MATH325 - Honours ODEs

A Course on Ordinary Differential Equations

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Contents

1 Introduction

1.1 Definitions

↪ **Definition 1.1: Diffferential equation**

A *diffferential equation* (DE) is an equation with derivatives. *Ordinary* DE's (ODE) will be covered in this course; other types (PDE's, SDE's, DDE's, FDE's, etc.) exist as well but won't be discussed. ODE's only have one independent variable (typically, $y = f(x)$ or $y = f(t)$).

⊛ **Example 1.1: A Trivial Example**

$\frac{dy}{dx} = 6x$. Integrating both sides:

$$\int \frac{dy}{dx}\,dx = \int 6x\,dx \implies y(x) = 3x^2 + C.$$

⊛ **Example 1.2: Another One**

$$\frac{d^2u}{dt^2} = 0 \implies y = at + b.$$

↪ **Definition 1.2: Order**

The order of a differential equation is defined as the order of the highest derivative in the equation.

1.2 Initival Values

Remark 1.1. *Note the existence of arbitrary constants in the previous examples, indicating infinite solutions. We often desire unique solutions by fixing these coefficients. For first order ODEs, we simply specify a single initial condition (say, some $y(x_0) = \alpha_0$). For higher order ODEs of degree n, we can either specify $n-1$ initial conditions for $n-1$ derivatives (say, $y(x_0) = \alpha_0, y'(x_0) = \beta_0$), or boundary conditions (say, $y(x_0) = \alpha_0, y(x_1) = \alpha_1$) where values for the solution itself are specified.*

⊛ **Example 1.3: A Less Trivial Example**

$\frac{dy}{dx} = y$. We cannot simply integrate both sides as before, as we have no way to know what $\int y\,dx$ (the RHS) is equal to. We can fairly easily guess that $y = e^x$ is a solution; its derivative is equal to itself, hence it does indeed solve the equation. This is not the only solution; indeed, given $y = ce^x$, we have

$$\frac{dy}{dx} = ce^x = y = ce^x.$$

Luckily, we were rather limited in how many places constants could appear; this doesn't always hold.

1.3 Physical Applications

⊛ Example 1.4: Simple Pendulum

Let θ be the angle of a pendulum of mass m from vertical and length l. Then, we have the equation of motion

$$ml\ddot{\theta} = -mg\sin\theta \implies \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \implies \ddot{\theta} + \omega^2\sin\theta = 0.$$

Take θ small, then, $\sin\theta \approx \theta$. Then, $\ddot{\theta} + \omega^2\theta = 0$. This is linear simple harmonic motion, and has periodic solutions; how do we know this is a valid solution to the non-linear model?

↪ Lecture 01; Last Updated: Thu Jan 4 15:16:18 EST 2024

⊛ Example 1.5: Lorenz Equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma(y - x)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = rx - y - xz$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz$$

These are a famous set of equations originally derived from atmospheric modeling, known for its chaotic behavior for particular parameters. This is a nonlinear system of de's, and beyond the scope of this class (indeed, it is not solvable exactly).

1.4 Uniqueness

Given an ODE of the general form $y^{(n)} = f(t, y, y', \dots, y^{n-1})$, if we wish to determine $y^{(n)}(t_0)$ uniquely, we need to specify the initial conditions

$$y(t_0), y'(t_0), \dots, y^{(n-1)}(t_0).$$

Moreover, this not only determines uniqueness of $y^{(n)}(t_0)$, byt the uniqueness of solution y for $t \in I$ for some "interval of validity" I.

↪ **Definition 1.3: Autonomous/Nonautonomous**

An ODE of the form
$$y^{(n)} = f(y, y', \dots, y^{(n-1)})$$
is called *autonomous*; that is, if it has no explicit dependence on the independent variable. Otherwise, the system is called *nonautonomous*.

↪ **Definition 1.4: Linear/Nonlinear**

Linear ODEs of dimension n have a solution space which is a vector space of dimension n. As a result, solutions can be written as a linear combination of n basis solutions (or "fundamental set of solutions"). Solutions to nonlinear ODEs cannot be written this way (except locally).

Alternatively (but equivalently), if we can write an nth order ODE in the form
$$a_n(t)y^n(t) + \cdots a_1(t)y'(t) + a_0(t)y(t) = g(t),$$
or equivalently,
$$\sum_{i=0}^{n} a_i(t)y^i(t) = g(t), \quad \circledast$$
where each $a_i(t)$ and $g(t)$ are known functions of t, then we say that the ODE is linear. Otherwise, it is nonlinear.

⊛ **Example 1.6**

The pendulum
$$\ddot{\theta} + \omega^2 \sin\theta = 0$$
is autonomous and linear;
$$\ddot{\theta} + \omega^2 \sin\theta = 0$$
is autonomous and nonlinear, due to the $\sin\theta$ term (indeed, this is a nonlinear oscillator equation); a damped-forced oscillator
$$\ddot{\theta} + k^2\dot{\theta} + \omega^2\theta = A\sin(\mu t)$$
is nonautonomous and linear.

Remark 1.2. *Note that the following definitions apply only to linear ODEs.*

↪ **Definition 1.5: Homogeneous/Nonhomogeneous**

A linear ODE of the form ⊛ is *homogeneous* if $g(t) = 0$; otherwise it is *nonhomogeneous*.

↪ **Definition 1.6: Constant/Variable**

A linear ODE of the form $*$ is *constant coefficient* if $a_j(t) = \text{constant}\ \forall j$; if at least one a_j not constant, it is *non-constant* or *variable coefficient*.

Remark 1.3. *Note that while we define linearity of ODEs in terms of the form of $y^{(n)} = f(t, y, \ldots)$, this more "helpfully" relates to the form of the solution of such an ODE, which is indeed linear.*

1.5 Solutions

Given an n order ODE $y^{(n)} = f(t, y, \ldots)$, and assuming f continuous, then for $y(t)$ to be a solution, we need y to be n-times differentiable; hence, $y, \ldots, y^{(n-1)}$ must all exist and be continuous. Then, $y^{(n)}$, being a continuous function of continuous functions, is, itself, continuous.

↪ **Definition 1.7: Solution**

The function $y(t) : I \to \mathbb{R}$ is a solution to an ODE on an interval $I \subseteq \mathbb{R}$ if it is n-times differentiable on I, and satisfies the ODE on this interval.

Given an well-defined IVP with $n-1$ initial values defined at t_0, then $y(t)$ is a solution if $t_0 \in I$, y satisfies the initial values, and $y(t)$ is a solution on the interval.

↪ **Definition 1.8: Interval of Validity**

The largest I on which $y(t) : I \to \mathbb{R}$ solves an ODE is called the *interval of validity* of the problem.

↪ *Lecture 02; Last Updated: Thu Jan 11 11:05:26 EST 2024*

2 First Order ODEs

2.1 Separable ODEs

↪ **Definition 2.1: Separable ODE**

An ODE of the form

$$y' = P(t)Q(y)$$

is called *separable*. We solve them:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= P(t)Q(y) \\ \implies \int \frac{1}{Q(y)} \, \mathrm{d}y &= \int P(t) \, \mathrm{d}t \,. \end{aligned}$$

Finish by evaluating both sides.

⊛ **Example 2.1**

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= ty && (1)\\
\implies \frac{1}{y}\,\mathrm{d}y &= t\,\mathrm{d}t && (2)\\
\implies \ln|y| &= \frac{t^2}{2} + C && (3)\\
\implies |y| &= Ke^{\frac{t^2}{2}} \quad \text{where } K = e^C && (4)\\
\implies y &= Be^{\frac{t^2}{2}} \quad \text{where } B = \pm K = \pm e^C && (5)
\end{aligned}$$

Note that we call line (3) an *implicit solution*. In this case, we could easily turn this into an explicit solution by solving for $y(t)$; this won't always be possible.

Note that it would appear, based on the definition, that $B \neq 0$ (as $e^{\cdots} \neq 0$); however, plugging $y = 0$ into (1) shows that this is indeed a solution. It is quite easy to verify that (5) is a valid solution;

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Be^{\frac{t^2}{2}}\right) = Bte^{\frac{t^2}{2}} = t \cdot y,$$

as desired; this holds $\forall B \in \mathbb{R}$.

Remark 2.1. *Is it valid to split the differentials like this?*

$$\begin{aligned}
\frac{1}{Q(y)}\frac{\mathrm{d}y}{\mathrm{d}t} &= P(t)\\
\implies \int \frac{1}{Q(t)}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t &= \int P(t)\,\mathrm{d}t
\end{aligned}$$

Let $g(y) = \frac{1}{Q}(y)$ and $G(y) = \int g(y)\,\mathrm{d}y$. By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}(G(y(t))) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}}{\mathrm{d}y}G(y(t)) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot g(y(t)) = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{1}{Q(y(t))}.$$

Integrating both sides with respect to time, we have

$$\begin{aligned}
G(y(t)) = \int \frac{1}{Q(y(t))}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t &= \int P(t)\,\mathrm{d}t + C\\
\implies \int g(y)\,\mathrm{d}y &= \int P(t)\,\mathrm{d}t + C\\
\implies \int \frac{1}{Q(y)}\,\mathrm{d}y &= \int P(t)\,\mathrm{d}t + C
\end{aligned}$$

This was our original expression obtaining by "splitting", hence it is indeed "valid".

⊛ **Example 2.2**

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2}{1-y^2} \\
\implies \int (1-y^2)\,\mathrm{d}y &= \int x^2\,\mathrm{d}x \\
\implies y - \frac{y^3}{3} &= \frac{x^3}{x} + C \\
\implies y - \frac{1}{3}(y^3 + x^3) &= C
\end{aligned}
$$

Suppose we have the same ODE but now with an IVP $y(0) = 4$. Then, plugging this into our implicit solution:

$$4 - \frac{1}{3}(64+0) = C \implies C = 4 - \frac{64}{3} = -\frac{52}{3},$$

so our IVP solution is

$$y - \frac{1}{3}(y^3 + x^3) = -\frac{52}{3}.$$

2.2 Linear First Order ODEs

↪ **Definition 2.2: Integrating Factor**

A linear first order ODE of the form

$$\begin{aligned} a_1(t)y'(t) + a_0(t)y(t) &= g(t) \\ \implies y' + \frac{a_0}{a_1}y &= \frac{g}{a_1} \\ \implies y' + p(t)y &= q(t). \end{aligned}$$

To solve, we multiply by some integrating factor $\mu(t)$;

$$\mu(t)y'(t) + p(t)\mu(t)y(t) = \mu(t)q(t)$$

It would be quite convenient if $p(t)\mu(t) = \mu'(t)$; in this case, we'd have

$$\begin{aligned} \mu(t)y' + \mu'(t)y &= \mu(t)q(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)y(t)) &= \mu(t)q(t) \\ \implies \mu(t)y(t) &= \int \mu(t)q(t)\,\mathrm{d}t + C \\ \implies y(t) = \frac{1}{\mu(t)} \int &\mu(t)q(t)\,\mathrm{d}t + \frac{C}{\mu(t)} \end{aligned}$$

Now, what is $\mu(t)$? We required that

$$\begin{aligned} \mu'(t) &= p(t)\mu \\ \frac{\mathrm{d}\mu}{\mathrm{d}t} &= p(t)\mu \\ \implies \int \frac{\mathrm{d}\mu}{\mu} = \int p(t)\,\mathrm{d}t \implies \ln|\mu| &= \int p(t)\,\mathrm{d}t \\ \implies \mu(t) &= Ke^{\int p(t)\mathrm{d}t} \end{aligned}$$

However, note in our whole process earlier, we need only one μ; hence, for convenience, we can disregard any constants of integration and simply take

$$\boxed{\text{Integrating Factor:} \quad \mu(t) := e^{\int p(t)\mathrm{d}t}}$$

Then, our original linear ODE has general solution

$$y(t) = Ce^{-\int p(t)\mathrm{d}t} + e^{-\int p(t)\mathrm{d}t} \int e^{\int p(t)\mathrm{d}t} q(t)\,\mathrm{d}t\,.$$

⊛ **Example 2.3**

$$
\begin{aligned}
ty' + 3y - t^2 &= 0 \\
y' + \frac{3}{t}y &= t \\
&\implies \mu(t) = e^{\int \frac{3}{t}\,\mathrm{d}t} = e^{3\ln|t|} = t^3 \\
&\implies t^3y' + 3t^2y = t^4 \\
&\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(yt^3\right) = t^4 \\
&\implies yt^3 = \int t^4\,\mathrm{d}t \\
&\implies y = \frac{1}{t^3}\cdot\frac{t^5}{5} + \frac{C}{t^3} = \frac{t^2}{5} + \frac{C}{t^3}
\end{aligned}
$$

Note the division by zero issue when $t = 0$; this is not an issue with the solution method, but indeed with the ODE itself. The ODE breaks down when $t = 0$ for the same reason.

Thus, this solution is valid for $t \in (-\infty, 0) \cup (0, \infty) =: I_1 \cup I_2$; if we are given an IVP $y(t_0) = y_0$, if $t_0 < 0$, then the interval of validity is I_1, and if $t_0 > 0$, the interval of validity is I_2.

2.3 Exact Equations

↪ **Definition 2.3: Exact Equations**

A first order ODE of the form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \iff \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{M(x,y)}{N(x,y)}$$

is said to be *exact* if

$$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y) \iff M_y(x,y) = N_x(x,y).$$

Suppose we have a solution $f(x, y(x)) = C$. Then,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(f(x,y(x))) &= 0 \\ \implies \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \implies \frac{f_x}{f_y} &= -\frac{\mathrm{d}y}{\mathrm{d}x} \end{aligned}$$

Now, with $f_x(x,y) = M(x,y)$ and $f_y = N(x,y)$, then $M_y(x,y) = f_{xy}(x,y)$ and $N_x = f_{yx}(x,y)$. Assuming f continuous with existing, continuous partial derivatives, then $f_{xy} = f_{yx}$ and hence $M_y(x,y) = N_x(x,y)$. Thus, a function f such that $f_x = M$ and $f_y = N$ yields a solution to the ODE.

⊛ **Example 2.4**

$$\begin{aligned} 2xy^2\,\mathrm{d}x + 2x^2y\,\mathrm{d}y = 0 &\equiv M\,\mathrm{d}x + N\,\mathrm{d}y = 0 \\ \implies M_y = 4xy, &\quad \implies N_x = 4xy \\ f_x = M = 2xy^2 \implies f(x,y) &= x^2y^2 + C + F(y) \\ f_y = N = 2x^2y \implies f(x,y) &= x^2y^2 + C + F(x) \\ \implies f(x,y) &= x^2y^2 + C = K \end{aligned}$$

We can rearrange this as an explicit solution

$$y = \frac{k}{x}$$

for some constant k.

↪ *Lecture 03; Last Updated: Tue Jan 16 10:10:00 EST 2024*

↪ **Theorem 2.1**

This technique works generally.

Proof. Given an exact ODE of the form $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$, we need to show that $\exists f(x,y)$ s.t. $f(x,y) = c$ solves the ODE. Let

$$f(x,y) = \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + g(y)$$

for some function $g(y)$ to be chosen such that $f_y = N$. But we have

$$\begin{aligned}
N(x,y) = f_y(x,y) &= \frac{\partial}{\partial y}\left[\int_{x_0}^{x} M(s,y)\,\mathrm{d}s + g(y)\right] \\
&= g'(y) + \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s \\
\implies g'(y) &= N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\,.
\end{aligned}$$

But the LHS is a function of y only, while the RHS depends explicitly on x; hence, this technique will only work if the entire expression is actually independent of x. To show this, we take the partial of the RHS with respect to x:

$$\begin{aligned}
\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right] &= N_x(x,y) - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s \\
&= N_x(x,y) - \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right] \\
&= N_x(x,y) - \frac{\partial}{\partial y}\left[M(x,y)\right] \\
&= N_x - M_y = 0,
\end{aligned}$$

as the ODE is exact. Hence, the RHS is indeed a function of y alone. So, integrating both sides with respect to y:

$$g(y) = \int \left[N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right]\mathrm{d}y\,,$$

which gives us a $f(x,y)$ of

$$\begin{aligned}
f(x,y) &= \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + \int \left[N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y)\,\mathrm{d}s\right]\mathrm{d}y\,, \\
\implies f(x,y) &= \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + \int_{y_0}^{y} N(x,t)\,\mathrm{d}t - \int_{y_0}^{y}\int_{x_0}^{x} M_y(s,t)\,\mathrm{d}s\,\mathrm{d}t \qquad \star
\end{aligned}$$

which satisfies $f_x = M$ and $f_y = N$. Then, for $f(x,y) = C$, we have

$$\frac{\partial f}{\partial x} + \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\partial f}{\partial y} = M + \frac{\mathrm{d}y}{\mathrm{d}x}N = 0 \implies M\,\mathrm{d}x + N\,\mathrm{d}y = 0,$$

as desired.

Note that $\star$ is evaluated over a rectangle $[x_0, x] \times [y_0, y]$, but holds for any connected domain containing (x_0, y_0) and (x, y).

