

MATH255 - Analysis 2

Basic point-set topology; metric spaces; Hölder-Minkowski Inequalities; compactness.

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Contents

1	Introduction	2
1.1	Metric Spaces	2
2	Point-Set Topology	6
2.1	Definitions	6
2.2	Basis	8
2.3	Subspaces	9
2.4	Continuous Functions	10
2.5	Product Spaces	11
2.6	Metrizability	14
2.7	Compactness, Connectedness	15
2.8	Path Components, Connected Components	19
2.8.1	Cantor Staircase Function	20
3	L^p Spaces	22
3.1	Review of ℓ^p Norms	22
3.2	ℓ^p Norms, Hölder-Minkowski Inequalities	22
3.3	An Aside on Complete Metric Spaces	26
3.4	Contraction Mapping Theorem	28
3.5	Compactness in Metric Spaces	30

1 Introduction

1.1 Metric Spaces

↪ Definition 1.1: Metric Space

A set X is a *metric space* with distance d if

1. (symmetric) $d(x, y) = d(y, x) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. (triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$

Remark 1.1. If 1., 3. are satisfied but not 2., d can be called a “pseudo-distance”.

↪ Definition 1.2: Open Metric Space

Let (X, d) be a metric space. A subset $A \subseteq X$ is open $\iff \forall x \in A, \exists r = r(x) > 0$ s.t. $B(x, r(x)) \subseteq A$.

↪ Definition 1.3: Normed Space

Let X be a vector space over \mathbb{R} . The norm on X , denoted $\|x\| \in \mathbb{R}$, is a function that satisfies

1. $\|x\| \geq 0$
2. $\|x\| = 0 \iff x = 0$
3. $\|c \cdot x\| = |c| \cdot \|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

If X is a normed vector space over \mathbb{R} , we can define a distance d on X by $d(x, y) = \|x - y\|$.

↪ Proposition 1.1

If X is a normed vector space over \mathbb{R} , a distance d on X by $d(x, y) = \|x - y\|$ makes (X, d) a metric space.

Proof. 1. $d(x, y) = \|x - y\| \geq 0$

2. $d(x, y) = 0 \iff \|x - y\| = 0 \iff x - y = 0 \iff x = y$

3. $d(x, y) + d(y, z) = \|x - y\| + \|y - z\| \geq \|(x - y) + (y - z)\| = \|x - z\| := d(x, z)$

■

⊛ **Example 1.1: L^p distance in \mathbb{R}^n**

Let $\bar{x} \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$. The L^p norm is defined

$$\|x\|_p := (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{\frac{1}{p}}.$$

In the case $p = 2, n = 2$, we simply have the standard Euclidean distance over \mathbb{R}^2 .

Unit Balls: consider when $\|x\|_p \leq 1$, over \mathbb{R}^2 .

- $p = 1$: $|x_1| + |x_2| \leq 1$; this forms a “diamond ball” in the plane.
- $p = 2$: $\sqrt{|x_1|^2 + |x_2|^2} \leq 1$; this forms a circle of radius 1. Clearly, this surrounds a larger area than in $p = 1$.

A natural question that follows is what happens as $p \rightarrow \infty$? Assuming $|x_1| \geq |x_2|$:

$$\begin{aligned} \|x\|_p &= (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \\ &= \left[|x_1|^p \left(1 + \left| \frac{x_2}{x_1} \right|^p \right) \right]^{\frac{1}{p}} \\ &= |x_1| \left(1 + \left| \frac{x_2}{x_1} \right|^p \right)^{\frac{1}{p}} \end{aligned}$$

If $|x_1| > |x_2|$, this goes to $|x_1|$. If they are instead equal, then $\|x\|_p = |x_1| \cdot 2^{\frac{1}{p}} \rightarrow |x_1| \cdot 1$ as well. Hence, $\lim_{p \rightarrow \infty} \|x\|_p = \max\{|x_1|, |x_2|\}$. Thus, the unit ball will approach $\max\{|x_1|, |x_2|\} \leq 1$, that is, the unit square.

↪ **Proposition 1.2**

Let $x \in \mathbb{R}^n$. Then, $\|x\|_p \rightarrow \max\{|x_1|, \dots, |x_n|\}$ as $p \rightarrow \infty$.

Remark 1.2. This is an extension of the previous example to arbitrary real space; the proof follows nearly identically.

↪ **Definition 1.4: Convex Set**

Let X be a normed space, and take $x, y \in X$. The line segment from x to y is the set

$$\{t \cdot x + (1 - t) \cdot y : 0 \leq t \leq 1\}.$$

Let $A \subseteq X$. A is *convex* $\iff \forall x, y \in A$, we have that

$$(t \cdot x + (1 - t) \cdot y) \in A \forall 0 \leq t \leq 1.$$

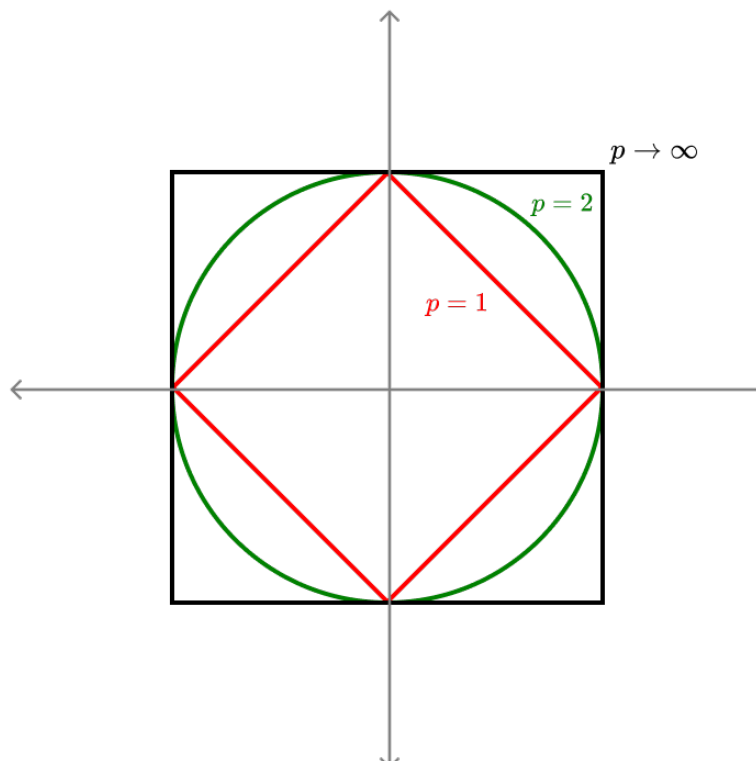


Figure 1: Regions of \mathbb{R}^2 where $\|x\|_p \leq 1$ for various values of p .

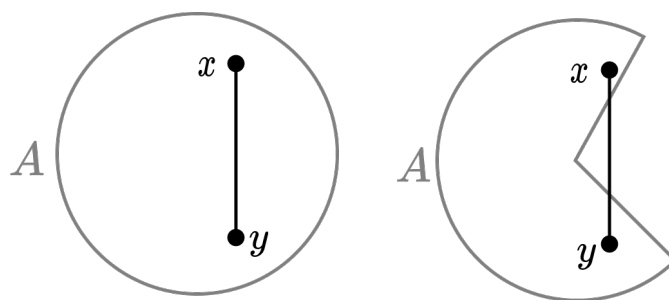


Figure 2: Convex (left) versus not convex (right) sets.

Remark 1.3. Think of this as saying “a set is convex iff every point on a line segment connected any two points is in the set”.

↪ **Definition 1.5:** ℓ_p

The space ℓ_p of sequences is defined as

$$\{x = (x_1, x_2, \dots, x_n, \dots) : \sum_{n=1}^{\infty} |x_n|^p < +\infty\} \quad *.$$

Then, $*$ defines the ℓ^p norm on the space of sequences; that is, $\|x\|_p := (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$.

⊛ **Example 1.2:** ℓ_p , $x_n = \frac{1}{n}$

. Let $x_n = \frac{1}{n}$. For which p is $x \in \ell_p$? We have, raising the norm to the power of p for ease:

$$\begin{aligned} \|x\|_p^p &= |x_1|^p + |x_2|^p + \cdots + |x_n|^p + \cdots \\ &= 1^p + \left(\frac{1}{2}\right)^p + \cdots < \infty \iff p > 1. \end{aligned}$$

In the case that $p = 1$, this becomes a harmonic sum, which diverges.

⊛ **Example 1.3:** L^p space of functions

Let $f(x)$ be a continuous function. We define the norm of f over an interval $[a, b]$

$$\|f\|_p = \left[\int_a^b |f(x)|^p dx \right]^{\frac{1}{p}}.$$

Remark 1.4. Triangle inequality for $\|x\|_p$ or $\|f\|_p$ is called *Minkowski inequality*; $\|x\|_p + \|y\|_p \geq \|x + y\|_p$. This will be discussed further.

⊛ **Example 1.4:** Distances between sets in \mathbb{R}^2

Let A, B be bounded, closed, “nice” sets in \mathbb{R}^2 . We define

$$d(A, B) := \text{Area}(A \triangle B),$$

where

$$A \triangle B : (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

It can be shown that this is a “valid” distance.

Remark 1.5. \triangle denotes the “symmetric difference” of two sets.

