MATH378 - Nonlinear Optimization

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§1 Preliminaries

§1.1 Terminology

We consider problems of the form

$$\text{minimize } f(x) \text{ subject to } x \in X, \qquad (\dagger)$$

with $X \subset \mathbb{R}^n$ the *feasible region* with x a *feasible point*, and $f : X \to \mathbb{R}$ the *objective (function)*; more concisely we simply write

$$\min_{x \in X} f(x).$$

When $X = \mathbb{R}^n$, we say the problem $(\dagger)$ is *unconstrained*, and conversely *constrained* when $X \subsetneq \mathbb{R}^n$.

⊛ **Example 1.1** (Polynomial Fit): Given $y_1, ..., y_m \in \mathbb{R}$ measurements taken at m distinct points $x_1, ..., x_m \in \mathbb{R}$, the goal is to find a degree $\leq n$ polynomial $q : \mathbb{R} \to \mathbb{R}$, of the form

$$q(x) = \sum_{k=0}^{n} \beta_k x^k,$$

"fitting" the data $\{(x_i, y_i)\}_i$, in the sense that $q(x_i) \approx y_i$ for each i. In the form of $(\dagger)$, we can write this precisely as

$$\min_{\beta \in \mathbb{R}^{n+1}} \frac{1}{2} \sum_{i=0}^{n} \left(\underbrace{\beta_n x_i^n + \cdots + \beta_1 x_i + \beta_0}_{q(x_i)} - y_i \right)^2;$$

namely, we seek to minimize the ℓ^2-distance between $(q(x_i))$ and (y_i). If we write

$$X := \begin{pmatrix} 1 & x_1 & \dots & x_1^n \\ \vdots & \dots & \dots & \vdots \\ 1 & x_m & \dots & x_m^n \end{pmatrix} \in \mathbb{R}^{m \times (n+1)}, \qquad y := \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{R}^m,$$

then concisely this problem is equivalent to

$$\min_{\beta \in \mathbb{R}^{n+1}} \frac{1}{2} \| X \cdot \beta - y \|_2^2,$$

a so-called *least-squares problem.*

We have two related tasks:

1. Find the optimal value asked for by $(\dagger)$, that is what $\inf_X f$ is;
2. Find a specific point $\overline{x}$ such that $f(\overline{x}) = \inf_X f$, i.e. the value of a point

$$\overline{x} \in \operatorname{argmin}_X f := \left\{ x \in X \mid f(x) = \inf_X f \right\}.$$

(noting that argmin should be viewed as a set-valued function, as there may be multiple admissible minimizers) Notice that if we can accomplish 2., we've accomplished 1. by computing $f(\overline{x})$.

Note that $\overline{x} \in \operatorname{argmin}_X f \Rightarrow f(\overline{x}) = \inf_X f$, *but* $\inf_X f \in \mathbb{R}$ does *not* necessarily imply $\operatorname{argmin}_X f \neq \emptyset$, that is, there needn't be a feasible minimum; for instance $\inf_{x \in \mathbb{R}} e^x = 0$, but $\operatorname{argmin}_{\mathbb{R}} f = \emptyset$ (there is no x for which $e^x = 0$).

↪**Definition 1.1** (Minimizers): Let $X \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$. Then $\overline{x} \in X$ is called a

- *global minimizer (of f over X)* if $f(\overline{x}) \leq f(x) \forall x \in X$, or equivalently if $\overline{x} \in \operatorname{argmin}_X f$;
- *local minimizer (of f over X)* if $f(\overline{x}) \leq f(x) \forall x \in X \cap B_\varepsilon(\overline{x})$ for some $\varepsilon > 0$.

In addition, we have *strict* versions of each by replacing "$\leq$" with "$<$".

↪**Definition 1.2** (Some Geometric Tools): Let $f : \mathbb{R}^n \to \mathbb{R}$.

