

# MATH325 - ODEs

A Course on Ordinary Differential Equations

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# 1 Introduction

## 1.1 Definitions

### ↪ Definition 1.1: Differential equation

A *differential equation* (DE) is an equation with derivatives. *Ordinary* DE's (ODE) will be covered in this course; other types (PDE's, SDE's, DDE's, FDE's, etc.) exist as well but won't be discussed. ODE's only have one independent variable (typically,  $y = f(x)$  or  $y = f(t)$ ).

### ⊗ Example 1.1: A Trivial Example

$\frac{dy}{dx} = 6x$ . Integrating both sides:

$$\int \frac{dy}{dx} dx = \int 6x dx \implies y(x) = 3x^2 + C.$$

### ⊗ Example 1.2: Another One

$$\frac{d^2u}{dt^2} = 0 \implies y = at + b.$$

### ↪ Definition 1.2: Order

The order of a differential equation is defined as the order of the highest derivative in the equation.

## 1.2 Initial Values

**Remark 1.1.** Note the existence of arbitrary constants in the previous examples, indicating infinite solutions. We often desire unique solutions by fixing these coefficients. For first order ODEs, we simply specify a single initial condition (say, some  $y(x_0) = \alpha_0$ ). For higher order ODEs of degree  $n$ , we can either specify  $n - 1$  initial conditions for  $n - 1$  derivatives (say,  $y(x_0) = \alpha_0$ ,  $y'(x_0) = \beta_0$ ), or boundary conditions (say,  $y(x_0) = \alpha_0$ ,  $y(x_1) = \alpha_1$ ) where values for the solution itself are specified.

### ⊗ Example 1.3: A Less Trivial Example

$\frac{dy}{dx} = y$ . We cannot simply integrate both sides as before, as we have no way to know what  $\int y dx$  (the RHS) is equal to. We can fairly easily guess that  $y = e^x$  is a solution; its derivative is equal to itself, hence it does indeed solve the equation. This is not the only solution; indeed, given  $y = ce^x$ , we have

$$\frac{dy}{dx} = ce^x = y = ce^x.$$

Luckily, we were rather limited in how many places constants could appear; this doesn't always hold.

## 1.3 Physical Applications

### ⊛ Example 1.4: Simple Pendulum

Let  $\theta$  be the angle of a pendulum of mass  $m$  from vertical and length  $l$ . Then, we have the equation of motion

$$ml\ddot{\theta} = -mg \sin \theta \implies \ddot{\theta} + \frac{g}{l} \sin \theta = 0 \implies \ddot{\theta} + \omega^2 \sin \theta = 0.$$

Take  $\theta$  small, then,  $\sin \theta \approx \theta$ . Then,  $\ddot{\theta} + \omega^2 \theta = 0$ . This is linear simple harmonic motion, and has periodic solutions; how do we know this is a valid solution to the non-linear model?

↪ Lecture 01; Last Updated: Thu Jan 4 15:16:18 EST 2024

### ⊛ Example 1.5: Lorenz Equations

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= rx - y - xz \\ \frac{dz}{dt} &= xy - bz\end{aligned}$$

These are a famous set of equations originally derived from atmospheric modeling, known for its chaotic behavior for particular parameters. This is a nonlinear system of de's, and beyond the scope of this class (indeed, it is not solvable exactly).

## 1.4 Uniqueness

Given an ODE of the general form  $y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$ , if we wish to determine  $y^{(n)}(t_0)$  uniquely, we need to specify the initial conditions

$$y(t_0), y'(t_0), \dots, y^{(n-1)}(t_0).$$

Moreover, this not only determines uniqueness of  $y^{(n)}(t_0)$ , byt the uniqueness of solution  $y$  for  $t \in I$  for some “interval of validity”  $I$ .

### ↪ Definition 1.3: Autonomous/Nonautonomous

An ODE of the form

$$y^{(n)} = f(y, y', \dots, y^{(n-1)})$$

is called *autonomous*; that is, if it has no explicit dependence on the independent variable. Otherwise, the system is called *nonautonomous*.

↪ **Definition 1.4: Linear/Nonlinear**

Linear ODEs of dimension  $n$  have a solution space which is a vector space of dimension  $n$ . As a result, solutions can be written as a linear combination of  $n$  basis solutions (or “fundamental set of solutions”). Solutions to nonlinear ODEs cannot be written this way (except locally).

Alternatively (but equivalently), if we can write an  $n$ th order ODE in the form

$$a_n(t)y^n(t) + \cdots a_1(t)y'(t) + a_0(t)y(t) = g(t),$$

or equivalently,

$$\sum_{i=0}^n a_i(t)y^i(t) = g(t), \quad (*)$$

where each  $a_i(t)$  and  $g(t)$  are known functions of  $t$ , then we say that the ODE is linear. Otherwise, it is nonlinear.

⊛ **Example 1.6**

The pendulum

$$\ddot{\theta} + \omega^2 \sin \theta = 0$$

is autonomous and linear;

$$\ddot{\theta} + \omega^2 \sin \theta = 0$$

is autonomous and nonlinear, due to the  $\sin \theta$  term (indeed, this is a nonlinear oscillator equation); a damped-forced oscillator

$$\ddot{\theta} + k^2 \dot{\theta} + \omega^2 \theta = A \sin(\mu t)$$

is nonautonomous and linear.

**Remark 1.2.** Note that the following definitions apply only to linear ODEs.

↪ **Definition 1.5: Homogeneous/Nonhomogeneous**

A linear ODE of the form  $(*)$  is *homogeneous* if  $g(t) = 0$ ; otherwise it is *nonhomogeneous*.

↪ **Definition 1.6: Constant/Variable**

A linear ODE of the form  $(*)$  is *constant coefficient* if  $a_j(t) = \text{constant} \quad \forall j$ ; if at least one  $a_j$  not constant, it is *non-constant* or *variable coefficient*.

**Remark 1.3.** Note that while we define linearity of ODEs in terms of the form of  $y^{(n)} = f(t, y, \dots)$ , this more “helpfully” relates to the form of the solution of such an ODE, which is indeed linear.

## 1.5 Solutions

Given an  $n$  order ODE  $y^{(n)} = f(t, y, \dots)$ , and assuming  $f$  continuous, then for  $y(t)$  to be a solution, we need  $y$  to be  $n$ -times differentiable; hence,  $y, \dots, y^{(n-1)}$  must all exist and be continuous. Then,  $y^{(n)}$ , being a continuous function of continuous functions, is, itself, continuous.

### ↪ Definition 1.7: Solution

The function  $y(t) : I \rightarrow \mathbb{R}$  is a solution to an ODE on an interval  $I \subseteq \mathbb{R}$  if it is  $n$ -times differentiable on  $I$ , and satisfies the ODE on this interval.

Given an well-defined IVP with  $n - 1$  initial values defined at  $t_0$ , then  $y(t)$  is a solution if  $t_0 \in I$ ,  $y$  satisfies the initial values, and  $y(t)$  is a solution on the interval.

### ↪ Definition 1.8: Interval of Validity

The largest  $I$  on which  $y(t) : I \rightarrow \mathbb{R}$  solves an ODE is called the *interval of validity* of the problem.