⊛ **Example 1.5:** p -adic distance

Let p be a prime number. Let $x = \frac{a}{b} \in \mathbb{Q}$, and write $x = p^k \cdot \left(\frac{c}{d}\right)$, where c, d are not divisible by p . Then, the p -adic norm is defined $\|x\|_p := p^{-k}$. It can be shown that this is a norm.

Suppose $p = 2$, $x = 28 = 4 \cdot 7 = 2^2 \cdot 7$. Then, $\|28\|_2 = 2^{-2} = \frac{1}{4}$; similarly, $\|1024\|_2 = \|2^{10}\|_2 = 2^{-10}$.

More generally, we have that $\|2^k\|_2 = 2^{-k}$; conversely, $\|2^{-k}\| = 2^k$. That is, the closer to 0, the larger the distance, and vice versa, contrary to our notion of Euclidean distance.

↪ **Proposition 1.3**

$\|x\|_p$ as defined above is a well-defined norm over \mathbb{Q} .

2 Point-Set Topology

2.1 Definitions

↪ **Definition 2.1: Topological space**

A set X is a topological space if we have a collection of subsets τ of X called *open sets* s.t.

1. $\emptyset \in \tau, X \in \tau$
2. Consider $\{A_\alpha\}_{\alpha \in I}$ where A_α an open set for any α ; then, $\bigcup_{\alpha \in I} A_\alpha \in \tau$, that is, it is also an open set.
3. If J is a finite set, and A_β open for all $\beta \in J$, then $\bigcap_{\beta \in J} A_\beta \in \tau$ is also open.

In other words, 2.: arbitrary unions of open sets are open, and 3.: finite intersections of open sets are open.

↪ **Definition 2.2: Closed sets**

Closed sets are complements of open sets; hence, axioms for closed sets follow appropriately;

- 1.* X, \emptyset closed;
- 2.* B_α closed $\forall \alpha \in I \implies \bigcap_{\alpha \in I} B_\alpha$ closed.
- 3.* B_β closed $\forall \beta \in J, J$ finite, then $\bigcup_{\beta \in J} B_\beta$ also closed.

↪ **Definition 2.3: Equivalence of Metrics**

Suppose we have a metric space X with two distances d_1, d_2 ; will these necessarily admit the same topology?

A sufficient condition is that, if $\forall x \neq y \in X, \exists 1 < C < +\infty$ s.t.

$$\frac{1}{C} < \frac{d_1(x, y)}{d_2(x, y)} < C.$$

That is, the distances are equivalent, up to multiplication by a constant.

Indeed, this condition gives that $d_2 < C d_1$ and $d_2 > \frac{d_1}{C}$; this gives

$$B_{d_1}(x, \frac{r}{C}) \subseteq B_{d_2}(x, r) \subseteq B_{d_1}(x, C \cdot r).$$

Hence, d_1, d_2 define the same open/closed sets on X thus admitting the same topologies. We write $d_1 \asymp d_2$.

Remark 2.1. If $d_1 \asymp d_2$ and $d_2 \asymp d_3$, then also $d_1 \asymp d_3$. Moreover, clearly, $d_1 \asymp d_1$ and $d_1 \asymp d_2 \implies d_2 \asymp d_1$, hence this is a well-defined equivalence relation.

Hence, it's enough to show that $\forall 1 < p < +\infty$, we have $\|x\|_p \asymp \|x\|_\infty$ to show that any $\|x\|_q$ norm are equivalent for all q on \mathbb{R}^n .

↪ **Definition 2.4: Interior, Boundary of a Topological Set**

Let X be a topological space, $A \subseteq X$ and let $x \in X$. We have the following possibilities

1. $\exists U$ -open : $x \in U \subseteq A$. In this case, we say $x \in$ the *interior* of A , denoted

$$x \in \text{Int}(A).$$

2. $\exists V$ -open : $x \in V \subseteq X \setminus A = A^C$. In this case, we write

$$x \in \text{Int}(X^C).$$

3. $\forall U$ -open : $x \in U, U \cap A \neq \emptyset$ AND $U \cap A^C \neq \emptyset$. In this case, we say x is in the *boundary* of A , and denote

$$x \in \partial A.$$

↪ **Definition 2.5: Closure**

$x \in \text{Int}(A)$ or $x \in \partial A$ (that is, $x \in \text{Int}(A) \cup \partial A$) \iff every open set U that contains x intersects A .¹ Such points are called *limit points* of A . The set of all limit points of A is called the *closure* of A , denoted \bar{A} .

¹"Requires" proof.

Remark 2.2. We have that

$$\text{Int}(A) \subseteq A \subseteq \overline{A} = \text{Int}(A) \cup \partial A.$$

↪ **Proposition 2.1: Properties of $\text{Int}(A)$**

$\text{Int}(A)$ is open, and it is the largest open set contained in A . It is the union of all U -open s.t. $U \subseteq A$. Moreover, we have that

$$\text{Int}(\text{Int}(A)) = \text{Int}(A).$$

↪ **Proposition 2.2: Properties of \overline{A}**

\overline{A} is closed; \overline{A} is the smallest closed set that contains A , that is, $\overline{A} = \bigcap B$ where B closed and $A \subseteq B$. We have too that

$$\overline{(\overline{A})} = \overline{A}.$$

↪ **Proposition 2.3**

1. A is open $\iff A = \text{Int}(A)$
2. A is closed $\iff A = \overline{A}$

2.2 Basis

↪ **Definition 2.6: Basis for a Topology**

Let τ be a topology on X . Let $\mathcal{B} \subseteq \tau$ be a collection of open sets in X such that every open set is a union of open sets in \mathcal{B} .

⊛ **Example 2.1: Example Basis**

$X = \mathbb{R}$, and $\mathcal{B} = \{\text{all open intervals } (a, b) : -\infty < a < b < +\infty\}$.

↪ **Proposition 2.4**

Let \mathcal{B} be a collection of open sets in X . Then, \mathcal{B} is a basis \iff

1. $\forall x \in X, \exists U\text{-open} \in \mathcal{B} \text{ s.t. } x \in U$.
2. If $U_1 \in \mathcal{B}$ and $U_2 \in \mathcal{B}$, and $x \in U_1 \cap U_2$, then $\exists U_3 \in \mathcal{B} \text{ s.t. } x \in U_3 \subseteq U_1 \cap U_2$.

⊛ **Example 2.2**

Consider $X = \mathbb{R}$. Requirement 1. follows from taking $U = (x - \varepsilon, x + \varepsilon)$ for any $\varepsilon > 0$. For 2., suppose $x \in (a, b) \cap (c, d) =: U_1 \cap U_2$. Let $U_3 = (\max\{a, c\}, \min\{b, d\})$; then, we have that $U_3 \subseteq U_1 \cap U_2$, while clearly $x \in U_3$.

↪ **Proposition 2.5**

In a metric space, a basis for a topology is a collection of open balls,

$$\{B(x, r) : x \in X, r > 0\} = \{\{y \in X : d(x, y) < r\} : x \in X, r > 0\}.$$

Proof. We prove via proposition 2.4. Property 1. holds clearly; $x \in B(x, \varepsilon)$ -open $\subseteq \mathcal{B}$.

For property 2., let $x \in B(y_1, r_1) \cap B(y_2, r_2)$, that is, $d(x, y_1) < r_1$ and $d(x, y_2) < r_2$. Let

$$\delta := \min\{r_1 - d(x, y_1), r_2 - d(x, y_2)\}.$$

We claim that $B(x, \delta) \subseteq U_1 \cap U_2$.

Let $z \in B(x, \delta)$. Then,

$$d(z, y_1) \stackrel{\Delta \neq}{\leq} d(z, x) + d(x, y_1) < \delta + d(x, y_1) \leq r_1 - d(x, y_1) + d(x, y_1) = r_1,$$

hence, as $d(z, y_1) < r_1 \implies z \in B(y_1, r_1) = U_1$. Replacing each occurrence of y_1, r_1 with y_2, r_2 respectively gives identically that $z \in B(y_2, r_2) = U_2$. Hence, we have that $B(x, \delta) \subseteq U_1 \cap U_2$ and 2. holds. ■

2.3 Subspaces

↪ **Definition 2.7**

Let X be a topological space and let $Y \subseteq X$. We define the subspace topology on Y :

1. Open sets in $Y = \{Y \cap \text{open sets in } X\}$

↪ **Proposition 2.6: Consequences of Subspace Topologies**

Suppose \mathcal{B} is a basis for a topology in X . Then, $\{U \cap Y : U \in \mathcal{B}\}$ forms a basis for the subspace $Y \subseteq X$.

Suppose X a metric space. Then, Y is also a metric space, with the same distance.

↪ **Proposition 2.7**

Let $Y \subseteq X$ - a metric space. Then, the metric space topology for (Y, d) is the same as the subspace topology.

Proof. (Sketch) A basis for the open sets in X can be written $\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha)$; hence

$$Y \cap \left(\bigcup_{\alpha \in I} B(x_\alpha, r_\alpha) \right) = \bigcup_{\alpha \in I} (Y \cap B(x_\alpha, r_\alpha))$$

is an open set topology for Y . ■

↪ **Lemma 2.1**

Let $A \subseteq X$ -open, $B \subseteq A$; B -open in subspace topology for $A \iff B$ -open in X .

