MATH574 - Dynamical Systems

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Contents

§1 Examples of Dynamical Systems

Roughly speaking, a dynamical system is a system that evolves in time, with common examples being a differential equation, in the continuous case, or a map, in the discrete case.

⊛ **Example 1.1** (The Logistic Map):

§2 Existence-Uniqueness Theory

↪**Definition 2.1** (Lipschitz): We say a function $f : \mathbb{R}^p \to \mathbb{R}^p$ is Lipschitz on $B \subseteq \mathbb{R}^p$ if there is a constant $L > 0$ such that $\|f(x) - f(y)\| \leq L \|x - y\|$ for every $x, y \in B$. We call L a "Lipschitz" constant. It is certainly not unique in general.

We say f *globally Lipschitz* if it is Lipschitz on $B = \mathbb{R}^p$, and f *locally Lipschitz* if f Lipschitz on every bounded domain $B \subseteq \mathbb{R}^p$ (note: the L will in general depend on the domain).

↪**Theorem 2.1**: Let $f : \mathbb{R}^p \to \mathbb{R}^p$ be a locally Lipschitz function. Then, there exists a unique solution to the initial value problem $\dot{u} = f(u), u(0) = u_0$ on some interval $t \in (-T_1(u_0), T_2(0))$, where $-T_1(u_0) < 0 < T_2(u_0)$ and

- either $T_2(u_0) = +\infty$ or $\|u(t)\| \to \infty$ as $t \to T_2(u_0)$, and
- either $T_1(u_0) = \infty$ or $\|u(t)\| \to -\infty$ as $t \to -T_1(u_0)$.

Heuristically, this first condition states that either our solution exists for all (forward) time after $-T_1(u_0)$, or it blows up in finite time, with a similar interpretation for the second, going backwards.

↪**Proposition 2.1**: Let $\dot{u} = f(u)$ where f locally Lipschitz. Let $B \subseteq \mathbb{R}^p$ be a bounded subset such that initial conditions $u_0, v_0 \in B$ define solutions $u(t), v(t)$ with $u(t), v(t) \in B$ for all $t \in [0, T]$. Let L be a Lipschitz constant for f on B. Then,

$$e^{-Lt} \|u_0 - v_0\| \leq \|u(t) - v(t)\| \leq e^{Lt} \|u_0 - v_0\| \qquad \forall\, t \in [0, T].$$

This provides a bound on how quickly solutions grow, decay in B.

↪**Corollary 2.1**: Let f locally Lipschitz and $u_0 \neq v_0$. Then, $u(t) \neq v(t)$ for all time such that the solutions both exist.

§3 Limit Sets and the Evolution Operator

We state definitions in this section first for ODEs, but they generalize.

↪**Definition 3.1** (Evolution Operator): Given $\dot{u} = f(u)$, the *evolution operator* is the map

$$S(t) : \mathbb{R}^p \to \mathbb{R}^p, \qquad t \geq 0$$

such that $u(t) = S(t)u_0$.

Such a map also defines a *semi-group* $\{S(t) : t \geq 0\}$ under composition, namely it is closed under repeated composition and this operator is associative.

For $B \subseteq \mathbb{R}^p$ define

$$S(t)B := \bigcup_{u \in B} S(t)u = \{u(t) = S(t)u_0 : u_0 \in B\}.$$

↪**Definition 3.2** (Forward/Positive Orbit): We define the *forward orbit* of a point u_0 as

$$\Gamma^+(u_0) := \bigcup_{t \geq 0} S(t)u_0,$$

i.e. the set of all points u_0 may "visit" as time increases.

↪**Definition 3.3** (Backwards/Negative Orbit): Similarly, define a *backwards orbit* (if one exists)

$$\Gamma^-(u_0) := \{u(t) : t \leq 0\},$$

s.t. $\forall\, t \leq s \leq 0, S(-t)u(t) = u_0$ and $S(s-t)u(t) = u(s)$.

Note that a negative orbit won't be unique in general, eg in maps, periodic points may multiple preimages.

↪**Definition 3.4** (Complete Orbit): If a negative orbit for u_0 exists, define the *complete orbit* through u_0 as

$$\Gamma(u_0) := \Gamma^+(u_0) \cup \Gamma^-(u_0).$$

Notice that if $v \in \Gamma(u_0)$, then $\Gamma(v) = \Gamma(u_0)$; namely a complete orbit through v exists.

↪**Definition 3.5** (Invariance): The set B is said to be *positively invariant* if $S(t)B \subseteq B$ for all $t \geq 0$. Similarly, B is said to be *negatively invariant* if $B \subseteq S(t)B$ for all $t \geq 0$.