- $\operatorname{gph} f := \{(x, f(x)) \mid x \in \mathbb{R}^n\} \subseteq \mathbb{R}^n \times \mathbb{R}$
- $f^{-1}(\{c\}) := \{x \mid f(x) = c\} \equiv$ *contour/level set at c*
- $\operatorname{lev}_c f := f^{-1}((-\infty, c]) = \{x \mid f(x) \leq c\} \equiv$ *lower level/sublevel set at c*

Remark 1.1:

- $\operatorname{lev}_{\inf f} f = \operatorname{argmin} f$
- assume f continuous; then all (sub)level sets are closed (possibly empty)

We recall the following result from calculus/analysis:

↪**Theorem 1.1** (Weierstrass): Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and $X \subset \mathbb{R}^n$ compact. Then, $\operatorname{argmin}_X f \neq \emptyset$.

From, we immediately have the following:

↪**Proposition 1.1**: Let $f : \mathbb{R}^n \to \mathbb{R}$ continuous. If there exists a $c \in \mathbb{R}$ such that $\operatorname{lev}_c f$ is nonempty and bounded, then $\operatorname{argmin}_{\mathbb{R}^n} f \neq \emptyset$.

Proof. Since f continuous, $\operatorname{lev}_c f$ is closed (being the inverse image of a closed set), thus $\operatorname{lev}_c f$ is compact (and in particular nonempty). By Weierstrass, f takes a minimimum over $\operatorname{lev}_c f$, namely there is $\overline{x} \in \operatorname{lev}_c f$ with $f(\overline{x}) \leq f(x) \leq c$ for each $x \in \operatorname{lev}_c f$. Also, $f(x) > c$ for each $x \notin \operatorname{lev}_c f$ (by virtue of being a level set), and thus $f(\overline{x}) \leq f(x)$ for each $x \in \mathbb{R}^n$. Thus, $\overline{x}$ is a global minimizer and so the theorem follows. ■

§1.2 Convex Sets and Functions

↪**Definition 1.3** (Convex Sets): $C \subset \mathbb{R}^n$ is *convex* if for any $x, y \in C$ and $\lambda \in (0,1)$, $\lambda x + (1-\lambda)y \in C$; that is, the entire line between x and y remains in C.

↪**Definition 1.4** (Convex Fucntions): Let $C \subset \mathbb{R}^n$ be convex. Then, $f : C \to \mathbb{R}$ is called

1. *convex (on C)* if
$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$
for every $x, y \in C$ and $\lambda \in (0,1)$;
2. *strictly convex (on C)* if the inequality $\leq$ is replaced with $<$;
3. *strongly convex (on C)* if there exists a $\mu > 0$ such that
$$f(\lambda x + (1-\lambda)y) + \mu\lambda(1-\lambda)\|x-y\|^2 \leq \lambda f(x) + (1-\lambda)f(y),$$
for every $x, y \in C$ and $\lambda \in (0,1)$; we call μ the *modulus of strong convexity.*

Remark 1.2: 3. ⇒ 2. ⇒ 1.

Remark 1.3: A function is convex iff its epigraph is a convex set.

⊛ **Example 1.2**: $\exp : \mathbb{R} \to \mathbb{R}$, $\log : (0,\infty) \to \mathbb{R}$ are convex. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ of the form $f(x) = Ax - b$ for $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ is called *affine linear.* For $m = 1$, every affine linear function is convex. All norms on $\mathbb{R}^n$ are convex.

↪**Proposition 1.2**:

1. *(Positive combinations)* Let f_i be convex on $\mathbb{R}^n$ and $\lambda_i > 0$ scalars for $i = 1, ..., m$, then $\sum_{i=1}^m \lambda_i f_i$ is convex; as long as one is strictly (resp. strongly) convex, the sum is strictly (resp. strongly) convex as well.
2. *(Composition with affine mappings)* Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and $G : \mathbb{R}^m \to \mathbb{R}^n$ be affine. Then, $f \circ G$ is convex on $\mathbb{R}^m$.

