A_1 \cup A_2)$ , hence by subadditivity,

$$m^*(B) \geq m^*(B \cap (A_1 \cup A_2)) + m^*(B \cap (A_1 \cup A_2)^c),$$

and since the other direction of the inequality comes for free, we conclude  $A_1 \cup A_2 \in \mathcal{M}$ .

Let now  $\{A_n\} \subseteq \mathcal{M}$ . We “disjointify”  $\{A_n\}$ ; put  $B_1 := A_1$ ,  $B_n := \frac{A_n}{\bigcup_{i=1}^{n-1} A_i}$ ,  $n \geq 2$ , noting  $\bigcup_n A_n = \bigcup_n B_n$ , and each  $B_n \in \mathcal{M}$ , as each is but a finite number of set operations applied to the  $A_n$ ’s, and thus in  $\mathcal{M}$  as demonstrated above. Put  $E_n := \bigcup_{i=1}^n B_i$ , noting again  $E_n \in \mathcal{M}$ . Then, for all  $B \subseteq \mathbb{R}$ ,

$$\begin{aligned} m^*(B) &= m^*\left(\underbrace{B \cap E_n}_{\text{chop up } B_n}\right) + m^*\left(\underbrace{B \cap E_n^c}_{E_n \subseteq \bigcup B_n \Rightarrow E_n^c \supseteq (\bigcup B_n)^c}\right) \\ &\geq m^*\left(B \cap \underbrace{E_n \cap B_n}_{=B_n}\right) + m^*\left(B \cap \underbrace{E_n \cap B_n^c}_{=E_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\ &\geq m^*(B \cap B_n) + m^*\left(\underbrace{B \cap E_{n-1}}_{\text{chop up } B_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\ &\geq m^*(B \cap B_n) + m^*(B \cap E_{n-1} \cap B_{n-1}) \\ &\quad + m^*(B \cap E_{n-1} \cap B_{n-1}^c) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right). \end{aligned}$$

Notice that the last line is essentially the second applied to  $B_{n-1}$ ; hence, we have a repeating (essentially, “descending”) pattern in this manner, which we repeat until  $n \rightarrow 1$ . We have, thus, that

$$m^*(B) \geq \sum_{i=1}^n [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right),$$

so taking  $n \rightarrow \infty$ ,

$$\begin{aligned} m^*(B) &\geq \sum_{i=1}^{\infty} [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\ &\geq m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right). \end{aligned}$$

As usual, the inverse inequality comes for free, and thus we can conclude  $\bigcup_{n=1}^{\infty} B_n$  also  $m^*$ -measurable, and thus so is  $\bigcup_{n=1}^{\infty} A_n$ . This proves  $\mathcal{M}$  a  $\sigma$ -algebra.

We show now  $m$  a measure. By previous propositions, we have that  $m \geq 0$  and  $m(\emptyset) = 0$  (since  $m = m^*|_{\mathcal{M}}$ ), so it remains to prove countable subadditivity.

Let  $\{A_n\} \subseteq \mathcal{M}$ -disjoint. Following precisely the same argument as above, used to prove that  $\mathcal{M}$  closed under countable unions, shows that for any  $n \geq 1$

$$m\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n m(A_i),$$

that is, finite additivity holds, and thus by subadditivity

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq m\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n m(A_i),$$

and so taking the limit of  $n \rightarrow \infty$ , we have

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} m(A_i),$$

with the converse inequality coming for free. Thus,  $m$  indeed a measure on  $\mathcal{M}$ . ■

↪ **Proposition 1.6:**  $\mathcal{M}, m$  translation invariant; for all  $A \in \mathcal{M}, x \in \mathbb{R}, x + A = \{x + a : a \in A\} \in \mathcal{M}$  and  $m(A) = m(A + x)$ .

**Remark 1.4:** We would like this to hold, heuristically, since if we shift sets on the real line, we should expect their length to remain constant.

PROOF. For all  $B \subseteq \mathbb{R}$ , we have (since  $m^*$  translation invariant)

$$\begin{aligned} m^*(B) &= m^*(B - x) = m^*\left(\underbrace{(B - x) \cap A}_{=B \cap (A+x)}\right) + m^*\left(\underbrace{(B - x) \cap A^c}_{=B \cap (A^c+x)=B \cap (A+x)^c}\right) \\ &= m^*(B \cap (A + x)) + m^*(B \cap (A + x)^c), \end{aligned}$$

thus  $A + x \in \mathcal{M}$ , and since  $m^*$  translation invariant, it follows that  $m$  is. ■

↪ **Theorem 1.3:**  $\forall a, b \in \mathbb{R}$  with  $a < b$ ,  $(a, b) \in \mathcal{M}$ , and  $m((a, b)) = b - a$ .

**Remark 1.5:** Again, we'd like this to hold, heuristically, since we would like the measure of an interval to simply be its length; we'd moreover like to be able to measure intervals, i.e. have intervals be contained in  $\mathcal{M}$ .

↪ **Corollary 1.1:**  $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M}$

PROOF.  $\mathfrak{B}_{\mathbb{R}}$  is generated by open intervals of the form  $(a, b)$ . All such intervals are in  $\mathcal{M}$  by the previous theorem, and hence the proof. ■

## §1.6 Properties of the Lebesgue Measure

↪ **Proposition 1.7** (Regularity Assumptions on  $m$ ): For all  $A \in \mathcal{M}$ , the following hold.

- For all  $\varepsilon > 0$ ,  $\exists G$  open such that  $A \subseteq G$  and  $m(G \setminus A) < \varepsilon$ .
- For all  $\varepsilon > 0$ ,  $\exists F$ -closed such that  $F \subseteq A$  and  $m(A \setminus F) \leq \varepsilon$ .
- $m(A) = \inf\{m(G) : G \text{ open}, G \supseteq A\}$ .
- $m(A) = \sup\{m(K) : K \text{ compact}, K \subseteq A\}$ .
- If  $m(A) < \infty$ , then for all  $\varepsilon > 0$ ,  $\exists K \subseteq A$  compact, such that  $m(A \setminus K) < \varepsilon$ .
- If  $m(A) < \infty$ , then for all  $\varepsilon \geq 0$ ,  $\exists$  finite collection of open intervals  $I_1, \dots, I_N$  such that  $m\left(A \Delta \left(\bigcup_{n=1}^N I_n\right)\right) \leq \varepsilon$ .

↪ **Proposition 1.8** (Completeness of  $m$ ):  $(\mathbb{R}, \mathcal{M}, m)$  is *complete*, in the sense that for all  $A \subseteq \mathbb{R}$ , if  $\exists B \in \mathcal{M}$  such that  $A \subseteq B$  and  $m(B) = 0$ , then  $A \in \mathcal{M}$  and  $m(A) = 0$ .

Equivalently, any subset of a null set is again a null set.

**Remark 1.6:** In general,  $A \in \mathcal{F}, B \subseteq A \not\Rightarrow B \in \mathcal{F}$ .

↪ **Proposition 1.9:** Up to rescaling,  $m$  is the unique, nontrivial measure on  $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$  that is finite on compact sets and is translation invariant, i.e. if  $\mu$  another such measure on  $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$  with  $\mu = c \cdot m$  for  $c > 0$ , then  $\mu = m$ .

**Remark 1.7:** Such a  $c$  is simply  $c = \mu((0, 1))$ .

To prove this proposition, we first introduce some helpful tooling:

↪ **Theorem 1.4** (Dynkin's  $\pi$ -d): Given a space  $X$ , let  $\mathcal{C}$  be a collection of subsets of  $X$ .  $\mathcal{C}$  is called a  $\pi$ -system if  $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$  (that is, it is closed under finite intersections).

Let  $\mathcal{F} = \sigma(\mathcal{C})$ , and suppose  $\mu_1, \mu_2$  are two finite measures on  $(X, \mathcal{F})$  such that  $\mu_1(X) = \mu_2(X)$  and  $\mu_1 = \mu_2$  when restricted to  $\mathcal{C}$ . Then,  $\mu_1 = \mu_2$  on all of  $\mathcal{F}$ .

↪ **Proposition 1.10:**  $\{\emptyset\} \cup \{(a, b) : a < b \in \mathbb{R}\}$  a  $\pi$ -system.

↪ **Proposition 1.11:** If  $\mu$  a measure on  $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$  such that for all intervals  $I$ ,  $\mu(I) = \ell(I)$ , then  $\mu = m$ .

PROOF. Consider for all  $n \geq 1$   $\mu|_{\mathfrak{B}_{[-n, n]}}$ . Clearly,  $\mu([-n, n]) = m([-n, n]) = 2n$ , and for all  $a, b \in \mathbb{R}$ ,  $\mu((a, b) \cap [-n, n]) = \ell((a, b) \cap [-n, n]) = m((a, b) \cap [-n, n])$ . Thus, by the previous theorem,  $\mu$  must match  $m$  on all of  $\mathfrak{B}_{[-n, n]}$ .

Let now  $A \in \mathfrak{B}_{\mathbb{R}}$ . Let  $A_n := A \cap [-n, n] \in \mathfrak{B}_{[-n, n]}$ . By continuity of  $m$  from below,

$$\begin{aligned}\mu(A) &= \lim_{n \rightarrow \infty} \mu(A_n) \\ &= \lim_{n \rightarrow \infty} m(A_n) \\ &= m(A),\end{aligned}$$

hence  $\mu = m$ . ■

↪ **Proposition 1.12:** If  $\mu$  a measure on  $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$  assigning finite values to compact sets and is translation invariant, then  $\mu = cm$  for some  $c > 0$ .

**Remark 1.8:** This proposition is also tacitly stating that  $\mathfrak{B}_{\mathbb{R}}$  translation invariant; this needs to be shown.

↪ **Lemma 1.1:**  $\mathfrak{B}_{\mathbb{R}}$  translation invariant; for any  $A \in \mathfrak{B}_{\mathbb{R}}, x \in \mathbb{R}, A + x \in \mathfrak{B}_{\mathbb{R}}$ .

PROOF. We employ the “good set strategy”; fix some  $x \in \mathbb{R}$  and let

$$\Sigma := \{B \in \mathfrak{B}_{\mathbb{R}} : B + x \in \mathfrak{B}_{\mathbb{R}}\}.$$

One can check that  $\Sigma$  a  $\sigma$ -algebra, and so  $\Sigma \subseteq \mathfrak{B}_{\mathbb{R}}$ . But in addition, its easy to see that  $\{(a, b) : a < b \in \mathbb{R}\} \subseteq \Sigma$ , since a translated interval is just another interval, and since these sets generate  $\mathfrak{B}_{\mathbb{R}}$ , it must be further that  $\mathfrak{B}_{\mathbb{R}} \subseteq \Sigma$ , completing the proof. ■

PROOF. (of the proposition) Let  $c = \mu((0, 1])$ , noting that  $c > 0$  (why? Consider what would happen if  $c = 0$ ).

This implies that  $\forall n \geq 1, \mu\left(\left(0, \frac{1}{n}\right]\right) = \frac{c}{n}$  (obtained by “chopping up”  $(0, 1]$  into  $n$  disjoint intervals); from here we can draw many further conclusions:

$$\forall m = 1, \dots, n-1, \mu\left(\left(0, \frac{m}{n}\right]\right) = \frac{m}{n}c$$

$$\Rightarrow \forall q \in \mathbb{Q} \cap (0, 1], \mu((0, q]) = qc$$

$$\Rightarrow \forall q \in \mathbb{Q}^+, \mu((0, q]) = q \cdot c \text{ (translate)}$$

$$\Rightarrow \forall a \in \mathbb{R}, \mu((a, a+q]) = q \cdot c$$

$$\Rightarrow \forall \text{ intervals } I, \mu(I) = c \cdot \ell(I) \text{ (continuity)}$$

$$\Rightarrow \forall n \geq 1, a, b \in \mathbb{R}, \mu((a, b) \cap [-n, n]) = c \cdot \ell((a, b) \cap [-n, n]) = c \cdot m((a, b) \cap [-n, n]),$$

but then,  $\mu = c \cdot m$  on  $\mathfrak{B}_{\mathbb{R}[-n, n]}$ , and by appealing again the Dynkin's,  $\mu = c \cdot m$  on all of  $\mathfrak{B}_{\mathbb{R}}$ . ■

↪ **Proposition 1.13 (Scaling):**  $m$  has the *scaling property* that  $\forall A \in \mathcal{M}, c \in \mathbb{R}, c \cdot A = \{cx : x \in A\} \in \mathcal{M}$ , and  $m(c \cdot A) = |c| m(A)$ .