↪ Lecture 02; Last Updated: Thu Jan 11 11:05:26 EST 2024

## 2 First Order ODEs

### 2.1 Separable ODEs

#### ↪ Definition 2.1: Separable ODE

An ODE of the form

$$y' = P(t)Q(y)$$

is called *separable*. We solve them:

$$\begin{aligned} \frac{dy}{dt} &= P(t)Q(y) \\ \Rightarrow \int \frac{1}{Q(y)} dy &= \int P(t) dt. \end{aligned}$$

Finish by evaluating both sides.

⊛ **Example 2.1**

$$\frac{dy}{dt} = ty \tag{1}$$

$$\implies \frac{1}{y} dy = t dt \tag{2}$$

$$\implies \ln |y| = \frac{t^2}{2} + C \tag{3}$$

$$\implies |y| = Ke^{\frac{t^2}{2}} \quad \text{where } K = e^C \tag{4}$$

$$\implies y = Be^{\frac{t^2}{2}} \quad \text{where } B = \pm K = \pm e^C \tag{5}$$

Note that we call line (3) an *implicit solution*. In this case, we could easily turn this into an explicit solution by solving for  $y(t)$ ; this won't always be possible.

Note that it would appear, based on the definition, that  $B \neq 0$  (as  $e^{\dots} \neq 0$ ); however, plugging  $y = 0$  into (1) shows that this is indeed a solution. It is quite easy to verify that (5) is a valid solution;

$$\frac{d}{dt} \left( Be^{\frac{t^2}{2}} \right) = Bte^{\frac{t^2}{2}} = t \cdot y,$$

as desired; this holds  $\forall B \in \mathbb{R}$ .

**Remark 2.1.** *Is it valid to split the differentials like this?*

$$\begin{aligned} \frac{1}{Q(y)} \frac{dy}{dt} &= P(t) \\ \implies \int \frac{1}{Q(y)} \frac{dy}{dt} dt &= \int P(t) dt \end{aligned}$$

Let  $g(y) = \frac{1}{Q}(y)$  and  $G(y) = \int g(y) dy$ . By the chain rule,

$$\frac{d}{dt}(G(y(t))) = \frac{dy}{dt} \cdot \frac{d}{dy}G(y(t)) = \frac{dy}{dt} \cdot g(y(t)) = \frac{dy}{dt} \cdot \frac{1}{Q(y(t))}.$$

Integrating both sides with respect to time, we have

$$\begin{aligned} G(y(t)) &= \int \frac{1}{Q(y(t))} \frac{dy}{dt} dt = \int P(t) dt + C \\ \implies \int g(y) dy &= \int P(t) dt + C \\ \implies \int \frac{1}{Q(y)} dy &= \int P(t) dt + C \end{aligned}$$

This was our original expression obtaining by “splitting”, hence it is indeed “valid”.

⊛ Example 2.2

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2}{1-y^2} \\ \implies \int (1-y^2) dy &= \int x^2 dx \\ \implies y - \frac{y^3}{3} &= \frac{x^3}{3} + C \\ \implies y - \frac{1}{3}(y^3 + x^3) &= C\end{aligned}$$

Suppose we have the same ODE but now with an IVP  $y(0) = 4$ . Then, plugging this into our implicit solution:

$$4 - \frac{1}{3}(64 + 0) = C \implies C = 4 - \frac{64}{3} = -\frac{52}{3},$$

so our IVP solution is

$$y - \frac{1}{3}(y^3 + x^3) = -\frac{52}{3}.$$

## 2.2 Linear First Order ODEs

↪ **Definition 2.2: Integrating Factor**

A linear first order ODE of the form

$$\begin{aligned}a_1(t)y'(t) + a_0(t)y(t) &= g(t) \\ \implies y' + \frac{a_0}{a_1}y &= \frac{g}{a_1} \\ \implies y' + p(t)y &= q(t).\end{aligned}$$

To solve, we multiply by some integrating factor  $\mu(t)$ ;

$$\mu(t)y'(t) + p(t)\mu(t)y(t) = \mu(t)q(t)$$

It would be quite convenient if  $p(t)\mu(t) = \mu'(t)$ ; in this case, we'd have

$$\begin{aligned}\mu(t)y' + \mu'(t)y &= \mu(t)q(t) \\ \frac{d}{dt}(\mu(t)y(t)) &= \mu(t)q(t) \\ \implies \mu(t)y(t) &= \int \mu(t)q(t) dt + C \\ \implies y(t) &= \frac{1}{\mu(t)} \int \mu(t)q(t) dt + \frac{C}{\mu(t)}\end{aligned}$$

Now, what is  $\mu(t)$ ? We required that

$$\begin{aligned}\mu'(t) &= p(t)\mu \\ \frac{d\mu}{dt} &= p(t)\mu \\ \implies \int \frac{d\mu}{\mu} &= \int p(t) dt \implies \ln |\mu| = \int p(t) dt \\ \implies \mu(t) &= Ke^{\int p(t) dt}\end{aligned}$$

However, note in our whole process earlier, we need only one  $\mu$ ; hence, for convenience, we can disregard any constants of integration and simply take

**Integrating Factor:**  $\mu(t) := e^{\int p(t) dt}$

Then, our original linear ODE has general solution

$$y(t) = Ce^{-\int p(t) dt} + e^{-\int p(t) dt} \int e^{\int p(t) dt} q(t) dt.$$



⊛ **Example 2.3**

$$ty' + 3y - t^2 = 0$$

$$y' + \frac{3}{t}y = t$$

$$\implies \mu(t) = e^{\int \frac{3}{t} dt} = e^{3 \ln|t|} = t^3$$

$$\implies t^3 y' + 3t^2 y = t^4$$

$$\implies \frac{d}{dt}(yt^3) = t^4$$

$$\implies yt^3 = \int t^4 dt$$

$$\implies y = \frac{1}{t^3} \cdot \frac{t^5}{5} + \frac{C}{t^3} = \frac{t^2}{5} + \frac{C}{t^3}$$

Note the division by zero issue when  $t = 0$ ; this is not an issue with the solution method, but indeed with the ODE itself. The ODE breaks down when  $t = 0$  for the same reason.

Thus, this solution is valid for  $t \in (-\infty, 0) \cup (0, \infty) =: I_1 \cup I_2$ ; if we are given an IVP  $y(t_0) = y_0$ , if  $t_0 < 0$ , then the interval of validity is  $I_1$ , and if  $t_0 > 0$ , the interval of validity is  $I_2$ .

## 2.3 Exact Equations

### ↪ **Definition 2.3: Exact Equations**

A first order ODE of the form

$$M(x, y) dx + N(x, y) dy = 0 \iff \frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)}$$

is said to be *exact* if

$$\frac{\partial}{\partial y} M(x, y) = \frac{\partial}{\partial x} N(x, y) \iff M_y(x, y) = N_x(x, y).$$

Suppose we have a solution  $f(x, y(x)) = C$ . Then,

$$\begin{aligned} \frac{d}{dx}(f(x, y(x))) &= 0 \\ \implies \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} &= 0 \\ \implies \frac{f_x}{f_y} &= -\frac{dy}{dx} \end{aligned}$$

Now, with  $f_x(x, y) = M(x, y)$  and  $f_y = N(x, y)$ , then  $M_y(x, y) = f_{xy}(x, y)$  and  $N_x = f_{yx}(x, y)$ . Assuming  $f$  continuous with existing, continuous partial derivatives, then  $f_{xy} = f_{yx}$  and hence  $M_y(x, y) = N_x(x, y)$ . Thus, a function  $f$  such that  $f_x = M$  and  $f_y = N$  yields a solution to the ODE.