↪ **Lemma 2.2**

Let $Y \subseteq X$, $A \subseteq Y$. Then, \overline{A} in $Y = Y \cap \overline{A}$ in X . We can denote this

$$\overline{A}_Y = \overline{A}_X \cap Y.$$

2.4 Continuous Functions

↪ **Definition 2.8: Continuous Function**

Let X, Y be topological spaces. Let $f : X \rightarrow Y$. f is *continuous* $\iff \forall$ open $V \in Y$, $f^{-1}(V)$ -open in X .

↪ **Proposition 2.8**

This definition is consistent with the normal ε - δ definition on the real line.

Proof. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, continuous; that is, $\forall \varepsilon > 0, \forall x \in \mathbb{R} \exists \delta > 0$ s.t. $|x_1 - x| < \delta$, then $|f(x_1) - f(x)| < \varepsilon$.

Let $V \subseteq \mathbb{R}$ open. Let $y \in V$. Then, $\exists \varepsilon : (y - \varepsilon, y + \varepsilon) \subseteq V$. Let $y = f(x)$, hence $y \in f^{-1}(V)$. Now, if $d(x, x_1) < \delta$, we have that $d(f(x_1), f(x)) < \varepsilon$ (by continuity of f), hence $f(x_1) \in (y - \varepsilon, y + \varepsilon) \subseteq V$; moreover, $(x - \delta, x + \delta) \subseteq f^{-1}(V)$, thus $f^{-1}(V)$ is open as required.

The inverse of this proof follows identically. ■

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↪ **Proposition 2.9**

Suppose \mathcal{B} forms a basis of topology for Y . Then, $f : X \rightarrow Y$ is continuous if $f^{-1}(U)$ open $\forall U \in \mathcal{B}$.

Proof. If U -open set in Y , then $\exists I$ -index set and a collection of open sets $\{A_\alpha\}_{\alpha \in I}$, $A_\alpha \in \mathcal{B}$, s.t. $U = \cup_{\alpha \in I} A_\alpha$. Then, we have

$$f^{-1}(U) = f^{-1}(\cup_{\alpha \in I} (A_\alpha)) = \cup_{\alpha \in I} \underbrace{f^{-1}(A_\alpha)}$$

Hence, if each $f^{-1}(A_\alpha)$ open, then $\cup_{\alpha \in I} f^{-1}(A_\alpha)$ open; hence it suffices to check if $f^{-1}(U) \forall U$ -open in V is open to see if f continuous. ■

↪ **Theorem 2.1: Continuity of Composition**

If $f : X \rightarrow Y$ continuous and $g : Y \rightarrow Z$ continuous, then $g \circ f$ continuous as well.

Proof. Let U -open in Z . Then

$$(g \circ f)^{-1}(U) = f^{-1}(\underbrace{g^{-1}(U)}_{\substack{\text{open in } Y \\ \text{open in } X}})$$

■

↪ **Proposition 2.10**

If $f : X \rightarrow Y$ continuous and $A \subseteq X$, A has subspace topology, then $f|_A : A \rightarrow Y$ is also continuous.²

Proof. Let U -open in Y . Then

$$(f|_A)^{-1}(U) = \underbrace{f^{-1}(U)}_{\text{open}} \cap \underbrace{A}_{\text{open}}$$

By the definition of subspace topology, this is an open set and hence $f|_A$ is continuous. ■

2.5 Product Spaces

↪ **Definition 2.9: Finite Product Spaces**

Let X_1, \dots, X_n be topological spaces. We define

$$(X_1 \times X_2 \times \dots \times X_n),$$

and aim to define a *product topology*; a basis of which consists of cylinder sets.

↪ **Definition 2.10: Cylinder Set**

A *cylinder set* has the form

$$A_1 \times A_2 \times \dots \times A_n$$

where each A_j -open in X_j .

⊗ **Example 2.3**

Given an open interval $(a_1, b_1), (a_2, b_2) \subset \mathbb{R}$, the set $(a_1, b_1) \times (a_2, b_2) \subset \mathbb{R}^2$ is a basis for the topology on \mathbb{R}^2 .

²We denote $f|_A$ as the restriction of the domain of f to A .

↪ **Definition 2.11: Projection**

Let $X_1 \times X_2 \times \cdots \times X_n =: X$. The *projection* $\pi_j : X \rightarrow X_j$ maps $(x_1, \dots, x_n) \rightarrow x_j \in X_j$.

Remark 2.3. One can show π_j continuous.

↪ **Definition 2.12: Coordinate Function**

Given a function $f : Y \rightarrow X_1 \times \cdots \times X_n = (x_1(y), x_2(y), \dots, x_n(y))$. The *coordinate function* is

$$f_j = \pi_j \circ f; \quad f_j = x_j(y).$$

↪ **Proposition 2.11**

$f : Y \rightarrow X = X_1 \times \cdots \times X_n$ continuous $\iff f_j : Y \rightarrow X_j$ continuous.

Proof. Its enough to show that $\forall U \in \mathcal{B}$ -basis for X -product space, $f^{-1}(U)$ -open in Y . Take $U = A_1 \times \cdots \times A_n$ -open. Then, we claim that

$$f^{-1}(U) = f^{-1}(A_1 \times \cdots \times A_n) = f_1^{-1}(A_1) \cap f_2^{-1}(A_2) \cap \cdots \cap f_n^{-1}(A_n). \quad \star$$

If this holds, then as each f_i continuous (being a composition of continuous functions) and each A_i open in X_i , then each $f_i^{-1}(A_i)$ open in Y and hence \star , being the finite intersection of open sets in Y , is itself open in Y . ■

⊛ **Example 2.4: Fourier Transform: Motivation for Infinite Product Topologies**

Let $f \in C([0, 2\pi])$ is real-valued. We write the n th Fourier coefficients

$$\begin{aligned}\hat{f}(n) &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos(nx) dx - i \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin(nx) dx.\end{aligned}$$

And the Fourier transform of f as the infinite product

$$f(x) \mapsto (\dots, \hat{f}(-n), \hat{f}(-n+1), \dots, \hat{f}(-1), \hat{f}(0), \hat{f}(1), \dots, \hat{f}(n), \dots) \in \prod_{n \in \mathbb{Z}} (\mathbb{C})_n.$$

Hence, this is an (countably, as indexed by integers) infinite product space.

Now, let $f : \mathbb{R} \rightarrow \mathbb{R}$. Suppose $f(x) \rightarrow 0$ “fast enough” as $|x| \rightarrow \infty$ and f continuous. Then, we can define the Fourier coefficients

$$\hat{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-itx} dx,$$

where $t \in \mathbb{R}$. We then have the transform

$$f \mapsto \{\hat{f}(t)\}_{t \in \mathbb{R}}.$$

In this case, our index set is \mathbb{R} is (uncountably) infinite.

↪ **Definition 2.13: Product Topology/Cylinder Sets for ∞ Products**

Let $X = \prod_{\alpha \in I} X_\alpha$. Then, a basis for X is given by cylinder sets of the form $A = \prod_{\alpha \in I} A_\alpha$ where A_α -open in X_α , AND $A_\alpha = X_\alpha$ except for finitely many indices α .

That is, there exists a finite set $J = (\alpha_1, \dots, \alpha_k) \subseteq I$, such that we can write $A = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$ (where A_α open in X_α).

↪ **Proposition 2.12**

Given $f : Y \rightarrow \prod_{\alpha \in I} X_\alpha = X$, then (taking $f_\alpha = \pi_\alpha \circ f$ as before) we have that f is continuous in $X \iff f_\alpha : Y \rightarrow X_\alpha$ continuous in $X_\alpha \forall \alpha \in I$.

Remark 2.4. Extension of proposition 2.11 to infinite product space.

Proof. Write $U = \prod_{\alpha \in J} A_\alpha \times \prod_{\alpha \notin J} X_\alpha$. Then,

$$f^{-1}(U) = \bigcap_{\alpha \in J} f_\alpha^{-1}(A_\alpha)$$

which is open in Y , hence f continuous. ■

Remark 2.5. The intersection of the entire spaces give no restriction.

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2.6 Metrizability

↪ Proposition 2.13

Different metrics can define the same topology.

⊗ Example 2.5

1. Different ℓ_p metrics in \mathbb{R}^n (PSET 1)
2. Let (X, d) be a metric space. Then,

$$\tilde{d}(x, y) := \frac{d(x, y)}{d(x, y) + 1}$$

is also a metric (the first two axioms are trivial), and defines the same topology. Note, moreover, that $\tilde{d}(x, y) \leq 1 \forall x, y$; this distance is bounded, and can often be more convenient to work with in particular contexts.

↪ Question 2.1

Suppose (X_k, d_k) are metric spaces $\forall k \geq 1$. Then, we can define the product topology τ on

$$X := \prod_{k=1}^{\infty} X_k.$$

Does the product topology τ come from a metric? That is, is τ metrizable?

Remark 2.6. There do indeed exist examples of non-metrizable topological spaces; this question is indeed well-founded.