PROOF. Assume  $c \neq 0$ . Given  $A \subseteq \mathbb{R}$ , remark that  $\{I_n\}$  an open interval cover of  $A$  iff  $\{cI_n\}$  and open interval cover of  $cA$ , and  $\ell(cI_n) = |c| \ell(I_n)$ , and thus  $m^*(cA) = |c| m^*(A)$ .

Now, suppose  $A \in \mathcal{M}$ . Then, we have for any  $B \subseteq \mathbb{R}$ ,

$$\begin{aligned} m^*(B) &= |c| m^*\left(\frac{1}{c}B\right) = |c| m^*\left(\frac{1}{c}B \cap A\right) + |c| m^*\left(\frac{1}{c}B \cap A^c\right) \\ &= m^*(B \cap cA) + m^*(B \cap (cA)^c), \end{aligned}$$

so  $cA \in \mathcal{M}$ . ■

### §1.7 Relationship between $\mathfrak{B}_{\mathbb{R}}$ and $\mathcal{M}$

↪ **Definition 1.8:** Given  $(X, \mathcal{F}, \mu)$ , consider the following collection of subsets of  $X$ ,

$$\mathcal{N} := \{B \subseteq X : \exists A \in \mathcal{F} \text{ s.t. } \mu(A) = 0, B \subseteq A\}.$$

Put  $\overline{\mathcal{F}} := \sigma(\mathcal{F} \cup \mathcal{N})$ ; this is called the *completion* of  $\mathcal{F}$  with respect to  $\mu$ .

↪ **Proposition 1.14:**  $\overline{\mathcal{F}} = \{F \subseteq X : \exists E, G \in \mathcal{F} \text{ s.t. } \exists E \subseteq F \subseteq G \text{ and } m(G \setminus E) = 0\}$ .

PROOF. Put  $\mathcal{G}$  the set on the right; one can check  $\mathcal{G}$  a  $\sigma$ -algebra. Since  $\mathcal{F} \subseteq \mathcal{G}$  and  $\mathcal{N} \subseteq \mathcal{G}$ , we have  $\overline{\mathcal{F}} \subseteq \mathcal{G}$ .

Conversely, for any  $F \in \mathcal{G}$ , we have  $E, G \in \mathcal{F}$  such that  $E \subseteq F \subseteq G$  with  $m(G \setminus E) = 0$ . We can rewrite

$$F = \underbrace{E}_{\in \mathcal{F}} \cup \underbrace{(F \setminus E)}_{\substack{\subseteq G \setminus E \\ \Rightarrow \mu(F \setminus E) = 0 \\ \Rightarrow G \setminus E \in \mathcal{N}}},$$

hence  $F \in \mathcal{F} \cup \mathcal{N}$  and thus in  $\overline{\mathcal{F}}$ , and equality holds. ■

↪ **Definition 1.9:** Given  $(X, \mathcal{F}, \mu)$ ,  $\mu$  can be *extended* to  $\overline{\mathcal{F}}$  by, for each  $F \in \overline{\mathcal{F}}$  with  $E \subseteq F \subseteq G$  s.t.  $\mu(G \setminus E) = 0$ , put

$$\mu(F) = \mu(E) = \mu(G).$$

We call then  $(X, \overline{\mathcal{F}}, \mu)$  a *complete measure space*.

**Remark 1.9:** It isn't obvious that this is well defined a priori; in particular, the  $E, G$  sets are certainly not guaranteed to be unique in general, so one must check that this definition is valid regardless of choice of "sandwich sets".

↪ **Theorem 1.5:**  $(\mathbb{R}, \mathcal{M}, m)$  is the completion of  $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, m)$ .

PROOF. Given  $A \in \mathcal{M}$ , then  $\forall n \geq 1, \exists G_n$ -open with  $A \subseteq G_n$  s.t.  $m^*(G_n \setminus A) \leq \frac{1}{n}$  and  $\exists F_n$ -closed with  $F_n \subseteq A$  s.t.  $m^*(A \setminus F_n) \leq \frac{1}{n}$ .

Put  $C := \bigcap_{n=1}^{\infty} G_n, B := \bigcap_{n=1}^{\infty} F_n$ , remarking that  $C, B \in \mathfrak{B}_{\mathbb{R}}, B \subseteq A \subseteq C$ , and moreover

$$\begin{aligned} m(C \setminus A) &\leq \frac{1}{n}, m(A \setminus B) \leq \frac{1}{n} \\ \Rightarrow m(C \setminus B) &= m(C \setminus A) + m(A \setminus B) \leq \frac{2}{n}, \end{aligned}$$

but  $n$  can be arbitrarily large, hence  $m(C \setminus B) = 0$ ; in short, given a measurable set, we can “sandwich it” arbitrarily closely with Borel sets. Thus,  $A \in \mathfrak{B}_{\mathbb{R}} \Rightarrow \mathcal{M} \subseteq \mathfrak{B}_{\mathbb{R}}$ . But recall that  $\mathcal{M}$  complete, so  $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M} \Rightarrow \overline{\mathfrak{B}_{\mathbb{R}}} \subseteq \overline{\mathcal{M}} = \mathcal{M}$ , and thus  $\overline{\mathfrak{B}_{\mathbb{R}}} = \mathcal{M}$  indeed.

Heuristically, this means that any measurable set is “different” from a Borel set by at most a null set. ■

## §1.8 Some Special Sets

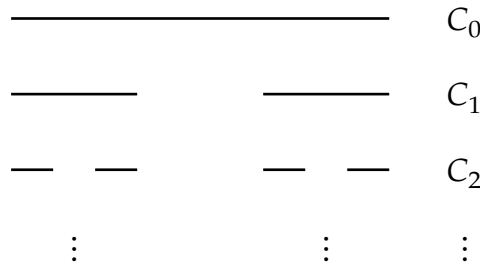
### 1.8.1 Uncountable Null Set?

Remark that for any countable set  $A \in \mathcal{M}, m(A) = 0$ ; indeed, one may write  $A = \bigcup_{n=1}^{\infty} \{a_n\}$  for singleton sets  $\{a_n\}$ , and so

$$m(A) = \sum_{n=1}^{\infty} m(a_n) = 0.$$

One naturally asks the opposite question, does there exist a measurable, *uncountable* set with measure 0? We construct a particular one here, the Cantor set,  $C$ .

This requires an “inductive” construction. Define  $C_0 = [0, 1]$ , and define  $C_k$  to be  $C_{k-1}$  after removing the middle third from each of its disjoint components. For instance  $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ , then  $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$ , and so on. This may be clearest graphically:



Remark that the  $C_n \downarrow$ . Put finally

$$C := \bigcap_{n=1}^{\infty} C_n.$$

↪ **Proposition 1.15:** The following hold for the Cantor set  $C$ :

1.  $C$  is closed (and thus  $C \in \mathfrak{B}_{\mathbb{R}}$ );
2.  $m(C) = 0$ ;
3.  $C$  is uncountable.

PROOF.

1. For each  $n$ ,  $C_n$  is the countable (indeed, finite) union of  $2^n$ -many disjoint, closed intervals, hence each  $C_n$  closed.  $C$  is thus a countable intersection of closed sets, and is thus itself closed.
2. For each  $n$ , each of the  $2^n$  disjoint closed intervals in  $C_n$  has length  $\frac{1}{3^n}$ , hence

$$m(C_n) = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$

Since  $\{C_n\} \downarrow$ , by continuity of  $m$  we have

$$m(C) = \lim_{n \rightarrow \infty} m(C_n) = \lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = 0.$$

3. This part is a little trickier. Notice that for any  $x \in [0, 1]$ , we can define a sequence  $(a_n)$  where each  $a_n \in \{0, 1, 2\}$ , and such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n};$$

in particular, this is just the base-3 representation of  $x$ , which we denote  $(x)_3 = (a_1 a_2 \dots)$ .

I claim now that

$$C = \{x \in [0, 1] : (x)_3 \text{ has no 1's}\}.$$

Indeed, at each stage  $n$  of the construction of the Cantor set, we get rid of the segment of the real line that would correspond to the  $a_n = 1$ . One should note that  $(x)_3$  not necessarily unique; for instance  $\left(\frac{1}{3}\right)_3 = (1, 0, 0, \dots) = (0, 2, 2, \dots)$ , but if we specifically consider all  $x$  such that there *exists* a base three representation with no 1's, i.e. like  $\frac{1}{3}$ , then  $C$  indeed captures all the desired numbers.

Thus, we have that

$$\text{card}(C) = \text{card}(\{(a_n) : a_n = 0, 2\}).$$

Define now the function

$$f : C \rightarrow [0, 1], \quad x \mapsto \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot \frac{1}{2^n}, \text{ where } (x)_3 = (a_n)$$

i.e., we “squish” the base-3 representation into a base-2 representation of a number.

This is surjective; for any  $y \in [0, 1]$ ,  $(b_n) := (y)_2$  contains only 0's and 1's, hence  $(2b_n)$



contains only 0's and 1's, so let  $x$  be the number such that  $(x)_3 = (2b_n)$ . This necessarily exists, indeed, we simply take our definitions backwards:

$$x := \sum_{n=1}^{\infty} \frac{2b_n}{3^n},$$

which maps to  $y$  under  $f$  and is contained in  $C$ . Hence,  $\text{card}(C) \geq \text{card}([0, 1])$ ; but  $[0, 1]$  uncountable, and thus so is  $C$ . ■

We can naturally extend the function  $f$  used here to map the entire interval  $[0, 1] \rightarrow [0, 1]$  as follows

$$f(x) := \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot \frac{1}{2^n} & \text{if } x \in C, (x)_3 = (a_n) \\ f(a) & \text{if } x \notin C \text{ then } x \in (a, b) \text{ s.t. } (a, b) \text{ removed from } [0, 1] \end{cases}.$$

This function is often called the *Devil's Staircase* or *Cantor-Lebesgue function*.

↪ **Proposition 1.16:**

1.  $f(0) = 0, f(1) = 1, f \equiv \frac{1}{2}$  on  $(\frac{1}{3}, \frac{2}{3}), f \equiv \frac{1}{4}$  on  $(\frac{1}{9}, \frac{2}{9})$
2.  $f : [0, 1] \rightarrow [0, 1]$  a surjection
3.  $f$  is nondecreasing
4.  $f$  is continuous

PROOF. 1., 2., clear from construction.

For 3., let  $x_1 < x_2 \in C$ , and suppose  $(x_1)_3 = (a_n), (x_2)_3 = (b_n)$ . Then, since  $x_1 < x_2$ , it must be that  $a_n, b_n$  can only be equal up to some finite  $N$ ; then the next  $0 = a_{N+1} < b_{N+1} = 2$ . Hence, it follows that the “modified binary expansion” that arises from  $f$  gives directly that  $f(x_1) \leq f(x_2)$ .

For 4.,  $f$  is clearly continuous on  $[0, 1] - C$ , since it is piecewise-constant here. Also,  $f$  is “one-sided continuous” at each of the “boundary points”  $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \dots$ . If  $x \in C$ , for any  $n \geq 1$ , there must be  $x_n, x_n'$  such that  $x_n < x < x_n'$  (if  $x = 0$ , only need  $x_n'$ , if  $x = 1$ , only need  $x_n$ ) and  $f(x_n') - f(x_n) \leq \frac{1}{2^n}$ . Then,  $f$  is continuous at  $x$  by monotonicity of  $f$ . ■

### 1.8.2 Non-Measurable Sets?

We've shown then that there is indeed an uncountable set of measure 0. Another question we may ask ourselves is, is there a  $A \subseteq \mathbb{R}$  that is non-measurable? The answer to this turns out to be yes, but the construction requires invoking the axiom of choice:

**Axiom 1** (Of Choice): If  $\Sigma$  a collection of nonempty sets, then  $\exists$  a function

$$S : \Sigma \rightarrow \bigcup_{A \in \Sigma} A,$$

such that  $A \in \sigma, S(A) \in A$ . Such a function is called a *selection function*, and  $S(A)$  a *representative* of  $A$ .

We construct now a non-measurable set, assuming the above. Consider  $[0, 1]$ , and define an equivalence relation  $\sim$  on  $[0, 1]$  by

$$a \sim b \Leftrightarrow a - b \in \mathbb{Q}.$$

Its easy to check that this is indeed an equivalence relation. Denote by  $E_a$  the equivalence class containing  $a$ , and set  $\Sigma = \{E_a : a \in [0, 1]\}$ . Note that for any  $E_a \in \Sigma, E_a \neq \emptyset$ .