⊛ **Example 2.4**

$$\begin{aligned}
 2xy^2 dx + 2x^2y dy &= 0 \equiv M dx + N dy = 0 \\
 \implies M_y &= 4xy, \quad \implies N_x = 4xy \\
 f_x = M = 2xy^2 &\implies f(x, y) = x^2y^2 + C + F(y) \\
 f_y = N = 2x^2y &\implies f(x, y) = x^2y^2 + C + F(x) \\
 &\implies f(x, y) = x^2y^2 + C = K
 \end{aligned}$$

We can rearrange this as an explicit solution

$$y = \frac{k}{x}$$

for some constant  $k$ .

↪ Lecture 03; Last Updated: Tue Jan 16 10:10:00 EST 2024

↪ **Theorem 2.1**

This technique works generally.

Proof. Given an exact ODE of the form  $M(x, y) dx + N(x, y) dy = 0$ , we need to show that  $\exists f(x, y)$  s.t.  $f(x, y) = c$  solves the ODE. Let

$$f(x, y) = \int_{x_0}^x M(s, y) ds + g(y)$$

for some function  $g(y)$  to be chosen such that  $f_y = N$ . But we have

$$\begin{aligned}
 N(x, y) = f_y(x, y) &= \frac{\partial}{\partial y} \left[ \int_{x_0}^x M(s, y) ds + g(y) \right] \\
 &= g'(y) + \frac{\partial}{\partial y} \int_{x_0}^x M(s, y) ds \\
 \implies g'(y) &= N(x, y) - \frac{\partial}{\partial y} \int_{x_0}^x M(s, y) ds.
 \end{aligned}$$

But the LHS is a function of  $y$  only, while the RHS depends explicitly on  $x$ ; hence, this technique will only work if the entire expression is actually independent of  $x$ . To show this, we take the partial of the RHS with respect to  $x$ :

$$\begin{aligned}
 \frac{\partial}{\partial x} \left[ N(x, y) - \frac{\partial}{\partial y} \int_{x_0}^x M(s, y) ds \right] &= N_x(x, y) - \frac{\partial}{\partial x} \frac{\partial}{\partial y} \int_{x_0}^x M(s, y) ds \\
 &= N_x(x, y) - \frac{\partial}{\partial y} \left[ \frac{\partial}{\partial x} \int_{x_0}^x M(s, y) ds \right] \\
 &= N_x(x, y) - \frac{\partial}{\partial y} [M(x, y)] \\
 &= N_x - M_y = 0,
 \end{aligned}$$

as the ODE is exact. Hence, the RHS is indeed a function of  $y$  alone. So, integrating both sides with respect to  $y$ :

$$g(y) = \int \left[ N(x, y) - \frac{\partial}{\partial y} \int_{x_0}^x M(s, y) \, ds \right] dy,$$

which gives us a  $f(x, y)$  of

$$\begin{aligned} f(x, y) &= \int_{x_0}^x M(s, y) \, ds + \int \left[ N(x, y) - \frac{\partial}{\partial y} \int_{x_0}^x M(s, y) \, ds \right] dy, \\ \implies f(x, y) &= \int_{x_0}^x M(s, y) \, ds + \int_{y_0}^y N(x, t) \, dt - \int_{y_0}^y \int_{x_0}^x M_y(s, t) \, ds \, dt \quad \star \end{aligned}$$

which satisfies  $f_x = M$  and  $f_y = N$ . Then, for  $f(x, y) = C$ , we have

$$\frac{\partial f}{\partial x} + \frac{dy}{dx} \frac{\partial f}{\partial y} = M + \frac{dy}{dx} N = 0 \implies M \, dx + N \, dy = 0,$$

as desired.

Note that  $\star$  is evaluated over a rectangle  $[x_0, x] \times [y_0, y]$ , but holds for any connected domain containing  $(x_0, y_0)$  and  $(x, y)$ .

Also note that, as described,  $g(y)$  is not a function of  $x$ ; hence, we can pick  $x$  arbitrarily. Suppose we take  $x = x_0$ , then

$$f(x, y) = \int_{x_0}^x M(s, y) \, ds + \int_{y_0}^y N(x_0, t) \, dt.$$

■

**Remark 2.2.** We could have taken  $g(x)$  and started from  $f_y = N$ . Then, we would have had the formula

$$f(x, y) = \int_{y_0}^y N(x, t) \, dt + \int_{x_0}^x M(s, y_0) \, ds.$$

⊛ Example 2.5

$$2xy \, dx + (x^2 - 1) \, dy = 0.$$

We have  $M(x, y) = 2xy$  and  $N(x, y) = x^2 - 1$ , so  $M_y = 2x = N_x$  and the ODE is exact; hence, a solution exists of the form  $f(x, y) = c$  where  $f_x = M$ ,  $f_y = N$ .

$$\begin{aligned} f(x, y) &= \int M(x, y) \, dx = \int 2xy \, dx = x^2y + k_1(y) \\ f(x, y) &= \int N(x, y) \, dy = \int (x^2 - 1) \, dy = x^2y - y + k_2(x) \end{aligned}$$

Hence  $k_1(y) = -y$  and  $k_2(x) = 0$ , so

$$f(x, y) = x^2y - y = y(x^2 - 1),$$

so solutions to the original ODE are

$$y(x^2 - 1) = C \implies y = \frac{C}{x^2 - 1}.$$

## 2.4 Exact ODEs Via Integrating Factors

Suppose

$$M(x, y) \, dx + N(x, y) \, dy = 0$$

but  $M_y \neq N_x$ , that is, the ODE is not exact. Can we find an integrating factor  $\mu(x, y)$  s.t.

$$[\mu(x, y)M(x, y)] \, dx + [\mu(x, y)N(x, y)] \, dy = 0$$

is exact? If so, such a  $\mu$  must satisfy

$$\begin{aligned} \frac{\partial}{\partial y} [\mu(x, y)M(x, y)] &= \frac{\partial}{\partial x} [\mu(x, y)N(x, y)] \\ \implies \mu_y M + \mu M_y &= \mu_x N + \mu N_x \\ \implies N\mu_x - M\mu_y &= (M_y - N_x)\mu \quad \text{⊛} \end{aligned}$$

This is not a generally easily soluble PDE; we will consider cases where  $\mu$  is a function of only one independent variable, which greatly simplifies the expression; this could be simply  $\mu(x)$ ,  $\mu(y)$ , or even  $\mu(x \cdot y)$ .

Suppose  $\mu = \mu(x) \implies \mu_y = 0$ . Then, ⊛ becomes

$$N\mu' = (M_y - N_x)\mu \implies \mu' = \left( \frac{M_y - N_x}{N} \right) \mu.$$

This is valid, provided the expression  $\left( \frac{M_y - N_x}{N} \right)$  is a function solely of  $x$ . In this case, this becomes a linear first order

ODE, with solution

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}.$$

OTOH, if  $\mu = \mu(y)$ , we can similarly derive

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy},$$

with a similar stipulation on the expression  $\left(\frac{N_x - M_y}{M}\right)$  being a function of  $y$  solely.