§2 Unconstrained Optimization

§2.1 Theoretical Foundations

We focus on the problem
$$\min_{x\in\mathbb{R}^n} f(x),$$
where $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable.

↪**Definition 2.1** (Directional derivative): Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$. We say f *directionally differentiable* at $\overline{x} \in D$ in the direction $d \in \mathbb{R}^n$ if
$$\lim_{t\to 0^+} \frac{f(\overline{x} + td) - f(\overline{x})}{t}$$
exists, in which case we denote the limit by $f'(\overline{x}; d)$.

↪**Lemma 2.1**: Let $D \subset \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$ differentiable at $x \in D$. Then, f is directionally differentiable at x in every direction d, with
$$f'(x; d) = \nabla f(x)^T d = \langle \nabla f(x), d\rangle.$$

⊛ **Example 2.1** (Directional derivatives of the Euclidean norm): Let $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \|x\|$ the usual Euclidean norm. Then, we claim

$$f'(x; d) = \begin{cases} \frac{x^T d}{\|x\|} & x \neq 0 \\ \|d\| & x = 0 \end{cases}.$$

For $x \neq 0$, this follows from the previous lemma and the calculation $\nabla f(x) = \frac{x}{\|x\|}$. For $x = 0$, we look at the limit

$$\lim_{t \to 0^+} \frac{f(0 + td) - f(0)}{t} = \lim_{t \to 0^+} \frac{t\|d\| - 0}{t} = \|d\|,$$

using homogeneity of the norm.

↪**Lemma 2.2** (Basic Optimality Condition): Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}$. If $\overline{x}$ is a *local minimizer* of f over X and f is directionally differentiable at $\overline{x}$, then $f'(\overline{x}; d) \geq 0$ for all $d \in \mathbb{R}^n$.

Proof. Assume otherwise, that there is a direciton $d \in \mathbb{R}^n$ for which the $f'(\overline{x}; d) < 0$, i.e.

$$\lim_{t \to 0^+} \frac{f(\overline{x} + td) - f(\overline{x})}{t} < 0.$$

Then, for all sufficiently small $t > 0$, we must have

$$f(\overline{x} + td) < f(\overline{x}).$$

Moreover, since X open, then for t even smaller (if necessary), $\overline{x} + td$ remains in X, thus $\overline{x}$ cannot be a local minimizer. ■

↪**Theorem 2.1** (Fermat's Rule): In addition to the assumptions of the previous lemma, assume further that f is differentiable at $\overline{x}$. Then, $\nabla f(\overline{x}) = 0$.

Proof. From the previous, we know $0 \leq f'(\overline{x}; d)$ for any d. Take $d = -\nabla f(\overline{x})$, then using the representation of a directional derivative for a differentiable function, and the fact that norms are nonnegative,

$$0 \leq -\|\nabla f(\overline{x})\|^2 \leq 0,$$

which can only hold if $\|\nabla f(\overline{x})\| = 0$ hence $\nabla f(\overline{x}) = 0$ ■

We recall the following from Calculus:

↪**Theorem 2.2** (Taylor's, Second Order): Let $f : D \to \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable, then for each $x, y \in D$, there is an η lying on the line between x and y such that

$$f(y) = f(x) + \nabla f(x)^T (y - x) + \frac{1}{2}(y - x)^T \nabla^2 f(\eta)(y - x).$$

↪**Theorem 2.3** (2nd-order Optimality Conitions): Let $X \subseteq \mathbb{R}^n$ open and $f : X \to \mathbb{R}$ twice continuously differentiable. Then, if x a local minimizer of f over X, then the Hessian matrix $\nabla^2 f(x)$ is positive semi-definite.

