MATH580 - Advanced PDEs 1

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§1 Local Existence Theory

§1.1 Terminology

↪**Definition 1.1** (Multiindex): We'll use *multiindex* notation throughout; if working in $\mathbb{R}^n$, we have a multiindex

$$\alpha \equiv (\alpha_1, \ldots, \alpha_n), \qquad \alpha_i \in \mathbb{Z}_+.$$

The *length* of a multiindex is given

$$|\alpha| \equiv \sum_i \alpha_i,$$

and we'll also write, for $x \in \mathbb{R}^n$,

$$x^\alpha \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

Finally, we'll write

$$\partial^\alpha \equiv \partial_{x_1}^{\alpha_1} \circ \cdots \circ \partial_{x_n}^{\alpha_n}$$

for higher-order partial derivatives in mixed directions.

Thus, the most general form of a k-th order PDE in independent variables $x \in \Omega \subset \mathbb{R}^n$ can be written succinctly by

$$F\left(x, (\partial^\alpha u)_{|\alpha| \leq k}\right) = 0, \qquad F : \Omega \times \mathbb{R}^{N(k)} \to \mathbb{R}, \qquad (\dagger)$$

with $N(k) \equiv \#\{\alpha \mid |\alpha| \leq k\}$.

↪**Definition 1.2** (Solution): We'll define a *(classical/strong) solution* to $(\dagger)$ to be a C^k-map $u : \Omega \to \mathbb{R}$ for which $(\dagger)$ is satisfied for all $x \in \Omega$.

↪**Definition 1.3** (Linearity/Quasilinearity): We say $(\dagger)$ is *linear* if F is affine-linear in $\partial^\alpha u$ for each multiindex, i.e. we may write equivalently

$$L[u] := \sum_{|\alpha| \leq k} a_\alpha(x) \partial^\alpha u = f(x),$$

where $L[u] = f(x) \Leftrightarrow F[x, u] = 0$. Similarly, $(\dagger)$ is said to be *quasilinear* if F is affine-linear in the highest order derivatives, i.e. $\partial^\alpha u$ for $|\alpha| = k$. An equivalent form is given by

$$\sum_{|\alpha| = k} a_\alpha\left(x, \left(\partial^\beta u\right)_{|\beta| \leq k-1}\right) \partial^\alpha u = b\left(x, \left(\partial^\beta u\right)_{|\beta| \leq k-1}\right).$$

↪**Definition 1.4** (Weak Solution): A *weak solution* to a linear PDE $L[u] = f$ is a function $u : \Omega \to \mathbb{R}$ such that

$$\sum_{|\alpha| \leq k} (-1)^{|\alpha|} \langle u, \partial^\alpha a_\alpha \varphi \rangle = \langle f, \varphi \rangle \qquad \forall \varphi \in C_c^\infty(\Omega),$$

with $\langle \cdot, \cdot \rangle$ the regular $L^2(\Omega)$-inner product.

Remark 1.1: Such a notation allows for non-C^k "solutions" to $(\dagger)$ which still have certain properties akin to those described by F. For a motivation of the definition, one need only integrate by parts $L[u] = f$ multiple times, hitting against $\varphi \in C_c^\infty(\Omega)$; if u were a strong solution, one would find the above equation as a result.

↪**Definition 1.5** (Characteristics): Let L be a linear operator associated to a kth-order linear PDE. The *characteristic form* of L is the kth degree homogeneous polynomial defined by

$$\chi_L(x,\xi) := \sum_{|\alpha| = k} a_\alpha(x)\xi^\alpha.$$

The *characteristic variety* is defined, for a fixed x, as the set of ξ for whcih χ_L vanishes, i.e.

$$\mathrm{char}_x(L) := \{\xi \neq 0 \mid \chi_L(x,\xi) = 0\}.$$

Remark 1.2: Suppose $\overline{\xi} = \xi_j e_j \neq 0 \in \mathrm{char}_x(L)$; then since

$$\chi_L\left(x,\overline{\xi}\right) = a_{\overline{\alpha}}\partial_{x_j}^k \xi_j, \qquad \overline{\alpha} \equiv k e_j,$$

then it must be that $a_{\overline{\alpha}} = 0$ at x. Heuristically, one has that L is not "genuinely" kth order in the direction of $\overline{\xi}$.