⊛ **Example 2.6**

$$xy \, dx + (2x^2 + 3y^2 - 20) \, dy = 0,$$

with  $M(x, y) = xy \implies M_y = x$  and  $N(x, y) = 2x^2 + 3y^2 - 20 \implies N_x = 4x$ . We have  $M_y - N_x = x - 4x = -3x$  (so the ODE is not exact). We write

$$\frac{M_y - N_x}{M} = \frac{-3x}{xy} = \frac{-3}{y},$$

which is a function solely of  $y$ ; hence, can find a  $\mu(y)$ :

$$\mu(y) = e^{-\int \frac{M_y - N_x}{M} dy} = e^{-\int -\frac{3}{y} dy} = e^{3 \ln y} = y^3,$$

noting that we, as before, do not care about any integrating factors; we are seeking a single function that works. Multiplying this into our original ODE:

$$\underbrace{xy^4 \, dx}_{:=\tilde{M}} + \underbrace{(2x^2 + 3y^2 - 20)y^3 \, dy}_{:=\tilde{N}} = 0.$$

And indeed, we have

$$\tilde{M}_y = 4xy^3; \quad \tilde{N}_x = 4xy^3 \implies \tilde{M}_y = \tilde{N}_x,$$

as desired.

↪ Lecture 04; Last Updated: Tue Jan 23 10:02:55 EST 2024

↪ Lecture 05; Last Updated: Tue Jan 23 10:23:37 EST 2024

## 2.5 Qualitative Methods and Theory

**Remark 2.3.** Read the first few chapters of Strogatz's *Nonlinear Dynamics and Chaos* book and you should be all good.

⊛ **Example 2.7**

Show that  $y' = y^{\frac{1}{3}}$  with  $y(0) = 0$  has infinite solutions.

↪ Lecture 06; Last Updated: Tue Jan 23 11:21:04 EST 2024

## 2.6 Existence and Uniqueness

### ↪ Definition 2.4: Lipschitz Continuity

A function  $f(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}$  is said to be *Lipschitz continuous* in  $y$  on the rectangle  $R = \{(x, y) : x \in [a, b], y \in [c, d]\} = [a, b] \times [c, d]$  if there exists a constant  $L > 0$  s.t.

$$|f(x, y_1) - f(x, y_2)| \leq L |y_1 - y_2|, \quad \forall (x, y_1), (x, y_2) \in R.$$

$L$  is called the *Lipschitz constant*.

**Remark 2.4.** Note that we define in terms on continuity in  $y$ ; the  $x$  variable in each coordinate is kept constant.

### ↪ Lemma 2.1

If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is such that  $f(x, y)$  and  $\frac{\partial f}{\partial y}$  are both continuous in  $x, y$  in the rectangle  $R$ , then  $f$  is Lipschitz in  $y$  on  $R$ .

**Remark 2.5.** This result gives *Differentiable*  $\implies$  *Lipschitz Continuous*  $\implies$  *Continuous*.

Proof. Using FTC, we have

$$\begin{aligned} f(x, y_2) &= f(x, y_1) + \int_{y_1}^{y_2} f_y(x, y) \, dy \\ \implies |f(x, y_2) - f(x, y_1)| &= \left| \int_{y_1}^{y_2} f_y(x, y) \, dy \right| \leq \int_{y_1}^{y_2} |f_y(x, y)| \, dy \\ &\leq |y_2 - y_1| \cdot \max_{(x, y) \in R} |f_y(x, y)|, \end{aligned}$$

noting that this maximum exists, and is attained, because  $f_y$  is continuous on a compact set. This gives, then, that  $f$  is Lipschitz in  $y$  with  $L = \max_{(x, y) \in R} |f_y(x, y)|$ . ■

↪ **Theorem 2.2: Existence and Uniqueness for Scalar First Order IVPs**

If  $f(t, y), f_y(t, y)$  are continuous in  $t$  and  $y$  on a rectangle  $R = \{(t, y) : t \in [t_0 - a, t_0 + a], y \in [y_0 - b, y_0 + b]\} = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$ , then  $\exists h \in (0, a]$  s.t. the IVP

$$y' = f(t, y), y(t_0) = y_0$$

has a unique solution, defined for  $t \in [t_0 - h, t_0 + h]$ . Moreover, this solution satisfies  $y(t) \in [y_0 - b, y_0 + b] \forall t \in [t_0 - h, t_0 + h]$ .

**Remark 2.6.** A stronger theorem also holds with a weakened condition on  $f$  that requires only  $f$  Lipschitz. Clearly,  $f_y$  continuous  $\implies f$  Lipschitz, so we will use this fact to prove the statement, but won't prove it for the only Lipschitz case for sake of conciseness.

Proof. Rewrite the IVP as

$$y(t) = y(t_0) + \int_{t_0}^t f(s, y(s)) \, ds.$$

We will show this form has a unique solution, using an iteration method (namely, Picard Iteration).

We will begin by guessing a solution of the IVP,  $y_0(t) = y_0, \forall t \in [t_0 - a, t_0 + a]$ . This clearly satisfies the initial condition, but not the ODE itself.

Now, given  $y_n(t)$ , we define

$$y_{n+1}(t) = y(t_0) + \int_{t_0}^t f(s, y_n(s)) \, ds.$$

If this terminates, that is,  $y_{n+1}(t) = y_n(t) \forall t \in [t_0 - a, t_0 + a]$ , then  $y_n(t)$  solves the IVP.

We now show that this iteration is both well-defined, and converges to unique solution.

By construction,  $y_0 : [t_0 - a, t_0 + a] \rightarrow [y_0 - b, y_0 + b]$ , and is continuous. As a bounded function on a bounded interval, it is integrable, and the first step of our step is well-defined.

Now suppose  $y_n(t) : [t_0 - a, t_0 + a] \rightarrow [y_0 - b, y_0 + b]$  is continuous and integrable. Then,

$$y_{n+1}(t) = y(t_0) + \int_{t_0}^t f(s, y_n(s)) \, ds$$

is continuous as well, as  $f$  is continuous and  $y_n(s)$  is as well. It is not guaranteed to be restricted to  $[y_0 - b, y_0 + b]$ , however.

Since  $f$  continuous and attains its maximum on  $R$ , let

$$M := \max_{(t,y) \in R} |f(t, y)| < \infty.$$

We have, then, that

$$\begin{aligned} y_{n+1}(t) - y(t_0) &= \int_{t_0}^t f(s, y_n(s)) \, ds \\ \implies |y_{n+1}(t) - y(t_0)| &\leq |t - t_0| M \end{aligned}$$

Hence, if we choose  $h : Mh \leq b$ , and then  $y_{n+1}(t) : [t_0 - h, t_0 + h] \rightarrow [y_0 - b, y_0 + b]$  and we can iterate inductively,  $y_n(t) : [t_0 - h, t_0 + h] \rightarrow [y_0 - b, y_0 + b] \forall n$ . Here, we take  $h = \min\{\frac{b}{M}, a\}$ .

Now, let  $I = [t_0 - h, t_0 + h]$ , then  $y_n(t) : I \rightarrow [y_0 - b, y_0 + b]$  for all  $n$ . Each iterate satisfies  $y_n(t_0) = y(t_0) = y_0$ ; it remains to show that the iteration converges.