Proof. Suppose not, then there exists a d such that $d^T \nabla^2 f(x) d < 0$. By Taylor's, for every $t > 0$, there is an η_t on the line between x and $x + td$ such that

$$\begin{aligned} f(x + td) &= f(x) + t\underbrace{\nabla f(x)^T}_{=0} d + \frac{1}{2} t^2 d^T \nabla^2 f(\eta_t) d \\ &= f(x) + \frac{t^2}{d^T} \nabla^2 f(\eta_t) d. \end{aligned}$$

As $t \to 0^+$, $\nabla^2 f(\eta_t) \to \nabla^2 f(x) < 0$. By continuity, for t sufficiently small, $\frac{t^2}{2} d^T \nabla^2 f(\eta_t) d < 0$ for t sufficiently small, whence we find

$$f(x + td) < f(x),$$

for sufficiently small t, a contradiction. ■

↪**Lemma 2.3**: Let $X \subset \mathbb{R}^n$ open, $f : X \to \mathbb{R}$ in C^2. If $\overline{x} \in \mathbb{R}^n$ is such that $\nabla^2 f(\overline{x}) > 0$ (i.e. is positive definite), then there exists $\varepsilon, \mu > 0$ such that $B_\varepsilon(\overline{x}) \subset X$ and

$$d^T \nabla^2 f(x) d \geq \mu \|d\|^2, \qquad \forall d \in \mathbb{R}^n, x \in B_\varepsilon(\overline{x}).$$

Combining this and Taylor's Theorem, we can deduce the following (our first "sufficient" result of this section):

↪**Theorem 2.4** (Sufficient Optimality Condition): Let $X \subset \mathbb{R}^n$ open and $f \in C^2(X)$. Let $\overline{x}$ be a stationary point of f such that $\nabla^2 f(\overline{x}) > 0$. Then, $\overline{x}$ is a *strict* local minimizer of f.

2.1.1 Quadratic Approximation

Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2 and $\overline{x} \in \mathbb{R}^n$. By Taylor's, we can approximate

$$f(y) \approx g(y) \coloneqq f(\overline{x}) + \nabla f(\overline{x})^T(y - \overline{x}) + \frac{1}{2}(y - \overline{x})^T \nabla^2 f(\overline{x})(y - \overline{x}).$$

⊛ **Example 2.2** (Quadratic Functions): For $Q \in \mathbb{R}^{n\times n}$ symmetric, $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, let

$$f : \mathbb{R}^n \to \mathbb{R}, \qquad f(x) = \frac{1}{2}x^T Q x + c^T x + \gamma,$$

a typical quadratic function. Then,

$$\nabla f(x) = \frac{1}{2}(Q + Q^T)x + c = Qx + c, \qquad \nabla^2 f(x) = Q.$$

We find that f has *no* minimizer if $c \notin \operatorname{rge}(Q)$ or Q is not positive semi-definite, combining our previous two results. In turn, if Q is positive definite (and thus invertible), there is a unique local minimizer $\overline{x} = -Q^{-1}c$ (*and* global minimizer, as we'll see).

§2.2 Differentiable Convex Functions

↪**Theorem 2.5**: Let $C \subset \mathbb{R}^n$ be open and convex and $f : C \to \mathbb{R}$ differentiable on C. Then:

1. f is convex (on C) iff
$$f(x) \geq f(\overline{x}) + \nabla f(\overline{x})^T(x - \overline{x}) \qquad \star_1$$
for every $x, \overline{x} \in C$;
2. f is *strictly* convex iff same inequality as 1. with strict inequality;
3. f is *strongly* convex with modulus $\sigma > 0$ iff
$$f(x) \geq f(\overline{x}) + \nabla f(\overline{x})^T(x - \overline{x}) + \frac{\sigma}{2}\|x - \overline{x}\|^2 \qquad \star_2$$
for every $x, \overline{x} \in C$.

Proof. (1., $\Rightarrow$) Let $x, \overline{x} \in C$ and $\lambda \in (0, 1)$. Then,

$$f(\lambda x + (1 - \lambda)\overline{x}) - f(\overline{x}) \leq \lambda(f(x) - f(\overline{x})),$$

which implies

$$\frac{f(\overline{x} + \lambda(x - \overline{x})) - f(\overline{x})}{\lambda} \leq f(x) - f(\overline{x}).$$

Letting $\lambda \to 0^+$, the LHS $\to$ the directional derivative of f at $\overline{x}$ in the direction $x - \overline{x}$, which is equal to, by differentiability of f, $\nabla f(\overline{x})^T(x - \overline{x})$, thus the result.