Also note that, as described, $g(y)$ is not a function of x; hence, we can pick x arbitrarily. Suppose we take $x = x_0$, then

$$f(x,y) = \int_{x_0}^{x} M(s,y)\,\mathrm{d}s + \int_{y_0}^{y} N(x_0,t)\,\mathrm{d}t\,.$$

$\blacksquare$

Remark 2.2. *We could have taken $g(x)$ and started from $f_y = N$. Then, we would have had the formula*

$$f(x,y) = \int_{y_0}^{y} N(x,t)\,\mathrm{d}t + \int_{x_0}^{x} M(s,y_0)\,\mathrm{d}y\,.$$

⊛ **Example 2.5**

$$2xy\,\mathrm{d}x + (x^2-1)\,\mathrm{d}y = 0.$$

We have $M(x,y) = 2xy$ and $N(x,y) = x^2 - 1$, so $M_y = 2x = N_y$ and the ODE is exact; hence, a solution exists of the form $f(x,y) = c$ where $f_x = M$, $f_y = N$.

$$f(x,y) = \int M(x,y)\,\mathrm{d}x = \int 2xy\,\mathrm{d}x = x^2y + k_1(y)$$
$$f(x,y) = \int N(x,y)\,\mathrm{d}y = \int (x^2-1)\,\mathrm{d}y = x^2y - y + k_2(x)$$

Hence $k_1(y) = -y$ and $k_2(x) = 0$, so

$$f(x,y) = x^2y - y = y(x^2-1),$$

so solutions to the original ODE are

$$y(x^2-1) = C \implies y = \frac{C}{x^2-1}.$$

2.4 Exact ODEs Via Integrating Factors

Suppose

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$

but $M_y \neq N_x$, that is, the ODE is not exact. Can we find an integrating factor $\mu(x,y)$ s.t.

$$[\mu(x,y)M(x,y)]\,\mathrm{d}x + [\mu(x,y)N(x,y)]\,\mathrm{d}y = 0$$

is exact? If so, such a μ must satisfy

$$\begin{aligned}
\frac{\partial}{\partial y}\left[\mu(x,y)M(x,y)\right] &= \frac{\partial}{\partial x}\left[\mu(x,y)N(x,y)\right] \\
\implies \mu_y M + \mu M_y &= \mu_x N + \mu N_x \\
\implies N\mu_x - M\mu_y &= \left(M_y - N_x\right)\mu \quad \circledast
\end{aligned}$$

This is not a generally easily soluble PDE; we will consider cases where μ is a function of only one independent variable, which greatly simplifies the expression; this could be simply $\mu(x)$, $\mu(y)$, or even $\mu(x \cdot y)$.

Suppose $\mu = \mu(x) \implies \mu_y = 0$. Then, $\circledast$ becomes

$$N\mu' = (M_y - N_x)\mu \implies \mu' = \left(\frac{M_y - N_x}{N}\right)\mu.$$

This is valid, provided the expression $\left(\frac{M_y - N_x}{N}\right)$ is a function solely of x. In this case, this becomes a linear first order ODE, with solution

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,\mathrm{d}x}.$$

OTOH, if $\mu = \mu(y)$, we can similarly derive

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\,\mathrm{d}y},$$

with a similar stipulation on the expression $\left(\frac{N_x - M_y}{M}\right)$ being a function of y solely.

$\circledast$ **Example 2.6**

$$xy\,\mathrm{d}x + (2x^2 + 3y^2 - 20)\,\mathrm{d}y = 0,$$

with $M(x,y) = xy \implies M_y = x$ and $N(x,y) = 2x^2 + 3y^2 - 20 \implies N_x = 4x$. We have $M_y - N_x = x - 4x = -3x$ (so the ODE is not exact). We write

$$\frac{M_y - N_x}{M} = \frac{-3x}{xy} = \frac{-3}{y},$$

which is a function solely of y; hence, can find a $\mu(y)$:

$$\mu(y) = e^{-\int \frac{M_y - N_x}{M}\mathrm{d}y} = e^{-\int -\frac{3}{y}\mathrm{d}y} = e^{3\ln y} = y^3,$$

noting that we, as before, do not care about any integrating factors; we are seeking a single function that works. Multiplying this into our original ODE:

$$\underbrace{xy^4}_{:=\tilde{M}}\,\mathrm{d}x + \underbrace{(2x^2+3y^2-20)y^3}_{:=\tilde{N}}\,\mathrm{d}y = 0.$$

And indeed, we have

$$\tilde{M}_y = 4xy^3; \quad \tilde{N}_x = 4xy^3 \implies \tilde{M}_y = \tilde{N}_x,$$

as desired.

↪ *Lecture 04; Last Updated: Tue Jan 23 10:02:55 EST 2024*

2.5 Substitutions

↪ **Definition 2.4: Homogeneous**

A function $f(x,y)$ is said to be homogeneous of degree d if $f(tx,ty) = t^d f(x,y)$.

Many ODEs can benefit from appropriate substitutions to make the proceeding solution method for clear. We present by example the following three types of substitutions, though naturally many other exist:

1. Homogeneous Equations, $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ where M, N homogeneous to the same degree.
2. Bernoulli Equations, $y' + f(x)y + g(x)y^n$.
3. $y' = f(Ax + By + C)$.

⊛ **Example 2.7: 1. Homogeneous Equations**

Consider

$$(x^2+y^2)\,\mathrm{d}x + (x^2 - xy)\,\mathrm{d}y = 0, \qquad x \neq 0$$

Dividing both sides by x^2, the correct substitution becomes obvious:

$$\begin{aligned} (x + (\frac{y}{x})^2)\,\mathrm{d}x + (1 - \frac{y}{x})\,\mathrm{d}y = 0 \\ u := \frac{y}{x} \implies y' = xu' + u \\ \implies (1+u^2) = (u-1)y' = (u-1)(u+xu') \\ \implies xu' = \frac{1+u^2}{u-1} - u = \frac{u+1}{u-1}, \end{aligned}$$

which is just linear in u.

⊛ **Example 2.8: 2. Bernoulli Equations**

Generally, let $v(x) = y^{1-n}$ to make the equation linear and solve. For instance, consider

$$xy' + y = x^2y^2.$$

⊛ **Example 2.9: 3.** $f(Ax + By + C)$

Let $u = Ax + By + C$.

↪ Lecture 05; Last Updated: Mon Feb 26 16:42:35 EST 2024

2.6 Qualitative Methods and Theory

Remark 2.3. *Read the first few chapters of Strogatz's Nonlinear Dynamics and Chaos book and you should be all good.*

⊛ **Example 2.10**

Show that $y' = y^{\frac{1}{3}}$ with $y(0) = 0$ has infinite solutions.

↪ Lecture 06; Last Updated: Wed Feb 14 15:27:47 EST 2024

2.7 Existence and Uniqueness

↪ **Definition 2.5: Lipschitz Continuity**

A function $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is said to be *Lipschitz continuous* in y on the rectangle $R = \{(x, y) : x \in [a, b], y \in [c, d]\} = [a, b] \times [c, d]$ if there exists a constant $L > 0$ s.t.

$$|f(x, y_1) - f(x, y_2)| \leqslant L\, |y_1 - y_2|\,, \quad \forall\, (x, y_1), (x, y_2) \in R.$$

L is called the *Lipschitz constant*.

Remark 2.4. *Note that we define in terms on continuity in y; the x variable in each coordinate is kept constant.*

↪ **Lemma 2.1**

If $f : \mathbb{R}^2 \to \mathbb{R}$ is such that $f(x, y)$ and $\frac{\partial f}{\partial y}$ are both continuous in x, y in the rectangle R, then f is Lipschitz in y on R.

Remark 2.5. *This result gives Differentiable $\implies$ Lipschitz Continuous $\implies$ Continuous.*

Proof. Using FTC, we have

$$\begin{aligned} f(x,y_2) &= f(x,y_1) + \int_{y_1}^{y_2} f_y(x,y)\,\mathrm{d}y \\ \implies |f(x,y_2)-f(x,y_1)| &= \left|\int_{y_1}^{y_2} f_y(x,y)\right| \leqslant \int_{y_1}^{y_2} \left|f_y(x,y)\right|\mathrm{d}y \\ &\leqslant |y_2-y_1| \cdot \max_{(x,y)\in R} \left|f_y(x,y)\right|, \end{aligned}$$

noting that this maximum exists, and is attained, because f_y is continuous on a compact set. This gives, then, that f is Lipschitz in y with $L = \max_{(x,y)\in\mathbb{R}} \left|f_y(x,y)\right|$. ■

↪ **Theorem 2.2: Existence and Uniqueness for Scalar First Order IVPs**

If $f(t,y), f_y(t,y)$ are continuous in t and y on a rectangle $R = \{(t,y) : t \in [t_0-a, t_0+a], y \in [y_0-b, y_0+b]\} = [t_0-a, t_0+a] \times [y_0-b, y_0+b]$, then $\exists h \in (0,a]$ s.t. the IVP

$$y' = f(t,y), y(t_0) = y_0$$

has a unique solution, defined for $t \in [t_0-h, t_0+h]$. Moreover, this solution satisfies $y(t) \in [y_0-b, y_0+b] \,\forall\, t \in [t_0-h, t_0+h]$.

Remark 2.6. *A stronger theorem also holds with a weakened condition on f that requires only f Lipschitz. Clearly, f_y continuous $\implies$ f Lipschitz, so we will use this fact to prove the statement, but won't prove it for the only Lipschitz case for sake of conciseness.*

Proof. Rewrite the IVP as

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s\,.$$

We will show this form has a unique solution, using an iteration method (namely, Picard Iteration).

We will begin by guessing a solution of the IVP, $y_0(t) = y_0$, $\forall\, t \in [t_0-a, t_0+a]$. This clearly satisfies the initial condition, but not the ODE itself.

Now, given $y_n(t)$, we define

$$y_{n+1}(t) = y(t_0) + \int_{t_0}^{t} f(s, y_n(s))\,\mathrm{d}s\,.$$

If this terminates, that is, $y_{n+1}(t) = y_n(t) \,\forall\, t \in [t_0-a, t_0+a]$, then $y_n(t)$ solves the IVP.

We now show that this iteration is both well-defined, and converges to unique solution.

By construction, $y_0 : [t_0-a, t_0+a] \to [y_0-b, y_0+b]$, and is continuous. As a bounded function on a bounded interval, it is integrable, and the first step of our step is well-defined.

Now suppose $y_n(t) : [t_0 - a, t_0 + a] \longrightarrow [y_0 - b, y_0 + b]$ is continuous and integrable. Then,

$$y_{n+1}(t) = y(t_0) + \int_{t_0}^{t} f(s, y_n(s))\,\mathrm{d}s$$

is continuous as well, as f is continuous and $y_n(s)$ is as well. It is not guaranteed to be restricted to $[y_0-b, y_0+b]$, however.

Since f continuous and attains its maximum on R, let

$$M := \max_{(t,y)\in R} |f(t,y)| < \infty.$$

We have, then, that

$$\begin{aligned} y_{n+1}(t) - y(t_0) &= \int_{t_0}^{t} f(s, y_n(s))\,\mathrm{d}s \\ \implies |y_{n+1}(t) - y(t_0)| &\leqslant |t - t_0|\, M \end{aligned}$$

Hence, if we choose $h : Mh \leqslant b$, and then $y_{n+1}(t) : [t_0 - h, t_0 + h] \longrightarrow [y_0 - b, y_0 + b]$ and we can iterative inductively, $y_n(t) : [t_0 - h, t_0 + h] \longrightarrow [y_0 - b, y_0 + b]\,\forall\, n$. Here, we take $h = \min\{\frac{b}{M}, a\}$.

Now, let $I = [t_0 - h, t_0 + h]$, then $y_n(t) : I \longrightarrow [y_0 - b, y_0 + b]$ for all n. Each iterate satisfies $y_n(t_0) = y(t_0) = y_0$; it remains to show that the iteration converges.

Let $C(I, [y_0 - b, y_0 + b])$ be the space of continuous functions $f : I \longrightarrow [y_0 - b, y_0 + b]$, noting that $y_n \in C\,\forall\, n$. We define a mapping on C, $T : C \longrightarrow C$ by

$$v = Tu, v(t) = y_0(t_0) + \int_{t_0}^{t} f(s, u(s))\,\mathrm{d}s\,.$$

Then, $y_{n+1} = Ty_n$. We aim to show that this iteration converges uniquely; we will do this by showing T is a contraction mapping.

For $y \in C$ define the norm $||y||_\infty$ by $||y||_\infty := \max_{t\in I} |y(t)|$. This is a norm;

1. $\forall\, k \in \mathbb{R}, ||ky||_\infty = |k|\, ||y||_\infty$.
2. $||y||_\infty = 0 \iff \max_{t\in I} |y(t)| = 0 \iff y(t) = 0\,\forall\, t \in I$.
3. $||y_1 + y_2||_\infty = \max_{t\in I} |y_1 + y_2| \leqslant \max_{t\in I}(|y_1| + |y_2|) \leqslant \max_{t\in I} |y_1| + \max_{t\in I} |y_2| = ||y_1||_\infty + ||y_2||_\infty$.

Now let $u, v \in C$. Then,

$$\begin{aligned}
||Tu - Tv||_\infty &= \max_{t\in I} |Tu(t) - Tv(t)| \\
&= \max_{t\in I} \left| y(t_0) + \int_{t_0}^{t} f(s, u(s)\, \mathrm{d}s) - y_0 + \int_{t_0}^{t} f(s, v(s))\, \mathrm{d}s \right| \\
&= \max_{t\in I} \left| \int_{t_0}^{t} f(s, u(s)) - f(s, v(s))\, \mathrm{d}s \right| \\
&\leqslant \max_{t\in I} \int_{t_0}^{t} |f(s, u(s)) - f(s, v(s))|\, \mathrm{d}s \\
&\leqslant \max_{t\in I} |t - t_0| \cdot \max_{s\in I} |f(s, u(s)) - f(s, v(s))| \\
&\leqslant hL \cdot \max_{s\in I} |u(s) - v(s)| \\
&= hL \cdot ||u - v||_\infty,
\end{aligned}$$

hence, we have a contraction mapping if $hL < 1$; if $hL \geqslant 1$, let $h < \min\{a, \frac{b}{m}, \frac{1}{L}\} > 0$. With such an h, $\exists \mu \in (0, 1) : hL \leqslant \mu < 1$, and $||Tu - Tv||_\infty \leqslant \mu ||u - v||_\infty$, hence, a contraction mapping.

The contractive mapping theorem, which will not be proven, states that any contraction mapping has a unique fixed point $y = Ty$; moreover, for any $y_0 \in C$, the iteration $y_{n+1} = Ty_n$ converges to y.

To see this, suppose $u = Tn, v = Tv$ are two solutions of our IVP. Then, by the contraction quality,

$$||u - v||_\infty = ||Tu - Tv||_\infty \leqslant \mu ||u - v||_\infty,$$

a contradiction unless $||u - v||_\infty = 0 \iff u = v$, hence, we have uniqueness of our solution; that is, our IVP has at most one solution. It remains to show that this solution exists.

Consider a sequence y_n, with $y_{n+1} = Ty_n$. Then,

$$\sum_{i=0}^{N} ||y_{i+1} - y_i||_\infty \leqslant \mu^N ||y_1 - y_0||_\infty,$$

by the contractive property, thus,

$$\sum_{i=0}^{\infty} ||y_{i+1} - y_i|| \leqslant (\sum_{i=0}^{\infty} \mu^j) ||y_1 - y_0||_\infty = \frac{1}{1-\mu} ||y_1 - y_0||_\infty = R_0,$$

for some radius (real number) R_0. Similarly, looking only at the tail of the series,

$$\sum_{j=n}^{\infty} ||y_{j+1} - y_j||_\infty \leqslant \frac{\mu^n}{1-\mu} ||y_1 - y_0||_\infty = \mu^n R_0,$$

that is, a "smaller" radius. We could, but won't, show that this sequence is Cauchy, and space C we are working in is complete and hence this sequence converges to some limit in the space; moreover, the limit of this sequence satisfies the IVP by construction. This is beyond the scope of this course. ■

↪ *Lecture 07; Last Updated: Thu Mar 28 20:55:03 EDT 2024*

⊛ Example 2.11: Using Picard Iteration

$$y' = 2t(1+y) =: f(t,y), \quad y(0) = 0.$$

This ODE is linear and separable, and has solution $y(t) = e^{t^2} - 1$ (solving whichever way you like). We can alternatively solve this using Picard Iteration.

Let $y_0(t) = 0 \, \forall \, t$, noting that the IC is satisfied. We define

$$y_{n+1}(t) = \cancelto{0}{y(0)} + \int_{\cancelto{0}{t_0}}^{t} f(s, y_n(s)) \, \mathrm{d}s \,,$$

where $f(s, y_n(s)) = 2s(1 + y(s))$. This gives

$$\begin{aligned}
y_{n+1}(t) = \int_0^t 2s(1 + y_n(s)) \, \mathrm{d}s \,. \\
\implies y_1(t) &= \int_0^t 2s(1 + y_0(s)) \, \mathrm{d}s = \int_0^t 2s \, \mathrm{d}s = t^2 \\
\implies y_2(t) &= \int_0^t 2s(1 + s^2) \, \mathrm{d}s = t^2 + \frac{1}{2}t^4 \\
\implies y_3(t) &= \cdots = t^2 + \frac{1}{2!}t^4 + \frac{1}{3!}t^6 \\
\cdots \implies y_n(t) &= \sum_{k=1}^{n} \frac{t^{2k}}{k!} \\
\implies \lim_{n \to \infty} y_n(t) &= \sum_{k=1}^{\infty} \frac{(t^2)^k}{k!} = e^{t^2} - 1,
\end{aligned}$$

the same solution as previously shown.

Remark 2.7. *The previous example worked nicely due to $y_n(t)$ always being a simple polynomial with a familiar convergence. This is not always (nor often) the case.*

Remark 2.8. *Recall the example $y' = y^{\frac{1}{3}}$ with multiple solutions. In the language of the theorem, $f(t,y) = y^{\frac{1}{3}}$ is continuous, but $f_1(t,y) = \frac{1}{3}y^{-\frac{2}{3}}$ becomes unbounded as $y \to 0$, and the function is thus not Lipschitz in a neighborhood of $y = 0$.*

Remark 2.9. *Recall that this theorem guarantees solutions in a closed rectangular region; it is possible, under certain conditions, to extend the solution beyond the bounds. But how far?*

⊛ **Example 2.12**

$$y' = y^2, \quad y(0) = 1.$$

This has a solution $y(t) = \frac{1}{c-t} = \frac{1}{1-t}$ (with IC). Notice that $y(t) \to +\infty$ as $t \to 1$. By this observation, we have that, if we were to repeat Picard iteration for increasing time t, the rectangular domains of our validity of each piecewise solution would be bounded by 1.

↪ **Corollary 2.1**

If $f(t,y)$ and $f_y(t,y)$ are continuous for all $t, y \in \mathbb{R}$, then $\exists t_- < t_0 < t_+$ such that the IVP

$$y' = f(t,y), \quad y(t_0) = y_0$$

has a unique solution $y(t)\, \forall\, t \in (t_-, t_+)$, and moreover, either $t_+ = +\infty$ or $\lim_{t\to t_+} |y(t)| = \infty$, and either $t_- = -\infty$ or $\lim_{t\to t_-} |y(t)| = \infty$.