Let  $C(I, [y_0 - b, y_0 + b])$  be the space of continuous functions  $f : I \rightarrow [y_0 - b, y_0 + b]$ , noting that  $y_n \in C \forall n$ . We define a mapping on  $C$ ,  $T : C \rightarrow C$  by

$$v = Tu, v(t) = y_0(t_0) + \int_{t_0}^t f(s, u(s)) \, ds.$$

Then,  $y_{n+1} = Ty_n$ . We aim to show that this iteration converges uniquely; we will do this by showing  $T$  is a contraction mapping.

For  $y \in C$  define the norm  $\|y\|_\infty$  by  $\|y\|_\infty := \max_{t \in I} |y(t)|$ . This is a norm;

1.  $\forall k \in \mathbb{R}, \|ky\|_\infty = |k| \|y\|_\infty$ .
2.  $\|y\|_\infty = 0 \iff \max_{t \in I} |y(t)| = 0 \iff y(t) = 0 \forall t \in I$ .
3.  $\|y_1 + y_2\|_\infty = \max_{t \in I} |y_1 + y_2| \leq \max_{t \in I} (|y_1| + |y_2|) \leq \max_{t \in I} |y_1| + \max_{t \in I} |y_2| = \|y_1\|_\infty + \|y_2\|_\infty$ .

Now let  $u, v \in C$ . Then,

$$\begin{aligned} \|Tu - Tv\|_\infty &= \max_{t \in I} |Tu(t) - Tv(t)| \\ &= \max_{t \in I} \left| y(t_0) + \int_{t_0}^t f(s, u(s)) \, ds - y_0 + \int_{t_0}^t f(s, v(s)) \, ds \right| \\ &= \max_{t \in I} \left| \int_{t_0}^t f(s, u(s)) - f(s, v(s)) \, ds \right| \\ &\leq \max_{t \in I} \int_{t_0}^t |f(s, u(s)) - f(s, v(s))| \, ds \\ &\leq \max_{t \in I} |t - t_0| \cdot \max_{s \in I} |f(s, u(s)) - f(s, v(s))| \\ &\leq hL \cdot \max_{s \in I} |u(s) - v(s)| \\ &= hL \cdot \|u - v\|_\infty, \end{aligned}$$

hence, we have a contraction mapping if  $hL < 1$ ; if  $hL \geq 1$ , let  $h < \min\{a, \frac{b}{m}, \frac{1}{L}\} > 0$ . With such an  $h$ ,  $\exists \mu \in (0, 1) : hL \leq \mu < 1$ , and  $\|Tu - Tv\|_\infty \leq \mu \|u - v\|_\infty$ , hence, a contraction mapping.

The contractive mapping theorem, which will not be proven, states that any contraction mapping has a unique fixed point  $y = Ty$ ; moreover, for any  $y_0 \in C$ , the iteration  $y_{n+1} = Ty_n$  converges to  $y$ .



To see this, suppose  $u = Tu, v = Tv$  are two solutions of our IVP. Then, by the contraction quality,

$$\|u - v\|_\infty = \|Tu - Tv\|_\infty \leq \mu \|u - v\|_\infty,$$

a contradiction unless  $\|u - v\|_\infty = 0 \iff u = v$ , hence, we have uniqueness of our solution; that is, our IVP has at most one solution. It remains to show that this solution exists.

Consider a sequence  $y_n$ , with  $y_{n+1} = Ty_n$ . Then,

$$\sum_{i=0}^N \|y_{i+1} - y_i\|_\infty \leq \mu^N \|y_1 - y_0\|_\infty,$$

by the contractive property, thus,

$$\sum_{i=0}^{\infty} \|y_{i+1} - y_i\|_\infty \leq \left( \sum_{i=0}^{\infty} \mu^i \right) \|y_1 - y_0\|_\infty = \frac{1}{1 - \mu} \|y_1 - y_0\|_\infty = R_0,$$

for some radius (real number)  $R_0$ . Similarly, looking only at the tail of the series,

$$\sum_{j=n}^{\infty} \|y_{j+1} - y_j\|_\infty \leq \frac{\mu^n}{1 - \mu} \|y_1 - y_0\|_\infty = \mu^n R_0,$$

that is, a “smaller” radius. We could, but won’t, show that this sequence is Cauchy, and space  $C$  we are working in is complete and hence this sequence converges to some limit in the space; moreover, the limit of this sequence satisfies the IVP by construction. This is beyond the scope of this course. ■

⊛ **Example 2.8: Using Picard Iteration**

$$y' = 2t(1 + y) =: f(t, y), \quad y(0) = 0.$$

This ODE is linear and separable, and has solution  $y(t) = e^{t^2} - 1$  (solving whichever way you like). We can alternatively solve this using Picard Iteration.

Let  $y_0(t) = 0 \forall t$ , noting that the IC is satisfied. We define

$$y_{n+1}(t) = y_0(t) + \int_0^t f(s, y_n(s)) \, ds,$$

where  $f(s, y_n(s)) = 2s(1 + y_n(s))$ . This gives

$$\begin{aligned} y_{n+1}(t) &= \int_0^t 2s(1 + y_n(s)) \, ds. \\ \implies y_1(t) &= \int_0^t 2s(1 + y_0(s)) \, ds = \int_0^t 2s \, ds = t^2 \\ \implies y_2(t) &= \int_0^t 2s(1 + s^2) \, ds = t^2 + \frac{1}{2}t^4 \\ \implies y_3(t) &= \dots = t^2 + \frac{1}{2!}t^4 + \frac{1}{3!}t^6 \\ \dots \implies y_n(t) &= \sum_{k=1}^n \frac{t^{2k}}{k!} \\ \implies \lim_{n \rightarrow \infty} y_n(t) &= \sum_{k=1}^{\infty} \frac{(t^2)^k}{k!} = e^{t^2} - 1, \end{aligned}$$

the same solution as previously shown.

**Remark 2.7.** The previous example worked nicely due to  $y_n(t)$  always being a simple polynomial with a familiar convergence. This is not always (nor often) the case.

**Remark 2.8.** Recall the example  $y' = y^{\frac{1}{3}}$  with multiple solutions. In the language of the theorem,  $f(t, y) = y^{\frac{1}{3}}$  is continuous, but  $f_1(t, y) = \frac{1}{3}y^{-\frac{2}{3}}$  becomes unbounded as  $y \rightarrow 0$ , and the function is thus not Lipschitz in a neighborhood of  $y = 0$ .

**Remark 2.9.** Recall that this theorem guarantees solutions in a closed rectangular region; it is possible, under certain conditions, to extend the solution beyond the bounds. But how far?

⊛ **Example 2.9**

$$y' = y^2, \quad y(0) = 1.$$

This has a solution  $y(t) = \frac{1}{c-t} = \frac{1}{1-t}$  (with IC). Notice that  $y(t) \rightarrow +\infty$  as  $t \rightarrow 1$ . By this observation, we have that, if we were to repeat Picard iteration for increasing time  $t$ , the rectangular domains of our validity of each piecewise solution would be bounded by 1.

### ↪ **Corollary 2.1**

If  $f(t, y)$  and  $f_y(t, y)$  are continuous for all  $t, y \in \mathbb{R}$ , then  $\exists t_- < t_0 < t_+$  such that the IVP

$$y' = f(t, y), \quad y(t_0) = y_0$$

has a unique solution  $y(t) \forall t \in (t_-, t_+)$ , and moreover, either  $t_+ = +\infty$  or  $\lim_{t \rightarrow t_+} |y(t)| = \infty$ , and either  $t_- = -\infty$  or  $\lim_{t \rightarrow t_-} |y(t)| = \infty$ .