(1., $\Leftarrow$) Let $x_1, x_2 \in C$ and $\lambda \in (0, 1)$. Let $\overline{x} \coloneqq \lambda x_1 + (1 - \lambda)x_2$. $\star_1$ implies

$$f(x_i) \geq f(\overline{x}) + \nabla f(\overline{x})^T(x_i - \overline{x}),$$

for each of $i = 1, 2$. Taking "a convex combination of these inequalities", i.e. multiplying them by $\lambda, 1 - \lambda$ resp. and adding, we find

$$\lambda f(x_1) + (1 - \lambda)f(x_2) \geq f(\overline{x}) + \nabla f(\overline{x})^T(\lambda x_1 + (1 - \lambda)x_2 - \overline{x}) = f(\lambda x_1 + (1 - \lambda)x_2),$$

thus proving convexity.

(2., $\Rightarrow$) Let $x \neq \overline{x} \in C$ and $\lambda \in (0, 1)$. Then, by 1., as we've just proven,

$$\lambda \nabla f(\overline{x})^T(x - \overline{x}) \leq f(\overline{x} + \lambda(x - \overline{x})) - f(\overline{x}).$$

But $f(\overline{x} + \lambda(x - \overline{x})) < \lambda f(x) + (1 - \lambda)f(\overline{x})$ by strict convexity, so we have

$$\lambda \nabla f(\overline{x})^T(x - \overline{x}) < \lambda(f(x) - f(\overline{x})),$$

and the result follows by dividing both sides by λ.

(2., $\Leftarrow$) Same as (1., $\Leftarrow$) replacing "$\leq$" with "$<$".

(3.) Apply 1. to $f - \frac{\sigma}{2}\|\cdot\|^2$, which is still convex if f σ-strongly convex, as one can check. ■

↪**Corollary 2.1**: Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable. Then,

a) there exists an *affine function* $g : \mathbb{R}^n \to \mathbb{R}$ such that $g(x) \leq f(x)$ everywhere;

b) if f strongly convex, then it is coercive, i.e. $\lim_{\|x\|\to\infty} f(x) = \infty$.

↪**Corollary 2.2**: Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and differentiable, then TFAE:

1. $\overline{x}$ is a global minimizer of f;
2. $\overline{x}$ is a local minimizer of f;
3. $\overline{x}$ is a stationary point of f.

Proof. 1. $\Rightarrow$ 2. is trivial and 2. $\Rightarrow$ 3. was already proven and 3. $\Rightarrow$ 1. follows from the fact that differentiability gives

$$f(x) \geq f(\overline{x}) + \cancel{\nabla(f)(\overline{x})^T(x - \overline{x})}$$

for any $x \in \mathbb{R}^n$. ■

↪**Corollary 2.3**: (2.2.4)

↪**Theorem 2.6** (Twice Differentiable Convex Functions): Let $\Omega \subset \mathbb{R}^n$ open and convex and $f \in C^2(\Omega)$. Then,

1. f is convex on Ω iff $\nabla^2 f \geq 0$;
2. f is strictly convex on $\Omega \Leftarrow \nabla^2 f > 0$;
2. f is σ-strongly convex on $\Omega \Leftrightarrow \sigma \leq \lambda_{\min}(\nabla^2 f(x))$ for all $x \in \Omega$.

↪**Corollary 2.4**: Let $A \in \mathbb{R}^{n \times n}$ be symmetric, $b \in \mathbb{R}^n$ and $f(x) := \frac{1}{2} x^T A x + b^T x$. Then,

1. f convex $\Leftrightarrow A \geq 0$;
2. f strongly convex $\Leftrightarrow A > 0$.