Remark 2.10. *Finding t_-, t_+ requires the solution. In example 2.12, $t_- = -\infty$, $t_+ = 1$. Changing the IC will naturally change these values.*

↪ **Theorem 2.3**

If $p(t)$, $g(t)$ continuous on an open interval $I = (\alpha, \beta)$ and $t_0 \in I$, then the IVP

$$y'(t) + p(t)y = g(t), \quad y(t_0) = y_0$$

has a unique solution $y(t) : I \to \mathbb{R}$.

Remark 2.11. *In other words, this is a special case of the corollary above for linear ODEs; any "misbehavior" of the solutions would be solely due to discontinuities in the defining ODE.*

3 Second Order ODEs

3.1 Introduction

Second Order ODEs are of the form

$$y'' = f(t, y, y').$$

There is no general technique to solving these; we will be looking at special classes throughout.

Specifically in the case of nonlinear odes, there are two special cases we can solve,

1. f does not depend on y; ie $y'' = f(t, y')$. A substitution $u = y'$ yields $u' = f(t, u)$, hence this is just a first order ODE, with corresponding $y(t) = k_1 + \int u(t)\,\mathrm{d}t$.

2. f does not depend on t; ie $y'' = f(y, y')$. Let $u = y'$, so $u' = y'' = f(y, u)$. Consider $u = u(y(t))$, then,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}u}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = u\frac{\mathrm{d}u}{\mathrm{d}y},$$

and so

$$u\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{\mathrm{d}u}{\mathrm{d}t} = f(y, u) \implies \frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{y}f(y, u),$$

which again yields a first order ODE, in $u = u(y)$.

⊛ **Example 3.1: Of Case 2.**

$$y'' + \omega^2 y = 0^a$$

Rewrite this as $y'' = -\omega^2 y = f(y, y')$, and let $u = y'$, then $\frac{\mathrm{d}u}{\mathrm{d}y} = \frac{1}{u}f(y, u) = \frac{1}{u}[-\omega^2 y]$. This is a separable equation:

$$\begin{aligned} u\,\mathrm{d}u &= -\omega^2 y\,\mathrm{d}y \\ \frac{1}{2}u^2 &= -\frac{1}{2}\omega^2 y^2 + c \\ \implies u^2 &= -\omega^2 y^2 + c' \\ \implies u = \pm\sqrt{k^2 - \omega^2 y^2} \implies \frac{\mathrm{d}y}{\mathrm{d}t} &= \pm\sqrt{k^2 - \omega^2 y^2} \end{aligned}$$

Which is just another separable equation[b]:

$$\begin{aligned} \pm\int \mathrm{d}t &= \frac{1}{\omega}\int \frac{\mathrm{d}y}{\sqrt{\frac{k^2}{\omega^2} - y^2}} \\ \implies \frac{1}{\omega}\arcsin\left(\frac{\omega y}{k}\right) &= \pm t + C \\ \implies \frac{\omega y}{k} = \sin\left(\pm\omega t \pm \omega\tilde{C}\right) &= \pm\sin\left(\omega t + \omega\tilde{C}\right) \\ \implies y(t) &= \pm\frac{k}{\omega}\sin\left(\omega t + \omega\tilde{C}\right) \\ \implies y(t) &= K\sin(\omega t + C), \end{aligned}$$

which can be rewritten $y(t) = k_1\sin(\omega t) + k_2\cos(\omega t)$ with the appropriate substitutions.

[a] This is the equation for a simple harmonic oscillator.
[b] Please excuse the sloppy use of constants, it doesn't really matter.

Remark 3.1. *This is not the easiest way to solve this equation. More generally, this technique can lead to intractable integrals.*

⊛ **Example 3.2: Nonlinear Pendulum**

$$y'' + \omega^2 \sin y = 0.$$

Making the same substitution as before, $u = y'$, we have

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}y} &= -\frac{1}{u}\omega^2 \sin y \\
\int u \,\mathrm{d}u &= \int -\omega^2 \sin y \,\mathrm{d}y \\
\frac{1}{2}u^2 &= \omega^2 \cos y + c_1 \\
\frac{1}{2}(y')^2 &= \omega^2 \cos y + c_1 \\
y' &= \pm\sqrt{2c_1 + 2\omega^2 \cos y} \\
\pm \int \mathrm{d}t &= \int \frac{\mathrm{d}y}{\sqrt{2c + 2\omega^2 \cos y}},
\end{aligned}$$

where the integral on the RHS is some type of elliptic integral.

3.2 Linear, Homogeneous

We will solve a general form

$$a(t)y'' + b(t)y' + c(t)y = 0 \quad \circledast \,.$$

3.2.1 Principle of Superposition

↪ **Theorem 3.1: Superposition of Solutions to Linear Second Order ODEs**

If $y_1(t), y_2(t)$ solve ⊛ for $t \in I$-interval, then $y(t) = k_1 y_1(t) + k_2 y_2(t)$, for constants k_1, k_2 solves ⊛ on I as well. In other words, linear combinations of solutions are themselves solutions.

Remark 3.2. *This can be extended quite naturally to any linear order of ODE.*

Proof. This is clear by just plugging into the problem; let $y(t) = k_1 y_1(t) + k_2 y_2(t)$. Then:

$$\begin{aligned}
a(t)y''(t) + b(t)y'(t) + c(t)y(t) &= a(t)(k_1 y_1'' + k_2 y_2'') + b(t)(k_1 y_1' + k_2 y_2') + c(t)(k_1 y_1 + k_2 y_2) \\
&= k_1(a y_1'' + b y_1' + c y_1) + k_2(a y_2'' + b y_2' + c y_2) \\
&= k_1 \cdot 0 + k_2 \cdot 0 = 0,
\end{aligned}$$

as desired. ■

↪ *Lecture 08; Last Updated: Thu Feb 15 10:05:56 EST 2024*

↪ **Definition 3.1: Linear Independence of Functions**

If $y_1(t), y_2(t)$ are defined $\forall\, t \in I$ for some interval $I \subseteq \mathbb{R}$, then $y_1(t), y_2(t)$ are *linearly dependent on* I if $\exists k_1, k_2$ constants (not both zero) so that $k_1 \cdot y_1(t) + k_2 \cdot y_2(t) = 0 \,\forall\, t \in I$.

If the only constants which solve this are $k_1 = k_2 = 0$, then $y_1(t), y_2(t)$ are linearly independent on I.

Remark 3.3. *If $y_j(t)$ is the zero function, then take $k_j = 1$ and the other constant zero; ie, the zero function is always linearly dependent.*

3.3 Reduction of Order

Suppose $y_1(t)$ solves the homogeneous ODE $0 = a(t)y'' + b(t)y' + c(t)y$. Let $y(t) = u(t)y_1(t)$ for some unknown $u(t)$, and assume it solves the ODE. Then:

$$y = uy_1 \implies y' = u'y_1 + uy_1' \implies y'' = u''y_1 + u'y_1' + u'y_1' + uy_1'' = uy_1'' + 2u'y_1' + u''y_1.$$

Substituting this into the original ODE:

$$\begin{aligned} 0 =& a(u''y_1 + 2u'y_1' + uy_1'') + b(u'y_1 + uy_1') + c(uy_1') \\ &= [ay_1]u'' + [2ay_1' + by_1]u' + \underbrace{[ay_1'' + by_1' + cy_1]}_{=0}u \end{aligned}$$

Let $v = u' \implies v' = u''$, and we have reduced to a first-order ODE

$$0 = [ay_1]v' + [2ay_1' + by_1]v$$

which we can solve for v, then conclude by integrating v to solve for u.

3.4 Constant Coefficient Linear Homogeneous Second Order ODEs

We consider the case

$$ay'' + by' + cy = 0,$$

where a, b, c are constants. If $a = 0$, this is simply first order with an exponential solution; so, suppose (guess) that this ODE has solution $y = e^{rt}$ for $a \neq 0$. This gives

$$\begin{aligned} a(e^{rt})'' + b(e^{rt})' + c(e^{rt}) &= 0 \\ \implies ar^2 e^{rt} + bre^{rt} + ce^{rt} &= 0 \\ \implies ar^2 + br + c = 0 \implies r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

and we thus have just to solve a quadratic equation. We call this the *auxiliary equation* or *characteristic equation* for the ODE.

We thus have three cases to consider:

1. $b^2 > 4ac$: r has two real roots, giving two real solutions to the original ODE of the form

$$y_1(t) = e^{r_+ t}, \quad y_2(t) = e^{r_- t},$$

where $r_\pm := r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Note that $\frac{y_2}{y_2} = e^{(r_- - r_+)t}$ is non-constant hence these solutions are independent. It follows that we have a general solution

$$y(t) = k_1 e^{r_+ t} + k_2 e^{r_- t}$$

for arbitrary constants k_1, k_2.

2. $b^2 = 4ac$: r has one real (repeated) solution, $r = \frac{-b}{2a}$. This gives only one solution $y_1 = e^{r_1 t}$: we find another by reduction of order. Let $y = uy_1 = ue^{r_1 t} = ue^{\frac{-bt}{2a}}$. We have:

$$\begin{aligned} 0 &= ay'' + by' + cy \\ 0 &= a(u''y_1 + 2u'y_1' + uy_1'') + b(u'y_1 + uy_1') + cuy' \\ 0 &= ay_1u'' + (2ay_1' + by_1)u' + \cancelto{0}{(ay_1'' + by_1' + cy_1)}u \\ 0 &= ae^{rt}u'' + (2are^{rt} + be^{rt})u' \\ 0 &= au'' + (2ar + b)u' \\ 0 &= au'' + (-\frac{2ab}{2a} + b)u' \\ 0 &= au'' \\ 0 &= u'' \implies u' = k_1 \implies u = k_1 t + k_2, \end{aligned}$$

and so we have another solution $y = uy_1 = (k_1 t + k_2)e^{rt}$; these constants k_1, k_2 are arbitrary (as long as $k_1 \neq 0$, which would just give a linearly dependent solution to the original), so take $k_1 = 1, k_2 = 0$. This gives a general solution

$$y(t) = c_1 e^{rt} + c_2 te^{rt} = (c_1 + c_2 t)e^{rt},$$

which is actually just the "second" solution we found before (and thus this one was indeed the general solution by itself).

3. $b^2 < 4ac$: r has two complex conjugate roots $r_\pm = -\frac{b}{2a} \pm \frac{\sqrt{4ac-b^2}}{2a} i := \alpha \pm i\beta$. This gives solutions

$$y_+ = e^{(\alpha+i\beta)t}, \quad y_- = e^{(\alpha-i\beta)t}.$$

While valid, these complex solutions are not necessarily useful in this form; we can rewrite them using Euler's formula to take only the real parts.

$$\begin{aligned} y_+ &= e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)] \\ y_- &= e^{(\alpha-i\beta)t} = e^{\alpha t}e^{-i\beta t} = e^{\alpha t}[\cos(-\beta t) + i\sin(-\beta t)] = e^{\alpha t}[\cos(\beta t) - i\sin(\beta t)] \end{aligned}$$

Let $y_1 = \frac{1}{2}(y_+ + y_-) = e^{\alpha t}\cos(\beta t)$; this is a first, purely real solution to our ODE. To find a second, we could use reduction of order, or just take another linear combination of y_+, y_-

$$y_2 = \frac{1}{2i}[y_+ - y_-] = e^{\alpha t}\sin(\beta t).$$

y_1, y_2 are linearly independent, since $\frac{y_2}{y_1} = \tan(\beta t) = 0\,\forall\, t \iff \beta = 0$, which we assumed was not the case (otherwise, we'd be in case 2.). Together, we have a general, purely real solution

$$y(t) = e^{\alpha t}(k_1\sin(\beta t) + k_2\cos(\beta t)),$$

where k_1, k_2 arbitrary and $r = \alpha \pm i\beta$.

Harding once said: that "there is no permanent place in the world for ugly mathematics"; that means that there is a temporary place in the world for ugly mathematics. Make it pretty later.

⊛ **Example 3.3**

1. $y'' - 3y' + 2y = 0$

   This gives an auxiliary equation $r^2 - 3r + 2 = 0$ with solution $r = \frac{3\pm\sqrt{9-8}}{2} = 2, 1$. These are both positive and real, and we thus have a general solution

$$y(t) = k_1e^t + k_2e^{2t}.$$

2. $y'' - 2y' + y = 0$

$$\begin{aligned} r^2 - 2r + 1 = 0 &\implies (r-1)(r-1) = 0 \implies r = 1 \\ &\implies y(t) = (k_1 t + k_2)e^t \end{aligned}$$

3. $y'' + 4y' + 7y = 0$

$$\begin{aligned} r^2 + 4r + 7 = 0 &\implies r = \frac{-4 \pm \sqrt{16-28}}{2} = -2 \pm i\frac{1}{2}\sqrt{12} = -2 \pm i\sqrt{3} \\ &\implies y(t) = e^{-2t}\left(k_1 \sin\left(\sqrt{3}t\right) + k_2 \cos\left(\sqrt{3}t\right)\right) \end{aligned}$$

↪ *Lecture 09; Last Updated: Tue Feb 6 10:07:06 EST 2024*

3.5 Nonhomogeneous Second Order ODEs

We consider equations of the form

$$a(t)y'' + b(t)y' + cy = g(t).$$

Let's look for solutions of the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t),$$

where y_1, y_2 are linearly independent solutions of the homogenous equation ($g = 0$) and y_p is a particular solution to the ODE. Plugging this into the original equation:

$$\begin{aligned} ay'' + by' + cy &= a(c_1 y_1'' + c_2 y_2'' + y_p'') + b(c_1 y_1' + c_2 y_2' + y_p') + c(c_1 y_1 + c_2 y_2 + y_p) \\ &= c_1\cancelto{0}{(ay_1'' + by_1' + cy_1)} + c_2\cancelto{0}{(ay_2'' + by_2' + cy_2)} + \cancelto{g}{ay_p'' + by_p' + cy_p} \\ &= g, \end{aligned}$$

as desired. Indeed, all solutions are of this form; we will show this later.

Remark 3.4. *Note that c_1, c_2 are arbitrary constants; y_p is not multiplied by a constant, and should not be.*

Remark 3.5. *y_1, y_2 are called a fundamental set of solutions; $y_c = c_1 y_1 + c_2 y_2$, the general solution to the homogeneous equation, is called the complementary solution of the nonhomogeneous equation. $y = y_c + y_p$ is the general solution of the nonhomogeneous equation.*

3.5.1 Linear Operator Notation

We denote $C(\mathbb{R})$ to be the space of continuous functions on $\mathbb{R}$. Let $C^p(\mathbb{R})$ be the space of p-times differentiable functions on $\mathbb{R}$; ie, $y \in C^p(\mathbb{R}) \implies y^{(j)} \in C(\mathbb{R}), j = 0, 1, \ldots, p$. Notice that $C^{p+1}(\mathbb{R}) \subsetneq C^p(\mathbb{R})$. It follows that $C^\infty(\mathbb{R}) \subsetneq \cdots \subsetneq C^n(\mathbb{R}) \subsetneq \cdots \subsetneq C(\mathbb{R})$.

Let $D : C^n(\mathbb{R}) \to C^{(n-1)}(\mathbb{R})$ be the differentiation operator, ie $Dy = y'$, noting that Dy less differentiable than y unless $y \in C^\infty(\mathbb{R})$. Its clear that D is a linear operator.

Now, define the operator $L = a(x)D^2 + b(x)D + c(x)$. Then, $L[y] = a(x)y'' + b(x)y' + c(x)y$; hence, $L[y] = 0$ and $L[y] = g$ are equivalent to our homogeneous and nonhomogeneous equations. It is clearly linear.

We explore two methods for finding the particular solution.

3.5.2 Finding y_p: Method of Undetermined Coefficients

This method only applies to ODEs with constant coefficients, and only for certain functions g.

⊛ Example 3.4

Consider $g(t) = L[y](t)$. Suppose $g(t) = \mu e^{\gamma t}$. Let's guess that $y_p = Ae^{\gamma t}$. Then:

$$L[y_p] = aA\gamma^2 e^{\gamma t} + bA\gamma e^{\gamma t} + cAe^{\gamma t} = (a\gamma^2 + b\gamma + c)Ae^{\gamma t},$$

hence, for $L[y_p] = g = \mu e^{\gamma t}$, we need $\mu = A(a\gamma^2 + b\gamma + c) \implies A = \frac{\mu}{a\gamma^2+b\gamma+c}$. Provided $a\gamma^2 + b\gamma + c \neq 0 \iff \gamma$ does not solve auxiliary equation, this A as defined will provide y_p.

Remark 3.6. *This example worked* because differentiating the exponential yields another exponential, which cancel nicely. The same idea can be applied for polynomials and trig functions.*

⊛ Example 3.5: With trig

Suppose $L[y] = y'' - y' + y = g(t) = 2\sin(3t)$, with auxiliary equation $r^2 - r + 1 = 0 \implies r = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. This gives complementary solution

$$y_c = e^{\frac{t}{2}}\left(k_1 \sin\left(\frac{\sqrt{3}}{2}t\right) + k_2 \cos\left(\frac{\sqrt{3}}{2}t\right)\right).$$

Suppose $y_p = A\sin(3t)$; this would give

$$-9A\sin(3t) - 3A\cos(3t) + A\sin(3t) = 2\sin(3t),$$

which implies $2 = -8A$ and $0 = -3A$, which has no solution. This does not necessarily mean that no y_p exists; at least in this case, we made a wrong guess at the beginning.

Suppose instead that $y_p = A\sin(3t) + B\cos(3t)$. This gives

$$\begin{aligned}
-9A\sin(3t) - 9B\cos(3t) - 3A\cos(3t) + 3B\sin(3t) + A\sin(3t) + B\cos(3t) &= 2\sin(3t) \\
2\sin(3t) &= (-3B - 8A)\sin(3t) + (-8B - 3A)\cos(3t) \\
\implies 2 = -3B - 8A, \quad 0 &= -8B - 3A
\end{aligned}$$

Solving this equation gives $A = -\frac{16}{73}$ and $B = \frac{6}{73}$. This gives $y_p = \frac{-16}{73}\sin(3t) + \frac{6}{73}\cos(3t)$.