Invoking the axiom of choice, we can select exactly one element  $S_a$  from  $E_a$  for each  $E_a \in \Sigma$ . Set

$$N := \{S_a : S_a \text{ is a representative of } E_a, E_a \in \Sigma\}.$$

**Proposition 1.17:**  $N$ , called a *Vitali set*, is non-measurable.

PROOF. Assume towards a contradiction that  $N$  indeed measurable,  $N \in \mathcal{M}$ . Consider  $[-1, 1] \cap \mathbb{Q}$ ; this is countable, so we can enumerate it  $\{q_k\}, k \geq 1$ . For each  $k$ , put

$$N_k := N + q_k.$$

By the assumption of measurability and translation invariance of  $m$ , it must be that each  $N_k$  measurable and has the same measure as  $N$ .

We claim each  $N_k$  disjoint. Assume not, then  $\exists k \neq \ell$  (i.e.  $q_k \neq q_\ell$ ) and  $S_a, S_b \in N$  such that  $S_a + q_k = S_b + q_\ell$ . But then  $S_a - S_b = q_\ell - q_k \in \mathbb{Q}$ , hence  $S_a \sim S_b$ . But we constructed  $N$  to have only one representative from each equivalence class, hence it must be that  $S_a = S_b$ , and so  $S_a + q_k = S_a + q_\ell \Rightarrow q_k = q_\ell$ , contradicting the assumed distinctness of the  $q$ 's; hence, the  $N_k$ 's indeed disjoint.

We claim next that  $[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k$ . Let  $x \in [0, 1]$ . Then,  $x \sim S_a$  for some unique  $S_a \in N$  and so  $x - S_a \in \mathbb{Q}$ . But also,  $x, S_a \in [0, 1]$ , hence  $x - S_a \in [-1, 1]$  (moreover,  $x - S_a \in [-1, 1] \cap \mathbb{Q}$ ) and there must exist a  $k$  such that  $x - S_a = q_k$ , since the  $q_k$ 's enumerate the entire  $[-1, 1] \cap \mathbb{Q}$ . Thus,  $x \in N_k$  by the construction of the  $N_k$ 's. Thus,  $[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k$  indeed.

On the other hand,  $\bigcup_{k=1}^{\infty} N_k \subseteq [-1, 2]$  and so we have the "bound"

$$[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k \subseteq [-1, 2].$$

Taking the measure of all sides then, we have the bound

$$1 \leq \mu\left(\bigcup_{n=1}^{\infty} N_k\right) \leq 3.$$

Invoking the disjointness of the  $N_k$ 's, we can also use countable additivity to write

$$\mu\left(\bigcup_{n=1}^{\infty} N_k\right) = \sum_{k=1}^{\infty} m(N_k) = \sum_{k=1}^{\infty} m(N),$$

but this final line is a sequence of positive, constant real numbers; hence, it is impossible for it to be within 1 and 3, and we have a contradiction. Hence,  $N$  indeed not measurable.

Remark that this proof also shows that  $m^*(N_k) > 0$  so  $m^*(N) > 0$  (given the interval bound on  $N$  we've found). ■

↪ **Proposition 1.18:** For every  $A \in \mathcal{M}$  such that  $m(A) > 0$ , there exists  $B \subseteq A$  such that  $B$  is non-measurable.

PROOF. Assume otherwise, that there is a  $A \in \mathcal{M}$  with  $m(A) > 0$  such that any subset  $B$  of  $A$  is also measurable.

Remark that  $A \subseteq \bigcup_{n \in \mathbb{Z}} A \cap [n, n+1]$ . Then, there exists an  $n$  such that  $m(A \cap [n, n+1]) > 0$  and thus, translating  $A' := A \cap [n, n+1] - n$ ,  $m(A') > 0$ , noting that  $A' \subseteq [0, 1]$ . Now, for any  $B' \subseteq A'$ ,  $B' + n \subseteq A$ . By assumption, then  $B' + n$  must be measurable so  $B'$  measurable.

In summary, then, we have  $A' \subseteq [0, 1]$  with  $m(A') > 0$  such that (by assumption)  $B'$  measurable for all  $B' \subseteq A'$ .

Let  $N, \{q_k\}, N_k$  be as in the previous proof. Set

$$A_k' := A' \cap N_k, k \geq 1.$$

Then,  $A_k'$  disjoint, and

$$A' = [0, 1] \cap A' \subseteq \bigcup_{k=1}^{\infty} (N_k \cap A') = \bigcup_{k=1}^{\infty} A_k'.$$

Since  $m(A') > 0$ , there exists a  $k$  such that  $m(A_k') > 0$ . Set, for this  $k$ ,

$$L := \{\ell \geq 1 : q_\ell + q_k \in [-1, 1]\}.$$

This set is again countably infinite. We translate, obtaining a disjoint sequence of sets  $\{q_\ell + A_k' : \ell \in L\}$ ; since  $q_\ell + q_k \in [-1, 1] \cap \mathbb{Q}$ , then  $q_\ell + q_k = q_m$  for some unique  $m$ , and so  $q_\ell + A_k' = q_\ell + A' \cap (N + q_k) \subseteq N_m$ . Hence, we have on the one hand that by countable additivity

$$\bigcup_{\ell \in L} (q_\ell + A_k') \subseteq [-1, 2] \Rightarrow \sum_{\ell \in L} m(q_\ell + A_k') \leq 3,$$

and so it must be that  $m(q_\ell + A_k') = m(A_k') = 0$  (else the series couldn't be finite), contradicting the finiteness assumption on  $m(A_k')$ . ■

### 1.8.3 Non-Borel Measurable Set?

We may ask, is there  $A \in \mathcal{M}$  such that  $A \notin \mathfrak{B}_{\mathbb{R}}$ ?

Let  $f : [0, 1] \rightarrow [0, 1]$  be the Cantor-Lebesgue function, and put  $g(x) = f(x) + x$ ; note that  $g$  is continuous and strictly increasing, and is defined  $g : [0, 1] \rightarrow [0, 2]$ . Remark that  $g$  bijective; the strictly increasing gives injective, and moreover  $g(0) = 0, g(1) = 2$  hence by intermediate value theorem it is surjective. Hence,  $g^{-1} : [0, 2] \rightarrow [0, 1]$  exists, and is also continuous, so in short  $g$  is a homeomorphism; it maps open to open, closed to closed. In particular, if  $A \in \mathfrak{B}_{\mathbb{R}}$ , then  $g(A) \in \mathfrak{B}_{\mathbb{R}}$ .

Recall that if  $(a, b)$  an open interval that gets removed from the construction of  $C$ , then  $f$  is constant and so  $g$  will map  $(a, b)$  to another open interval of the same length  $b - a$ . Thus,

$$m(g([0, 1] \setminus C)) = m([0, 1] \setminus C) = 1.$$

Hence,  $m(g(C)) = 2 - 1 = 1 > 0$ , since  $g(C \cup [0, 1] \setminus C) = [0, 2]$ . Hence, there exists a  $B \subseteq g(C)$  such that  $B \notin \mathcal{M}$ , as per the previous proposition.

Let  $A := g^{-1}(B)$ ; then  $A \subseteq g^{-1}(g(C)) = C$ . Since  $m(C) = 0$ ,  $A \in \mathcal{M}$  and  $m(A) = 0$ . But,  $A \notin \mathfrak{B}_{\mathbb{R}}$ ; if it were, then  $g(A) = B \in \mathfrak{B}_{\mathbb{R}}$ , since  $g$  “maintains” Borel sets, but  $B$  is not even Lebesgue measurable and so this is a contradiction).

## §2 INTEGRATION THEORY

### §2.1 Measurable Functions

We will be considering functions  $f$  defined on  $\mathbb{R}$  or some subset of  $\mathbb{R}$  that could take positive or negative infinity as its value i.e.

$$f : \mathbb{R} \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\},$$

where  $\overline{\mathbb{R}}$  the *extended real line*; we say  $f$  is  $\overline{\mathbb{R}}$ -valued. If  $f$  never takes  $\infty, -\infty$  for any  $x \in \mathbb{R}$ , we say  $f$  finite-valued, or just  $\mathbb{R}$ -valued.

For all  $a \in \mathbb{R}$ , we consider inverse images

$$f^{-1}([-\infty, a)) := \{x \in \mathbb{R} : f(x) \in [-\infty, a)\} = \{f < a\},$$

remarking the inclusion of  $-\infty$ ; similarly

$$f^{-1}((a, \infty]) := \{x \in \mathbb{R} : f(x) \in (a, \infty]\} = \{f > a\},$$

and so on, for any  $B \subseteq \mathbb{R}$ ,

$$f^{-1}(B) := \{x \in \mathbb{R} : f(x) \in B\} = \{f \in B\}.$$

Remark that

$$f^{-1}(B^c) = (f^{-1}(B))^c$$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B),$$

which extend naturally for countable unions/intersections.

↪ **Definition 2.1** (Measurable Function):  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  is *measurable* if  $\forall a \in \mathbb{R}$ ,

$$f^{-1}([-\infty, a)) \in \mathcal{M}.$$

↪ **Proposition 2.1** (Equivalent Definitions of Measurability):

$$\begin{aligned} f \text{ is measurable} &\Leftrightarrow \forall a \in \mathbb{R}, f^{-1}([a, \infty]) \in \mathcal{M} \\ &\Leftrightarrow \forall a \in \mathbb{R}, f^{-1}((a, \infty]) \in \mathcal{M} \\ &\Leftrightarrow \forall a \in \mathbb{R}, f^{-1}([-\infty, a]) \in \mathcal{M} \end{aligned}$$

PROOF. We prove just the last equivalence. Notice that  $\forall a \in \mathbb{R}$ , we can use the commuting of inverse images with countable unions, intersections, complement to write

$$f^{-1}([-\infty, a)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a - \frac{1}{n}\right)\right)$$

and

$$f^{-1}([-\infty, a]) = \bigcap_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a + \frac{1}{n}\right]\right).$$

■

↪ **Proposition 2.2**: If  $f$  finite-valued, Then

$$\begin{aligned} f \text{ is measurable} &\Leftrightarrow \forall a < b \in \mathbb{R}, f^{-1}((a, b)) \in \mathcal{M} \\ &\Leftrightarrow \dots \quad f^{-1}((a, b]) \in \mathcal{M} \\ &\Leftrightarrow \dots \quad f^{-1}([a, b)) \in \mathcal{M} \\ &\Leftrightarrow \dots \quad f^{-1}([a, b]) \in \mathcal{M}. \end{aligned}$$

↪ **Definition 2.2** (Extended Borel Sigma Algebra): Define the Borel “extended” algebra  $\mathfrak{B}_{\overline{\mathbb{R}}}$  of subsets of  $\overline{\mathbb{R}}$ , defined by

$$\mathfrak{B}_{\overline{\mathbb{R}}} := \sigma(\mathfrak{B}_{\mathbb{R}} \cup \{-\infty\}, \{\infty\}).$$

↪ **Proposition 2.3**:  $\mathfrak{B}_{\overline{\mathbb{R}}} = \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ .

PROOF. For every  $a \in \mathbb{R}$ , we may write

$$[-\infty, a) = \underbrace{(-\infty, a)}_{\in \mathfrak{B}_{\mathbb{R}}} \cup \{-\infty\} \in \mathfrak{B}_{\mathbb{R}},$$

so  $\sigma(\{[-\infty, a) : a \in \mathbb{R}\}) \subseteq \mathfrak{B}_{\mathbb{R}}$ .

Conversely, notice that

$$\{-\infty\} = \bigcap_{n=1}^{\infty} [-\infty, -n),$$

and

$$\{\infty\} = \overline{\mathbb{R}} - \left( \bigcup_{n=1}^{\infty} [-\infty, n) \right),$$

so  $\{-\infty\}, \{\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ . Hence, for any  $a \in \mathbb{R}$ ,

$$(-\infty, a) = [-\infty, a) - \{-\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\}),$$

and so  $\mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ .  $\{-\infty\}, \{\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$  already, and thus  $\mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ . ■

↪ **Proposition 2.4:**  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  measurable  $\Leftrightarrow$  for all  $B \in \mathfrak{B}_{\mathbb{R}}, f^{-1}(B) \in \mathcal{M}$ .

