

MATH251 - Algebra 2

Vector spaces, linear (in)dependence, span, bases; linear transformations, kernel, image, isomorphisms, nilpotent operators.

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1 INTRODUCTION

Remark 1.1. *This course is about vector spaces and linear transformations between them; a vector space involves multiplication by scalars, where the scalars come from some field. We recall first examples of fields, then vector spaces, as a motivation, before presenting a formal definition.*

1.1 Vector Spaces

Remark 1.2. *Much of this is recall from [Algebra 1](#).*

⊗ Example 1.1: Examples of Fields

1. \mathbb{Q} ; the field of rational numbers.
2. \mathbb{R} ; the field of real numbers; $\mathbb{Q} \subseteq \mathbb{R}$.
3. \mathbb{C} ; the field of complex numbers; $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.
4. $\mathbb{F}_p \equiv \mathbb{Z}/p\mathbb{Z} \equiv \{0, 1, \dots, p-1\}$; the (unique) field of p elements, where p prime.^a
 - (a) $p = 2$; $\mathbb{F}_2 \equiv \{0, 1\}$.
 - (b) $p = 3$; $\mathbb{F}_3 \equiv \{0, 1, 2\}$.
 - (c) \dots

^awhere $a +_p b := \text{remainder of } \frac{a+b}{p}$, $a \cdot_p b := \text{remainder of } \frac{a \cdot b}{p}$.

Remark 1.3. *Throughout the course, we will denote an abstract field as \mathbb{F} .*

⊗ Example 1.2: Examples of Vector Spaces

1. $\mathbb{R}^3 := \{(x, y, z) : x, y, z \in \mathbb{R}\}$. We can add elements in \mathbb{R}^3 , and multiply them by real scalars.
2. $\mathbb{F}^n := \underbrace{\mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F}}_{n \text{ times}} := \{(a_1, a_2, \dots, a_n) : a_i \in \mathbb{F}\}$, where $n \in \mathbb{N}^1$; this is a generalization of the previous example, where we took $n = 3$, $\mathbb{F} = \mathbb{R}$. Operations follow identically; addition:

$$(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) := (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

and, taking a scalar $\lambda \in \mathbb{F}$, multiplication:

$$\lambda \cdot (a_1, a_2, \dots, a_n) := (\lambda \cdot a_1, \lambda \cdot a_2, \dots, \lambda \cdot a_n).$$

We refer to these elements (a_1, \dots, a_n) as *vectors* in \mathbb{F}^n ; the vector for which $a_i = 0 \forall i$ is the *0 vector*, and is the additive identity, making \mathbb{F}^n an abelian group under addition, that admits multiplication by scalars from \mathbb{F} .

3. $C(\mathbb{R}) := \{f : \mathbb{R} \rightarrow \mathbb{R} : f \text{ continuous}\}$. Here, we have the constant zero function as our additive identity ($x \mapsto 0 \forall x$), and addition/scalar multiplication of two continuous real functions are continuous.
4. $\mathbb{F}[t] := \{a_0 + a_1t + a_2t^2 + \cdots + a_nt^n : a_i \in \mathbb{F} \forall i, n \in \mathbb{N}\}$, ie, the set of all polynomials in t with coefficients from \mathbb{F} . Here, we can add two polynomials;

$$(a_0 + a_1t + \cdots + a_nt^n) + (b_0 + b_1t + \cdots + b_mt^m) := \sum_{i=0}^{\max\{n,m\}} (a_i + b_i)t^i,$$

(where we “take” undefined a_i/b_i ’s as 0; that is, if $m > n$, then $a_{m-n}, a_{m-n+1}, \dots, a_m$ are taken to be 0). Scalar multiplication is defined

$$\lambda \cdot (a_0 + a_1t + a_2t^2 + \cdots + a_nt^n) := \lambda a_0 + \lambda a_1t + \lambda a_2t^2 + \cdots + \lambda a_nt^n.$$

Here, the zero polynomial is simply 0 (that is, $a_i = 0 \forall i$).