**Remark 2.10.** Finding  $t_-, t_+$  requires the solution. In example 2.9,  $t_- = -\infty, t_+ = 1$ . Changing the IC will naturally change these values.

### ↪ **Theorem 2.3**

If  $p(t), g(t)$  continuous on an open interval  $I = (\alpha, \beta)$  and  $t_0 \in I$ , then the IVP

$$y'(t) + p(t)y = g(t), \quad y(t_0) = y_0$$

has a unique solution  $y(t) : I \rightarrow \mathbb{R}$ .

**Remark 2.11.** In other words, this is a special case of the corollary above for linear ODEs; any “misbehavior” of the solutions would be solely due to discontinuities in the defining ODE.

## 3 Second Order ODEs

### 3.1 Introduction

Second Order ODEs are of the form

$$y'' = f(t, y, y').$$

There is no general technique to solving these; we will be looking at special classes throughout.

Specifically in the case of nonlinear odes, there are two special cases we can solve,

1.  $f$  does not depend on  $y$ ; ie  $y'' = f(t, y')$ . A substitution  $u = y'$  yields  $u' = f(t, u)$ , hence this is just a first order ODE, with corresponding  $y(t) = k_1 + \int u(t) dt$ .
2.  $f$  does not depend on  $t$ ; ie  $y'' = f(y, y')$ . Let  $u = y'$ , so  $u' = y'' = f(y, u)$ . Consider  $u = u(y(t))$ , then,

$$\frac{du}{dt} = \frac{du}{dy} \frac{dy}{dt} = u \frac{du}{dy},$$

and so

$$u \frac{du}{dy} = \frac{du}{dt} = f(y, u) \implies \frac{du}{dy} = \frac{1}{y} f(y, u),$$

which again yields a first order ODE, in  $u = u(y)$ .

⊛ Example 3.1: Of Case 2.

$$y'' + \omega^2 y = 0^a$$

Rewrite this as  $y'' = -\omega^2 y = f(y, y')$ , and let  $u = y'$ , then  $\frac{du}{dy} = \frac{1}{u} f(y, u) = \frac{1}{u} [-\omega^2 y]$ . This is a separable equation:

$$\begin{aligned} u \, du &= -\omega^2 y \, dy \\ \frac{1}{2} u^2 &= -\frac{1}{2} \omega^2 y^2 + c \\ \implies u^2 &= -\omega^2 y^2 + c' \\ \implies u = \pm \sqrt{k^2 - \omega^2 y^2} &\implies \frac{dy}{dt} = \pm \sqrt{k^2 - \omega^2 y^2} \end{aligned}$$

Which is just another separable equation<sup>b</sup>:

$$\begin{aligned} \pm \int dt &= \frac{1}{\omega} \int \frac{dy}{\sqrt{\frac{k^2}{\omega^2} - y^2}} \\ \implies \frac{1}{\omega} \arcsin\left(\frac{\omega y}{k}\right) &= \pm t + C \\ \implies \frac{\omega y}{k} = \sin\left(\pm \omega t \pm \omega \tilde{C}\right) &= \pm \sin\left(\omega t + \omega \tilde{C}\right) \\ \implies y(t) = \pm \frac{k}{\omega} \sin\left(\omega t + \omega \tilde{C}\right) \\ \implies y(t) &= K \sin(\omega t + C), \end{aligned}$$

which can be rewritten  $y(t) = k_1 \sin(\omega t) + k_2 \cos(\omega t)$  with the appropriate substitutions.

<sup>a</sup>This is the equation for a simple harmonic oscillator.

<sup>b</sup>Please excuse the sloppy use of constants, it doesn't really matter.

**Remark 3.1.** *This is not the easiest way to solve this equation. More generally, this technique can lead to intractable integrals.*

⊗ **Example 3.2: Nonlinear Pendulum**

$$y'' + \omega^2 \sin y = 0.$$

Making the same substitution as before,  $u = y'$ , we have

$$\begin{aligned} \frac{du}{dy} &= -\frac{1}{u} \omega^2 \sin y \\ \int u \, du &= \int -\omega^2 \sin y \, dy \\ \frac{1}{2} u^2 &= \omega^2 \cos y + c_1 \\ \frac{1}{2} (y')^2 &= \omega^2 \cos y + c_1 \\ y' &= \pm \sqrt{2c_1 + 2\omega^2 \cos y} \\ \pm \int dt &= \int \frac{dy}{\sqrt{2c_1 + 2\omega^2 \cos y}}, \end{aligned}$$

where the integral on the RHS is some type of elliptic integral.

## 3.2 Linear, Homogeneous

We will solve a general form

$$a(t)y'' + b(t)y' + c(t)y = 0 \quad \text{⊗}.$$

### 3.2.1 Principle of Superposition

↪ **Theorem 3.1: Superposition of Solutions to Linear Second Order ODEs**

If  $y_1(t)$ ,  $y_2(t)$  solve ⊗ for  $t \in I$ -interval, then  $y(t) = k_1 y_1(t) + k_2 y_2(t)$ , for constants  $k_1, k_2$  solves ⊗ on  $I$  as well. In other words, linear combinations of solutions are themselves solutions.

**Remark 3.2.** This can be extended quite naturally to any linear order of ODE.

Proof. This is clear by just plugging into the problem; let  $y(t) = k_1 y_1(t) + k_2 y_2(t)$ . Then:

$$\begin{aligned} a(t)y''(t) + b(t)y'(t) + c(t)y(t) &= a(t)(k_1 y_1'' + k_2 y_2'') + b(t)(k_1 y_1' + k_2 y_2') + c(t)(k_1 y_1 + k_2 y_2) \\ &= k_1 (a y_1'' + b y_1' + c y_1) + k_2 (a y_2'' + b y_2' + c y_2) \\ &= k_1 \cdot 0 + k_2 \cdot 0 = 0, \end{aligned}$$

as desired. ■

### ↪ Definition 3.1: Linear Independence of Functions

If  $y_1(t), y_2(t)$  are defined  $\forall t \in I$  for some interval  $I \subseteq \mathbb{R}$ , then  $y_1(t), y_2(t)$  are *linearly dependent on  $I$*  if  $\exists k_1, k_2$  constants (not both zero) so that  $k_1 \cdot y_1(t) + k_2 \cdot y_2(t) = 0 \forall t \in I$ .

If the only constants which solve this are  $k_1 = k_2 = 0$ , then  $y_1(t), y_2(t)$  are linearly independent on  $I$ .

**Remark 3.3.** If  $y_j(t)$  is the zero function, then take  $k_j = 1$  and the other constant zero; ie, the zero function is always linearly dependent.

## 3.3 Reduction of Order

Suppose  $y_1(t)$  solves the homogeneous ODE  $0 = a(t)y'' + b(t)y' + c(t)y$ . Let  $y(t) = u(t)y_1(t)$  for some unknown  $u(t)$ , and assume it solves the ODE. Then:

$$y = uy_1 \implies y' = u'y_1 + uy_1' \implies y'' = u''y_1 + u'y_1' + u'y_1' + uy_1'' = uy_1'' + 2u'y_1' + u''y_1.$$

Substituting this into the original ODE:

$$\begin{aligned} 0 &= a(u''y_1 + 2u'y_1' + uy_1'') + b(u'y_1 + uy_1') + c(uy_1') \\ &= [ay_1]u'' + [2ay_1' + by_1]u' + \underbrace{[ay_1'' + by_1' + cy_1]}_{=0}u \end{aligned}$$

Let  $v = u' \implies v' = u''$ , and we have reduced to a first-order ODE

$$0 = [ay_1]v' + [2ay_1' + by_1]v$$

which we can solve for  $v$ , then conclude by integrating  $v$  to solve for  $u$ .

