MATH457 - Algebra 4

Representation Theory; Galois Theory

Based on lectures from Winter 2025 by Prof. Henri Darmon.
Notes by Louis Meunier

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§1 Representation Theory

§1.1 Introduction

↪**Definition 1.1** (Linear Representation): A *linear representation* of a group G is a vector space V over a field $\mathbb{F}$ equipped with a map $G \times V \to V$ that makes V a G-set in such a way that for each $g \in G$, the map $v \mapsto gv$ is a linear homomorphism of V.

This induces a homomorphism

$$\rho : G \to \mathrm{Aut}_{\mathbb{F}}(V),$$

or, in particular, when $n = \dim_{\mathbb{F}} V < \infty$, a homomorphism

$$\rho : G \to \mathrm{GL}_n(\mathbb{F}).$$

Alternatively, a linear representation V can be viewed as a module over the group ring $\mathbb{F}[G] = \left\{\sum_{g \in G} : \lambda_g g : \lambda_g \in \mathbb{F}\right\}$ (where we require all but finitely many scalars λ_g to be zero).

↪**Definition 1.2** (Irreducible Representation): A linear representation V of a group G is called *irreducible* if there exists no proper, nontrivial *subspace* $W \subsetneq V$ such that W is G-stable.

⊛ **Example 1.1**:

1. Consider $G = \mathbb{Z}/2 = \{1, \tau\}$. If V a linear representation of G and $\rho : G \to \mathrm{Aut}(V)$. Then, V uniquely determined by $\rho(\tau)$. Let $p(x)$ be the minimal polynomial of $\rho(\tau)$. Then, $p(x) \mid x^2 - 1$. Suppose $\mathbb{F}$ is a field in which $2 \neq 0$. Then, $p(x) \mid (x-1)(x+1)$ and so $p(x)$ has either $1, -1$, or both as eigenvalues and thus we may write
$$V = V_+ \oplus V_-,$$
where $V_\pm := \{v \mid \tau v = \pm v\}$. Hence, V is irreducible only if one of V_+, V_- all of V and the other is trivial, or in other words τ acts only as multiplication by 1 or -1.
2. Let $G = \{g_1, ..., g_N\}$ be a finite abelian group, and suppose $\mathbb{F}$ an algebraically closed field of characteristic 0 (such as $\mathbb{C}$). Let $\rho : G \to \mathrm{Aut}\,(V)$ and denote $T_j := \rho\left(g_j\right)$ for $j = 1, ..., N$. Then, $\{T_1, ..., T_N\}$ is a set of mutually commuting linear transformations. Then, there exists a simultaneous eigenvector, say v, for $\{T_1, ..., T_N\}$, and so span (v) a G-stable subspace of V. Thus, if V irreducible, it must be that $\dim_{\mathbb{F}} V = 1$.

↪**Theorem 1.1**: If G a finite abelian group and V an irreducible finite dimensional representation over an algebraically closed field of characteristic 0, then $\dim V = 1$.

Proof. Let $\rho : G \to \mathrm{Aut}(V)$, label $G = \{g_1, ..., g_N\}$ and put $T_j := \rho\left(g_j\right)$ for $j = 1, ..., N$. Then, $\{T_1, ..., T_N\}$ a family of mutually commuting linear transformations on V. Then, there is a simultaneous eigenvector v for $\{T_1, ..., T_N\}$ and thus $\operatorname{span}(v)$ is $T_1, ..., T_N$-stable and so $V = \operatorname{span}(v)$. ∎

↪**Lemma 1.1**: Let V be a finite dimensional vector space over $\mathbb{C}$ and let $T_1, ..., T_N : V \to V$ be a family of mutually commuting linear automorphisms on V. Then, there is a simultaneous eigenvector for $T_1, ..., T_N$.

↪**Proposition 1.1**: Let $\mathbb{F}$ a field where $2 \neq 0$ and V an irreducible representation of S_3. Then, there are three distinct (i.e., up to homomorphism) possibilities for V.

Proof. Let $\rho : G \to \operatorname{Aut}(V)$ and let $T = \rho((23))$. Then, notice that $p_T(x) \mid \left(x^2 - 1\right)$ so T has eigenvalues in $\{-1, 1\}$.

If the only eigenvalue of T is -1, we claim that V one-dimensional.

If T has 1 as an eigenvalue. ∎

↪**Proposition 1.2**: D_8 has a unique faithful irreducible representation, of dimension 2 over a field F in which $0 \neq 2$.