Answer. Let $\underline{x} = (x_1, x_2, \dots, x_n, \dots), \underline{y} = (y_1, y_2, \dots, y_n, \dots) \in \prod_{k=1}^{\infty} X_k$ (where $x_i, y_i \in X_i$) be infinite sequences of elements. Then, for each metric space X_k take the metric

$$\tilde{d}_k(x_k, y_k) = \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)}$$

(as in the example above). Then, we define

$$D(\underline{x}, \underline{y}) = \sum_{k=1}^{\infty} \frac{\tilde{d}_k(x_k, y_k)}{2^k},$$

noting that $D(\underline{x}, \underline{y}) \leq \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ (by our construction, “normalizing” each metric), hence this is a valid, *converging* metric (which wouldn’t otherwise be guaranteed if we didn’t normalize the metrics). It remains to show whether this metric omits the same topology as τ . ■

2.7 Compactness, Connectedness

↪ Definition 2.14: Compact

A set A in a topological space is said to be *compact* if every cover has a finite subcover. That is, if

$$A \subseteq \bigcup_{\alpha \in I} U_{\alpha} - \text{open},$$

then $\exists \{\alpha_1, \dots, \alpha_n \in I\}$ such that $A \subseteq \bigcup_{i=1}^n U_{\alpha_i}$.

↪ Proposition 2.14

A closed interval $[a, b]$ is compact.

Proof. If³ $a = b$, this is clear. Suppose $a < b$, and let $[a, b] \subseteq \bigcup_{i \in I} U_i =: \mathcal{U}$ be an arbitrary cover. Then, we proceed in the following steps:

1. **Claim:** Given $x \in [a, b]$, $x \neq b$, $\exists y \in [a, b]$ s.t. $[x, y]$ has a finite subcover.

Let $x \in [a, b]$, $x \neq b$. Then, $\exists U_{\alpha} \in \mathcal{U} : x \in U_{\alpha}$. Since U_{α} open, and $x \neq b$, we further have that $\exists c \in [a, b]$ s.t. $[x, c] \subseteq U_{\alpha}$.

Now, let $y \in (x, c)$; then, the interval $[x, y] \subseteq [x, c] \subseteq U_{\alpha}$, that is, $[x, y]$ has a finite subcover.

2. Define $C := \{y \in [a, b] : [a, y] \text{ has a finite subcover}\}$. We note that

- $C \neq \emptyset$; taking $x = a$ in Step 1. above, we have that $\exists y \in [a, b]$ such that $[a, y]$ has a finite step cover, so this $y \in C$.
- C bounded; by construction, $\forall y \in C, a < y \leq c$.

Thus, we can validly define $c := \sup C$, noting that $a < c \leq b$. Ultimately, we wish to prove that $c = b$, completing the proof that $[a, b]$ has a finite subcover.

3. **Claim:** $c \in C$.

Let $U_{\beta} \in \mathcal{U} : c \in U_{\beta}$. Then, by the openness of U_{β} , $\exists d \in [a, b]$ s.t. $(d, c] \subseteq U_{\beta}$.

Supposing $c \notin C$, then $\exists z \in C$ such that $z \in (d, c)$; if one did not exist, then this would imply that d was a smaller upper bound than c , a contradiction. Thus, $[z, c] \subseteq (d, c] \subseteq U_{\beta}$.

Moreover, we have that, given $z \in C$, $[a, z]$ has a finite subcover; call it $U_z \subseteq \mathcal{U}$. This gives, then:

$$[a, c] = [a, z] \cup [z, c] \subseteq U_z \cup U_{\beta}.$$

But this is a finite subcover of $[a, c]$, contradicting the fact that $c \notin C$. We conclude, then, that $c \in C$ after all.

4. **Claim:** $c = b$.

Suppose not; then, since we have $c \leq b$, then assume $c < b$. Then, applying Step 1. with $x = c$ (which we can do, by our assumption of $c \neq b$!), then we have that $\exists y > c$ s.t. $[c, y]$ has a finite subcover, call this $U_y \subseteq \mathcal{U}$.

Moreover, we had $c \in C$, hence $[a, c]$ has a finite subcover, call this $U_c \subseteq \mathcal{U}$.

³This proof is adapted from that of Theorem 27.1 in Munkre's Topology, an identical theorem but applied to more general ordered topologies.

Then, this gives us that

$$[a, y] = [a, c] \cup [c, y] \subseteq U_c \cup U_y,$$

that is, $[a, y]$ has a finite subcover, and so $y \in C$. But recall that $y > c$; hence, this a contradiction to c being the least upper bound of C . We conclude that $c = b$, and thus $[a, b]$ has a finite subcover, and is thus compact. ■

Remark 2.7. A similar proof shows that $[a, b]$ is connected; we cannot cover it by two disjoint open sets.

↪ **Theorem 2.2: On Compactness**

Let $A \subseteq \mathbb{R}^n$. Then, A compact $\iff A$ closed and bounded.

↪ **Proposition 2.15**

If X, Y are compact topological spaces, then $X \times Y$ is compact.

Remark 2.8. By induction, if X_1, \dots, X_n compact, so is $\prod_{i=1}^n X_i$.

↪ **Proposition 2.16**

A closed subset of a compact topological space is compact in the subspace topology.

Proof. (Of theorem 2.2)

(\Leftarrow) If $A \subseteq \mathbb{R}^n$ closed and bounded, then $A \subseteq [-R, +R]^n$ for some $R > 0$ (it is contained in some “ n -cube”). Then, we have that $[-R, R]$ is compact, by proposition 2.14, proposition 2.15, and proposition 2.16, A itself compact.

(\Rightarrow) Suppose $A \subseteq \mathbb{R}^n$ is compact. Then, $\bigcup_{x \in A} B(x, \varepsilon)$ for some $\varepsilon > 0$ is an open cover of A . As A compact, there must exist a finite subcover of this cover, $A \subseteq \bigcup_{i=1}^N B(x_i, r_i)$. Let $R := \max_{i=1}^N (\|x_i\| + r_i)$. Then, $A \subseteq \overline{B(0, R)}$, that is, A is bounded.

Now, suppose x is a limit point of A . Then, any neighborhood of x contains a point in A , so $\forall r > 0, B(x, r) \cap A \neq \emptyset$, and so $\overline{B(x, r)}$ also contains a point of A for any $r > 0$.

Now, suppose $x \notin A$ (looking for a contradiction). Then,

$$U := \bigcup_{r>0} U_r := \bigcup_{r>0} (\mathbb{R}^n \setminus \overline{B(x, r)}) = \mathbb{R}^n \setminus \{x\}$$

is an open cover for the set A . A being compact implies that U has an finite subcover such that $A \subset U_{r_1} \cup U_{r_2} \cup \dots \cup U_{r_N}$. Let $r_0 = \min_{i=1}^N r_i$. Then, $A \subseteq U_{r_0}$, and $A \cap B(x, r_0) = \emptyset$; but this is a contradiction to the definition of a limit point, hence any limit point x is contained in A and A is thus closed by definition. ■

↪ **Proposition 2.17**

Compact \implies sequentially compact; that is, every sequence in a compact set has a convergent subsequence.

↪ **Definition 2.15: Connected**

A topological space X is *not connected* if $X = U \cup V$ for two open, nonempty, disjoint sets U, V .

If this does not hold, X is said to be *connected*.

A set $A \subseteq X$ is not connected if A is not connected in the subspace topology $\iff A = \subseteq U \cup V$, for U, V -open in X , $(U \cap A) \neq \emptyset$, $(V \cap A) \neq \emptyset$ and $U \cap V = \emptyset$.

↪ **Theorem 2.3**

Let X be a connected topological space. Let $f : X \rightarrow Y$ be a continuous function. Then, $f(X)$ is also connected.

Proof. Suppose, seeking a contradiction, that X is connected, but $f(X)$ is not. Then, we can write $f(X) \subseteq Y$ as $f(X) \subseteq U \cup V$, such that U, V open in Y and $U \cap V = \emptyset$. Then,

$$(U \cap f(X)) \cap (V \cap f(X)) = \emptyset.$$

We also have that

$$X \subseteq \underbrace{f^{-1}(U)}_{\text{open in } X, \neq \emptyset} \cup \underbrace{f^{-1}(V)}_{\text{open in } X, \neq \emptyset}.$$

$f^{-1}(U) \cap f^{-1}(V) = \emptyset$ (that is, they are disjoint) by our assumption; this is a contradiction to the connectedness of X , as we are able to write it as a subset of two disjoint open sets. Hence, $f(X)$ is indeed connected. ■

↪ **Lemma 2.3**

Any interval $(a, b), [a, b], [a, b), \dots, \subseteq \mathbb{R}$ is connected.

Proof. ■

↪ **Theorem 2.4: “Intermediate Value Theorem”**

Suppose X is connected and $f : X \rightarrow \mathbb{R}$ is a continuous function. Then, f takes intermediate values.

More precisely, let $a = f(x), b = f(y)$ for $x, y \in X$. Assume $a < b$. Then, $\forall a < c < b, \exists z \in X$ s.t. $f(z) = c$.

Proof. Suppose, seeking a contradiction, that $\exists c : a < c < b$ s.t. $c \notin f(X)$ (that is, there exists an intermediate value that is “not reached” by the function).