PROOF.  $\Leftarrow$  is immediate. For  $\Rightarrow$ , let  $\mathcal{C}$  be a collection of subsets of  $\overline{\mathbb{R}}$ , then put

$$f^{-1}(\mathcal{C}) := \{f^{-1}(B) : B \in \mathcal{C}\}.$$

By an assignment question (2.6),

$$f^{-1}(\sigma(\mathcal{C})) = \sigma(f^{-1}(\mathcal{C})).$$

Take  $\mathcal{C} = \{[-\infty, a) : a \in \mathbb{R}\}$ . Then,

$$f^{-1}(\sigma(\mathcal{C})) = f^{-1}(\mathfrak{B}_{\mathbb{R}}) = \sigma(f^{-1}(\{[-\infty, a) : a \in \mathbb{R}\})).$$

But  $f$  measurable, so  $f^{-1}([- \infty, a)) \in \mathcal{M}$  for each  $a \in \mathbb{R}$ , hence  $\sigma(f^{-1}(\{[-\infty, a) : a \in \mathbb{R}\})) \subseteq \mathcal{M}$  and so  $f^{-1}(\sigma(\mathcal{C})) \subseteq \mathcal{M}$  completing the proof. ■

↪ **Corollary 2.1:** If  $f$  finite-valued, then  $f$  is measurable  $\Leftrightarrow$  for every  $B \in \mathfrak{B}_{\mathbb{R}}, f^{-1}(B) \in \mathcal{M}$ .

↪ **Proposition 2.5:** Given  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ , define the *finite valued component* of  $f$  given by

$$f_{\mathbb{R}}(x) := \begin{cases} f(x) & : -\infty < f(x) < \infty \\ 0 & \text{otherwise} \end{cases}.$$

Then,  $f$  measurable  $\Leftrightarrow \forall B \in \mathfrak{B}_{\mathbb{R}}, f_{\mathbb{R}}^{-1}(B) \in \mathcal{M}$  AND  $\{f = \infty\}, \{f = -\infty\}$  both in  $\mathcal{M}$ .

PROOF. ( $\Leftarrow$ ) For any  $a \in \mathbb{R}$ ,

$$f^{-1}([- \infty, a)) = \{f = -\infty\} \cup f^{-1}((-\infty, a)) = \{f = -\infty\} \cup f_{\mathbb{R}}^{-1}((-\infty, a)),$$

a union of measurable sets and hence is itself measurable.

( $\Rightarrow$ ) Remark that  $\{f = \infty\}, \{f = -\infty\} \in \mathcal{M}$  automatically. For any  $B \in \mathfrak{B}_{\mathbb{R}}$ , we have  $f_{\mathbb{R}}^{-1}(B) = \{x \in \mathbb{R} : f_{\mathbb{R}}(x) \in B\} = \{x \in \mathbb{R} : f(x) \in B, -\infty < f < \infty\} \cup \{x \in \mathbb{R} : 0 \in B, f(x) = \pm\infty\} \in \mathcal{M}$ .

■

$\hookrightarrow$  **Definition 2.3:** If a statement is true for every  $x \in A$  where  $A \in \mathcal{M}$  s.t.  $m(A^c) = 0$ , then we say the statement is true a.e. (almost everywhere).

$\hookrightarrow$  **Proposition 2.6:** If  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  is measurable and  $f = g$  a.e. then  $g$  is measurable.

$\hookrightarrow$  **Corollary 2.2:** If  $f$  is finite-valued a.e., then  $f$  is measurable  $\Leftrightarrow f_{\mathbb{R}}$  is measurable  $\Leftrightarrow \forall a < b \in \mathbb{R}, f^{-1}((a, b)) \in \mathcal{M}$ .

$\hookrightarrow$  **Proposition 2.7:** If  $f \equiv c$  then  $f$  measurable.

If  $f = \mathbb{1}_A$  for some  $A \subseteq \mathbb{R}$ , then  $f$  is measurable  $\Leftrightarrow A \in \mathcal{M}$ .

PROOF. Assume  $f \equiv c$ . Then

$$f^{-1}([-\infty, a)) = \begin{cases} \mathbb{R} & \text{if } c < a \\ \emptyset & \text{if } c \geq a \end{cases} \in \mathcal{M}.$$

Assume now  $f = \mathbb{1}_A$ . For all  $a \in \mathbb{R}$ ,

$$f^{-1}([-\infty, a)) = \begin{cases} \mathbb{R} & \text{if } a > 1 \\ A^c & \text{if } 0 < a \leq 1 \in \mathcal{M} \Leftrightarrow A \in \mathcal{M}. \\ \emptyset & \text{if } a \leq 0 \end{cases}$$

■

$\hookrightarrow$  **Proposition 2.8:** If  $f$  is (finite-valued) continuous, then  $f$  is measurable.

PROOF.  $f : \mathbb{R} \rightarrow \mathbb{R}$  continuous  $\Leftrightarrow$  for all  $G \subseteq \mathbb{R}$  open,  $f^{-1}(G)$  open. For all  $a < b \in \mathbb{R}$ , then  $f^{-1}((a, b))$  open so  $f^{-1}((a, b)) \in \mathcal{M}$  so  $f$  measurable.

In fact, if  $f : \mathbb{R} \rightarrow \mathbb{R}$  continuous, then for all  $B \in \mathfrak{B}_{\mathbb{R}}, f^{-1}(B) \in \mathfrak{B}_{\mathbb{R}}$ ;

$$f^{-1}(\mathfrak{B}_{\mathbb{R}}) = f^{-1}(\sigma(\{\text{open sets}\})) = \sigma\left(\underbrace{f^{-1}(\{\text{open sets}\})}_{\text{all open}}\right) \subseteq \sigma(\{\text{open sets}\}) = \mathfrak{B}_{\mathbb{R}}.$$

Moreover, if  $f^{-1}$  (inverse) exists and is continuous, then for any  $B \in \mathfrak{B}_{\mathbb{R}}, f(B) \in \mathfrak{B}_{\mathbb{R}}$ . ■

↪ **Proposition 2.9:** If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is measurable and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, then  $g \circ f$  is measurable.

**Remark 2.1:** The order matters! The converse doesn't hold in general.

PROOF. For all  $a \in \mathbb{R}$ ,

$$\begin{aligned} (g \circ f)^{-1}((-\infty, a)) &= \{x \in \mathbb{R} : g(f(x)) < a\} \\ &= \{x \in \mathbb{R} : f(x) \in g^{-1}((-\infty, a))\} \\ &= f^{-1}(g^{-1}((-\infty, a))) \in \mathcal{M}. \end{aligned}$$

■

↪ **Proposition 2.10:** If  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$  is measurable, then:

1. for every  $c \in \mathbb{R}$ ,  $cf$  is measurable (in particular  $-f$  measurable);
2.  $|f|$  is measurable;
3. for every  $k \in \mathbb{N}$ ,  $f^k$  is a measurable.

PROOF. We prove just 3. If  $k = 0$  this is trivial. For any  $a \in \mathbb{R}$ ,

$$(f^k)^{-1}([-\infty, a)) = \begin{cases} f^{-1}\left([-\infty, a^{\frac{1}{k}})\right) & \text{if } k \text{ is odd} \\ \emptyset & \text{if } k \text{ is even and } a \leq 0 \in \mathcal{M}. \\ f^{-1}\left([-a^{\frac{1}{k}}, a^{\frac{1}{k}})\right) & \text{if } k \text{ is even and } a > 0 \end{cases}$$

■

↪ **Proposition 2.11:** If  $f, g$  are two finite-valued measurable functions, then  $f + g, f \cdot g, f \vee g := \max\{f, g\}, f \wedge g := \min\{f, g\}$  are measurable functions, where

$$(f \vee g)(x) = \max\{f(x), g(x)\}.$$

PROOF. For all  $a \in \mathbb{R}$ ,

$$\begin{aligned} (f + g)^{-1}([-\infty, a)) &= \{x \in \mathbb{R} : f(x) + g(x) < a\} \\ &= \{x \in \mathbb{R} : f(x) < a - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \{x \in \mathbb{R} : f(x) < q < a - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \underbrace{\{x \in \mathbb{R} : f(x) < q\}}_{\in \mathcal{M}} \cap \underbrace{\{x \in \mathbb{R} : g(x) < a - q\}}_{\in \mathcal{M}} \in \mathcal{M}. \end{aligned}$$

This implies, then, that  $f - g$  measurable, as are  $(f + g)^2$  and  $(f - g)^2$ , and thus



$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$$

is measurable.

We have too that

$$f \vee g = \frac{1}{2}(|f - g| + (f + g))$$

and so is measurable, and so

$$f \wedge g = -\max\{-f, -g\} = -(-f \vee -g)$$

is measurable. ■

↪ **Corollary 2.3:** If  $f$  is measurable, then  $f^+ := f \vee 0 = \max\{f, 0\}$  and  $f^- := -(f \wedge 0) = \max\{-f, 0\}$  are measurable, as is  $f \wedge k$  for any  $k \in \mathbb{R}$ .

**Remark 2.2:** Notice that  $f = f^+ - f^-$ , even with “infinities”, and  $|f| = f^+ + f^-$ .

↪ **Proposition 2.12:** Let  $\{f_n\}$  be a sequence of measurable functions. Then,  $\sup_n f_n$ ,  $\inf_n f_n$ ,  $\limsup_{n \rightarrow \infty} f_n$ , and  $\liminf_{n \rightarrow \infty} f_n$  are all measurable (where  $(\limsup_{n \rightarrow \infty} f_n)(x) := \limsup_{n \rightarrow \infty} f_n(x) = \inf_{m \geq 1} \sup_{n \geq m} f_n(x) = \lim_{m \rightarrow \infty} \sup_{n \geq m} f_n(x)$ ).

PROOF. To show  $\sup_n f_n$  measurable, we will show for all  $a \in \mathbb{R}$   $\{\sup_n f_n \leq a\} \in \mathcal{M}$ .

$$x \in \left\{ \sup_n f_n \leq a \right\} \Leftrightarrow \sup_n f_n(x) \leq a \Leftrightarrow f_n(x) \leq a \forall n \geq 1 \Leftrightarrow x \in \bigcap_{n=1}^{\infty} \{f_n \leq a\},$$

hence  $\{\sup_n f_n \leq a\} = \bigcap_{n=1}^{\infty} \underbrace{\{f_n \leq a\}}_{\in \mathcal{M}} \in \mathcal{M}$  and hence  $\sup_n f_n$  is measurable. Note that using  $\leq$  was important;  $\{\sup_n f_n < a\} \subsetneq \bigcap_{n=1}^{\infty} \{f_n < a\}$ , since the  $\sup_n f_n$  could equal  $a$ . We could say the following, however:

$$\left\{ \sup_n f_n < a \right\} = \bigcup_{k=1}^{\infty} \left\{ \sup_n f_n \leq a - \frac{1}{k} \right\} = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{ f_n \leq a - \frac{1}{k} \right\} \in \mathcal{M}.$$

Next, we have  $\inf_n f_n = -\sup_n (-f_n)$  so we are done.

For  $\limsup$ ,  $\liminf$ , we have

$$\limsup_n f_n = \inf_{m \geq 1} \underbrace{\sup_{n \geq m} f_n}_{:= g_m}.$$

$g_m$  is measurable for each  $m \geq 1$ , hence  $\inf_m g_m$  is measurable, hence  $\limsup_n f_n$  is measurable. Similar logic follows for  $\liminf$ .

We could have show, more directly, that

$$\begin{aligned}
\left\{ \limsup_n f_n < a \right\} &= \left\{ \inf_{m \geq 1} \sup_{n \geq m} f_n < a \right\} \\
&= \bigcup_{m=1}^{\infty} \left\{ \sup_{n \geq m} f_n < a \right\} \\
&= \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \sup_{n \geq m} f_n \leq a - \frac{1}{k} \right\} \\
&= \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ f_n \leq a - \frac{1}{k} \right\}.
\end{aligned}$$

■

↪ **Proposition 2.13:** Let  $\{f_n\}$  be a sequence of measurable functions. Then, all of the following sets are also measurable:

$$\begin{aligned}
\{x \in \mathbb{R} : \lim_{n \rightarrow \infty} f_n(x) \text{ exists in } \mathbb{R}\} &=: \{\lim_{n \rightarrow \infty} f_n \text{ exists in } \mathbb{R}\}, \\
\{\lim f_n = \infty\}, \{\lim f_n = -\infty\}, \{\lim f_n = c \in \mathbb{R}\}.
\end{aligned}$$

Moreover, if  $\lim_{n \rightarrow \infty} f_n$  exists (in  $\mathbb{R}$  or as  $\pm\infty$ ) a.e. with  $f = \lim_{n \rightarrow \infty} f_n$  a.e. then  $f$  is measurable.