¹Where we take $0 \in \mathbb{N}$, for sake of consistency. Moreover, by convention, we define \mathbb{F}^0 (that is, when $n = 0$) to be $\{0\}$; the trivial vector space.

↪ Definition 1.1: Vector Space

A *vector space* V over a field \mathbb{F} is an *abelian group* with an operation denoted $+$ (or $+_V$) and identity element² denoted 0_V , equipped with *scalar multiplication* for each scalar $\lambda \in \mathbb{F}$ satisfying the following axioms:

1. $1 \cdot v = v$ for $1 \in \mathbb{F}, \forall v \in V$.
2. $\alpha \cdot (\beta \cdot v) = (\alpha \cdot \beta)v, \forall \alpha, \beta \in \mathbb{F}, v \in V$.
3. $(\alpha + \beta) \cdot v = \alpha \cdot v + \beta \cdot v, \forall \alpha, \beta \in \mathbb{F}, v \in V$.
4. $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v, \forall \alpha \in \mathbb{F}, u, v \in V$.

We refer to elements $v \in V$ as *vectors*.

↪ Proposition 1.1

For a vector space V over a field \mathbb{F} , the following holds:

1. $0 \cdot v = 0_V, \forall v \in V$ (where $0 := 0_{\mathbb{F}}$)
2. $-1 \cdot v = -v, \forall v \in V$ (where $1 := 1_{\mathbb{F}}$)³
3. $\alpha \cdot 0_V = 0_V, \forall \alpha \in \mathbb{F}$

Proof. 1. $0 \cdot v = (0 + 0) \cdot v = 0 \cdot v + 0 \cdot v \implies 0 \cdot v = 0_V$ (by “cancelling” one of the $0 \cdot v$ terms on each side).
2. $v + (-1 \cdot v) = (1 \cdot v + (-1) \cdot v) = (1 - 1) \cdot v = 0 \cdot v = 0_V \implies (-1 \cdot v) = -v$.
3. $\alpha \cdot 0_V = \alpha \cdot (0_V + 0_V) = \alpha \cdot 0_V + \alpha \cdot 0_V \implies \alpha \cdot 0_V = 0_V$ (by, again, cancelling a term on each side).

■

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1.2 Creating Spaces from Other Spaces

²The “zero vector”.

³NB: “additive inverse”

↪ Definition 1.2: Product/Direct Sum of Vector Spaces

For vector spaces U, V over the same field \mathbb{F} , we define their *product* (or *direct sum*) as the set

$$U \times V = \{(u, v) : u \in U, v \in V\},$$

with the operations:

$$(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2)$$

$$\lambda \cdot (u, v) := (\lambda \cdot u, \lambda \cdot v)$$

⊗ Example 1.3: \mathbb{F}^2

$\mathbb{F}^2 = \mathbb{F} \times \mathbb{F}$, where \mathbb{F} is considered as the vector space over \mathbb{F} (itself).

↪ Definition 1.3: Subspace

For a vector space V over a field \mathbb{F} , a *subspace* of V is a subset $W \subseteq V$ s.t.

1. $0_V \in W$ ⁴
2. $u + v \in W \forall u, v \in W$ (closed under addition)
3. $\alpha \cdot u \in W \forall u \in W, \alpha \in \mathbb{F}$ ⁵

Then, W is a vector space in its own right.

⁴This is equivalent to requiring that $W \neq \emptyset$; stated this way, axiom 3. would necessitate that $0 \cdot w = 0_V \in W$.

⁵Note that these axioms are equivalent to saying that W is a subgroup of V with respect to vector addition; 2. ensures closed under addition, and 3. ensures the existence of additive inverses (as per $-1 \cdot v = -v$).