↪**Definition 1.6** (Elliptic): We say L is *elliptic* at x if $\mathrm{char}_x(L) = \emptyset$.

↪**Proposition 1.1**: $\mathrm{char}_x(L)$ is independent of choice of coordinates.

§1.2 First Order Scalar PDEs

We consider the quasilinear first-order PDE of the form

$$\sum_{i=1}^n a_i(x,u)\partial_i u = b(x,u), \qquad (*)$$

subject to the initial condition $u|_S = \varphi$ where $S \subseteq \mathbb{R}^n$ some hypersurface with φ given. We assume a_i, b C^1 in all arguments.

↪**Theorem 1.1**: Let $A(x) = (a_1(x,u), \ldots, a_n(x,u), b(x,u))$ and $S^* = \{(x,\varphi(x)) : x \in S\} \subseteq \mathbb{R}^{n+1}$. Then, if A nowhere tangent to S^*, then for any sufficiently small neighborhood Ω on S, there exists a unique solution to $(*)$ on Ω.

Proof. Locally, S can be parametrized by

$$(s_1, \ldots, s_{n-1}) \mapsto g(s) = (g_1(s), \ldots, g_n(s)).$$

Then, the "transversality condition" (about the tangency of A) can equivalently be written as

$$\det\begin{pmatrix} \partial g_1/\partial s_1 & \ldots & \partial g_1/\partial s_{n-1} & a_1(g(s)) \\ \vdots & & \vdots & \vdots \\ \partial g_n/\partial s_1 & \ldots & \partial g_n/\partial s_{n-1} & a_n(g(s)) \end{pmatrix} \neq 0.$$

Remark 1.3: In the linear case, one sees that this equivalently means that the normal ν of S is not in $\mathrm{char}_x(L)$; in particular, it is independent of the choice of initial data.

Remark that if we write coordinates $(x_1, ..., x_n, y) \in \mathbb{R}^{n+1}$ and define $F(x, y) = u(x) - y$, then the PDE can be written succinctly as the statement $A \cdot \nabla F = 0$, and that the zero set $F = 0$ gives the graph of the solution u; hence, we essentially need that the vector field A everywhere tangent to the graph of any solution. The idea of our solution is to consider A "originating" at S^*, and "flowing" our solution along the integral curves defined by A to obtain a solution locally.

The integral curves of A are defined by the system of ODEs

$$\begin{cases} \frac{\mathrm{d}x_j}{\mathrm{d}t} = a_j(x, y), \frac{\mathrm{d}y}{\mathrm{d}t} = b(x, y) \\ x_j(s, 0) = g_j(s), y(s, 0) = \varphi(g(s)) \end{cases} \qquad j = 1, ..., n.$$

By existence/uniqueness theory of ODEs, there is a local solution to this ODE, viewing s as a parameter, inducing a map

$$(s, t) \mapsto (x_1(s, t), ..., x_n(s, t)),$$

which is at least C^1 in s, t by smooth dependence on initial data. By the transversality condition, we may apply inverse function theorem to this mapping to find C^1-inverses $s = s(x), t = t(x)$ with $t(x) = 0$ and $g(s(x)) = 0$ whenever $x \in S$. Define now

$$u(x) := y(t(x), s(x)).$$

We claim this a solution. By the inverse function theorem argument, it certainly satisfies the initial condition, and repeated application of the chain rule shows that the solution satisfies the PDE. ■

We briefly discuss, but don't prove in detail, the fully nonlinear case, i.e.