PROOF. We have

$$\begin{aligned}
\{\lim f_n \text{ exists in } \mathbb{R}\} &= \{\limsup f_n = \liminf f_n \text{ and } -\infty < \limsup f_n < \infty\} \\
&= \{-\infty < \liminf f_n < \infty\} \cap \{-\infty < \limsup f_n < \infty\} \cap \{\limsup f_n - \liminf f_n = 0\} \in \mathcal{M}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\{\lim f_n = c\} &= \left\{ x \in \mathbb{R} : \forall k \geq 1, \exists n \geq 1 \text{ s.t. } \forall m \geq n, |f_m(x) - c| \leq \frac{1}{k} \right\} \\
&= \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \left\{ |f_m(x) - c| \leq \frac{1}{k} \right\}.
\end{aligned}$$

■

## §2.2 Approximation by Simple Functions

Given a function  $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ , measurable, we may write

$$f = f^+ - f^-,$$

where  $f^+, f^-$  are non-negative measurable functions; so, it suffices to study non-negative measurable functions. For any  $n \geq 1$ , we have

$$f_n^+ := (f^+ \wedge n) \cdot \mathbb{1}_{[-n, n]},$$

i.e., we cap  $f^+$  at  $n$ , and disregard values of  $f^+$  outside of  $[-n, n]$ ; hence we limit our view to a  $2n \times n$  “box”. Then,  $f_n^+$  is non-negative, measurable, bounded (by  $n$ ), compactly supported (zero outside a bounded set), and in particular  $f_n^+ \uparrow$ , with limit

$$\lim_{n \rightarrow \infty} f_n^+ = f^+.$$

An identical construction follows for  $f^-$  with

$$f_n^- := (f^- \wedge n) \mathbb{1}_{[-n, n]},$$

with  $f_n^- \uparrow$  and

$$\lim_{n \rightarrow \infty} f_n^- = f^-.$$

Fix some  $n$  and consider  $f_n^+$ . For  $k = 0, 1, 2, \dots, 2^n n$ , define

$$A_{n,k} := \left\{ x \in [-n, n] : \frac{k}{2^n} \leq f_n^+(x) < \frac{k+1}{2^n} \right\} = \left\{ \frac{k}{2^n} \leq f_n^+ < \frac{k+1}{2^n} \right\} \cap [-n, n] \in \mathcal{M},$$

noting that  $A_{n,k} \cap A_{n,\ell} = \emptyset$  if  $k \neq \ell$ . Set now

$$\varphi_n := \sum_{k=0}^{n \cdot 2^n} \mathbb{1}_{A_{n,k}} \frac{k}{2^n} = \sum_{k=0}^{n \cdot 2^n} \begin{cases} \frac{k}{2^n} & \text{if } x \in A_{n,k} \\ 0 & \text{else} \end{cases}.$$

We call  $\varphi_n$  a “simple function”; more generally:

↪ **Definition 2.4:**  $\varphi$  is a *simple function* if  $\varphi = \sum_{k=1}^L \mathbb{1}_{E_k} \cdot a_k$  where  $L$  a positive integer,  $a_k$ 's are constant,  $E_k$ 's are measurable sets of finite measure.

Moreover, note that  $\varphi_n \uparrow$ ; at each new stage  $n \rightarrow n+1$ , the regions are cut in two,  $A_{n,k} = A_{n+1,2k} \cup A_{n+1,2k+1}$ . In addition, we have  $\varphi_n \leq f_n^+ \leq f^+$  for all  $n$ . Moreover, we have the following:

↪ **Proposition 2.14:**

$$\lim_{n \rightarrow \infty} \varphi_n(x) = f^+(x)$$

for all  $x \in \mathbb{R}$ .

PROOF. For all  $x \in \mathbb{R}$ , for sufficiently large  $n$  we have that  $x \in [-n, n]$  and so  $f^+(x) = f^+(x) \mathbb{1}_{[-n, n]}(x)$ . Assume for now  $f^+ < \infty$ . Then, for sufficiently large  $n$ , we can ensure  $f^+(x) < n$  and so  $f^+(x) = f_n^+(x)$  for such an  $x$ . Further, we have that  $x \in A_{n,k}$  for some  $k$  so  $\varphi_n(x) = \frac{k}{2^n}$  and  $f_n^+(x) < \frac{k+1}{2^n}$  and thus

$$0 \leq f_n^+(x) - \varphi_n(x) < \frac{k+1}{2^n} - \frac{k}{2^n} = 2^{-n}$$

by construction and so  $0 \leq f^+(x) - \varphi_n(x) \leq 2^{-n}$  and thus  $\lim_{n \rightarrow \infty} \varphi_n(x) = f^+(x)$ .

In the case that  $f^+(x) = \infty$ , then  $\varphi_n(x) = n$  for all sufficiently large  $n$  hence

$$\lim_{n \rightarrow \infty} \varphi_n(x) = \lim_{n \rightarrow \infty} n = \infty = f^+(x).$$

■

↪ **Theorem 2.1:** If  $g$  is measurable and non-negative, there exists a sequence of simply functions  $\{\varphi_n\}$  such that  $\varphi_n \uparrow$  and  $\lim_{n \rightarrow \infty} \varphi_n(x) = g(x)$  for every  $x \in \mathbb{R}$ .

We can repeat this same construction and proof for  $f^-$  with a sequence  $\tilde{\varphi}_n$ . Even better:

↪ **Theorem 2.2:** If  $f$  is measurable, then  $\exists$  a sequence of simple functions  $\{\psi_n\}$  such that  $|\psi_n| \uparrow$  and  $|\psi_n| \leq |f|$  for all  $n$  and for all  $x \in \mathbb{R}$ ,  $\lim_{n \rightarrow \infty} \psi_n(x) = f(x)$ .

PROOF. Take  $\psi_n = \varphi_n - \tilde{\varphi}_n$  as above; then for all  $x \in \mathbb{R}$ , at least one of  $\varphi_n(x), \tilde{\varphi}_n(x)$  equals zero. Then

$$|\psi_n| = \varphi_n + \tilde{\varphi}_n < f^+ + f^- = |f|,$$

and

$$\lim_{n \rightarrow \infty} \psi_n(x) = \lim_{n \rightarrow \infty} \varphi_n(x) - \lim_{n \rightarrow \infty} \tilde{\varphi}_n(x) = f^+ - f^- = f.$$

■

↪ **Definition 2.5** (Step Function):  $\theta$  a *step function* if it takes the form

$$\theta(x) = \sum_{k=1}^L a_k \mathbb{1}_{I_k}(x),$$

where  $L \in \mathbb{N}$ ,  $a_k$ 's constant, and  $I_k$  finite, open intervals.

↪ **Theorem 2.3:** If  $f$  is measurable, then there exists a sequence of step functions  $\{\theta_n\}$  such that

$$\lim_{n \rightarrow \infty} \theta_n(x) = f(x) \text{ for **almost every** } x \in \mathbb{R}.$$

In particular, we do not have pointwise convergence as for general simple functions, but we have convergence outside a zero-measure set.

PROOF. Assume, wlog, that  $f$  non-negative (by the previous construction, we can “split”  $f$  if not and approximate its positive, negative parts). Given  $A \in \mathcal{M}$  with finite measure, recall that for every  $\varepsilon > 0$ , there exists finitely many finite open intervals  $I_1, \dots, I_N$  such that

$$m\left(A \triangle \left(\bigcup_{i=1}^N I_i\right)\right) < \varepsilon.$$

By renaming/rearranging  $I_i$ 's if necessary, we may assume that  $I_i$ 's are disjoint; hence

$$\mathbb{1}_{\bigcup_{i=1}^N I_i} = \sum_{i=1}^N \mathbb{1}_{I_i}.$$

Put

$$\theta_A := \sum_{i=1}^N \mathbb{1}_{I_i},$$

noting this is indeed a step function as the name suggests. Then, remark that

$$m(\underbrace{\{x \in \mathbb{R} : \mathbb{1}_A(x) \neq \theta_A(x)\}}_{=A \Delta (\bigcup_{i=1}^N I_i)}) < \varepsilon.$$

Since  $f$  measurable and non-negative,  $\exists \{\varphi_n\}$  sequence of simple functions with limit  $f$ . In particular,

$$\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}}.$$

Applying our above analysis to each  $A_{n,k}$ , then, we have that for any  $n \geq 1$  and  $k = 0, 1, \dots, n2^n$  we can find a step function  $\theta_{n,k}$  such that

$$m(\{x \in \mathbb{R} : \mathbb{1}_{A_{n,k}} \neq \theta_{n,k}(x)\}) < \frac{1}{2^n(n2^n + 1)} \quad (" = \varepsilon").$$

Put then

$$\theta_n := \sum_{k=0}^{n2^n} \frac{k}{2^n} \theta_{n,k},$$

which is itself a step function. Put

$$E_n := \{x \in \mathbb{R} : \theta_n(x) \neq \varphi_n(x)\}.$$

Then,

$$m(E_n) \leq m\left(\bigcup_{k=0}^{n2^n} \{\theta_{n,k} \neq \mathbb{1}_{A_{n,k}}\}\right) \leq \sum_{k=0}^{n2^n} m(\{\theta_{n,k} \neq \mathbb{1}_{A_{n,k}}\}) \leq 2^{-n}.$$

The  $\varphi_n$ 's are chosen such that  $\forall x \in \mathbb{R}, |\varphi_n(x) - f_n(x)| \leq \frac{1}{2^n}$ . Putting

$$F_n := \{x \in \mathbb{R} : |\theta_n(x) - f_n(x)| > 2^{-n}\},$$

then remark that  $F_n \subseteq E_n$  so  $m(F_n) \leq \frac{1}{2^n}$ .

We claim now that for a.e.  $x \in \mathbb{R}$ ,  $\exists m \geq 1$  such that  $\forall n \geq m, |\theta_n(x) - f_n(x)| \leq \frac{1}{2^n}$ , remarking that such an  $m$  is *dependent* on  $x$ . Consider the complement of this statement; if this set has measure 0, we are done. The logical negation would be "for every  $m \geq 1$ , exist  $n \geq m$  such that  $|\theta_n(x) - f_n(x)| > 2^{-n}$ ", which is equivalent to the set

$$\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in \mathbb{R} : |\theta_n(x) - f_n(x)| > 2^{-n}\} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n.$$

Let  $B_m := \bigcup_{n=m}^{\infty} F_n$ ; notice  $B_m \downarrow$ . Then, by continuity from above \*\*\*\*

$$m\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n\right) = \lim_{m \rightarrow \infty} m(B_m) \leq \lim_{m \rightarrow \infty} \sum_{n=m}^{\infty} m(F_n) \leq \lim_{m \rightarrow \infty} \sum_{n=m}^{\infty} \frac{1}{2^n} = 0,$$

since the tail of a convergent series must converge to zero. Hence, the set has measure 0 as desired so for almost every  $x \in \mathbb{R}$  there exists  $m \geq 1$  such that for all  $n \geq m$ ,  $|\theta_n - f_n| \leq \frac{1}{2^n}$ , hence almost everywhere  $\lim_{n \rightarrow \infty} (\theta_n - f_n) = 0$ . Therefore, almost everywhere,

$$\theta_n = (\theta_n - f_n) + f_n \xrightarrow{n \rightarrow \infty} f.$$

■

In this proof, we have proven (and then used) more generally:

↪ **Lemma 2.1** (Borel-Cantelli Lemma): If  $\{F_n\} \subseteq \mathcal{M}$  such that  $\sum_{n=1}^{\infty} m(F_n) < \infty$ , then

$$m\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n\right) = 0.$$

### §2.3 Convergence Almost Everywhere vs Convergence in Measure

↪ **Definition 2.6** (Convergence Almost Everywhere): For measurable functions  $\{f_n\}, f$  we say  $f_n$  converges to  $f$  a.e. and write  $f_n \rightarrow f$  a.e. if for almost every  $x \in \mathbb{R}$ ,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

Similarly, we say  $f_n \rightarrow f$  a.e. on  $A$  if  $\exists B \subseteq A$  with  $m(B) = 0$  such that  $\forall x \in A - B$ ,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

↪ **Definition 2.7** (Convergence in Measure): For measurable, finite-valued functions  $\{f_n\}, f$  we say  $f_n$  converges to  $f$  in measure and write  $f_n \rightarrow f$  in measure if for every  $\delta > 0$ ,

$$\lim_{n \rightarrow \infty} m(\{x \in \mathbb{R} : |f_n(x) - f(x)| \geq \delta\}) = 0.$$

Similarly, we say  $f_n \rightarrow f$  in measure on  $A$  if  $\forall \delta > 0$ ,  $\lim_{n \rightarrow \infty} m(\{x \in A : |f_n(x) - f(x)| \geq \delta\}) = 0$ .

↪ **Proposition 2.15**: Given finite-valued measurable functions  $\{f_n\}, f$  and  $A \in \mathcal{M}$  with finite measure, then if  $f_n \rightarrow f$  a.e. on  $A$ , then  $f_n \rightarrow f$  in measure on  $A$ .