Proof. Write $G = D_8 = \left\{1, r, r^2, r^3, v, h, d_1, d_2\right\}$ as standard. Let ρ be our irreducible, faithful representation and let $T = \rho\left(r^2\right)$. Then, $p_T(x) \mid x^2 - 1 = (x - 1)(x + 1)$ and so $V = V_+ \oplus V_-$, the respective eigenspaces for $\lambda = +1, -1$ respectively for T. Then, notice that since r^2 in the center of G, both V_+ and V_- are preserved by the action of G, hence one must be trivial and the other the entirety of V. V can't equal V_+, else $T = I$ on all of V hence ρ not faithful so $V = V_-$.

Next, it must be that $\rho(h)$ has both eigenvalues 1 and -1. Let $v_1 \in V$ be such that $hv_1 = v_1$ and $v_2 = rv_1$. We claim that $W := \operatorname{span}\{v_1, v_2\}$, namely $V = W$ 2-dimensional.

We simply check each element. $rv_1 = v_2$ and $rv_2 = r^2 v_1 = -v_1$ which are both in W hence r and thus $\langle r \rangle$ fixes W. Next, $hv_1 = v_1$ and $vv_2 = vrv_1 = rhv_1 = rv_1 = v_2$ (since $rhr^{-1} = v$) and so $hv_2 = -v_2$ and $vv_1 = -v_1$ and so W G-stable. Finally, d_1 and d_2 are just products of these elements and so W G-stable. ∎

↪**Definition 1.3** (Isomorphism of Representations): Given a group G and two representations $\rho_i : G \to \operatorname{Aut}_{\mathbb{F}}(V_i)$, $i = 1, 2$ an isomorphism of representations is a vector space isomorphism $\varphi : V_1 \to V_2$ that respects the group action, namely

$$\varphi(gv) = g\varphi(v)$$

for every $g \in G, v \in V_1$.

§1.2 Maschke's Theorem

↪**Theorem 1.2** (Maschke's): Any representation of a finite group G over $\mathbb{C}$ can be written as a direct sum of irreducible representations, i.e.

$$V = V_1 \oplus \cdots \oplus V_t,$$

where V_j irreducible.

Remark 1.1: $|G| < \infty$ essential. For instance, consider $G = (\mathbb{Z}, +)$ and 2-dimensional representation given by $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. Then, $n \cdot e_1 = e_1$ and $n \cdot e_2 = ne_1 + e_2$. We have that $\mathbb{C}e_1$ irreducible then. But if $v = ae_1 + e_2 \in W := V \setminus \mathbb{C}e_1$, then $Gv = (a+1)e_1 + e_2$ so $Gv - v = e_1 \in W$, contradiction.

Remark 1.2: $|\mathbb{C}|$ essential. Suppose $F = \mathbb{Z}/3\mathbb{Z}$ and $V = Fe_1 \oplus Fe_2 \oplus Fe_3$, and $G = S_3$ acts on V by permuting the basis vectors e_i. Then notice that $F(e_1 + e_2 + e_3)$ an irreducible subspace in V. Let $W = F(w)$ with $w := ae_1 + be_2 + ce_3$ be any other G-stable subspace. Then, by applying (123) repeatedly to w and adding the result, we find that $(a+b+c)(e_1 + e_2 + e_3) \in W$. Similarly, by applying $(12), (23), (13)$ to w, we find $(a-b)(e_1 - e_2), (b-c)(e_2 - e_3), (a - c)(e_1 - e_3)$ all in W. It must be that at least one of $a-b, a-c, b-c$ nonzero, else we'd have $w \in F(e_1 + e_2 + e_3)$. Assume wlog $a - b \neq 0$. Then, we may apply $(a-b)^{-1}$ and find $e_1 - e_2 \in W$. By applying $(23), (13)$ to this vector and scaling, we find further $e_2 - e_3$ and $e_1 - e_3 \in W$. But then,

$$2(e_1 - e_2) + 2(e_1 - e_3) = e_1 + e_2 + e_3 \in W,$$

so $F(e_1 + e_2 + e_3)$ a subspace of W, a contradiction.

↪**Proposition 1.3**: Let V be a representation of $|G| < \infty$ over $\mathbb{C}$ and let $W \subseteq V$ a sub-representation. Then, W has a G-stable complement W', such that $V = W \oplus W'$.