⊛ **Example 3.6: With polynomials**

Consider $L[y] = y'' + 2y' + y = t^3 = g$. Suppose $y_p = At^3 + Bt^2 + Ct + D$. Then:

$$\begin{aligned}
L[y_p] = 6At + 2B + 2(3At^2 + 2Bt + C) + At^3 + Bt^2 + Ct + D &= t^3 \\
At^3 + (6A + B)t^2 + (6A + B)t^2 + (6A + 4B + C)t + (2B + C + D) &= t^3 \\
\implies \begin{matrix} 1 = A \\ 0 = 6A + B \\ 0 = 6A + 4B + C \\ 0 = 2B + C + D \end{matrix} \implies \begin{matrix} A = 1 \\ B = -6 \\ C = 18 \\ D = -24 \end{matrix},
\end{aligned}$$

so $y_p = t^3 - 6t^2 + 18t - 24$.

⊛ **Example 3.7: Exponential**

Take $L[y] = y'' - 2y' + y = 4e^x$ with homogeneous auxiliary $r^2 - 2r + 1 = 0 \implies (r-1)^2 = 0$ so

$$y_1 = e^x, \quad y_2 = xe^x.$$

If we guessed, $y_p = Ae^x$ then we'd have $L[Ae^x] = AL[e^x] = 0$, so it will not work. The same happens with guessing Axe^x. Suppose, then, that Ax^2e^x. Then:

$$\begin{aligned}
L[Ax^2e^x] = A(x^2 + 4x + 2)e^x - 2A(x^2 + 2x)e^x + Ax^2e^x &= 4e^x \\
4e^x = 2Ae^x &\implies A = 2.
\end{aligned}$$

$y_p = 2x^2e^x$, with general solution $y = (k_1 + k_2 + 2x^2)e^x$.

We now generalize the method:

Let $p(x) = \sum_{j=0}^{n} a_j x^j$ and $q(x) = \sum_{j=0}^{n} b_j x^j$ be given polynomials. To solve $L[y](x) = g(x)$ for a constant coefficient ODE, we have the following cases:

- $s = 0$ if $\alpha + i\beta$ is not a root of the auxiliary equation.

$g(x)$ (given)	$y_{p(x)}$ (guess)
$p(x)$	$x^s(A_n x^n + \cdots + A_1 x + A_0)$
$e^{\alpha x}$	$x^s A e^{\alpha x}$
$p(x)e^{\alpha x}$	$x^s(A_n x^n + \cdots + A_1 x + A_0)e^{\alpha x}$
$p(x)e^{\alpha x}\cos\beta x + q(x)e^{\alpha x}\sin\beta x$	$x^s e^{\alpha x}\cos(\beta x)\sum_{i=0}^n A_i x^i + x^s e^{\alpha x}\sin(\beta x)\sum_{j=0}^n B_j x^j$.

- s = multiplicity of the root of $\alpha + i\beta$ if it is a root of the equation.

Remark 3.7. *First two cases are just special cases of the third; they are all just special cases of the last one.*

↪ Lecture 10; Last Updated: Mon Feb 19 21:16:13 EST 2024

Remark 3.8. *Linear combinations of the g's above can also be solved, ie if $L[y] = g_1 + g_2$, take $y_p = y_{p1} + y_{p2}$ where y_{pi} matches the "proper guess" for g_i.*

Remark 3.9. *The method fails if a, b, c not constants, or if g not of the required form.*

⊛ **Example 3.8**

1. Consider $y'' + y' - 2y = 3e^{2x}$. We have

$$r^2 + r - 2 = 0 \implies (r-1)(r+2) = 0 \implies y_1 = e^x, y_2 = e^{-2x}$$

   for the homogeneous equations. Let $y_p = Ae^{2x}$, since e^{2x} does solve the equation.

2. $y'' = 1 - x^2$. $r^2 = 0 \implies y_1 = 1, y_2 = x$. Guess $g(x) = p(x)e^{\alpha x}\cos(\beta x)$ for $\alpha = 0, \beta = 0$, $p(x) = 1 - x^2$. Guessing $y_p = Ax^2 + Bx + C$ won't work; instead, guess $x^2(Ax^2 + Bx + C)$. Forgetting the x^2 would yield an unsolvable equation.

3. $y'' + 4y = 3\cos x$. $r^2 + 4 = 0 \implies r = \pm 2i$ so $y_1 = \cos 2x, y_2 = \sin 2x$. Guess $y_p = A\cos x + B\sin x$. We don't need the sin, since it won't appear in the ODE; this isn't a problem anyways, as this way we'll just find that $B = 0$.

3.6 Variation of Parameters

This method works for non-constant coefficient ODEs, and (in principle) any g. To use it, we need first to know a fundamental set of solutions y_1, y_2 of the homogeneous equation.

Consider the nonhomogeneous equation

$$L[y](x) = g(x) = a(x)y'' + b(x)y' + c(x)y. \quad ⊛$$

Suppose $L[y_1] = L[y_2] = 0$, so $y_c = k_1 y_1 + k_2 y_2$ solves the homogeneous equation (constants k_i). Replace these k_i's with unknown functions, $u_i(x)$, and assume that $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ solves the ODE.

We have

$$y_p' = [u_1'y_1 + u_2'y_2] + [y_1u_1' + y_2u_2']$$
$$y_p'' = [u_1'y_1 + u_2'y_2]' + [u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'']$$

Substituting this into ⊛, we have that

$$\begin{aligned}
g = L[y_p] &= a(x)([u_1'y_1 + u_2'y_2]') + a(x)[u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''] \\
&\quad + b(x)[u_1'y_1 + u_2'y_2] + b(x)[u_1y_1' + u_2y_2'] \\
&\quad + c(x)[u_1y_1 + u_2y_2] \\
&= u_1\underbrace{[ay_1'' + by_1' + cy_1]}_{0} + u_2\underbrace{[ay_2'' + by_2' + cy_2]}_{0} \qquad \text{(solve ODE by assumption)} \\
&\quad + a[u_1'y_1 + u_2'y_2]' + a[u_1'y_1' + u_2'y_2'] + b[u_1'y_1 + u_2'y_2].
\end{aligned}$$

But this is a single equation "trying" to define two unknown functions u_1, u_2; it is undetermined. We introduce an extra constraint to make it solvable. Let us state, for convenience, $u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0 \,\forall\, x$, implying $[u_1'y_1 + u_2'y_2]' = 0 \,\forall\, x$.[1] This assumption yields $g = a[u_1'y_1' + u_2'y_2']$, so we write

$$\boxed{f(x) := \frac{g}{a} = u_1'y_1' + u_2'y_2'. \quad 0 = u_1'y_1 + u_2'y_2,}$$

a system of two differential equations for u_1, u_2. We can solve these:

$$\begin{aligned}
\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} &= \begin{pmatrix} 0 \\ f(x) \end{pmatrix} \\
\implies \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} &= \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ f(x) \end{pmatrix} \\
&= \frac{1}{y_1y_2' - y_1'y_2}\begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix}\begin{pmatrix} 0 \\ f \end{pmatrix}.
\end{aligned}$$

This can be problematic if $y_1y_2' - y_1'y_2 = 0$; define $W(y_1, y_2)(x) := y_1y_2' - y_1'y_2$. Then, assuming $W(y_1, y_2)(x) \neq 0$, we have

$$u_1'(x) = \frac{-y_2(x)f(x)}{W(y_1, y_2)(x)} \quad u_2'(x) = \frac{y_1(x)f(x)}{W(y_1, y_2)(x)},$$

which we can then integrate to find u_1, u_2 appropriately. We call $W(y_1, y_2)(x)$ the *Wronskian* of y_1, y_2 wrt x.

Note that, if y_1, y_2 are linearly dependent with $y_2 = cy_1$, then $W(y_1, y_2)(x) = y_1(cy_1') - y_1'(cy_1) = 0$; that is, a necessary condition for $W(y_1, y_2) \neq 0$ is for y_1, y_2 to be linearly independent; it is not sufficient. However, we'll

[1] This is a "trust me for now" instance.

only use W when y_1, y_2 both solve the same ODE; in this case, it can be shown that $W(y_1, y_2)(x) \neq 0 \iff y_1, y_2$ are linearly independent[2].

⊛ Example 3.9

$4y'' + 36y = \frac{1}{\sin(3x)} \implies y'' + 9y = \frac{1}{4\sin(3x)} = \frac{1}{4}\csc(3x)$.

Solving the homogeneous equation: $r^2 + 9 = 0 \implies r = \pm 3i$. This gives us $y_1 = \cos(3x), y_2 = \sin(3x)$. Let $y_p = u_1\cos(3x) + u_2\sin(3x)$. We have $W(y_1, y_2) = (\cos 3x)3\cos(3x) + (3\sin(3x))(\sin(3x)) = 3$, yielding

$$u_1' = \frac{-y_2 f}{W(y_1, y_2)(x)} = \frac{-\sin(3x)\frac{1}{4\sin(3x)}}{3} = -\frac{1}{12} \implies u_1 = -\frac{x}{12}$$
$$u_2' = \frac{\cos(3x)\frac{1}{4\sin(3x)}}{3} = \frac{1}{36}\left(\frac{3\cos(3x)}{\sin(3x)}\right) = \frac{1}{36}\frac{h'}{h} \implies u_1 = \frac{1}{36}\ln(|\sin 3x|)$$

We have

$$y_p = -\frac{x}{12}\cos(3x) + \frac{1}{36}(\ln|\sin 3x|)\sin(3x),$$

with a general solution

$$y(x) = \left(k_1 - \frac{x}{12}\right)\cos(3x) + \sin(3x)\left(k_2 + \frac{1}{36}\ln|\sin(3x)|\right).$$

4 Nth Order ODEs

4.1 A Little Theory

Consider a nonlinear nth order IVP,

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x)) \quad (i)$$
$$y(x_0) = \alpha_1, \dots, y^{(n-1)}(x_0) = \alpha_n \quad (ii),$$

noting that this is sufficient to specify a unique solution.

↪ Theorem 4.1

If $f(x, y_1, y_2, \dots, y_n)$ and $\frac{\partial f}{\partial y_j}$ are continuous on the box $R = \{(x, y_1, \dots, y_n) : |x - x_0| \leqslant a, |y_i - \alpha_i| \leqslant b, i = 1, \dots, n\}$, then the initial value problem $(i), (ii)$ has a unique solution $y(x)$ for $x \in [x_0 - h, x + 0 + h]$ for some $h \in (0, a]$, with solution satisfying $|y(x) - \alpha_1| \leqslant b \,\forall\, x \in [x_0 - h, x_0 + h]$.

[2] Abel's Identity

Remark 4.1. *The proof is very similar to the case $n = 1$; the key step is to rewrite the nth order ODE as a system of first order ODEs.*

Let $u_1 = y, u_2 = y', \ldots, u_n = y^{(n-1)}$, and define $\underline{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}$. The ODE, then, can be written

$$\underline{u}'(t) = \begin{pmatrix} u_1'(t) \\ \vdots \\ u_n'(t) \end{pmatrix} = \begin{pmatrix} y' \\ \vdots \\ y^{(n)} \end{pmatrix} = \begin{pmatrix} u_2 \\ \vdots \\ u_n \end{pmatrix} =: \underline{F}(x, \underline{u}),$$

"vectorally".

4.2 Linear nth Order ODEs

We consider

$$y^{(n)} + \sum_{i=1}^{n} p_i(x) y^{(n-1)} = g(x) =: L[y],$$

with ICs

$$y(x_0) = \alpha_1, \ldots, y^{(n-1)}(x_0)\alpha_n.$$

We would like to show that the general solution is as before with second order ODEs, ie

$$y(x) = \sum_{j=1}^{n} k_j y_j + y_p,$$

where y_p is a particular solution of $L[y] = g$, and $y_1, \ldots, y_n$ a fundamental set of solutions (of $L[y] = 0$, eg). We want to show "both directions" of this equality; this form defines solutions, and any solution is of this form. This implies, then, that the solution space has exactly dimension n.

↪ **Lemma 4.1**

Let $\varphi(x)$ be any solution of the homogeneous ODE $L[y](x) = 0$ on I. Let $u(x) \geqslant 0$ be defined by $(u(x))^2 = \varphi(x)^2 + \varphi'(x)^2 + \cdots + \varphi^{(n-1)}(x)^2$. Then, $\forall\, x \in I$,

$$u(x_0)e^{-k|x-x_0|} \leqslant u(x) \leqslant u(x_0)e^{k|x-x_0|},$$

where $k = 1 + \sum_{i=1}^{n} \beta_i$, $\beta = \max_{x \in I} |p_i(x)|$.

↪ *Lecture 11; Last Updated: Tue Feb 27 13:01:32 EST 2024*

↪ **Proposition 4.1**

Let $I \subseteq \mathbb{R}$, $x_0 \in I$ and let $p_i(x)$, $i = 1, \ldots, n$ and $g(x)$ be continuous on I. Then, the IVP

$$L[y](x) = g(x) \quad y^{(j)}(x_0) = \alpha_{j+1}, j = 0, \ldots, n-1$$

has at most one solution $y(x)$ defined on I.

Proof. Let y_1, y_2 be two such solutions and let $\varphi(x) = y_1(x) - y_2(x)$. Then,

$$L[\varphi] = L[y_1 - y_2] = L[y_1] - L[y_2] = g(x) - g(x) = 0 \,\forall\, x \in I,$$

so $L[\varphi] = 0 \,\forall\, x \in I$. Moreover, $\varphi(x_0) = y_1(x_0) - y_2(x_0) = \alpha_1 - \alpha_1 = 0$ (with similar computations for the other ICs wrt derivatives of φ). Let $u(x) = \varphi(x)^2 + \varphi'(x)^2 + \cdots + (\varphi^{(n-1)}(x))^2$. Then, $\varphi(x_0) = 0$, so by the previous lemma $u(x) = 0 \,\forall\, x \in I$, and thus $y_1(x) = y_2(x) \,\forall\, x \in I$, and thus there is at most one solution of the IVP. ■

4.3 Linear Homogeneous Nth Order ODES

Consider $L[y] = y^{(n)} + \sum_{j=1}^{n} p_j(x) y^{(nj)} = 0$; in this section, we aim to find the exact dimension of the solution space of L.

↪ **Theorem 4.2: Principle of Superposition**

If $y_1, \ldots, y_m$ are solutions of $L[y] = 0$ for some $I \subseteq \mathbb{R}$ then $y(t) = \sum_{j}^{m} k_j y_j(t)$ is also a solution for arbitrary constants k_j.

↪ **Definition 4.1: Fundamental Set of Solutions**

A set of n functions $\{y_i(x) : L[y_i] = 0, i = 1, \ldots, n\}$ on an interval $I \subseteq \mathbb{R}$ is called a *fundamental set of solutions*if $y_1, \ldots, y_n$ are linearly independent on I.

This necessitates the need to test for linear independence of solutions, which is far harder in $\mathbb{R}^n, n \geqslant 3$ than $n = 2$.

↪ **Definition 4.2: Wronskian**

We define

$$W(y_1, \ldots, y_n)(x) := \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}.$$

↪ **Theorem 4.3**

Let $y_1, \ldots, y_n \in C^{n-1}(I)$. If $W(y_1, \ldots, y_n)(x_0) \neq 0$ for some $x_0 \in I$, then $y_1, \ldots, y_n$ are linearly independent on I. Consequently, if $y_1, \ldots, y_n$ are linearly dependent on I, then $W(y_1, \ldots, y_n)(x) = 0 \,\forall\, x \in I$.

Remark 4.2. *This does not mean that $W(y_1, \ldots, y_n)(x) = 0$ implies the functions are linearly dependent; it does not hold iff.*

Proof. We show the contrapositive. Assume $y_1, \ldots, y_n$ are linearly dependent on I. Then, $\exists k_i, i = 1, \ldots, n$, not all zero, such that $\sum_{j=1}^{n} k_j y_j(x) 0 \,\forall\, x \in I$, assuming wlog that $k_n \neq 0$. Then

$$
\begin{aligned}
y_n(x) &= -\frac{k_1}{k_n} y_1(x) - \frac{k_2}{k_n} y_2(x) - \cdots - \frac{k_{n-1}}{k_n} y_{n-1}(x) \\
&\implies y_n'(x) = -\frac{k_1}{k_n} y_1'(x) - \cdots \frac{k_{n-1}}{k_n} y_{n-1}'(x) \\
&\vdots \\
&\implies y_n^{(n-1)}(x) = -\frac{k_1}{k_n} y_1^{(n-1)}(x) - \cdots - \frac{k_{n-1}}{k_n} y_{n-1}^{(n-1)}(x) \\
&\implies \begin{pmatrix} y_n(x) \\ \vdots \\ y_n^{(n-1)}(x) \end{pmatrix} = -\frac{k_1}{k_n} \begin{pmatrix} y_1(x) \\ \vdots \\ y_1^{(n-1)}(x) \end{pmatrix} - \cdots - \frac{k_{n-1}}{k_n} \begin{pmatrix} y_{n-1}(x) \\ \vdots \\ y_{n-1}^{(n-1)}(x) \end{pmatrix},
\end{aligned}
$$

but each of these column vectors are just rows of the Wronskian (times constants), and we thus have that the Wronskian has linearly dependent columns, ie is singular, ie has zero determinant, as we aimed to show. ■

⊛ **Example 4.1**

Let $y_1(x) = x^2$ and $y_2(x) = \begin{cases} x^2 & x \geqslant 0 \\ -x^2 & x < 0 \end{cases}$, where both are continuously differentiable on $\mathbb{R}$, but $y_2''(x)$ is discontinuous at $x = 0$.

$$
W(y_1, y_2)(x) = \begin{cases} \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0 & \forall\, x \geqslant 0 \\ \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0 & \forall\, x < 0 \end{cases} = 0 \,\forall\, x.
$$

Notice too that for $I = [0, \infty)$, $y_1 = y_2$ and are thus linearly dependent. However, y_1, y_2 are linearly independent on $\mathbb{R}$. Clearly, our choice of interval changes the dependence/independence of our functions, and moreover, this is an example of functions with Wronskian 0 but are not linearly dependent.