PROOF. For all  $\delta > 0$ ,

$$\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in A : |f_n(x) - f(x)| > \delta\} \subseteq \{x \in A : \lim_{n \rightarrow \infty} f_n(x) \neq f(x)\}.$$

The set on the RHS has measure zero and thus so does the left one. Then,

$$\lim_{m \rightarrow \infty} m\left(\bigcup_{n=m}^{\infty} \{x \in A : |f_n(x) - f(x)| > \delta\}\right) = 0$$

by continuity, and

$$\{|f_m - f| > \delta\} \subseteq \bigcup_{n=m}^{\infty} \{|f_n - f| > \delta\}$$

hence  $m(\{|f_m - f| > \delta\}) \leq m(\bigcup_{n=m}^{\infty} \{|f_n - f| > \delta\}) \xrightarrow{m \rightarrow \infty} 0$ . ■

⊗ **Example 2.1:** We give an example of why the assumption that  $m(A) < \infty$  is necessary. Let,  $f_n = \mathbb{1}_{[n, \infty)}$  and  $f \equiv 0$ . Then,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for every  $x \in \mathbb{R}$ . But  $m(\{x \in \mathbb{R} : |f_n(x) - f(x)| = 1\}) = m([n, \infty)) = \infty$ .

In general, the converse statement  $f_n \rightarrow f$  in measure does *not* imply that  $f_n \rightarrow f$  almost everywhere, even on finite measure sets. Put  $\varphi_{1,1} = \mathbb{1}_{[0,1)}$ ,  $\varphi_{2,1} = \mathbb{1}_{[0, \frac{1}{2})}$ ,  $\varphi_{2,2} = \mathbb{1}_{[\frac{1}{2}, 1)}$ ,  $\varphi_{3,1} = \mathbb{1}_{[0, \frac{1}{3})}$ ,  $\varphi_{3,2} = \mathbb{1}_{[\frac{1}{3}, \frac{2}{3})}$ ,  $\varphi_{3,3} = \mathbb{1}_{[\frac{2}{3}, 1)}$ , or in general  $\varphi_{k,j} = \mathbb{1}_{[\frac{j-1}{k}, \frac{j}{k})}$  for  $j = 1, \dots, k$ . Reorder  $\varphi_{k,j}$  “lexicographically” into  $\{f_n\}$ . Then, we claim  $f_n \rightarrow 0$  in measure on  $[0, 1)$ ; for any  $\delta \in (0, 1)$ ,

$$m(\{|f_n - 0| > \delta\}) = \frac{1}{k(n)} \rightarrow 0,$$

where  $k(n)$  the “row” that  $f_n$  comes from. Hence,  $f_n$  converges in measure. However,  $f_n$  does not converge almost everywhere on  $[0, 1)$ . Indeed, for each  $x \in \mathbb{R}$  and  $k \geq 1$ , there exists a *unique*  $j$  such that  $x \in [\frac{j-1}{k}, \frac{j}{k})$  hence  $\varphi_{k,j}(x) = 1$ , so in other notation there always exists an  $n$  such that  $f_n(x) = 1$ , and so precisely  $f_n(x) = 1$  for infinitely many  $n$ . Hence, we do not have convergence everywhere (in fact, anywhere).

↔ **Proposition 2.16:** Given  $\{f_n\}$ ,  $f$  measurable, finite-valued functions, if  $f_n \rightarrow f$  in measure, then there exists a subsequence  $\{f_{n_k}\}$  such that  $f_{n_k} \rightarrow f$  a.e. as  $k \rightarrow \infty$ .

PROOF. Assume  $f_n \rightarrow f$  in measure, that is for every  $\delta > 0$ ,  $m(\{|f_n - f| > \delta\}) \rightarrow 0$ .

Hence, for all  $k \geq 1$ , with  $\delta = \frac{1}{k}$ , we have that for some sufficiently large  $n_k$ , we have

that  $m\left(\underbrace{\{|f_{n_k} - f| > \frac{1}{k}\}}_{:=A_k}\right) \leq \frac{1}{k^2}$ , hence  $\sum_{k=1}^{\infty} m(A_k) < \infty$ . Hence,

$$m\left(\bigcap_{\ell=1}^{\infty} \bigcup_{k=\ell}^{\infty} A_k\right) = \lim_{\ell \rightarrow \infty} m\left(\bigcup_{k=\ell}^{\infty} A_k\right) \leq \lim_{\ell \rightarrow \infty} \sum_{k=\ell}^{\infty} m(A_k) = 0,$$

since  $\sum_{k=\ell}^{\infty} m(A_k)$  the tail of a converging series. Hence, complementing the above, a.e. there  $\exists \ell$  such that for every  $k \geq \ell$ ,  $|f_{n_k} - f| \leq \frac{1}{k}$  and so  $\lim_{k \rightarrow \infty} |f_{n_k} - f| = 0$  almost everywhere, and so  $f_{n_k} \rightarrow f$  a.e. (as  $k \rightarrow \infty$ ). ■

↔ **Proposition 2.17 (Subsequence Test):** Given  $\{f_n\}$ ,  $f$  measurable, finite-valued functions,  $f_n \rightarrow f$  in measure  $\Leftrightarrow$  for every subsequence  $\{n_k\}$ , there exists a subsubsequence  $\{n_{k_\ell}\} \subset \{n_k\}$  such that  $f_{n_{k_\ell}} \rightarrow f$  in measure as  $\ell \rightarrow \infty$ .

PROOF.  $\Rightarrow$  is clear. For  $\Leftarrow$ , suppose towards a contradiction that  $f_n \not\rightarrow f$  in measure. Then,  $\exists \delta > 0$  and subsequence  $\{n_k\}$   $m(\{|f_{n_k} - f| > \delta\}) > \delta$  for every  $k$ . By the assumption of the RHS, there exists a further subsequence  $\{n_{k_\ell}\}$  such that  $f_{n_{k_\ell}} \rightarrow f$  in measure. This is a contradiction. ■

⊗ **Example 2.2** (Assignment Exercise): Prove that if  $f_n \rightarrow f$  in measure and  $g_n \rightarrow g$  in measure,  $f_n g_n \rightarrow fg$  in measure (everything finite valued, measurable).

## §2.4 Egorov's Theorem and Lusin's Theorem

Recall that if  $f$  is measurable, then  $\exists \{\theta_n\}$  sequence of step functions such that  $\theta_n \rightarrow f$  almost everywhere.

↪ **Theorem 2.4** (Egorov's): Given  $\{f_n\}$ ,  $f$  measurable functions and  $A \in \mathcal{M}$  with  $m(A) < \infty$ , if  $f_n \rightarrow f$  a.e. on  $A$ , then  $\forall \varepsilon > 0$ , there exists a closed subset  $A_\varepsilon \subseteq A$  with  $m(A \setminus A_\varepsilon) \leq \varepsilon$  such that  $f_n \rightarrow f$  uniformly on  $A_\varepsilon$ .

PROOF. We assume first  $f$  is finite-valued on  $A$  (otherwise, replace  $A$  with  $A \cap \{-\infty < f < \infty\}$ ; we'll deal with  $\{f = \pm\infty\}$  later). We want to show that  $\forall \varepsilon > 0, \exists$  closed  $A_\varepsilon \subseteq A$  s.t.  $m(A \setminus A_\varepsilon) < \varepsilon$  and  $\sup_{x \in A_\varepsilon} |f_n(x) - f(x)| \rightarrow 0$  as  $n \rightarrow \infty$ .

For each  $k \geq 1$  and  $n \geq 1$ , put

$$E_n^{(k)} := \left\{ x \in A : |f_j(x) - f(x)| \leq \frac{1}{k} \forall j \geq n \right\}.$$

For fixed  $k$ , remark that  $E_n^{(k)} \subseteq E_{n+1}^{(k)}$ , i.e.  $E_n^{(k)}$  increasing (wrt  $n$ ), so we may consider

$$\bigcup_{n=1}^{\infty} E_n^{(k)} = \left\{ x \in A : \exists n \geq 1 \text{ s.t. } \forall j \geq n, |f_j(x) - f(x)| \leq \frac{1}{k} \right\} \supseteq \left\{ x \in A : \lim_{n \rightarrow \infty} f_n(x) = f(x) \right\} =: A'.$$

By assumption,  $m(A') = m(A)$ , so by continuity and the superset relation above,  $m(A) = m(A') \leq m\left(\bigcup_{n=1}^{\infty} E_n^{(k)}\right) = \lim_{n \rightarrow \infty} m(E_n^{(k)}) \leq m(A)$ , and thus  $\lim_{n \rightarrow \infty} m(E_n^{(k)}) = m(A)$  for every  $k \geq 1$ .

Given, then, any  $\varepsilon > 0$ , there exists a  $n_k$  such that  $m(A \setminus E_{n_k}^{(k)}) = m(A) - m(E_{n_k}^{(k)}) < \frac{1}{2^k} \frac{\varepsilon}{2}$ . Set

$$B := A \setminus \left( \bigcap_{k=1}^{\infty} E_{n_k}^{(k)} \right),$$

then

$$m(B) = m\left(\bigcup_{k=1}^{\infty} A \setminus E_{n_k}^{(k)}\right) \leq \sum_{k=1}^{\infty} m(A \setminus E_{n_k}^{(k)}) \leq \frac{\varepsilon}{2}.$$

Put



$$\tilde{A} := A \setminus B = \bigcap_{k=1}^{\infty} E_{n_k}^{(k)}.$$

Then, if  $x \in \tilde{A}$ , then  $x \in E_{n_k}^{(k)}$  for every  $k$ , and hence for every  $k \geq 1$  and  $j \geq n_k$ ,  $|f_j(x) - f(x)| \leq \frac{1}{k}$ . This shows then that  $f_n \rightarrow f$  uniformly on  $\tilde{A}$ . By regularity of  $m$ , there exists a closed  $A_\varepsilon \subseteq \tilde{A}$  such that  $m(\tilde{A} \setminus A_\varepsilon) \leq \frac{\varepsilon}{2}$ . Then,  $f_n \rightarrow f$  uniformly on  $A_\varepsilon$ , and  $m(A \setminus A_\varepsilon) = m(A \setminus \tilde{A}) + m(\tilde{A} \setminus A_\varepsilon) < \varepsilon$ .

Now, if  $f = \infty / -\infty$  on  $A$ , then  $A = A^\infty \cup A^{-\infty} \cup A^\mathbb{R}$  (with  $A^\bullet := \{f = \bullet\} \cap A$ ). The last case is done. For  $A^\infty$  (similar construction for  $A^{-\infty}$ ), define for every  $k, n \geq 1$ ,

$$E_n^{(k)} := \{x \in A : f_j(x) > k \forall j \geq n\}.$$

Then, the remainder of the proof follows precisely the same for the sequence of sets  $E_n^{(k)}$ . ■

**Remark 2.3:**

1. The assumption  $m(A) < \infty$  is necessary. For instance  $f_n = \mathbb{1}_{[n, \infty)} \rightarrow 0$  pointwise, but for any  $a \in \mathbb{R}$ ,  $f_n$  does not converge to 0 uniformly on  $(a, \infty)$ .
2. In general, Egorov's  $\nRightarrow f_n \rightarrow f$  uniformly a.e.. For instance, on  $[0, 1]$ , let  $f_n(x) = x^n$  and  $f(x) \equiv 0$ . For every  $x \in [0, 1)$ ,  $f_n(x) \rightarrow f(x)$  as  $n \rightarrow \infty$ . Hence,  $f_n \rightarrow f$  a.e. on  $[0, 1]$  (the only point that doesn't converge, indeed, is at 1). If  $A \subseteq [0, 1]$  is closed such that  $1 \in A$ , then  $f_n \nrightarrow f$  uniformly on  $A$ . To see this, let  $\{x_m\} \subseteq A$  such that  $x_m \uparrow$  and  $\lim_{m \rightarrow \infty} x_m = 1$ . Then, for any fixed  $n$ ,

$$\sup_{x \in A} |f_n(x) - f(x)| \geq \sup_m |f_n(x_m) - f(x_m)| = \sup_m x_m^n = 1,$$

hence  $f_n$  does not converge uniformly on  $A$ .

↪ **Theorem 2.5** (Lusin's Theorem): Given  $f$  measurable and finite-valued and  $A \in \mathcal{M}$  with  $m(A) < \infty$ , for all  $\varepsilon > 0$ , there exists a closed  $A_\varepsilon \subseteq A$  with  $m(A \setminus A_\varepsilon) < \varepsilon$  such that  $f|_{A_\varepsilon}$  is continuous.