Let $U = (-\infty, c)$ and $V = (c, +\infty)$; note that these are disjoint open sets. Then, we have that

$$X = f^{-1}(U) \cup f^{-1}(V),$$

by our assumption of $c \notin f(X)$. But this gives that X is not connected, as the union of two open (by continuity), disjoint, nonempty ($f(x) = a \in U \implies x \in f^{-1}(U)$, and $f(y) = b \in V \implies y \in f^{-1}(V)$) sets, a contradiction. ■

↪ **Theorem 2.5**

Suppose X is compact, Y -topological space, $f : X \rightarrow Y$ is a continuous function. Then, $f(X)$ is also compact.

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of $f(X) \subseteq Y$, that is,

$$f(X) \subseteq \bigcup_{\alpha \in I} U_\alpha \implies X \subseteq f^{-1}\left(\bigcup_{\alpha \in I} U_\alpha\right) = \bigcup_{\alpha \in I} f^{-1}(U_\alpha) =: \bigcup_{\alpha \in I} V_\alpha - \text{open.}$$

Then, this is an open cover of X ; X is compact, thus there exists a finite subcover, that is, indices $\{\alpha_1, \dots, \alpha_n\} \subseteq I$ such that $X = \bigcup_{i=1}^n V_{\alpha_i}$. Thus,

$$f(X) \subseteq \bigcup_{i=1}^n U_{\alpha_i},$$

which is a finite subcover of $f(X)$. Thus, $f(X)$ is compact. ■

Remark 2.9. Recall the “extreme value theorem”: let $f : [a, b] \rightarrow \mathbb{R}$ a continuous function; then, a minimum and maximum is obtained for $f(x)$ on this interval for values in this interval.

↪ **Theorem 2.6**

Let X compact, and $f : X \rightarrow \mathbb{R}$ a continuous function. Then,

$$\max_{x \in X} f(x) \text{ and } \min_{x \in X} f(x)$$

are both attained.

Proof. $f(X) \subseteq \mathbb{R}$ is compact by theorem 2.5, and so by theorem 2.2, $f(X)$ is closed and bounded. Let, then, $m = \inf f(X)$ and $M = \sup f(X)$; these necessarily exist, since $f(X)$ is bounded. Both m and M are limit points of $f(X)$. But $f(X)$ is closed, and hence contains all of its limit points, and thus $m \in f(X)$ and $M \in f(X)$, and thus $\exists y_m : f(y_m) = m$ and $y_M : f(y_M) = M$. ■

↪ **Definition 2.16: Path Connected**

A set $A \subseteq X$ is called *path connected* if $\forall x, y \in A, \exists f : [a, b] \rightarrow X$, continuous, s.t. $f(a) = x, f(b) = y$ and $f([a, b]) \subseteq A$.

The set $\{f(t) : a \leq t \leq b\}$ is called a *path* from x to y .

↪ **Theorem 2.7: Path connected \implies connected**

If $A \subseteq X$ is path connected, then A is connected.

Proof. Suppose, seeking a contradiction, that A is path connected, but not connected. Then, we can write $A \subseteq U \cup V$, for open, disjoint, nonempty subsets $U, V \subseteq X$.

Let $x \in U \cap A$ and $y \in V \cap A$. Then, $\exists f : [a, b] \rightarrow A$ s.t. $f(a) = x, f(b) = y$, and $f([a, b]) \subseteq A$, by the path connectedness of A . Then,

$$[a, b] \subseteq f^{-1}(A) \subseteq \underbrace{f^{-1}(U \cap A)}_{\text{open}} \cup \underbrace{f^{-1}(V \cap A)}_{\text{open}} =: \underbrace{U_1}_{a \in} \cup \underbrace{U_2}_{b \in},$$

that is, $[a, b]$ is contained in a union of open, nonempty, disjoint sets, contradicting $[a, b]$ the connectedness of $[a, b]$ by lemma 2.3. Thus, A is connected. ■

Remark 2.10. A counterexample to the opposite side of the implication is the Topologist's sine curve, the set

$$\{(x, \sin(\frac{1}{x})) : x \in (0, 1]\} \cup \{0\} \times [-1, 1].$$

This set is connected in \mathbb{R}^2 , but is not path connected.

↪ **Proposition 2.18**

For open sets in \mathbb{R}^n , path connected \iff connected.

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2.8 Path Components, Connected Components

Remark 2.11. Remark that if a metric space X is not connected, then we can write $X = U \cup V$ where U, V are open, nonempty and disjoint. It follows, then, that $U = V^C$ (and vice versa) and hence U, V are both open and closed.

↪ **Definition 2.17: Connected Component**

A connected component of $x \in X$ is the largest connected subset of X that contains x .

⊛ **Example 2.6**

Let $X = (0, 1) \cup (1, 2)$. Here, we have two connected components, $(0, 1)$ and $(1, 2)$

⊛ **Example 2.7: Middle Thirds Cantor Set**

Let $C_0 := [0, 1]$, and given C_n , define $C_{n+1} := \frac{1}{3}(C_n \cup (2 + C_n))$ for $n \geq 0$. C_∞ is totally disconnected.

↪ **Definition 2.18: Path Component**

A path component $P(x)$ of $x \in X$ is the largest path connected subset of X that contains x .

↪ **Proposition 2.19**

$P(x) = \{x \in X : \exists \text{ continuous path } \gamma : [0, 1] \rightarrow X : \gamma(0) = x, \gamma(1) = y\}.$

Remark 2.12. Where we “start” a path does not matter. We write $x \sim y$ if $\exists \gamma$ from x to y ; this is an equivalence relation on the elements of X .

Remark 2.13. The choice of $[0, 1]$ here is arbitrary; any closed interval is homeomorphic.

↪ **Lemma 2.4**

If $P(x) \cap P(y) \neq \emptyset$, then $P(x) = P(y)$.

Proof. $P(x) \cap P(y) \neq \emptyset \implies \exists z : x \sim z \wedge y \sim z \implies x \sim y.$ ■

↪ **Lemma 2.5**

If $A \subseteq X$ is connected, then \bar{A} is also connected.

↪ **Lemma 2.6**

Suppose $A \subseteq X$ is both open and closed. Then, if $C \subseteq X$ is connected and $C \cap A \neq \emptyset$, then $C \subseteq A$.

Proof. If A is both open and closed, then $C \cap A$ is both open and closed in C . If $C \cap A^c \neq \emptyset$, then this is also open and closed in C . Hence, we can write $C = (C \cap A) \cup (C \cap A^c)$, that is, a disjoint union of two nonempty open sets, contradicting the connectedness of C . Hence, $C \cap A^c = \emptyset$, and so $C \subseteq A$. ■

↪ **Proposition 2.20**

Let $\{C_\alpha\}_{\alpha \in I}$ be a collection of nonempty connected subspaces of X s.t. $\forall \alpha, \beta \in I, C_\alpha \cap C_\beta \neq \emptyset$. Then, $\cup_{\alpha \in I} C_\alpha$ is connected.

↪ **Proposition 2.21**

Suppose each $x \in X$ has a path-connected neighborhood. Then, the path components in X are the same as the connected components in X .

2.8.1 Cantor Staircase Function

↪ **Definition 2.19: An Explicit Definition**

Let $x \in C : x = 0.a_1a_2a_3 \dots$ (base 3), ie $a_j = \begin{cases} 0 \\ 2 \end{cases}$. Define

$$f(x) = \begin{cases} \sum \frac{a_j/2}{2^j} & x \in C \\ \text{extend by continuity} & x \notin C. \end{cases}$$

That is, if $x \notin C$, set $f(y) = \sup_{x \in C, x < y} f(x) = \inf_{x \in C, x > y} f(x)$.

↪ **Definition 2.20: Complement Definition**

To construct the complement of the Cantor set, begin with $[0, 1]$ and at a step n , we remove 2^n open intervals from this interval. $f(x)$ will be constant on each of these intervals with values $\frac{k}{2^n}$ where k odd and $0 < k < 2^n$. Extend by continuity to all $x \in C$.

Remark 2.14. *Wikipedia's explanation of this is far better than whatever this definition is trying to say.*

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3 L^p Spaces

3.1 Review of ℓ^p Norms

Remark 3.1. Recall that for $1 \leq p \leq +\infty$, we define for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the norm

$$\|x\|_p = (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}, \quad \|x\|_\infty = \max_{i=1}^n |x_i|.$$

Similarly, for infinite vector spaces, we had, for $x = (x_1, \dots, x_n, \dots)$, the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{\frac{1}{p}}, \quad \|x\|_\infty = \sup_{i \geq 1} |x_i|.$$

Here, we define

$$\ell_p := \{x = (x_1, \dots, x_n) : \|x\|_p < +\infty\}.$$

3.2 ℓ^p Norms, Hölder-Minkowski Inequalities

↪ **Definition 3.1: Hölder Conjugates**

For $1 \leq p, q \leq +\infty$, we say that p, q are said to be *Hölder conjugates* if

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Remark 3.2. We refer to these simply as “conjugates” throughout as no other conception of conjugate numbers will be discussed.

Further, we take by convention $\frac{1}{\infty} = 0$.