This example seems to show that the use of the Wronksian to determine independence of solutions is not reliable; however, we are not particularly interested in this in general, rather, we are concerned with solutions to an nth order ODE. In the previous example, y_2 was not twice continuously differentiable, and so wouldn't even solve a second order ODE.

↪ **Theorem 4.4: Abel's**

Let $y_1, \dots, y_n$ be solutions of the nth order homogeneous ODE $L[y] = 0$ on I with continuous $p_j(x)$ on I. Then,

$$W(x) := W(y_1, \dots, y_n)(x)$$

satisfies the ODE

$$W'(x) + p_1(x)W(x) = 0 \quad \forall\, x \in I,$$

and hence

$$W(x) = Ce^{-\int p_1(x) \mathrm{d}x}.$$

Moreover, either

1. $c = 0$, and $W(y_1, \dots, y_n)(x) = 0 \,\forall\, x \in I$ and $y_1, \dots, y_n$ are linearly dependent on I.
2. c $\neq 0$, and $W(y_1, \dots, y_n)(x) \neq 0 \,\forall\, x \in I$ and $y_1, \dots, y_n$ are linearly independent on I, forming a fundamental set of solutions.

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Proof. We show first that W satisfies the required ODE.

Consider first the $n = 2$ case. We have, $\forall\, x \in I$

$$\begin{aligned} 0 = L[y_1] = y_1'' + p_1(x)y_1' + p_2(x)y_1 \\ 0 = L[y_2] = y_2'' + p_1(x)y_2' + p_2(x)y_2 \end{aligned}$$

Consider:

$$\begin{aligned} y_2(y_1'' + p_1y_1' + p_2y_1) - y_1(y_2'' + p_1y_2' + p_2y_2) = 0 \\ \implies y_1y_2'' - y_2y_1'' + p_1(y_1y_2' - y_2y_1') = 0 \quad *^1 \end{aligned}$$

But recall that $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$, hence

$$W'(x) = y_1y_2'' + y_1'y_2' - y_1'y_2' - y_1''y_2 = y_1y_2'' - y_1''y_2,$$

and thus, as this matches the left-hand terms of $*^1$, $W'(x)+p_1W(x)=0$ as desired.

For general n,

$$W(y_1,\dots,y_n)(x)=\begin{vmatrix} y_1(x) & \cdots & y_n(x)\\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x)\end{vmatrix}$$

$$W'(x)=\underbrace{\begin{vmatrix} y_1' & \cdots & y_n'\\ y_1' & \cdots & y_n'\\ y_1'' & \cdots & y_n''\\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)}\end{vmatrix}+\begin{vmatrix} y_1 & \cdots & y_n\\ y_1'' & \cdots & y_n''\\ y_1'' & \cdots & y_n''\\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)}\end{vmatrix}+\cdots+\begin{vmatrix} y_1 & \cdots & y_n\\ y_1' & \cdots & y_n'\\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)}\\ y_1^{(n-1)} & \cdots & y_n^{(n-1)}\end{vmatrix}}_{=0;\text{ have a repeated row}}+\begin{vmatrix} y_1 & \cdots & y_n\\ y_1' & \cdots & y_n'\\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{n-1}\\ y_1^{(n)} & \cdots & y_n^{(n)}\end{vmatrix} \qquad *^2$$

But we have that $y_j^{(n)}=-p_1y_j^{(n-1)}-p_2y_j^{(n-2)}-\dots-p_ny_j$, $j=1,\dots,n$, so we can substitute this into $*^2$. This will simplify:

$$W'=-p_1\begin{vmatrix} y_1 & \cdots & y_n\\ \vdots & \ddots & \vdots \\ y_1^{(n-2)} & \cdots & y_n^{(n-2)}\\ y_1^{(n-1)} & \cdots & y_n^{(n-1)}\end{vmatrix}\underbrace{-p_2\begin{vmatrix} y_1 & \cdots & y_n\\ \vdots & \ddots & \vdots \\ y_1^{(n-2)} & \cdots & y_n^{(n-2)}\\ y_1^{(n-2)} & \cdots & y_n^{(n-2)}\end{vmatrix}-\cdots-p_n\begin{vmatrix} y_1 & \cdots & y_n\\ \vdots & \ddots & \vdots \\ y_1^{(n-2)} & \cdots & y_n^{(n-2)}\\ y_1 & \cdots & y_n\end{vmatrix}}_{=0}$$

$$=-p_1W,$$

as required.

In the case $c\neq 0$, case 2., then $W(x)\neq 0\,\forall\,x\in I$, and we've already shown that $y_1,\dots,y_n$ are linearly independent on I.

If $c=0$, case 2., and $W(x)=0\,\forall\,x\in I$, then it remains to show that $y_1,\dots,y_n$ are linearly dependent.

Let $\varphi(x)=\sum_{j=1}^n c_jy_j(x)$, with c_j such that φ solves the IVP; ie

$$L[\varphi]=0;\quad \varphi(x_0)=\dots=\varphi^{(n-1)}(x_0)=0.$$

We must have:

$$\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}=\begin{pmatrix}\varphi(x_0)\\ \vdots\\ \varphi^{(n-1)}(x_0)\end{pmatrix}=\underbrace{\begin{pmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0)\\ \vdots & \ddots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & \cdots & \cdots & y_n^{(n-1)}(x_0)\end{pmatrix}}_{:=A}\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix}$$

Since $W(x) = 0 \,\forall\, x \in I$, $W(x_0) = 0$ and thus this matrix A has determinant 0, is singular, and has a non-trivial kernel.

Let $(c_1, \ldots, c_n)^T \in \text{Ker}(A)$, not equal to the zero vector; then, these c_j make φ satisfy the IVP as desired:

$$L[\varphi] = \sum_{j=1}^{n} c_j L[y_j] = 0,$$

as y_j solutions and c_j chosen appropriately to satisfy IVP.

We clearly have, as well, that $y(x) = 0$ will solve the IVP; but by uniqueness, it must be that

$$\begin{aligned} 0 = y(x) &= \varphi(x) \,\forall\, x \in I \\ \implies 0 &= \sum_{j=1}^{n} c_j y_j(x), \end{aligned}$$

but by construction the c_js are not all zero, hence, $y_1, \ldots, y_n$ must be linearly dependent. ∎

↪ Corollary 4.1

If $L[y_j] = 0 \,\forall\, x \in I, j = 1, \ldots, n$, where p_j are continuous for all $x \in I$, and let $Y := \{y_j : 1 \leqslant j \leqslant 1\}$. TFAE:

1. Y form a fundamental set of solutions on I;
2. Y are linearly independent on I;
3. $W(Y)(x_0) \neq 0$ for some $x_0 \in I$;
4. $W(Y)(x) \neq 0 \,\forall\, x \in I$.

$\hookrightarrow$ **Theorem 4.5**

Let $y_1, \ldots, y_n$ be a fundamental set of solutions for $L[y] = 0$ on I, where $p_j(x)$-continuous on I.

1. The IVP
$$L[y] = 0, \quad y(x_0) = \alpha_1, \ldots, y^{(n-1)}(x_0) = \alpha_n$$
has a unique solution $y(x)$ for $x \in I$, which can be written as
$$y(x) = \sum_{j=1}^{n} c_j y_j(x), \quad \dagger$$
for a unique choice of the constants $c_1, \ldots, c_n$.

2. Every solution $y(x)$ of the ODE $L[y] = 0$ defined on I can be written in the form † for some choice of the parameters $c_1, \ldots, c_n$.

Remark 4.3. *This theorem does not guarantee existence of the fundamental set of solutions for an arbitrary* $L[y] = 0$.

Part 2. shows that the fundamental set of solutions span the whole solution space: the space of solutions is exactly n*-dimensional.*

Proof. To prove 1., let $y(x)$ as defined by †. Then, $L[y] = 0$ trivially satisfies the ODE, by superposition, so it remains to show that there is a unique choice of (c_j) such that the IVP is satisfied. We need:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} y(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{pmatrix} = \underbrace{\begin{pmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ \vdots & \ddots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{pmatrix}}_{:=A} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

But now, $\det\{A\} = W(y_1, \ldots, y_n)(x_0) \neq 0$, hence A invertible, and we have

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = A^{-1} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Since A^{-1} is unique, then so are the (c_j)'s.

To prove 2., note that any $y(x)$ defined by † satisfies $L[y] = 0 \, \forall \, x \in I$ for any choice of c_j by superposition. To show that there are no other forms of solutions, suppose $\varphi(x)$ is a solution that cannot be written as such.

Suppose $L[\varphi](x) = 0 \, \forall \, x \in I$. For φ, let $x_0 \in I$ and find $y(x)$ that satisfies the IVP

$$L[y] = 0, \quad y(x_0) = \varphi(x_0), \cdots, y^{(n-1)}(x_0) = \varphi^{(n-1)}(x_0).$$

By 1. this IVP has a unique solution of the form †, and with the same IC as φ, we have thus that $\varphi = y$, a contradiction. ■

4.4 Nonhomogeneous Nth Order Linear ODEs

Consider $L[y] = g$. If $y_1, \ldots, y_n$ a fundamental set of solutions of $L[y] = 0$ and $L[y_p] = g$, then

$$y(x) = y_p(x) + \sum_{j=1}^{n} c_j y_j(x)$$

will satisfy the original $L[y] = g$. We need to show that we can construct such an y_p.

We will use variation of parameters to find y_p. Suppose $y_p(x) = \sum_{j=1}^{n} u_j(x) y_j(x)$ for TBD $u_j(x)$, and suppose $L[y_p] = g$. This gives

$$y_p'(x) = \sum_j u_j(x) y_j'(x) + \sum_j u_j'(x) y_j(x).$$

To simplify, we'll assume that $\sum_j u_j' y_j = 0 \,\forall\, x \in I$, so

$$y_p''(x) = \sum_j u_j y_j'' + \sum_j u_j' y_j',$$

and assume, similarly, $\sum_j u_j' y_j' = 0 \,\forall\, x$, remarking that at each of these steps we introduce a new constraint, and as such we will eventually have $n-1$ constraints to solve for.

↪ *Lecture 13; Last Updated: Thu Apr 4 17:53:29 EDT 2024*

↪ Theorem 4.6

Let $y_1, \ldots, y_n$ be a fundamental set of solutions of $L[y] = 0$ for $x \in I$ where p_j continuous on I. Suppose $g(x)$ continuous on I. Then

1. The IVP $L[y] = g$, $y(x_0) = \alpha_1, \ldots, y^{(n-1)}(x_0) = \alpha_n$ has a unique solution $y(x)$ for $x \in I$.
2. Every solution of the ODE $L[y] = g$ can be written in the form

$$y(x) = y_p(x) + \sum_{j=1}^{n} c_j y_j(x) \qquad \ddagger$$

where y_p a particular solution satisfying $L[y_p] = g$.

Proof. We show 2. first. Suppose y_{p_1} solves $L[y_{p_1}] = g$ (which exists by 1.). Then, $y_{p_1}(x)$ is of the form ‡ with $c_j = 0$ and $y_p = y_{p_1}$. Let y_{p_2} be a different solution of $L[y_{p_2}] = g$. Let $Y = y_{p_2} - y_{p_1}$. Then,

$$L[Y] = L[y_{p_2}] - L[y_{p_1}] = g - g = 0 \,\forall\, x \in I,$$

hence $Y(x)$ solves the corresponding homogeneous problem $L[Y] = 0$, and so by the previous theorem, can be written in the form $Y = \sum_{j=1}^{n} c_j y_j(x)$ for appropriate choice of c_j's. Thus,

$$y_{p_2}(x) = Y(x) + y_{p_1}(x) = \sum_{j=1}^{n} c_j y_j(x) + y_{p_1}(x),$$

as required.

We proceed to 1. We've already shown that this IVP has at most one solutions, so it suffices to find that there is exactly one. We will do so by variation of parameters. Suppose $y_p = \sum_{j=1}^{n} u_j(x) y_j(x)$ where y_p solves $L[y_p] = g$. Then,

$$y_p' = \sum_{j=1}^{n} u_j y_j' + \sum_{j=1}^{n} u_j' y_j,$$

and assume that $\sum_{j=1}^{n} u_j' y_j = 0 \,\forall\, x \in I$, hence

$$y_p'' = \sum u_j' y_j' + \sum u_j y_j''.$$

Let us assume too that $\sum u_j' y_j' = 0 \,\forall\, x \in I$. We can continue in this manner, differentiating $n-1$ times, yielding

$$y_p^{(j)} = \sum_{i=1}^{n} u_i y_i^{(j)}, \quad j = 0, \ldots, n-1,$$

and assuming appropriately $\sum u_i' y_i^{(j-1)} = 0$, for $j = 1, \ldots, n-1$. Finally, differentiating once more, we have

$$y_p^{(n)} = \sum u_i y_i^{(n)} + \sum u_i' y_i^{(n-1)},$$

this time, *not* assuming that the last term vanishes. Plugging into L, then we have

$$
\begin{aligned}
g = L[y_p] &= y_p^{(n)} + \sum_{j=1}^{n} p_j y_p^{(n-j)} \\
&= \sum u_i y_i^{(n)} + \sum u_i' y_i^{(n-1)} + \sum_{j=1}^{n} p_j(x) \sum_{i=1}^{n} u_i y_i^{(n-j)} \\
&= \sum u_i' y_i^{(n-1)} + \sum_i u_i \underbrace{\left[y_i^{(n)} + \sum_j p_j y_i^{(n-j)} \right]}_{=0,\text{ for each } i,\text{ solving } L[y_i]=0} \\
\implies g &= \sum_i u_i' y_i^{(n-1)}.
\end{aligned}
$$

This, along with our $n-1$ constraints, gives us n equations defining the $u_i'(x)$, giving us the linear system:

$$
\begin{pmatrix}
y_1 & y_2 & \cdots & y_n \\
y_1' & y_2' & \cdots & y_n' \\
\vdots & \ddots & \ddots & \vdots \\
y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\
y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{pmatrix}
\cdot
\begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix}
=
\begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(x) \end{pmatrix},
$$

where the first $n-1$ rows of the matrix follow from the constrains we imposed on u_i', the last follows from the previous line when we plugged in our y_p into $L[y_p] = g$. But this is just the Wronskian matrix, and $W(y_1, \ldots, y_n)(x) \neq 0 \,\forall\, x \in I$ by Abel's since y_i's form a fundamental set of solutions by assumption, thus, the matrix is invertible and we can therefore solve for u_i's:

$$
\begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix}
=
\begin{pmatrix}
y_1 & y_2 & \cdots & y_n \\
y_1' & y_2' & \cdots & y_n' \\
\vdots & \ddots & \ddots & \vdots \\
y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\
y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{pmatrix}^{-1}
\begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(x) \end{pmatrix}
=:
\begin{pmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{pmatrix},
$$

hence, $u_j'(x) = f_j(x)$ for some f_j as defined, and thus

$$
u_j(x) = \int_{x_0}^{x} f_j(s) \, \mathrm{d}s \,,
$$

and so our particular solution is

$$
y_p(x) = \sum_i y_i \int_{x_0}^{x} f_i(s) \, \mathrm{d}s \,.
$$

This is a solution to the ODE; it remains to show that the IVP can be solved by a unique choice of the c_j's. This is similar to the homogeneous case; left as a (homework) exercise. ■

↪ Theorem 4.7: Cramer's Rule

Let $A \in M_n(\mathbb{R})$ be invertible and x, b $n \times 1$ column vectors. Then for any $b \in \mathbb{R}^n$, $Ax = b$ has a unique solution $x \in \mathbb{R}^n$ given by

$$x_i = \frac{\det A_i}{\det A}, \quad i = 1, \ldots, n,$$

where A_i is the matrix obtained by replacing the ith column of A by the vector b.