## 3.4 Constant Coefficient Linear Homogeneous Second Order ODEs

We consider the case

$$ay'' + by' + cy = 0,$$

where  $a, b, c$  are constants. If  $a = 0$ , this is simply first order with an exponential solution; so, suppose (guess) that this ODE has solution  $y = e^{rt}$  for  $a \neq 0$ . This gives

$$\begin{aligned} a(e^{rt})'' + b(e^{rt})' + c(e^{rt}) &= 0 \\ \implies ar^2e^{rt} + bre^{rt} + ce^{rt} &= 0 \\ \implies ar^2 + br + c = 0 \implies r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

and we thus have just to solve a quadratic equation. We call this the *auxiliary equation* or *characteristic equation* for the ODE.

We thus have three cases to consider:

1.  $b^2 > 4ac$ :  $r$  has two real roots, giving two real solutions to the original ODE of the form

$$y_1(t) = e^{r_+t}, \quad y_2(t) = e^{r_-t},$$

where  $r_{\pm} := r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ . Note that  $\frac{y_2}{y_1} = e^{(r_- - r_+)t}$  is non-constant hence these solutions are independent. It follows that we have a general solution

$$y(t) = k_1 e^{r_+t} + k_2 e^{r_-t}$$

for arbitrary constants  $k_1, k_2$ .

2.  $b^2 = 4ac$ :  $r$  has one real (repeated) solution,  $r = \frac{-b}{2a}$ . This gives only one solution  $y_1 = e^{r_1t}$ : we find another by reduction of order. Let  $y = uy_1 = ue^{r_1t} = ue^{\frac{-bt}{2a}}$ . We have:

$$\begin{aligned} 0 &= ay'' + by' + cy \\ 0 &= a(u''y_1 + 2u'y_1' + uy_1'') + b(u'y_1 + uy_1') + cuy' \\ 0 &= ay_1u'' + (2ay_1' + by_1)u' + \overbrace{(ay_1'' + by_1' + cy_1)}^0 u \\ 0 &= ae^{rt}u'' + (2are^{rt} + be^{rt})u' \\ 0 &= au'' + (2ar + b)u' \\ 0 &= au'' + \left(-\frac{2ab}{2a} + b\right)u' \\ 0 &= au'' \\ 0 &= u'' \implies u' = k_1 \implies u = k_1t + k_2, \end{aligned}$$

and so we have another solution  $y = uy_1 = (k_1t + k_2)e^{rt}$ ; these constants  $k_1, k_2$  are arbitrary (as long as  $k_1 \neq 0$ , which would just give a linearly dependent solution to the original), so take  $k_1 = 1, k_2 = 0$ . This gives a general solution

$$y(t) = c_1 e^{rt} + c_2 t e^{rt} = (c_1 + c_2 t) e^{rt},$$

which is actually just the “second” solution we found before (and thus this one was indeed the general solution by itself).

3.  $b^2 < 4ac$ :  $r$  has two complex conjugate roots  $r_{\pm} = -\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a}i := \alpha \pm i\beta$ . This gives solutions

$$y_+ = e^{(\alpha + i\beta)t}, \quad y_- = e^{(\alpha - i\beta)t}.$$

While valid, these complex solutions are not necessarily useful in this form; we can rewrite them using Euler’s formula to take only the real parts.

$$\begin{aligned} y_+ &= e^{(\alpha + i\beta)t} = e^{\alpha t} e^{i\beta t} = e^{\alpha t} [\cos(\beta t) + i \sin(\beta t)] \\ y_- &= e^{(\alpha - i\beta)t} = e^{\alpha t} e^{-i\beta t} = e^{\alpha t} [\cos(-\beta t) + i \sin(-\beta t)] = e^{\alpha t} [\cos(\beta t) - i \sin(\beta t)] \end{aligned}$$

Let  $y_1 = \frac{1}{2}(y_+ + y_-) = e^{\alpha t} \cos(\beta t)$ ; this is a first, purely real solution to our ODE. To find a second, we could use reduction of order, or just take another linear combination of  $y_+, y_-$

$$y_2 = \frac{1}{2i}[y_+ - y_-] = e^{\alpha t} \sin(\beta t).$$

$y_1, y_2$  are linearly independent, since  $\frac{y_2}{y_1} = \tan(\beta t) = 0 \forall t \iff \beta = 0$ , which we assumed was not the case (otherwise, we'd be in case 2.). Together, we have a general, purely real solution

$$y(t) = e^{\alpha t}(k_1 \sin(\beta t) + k_2 \cos(\beta t)),$$

where  $k_1, k_2$  arbitrary and  $r = \alpha \pm i\beta$ .

Harding once said: that “there is no permanent place in the world for ugly mathematics”; that means that there is a temporary place in the world for ugly mathematics. Make it pretty later.

### ⊛ Example 3.3

1.  $y'' - 3y' + 2y = 0$

This gives an auxiliary equation  $r^2 - 3r + 2 = 0$  with solution  $r = \frac{3 \pm \sqrt{9-8}}{2} = 2, 1$ . These are both positive and real, and we thus have a general solution

$$y(t) = k_1 e^t + k_2 e^{2t}.$$

2.  $y'' - 2y' + y = 0$

$$\begin{aligned} r^2 - 2r + 1 = 0 &\implies (r-1)(r-1) = 0 \implies r = 1 \\ &\implies y(t) = (k_1 t + k_2) e^t \end{aligned}$$

3.  $y'' + 4y' + 7y = 0$

$$\begin{aligned} r^2 + 4r + 7 = 0 &\implies r = \frac{-4 \pm \sqrt{16-28}}{2} = -2 \pm i\frac{1}{2}\sqrt{12} = -2 \pm i\sqrt{3} \\ &\implies y(t) = e^{-2t}(k_1 \sin(\sqrt{3}t) + k_2 \cos(\sqrt{3}t)) \end{aligned}$$

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## 3.5 Nonhomogeneous Second Order ODEs

We consider equations of the form

$$a(t)y'' + b(t)y' + cy = g(t).$$

Let's look for solutions of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t),$$



where  $y_1, y_2$  are linearly independent solutions of the homogenous equation ( $g = 0$ ) and  $y_p$  is a particular solution to the ODE. Plugging this into the original equation:

$$\begin{aligned} ay'' + by' + cy &= a(c_1y_1'' + c_2y_2'' + y_p'') + b(c_1y_1' + c_2y_2' + y_p') + c(c_1y_1 + c_2y_2 + y_p) \\ &= c_1(\overbrace{ay_1'' + by_1' + cy_1}^0) + c_2(\overbrace{ay_2'' + by_2' + cy_2}^0) + \overbrace{ay_p'' + by_p' + cy_p}^g \\ &= g, \end{aligned}$$

as desired. Indeed, all solutions are of this form; we will show this later.

**Remark 3.4.** Note that  $c_1, c_2$  are arbitrary constants;  $y_p$  is not multiplied by a constant, and should not be.