↪ **Proposition 3.1: Hölder’s Inequality**

Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$. Suppose $p, q : 1 \leq p, q \leq +\infty$ are conjugate. Then,

$$\langle x, y \rangle_{\mathbb{R}^n} := \left| \sum_{i=1}^n x_i y_i \right| \leq \|x\|_p \cdot \|y\|_q$$

⊛ **Example 3.1**

For the case $p = 1$ or ∞ (functionally, the same case):

↪ **Lemma 3.1**

Let p, q be conjugates, and $x, y \geq 0$. Then,

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Remark 3.3. If the inequality holds, then, for some $t > 0$, let $\tilde{x} = t^{\frac{1}{p}} \cdot x$, $\tilde{y} = t^{\frac{1}{q}} y$. Substituting x for \tilde{x} and y for \tilde{y} , we have

$$\text{LHS: } \tilde{x}\tilde{y} = t^{\frac{1}{p}}x \cdot t^{\frac{1}{q}}y = t^{\frac{1}{p}+\frac{1}{q}} \cdot xy = xy$$

$$\text{RHS: } \dots = t\left(\frac{x^p}{p} + \frac{y^q}{q}\right)$$

That is, we have

$$t \cdot xy \leq t \left(\frac{x^p}{p} + \frac{y^q}{q} \right),$$

hence, the inequality is preserved under multiplication by a positive scalar; moreover, the original inequality holds iff this “scaled” version holds. Hence, choosing t such that $\tilde{y} = 1$ (let $t = \left(\frac{1}{y}\right)^q$), it suffices to prove the lemma for $y = 1$.

Proof. If $x = 0$ or $y = 0$, then the entire LHS becomes 0 and we are done; assume $x, y > 0$; by the previous remark, assume wlog $y = 1$. Then, we have

$$\begin{aligned} x \cdot y \leq \frac{x^p}{p} + \frac{y^q}{q} &\iff x \cdot 1 \leq \frac{x^p}{p} + \frac{1}{q} \\ &\iff \frac{x^p}{p} - x + \frac{1}{q} =: f(x) \geq 0. \end{aligned}$$

Taking the derivative, we have

$$\begin{aligned} f'(x) &= \frac{px^{p-1}}{p} - 1 = x^{p-1} - 1 \\ p > 1 &\implies p - 1 > 0 \implies \begin{cases} f'(x) > 0 & \forall x > 1 \\ f'(x) = 0 & x = 1 \\ f'(x) < 0 & \forall 0 < x < 1 \end{cases} \end{aligned}$$

Hence, $x = 1$ is a local minimum of the function, and thus $f(x) \geq f(1) \forall 0 < x \leq 1$. But $f(1) = \frac{1^p}{p} - 1 + \frac{1}{q} = 1 - 1 = 0$, hence $f(x) \geq 0 \forall x \geq 0$, as desired, and the inequality holds. ■

Proof. Assume $\|x\|_p = \|y\|_q = 1$. Then,

$$\begin{aligned}
\left| \sum_{i=1}^n x_i y_i \right| &\leq \sum_{i=1}^n |x_i y_i| && \text{(by triangle inequality)} \\
&\leq \sum_{i=1}^n \left| \frac{x_i^p}{p} + \frac{y_i^q}{q} \right| && \text{(by lemma 3.1)} \\
&= \frac{1}{p} \left(\sum_{i=1}^n |x_i|^p \right) + \frac{1}{q} \left(\sum_{i=1}^n |y_i|^q \right) \\
&= \frac{1}{p} \|x\|_p^p + \frac{1}{q} \|y\|_q^q && \text{(by staring)} \\
&= \frac{1}{p} \cdot 1^p + \frac{1}{q} \cdot 1^q = \frac{1}{p} + \frac{1}{q} = 1 && \text{(by assumption)} \\
&= \|x\|_p \cdot \|y\|_q,
\end{aligned}$$

and the proposition holds, in the special case $\|x\|_p = \|y\|_q = 1$.

If $\|x\|_p = 0$ or $\|y\|_q = 0$, then $x_1 = \dots = x_n = 0$ or $y_1 = \dots = y_n = 0$, resp, then we'd have ($\|x\|_p = 0$ case)

$$0 \cdot y_1 + \dots + 0 \cdot y_n \leq 0,$$

which clearly holds.

Assume, then, $\|x\|_p > 0, \|y\|_q > 0$. Let $\tilde{x} := \frac{x}{\|x\|_p}, \tilde{y} := \frac{y}{\|y\|_q}$. Then,

$$\|\tilde{x}\|_p^p = \frac{(\sum_{i=1}^n |x_i|^p)}{\|x\|_p^p} = \frac{\|x\|_p^p}{\|x\|_p^p} = 1 \implies \|\tilde{x}\|_p = 1.$$

The same case holds for \tilde{y} , hence $\|\tilde{y}\|_q = 1$; that is, we have “rescaled” both vectors. Hence, we can use the case we proved above for when the norms were identically 1 on \tilde{x}, \tilde{y} . We have:

$$\left| \sum_{i=1}^n \tilde{x}_i \tilde{y}_i \right| \leq 1$$

But by definition of \tilde{x}, \tilde{y} , we have

$$\left| \sum_{i=1}^n \tilde{x}_i \tilde{y}_i \right| = \left| \frac{1}{\|x\|_p \|y\|_q} \sum_{i=1}^n x_i y_i \right| \leq 1 \implies \left| \sum_{i=1}^n x_i y_i \right| \leq \|x\|_p \cdot \|y\|_q,$$

and the proof is complete. ■

↪ **Proposition 3.2: Minkowski Inequality**

Let $1 \leq p \leq \infty, x, y \in \mathbb{R}^n$. Then,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Remark 3.4. This is just the triangle inequality for ℓ_p norms.

Proof. The cases $p = 1, \infty$ are left as an exercise.

Assume $1 < p < \infty$. Then,

$$\begin{aligned}
 \|x + y\|_p^p &= \sum_{j=1}^n |x_j + y_j|^p = \sum_{j=1}^n |x_j + y_j| |x_j + y_j|^{p-1} \\
 &\leq \sum_{j=1}^{\infty} (|x_j| + |y_j|) \cdot |x_j + y_j|^{p-1} \\
 &= \underbrace{\sum_{j=1}^n |x_j| \cdot |x_j + y_j|^{p-1}}_{:=A} + \overbrace{\sum_{j=1}^n |y_j| \cdot |x_j + y_j|^{p-1}}^{:=B} \quad \circledast
 \end{aligned}$$

Let $\vec{u} = (|x_1|, \dots, |x_n|)$ and $\vec{v} = (|x_1 + y_1|^{p-1}, \dots, |x_n + y_n|^{p-1})$, then, $A = \vec{u} \cdot \vec{v} = \langle \vec{u}, \vec{v} \rangle_{\mathbb{R}^n}$. We have

$$\begin{aligned}
 \|\vec{u}\|_p &= \left(\sum_{i=1}^n (|x_i|^p) \right)^{\frac{1}{p}} = \|x\|_p \\
 \|\vec{v}\|_q &= \left(\sum_{i=1}^n (|x_i + y_i|^{p-1})^q \right)^{\frac{1}{q}} \\
 &= \left[\sum_{i=1}^n (|x_i + y_i|^{p-1})^{\frac{p}{p-1}} \right]^{\frac{p-1}{p}} \\
 &= \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\
 &= \|x + y\|_p^{p-1}
 \end{aligned}$$

where the second-to-last line follows from p, q being conjugate, hence $q = \frac{p}{p-1}$. Thus, by Hölder's Inequality, we have that

$$A = \langle \vec{u}, \vec{v} \rangle \leq \|\vec{u}\|_p \cdot \|\vec{v}\|_q = \|x\|_p \cdot \|x + y\|_p^{p-1}.$$

By a similar construction, we can show that

$$B \leq \|y\|_p \cdot \|x + y\|_p^{p-1}.$$

Thus, returning to our original inequality in \circledast , we have

$$\begin{aligned}
 \|x + y\|_p^p &\leq A + B \\
 &\leq \|x\|_p \cdot \|x + y\|_p^{p-1} + \|y\|_p \cdot \|x + y\|_p^{p-1} \\
 \implies \|x + y\|_p &\leq \|x\|_p + \|y\|_p,
 \end{aligned}$$

and the proof is complete. ■

3.3 An Aside on Complete Metric Spaces

↪ **Theorem 3.1**

The sequence of centers of balls with monotonically decreasing radii is a Cauchy sequence in X .

Proof. Let $\varepsilon > 0$ and let $N : \forall j > N, r_j < \varepsilon$. Then,

$$d(x_j, x_k) < r_{\min(j,k)} = r_j$$

■

↪ **Definition 3.2: Complete Metric Space**

A metric space is complete if every Cauchy sequence converges to a limit in that space.

⊗ **Example 3.2: Examples of Complete Metric Spaces**

1. \mathbb{R} , p -adic integers (\mathbb{Z}_p)/rationals (\mathbb{Q}_p).
2. $\ell_p = \{x = (x_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |x_i|^p < +\infty\}, 1 \leq p \leq +\infty$
3. $\ell_{\infty} = \{x = (x_i) : \sup_{i=1}^{\infty} |x_i| < +\infty\}$.

↪ **Proposition 3.3**

Hölder's Inequality and Minkowski Inequality inequalities hold for infinite sequences. that is,

1. if $x = (x_i) \in \ell_p$ and $y = (y_i) \in \ell_q$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left| \sum_{i=1}^{\infty} x_i y_i \right| \leq \|x\|_{\ell_p} \|y\|_{\ell_q}.$$

2. if $x, y \in \ell_p$, then

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Remark 3.5. 2. gives the triangle inequality for the $\|x\|_p$ norm on ℓ_p .