↪ Theorem 4.8: Variation of Parameters

Let $y_1, \ldots, y_n$ be a fundamental set of solutions of $L[y] = 0$, let $W(x) = W(y_1, \ldots, y_n)(x)$, let $W_i(x)$ be the determinant of the matrix obtained by replacing the ith column of W by $\begin{pmatrix} 0 \\ \vdots \\ g \end{pmatrix}$, and let $u_i = \int_{x_0}^{x} \frac{W_i(s)}{W(s)}\,\mathrm{d}s$, then

$$y_p = \sum_{i=1}^{n} u_i(x) y_i(x).$$

Proof. This follows from the work we showed in the proof of theorem 4.6 part 2. and Cramer's Rule. ■

⊛ Example 4.2

Find the general solution of $y''' + y' = \tan x$. We first find a fundamental set of solutions to

$$y''' + y' = 0.$$

Suppose $y = e^{rx}$, giving

$$0 = r^3 + r = r(r^2 + 1) \implies r = 0, \pm i,$$

giving us solutions

$$y_1(x) = 1, \quad y_2(x) = \cos x, \quad y_3(x) = \sin x.$$

To verify linear independence:

$$W(x) = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \sin^2(x) + \cos^2(x) = 1.$$

To solve $L[y] = \tan x$, we have

$$W_1(x) = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \cos^2 x \tan x + \sin^2 x \tan x = \tan x$$
$$W_2(x) = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = -\cos x \tan x = -\sin x$$
$$W_3(x) = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = -\sin x \tan x = \frac{-\sin^2 x}{\cos x}$$

Then, this gives

$$u_1 = \int \frac{W_1}{W}\,\mathrm{d}x = \int \tan x\,\mathrm{d}x = -\ln|\cos x|$$
$$u_2 = \int \frac{W_2}{W}\,\mathrm{d}x = \int -\sin x\,\mathrm{d}x = \cos x$$
$$u_3 = \int \frac{W_3}{W}\,\mathrm{d}x = \int \frac{-\sin^2 x}{\cos x}\,\mathrm{d}x = \int \frac{\cos^2 x - 1}{\cos x} = \sin x - \ln|\tan x + \sec x|$$

and so

$$\begin{aligned} y_p &= \sum_{j=1}^{3} u_j y_j = -\ln|\cos x| + \cos x \cdot \cos x + (\sin x - \ln|\tan x + \sec x|) \cdot \sin x \\ &= 1 - \ln|\cos x| - (\ln|\tan x + \sec x|) \sin x, \end{aligned}$$

giving us a general solution

$$y = y_c + y_p = c_1 + c_2 \cos x + c_3 \sin x + 1 - \ln|\cos x| - \sin x \ln|\tan x + \sec x|,$$

which can be simplified by absorbing the 1 into the constant c_1, and simplifying appropriately:

$$y = \tilde{c}_1 + c_2 \cos x + \sin x(c_3 - \ln|\tan x + \sec x|) - \ln|\cos x|$$

4.5 Fundamental Set of Solutions

↪ Lecture 14; Last Updated: Mon Mar 11 15:30:06 EDT 2024

↪ Theorem 4.9

Let $L[y] := \sum_{j=0}^{n} a_j y^{(j)}$ where a_j are real constants with $a_n \neq 0$. Let

$$\sum_{j=0}^{n} a_j r^j = 0 \qquad \textbf{(A)}$$

be the corresponding auxiliary equation, supposing it has roots r_j of multiplicity s_j. Then, the linear homogeneous $L[y] = 0$ has a fundamental set of solutions defined on $\mathbb{R}$ composed of

$$x^k e^{r_j x}, \qquad k = 0, \ldots, s_j - 1, r_j \in \mathbb{R} \text{ of mult. } s_j$$

and

$$x^k e^{\alpha_j x} \cos(\beta_j x), \quad x^k e^{\alpha_j x} \sin(\beta_j x), \qquad k = 0, 1, \ldots, s_j - 1, \text{ where } r_j = \alpha_j \pm \beta_j \text{ of mult. } s_j.$$

Proof. We won't prove this, but is just a generalization of the same idea for second-order equations. Difficulties in the proof arise when proving linear independence. ■

Remark 4.4. *Combined with the previous theorem, we thus have that all solutions of $L[y] = 0$ can be written in the form $y = \sum_{j=1}^{n} c_j y_j(x)$.*

4.6 Non-Constant Coefficient Linear ODEs

↪ **Theorem 4.10**

Let $L[y] = y^{(n)} + \sum_{j=1}^{n} p_j(x) y^{(n-j)}(x)$, where each $p_j(x)$ continuous on some $I \subseteq \mathbb{R}$, and let $x_0 \in I$. Let $y_i(x)$ solve the IVP

$$L[y_i](x) = 0 \qquad y_i^{(i-1)}(x_0) = 1,\ y_i^{(j)}(x_0), j = 0, \dots, n-1, j \neq i-1.$$

Then, $\{y_i : i = 1, \dots, n\}$ form a fundamental set of solutions for $L[y] = 0$ on I.

Proof. Each of these IVPs has a unique solution $y_i(x)$ on I by Picard's Theorem. Now,

$$W(y_1, \dots, y_n)(x_0) = \begin{vmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

so y_i are indeed linearly independent, by Abel's Theorem, on I. ■

⊛ **Example 4.3**

Consider the IVP

$$L[y] := y^{(4)} + y'' - 2y = \cos x, \quad y(0) = 1, y^{(i)}(0) = 0, \quad i = 1, 2, 3.$$

We first find $L[y_c] = 0$. We have auxiliary

$$r^4 + r^2 - 2 = 0 \implies (r^2 - 1)(r^2 + 2) = 0 \implies r = \pm 1, \pm i\sqrt{2}$$

and thus

$$y_1 = e^x, \quad y_2 = e^{-x}, \quad y_3 = \cos\sqrt{2}x, \quad y_4 = \sin\sqrt{2}x.$$

We seek now a particular solution, guessing

$$y_p = A\cos x \implies L[y_p] = A\cos x - A\cos x - 2A\cos x = \cos x \implies A = -\frac{1}{2}$$

and thus $y_p = -\frac{1}{2}\cos x$, giving general solution

$$y(x) = k_1 e^x + k_2 e^{-x} + k_3\cos\left(\sqrt{2}x\right) + k_4\sin\left(\sqrt{2}x\right) - \frac{1}{2}\cos(x).$$

Solving the IVP, we find

$$\begin{aligned}
1 = y(0) &= k_1 + k_2 + k_3 - \frac{1}{2} \qquad \text{(i)} \\
y'(x) &= k_1 e^x - k_2 e^{-x} - \sqrt{2}k_3\sin\left(\sqrt{2}x\right) + \sqrt{2}k_4\cos\left(\sqrt{2}x\right) + \frac{1}{2}\sin(x) \\
&\implies y'(0) = 0 = k_1 - k_2 + \sqrt{2}k_4 \qquad \text{(ii)} \\
y''(x) &= k_1 e^x + k_2 e^{-x} - 2k_3\cos\left(\sqrt{2}x\right) - 2k_4\sin\left(\sqrt{2}x\right) + \frac{1}{2}\cos(x) \\
&\implies y''(0) = 0 = k_1 + k_2 - 2k_3 + \frac{1}{2} \qquad \text{(iii)} \\
y'''(x) &= k_1 e^x - k_2 e^{-x} + 2\sqrt{2}k_3\sin\left(\sqrt{2}x\right) - 2\sqrt{2}k_4\cos\left(\sqrt{2}x\right) - \frac{1}{2}\sin(x) \\
&\implies y'''(0) = 0 = k_1 - k_2 - 2\sqrt{2}k_4 \qquad \text{(iv)}
\end{aligned}$$

$$\begin{aligned}
\text{(i)} - \text{(iii)} &\implies 1 = 3k_3 - 1 \implies k_3 = \frac{2}{3} \\
\text{(ii)} - \text{(iv)} &\implies 0 = (\sqrt{2} + 2\sqrt{2})k_4 \implies k_4 = 0 \\
\text{(iii)} + \text{(iv)} &\implies 0 = 2k_1 - 2k_3 + \frac{1}{2} - 2\sqrt{2}k_4 \implies k_1 = \frac{5}{12} \\
\text{(i)} &\implies 1 = \frac{5}{12} + k_2 + \frac{2}{3} - \frac{1}{2} \implies k_2 = \frac{5}{12}
\end{aligned}$$

So our IVP solution is

$$y(x) = \frac{5}{12}(e^x + e^{-x}) + \frac{2}{3}\cos\left(\sqrt{2}x\right) - \frac{1}{2}\cos(x).$$

5 Series Solutions

5.1 Review of Power Series

↪ Definition 5.1: Convergence

A power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges at a point x_0 if $\lim_{m\to\infty} \sum_{n=0}^{m} a_n(x-x_0)^n$ exists for that x. The series is absolutely convergent at x_0 if $\sum_{n=0}^{m} |a_n| \, |x-x_0|^n$ exists as $m \to \infty$.

The radius of convergence of a series is the minimal $\rho \geqslant 0$ such that the series is absolutely convergent for x such that $|x-x_0| < \rho$ and divergent for $|x-x_0| > \rho$.

Remark 5.1. *Absolutely convergent* $\implies$ *convergent.*

↪ Definition 5.2: Real Analytic

A function $f : I \to \mathbb{R}$ is (real) analytic at $x_0 \in I$ if $\exists \rho > 0$ s.t. $\forall\, x \in I : |x-x_0| < \rho$ we have

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

with power series having radius of convergence (at least) ρ.

Remark 5.2. *When f real analytic, it is continuous and has derivatives of all orders for $|x-x_0| < \rho$, and these derivatives can be found by differentiating the power series. Indeed, we have*

$$f^{(m)}(x) = \sum_{n=0}^{\infty} n(n-1)\cdots(n-m+1)a_n(x-x_0)^{n-m} = \sum_{n=m}^{\infty} n(n-1)\cdots(n-m+1)a_n(x-x_0)^{n-m}.$$

↪ Lecture 15; Last Updated: Tue Feb 27 10:08:23 EST 2024

↪ Proposition 5.1

Let $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ and $g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$.

1. $f(x) = g(x) \,\forall\, x$ s.t. $|x-x_0| < \rho$ iff $a_n = b_n \,\forall\, n$.
2. $f(x) \pm g(x) = \sum_{n=0}^{\infty}(a_n \pm b_n)(x-x_0)^n$. The resulting power series has radius of convergence at least as large as the minimum of the radii of convergence of f, g.
3. $f(x)g(x) = [\sum_{i=0}^{\infty} a_i(x-x_0)^i][\sum_{j=0}^{\infty} b_j(x-x_0)^j] = \sum_{n=0} c_n(x-x_0)^n$ where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$. This power series also has radius of convergence as least as large of the minimum of f, g.
4. We can divide power series (essentially long division of polynomials, but with infinite degrees) and can result in smaller radius of convergence, but won't.

↪ Proposition 5.2

If $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists then $\rho = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$.

Proof. We have by the ratio test that $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges if

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| < 1 &\iff \lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)}{a_n}\right| \\
&\iff |x-x_0| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \\
&\iff |x-x_0| < \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|
\end{aligned}$$

■

⊛ **Example 5.1**

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty}\frac{x^n}{n!} \implies e^{x-x_0} = \sum_{n=0}^{\infty}\frac{(x-x_0)^n}{n!} \\
\cos(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!} \\
\sin(x) &= \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}
\end{aligned}$$

These all have $\rho = +\infty$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

This series converges for $\rho < 1$ since

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 1.$$

Remark 5.3. *In the case that* $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ *does not exist, then the root test gives that*

$$\rho = \frac{1}{\limsup_{n\to\infty}|a_n|^{1/n}}.$$

↪ **Proposition 5.3**

If $P(x), Q(x)$ are polynomials, then $\frac{Q(x)}{P(x)}$ is analytic at x_0 if $P(x_0) \neq 0$. When analytic, the radius of convergence from x_0 is the distance from x_0 to the nearest zero of $P(x)$ in the complex plane.

⊛ **Example 5.2**

$\frac{Q(x)}{P(x)} = \frac{1}{1+x^2}$. In the complex plane, $P(x)$ has roots at $x = \pm i$, and so $\rho = \sqrt{1+x_0^2}$.

5.2 Series Solutions near Ordinary Points

↪ **Definition 5.3: Ordinary Point**

Let $L[y] = P(x)'' + Q(x)y' + R(x)y$ and $p(x) = \frac{Q(x)}{P(x)}, q(x) = \frac{R(x)}{P(x)}$. x_0 is an *ordinary point* of $L[y] = 0$ if p, q are both analytic at x_0; otherwise, x_0 is a *singular point*.

↪ **Theorem 5.1**

If x_0 an ordinary point for $L[y] = 0$ then the general solution can be written as

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 y_1(x) + a_1 y_2(x),$$

where a_0, a_1 arbitrary and the other a_i's are uniquely determined by choice of a_0, a_1. The functions y_1, y_2 will be two power series, analytic at x_0, and form a fundamental set of solutions with $W(y_1, y_2)(x_0) = 1$. The radius of convergence of y_1, y_2 and y is at least as large as the smaller of the radii of p, q.

⊛ **Example 5.3**

Consider $(1+x^2)y'' - 4xy' + 6y = 0$, with $p(x) = \frac{-4x}{1+x^2}, q(x) = \frac{6}{1+x^2}$; these are analytic $\forall\, x \in \mathbb{R}$, so we can expand about any $x_0 \in \mathbb{R}$. For convenience, take $x_0 = 0$. The radius of convergence of $y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ will then be $\rho = 1$. Then:

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ y'(x) &= \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n = \sum_{n=0}^{\infty} n a_n x^{n-1} \\ y''(x) &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} \end{aligned}$$

So

$$
\begin{aligned}
0 &= (1+x^2)y'' - 4xy' + 6y = y'' + x^2y'' - 4xy' + 6y \\
&= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x^2\sum_{n=0}^{\infty}n(n-1)a_nx^{n-2} - 4x\sum_{n=0}^{\infty}na_nx^{n-1} + 6\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4na_n + 6a_n\right]x^n,
\end{aligned}
$$

so, $\forall\, n \geqslant 0$, we need

$$
\begin{aligned}
(n+2)(n+1)a_{n+2} + n(n-1)a_n - 4na_n + 6a_n &= 0 \\
(n+2)(n+1)a_{n+2} + (n-2)(n-3)a_n &= 0 \\
\implies a_{n+2} &= \frac{-(n-2)(n-3)}{(n+2)(n+1)}a_n \\
n = 0 &\implies a_2 = a_2 = -3a_0 \\
n = 1 &\implies a_3 = -\frac{a_1}{3} \\
n = 2 &\implies a_4 = 0 \\
n = 3 &\implies a_5 = 0 \\
&\implies a_n = 0\,\forall\, n \geqslant 4,
\end{aligned}
$$

so

$$
y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 = a_0 + a_1x - 3a_0x^2 - \frac{a_1}{3}x^3 = a_0(1-3x^2) + a_1(x - \frac{x^3}{3}) =: a_0y_1 + a_1y_2.
$$

Remark that

$$
W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.
$$

⊛ **Example 5.4**

Consider $y'' - xy' - x^2 y = 0$, $p(x) = -x$, $q(x) = -x^2$ which are both analytic on all $\mathbb{R}$. Let $x_0 = 0$, so

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \implies x^2 y = \sum_{n=0}^{\infty} a_n x^{n+2} = \sum_{n=2}^{\infty} a_{n-2} x^n \\
y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} \implies xy' = \sum_{n=0}^{\infty} n a_n x^n \\
y'' &= \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \\
0 &= y'' - xy' - x^2 y = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=2}^{\infty} a_{n+2} x^n \\
0 &= 2a_2 + 3 \cdot 2 \cdot a_3 \cdot x - a_1 x + \sum_{n=2}^{\infty} [(n+2)(n+1) a_{n+2} - n a_n - a_{n-2}] x^n
\end{aligned}
$$

Matching powers of x^n yields

$$
\begin{aligned}
n &= 0] \qquad a_2 = 0 \\
n &= 1] \qquad a_3 = \frac{a_1}{6} \\
n &\geqslant 2] \qquad (n+2)(n+1) a_{n+2} n a_n - a_{n-2} = 0 \implies a_{n+2} = \frac{n a_n + a_{n-2}}{(n+2)(n+1)}
\end{aligned}
$$

From here, you can find as many terms of a_n as you really want. The important thing to notice is that if n odd, a_n will only depend on a_1, and if n even, a_n will only depend on a_0. This gives a final form

$$
y(x) = a_0 y_1(x) + a_1 y_2(x),
$$

where y_1, y_2 power series involving only event, odd terms resp. Remark too that $W(y_1, y_2)(0) = 1$ (why?).

↪ *Lecture 16; Last Updated: Thu Mar 28 14:19:24 EDT 2024*

Remark 5.4. *No lecture, in-class midterm.*

↪ *Lecture 17; Last Updated: Thu Mar 28 14:19:24 EDT 2024*

5.3 Analytic Coefficients

We consider now series solutions to

$$
L[y] = P(x)y'' + Q(x)y' + R(x)y = y'' + p(x)y' + q(x)y = 0
$$

where P, Q, R analytic but not necessarily polynomials. Similar theory holds; a power series solution $y(x)$ will have radius of convergence at least as large as that of p and q. We proceed by instructive example.

⊛ **Example 5.5**

$x_0 = 0, L[y] = y'' - e^x y$. Here, $q(x) = -e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ with infinite radius of convergence. $p(x) = 0$ also has infinite radius of convergence, hence we should find that our solution will as well. Letting $y(x) = \sum_{n=0}^{\infty} a_n x^n$, we compute as before.

$$L[y] = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}\left[\sum_{j=0}^{\infty}[\frac{a_{n-j}}{j!}]x^n\right]$$

Computation of the corresponding a_n follows very similarly to previous examples; the only difficulty is the fact that now a_n will rely on all a_n's less than it. Namely, one should find

$$a_{n+2} = \frac{1}{(n+2)(n+1)}\sum_{j=0}^{n}\frac{a_{n-j}}{j!}$$

5.4 Nonhomogeneous Series Solutions

We consider the case

$$L[y] := y'' + p(x)y' + q(x)y = g(x).$$

Writing $L[y] = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ where c_n dependent on a_m for $m \leqslant n$ and $g(x) = \sum_{n=0}^{\infty} g_n (x - x_0)^n$, we have that

$$L[y] = g(x) \iff c_n = g_n \,\forall\, n \geqslant 0.$$

So, we generally have a very similar method, only now we have to deal with a non-zero equivalence on the RHS.

⊛ **Example 5.6**

$y'' - xy = \frac{1}{6}x^3$; remark that any series solution will have infinite radius of convergence about $x_0 = 0$. We have

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = \frac{1}{6}x^3$$
$$\implies 2a_2 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} - a_{n-1}]x^n = \frac{1}{6}x^3.$$

We proceed by matching powers of x on the left, right hand sides.

$$\begin{aligned}
x^0] &\quad 2a_2 = 0 \implies a_2 = 0 \\
x^1] &\quad 3 \cdot 2 \cdot a_3 - a_0 = 0 \implies a_3 = \frac{a_0}{3 \cdot 2} \\
x^2] &\quad 4 \cdot 3 \cdot a_4 - a_1 = 0 \implies a_4 = \frac{a_1}{4 \cdot 3} \\
x^3] &\quad 5 \cdot 4 \cdot a_5 - a_2 = \frac{1}{6} \implies a_5 = \frac{1}{5!} \\
n \geqslant 4] &\quad a_{n+2}(n+2)(n+1) - a_{n-1} = 0 \implies a_{n+2} = \frac{a_{n-1}}{(n+1)(n+1)}
\end{aligned}$$

One can show that for $n \geqslant 0$,

$$\begin{aligned}
a_{3n} &= \frac{(3n-1)(3n-4)(\cdots)(7)(4)a_0}{(3n)!} \\
a_{3n+1} &= \frac{(3n-1)(3n-4)(\cdots)(8)(5)(2)a_1}{(3n+1)!} \\
a_{3n+2} &= \frac{3^{n-1}n!}{(3n+2)!},
\end{aligned}$$

remarking in particular that a_{3n+2} has no reliance on a_0 or a_1, and indeed serve as the coefficients of our particular solution. We find

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_{3n}x^{3n} + \sum_{n=0}^{\infty} a_{3n+1}x^{3n+1} + \sum_{n=0}^{\infty} a_{3n+2}x^{3n+2} \\
&= a_0 y_1(x) + a_1 y_2(x) + y_p(x).
\end{aligned}$$

5.5 Singular Points

What about finding solutions about non-ordinary points? We now need to be more careful.