**Remark 3.5.**  $y_1, y_2$  are called a fundamental set of solutions;  $y_c = c_1y_1 + c_2y_2$ , the general solution to the homogeneous equation, is called the complementary solution of the nonhomogeneous equation.  $y = y_c + y_p$  is the general solution of the nonhomogeneous equation.

### 3.5.1 Linear Operator Notation

We denote  $C(\mathbb{R})$  to be the space of continuous functions on  $\mathbb{R}$ . Let  $C^p(\mathbb{R})$  be the space of  $p$ -times differentiable functions on  $\mathbb{R}$ ; ie,  $y \in C^p(\mathbb{R}) \implies y^{(j)} \in C(\mathbb{R}), j = 0, 1, \dots, p$ . Notice that  $C^{p+1}(\mathbb{R}) \subsetneq C^p(\mathbb{R})$ . It follows that  $C^\infty(\mathbb{R}) \subsetneq \dots \subsetneq C^n(\mathbb{R}) \subsetneq \dots \subsetneq C(\mathbb{R})$ .

Let  $D : C^n(\mathbb{R}) \rightarrow C^{(n-1)}(\mathbb{R})$  be the differentiation operator, ie  $Dy = y'$ , noting that  $Dy$  less differentiable than  $y$  unless  $y \in C^\infty(\mathbb{R})$ . Its clear that  $D$  is a linear operator.

Now, define the operator  $L = a(x)D^2 + b(x)D + c(x)$ . Then,  $L[y] = a(x)y'' + b(x)y' + c(x)y$ ; hence,  $L[y] = 0$  and  $L[y] = g$  are equivalent to our homogeneous and nonhomogeneous equations. It is clearly linear.

We explore two methods for finding the particular solution.

### 3.5.2 Finding $y_p$ : Method of Undetermined Coefficients

This method only applies to ODEs with constant coefficients, and only for certain functions  $g$ .

#### ⊛ Example 3.4

Consider  $g(t) = L[y](t)$ . Suppose  $g(t) = \mu e^{\gamma t}$ . Let's guess that  $y_p = Ae^{\gamma t}$ . Then:

$$L[y_p] = aA\gamma^2 e^{\gamma t} + bA\gamma e^{\gamma t} + cAe^{\gamma t} = (a\gamma^2 + b\gamma + c)Ae^{\gamma t},$$

hence, for  $L[y_p] = g = \mu e^{\gamma t}$ , we need  $\mu = A(a\gamma^2 + b\gamma + c) \implies A = \frac{\mu}{a\gamma^2 + b\gamma + c}$ . Provided  $a\gamma^2 + b\gamma + c \neq 0 \iff \gamma$  does not solve auxiliary equation, this  $A$  as defined will provide  $y_p$ .

**Remark 3.6.** This example worked\* because differentiating the exponential yields another exponential, which cancel nicely. The same idea can be applied for polynomials and trig functions.

⊛ **Example 3.5: With trig**

Suppose  $L[y] = y'' - y' + y = g(t) = 2 \sin(3t)$ , with auxiliary equation  $r^2 - r + 1 = 0 \implies r = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ . This gives complementary solution

$$y_c = e^{\frac{t}{2}} \left( k_1 \sin\left(\frac{\sqrt{3}}{2}t\right) + k_2 \cos\left(\frac{\sqrt{3}}{2}t\right) \right).$$

Suppose  $y_p = A \sin(3t)$ ; this would give

$$-9A \sin(3t) - 3A \cos(3t) + A \sin(3t) = 2 \sin(3t),$$

which implies  $2 = -8A$  and  $0 = -3A$ , which has no solution. This does not necessarily mean that no  $y_p$  exists; at least in this case, we made a wrong guess at the beginning.

Suppose instead that  $y_p = A \sin(3t) + B \cos(3t)$ . This gives

$$\begin{aligned} -9A \sin(3t) - 9B \cos(3t) - 3A \cos(3t) + 3B \sin(3t) + A \sin(3t) + B \cos(3t) &= 2 \sin(3t) \\ 2 \sin(3t) &= (-3B - 8A) \sin(3t) + (-8B - 3A) \cos(3t) \\ \implies 2 &= -3B - 8A, \quad 0 = -8B - 3A \end{aligned}$$

Solving this equation gives  $A = -\frac{16}{73}$  and  $B = \frac{6}{73}$ . This gives  $y_p = \frac{-16}{73} \sin(3t) + \frac{6}{73} \cos(3t)$ .

⊛ **Example 3.6: With polynomials**

Consider  $L[y] = y'' + 2y' + y = t^3 = g$ . Suppose  $y_p = At^3 + Bt^2 + Ct + D$ . Then:

$$\begin{aligned} L[y_p] &= 6At + 2B + 2(3At^2 + 2Bt + C) + At^3 + Bt^2 + Ct + D = t^3 \\ At^3 + (6A + B)t^2 + (6A + B)t + (2B + C + D) &= t^3 \\ \implies \begin{aligned} 1 &= A & A &= 1 \\ 0 &= 6A + B & B &= -6 \\ 0 &= 6A + 4B + C & C &= 18 \\ 0 &= 2B + C + D & D &= -24 \end{aligned} \end{aligned}$$

so  $y_p = t^3 - 6t^2 + 18t - 24$ .

⊛ **Example 3.7: Exponential**

Take  $L[y] = y'' - 2y' + y = 4e^x$  with homogeneous auxiliary  $r^2 - 2r + 1 = 0 \implies (r - 1)^2 = 0$  so

$$y_1 = e^x, \quad y_2 = xe^x.$$

If we guessed,  $y_p = Ae^x$  then we'd have  $L[Ae^x] = AL[e^x] = 0$ , so it will not work. The same happens with guessing  $Axe^x$ . Suppose, then, that  $Ax^2e^x$ . Then:

$$\begin{aligned} L[Ax^2e^x] &= A(x^2 + 4x + 2)e^x - 2A(x^2 + 2x)e^x + Ax^2e^x = 4e^x \\ 4e^x &= 2Ae^x \implies A = 2. \end{aligned}$$

$y_p = 2x^2e^x$ , with general solution  $y = (k_1 + k_2 + 2x^2)e^x$ .

We now generalize the method:

Let  $p(x) = \sum_{j=0}^n a_j x^j$  and  $q(x) = \sum_{j=0}^n b_j x^j$  be given polynomials. To solve  $L[y](x) = g(x)$  for a constant coefficient ODE, we have the following cases:

$g(x)$ (given)	$y_{p(x)}$ (guess)
$p(x)$	$x^2(A_n x^n + \dots + A_1 x + A_0)$
$e^{\alpha x}$	$x^s A e^{\alpha x}$
$p(x)e^{\alpha x}$	$x^2(A_n x^n + \dots + A_1 x + A_0)e^{\alpha x}$
$p(x)e^{\alpha x} \cos \beta x + q(x)e^{\alpha x} \sin \beta x$	$x^s e^{\alpha x} \cos(\beta x) \sum_{i=0}^n A_i x^i + x^s e^{\alpha x} \sin(\beta x) \sum_{j=0}^n B_j x^j.$

- $s = 0$  if  $\alpha + i\beta$  is not a root of the auxiliary equation.
- $s =$  multiplicity of the root of  $\alpha + i\beta$  if it is a root of the equation.

**Remark 3.7.** First two cases are just special cases of the third; they are all just special cases of the last one.