Proof. Denote by ρ the homomorphism induced by the representation. Let $W_{0'}$ be any complementary subspace of W and let

$$\pi : V \to W$$

be a projection onto W along $W_{0'}$, i.e. $\pi^2 = \pi, \pi(V) = W$, and $\ker(\pi) = W_{0'}$. Let us "replace" π by the "average"

$$\tilde{\pi} := \frac{1}{\#G} \sum_{g \in G} \rho(g) \pi \rho(g)^{-1}.$$

Then the following hold:

(1) $\tilde{\pi}$ G-equivariant, that is $\tilde{\pi}(gv) = g\tilde{\pi}(v)$ for every $g \in G, v \in V$.

(2) $\tilde{\pi}$ a projection onto W.

Let $W' = \ker(\tilde{\pi})$. Then, W' G-stable, and $V = W \oplus W'$. ■

We present an alternative proof to the previous proposition by appealing to the existence of a certain inner product on complex representations of finite groups.

↪**Definition 1.4**: Given a vector space V over $\mathbb{C}$, a *Hermitian pairing/inner product* is a hermitian-bilinear map $V \times V \to \mathbb{C}$, $(v, w) \mapsto \langle v, w\rangle$ such that

- linear in the first coordinate;
- conjugate-linear in the second coordinate;
- $\langle v, v\rangle \in \mathbb{R}^{\geq 0}$ and equal to zero iff $v = 0$.

↪**Theorem 1.3**: Let V be a finite dimensional complex representation of a finite group G. Then, there is a hermitian inner product $\langle \cdot, \cdot \rangle$ such that $\langle gv, gw\rangle = \langle v, w\rangle$ for every $g \in G$ and $v, w \in V$.

Proof. Let $\langle \cdot, \cdot \rangle_0$ be any inner product on V (which exists by defining $\langle e_i, e_j\rangle_0 = \delta_i^j$ and extending by conjugate linearity). We apply "averaging":

$$\langle v, w\rangle := \frac{1}{\#G} \sum_{g \in G} \langle gv, gw\rangle.$$

Then, one can check that $\langle \cdot, \cdot \rangle$ is hermitian linear, positive, and in particular G-equivariant. ■

From this, the previous proposition follows quickly by taking $W' = W^{\perp}$, the orthogonal complement to W with respect to the G-invariant inner product that the previous theorem provides.

From this proposition, Maschke's follows by repeatedly applying this logic. Since at each stage V is split in two, eventually the dimension of the resulting dimensions will become zero since V finite dimensional. Hence, the remaining vector spaces $V_1, \ldots, V_t$ left will necessarily be irreducible, since if they weren't, we could apply the proposition further.

↪**Theorem 1.4** (Schur's Lemma): Let V, W be irreducible representations of a group G. Then,

$$\mathrm{Hom}_G(V, W) = \begin{cases} 0 \text{ if } V \underset{G}{\not\cong} W \\ \mathbb{C} \text{ if } V \underset{G}{\cong} W' \end{cases}$$

where $\mathrm{Hom}_G(V, W) = \{T : V \to W \mid T \text{ linear and } G - \text{equivariant}\}$.

Proof. Suppose $V \underset{G}{\not\cong} W$ and let $T \in \mathrm{Hom}_G(V, W)$. Then, notice that $\ker(T)$ a subrepresentation of V (a subspace that is a representation in its own right), but by assumption V irreducible hence either $\ker(T) = V$ or $\{0\}$.

If $\ker(T) = V$, then T trivial, and if $\ker(T) = \{0\}$, then this implies $T : V \to \operatorname{im}(T) \subset W$ a representation isomorphism, namely $\operatorname{im}(T)$ a irreducible subrepresentation of W. This implies that, since W irreducible, $\operatorname{im}(T) = W$, contradicting the original assumption.

Suppose now $V \underset{G}{\cong} W$. Let $T \in \operatorname{Hom}_G(V, W) = \operatorname{End}_G(V)$. Since $\mathbb{C}$ algebraically closed, T has an eigenvalue, λ. Then, notice that $T - \lambda I \in \operatorname{End}_G(V)$ and so $\ker(T - \lambda I) \subset V$ a, necessarily trivial because V irreducible, subrepresentation of V. Hence, $T - \lambda I = 0 \Rightarrow T = \lambda I$ on V. It follows that $\operatorname{Hom}_G(V, W)$ a one-dimensional vector space over $\mathbb{C}$, so namely $\mathbb{C}$ itself. ■

↪**Corollary 1.1**: Given a general representation $V = \bigoplus_{j=1}^{t} V_j^{m_j}$,

$$m_j = \dim_{\mathbb{C}} \operatorname{Hom}_G\left(V_j, V\right).$$

↪**Definition 1.5** (Trace): The trace of an endomorphism $T : V \to V$ is the trace of any matrix defining T. Since the trace is conjugation-invariant, this is well-defined regardless of basis.