↪ Definition 5.4: Regular Singular Point

A "not too singular point". If $L[y] = P(x)y'' + Q(x)y' + R(x)y$, then x_0 a regular singular point if it is a singular point of $L[y] = 0$, and also

$$(x - x_0)\frac{Q(x)}{P(x)} \qquad (x - x_0)^2\frac{R(x)}{P(x)}$$

are both analytic at x_0. In particular, if P, Q, R polynomials, x_0 a singular point iff $P(x_0) = 0$, and regular iff $\lim_{x \to x_0}(x - x_0)\frac{Q(x)}{P(x)}$, $\lim_{x \to x_0}(x - x_0)^2\frac{R(x)}{P(x)}$ are both finite.

↪ *Lecture 18; Last Updated: Thu Mar 28 14:17:59 EDT 2024*

5.6 Frobenius's Method

We consider $L[y] = P(x)y'' + Q(x)y' + R(x)y = 0$. Let x_0 be a regular singular point, and multiply both sides by $\frac{(x-x_0)^2}{P(x)}$:

$$(x-x_0)^2 y'' + (x-x_0)\underbrace{\left[(x-x_0)\frac{Q(x)}{P(x)}\right]}_{:=p(x)} y + \underbrace{\left[(x-x_0)^2\right]}_{:=q(x)} y = 0$$

Recall that, by definition of a regular singular point, we have that p, q analytic at x_0 and so can be represented as a local power series. We will seek a solution of the form

$$y(x) = |x-x_0|^r \sum_{n=0}^{\infty} a_n (x-x_0)^n,$$

for some $r \in R$ with $a_0 \neq 0$. For convenience, and wlog (by linearity, scaling appropriately) we take $a_0 = 1$ by convention. Also for simplicity, we often assume that $x > 0$ so we do not have to work with the absolute value.

After tedious computation, one can find that an appropriate such r must satisfy the *indicial equation*

$$F(r) = r(r-1) + rp_0 + q_0 = 0$$

where p_0, q_0 the x^0 coefficients of $p(x), q(x)$ resp.

From here, we can either 1) solve to find r (for which we need to do no more work than stare at p, q), plug in $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+r}$ with appropriate r into our ODE and solve for a_n, or 2) derive a general formula.

We find the general formula to be

$$a_n = \frac{-1}{F(n+r)} \cdot \sum_{k=0}^{n-1} a_k \left[(k+r)p_{n-k} + q_{n-k}\right], \quad \forall\, n \geqslant 1.$$

Remark 5.5. *This is a "worst case" general form, where a_n depends on $a_{n-1}, \ldots, a_1$; we will generally find in examples that much simplification occurs.*

Remark 5.6. *Remark that if $F(r) = 0$ has 2 real roots $r_1 < r_2$, we'll be dividing by $F(n+r_2), n = 1, 2, \ldots$; but $F(r_2) = 0 \implies F(n+r_2) \neq 0 \,\forall\, n \geqslant 1$, so there is no division by zero problem. But this does give that if $r_2 - r_1 = N \in \mathbb{N}$, then the formula will break (division by zero) at a_N. Similarly, if $F(r) = 0$ has repeated roots, $r_1 = r_2$, we can only derive one formula this way.*

⊛ **Example 5.7**

$0 = L[y] = 4xy'' + 2y' + 2y.$

↪ **Theorem 5.2: Frobenius**

Let $L[y] = (x-x_0)^2 y'' + (x-x_0)p(x)y' + q(x)y = 0$ where x_0 a regular singular point, p, q both analytic at x_0, with $p(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n$, $q(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n$, with $\rho :=$ min of the radii of convergence of p, q. Let r_1, r_2 be the roots of

$$0 = F(r) = r(r-1) + p_0 r + q_0,$$

where $r_1 \geqslant r_2$ if both real. Then, there exists a solution of the form

$$y_1(x) = |x-x_0|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)(x-x_0)^n\right],$$

with $a_n(r_1)$ s.t. $a_0 = 1, a_n = -\frac{1}{F(n+r)} \sum_{k=0}^{n-1} a_k(r)\left[(k+r)p_{n-k} + q_{n-k}\right]$, $n \geqslant 1$, with $r = r_1$. We define a second solution as follows:

(i) ($r_1 - r_2 \neq 0$ and $r_1 - r_2 \notin \mathbb{Z}$)

$$y_2(x) = |x-x_0|^{r_2}\left[1 + \sum_{n=1}^{\infty} a_n(r_2)(x-x_0)^n\right]$$

(ii) ($r_1 = r_2$)

$$y_2(x) = y_1(x) \cdot \ln|x-x_0| + |x-x_0|^{r_1} \cdot \sum_{n=1}^{\infty} b_n (x-x_0)^n,$$

where $b_n := a_n'(r_1), n \geqslant 1$.

(iii) ($r_1 - r_2 =: N \in \mathbb{N}$)

$$y_2(x) = a y_1(x) \ln|x-x_0| + |x-x_0|^{r_2} \cdot \left[1 + \sum_{n=1}^{\infty} c_n (x-x_0)^n\right],$$

where $a := \lim_{r\to r_2}(r-r_2)a_N(r)$ (possible zero) and

$$c_n := \frac{\mathrm{d}}{\mathrm{d}r}[(r-r_2)a_n(r)]|_{r=r_2} = \begin{cases} a_n(r_2) & a_n \text{ well-defined} \\ \text{something else} & \text{otherwise} \end{cases}.$$

In each case, each series converges absolutely for $|x-x_0| < \rho$, and y_1, y_2 define a fundamental set of solutions for $x \in (x_0 - \rho, x)$ and $x \in (x_0, x_0 + \rho)$.

Remark 5.7. *In practice, for cases (ii), (iii), it may be easier to manually find b_n, c_n rather than that the derivative of a recursive sequence.*

↪ Lecture 19; Last Updated: Thu Mar 28 14:50:27 EDT 2024

Remark 5.8. *Lecture cancelled this day because of a power outage or something.*

↪ Lecture 20; Last Updated: Thu Mar 28 14:31:57 EDT 2024

↪ Lecture 21; Last Updated: Thu Mar 28 14:39:31 EDT 2024

6 Laplace Transforms

6.1 Definitions

↪ Definition 6.1: Laplace Transform

Let $f : [0, \infty) \to \mathbb{R}$. The *Laplace transform* of f, denote $F(s)$ or $\mathsf{L}\{f(t)\}$, is defined by

$$\mathcal{L}\{f(T)\} := \int_0^\infty f(t)\,\mathrm{d}t\,.$$

↪ Definition 6.2: Piecewise Continuous

A function f is *piecewise continuous* (pw cont) for $t \in [\alpha, \beta]$ if $[\alpha, \beta]$ can be partitioned by a finite number of points

$$\alpha =: t_0 < t_1 < \cdots < t_n := \beta$$

such that

(i) f continuous on each (t_j, t_{j+1}),

(ii) for $t \in (t_j, t_{j+1})$, $\lim_{t \to t_j} f(t)$ and $\lim_{t \to t_{j+1}} f(t)$ both exist, are finite.

In particular, $\lim_{t \to t_j^+} f(t)$ does not necessarily have to equal $\lim_{t \to t_j^-} f(t)$.

We say f pw cont on $[\alpha, \infty)$ if pw cont on $[\alpha, \beta]$, $\forall \beta \in (\alpha, \infty)$.

↪ Definition 6.3: Exponential Order

We say a function $f(t)$ of *exponential order a* (only specifying a if relevant) if $\exists$ constants a, K, T such that

$$|f(t)| \leqslant Ke^{at}, \forall t \geqslant T.$$

↪ **Theorem 6.1**

Suppose $f(t)$ pw cont on $[0, \infty)$ and f has exponential order a. Then, $\mathcal{L}\{f(t)\}$ exists for $s > a$.

Proof. Remark that to show that $\lim_{\beta \to \infty} \int_0^\beta g(t) \, \mathrm{d}t$ exists, it suffices to show that $\lim_{\beta \to \infty} \int_0^\beta |g(t)| \, \mathrm{d}t$ exists and is finite.

We have that, for some $M > T$ in the definition of exponential order,

$$F(s) = \int_0^\infty e^{-st} f(t) \, \mathrm{d}t = \underbrace{\int_0^M e^{-st} f(t) \, \mathrm{d}t}_{\text{finite, since } f \text{ pw cont thus bounded}} + \int_M^\infty e^{-st} f(t) \, \mathrm{d}t$$

So, we need to show the RHS converges. Since $M > T$, we have that

$$\begin{aligned} \int_M^\infty e^{-st} |f(t)| \, \mathrm{d}t &\leqslant K \cdot \int_M^\infty e^{-st} e^{-at} \, \mathrm{d}t \\ &= K \int_M^\infty e^{(a-s)t} \, \mathrm{d}t \\ &= K \frac{e^{(a-s)M}}{s-a} < \infty, \end{aligned}$$

where the final line assumes that $s > a$. ■

⊛ **Example 6.1**

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} \, \mathrm{d}t = \left[\frac{e^{(a-s)t}}{a-s} \right]_0^\infty = \frac{1}{s-a},$$

valid for $s > a$. Remark that taking $a = 0$ gives us that $\mathcal{L}\{1\} = \frac{1}{s}$, again assuming that $s > 0$.

↪ **Proposition 6.1**

$\mathcal{L}\{\cdots\}$ linear.

Proof. Indeed, we have for $\alpha, \beta \in R$ and f, g pw cont functions,

$$\begin{aligned} \mathcal{L}\{\alpha f(t) + \beta g(t)\} &= \int_0^\infty e^{-st} [\alpha f(t) + \beta g(t)] \, \mathrm{d}t \\ &= \alpha \int_0^\infty e^{-st} f(t) \, \mathrm{d}T + \beta \int_0^\infty e^{-st} g(t) \, \mathrm{d}t \\ &= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\} \end{aligned}$$

■

Remark 6.1. *This gives, moreover, that* $\mathcal{L}\{K\} = K\mathcal{L}\{1\} = \frac{K}{s}$ *as before.*

⊛ Example 6.2

First, remark that e^{t^2} and $\tan t$ do not have Laplace transforms; the first is not of exponential order, and the second is unbounded at its discontinuities and thus not pw cont (indeed, it is also, as a result, not of exponential order).

Next, we compute some basic examples.

- $$\mathcal{L}\{t\} = \int_0^\infty te^{-st}\,\mathrm{d}t = \left[\frac{te^{-st}}{-s}\right]_0^\infty - \int_0^\infty e^{-st}{-s}\,\mathrm{d}t = \frac{1}{s}\int_0^\infty e^{-st}\,\mathrm{d}t = \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s^2}.$$

  Remark too that for any $\varepsilon > 0$, $t < e^{\varepsilon t}$ for sufficiently large t; we say t not only of exponential order, but of "exponential order 0".

- $$\begin{aligned} \mathcal{L}\{\cos(\omega t)\} &= \int_0^\infty e^{-st}\cos(\omega t)\,\mathrm{d}t = \left[\frac{1}{s}e^{-st}\cos(\omega t)\right]_0^\infty - \frac{\omega}{s}\int_0^\infty e^{-st}\sin(\omega t)\,\mathrm{d}t \\ &= \frac{1}{s} - \frac{\omega}{s}\left[\left[\sin(\omega t)\frac{e^{-st}}{-s}\right]_0^\infty + \frac{\omega}{s}\int_0^\infty e^{-st}\cos(\omega t)\,\mathrm{d}t\right] \\ &\implies \mathcal{L}\{\cos\omega t\} = \frac{1}{s} - \frac{\omega^2}{s^2}\mathcal{L}\{\cos(\omega t)\} \implies \mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}. \end{aligned}$$

  A similar computation gives $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$.

↪ Theorem 6.2: First Translation theorem

If $\mathcal{L}\{f(t)\} = F(s)$, $k \in \mathbb{R}$, then

$$\mathcal{L}\{e^{kt}f(t)\} = F(s-k).$$

Proof.

$$\mathcal{L}\{e^{kt}f(t)\} = \int_0^\infty e^{-st}e^{kt}f(t)\,\mathrm{d}t = \int_0^\infty e^{-(s-k)t}f(t)\,\mathrm{d}t = F(s-k).$$

■

Remark 6.2. *We often denote* $F(s-a) = \mathcal{L}\{f(t)\}_{s\to s-a}$

⊛ **Example 6.3**

$\mathcal{L}\{e^{at}\cos(\omega t)\} = \mathcal{L}\{\cos(\omega t)\}_{s\to s-a} = \frac{s}{s^2+\omega^2}|_{s\to s-a} = \frac{s-a}{(s-a)^2+\omega^2}$

6.2 Solving Constant Coefficient Linear ODE IVP's

↪ **Theorem 6.3**

Suppose $f, f', \ldots, f^{(n-1)}$ continuous on $[0,\infty)$ and $f^{(n)}$ pw cont on $[0,\infty)$ and all are of exponential order a. Then, $\mathcal{L}\{f^{(n)}(t)\}$ exists for $s > a$, and

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0).$$

↪ Lecture 22; Last Updated: Fri Mar 29 11:39:19 EDT 2024

Proof. For $n = 1$, suppose $f'(t)$ has discontinuities at $t_1, \ldots, t_{n-1}$ on $[0, A]$ for some $A > 0$; let $t_0 \coloneqq 0, t_n \coloneqq A$. Then

$$\begin{aligned}
\int_0^A e^{-st} f'(t)\,\mathrm{d}t &= \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} e^{-st} \cdot f'(t)\,\mathrm{d}t \\
\text{(integrate by parts)} \qquad &= \sum_{j=0}^{m-1} \left[\left[e^{-st} f(t)\right]_{t_j}^{t_{j+1}} + s \int_{t_j}^{t_{j+1}} e^{-st} f(t)\,\mathrm{d}t \right] \\
&= \sum_{j=0}^{m-1} \left[e^{-st} f(t)\right]_{t_j}^{t_{j+1}} + s \cdot \sum_{j=0}^{m-1} \int_{t_j}^{t_{j+1}} e^{-st} f(t)\,\mathrm{d}t \\
\text{(both terms telescope}^*\text{)} \qquad &= e^{-sA} f(A) - f(0) + s \int_0^A e^{-st} f(t)\,\mathrm{d}t
\end{aligned}$$

Remark in $*$, we use that f continuous on each (t_j, t_{j+1}), hence additivity applies.

Hence, for sufficiently large A, f being of exponential order gives us that

$$\left|e^{-sA} f(A)\right| \leqslant e^{-sA} \cdot Ke^{aA} = Ke^{-A(s-a)},$$

which $\to 0$ as $A \to \infty$, since $s > a$. Hence, taking $A \to \infty$, we find that the LHS of our original equation $\to \mathcal{L}\{f'(t)\}$, and thus $\mathcal{L}\{f'(t)\} \to f(0) + s \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = s\mathcal{L}\{f(t)\} - f(0)$ as $A \to \infty$. Hence, we have the desired form for $n = 1$.

For $n = 2$, we can simply use that $f''(t) = \frac{\mathrm{d}}{\mathrm{d}t}(f'(t))$, namely

$$\begin{aligned}
\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0) \qquad \text{(by } n = 1 \text{ case applied to } f'(t)) \\
&= s[s\mathcal{L}\{f(t)\} - f(0)] - f'(0) \qquad \text{(by } n = 1 \text{ case applied to } f(t)) \\
&= s^2\mathcal{L}\{f(t)\} - sf'(0) - f(0),
\end{aligned}$$

the desired form for $n = 2$; we explicitly computed these two cases as they are the ones we will encounter most frequently in application.

For the general case, we proceed by induction. We already proved the base case $n = 1$, so suppose the case for some $1, 2, \ldots,$ up to some $n \in \mathbb{N}$, ie $\mathcal{L}\{f^{(n)(t)}\} = s^n\mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)$. Then, we have that (under appropriate assumptions of exponential order, continuity, etc of $f^{(n+1)}$)

$$\begin{aligned}
\mathcal{L}\{f^{(n+1)}(t)\} &= s\mathcal{L}\{f^n(t)\} - f^n(0) \qquad \text{(by assumption, base case)} \\
&= s\left[s^n\mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0)\right] - f^{(n)}(0) \qquad \text{(by assumption, } n \text{ case)} \\
&= s^{n+1}\mathcal{L}\{f(t)\} - s\sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0) - f^{(n)}(0) \\
&= s^{n+1}\mathcal{L}\{f(t)\} - \sum_{k=0}^{(n+1)-1} s^{(n+1)-1k} f^{(k)}(0)
\end{aligned}$$

as desired; the inductive step is complete and thus the claim holds in general. ■

This theorem, combined with the linearity of $\mathcal{L}\{\ldots\}$, allows us to convert linear, constant coefficient ODES to algebraic expressions, encoding initial values into the problem directly. To see this, consider the nth order, constant coefficient, linear IVP

$$L[y] := \sum_{k=0}^{n} a_k y^{(k)}, \qquad y(0) = \alpha_1, y'(0) = \alpha_2, \cdots y^{(n-1)}(0) = \alpha_n,$$

where a_k constants with $a_n \neq 0$. We venture to solve $L[y] = f(t)$. Letting $F(s) := \mathcal{L}\{f(t)\}, Y(s) := \mathcal{L}\{y(t)\}$, then applying $\mathcal{L}\{\dots\}$ to both sides of our ODE, we find

$$\begin{aligned}
F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{L[y](t)\} &= \mathcal{L}\{\sum_{k=0}^{n} a_k y^{(k)}\} \\
&= \sum_{k=0}^{n} a_k \mathcal{L}\{y^{(k)}\} \qquad \text{(linearity)} \\
&= \sum_{k=0}^{n} a_n \left[s^k Y(s) - \sum_{j=0}^{k-1} s^{k-1-j} y^{(j)}(0) \right] \qquad \text{(by theorem 6.3)} \\
&= \underbrace{\left[\sum_{k=0}^{n} a_k s^k \right]}_{:=P(s)} Y(s) - \underbrace{\sum_{k=0}^{n} a_k \sum_{j=0}^{k-1} s^{k-1-j} y^{(j)}(0)}_{:=Q(s)} \\
&\implies F(s) = P(s)Y(s) + Q(s) \\
&\implies Y(s) = \frac{F(s)}{P(s)} + \frac{Q(s)}{P(s)}
\end{aligned}$$

Remark that $P(s)$ is a known (based on the ODE) polynomial in s of degree n, and moreover, is precisely the characteristic equation that we found when solving linear ODEs previously. $Q(s)$ on the other hand is a polynomial in s of degree $n-1$, defined by the ICs of the problem.