Moreover,

$$\begin{aligned} \|c \cdot x\|_p &= \|(c_1 x_1, \dots, c_n x_n, \dots)\|_p \\ &= \left(\sum_{i=1}^{\infty} |c x_i|^p \right)^{\frac{1}{p}} = \left(\sum_{i=1}^{\infty} |c|^p |x_i|^p \right)^{\frac{1}{p}} \\ &= (|c|^p)^{\frac{1}{p}} \|x\|_p = c \cdot \|x\|_p \end{aligned}$$

Proof. (of 2.) If $x, y \in \ell_p$, we have that $\sum_{i=1}^{\infty} |x_i|^p < +\infty$, $\sum_{i=1}^{\infty} |y_i|^p < +\infty$, so $\exists N > 0 : \sum_{i=N+1}^{\infty} |x_i|^p < \varepsilon$, $\sum_{i=N+1}^{\infty} |y_i|^p < \varepsilon$. Let $x_i^{(n)} = (x_1, \dots, x_n, 0, 0, \dots)$ be (x) truncated after n (finite) coordinates. This gives

$$\|(x_i + y_i)^{(n)}\|_p \leq \|x_i^{(n)}\|_p + \|y_i^{(n)}\|_p \leq \|x\|_p + \|y\|_p$$

by Minkowski on finite spaces. Taking $n \rightarrow \infty$ (ie, “detruncating”), we have $(x + y) \in \ell_p$, and thus $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

1. left as an exercise. ■

↪ Proposition 3.4

Let $1 \leq p \leq +\infty$, and $\|x\|_{\infty} = \sup_{i=1}^{\infty} |x_i| = A < +\infty$, $\|y\|_{\infty} = \sup_{i=1}^{\infty} |y_i| = B < +\infty$. Then, the triangle inequality $\|x + y\|_{\infty} \leq \|x\|_{\infty} + \|y\|_{\infty}$ holds.

Proof. We have

$$\sup_{i=1}^{\infty} |x_i + y_i| \leq \sup_{i=1}^{\infty} (|x_i| + |y_i|) \leq \sup_{i=1}^{\infty} |x_i| + \sup_{i=1}^{\infty} |y_i| = \|x\|_{\infty} + \|y\|_{\infty}.$$

↪ Proposition 3.5

$\|x\|_{\infty} := \sup_{i=1}^{\infty} |x_i|$ is a well-defined norm on ℓ_{∞} .

Proof. The triangle inequality is prove in proposition 3.4. The remainder of the requirements are left as an exercise. ■

↪ Proposition 3.6

$\ell_p \subseteq \ell_q$ if $p < q$.

Proof. Let $x \in \ell_p$. If $\sum_{i=1}^{\infty} |x_i|^p < +\infty$, then $\exists N : \forall i \geq N, |x_i| \leq 1$. Then,

$$\begin{aligned} \sum_{i \geq N} |x_i|^q &\leq \sum_{i \geq N} |x_i|^p < \infty \\ \Rightarrow \sum_{i=1}^{\infty} |x_i|^q &< +\infty \Rightarrow x \in \ell_q \\ &\Rightarrow \ell_p \subseteq \ell_q \end{aligned}$$

3.4 Contraction Mapping Theorem

↪ Definition 3.3: Contraction Mapping

Let (X, d) be a metric space. A *contraction mapping* on X is a function $f : X \rightarrow X$ for which \exists a constant $0 < c < 1$ such that

$$d(f(x), f(y)) \leq c \cdot d(x, y) \quad \forall x, y \in X.$$

↪ Theorem 3.2: Contraction Mapping Theorem

Let (X, d) be a complete metric space, and let $f : X \rightarrow X$ be a contraction. Then, there exists a unique fixed point z of f such that $f(z) = z$.

Moreover, $f^{[n]}(x) := f \circ f \circ \cdots \circ f(x) \rightarrow z$ as $n \rightarrow \infty$ for any $x \in X$.

Remark 3.6. The “functional construction” of the Cantor set is an example of a contraction mapping, with $f_1(x) = \frac{x}{3}$, $f_2(x) = \frac{x+2}{3}$. The first has a fixed point of 0, and the second a fixed point of 1.

Remark 3.7. This is a generalization of *this proof* done in Analysis I, an equivalent claim over the reals.

Proof. Fix $x \in X$. Consider the sequence $\{x_0, x_1, x_2, \dots, x_n, \dots\} := \{x, f(x), f \circ f(x), \dots, f^{[n]}(x), \dots\}$ (we call $f^{[n]}$ the *orbit* of x under iterations of f). We claim that this is a Cauchy sequence. Let $n \in \mathbb{N}$ arbitrary, then we have, by the property of the contraction mapping,

$$d(f^{[n+1]}(x) - f^{[n]}(x)) \leq c \cdot d(f^{[n]}(x) - f^{[n-1]}(x)) \leq c^2 d(f^{[n-1]}(x) - f^{[n-2]}(x)).$$

Arguing inductively, it follows that

$$d(f^{[n+1]}(x) - f^{[n]}(x)) \leq c^n d(f(x), x). \quad \star$$

Let now $m, k \in \mathbb{N}$, $m, k > 0$. It follows that

$$\begin{aligned} d(f^{[m]}, f^{[m+k]}(x)) &\leq d(f^{[m]}(x), f^{[m+1]}(x)) + d(f^{[m+1]}(x), f^{[m+2]}(x)) + \cdots + d(f^{[m+k-1]}(x), f^{[m+k]}(x)) \\ &\stackrel{\star}{\leq} d(x, f(x)) [c^m + c^{m+1} + \cdots + c^{m+k-1}] \\ &\leq c^m d(x, f(x)) [1 + c + \cdots + c^k + c^{k+1} + \cdots] = \frac{c^m d(x, f(x))}{1 - c} \end{aligned}$$

Now, given $\varepsilon > 0$, choose N such that $\frac{c^N d(x, f(x))}{1 - c} < \varepsilon$. It follows, then, that $\{f^{[n]}(x)\}_{n \in \mathbb{N}}$ a Cauchy sequence, and thus converges, $f^{[n]}(x) \rightarrow z$ as $n \rightarrow \infty$ for some z .

We further have to show that $f(z) = z$. It is easy to show that f continuous due to the contraction mapping (it is clearly Lipschitz with constant c), and it thus follows that

$$\lim_{n \rightarrow \infty} f(f^{[n]}(x)) = \lim_{n \rightarrow \infty} f^{[n+1]}(x) \implies f(z) = z,$$

by sequential characterization of continuous functions.

Finally, we need to show that this limit is unique. Suppose $\exists y_1 \neq y_2$, ie two fixed points with $f(y_1) = y_1$ and $f(y_2) = y_2$. Then, by the property of the contraction mapping,

$$d(f(y_1), f(y_2)) \leq c \cdot d(y_1, y_2),$$

but by assumption of being fixed points,

$$d(f(y_1), f(y_2)) = d(y_1, y_2),$$

implying $d(y_1, y_2) \leq c \cdot d(y_1, y_2)$. This is only possible if $d(y_1, y_2) = 0$, and thus $y_1 = y_2$ and the fixed point is indeed unique. ■

↪ **Theorem 3.3: ℓ_p complete**

The space ℓ_p is complete for all $1 \leq p \leq +\infty$.

Equivalently, if $(x^1), (x^2), \dots, (x^n)$ is a Cauchy sequence in ℓ^p , $\exists y \in \ell^p$ s.t. $x^n \rightarrow y$ as $n \rightarrow \infty$.

Proof. (Sketch) We suppose first $p < +\infty$. Consider an arbitrary number of Cauchy sequences in ℓ_p :

$$\begin{aligned} x^{(1)} &= (x_1^{(1)}, \dots, x_n^{(1)}, \dots) \\ x^{(2)} &= (x_1^{(2)}, \dots, x_n^{(2)}, \dots) \\ &\vdots \quad \quad \quad \vdots \\ x^{(k)} &= (x_1^{(k)}, \dots, x_n^{(k)}, \dots) \in \ell_p \end{aligned}$$

We claim that, for any $k \in \mathbb{N}$, the $(x_k^{(n)})_{n \in \mathbb{N}}$ is a Cauchy sequence; note that in this definition we are taking a *fixed-index* (namely, the k th) element from different sequences (namely, the n th sequence).

Since $x^{(1)}, x^{(2)}, \dots, x^{(n)}, \dots$ are Cauchy sequences in ℓ^p , we have for a fixed $\varepsilon > 0$, $\exists N \in \mathbb{N} : \forall m, n > N$, $d_p(x^{(m)}, x^{(n)}) < \varepsilon$:

$$\begin{aligned} d_p(x^{(m)}, x^{(n)})^p &= \|x^{(m)} - x^{(n)}\|_p^p = \sum_{i=1}^{\infty} |x_i^{(m)} - x_i^{(n)}|^p < \varepsilon^p \\ |x_k^{(m)} - x_k^{(n)}|^p &\leq \sum_{i=1}^{\infty} |x_i^{(m)} - x_i^{(n)}|^p \implies |x_k^{(m)} - x_k^{(n)}|^p < \varepsilon^p \\ &\implies |x_k^{(m)} - x_k^{(n)}| < \varepsilon, \end{aligned}$$

since we are taking “less of the summands in the second line”. It follows, then, that for each k , $\exists z_k : x_k^{(n)} \rightarrow z_k$ as $n \rightarrow \infty$. Let $z = (z_1, \dots, z_n, \dots)$. We claim that $x^{(n)} \rightarrow z \in \ell_p$ as $n \rightarrow \infty$.