**Remark 2.4:** Lusin's Theorem states that  $f|_{A_\varepsilon}$  is continuous as a function on  $\varepsilon$ , which is *not* the same as saying  $f$  as a function of  $A$  is continuous at points in  $A_\varepsilon$ .

For instance,  $f = \mathbb{1}_{\mathbb{Q} \cap [0, 1]}$  is not continuous anywhere on  $[0, 1]$ . However,  $f|_{\mathbb{Q} \cap [0, 1]}$  is constant and therefore continuous on  $\mathbb{Q} \cap [0, 1]$ .

PROOF. Let  $\{\theta_n\}$  be a sequence of step functions such that  $\theta_n \rightarrow f$  a.e. on  $A$ . Note that  $\theta_n$  piecewise constant and hence piecewise continuous. Given  $\varepsilon > 0$  and  $n \geq 1$ , we can find an open set  $E_n$  such that  $\theta_n|_{E_n^c}$  is continuous and  $m(E_n) \leq \frac{\varepsilon}{2^{2^n}}$ . Meanwhile, Egorov's implies that there exists a closed  $B \subseteq A$  such that  $m(A \setminus B) \leq \frac{\varepsilon}{2}$  such that  $\theta_n \rightarrow f$  uniformly on  $B$ . Set

$$A_\varepsilon = B \setminus \bigcup_{n=1}^{\infty} E_n,$$

noting that  $A_\varepsilon \subset A$  closed and

$$m(A \setminus A_\varepsilon) = m(A \setminus B) + m\left(\bigcup_{n=1}^{\infty} E_n\right) = \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} m(E_n) \leq \varepsilon.$$

Finally, on  $A_\varepsilon$ ,  $\theta_n \rightarrow f$  uniformly and  $\theta_n|_{A_\varepsilon}$  continuous, and hence  $f|_{A_\varepsilon}$  continuous (uniform limit of continuous functions is continuous). ■

**Remark 2.5:**

1. Lusin's Theorem  $\nRightarrow f$  is continuous almost everywhere in general. For instance, recall that fat Cantor set  $\tilde{C}$ , with  $m(\tilde{C}) = \frac{1}{2}$ . Let  $f = \mathbb{1}_{\tilde{C}}$ .  $f$  is NOT continuous a.e. on  $[0, 1]$ , i.e.  $\forall B \subseteq [0, 1]$  with  $m(B) = 1$ ,  $f|_B$  is NOT continuous. To see this, let  $\tilde{D} = [0, 1] \setminus \tilde{C}$ . Since  $m(B) = 1$ , then  $m(\tilde{C} \cap B) = m(\tilde{D} \cap B) = \frac{1}{2}$ . Then for any  $x \in \tilde{C} \cap B$ ,  $f|_B$  is NOT continuous at  $x$ . If it were at say some  $x_0 \in \tilde{C} \cap B$ , then there must exist some  $\delta > 0$  such that for any  $x \in (x_0 - \delta, x_0 + \delta) \cap B$ ,  $|f(x) - f(x_0)| < \frac{1}{2}$ . Hence, for any  $x \in (x_0 - \delta, x_0 + \delta) \cap B$ ,  $\frac{1}{2} \leq f(x) \leq \frac{3}{2}$ . However,  $m((x_0 - \delta, x_0 + \delta) \cap B \cap \tilde{D}) > 0$  so it must be that  $\exists y \in (x_0 - \delta, x_0 + \delta) \cap B \cap \tilde{D} \Rightarrow f(y) = 0$ , a contradiction. How, then, does one apply Lusin's; that is,  $\forall \varepsilon > 0$ , there must exist some  $A_\varepsilon \subseteq [0, 1]$  such that  $m([0, 1] \setminus A_\varepsilon) < \varepsilon$  and  $f|_{A_\varepsilon} < \varepsilon$  (exercise)?
2. (Exercise) The  $\{\theta_n\}$ 's are not continuous on  $\mathbb{R}$ , but you can choose a sequence  $\{\tilde{\theta}_n\}$  to be continuous on  $\mathbb{R}$  such that  $\tilde{\theta}_n \rightarrow f$  a.e..
3. Lusin's Theorem  $\Rightarrow \forall k$  sufficiently large,  $\exists A_k \subseteq A$  closed such that  $m(A \setminus A_k) \leq \frac{1}{k}$  and  $f|_{A_k}$  continuous on  $A_k$ . In fact, we can construct them such that  $A_k \uparrow$  (otherwise replace  $A_k$  with  $\bigcup_{i=1}^k A_i$ ).

## §2.5 Construction of Integrals

### 2.5.1 Integral of Simple Functions

↪ **Definition 2.8:** Given a simple function  $\varphi = \sum_{k=1}^L a_k \mathbb{1}_{E_k}$ , the (Lebesgue) integral of  $\varphi$  is defined as

$$\int_{\mathbb{R}} \varphi(x) dx = \int_{\mathbb{R}} \varphi := \sum_{k=1}^L a_k \cdot m(E_k).$$

For any  $A \in \mathcal{M}$ ,  $\mathbb{1}_A \varphi$  is again a simple function and we define

$$\int_A \varphi := \int_{\mathbb{R}} \mathbb{1}_A \varphi.$$

↪ **Proposition 2.18** (Properties of  $\int_{\mathbb{R}} \varphi$ ):

1. (Well-definedness) The written representation of  $\varphi$  is not necessarily unique, but if  $\varphi = \sum_{k=1}^L a_k \mathbb{1}_{E_k} = \sum_{\ell=1}^M b_{\ell} \mathbb{1}_{F_{\ell}}$ , then

$$\sum_{k=1}^L a_k m(E_k) = \sum_{\ell=1}^M b_{\ell} m(F_{\ell}).$$

2. (Linearity) If  $\varphi, \psi$  two simple functions and  $a, b \in \mathbb{R}$ , then  $a\varphi + b\psi$  a simple function, and

$$\int_{\mathbb{R}} a\varphi + b\psi = a \cdot \int_{\mathbb{R}} \varphi + b \cdot \int_{\mathbb{R}} \psi.$$

3. (Finite Additivity) If  $\varphi$  a simple function,  $A, B \in \mathcal{M}$  with  $A \cap B = \emptyset$ , then

$$\int_{A \cup B} \varphi = \int_A \varphi + \int_B \varphi.$$

4. (Monotonicity) If  $\varphi, \psi$  are two simple functions with  $\varphi \leq \psi$ , then  $\int_{\mathbb{R}} \varphi \leq \int_{\mathbb{R}} \psi$ .
5. If  $\varphi$  a simple function then so is  $|\varphi|$  and  $|\int_{\mathbb{R}} \varphi| \leq \int_{\mathbb{R}} |\varphi|$ .

PROOF.

1. wlog, we may assume  $E_k$  and  $F_{\ell}$  are respectively disjoint. Set  $a_0 = b_0 = 0$ ,  $E_0 := \left(\bigcup_{k=1}^L E_k\right)^c$ ,  $F_0 := \left(\bigcup_{\ell=1}^M F_{\ell}\right)^c$  for convenience. Now,  $\{E_0, \dots, E_L\}, \{F_0, \dots, F_M\}$  are two partitions of  $\mathbb{R}$ . In particular, then, for each  $k$ ,  $\mathbb{1}_{E_k} = \sum_{\ell=0}^M \mathbb{1}_{E_k \cap F_{\ell}}$ , since  $E_k = \bigsqcup_{\ell=0}^M (E_k \cap F_{\ell})$ . Now, we have

$$\varphi = \sum_{k=0}^L a_k \mathbb{1}_{E_k} = \sum_{k=0}^L \sum_{\ell=0}^M a_k \mathbb{1}_{E_k \cap F_{\ell}}.$$

Similarly partitioning, we have

$$\varphi = \sum_{\ell=0}^M b_{\ell} \mathbb{1}_{F_{\ell}} = \sum_{\ell=0}^M \sum_{k=0}^L b_{\ell} \mathbb{1}_{E_k \cap F_{\ell}}.$$

If  $E_k \cap F_{\ell} \neq \emptyset$ , then  $a_k = b_{\ell}$ , and thus on the one hand

$$\int_{\mathbb{R}} \varphi = \sum_{k=0}^L \sum_{\ell=0}^M a_k m(E_k \cap F_{\ell})$$

and on the other

$$\int_{\mathbb{R}} \varphi = \sum_{\ell=0}^M \sum_{k=0}^L b_{\ell} m(E_k \cap F_{\ell}),$$

(with summation convention  $0 \cdot \infty = 0$ ). If  $m(E_k \cap F_{\ell}) > 0$ , then  $E_k \cap F_{\ell} \neq \emptyset$  and so  $a_k = b_{\ell}$  and so the two sums agree.

4. Assume  $\varphi = \sum_{k=1}^L a_k \mathbb{1}_{E_k}$ ,  $\psi = \sum_{\ell=1}^M b_{\ell} \mathbb{1}_{F_{\ell}}$ . Repeat the partitioning/rewriting steps from part 1, then note that since  $\varphi \leq \psi$ , if  $E_k \cap F_{\ell} \neq \emptyset$ , it must be that  $a_k \leq b_{\ell}$ , so if  $m(E_k \cap F_{\ell}) > 0$   $a_k \leq b_{\ell}$  and thus the monotonicity follows.

■

## 2.5.2 Integral of Non-Negative Functions

↪ **Definition 2.9:** If  $f$  a non-negative, measurable function then the integral of  $f$  is given by

$$\int_{\mathbb{R}} f(x) dx = \int_{\mathbb{R}} f := \sup \left\{ \int_{\mathbb{R}} \varphi : \varphi \text{ is simple and } \varphi \leq f \right\}.$$

↪ **Proposition 2.19:** The definition above agrees with that for simple functions that are also non-negative, namely this definition is consistent with the previous.

PROOF. Let  $\varphi$  be non-negative. Then  $\varphi \leq \varphi$  certainly so the first definition  $\int_{\mathbb{R}} \varphi \leq \sup \{\dots\}$ . Conversely, it suffices to show that for any non-negative simple  $\psi \leq \varphi$ ,  $\int_{\mathbb{R}} \psi \leq \int_{\mathbb{R}} \varphi$ , using the first definition. But this simply follows from monotonicity of  $\int$ , and we are done. ■

**Remark 2.6:** Given  $f \geq 0$  and measurable, this definition implies that there exists a sequence  $\{\varphi_n\}$  of simple functions such that  $\varphi_n \leq f$  and  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n = \int_{\mathbb{R}} f$ . We would like to show that, in some sense, the choice of  $\{\varphi_n\}$  is arbitrary.

↪ **Theorem 2.6:** Suppose  $f \geq 0$  and measurable. If  $\{\varphi_n\}$  a sequence of simple functions such that  $\varphi_n \uparrow$  and  $\lim_{n \rightarrow \infty} \varphi_n = f$  pointwise, then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n = \int_{\mathbb{R}} f.$$

PROOF. Since  $\varphi_n \leq f$  for all  $n \geq 1$ , then  $\int_{\mathbb{R}} \varphi_n \leq \int_{\mathbb{R}} f$  and so  $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n \leq \int_{\mathbb{R}} f$  (nothing the limit on the LHS necessarily always exists by monotonicity). On the other hand, it suffices to show that  $\forall \psi \leq f$  simple, that  $\int_{\mathbb{R}} \psi \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n$ . Assume  $\psi = \sum_{k=1}^L a_k \mathbb{1}_{E_k} = \sum_{k=0}^L a_k \mathbb{1}_{E_k}$  where  $\{E_0, \dots, E_L\}$  forms a partition of  $\mathbb{R}$ . Since

$$\int_{\mathbb{R}} \psi = \sum_{k=0}^L a_k m(E_k)$$

and

$$\int_{\mathbb{R}} \varphi_n = \sum_{k=0}^L \int_{E_k} \varphi_n$$

by finite additivity. It suffices to show then that for each  $k = 0, \dots, L$ ,  $a_k m(E_k) \leq \lim_{n \rightarrow \infty} \int_{E_k} \varphi_n$ .