$$F(x, u, \partial u) = 0,$$

where we assume $F \in C^2$. We approach by analogy. Putting $\xi_i := \frac{\partial u}{\partial x_i}$, then we see F as a function $\mathbb{R}^{2n+1} \to \mathbb{R}$. We seek "characteristic" ODEs akin to those found for the integral curves in the quasilinear case. We naturally take, as in the previous, $\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial F}{\partial \xi_i}$. Applying chain rule, we find that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sum_i \frac{\partial u}{\partial x_i} \frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_i \xi_i \frac{\partial F}{\partial \xi_i}.$$

Finally, if we differentiate $F = 0$ w.r.t. x_j, we find

$$0 = \frac{\partial F}{\partial x_j} + \xi_j \frac{\partial F}{\partial y} + \sum_k \frac{\partial F}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_j}$$

whence

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}t} = \sum_k \frac{\partial \xi_j}{\partial x_k} \frac{\partial x_k}{\partial t} = -\frac{\partial F}{\partial x_j} - \xi_j \frac{\partial F}{\partial y}.$$

In summary, this gives a system of $2n+1$ ODEs in (x, y, ξ) variables

$$\frac{\mathrm{d}x_j}{\mathrm{d}t} = \frac{\partial F}{\partial \xi_j}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \sum_i \xi_i \frac{\partial F}{\partial \xi_i}$$

$$\frac{\mathrm{d}\xi_j}{\mathrm{d}t} = -\frac{\partial F}{\partial x_j} - \xi_j \frac{\partial F}{\partial y}.$$

After imposing a similar (but slightly more complex) transversality requirement, one can show similarly obtain a solution from this system by an inverse function theorem argument.

In terms of initial conditions, if u is specified on some hypersurface S, we need to lift it to $S^{**} \subseteq \mathbb{R}^{2n+1}$ to "encode" the information of the initial values of u and its derivatives on u.

⊛ **Example 1.1**: Show that

$$\partial_1 u \partial_2 u = u, \qquad u(0, x_2) = x_2^2$$

has solution

$$u(x_1, x_2) = \frac{(x_1 + 4x_2)^2}{16}.$$

⊛ **Example 1.2** (Geodesics): For an invertible matrix $g = \left(g^{ij}\right)$, solve

$$\sum_{ij} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = 0.$$

Solution *(To ⊛ Example 1.2)*.

■

§1.3 Cauchy-Kovaleskaya

We discuss the essential existence and uniqueness theorem related to the following general k-order Cauchy problem:

$$(*') \qquad \begin{cases} F(x, u, \partial^\alpha u) = 0 & |\alpha| \le k \\ \partial_\nu^j u|_S = \varphi_j & 0 \le j \le k-1' \end{cases}$$

in which S a given hypersurface with normal ν, and we assume F and φ_j to be analytic, for which we write that they are in C^ω. We aim to show that, for $x_0 \in S$, there exists a neighborhood of x_0 and unique solution to $(*')$ on that neighborhood.

We begin to rewriting $(*')$ in several ways to simplify things. First, since we are working locally, we can always change coordinates to $(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that S is locally given by the zero set $t = 0$, in which case our problem becomes

$$(*) \qquad \begin{cases} F\left(x, t, \partial_x^\alpha \partial_t^j u\right) = 0 & |\alpha| + j \le k \\ \partial_t^j u(x, 0) = \varphi_j(x) & 0 \le j \le k-1 \end{cases},$$

where now of course $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ a $n-1$ length multiindex.