⊗ Example 1.4: Examples of Subspaces

1. Let $V := \mathbb{F}^n$.

- $W := \{(x_1, x_2, \dots, x_n) \in \mathbb{F}^n : x_1 = 0\} = \{(0, x_2, x_3, \dots, x_n) : x_i \in \mathbb{F}\}.$
- $W := \{(x_1, x_2, \dots, x_n) \in \mathbb{F}^n : x_1 + 2 \cdot x_2 = 0\}$

Proof. Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in W$. Then, $x + y = (x_1 + y_1, \dots, x_n + y_n)$, and $x_1 + y_1 + 2 \cdot (x_2 + y_2) = x_1 + 2 \cdot x_2 + y_1 + 2 \cdot y_2 = 0 + 0 = 0 \implies x + y \in W$. Similar logic follows for axioms 2., 3. ■

- (More generally)

$$W := \{(x_1, \dots, x_n) \in \mathbb{F}^n : \begin{array}{l} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ a_{21}x_1 + \dots + a_{2n}x_n = 0 \\ \dots \\ a_{k1}x_1 + \dots + a_{kn}x_n = 0 \end{array} \},$$

that is, a linear combination of homogenous “conditions” on each term.

- $W^* := \{(x_1, \dots, x_n) : x_1 + x_2 = 1\}$ is *not* a subspace; it is not closed under addition, nor under scalar multiplication.

2. Let $\mathbb{F}[t]_n := \{a_0 + a_1t + \dots + a_nt^n : a_i \in \mathbb{F}\}$. Then, $\mathbb{F}[t]_n$ is a subspace of $\mathbb{F}[t]$, the more general polynomial space. *However*, the set of all polynomials of degree *exactly* n (all axioms fail, in fact) is not a subspace of $\mathbb{F}[t]_n$.

- $W := \{p(t) \in \mathbb{F}[t]_n : p(1) = 0\}.$
- $W := \{p(t) \in \mathbb{F}[t]_n : p''(t) + p'(t) + 2p(t) = 0\}.$

3. Let $V := C(\mathbb{R})$ be the space of continuous function $\mathbb{R} \rightarrow \mathbb{R}$.

- $W := \{f \in C(\mathbb{R}) : f(\pi) + 7f(\sqrt{2}) = 0\}.$
- $W := C^1(\mathbb{R}) :=$ everywhere differentiable functions.
- $W := \{f \in C(\mathbb{R}) : \int_0^1 f \, dx = 0\}.$

↪ Proposition 1.2

Let W_1, W_2 be subspaces of a vector space V over \mathbb{F} . Then, define the following:

1. $W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}$
2. $W_1 \cap W_2 := \{w \in V : w \in W_1 \wedge w \in W_2\}$

These are both subspaces of V .

- Proof.
1. (a) $0_V \in W_1$ and $0_V \in W_2 \implies 0_V = 0_V + 0_V \in W_1 + W_2$.
(b) $(u_1 + u_2) + (v_1 + v_2) = (u_1 + v_1) + (u_2 + v_2) \in W_1 + W_2$.
(c) $\alpha \cdot (u + v) = \alpha \cdot u + \alpha \cdot v \in W_1 + W_2$
 2. (a) $0_V \in W_1$ and $0_V \in W_2 \implies 0_V = 0_V + 0_V \in W_1 \cap W_2$.
(b) $u, v \in W_1 \cap W_2 \implies u + v \in W_1 \wedge u + v \in W_2 \implies u + v \in W_1 \cap W_2$.
(c) $\alpha \cdot u \in W_1 \wedge \alpha \cdot u \in W_2 \implies \alpha \cdot u \in W_1 \cap W_2$.

■

1.3 Linear Combinations and Span

↪ Definition 1.4: Linear Combination

Let V be a vector space over a field \mathbb{F} . For finitely many vectors v_1, v_2, \dots, v_n , their *linear combination* is a sum of the form

$$\sum_{i=1}^n a_i v_i = a_1 \cdot v_1 + \dots + a_n \cdot v_n,$$

where $a_i \in \mathbb{F} \forall i$.

A linear combination is called *trivial* if $a_i = 0 \forall i$, that is, all coefficients are 0.

If $n = 0$ (ie, we are “summing up” 0 vectors), we define the sum as the zero vector; $\sum_{i=1}^0 a_i v_i := 0_V$.