This gives a clear method to find $Y(s)$, that is, the Laplace transform of our solution; hence, we need to somehow invert this to find $y(t)$, ie $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Complex analysis gives us that the inverse Laplace is given by the Bronwich Integral formula

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s)e^{st} \, \mathrm{d}s \, .$$

We won't use this in practice, but rather make use of algebraic simplifications to bring our solution to a form recognizable as the Laplace transform of a (linear combination) of "elementary" functions. To do so, we first need the following proposition.

↪ **Proposition 6.2**

$\mathcal{L}^{-1}\{F(s)\}$ is linear.

Proof. Recall that $\mathcal{L}\{\dots\}$ linear, so

$$\begin{aligned}
&\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s) \\
&\qquad \implies \mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t) = \alpha \mathcal{L}^{-1}\{F(s)\} + \beta \cdot \mathcal{L}^{-1}\{G(s)\}.
\end{aligned}$$

■

⊛ Example 6.4: Computing $\mathcal{L}^{-1}\{\dots\}$

Consider $F(s) = \frac{2s+1}{s^2+4} = 2(\frac{s}{s^2+4}) + \frac{1}{2}(\frac{2}{s^2+4})$. Then, one can observe that

$$\begin{aligned}\mathcal{L}^{-1}\{F(s)\} &= 2\mathcal{L}^{-1}\{\frac{s}{s^2+4}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{s^2+4}\} \\ &= 2\cos(2t) + \frac{1}{2}\sin(2t).\end{aligned}$$

In essence, computing inverse Laplace is an exercise in algebraic manipulation and purposeful staring.

⊛ Example 6.5: Solving Second Order Linear ODE

We consider

$$y'' - 3y' + 2y = e^{-4t}, \qquad y(0) = 1, y'(0) = 5.$$

Taking $\mathcal{L}\{\dots\}$ of both sides:

$$\begin{aligned}&\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{e^{-4t}\} \\ &\qquad\implies [s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4} \\ &\qquad\implies (s^2 - 3s + 2)Y(s) - s - 5 + 3 = \frac{1}{s+4} \\ &\qquad\implies Y(s) = \frac{1}{s^2 - 3 + 2}[\frac{1}{s+4} + s + 2] \\ &\qquad\implies Y(s) = \frac{1}{(s-1)(s-2)(s+4)} + \frac{s+2}{(s-1)(s-2)}\end{aligned}$$

After applying "classical partial fractions theory", one finds

$$\begin{aligned}y(t) = \mathcal{L}^{-1}\{Y(t)\} &= -\frac{16}{5}\mathcal{L}^{-1}\{\frac{1}{s-1}\} + \frac{25}{6}\mathcal{L}^{-1}\{\frac{1}{s-2}\} + \frac{1}{30}\mathcal{L}^{-1}\{\frac{1}{s+4}\} \\ &= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}.\end{aligned}$$

Remark 6.3. *Many questions such as this end up with some kind of partial fractions to work out; as such, don't bother simplifying excessively to find a common denominator or anything like that.*

Remark 6.4. *We already know how to solve these problems; but one particular advantage of this method is the encoding of the ICs. In the typical characteristic method technique, we needed to differentiate our entire solution in order ot set the appropriate constants. Here, we never differentiated (explicitly).*

⊛ Example 6.6: First Order

Consider $y' + y = \sin t, y(0) = 1$. Taking the Laplace of both sides, we find

$$\begin{aligned} sY(s) - y(0) + Y(s) &= \frac{1}{s^2+1} \\ \implies Y(s) &= \frac{1}{s+1} + \frac{1}{(s^2+1)(s+1)}, \end{aligned}$$

and after partial fractioning,

$$\begin{aligned} Y(s) &= \frac{1}{s+1} + \frac{1/2}{s+1} - \frac{1}{2}\left(\frac{s-1}{s^2+1}\right) = \frac{3/2}{s+1} - \frac{1}{2}\left(\frac{s}{s^2+1}\right) + \frac{1}{2}\left(\frac{1}{s^2+1}\right) \\ \implies y(t) &= \frac{3}{2}\mathcal{L}^{-1}\{\frac{1}{s+1}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{s}{s^2+1}\} + \frac{1}{2}\mathcal{L}^{-1}\{\frac{1}{s^2+1}\} \\ \implies y(t) &= \frac{3}{2}e^{-t} - \frac{1}{2}\cos t + \frac{1}{2}\sin t \end{aligned}$$

6.3 Discontinuous Functions

↪ **Definition 6.4: Unit Step Function**

The function given by

$$\mathcal{U}(t-a) := \begin{cases} 0 & t < a \\ 1 & t \geqslant a \end{cases}$$

↪ **Theorem 6.4: Second Translation Theorem**

If $F(s) = \mathcal{L}\{f(t)\}$, then for $a > 0$,

$$\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} = e^{-as}F(s).$$

Proof.

$$\begin{aligned} \mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} &= \int_0^a e^{-st}\underbrace{\mathcal{U}(t-a)}_{=0} f(t-a)\,\mathrm{d}t + \int_a^\infty e^{-st}\overbrace{\mathcal{U}(t-a)}^{=1} f(t-a)\,\mathrm{d}t \\ &= \int_a^\infty e^{-st} f(t-a)\,\mathrm{d}t \qquad (w := t-a) \\ &= \int_0^\infty e^{-(a+w)s} f(w)\,\mathrm{d}w \\ &= e^{-as}\int_0^\infty e^{-ws} f(w)\,\mathrm{d}w = e^{-as}F(s) \end{aligned}$$

■

↪ **Corollary 6.1**

$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$.

Proof. $\mathcal{L}\{\mathcal{U}(t-a)\cdot 1\} \overset{*}{=} e^{-as}\mathcal{L}\{1\} = \frac{e^{-as}}{s}$, where we use the previous theorem at $*$. ■

↪ *Lecture 23; Last Updated: Tue Apr 2 12:16:34 EDT 2024*

⊛ **Example 6.7**

$y' + y = f(t) := \begin{cases} 0 & 0 \leqslant t < \pi \\ 3\cos t & t \geqslant \pi \end{cases}$, $y(0) = 2$. We can rewrite $f(t) = \mathcal{U}(t-\pi)g(t-\pi) = 3\mathcal{U}(t-\pi)\cos(t)$, remarking that $g(t) = 3\cos(t+\pi) = -3\cos(t)$, and so using the translation theorem we have

$$\begin{aligned} sY(s) - y(0) + Y(s) = (s+1)Y(s) - 2 = 3\mathcal{L}\{\mathcal{U}(t-\pi)\cos(t)\} &= -3\frac{s}{s^2+1}e^{-\pi s} \\ \implies Y(s) = \frac{2}{s+1} - 3e^{-\pi s}&\frac{s}{(s^2+1)(s+1)}. \end{aligned}$$

Now, we proceed as normal, ignore the exponential for now. We find that

$$\frac{s}{(s^2+1)(s+1)} = \frac{-1/2}{s+1} + \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1},$$

and so, applying the translation theorem in reverse,

$$\begin{aligned} y(t) &= 2e^{-t} - 3\mathcal{L}^{-1}\{e^{-\pi s}\left[\frac{-1/2}{s+1} + \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1}\right]\} \\ &= 2e^{-t} + \frac{3}{2}\mathcal{U}(t-\pi)\left[e^{-(t-\pi)} + \cos(t) + \sin(t)\right]. \end{aligned}$$

Remark that, as the ODE was discontinuous at $t = \pi$ with a jump of $|\lim_{t\to\pi^+} f(t) - \lim_{t\to\pi^-} f(t)| = 3$; we can show (1) $y(t)$ is continuous and (2) $y'(t)$ is discontinuous at precisely $t = \pi$ with the same jump; this occurs generally.

6.4 Derivatives of Transforms

↪ **Proposition 6.3**

$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}\{f(t)\}$.

Proof. Follows from easy induction. ■

⊛ **Example 6.8**

Show that the Laplace transform of the Euler equation $at^2y'' + bty' + cy = 0$ is itself an Euler equation.

6.5 Transforms of Integrals

↪ **Definition 6.5: Convolution**

$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau$.

⊛ **Example 6.9**

$$\begin{aligned} e^t * \sin t &= \int_0^t e^\tau \sin(t-\tau)\,\mathrm{d}\tau \\ &= \cdots - \sin t + e^t - \cos t - e^t * \sin t \\ \implies e^t * \sin t &= \frac{1}{2}[e^t - \sin t - \cos t]. \end{aligned}$$

↪ **Theorem 6.5: Convolution Theorem**

If f, g pw-cont on $[0, \infty)$ and are of exponential order, then

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s).$$

Proof. We should but won't show that the Laplace of f, g existing implies that the Laplace of their convolution exists, but won't.

$$\begin{aligned} \mathcal{L}\{f * g\} &= \int_0^\infty \int_0^t f(\tau)g(t-\tau)e^{-st}\,\mathrm{d}t \\ &= \int_0^\infty \int_\tau^\infty f(\tau)g(t-\tau)e^{-st}\,\mathrm{d}t\,\mathrm{d}\tau \\ &= \int_0^\infty f(\tau)e^{-s\tau} \int_\tau^\infty g(t-\tau)e^{-s(t-\tau)}\,\mathrm{d}t\,\mathrm{d}\tau \\ (w := t-\tau) \qquad &= \int_0^\infty f(\tau)e^{-s\tau}\,\mathrm{d}\tau \int_0^\infty g(w)e^{-sw}\,\mathrm{d}w \\ &= \mathcal{L}\{f\}\mathcal{L}\{g\} \end{aligned}$$

■

↪ **Corollary 6.2**

$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g$.

↪ **Proposition 6.4**

For f, g, h functions and α, β scalars,

(i) $(f * g)(t) = (g * f)(t)$

(ii) $((\alpha f * \beta g) * h)(t) = \alpha(f * h)(t) + \beta(g * h)(t)$

(iii) $0 * g = 0$

(iv) $(\mathrm{Id} * g)(t) \neq g(t)$

⊛ **Example 6.10**

Show that $\mathcal{L}\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$.

⊛ **Example 6.11**

Show that $\mathcal{L}^{-1}\{\frac{s}{(s^2+1)(s+3)}\}$ without using partial fractions.

↪ *Lecture 24; Last Updated: Tue Apr 2 13:10:02 EDT 2024*

6.6 Dirac Delta Function

↪ **Definition 6.6: Dirac Delta**

Denote $\delta(t - t_0) := \begin{cases} 0 & t \neq t_0 \\ \text{unbounded} & t = t_0 \end{cases}$, in such a way that for any $\varepsilon > 0$, $\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0) f(t) \, \mathrm{d}t = \int_{-\infty}^{\infty} f(t)\delta(t - t_0) \, \mathrm{d}t = f(t_0)$. Ie, $\delta(t - t_0)$ "picks out" the function's value at t_0.

In particular, letting $f(t) \equiv 1$, we see that

$$\int_0^t \delta(s - t_0) \, \mathrm{d}s = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}.$$

Remark 6.5. *This is not a very rigorous definition. Sorry.*

↪ **Theorem 6.6**

For $t_0 > 0$, $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$.

Proof. $\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,\mathrm{d}t = e^{-st_0}$. ■

↪ **Corollary 6.3**

$\mathcal{L}\{\delta(t)\} = 1$

6.7 Convolutions, Green's Function

Recall that we can write $L[y] = \sum_{k=0}^n a_k y^{(k)}(t) = f(t)$ (with IVPs) as $P(s)Y(s) - Q(s) = F(s)$, where $P(s) = \sum_{k=0}^n a_k s^k$, $Y(s) = \mathcal{L}\{y(t)\}$, $F(s) = \mathcal{L}\{f(t)\}$, and $Q(s)$ of degree $n-1$ and dependent on the ICs. Letting $G(s) := \frac{1}{P(s)}$, then, we can rewrite this as

$$y(t) = \mathcal{L}^{-1}\{F(s)G(s)\} + \mathcal{L}^{-1}\{\frac{Q(s)}{P(s)}\}.$$

$\deg(Q) < \deg(P)$ so we can find the RHS of this using typical partial fractions techniques, and we can solve the LHS using the convolution theorem, namely $\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$.

↪ **Definition 6.7: Green's function**

The function $g(t)$ that solves $L[g(t)] = \delta(t)$ with IC $g(0) = g'(0) = \cdots = g^{(n-1)}(0)$ is called the *Green's function* of L.

↪ **Theorem 6.7**

Let $g(t)$ be the Green's function of L. Then, $L[g(t)] = G(s) = \frac{1}{P(s)}$.

Proof. $L[g] = \delta(t) \implies P(s)G(s) - Q(s) = 1 \implies P(s)G(s) = 1$. ■

⊛ **Example 6.12**

Find an expression for $y(t)$ with respect to a convolution integral and $Q(s)/P(s)$ for the ODE

$$y'' + \omega^2 y = f(t),$$

for arbitrary $y(0) = \alpha_0$, $y'(0) = \alpha_1$, and then when $\alpha_0 = \alpha_1 = 0$.

6.8 Transforms of Periodic Functions

↪ **Definition 6.8: Periodic function**

We say a function $f(t)$ is periodic of period T if $f(t) = f(t+T)$ for some minimal $T > 0$ for all $t > 0$.

Remark 6.6. *This definition excludes the constant function as a periodic (why?).*

↪ **Theorem 6.8**

Let f-periodic of period T and pw-cont on $[0, \infty)$. Then,

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) \,\mathrm{d}t\,.$$

Proof. Straightforward computation (hint: split up the integral in $\mathcal{L}\{f(t)\}$ into two integrals with T as the upper, lower limits resp.). ■

⊛ **Example 6.13**

Find the Laplace transform of $f(t) := \sum_{n=0}^{\infty}(-1)^n \mathcal{U}(t-n)$ (remarking that f periodic with $T = 2$) using the previous theorem. Then find it using the linearity of f.

⊛ **Example 6.14: Cursed**

We consider $y'' + y' + y = f(t) = \delta(t-1) + \mathcal{U}(t-2)e^{-(t-2)}$ with $y(0) = 0$, $y'(0) = 1$. Taking the Laplace of both sides

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) + Y(s) &= e^{-s} + e^{-2s}\mathcal{L}\{e^{-t}\} \\ \implies Y(s)(s^2+s+1) - 1 &= e^{-s} + e^{-2s}\left(\frac{1}{s+1}\right) \\ \implies Y(s) &= \frac{1}{s^2+s+1} + e^{-s}\frac{1}{s^2+s+1} + e^{-2s}\frac{1}{(s^2+s+1)(s+1)} \end{aligned}$$

Unlike other examples, $s^2 + s + 1$ not reducible (over $\mathbb{R}$) so we have some difficulties. Completing the square, we find $s^2 + s + 1 = (s + \frac{1}{2})^2 + \frac{3}{4}$, and so

$$\frac{1}{s^2+s+1} = \frac{1}{(s+\frac{1}{2})^2 + \frac{3}{4}}, \qquad \frac{1}{(s^2+s+1)(s+1)} = \frac{1}{((s+\frac{1}{2})^2+\frac{3}{4})(s+1)}.$$

Using partial fractions on the second expression,

$$\begin{aligned}
\frac{1}{(s^2+s+1)(s+1)} &= \frac{As+B}{s^2+s+1} + \frac{C}{s+1} \\
\implies 1 &= (As+B)(s+1) + C(s^2+s+1) \\
s=-1] &\quad 1 = C \\
s^2] &\quad 0 = A + C \implies A = -1 \\
s^0] &\quad 1 = B + C \implies B = 0
\end{aligned}$$

Bring all the "simplifications" together, we have

$$Y(s) = \frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}} + e^{-s}[\frac{1}{(s+\frac{1}{2})^2+\frac{3}{4}}] + e^{-2s}[\frac{-s}{(s+\frac{1}{2})^2+\frac{3}{4}} + \frac{1}{s+1}]$$

For the first term, we need to use the first translation theorem, and for the other two we need to use both the first and second theorems.

$$\begin{aligned}
\frac{1}{(s+1/2)^2+3/4} = \frac{2}{\sqrt{3}}(\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}) &\overset{\mathcal{L}^{-1}\{\dots\}}{\rightsquigarrow} \frac{2}{\sqrt{3}} e^{-1/2t} \sin\left(\frac{\sqrt{3}}{2}t\right) \\
e^{-s}[\frac{1}{(s+1/2)^2+3/4}] &\overset{\mathcal{L}^{-1}\{\dots\}}{\rightsquigarrow} \mathcal{U}(t-1)\mathcal{L}^{-1}\{\frac{2}{\sqrt{3}}(\frac{\sqrt{3}/2}{(s+1/2)^2+3/4})\}_{t\mapsto t-1} \\
&\overset{\mathcal{L}^{-1}\{\dots\}}{\rightsquigarrow} \mathcal{U}(t-1)\frac{2}{\sqrt{3}} e^{-1/2(t-1)} \sin\left(\frac{\sqrt{3}}{2}(t-1)\right)
\end{aligned}$$

↪ *Lecture 25; Last Updated: Sun Apr 7 09:28:47 EDT 2024*

7 List of Theorems