First, we show that $d_p(x^{(n)}, z) \rightarrow 0$ as $n \rightarrow \infty$ (that is, $x^{(n)} \rightarrow z$ as $n \rightarrow \infty$). Fix $\varepsilon > 0$, and choose $N \in \mathbb{N}$ for

which $d_p(x^{(m)}, x^{(n)}) < \varepsilon \ \forall m, n \geq N$ (by Cauchy). Fix $K \in \mathbb{N}, K > 0$.

$$d_p^p(x^{(n)}, z) = \|x^{(n)} - z\|_p^p = \sum_{i=1}^{\infty} |x_i^{(n)} - z_i|^p$$

$$\|x^{(m)} - x^{(n)}\|_p^p < \varepsilon^p \implies \sum_{i=1}^K |x_i^{(m)} - x_i^{(n)}|^p \leq \varepsilon^p$$

Let $m \rightarrow \infty$; then $x_i^{(m)} \rightarrow z_i$ (note that i fixed!), and we have

$$\sum_{i=1}^K |z_i - x_i^{(n)}|^p \leq \varepsilon^p.$$

Let $K \rightarrow \infty$; then,

$$\sum_{i=1}^{\infty} |z_i - x_i^{(n)}|^p \leq \varepsilon^p \implies \|z - x^{(n)}\|_p \leq \varepsilon \implies d_p(z, x^{(n)}) \leq \varepsilon,$$

and thus $x^{(n)} \rightarrow z$ as $n \rightarrow \infty$.

It remains to show that $z \in \ell_p$, ie $\|z\|_p < +\infty$. We have:

$$\|z\|_p \leq \underbrace{\|z - x^{(n)}\|_p}_{\rightarrow 0} + \|x^{(n)}\|_p.$$

For sufficiently large n , $\|z - x^{(n)}\| \leq 1$ (for instance); $x^{(n)} \in \ell_p$, hence $\|x^{(n)}\|_p < +\infty$ (say, $\|x^{(n)}\|_p \leq M$). Thus:

$$\|z\|_p \leq 1 + M < +\infty \implies z \in \ell_p,$$

and the proof is complete. ■

3.5 Compactness in Metric Spaces

↪ **Definition 3.4: Totally Bounded**

Let (X, d) be a metric space. If for every $\varepsilon > 0$, $\exists x_1, \dots, x_n \in X, n = n(\varepsilon) : \bigcup_{i=1}^n B(x_i, \varepsilon) = X$, we say X is *totally bounded*.

↪ **Theorem 3.4**

Let (X, d) be a metric space. TFAE:

1. X is complete and totally bounded;
2. X is compact;
3. X is sequentially compact (every sequence has a convergent subsequence).

Proof. (1. \implies 2.) Suppose X complete and totally bounded. Assume towards a contradiction that X not compact, ie there exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of X with no finite subcover.

X being totally bounded gives that it can be covered by finitely many open balls of radius $\frac{1}{2}$. It must be that at least one of these open balls cannot be finitely covered, otherwise we would have a finite subcover. Let F_1 be the closure of this ball. F_1 closed, with diameter $\text{diam}(F_1) \leq 1$. X .

We also have that X can be covered by finitely many balls of radius $\frac{1}{4}$; again, there must be at least one ball B_1 such that $B_1 \cap F_1$ cannot be covered by finitely many open sets from the cover. Let $F_2 = \overline{B_1} \cap F_1$ -closed, with $\text{diam}(F_2) \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.⁴

Arguing inductively, at some step n , X can be covered by finitely many balls of radius $\frac{1}{2^n}$; at least one of these balls B cannot be covered by a finite subcover hence $B \cap F_{n-1}$ cannot be covered by finitely many U_α 's. Let $F_n = \overline{B} \cap F_{n-1}$ -closed, with $\text{diam}(F_n) \leq \frac{1}{2^{n-1}}$.

As such, we have a nested sequence $F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$ of closed sets, where $\text{diam}(F_k) \leq \frac{1}{2^{k-1}} \rightarrow 0$ as $k \rightarrow \infty$.

↪ **Lemma 3.1** (Cantor Intersection Theorem). $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$.

Proof. (Of Lemma) Let $x_k \in F_k$. Then, $\{x_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence, since

$$d(x_n, x_{n+k}) \leq \text{diam}(F_n) + \dots + \text{diam}(F_{n+k}) \leq \frac{1}{2^{n-1}},$$

by the nested property, which can be made arbitrarily small for sufficiently large n, k . Hence, $x_n \rightarrow y \in X$ for some y , as X complete. The tail of x_n lies in F_n for all sufficiently large n , and as each F_n closed, the limit must lie in F_n for all sufficiently large n . We conclude the intersection nonempty. ■

This y from the lemma is covered by some U_{α_0} -open for some $\alpha_0 \in I$. Being open, $\exists \varepsilon > 0 : B(y, \varepsilon) \subseteq U_{\alpha_0}$. Let $n : \frac{1}{2^{n-1}} < \varepsilon$. Then, $y \in F_n$, and as $\text{diam}(F_n) \leq \frac{1}{2^{n-1}}$, we have that $F_n \subseteq B(y, \frac{1}{2^{n-1}}) \subseteq B(y, \varepsilon) \subseteq U_{\alpha_0}$. But then, we have that F_n covered by a single open set U_{α_0} , a contradiction to our inductive construction of F_n . We conclude X compact.

(2. \implies 3.) Suppose X compact. Let $\{x_n\}_{n \in \mathbb{N}} \in X$. Let $F_n = \overline{\bigcup_{k \geq n} \{x_k\}}$ -closed; we have too that $F_1 \supseteq F_2 \supseteq \dots \supseteq F_n \supseteq \dots$.

↪ **Definition 3.5: Finite Intersection Property**

\mathcal{F} has finite intersection property provided any finite subcollection of sets in \mathcal{F} has a non-empty intersection.

⁴ B_1 has radius $\frac{1}{4}$ and hence diameter $\frac{1}{2}$. The intersection of B_1 with a set with a larger diameter must have diameter leq $\frac{1}{2}$

↪ **Lemma 3.2** (Finite Intersection Formulation of Compactness). X -compact \iff every collection \mathcal{F} of closed subsets of X with finite intersection property has non-empty intersection.

Proof. ■

This lemma directly gives that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, $\{F_n\}_{n \in \mathbb{N}}$ being a collection of closed subsets with any subset having nonempty intersection (by the nestedness). Let $y \in \bigcap_{n=1}^{\infty} F_n$. Take $B(y, \frac{1}{k})$, which thus has nonempty intersection with $\{x_k\}_{k \geq n} \forall n$, ie $\exists n_1 : d(y, x_{n_1}) < 1$ and $\exists n_2 > n_1 : d(y, x_{n_2}) < \frac{1}{2}$. Arguing inductively, $\exists n_j > n_{j-1} : d(y, x_{n_j}) < \frac{1}{j}$ for any given n_{j-1} . It follows that $\lim_{j \rightarrow \infty} x_{n_j} = y$, and thus $\{x_{n_j}\}$ is a convergent subsequence of $\{x_n\}$ that converges within X , and thus X is sequentially compact.

(3. \implies 1.) Suppose X sequentially compact. Let $\{x_n\} \in X$ be a Cauchy sequence in X , which thus have a convergent subsequence $\{x_{n_k}\} \rightarrow y$.

↪ **Lemma 3.3.** Let $\{x_n\}$ be a Cauchy sequence in X where X sequentially compact. Then, if $\{x_{n_k}\} \rightarrow y$, so does $\{x_n\} \rightarrow y$

Proof. ■

Then, $\{x_n\}_n \rightarrow y$ and so X complete.

Suppose X not totally bounded, ie $\exists \varepsilon > 0 : X$ cannot be covered by a finite union of balls of $B(x_j, \varepsilon)$. Let $x_1 \in X$ s.t. $B(x_1, \varepsilon) \not\supseteq X$; $\exists x_2 \in X \setminus B(x_1, \varepsilon)$, and so $X \not\subseteq B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$ by assumption. Then, choose $x_3 \in X \setminus (B(x_1, \varepsilon) \cup B(x_2, \varepsilon))$. Arguing inductively, we have that $\exists x_n \in X \setminus (\bigcup_{i=1}^n B(x_i, \varepsilon))$, noting that $d(x_n, x_j) \geq \varepsilon \forall 1 \leq j \leq n$.

Consider the sequence $\{x_j\}_{j \in \mathbb{N}}$:

↪ **Lemma 3.4.** $\{x_j\}$ cannot have a convergent subsequence.

Proof. Follows by $d(x_m, x_n) \geq \varepsilon \forall m, n$. ■

This contradicts our assumption that X sequentially compact, and we conclude X must be totally bounded. ■