First, if  $a_k = 0$  or  $m(E_k) = 0$ , then we are done. Assume  $a_k, m(E_k) > 0$ . For each fixed  $k$ ,  $\lim_{n \rightarrow \infty} \varphi_n = f \geq \psi$  so for every  $x \in E_k$ ,  $\lim_{n \rightarrow \infty} \varphi_n(x) \geq \psi(x) = a_k$ . For any  $\varepsilon > 0$ , put

$$C_n^\varepsilon := \{x \in E_k : \varphi_n(x) \geq (1 - \varepsilon)a_k\}.$$

Since  $\varphi_n \leq \varphi_{n+1}$ ,  $C_n^\varepsilon \uparrow$  wrt  $n$ . Then note

$$\bigcup_{n=1}^{\infty} C_n^\varepsilon = E_k.$$

Then,

$$\lim_{n \rightarrow \infty} \int_{E_k} \varphi_n = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \mathbb{1}_{E_k} \varphi_n \geq \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \mathbb{1}_{C_n^\varepsilon} \varphi_n \geq \lim_{n \rightarrow \infty} (1 - \varepsilon)a_k m(C_n^\varepsilon) = (1 - \varepsilon)a_k m(E_k),$$

where we use the fact that  $\mathbb{1}_{E_k} \varphi_n \geq \mathbb{1}_{C_n^\varepsilon} \varphi_n \geq (1 - \varepsilon)a_k \mathbb{1}_{C_k^\varepsilon}$  and  $\lim_{n \rightarrow \infty} m(C_n^\varepsilon) = m(\bigcup_{n=1}^{\infty} C_n^\varepsilon) = m(E_k)$ . Since  $\varepsilon$  arbitrary, then

$$\lim_{n \rightarrow \infty} \int_{E_k} \varphi_n \geq a_k m(E_k),$$

and we are done. ■

↪ **Corollary 2.4:** For any  $f \geq 0$  measurable, if  $\forall n \geq 1, k = 0, 1, \dots, n2^n$  with  $A_{n,k} := \left\{ \frac{k}{2^n} \leq f < \frac{k+1}{2^n} \right\}$ , then

$$\int_{\mathbb{R}} f = \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} m(A_{n,k}).$$

PROOF. Let  $\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}}$ , then  $\varphi_n \uparrow$  and  $\varphi_n \rightarrow f$ . ■

↪ **Proposition 2.20** (Properties of Integral of Non-Negative Functions):

1. (Well-definedness) If  $f, g \geq 0$  measurable such that  $f = g$  a.e., then  $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$ .
2. (Linearity) For any  $f, g \geq 0$  measurable and  $a, b \geq 0$ , then  $\int_{\mathbb{R}} (af + bg) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g$ .
3. (Monotonicity) If  $f, g \geq 0$  measurable and  $f \leq g$  a.e., then  $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$ .
4. i. Let  $f \geq 0$  measurable, then  $\int_{\mathbb{R}} f = 0 \Leftrightarrow f \equiv 0$  a.e.  
 ii. Let  $f \geq 0$  measurable,  $A \in \mathcal{M}$ . Then  $\int_A f = 0 \Leftrightarrow$  either  $f \equiv 0$  a.e. on  $A$  or  $m(A) = 0$ .  
 iii. Let  $f \geq 0$  measurable, then if  $\int_{\mathbb{R}} f < \infty$  then  $f$  is finite valued a.e.
5. (Markov's Inequality) Let  $f \geq 0$  measurable and  $0 < a < \infty$ . Then,  $m(\{f > a\}) \leq \frac{1}{a} \int_{\mathbb{R}} f$ . In particular, if the RHS is finite,  $\lim_{a \rightarrow \infty} m(\{f > a\}) = 0$ , in fact in  $O\left(\frac{1}{a}\right)$ .

PROOF.

1. Let  $\{\varphi_n\}, \{\psi_n\}$  sequences of simple functions such that both are monotonically increasing with  $\varphi_n \rightarrow f, \psi_n \rightarrow g$ . Put  $h_n := \varphi_n \mathbb{1}_{\{f=g\}} + \psi_n \mathbb{1}_{\{f \neq g\}}$ ; then  $h_n$  again simple,  $h_n \uparrow$ , and  $h_n \rightarrow g$  everywhere. Then,

$$\int_{\mathbb{R}} g = \lim_n \int_{\mathbb{R}} h_n = \lim_n \left( \int_{\{f=g\}} \varphi_n + \int_{\{f \neq g\}} \psi_n \right) = \lim_n \int_{\{f=g\}} \varphi_n.$$

Meanwhile,

$$\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} \varphi_n = \lim_n \left( \int_{\{f=g\}} \varphi_n + \int_{\{f \neq g\}} \varphi_n \right) = \lim_n \int_{\{f=g\}} \varphi_n,$$

and so  $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$ .

2. Take  $\{\varphi_n\}, \{\psi_n\}$  as in the previous proof. Then  $\{h_n : a\varphi_n + b\psi_n\}$  again a sequence of monotonically increasing simple functions with limit  $af + bg$ . Then

$$\int_{\mathbb{R}} (af + bg) = \lim_n \int_{\mathbb{R}} h_n = \lim_n \int_{\mathbb{R}} (a\varphi_n + b\psi_n) = \lim_n \left( a \int_{\mathbb{R}} \varphi_n + b \int_{\mathbb{R}} \psi_n \right) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g.$$

3. wlog, assume that  $f \leq g$  everywhere by replacing  $f$  with  $f \mathbb{1}_{\{f \leq g\}}$ . Then,  $\{\varphi : \text{simple}, \varphi \leq f\} \subseteq \{\varphi : \text{simple}, \varphi \leq g\}$  and so  $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$ .
4. i.  $\Leftarrow$  clear. Conversely, we would like to prove that if  $A = \{f > 0\}, m(A) = 0$ . Put  $A_n := \{f \geq \frac{1}{n}\}$  for  $n \geq 1$ . Then,  $A_n \uparrow$  and  $\bigcup_{n=1}^{\infty} A_n = A$ . By continuity of  $m$ ,

$$m(A) = \lim_n m(A_n).$$

Suppose towards a contradiction that  $m(A) = \delta > 0$ . Then,  $\delta = \lim_n m(A_n)$ , and so must exist  $N \geq 1$  such that  $m(A_N) \geq \frac{\delta}{2}$ . Since  $f \geq f \mathbb{1}_{A_N} \geq \frac{1}{N} \mathbb{1}_{A_N}$ . By monotonicity,  $\int_{\mathbb{R}} f \geq \int_{\mathbb{R}} \frac{1}{N} \mathbb{1}_{A_N} = \frac{1}{N} m(A_N) \geq \frac{1}{N} \frac{\delta}{2} > 0$ , a contradiction.

ii. By i.,  $\int_A f = 0 \Leftrightarrow \mathbb{1}_A f \equiv 0$  a.e. on  $\mathbb{R}$ . If  $m(A) = 0$ , then  $\mathbb{1}_A \equiv 0$  a.e. so  $\mathbb{1}_A f \equiv 0$  a.e.. Else, if  $m(A) > 0$ , then  $f \equiv 0$  a.e. on  $A$ .

iii. Put  $A := \{f = \infty\}$ . Assume towards a contradiction that  $m(A) = \delta > 0$ . Then, for every  $n \geq 1, f \geq f \mathbb{1}_A \geq n \mathbb{1}_A$  and so  $\int_{\mathbb{R}} f \geq \int_{\mathbb{R}} n \mathbb{1}_A = nm(A) = n\delta$ . But this holds for any arbitrary  $n$ , so  $\int_{\mathbb{R}} f = \infty$ , a contradiction.

5. Put  $A_a := \{f > a\}$ . Then  $f \geq f \mathbb{1}_{A_a} > a \mathbb{1}_{A_a}$  so  $\int_{\mathbb{R}} f \geq am(A_a)$ .

■

### 2.5.3 Integral of General Measurable, Integrable Functions

**Definition 2.10:** For  $f$  measurable,  $\int_{\mathbb{R}} f := \int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-$ , provided that at least one of  $\int_{\mathbb{R}} f^+, \int_{\mathbb{R}} f^-$  is finite; in particular,  $\int_{\mathbb{R}} f$  may be finite or infinite.

**Remark 2.7:** Only having  $\int_{\mathbb{R}} f$  being defined is not sufficient for the desirable properties (linearity, monotonicity) to hold.

**Definition 2.11 (Integrable):** A measurable function  $f$  is called *integrable*, denoted  $f \in L^1(\mathbb{R})$ , if both  $\int_{\mathbb{R}} f^+ < \infty$  and  $\int_{\mathbb{R}} f^- < \infty$ . Note that

$$\begin{aligned} f \in L^1(\mathbb{R}) &\Leftrightarrow \int_{\mathbb{R}} |f| < \infty \text{ (since } \int_{\mathbb{R}} |f| = \int_{\mathbb{R}} f^+ + \int_{\mathbb{R}} f^-) \\ &\Leftrightarrow \int_{\mathbb{R}} f \text{ finite valued.} \end{aligned}$$

↪ **Proposition 2.21** (Properties of Integrals of Integrable Functions):

1.  $|\int_{\mathbb{R}} f| \leq \int_{\mathbb{R}} |f|$
2.  $f \in L^1(\mathbb{R}) \Rightarrow f$  is finite valued a.e.
3. (Linearity) For  $f, g \in L^1(\mathbb{R})$  and  $a, b \in \mathbb{R}$ ,  $af + bg \in L^1(\mathbb{R})$  and  $\int_{\mathbb{R}} (af + bg) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g$
4. If  $f \in L^1(\mathbb{R})$  and  $A \in \mathcal{M}$  and  $m(A) = 0$  then  $\int_A f = 0$ ; in particular if  $f, g \in L^1(\mathbb{R})$  with  $f = g$  a.e. then  $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$
5. (Monotonicity) If  $f, g \in L^1(\mathbb{R})$  with  $f \leq g$  a.e., then  $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$

PROOF.

1.  $-\int_{\mathbb{R}} f^- \leq \int_{\mathbb{R}} f \leq \int_{\mathbb{R}} f^+$  and  $\int_{\mathbb{R}} f^{\pm} \leq \int_{\mathbb{R}} |f|$ .
2. We know  $\int_{\mathbb{R}} |f| < \infty$  so  $|f| < \infty$  a.e. by properties of integrals of non-negative functions so  $m(\{f = \pm\infty\}) = 0$
3.  $|af| \leq |a| |f|$  so by monotonicity of non-negative functions,  $\int_{\mathbb{R}} |af| \leq |a| \int_{\mathbb{R}} |f| < \infty$  so  $af$  in  $L^1(\mathbb{R})$ . Note then that

$$(af)^+ = \begin{cases} af^+ & \text{if } a \geq 0 \\ -af^- & \text{if } a < 0 \end{cases} \quad (af)^- = \begin{cases} af^- & \text{if } a \geq 0 \\ -af^+ & \text{if } a < 0 \end{cases}$$

so

$$\begin{aligned} \int_{\mathbb{R}} af &= \int_{\mathbb{R}} (af)^+ - \int_{\mathbb{R}} (af)^- \\ &= \begin{cases} \int_{\mathbb{R}} af^+ - \int_{\mathbb{R}} af^- & \text{if } a \geq 0 \\ \int_{\mathbb{R}} (-a)f^- - \int_{\mathbb{R}} (-a)f^+ & \text{if } a < 0 \end{cases} \\ &= \begin{cases} a(\int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-) & \text{if } a \geq 0 \\ (-a)(\int_{\mathbb{R}} f^- - \int_{\mathbb{R}} f^+) & \text{if } a < 0 \end{cases} = a \int_{\mathbb{R}} f. \end{aligned}$$

By the same argument  $bg \in L^1(\mathbb{R})$  and  $\int_{\mathbb{R}} (bg) = b \int_{\mathbb{R}} g$ . wlog,  $a = b = 1$ . We want to show  $f + g \in L^1(\mathbb{R})$ ; clearly  $|f + g| \leq |f| + |g| < \infty$  so it must be  $f + g \in L^1(\mathbb{R})$ . Set  $h := f + g$  then  $|h, f, g| < \infty$  a.e. and each of the integrals of  $|h, f, g| < \infty$ . Then,  $h^+ - h^- = f^+ - f^- + g^+ - g^-$ . Then  $h^+ + f^- + g^- = f^+ + g^+ + h^-$ , where now both sides are non-negative functions. By linearity of integrals of non-negative functions and since all terms finite a.e.,

$$\begin{aligned} \int h^+ + \int f^- + \int g^- &= \int f^+ + \int g^+ + \int h^- \\ \Rightarrow \int h^+ - \int h^- &= \int f^+ - \int f^- + \int g^+ - \int g^- \\ \Rightarrow \int (f + g) &= \int h = \int f + \int g. \end{aligned}$$

4.  $|\int_A f| \leq \int_A |f| = 0$ .
5. Put  $h = g - f$  (valid since  $f, g \in L^1(\mathbb{R})$ ) then  $h \geq 0$  a.e. Then  $\int_{\mathbb{R}} h \geq 0$  so by linearity  $\int_{\mathbb{R}} (g - f) = \int_{\mathbb{R}} g - \int_{\mathbb{R}} f \geq 0$ .