Remark that if u were a C^r solution for $r \geq k$, we can compute

$$\partial_x^\alpha \partial_t^j u(x,0) = \partial_x^\alpha \varphi_j(x)$$

for any $0 \leq j \leq k-1$ and $|\alpha| \leq r$. I.e., we can compute the mixed partial derivatives of u up to order $k-1$ in t along S in this way. To find those related to the kth order in t, we'd need to be able to use the equation $F = 0$ directly to solve for $\partial_t^k u(x,0)$ in terms of the other variables. However, this is not always possible, for arbitrary Cauchy data:

⊛ **Example 1.3**:
1. $\partial_x \partial_t u = 0$, $u(x,0) = \varphi_0(x)$, $\partial_t u(x,0) = \varphi_1(x)$ does not have unique solutions, and in fact the initial conditions dictate that φ_1 must be constant (which is already problematical). Moreover, $u(x,t) := \varphi_0(x) + f(t)$, with f any C^1 function with $f(0) = \varphi_1$, is a valid solution.
2. $\partial_x^2 u - \partial_t u = 0$ dictates that $\varphi_{0''}(x) = \varphi_1(x)$, so we can't choose arbitrary initial conditions again.

We enforce then this condition in the following:

↪**Definition 1.7** (Characteristic): We say S given by $t = 0$ is *non-characteristic* for $(*)$ if one can solve for $\partial_t^k u$ from the equation directly.

In this case, we may rewrite our equation as

$$(1) \qquad \begin{cases} \partial_t^k u = G\left(x, t, \left(\partial_x^\alpha \partial_t^j u\right)_{\substack{|\alpha|+j\leq k \\ 1\leq j\leq k-1}}\right) & \\ \partial_t^j u(x,0) = \varphi_j(x) & 0 \leq j \leq k-1 \end{cases}.$$

Moreover, we assume $x_0 = (0,0)$ in (x,t) space by translating. We write, for notational convenience, $y_{\alpha j} := \partial_x^\alpha \partial_t^j u$, noting that we will use this both as a separate coordinate system and for shorthand distinctly, so one should be careful with interpreting notation to follow.

Now, by differentiating (1) repeatedly with respect to t and evaluating when $t = 0$ (so on S), we can thus solve for the higher-order derivatives of $\partial_t^j u$ in terms of lower-order, known terms. For instance,

$$\partial_t^{k+1} u = \frac{\partial G}{\partial t} + \sum_{\substack{|\alpha|\ +j\leq k \\ 0\leq j\leq k-1}} \frac{\partial G}{\partial y_{\alpha j}} \partial_x^\alpha \partial_t^{j+1} u.$$

On S, everything on the right-hand side is determined, and so thus we know what $\partial_t^{k+1} u$ is as well here. We can repeat this process for any order derivative of u. This proves our first result:

↪**Proposition 1.2**: (1) has at most 1 analytic solution.

Proof. If (1) has an analytic solution u, then the discussion above demonstrates how to compute all of its derivatives at a specific point, i.e. on S. But these then just form the coefficients of a local power series representation of u, which must be unique, and hence u is unique as well, being determined by such coefficients. ■

↪**Theorem 1.2** (Cauchy-Kovaleskaya): (1) has a unique analytic solution.

The proof of the theorem is fairly constructive. Using similar ideas to above, we find the Taylor series coefficients of a solution. Then, we show that such a series actually converges with strictly positive radius of convergence, thus proving in turn existence and analyticity of the solution. The previous proposition give the uniqueness once this existence has been established.

First, we can rewrite (1) a couple of times:

↪**Lemma 1.1**: (1) is equivalent to

$$\begin{cases} \partial_t Y = \sum_{j=1}^{n-1} A_j(x,t,Y) \partial_j Y + B(t,x,Y) \\ Y(x,0) = \Phi(x) \end{cases},$$

where Y a vector $\left(y_{\alpha j}\right)_{|\alpha| + j \leq k}$, $A_j(x,t,Y)$ matrix-valued, $B(t,x,Y)$ vector valued, $\partial_j \equiv \partial_{x_j}$, and Φ determined by φ_j.

We can do even better:

↪**Lemma 1.2**: The problem (1) is equivalent to one in the same form as the previous lemma, but with A_j and B independent of t (and Y now of 1 higher dimension).