↪**Proposition 1.4**: Let $W \subseteq V$ a subspace and $\pi : V \to W$ a projection. Then, $\operatorname{tr}(\pi) = \dim(W)$.

↪**Theorem 1.5**: If $\rho : G \to \operatorname{Aut}_{\mathbb{F}}(V)$ a complex representation of G, then

$$\dim\left(V^G\right) = \frac{1}{\#G} \sum_{g \in G} \operatorname{tr}(\rho(g)),$$

where $V^G = \{v \in V : gv = v \,\forall\, g \in G\}$.

Proof. Let $\pi = \frac{1}{\#g} \sum_{g \in G} \rho(g)$. Then, notice that $\operatorname{im}(\pi) = V^G$ and $\pi^2 = \pi$ hence a projection from V onto V^G. Using the previous proposition and linearity of the trace completes the proof. ■

§1.3 Characters

↪**Definition 1.6**: Let $\dim(V) < \infty$ and G a group. The *character* of V is the function

$$\chi_V : G \to \mathbb{C}, \qquad \chi_V(g) := \operatorname{tr}(\rho(g)).$$

↪**Proposition 1.5**: Characters are class functions, namely constant on conjugacy classes.

↪**Theorem 1.6**: If V_1, V_2 are 2 representations of G, then $V_1 \underset{G}{\cong} V_2 \Leftrightarrow \chi_{V_1} = \chi_{V_2}$.

↪**Proposition 1.6**: Given two representations V, W of G, there is a natural action of G on $\mathrm{Hom}(V, W)$ given by $g * T = g \circ T \circ g^{-1}$. Then,

$$\mathrm{Hom}(V, W)^G = \{T : V \to W \mid g * T = T\},$$

so

$$\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W).$$

↪**Proposition 1.7**: Suppose $V = V_1^{m_1} \oplus \cdots \oplus V_t^{m_t}$ a representation of G written in irreducible form. Then,

$$\mathrm{Hom}_G\left(V_j, V\right) = \mathbb{C}_j^m.$$

Proof. "Hom is linear with respect to $\oplus$". ■

↪**Proposition 1.8**: If V, W are two representations, then so is $V \oplus W$ with point-wise action, and $\chi_{V \oplus W} = \chi_V + \chi_W$.

↪**Theorem 1.7**: $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V} \chi_W$.

Proof. Use an eigenbasis for V, W respectively to define a corresponding eigenbasis for $\mathrm{Hom}(V, W)$ such as to write any $g \in G$ as a diagonal matrix. The entries will contain an expression depending solely on the eigenvalues for g acting on V, W. ■

↪**Theorem 1.8** (Orthogonality of Irreducible Group Characters): Suppose $V_1, \ldots, V_t$ is a list of irreducible representations of G and $\chi_1, \ldots, \chi_t$ are their corresponding characters. Then, the χ_j's naturally live in the space $L^2(G) \simeq \mathbb{C}^{\#G}$, which we can equip with the inner product

$$\langle f_1, f_2 \rangle : \frac{1}{\#G} \sum_{g \in G} \overline{f_1(g)} f_2(g).$$

Then,

$$\langle \chi_i, \chi_j \rangle = \delta_i^j.$$

Proof.

$$
\begin{aligned}
\langle\chi_i,\chi_j\rangle &= \frac{1}{\#G}\sum_{g\in G}\overline{\chi_i(g)}\chi_j(g)\\
&= \frac{1}{\#G}\sum_{g\in G}\chi_{\mathrm{Hom}(V_i,V_j)}(g)\\
&= \dim_{\mathbb{C}}\left(\mathrm{Hom}\left(V_i,V_j\right)^G\right)\\
&= \begin{cases}\dim_{\mathbb{C}}(\mathbb{C}) i = j\\ \dim_{\mathbb{C}}(0) i \neq j\end{cases} = \delta_i^j.
\end{aligned}
$$

■

↪**Corollary 1.2**: $\chi_1, \ldots, \chi_t$ orthonormal vectors in $L^2(G)$.

↪**Corollary 1.3**: $\chi_1, \ldots, \chi_t$ linearly independent, so in particular $t \leq \#G = \dim L^2(G)$.

↪**Corollary 1.4**: $t \leq h(G) := \#$ conjugacy classes.