MATH578 - Numerical Analysis 1

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Contents

§1 Polynomial Interpolation

In general, the goal of interpolation is, given a function $f(x)$ on $[a, b]$ and a series of distinct ordered points (often called *nodes* or *collocation points*) $\{x_j\}_{j=1}^n \subseteq [a, b]$, to find a polynomial $P(x)$ such that $f(x_j) = P(x_j)$ for each j.

↪**Theorem 1.1** (Existence and Uniqueness of Lagrange Polynomial): Let $f \in C[a, b]$ and $\{x_j\}$ a set of n distinct points. Then, there exists a unique $P(x) \in \mathbb{P}_{n-1}$, the space of $n-1$-degree polynomials, such that $P(x_j) = f(x_j)$ for each j.

We call such a P the *Lagrange polynomial* associated to the points $\{x_j\}$ for f.

Proof. We define the following $n-1$ degree "fundamental polynomials" associated to $\{x_j\}$,

$$\ell_j(x) \equiv \prod_{\substack{1 \leq i \leq n \\ i \neq j}} \frac{x - x_i}{x_j - x_i}, \qquad j = 1, ..., n.$$

Then, one readily verifes $\ell_j(x_i) = \delta_{ij}$, and that the distinctness of the nodes guarantees the denominator in each term of the product is nonzero. Define

$$P(x) = \sum_{j=1}^n f(x_j)\ell(x),$$

which, being a linear combination of $n-1$ degree polynomials is also in $\mathbb{P}_{n-1}$. Moreover,

$$P(x_i) = \sum f(x_j)\delta_{i,j} = f(x_i),$$

as desired.

For uniqueness, suppose $\overline{P}$ another $n-1$ degree polynomial satisfying the conditions of the theorem. Then, $q(x) \equiv P(x) - \overline{P}(x)$ is also a degree $n-1$ polynomial with $q(x_i) = 0$ for each $i = 1, ...n$. Hence, q a polynomial with more distinct roots than its degree, and thus it must be identically zero, hence $P = \overline{P}$, proving uniqueness. ■

↪**Theorem 1.2** (Interpolation Error): Suppose $f \in C^n[a, b]$, and let $P(x)$ be the Lagrange polynomial for a set of n points $\{x_j\}$, with $x_1 = a, x_n = b$. Then, for each $x \in [a, b]$, there is a $\xi \in [a, b]$ such that

$$f(x) - P(x) = \frac{f^{(n)}(\xi)}{n!}(x - x_1)\cdots(x - x_n).$$

Moreover, if we put $h := \max_i (x_{i+1} - x_i)$, then

$$\|f - P\|_\infty \leq \frac{h^n}{4n} \left\|f^{(n)}\right\|_\infty.$$

Proof. We prove the first identity, and leave the second "Moreover" as a homework problem. Notice that it holds trivially for $x = x_j$ for any j, so assume $x \neq x_j$ for any j, and define the function

$$g(t) := f(t) - P(t) - \omega(t)\frac{f(x) - P(x)}{\omega(x)}, \qquad \omega(t) := (t - x_1)...(t - x_n) \in \mathbb{P}_n[t].$$

Then, we observe the following:

- $g \in C^n[a, b]$
- $g(x) = 0$
- $g(t = x_j) = 0$ for each j

Recall that by Rolle's Theorem, if a C^1 function has $\geq m$ roots, then its derivative has $\geq m - 1$ roots. Thus, applying this principle inductively to $g(t)$, we conclude that $g^{(n)}(t)$ has at least one root. Take ξ to be such a root. Then, one readily verifies that $P^{(n)} \equiv 0$ and $\omega^{(n)} \equiv n!$ (using polynomial properties), from which we may use the fact that $g^{(n)}(\xi) = 0$ to simplify to the required identity. ■

Remark 1.1: In general, larger n leads to smaller maximum step size h. However, it is *not* true that $n \to \infty$ implies $P \to f$ in L^∞. From the previous theorem, one would need to guarantee $\left\|f^{(n)}\right\| \to 0$ (or at least, doesn't grow faster than $\frac{h^n}{4n}$), which certainly won't hold in general; we have no control on the nth-derivative of an arbitrarily given function. However, we can try to optimize our choice of points $\{x_j\}$ for a given j.

We switch notation for convention's sake to $n+1$ points x_j. Our goal is the optimization problem

$$\min_{x_j} \max_{x \in [a,b]} \left|\prod_j (x - x_j)\right|,$$

the only term in the error bound above that we have control over. Remark that we can expand the product term:

$$\prod_j (x - x_j) = x^n - r(x),$$

where $r(x) \in \mathbb{P}_n$. So, really, we equivalently want to solve the problem

$$\min_{r \in \mathbb{P}_n} \left\|x^{n+1} - r(x)\right\|_\infty,$$

namely, what n-degree polynomial minimizes the max difference between x^{n+1}?

↪**Theorem 1.3** (De la Vallé-Poussin Oscillation Theorem): Let $f \in C([a, b])$, and suppose $r \in \mathbb{P}_n$ for which there exists $n + 2$ distinct points $\{x_j\}$ such that $a \leq x_0 < \cdots < x_{n+1} \leq b$ at which the error $f(x) - r(x)$ "oscillate" sign, i.e.

$$\operatorname{sign}(f(x_j) - r(x_j)) = -\operatorname{sign}(f(x_{j+1}) - r(x_{j+1})).$$

Then,

$$\min_{P \in \mathbb{P}_n} \|f - P\|_\infty \geq \min_{0 \leq j \leq n+1} |f(x_j) - r(x_j)|.$$