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↪ Definition 1.5: A More General Definition of Linear Combination

For a (possibly infinite) set S of vectors from V , a *linear combination* of vectors in S is a linear combination of $a_1 v_1 + \dots + a_n v_n$ for some finite subset $\{v_1, \dots, v_n\} \subseteq S$.⁶

⁶That is, we do not allow infinite sums.

↪ Definition 1.6: Span

For a subset $S \subseteq V$, we define its *span* as

$$\text{Span}(S) := \text{set of all linear combinations of } S := \{a_1v_1 + \cdots + a_nv_n : a_i \in \mathbb{F}, v_i \in S\}.$$

By convention, we set $\text{Span}(\emptyset) = \{0_V\}$.

⊗ Example 1.5

Let $S := \{(1, 0, -1), (0, 1, -1), (1, 1, -2)\} \subseteq \mathbb{R}^3$. Then,

$$0_{\mathbb{R}^3} = (0, 0, 0) = 1 \cdot (1, 0, -1) + 1 \cdot (0, 1, -1) + -1 \cdot (1, 1, -2).$$

We claim, moreover, that $\text{Span}(S) = U := \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$ (a plane through the origin).

Proof. Note that $S \subseteq U$, hence $S \subseteq \text{Span } S \subseteq U$. OTOH, if $(x, y, z) \in U$, we have $z = -x - y$, and so

$$(x, y, z) = (x, y, -x - y) = x \cdot (1, 0, -1) + y \cdot (0, 1, -1) \in \text{Span}(S)$$

hence $U \subseteq \text{Span}(S)$ and thus $\text{Span}(S) = U$. ■

Remark 1.4. We implicitly used the following claim in the proof above; we prove it more generally.

↪ Proposition 1.3

Let V be a vector space over \mathbb{F} and let $S \subseteq V$. Then, $\text{Span}(S)$ is always a subspace. Moreover, it is the smallest (minimal) subspace containing S (that is, for any subspace $U \supseteq S$, we have that $U \supseteq \text{Span } S$).

Proof. Because adding/scalar multiplying linear combinations of elements of S again results in a linear combination of elements of S , and $0_V \in \text{Span}(S)$ by definition, we have that $\text{Span}(S)$ is indeed a subspace.

If $U \supset S$ is a subspace of V containing S , then by definition U is closed under addition, that is, taking linear combinations of its elements (in particular, of elements of S); hence, $U \supset \text{Span}(S)$. ■

↪ Lemma 1.1

For $S \subseteq V$ and $v \in V$, $v \in \text{Span}(S) \iff \text{Span}(S \cup \{v\}) = \text{Span}(S)$.

Proof. (\implies) Let $v \in \text{Span}(S) \implies v = a_1v_1 + \cdots + a_nv_n, a_i \in \mathbb{F}, v_i \in V$. Then, for any linear combination

$$b_1u_1 + \cdots + b_mu_m + b \cdot v = b_1u_1 + \cdots + b_mu_m + b(a_1v_1 + \cdots + a_nv_n)$$

is a linear combination of vectors in $S \cup \{v\}$ (first equality) or equivalently, a combination of vectors in S (second equality) and thus $\text{Span}(S \cup \{v\}) \subseteq \text{Span } S$. The reverse inclusion follows trivially.

$$(\Leftarrow) \text{Span}(S \cup \{v\}) = \text{Span } S \implies v \in \text{Span}(S).$$

⊗ Example 1.6

(From the above example) We have

$$\text{Span}(\{(1, 0, -1), (0, 1, -1)\} \cup \{(1, 1, -2)\}) = \text{Span}(\{(1, 0, -1), (0, 1, -1)\}),$$

since $(1, 1, -2) \in \text{Span}(\{(1, 0, -1), (0, 1, -1)\})$ (it was redundant, as it could be generated by the other two vectors).

↪ Definition 1.7: Spanning Set

Let V be a vector space over a field \mathbb{F} . We call $S \subseteq V$ a *spanning set* for V if $\text{Span}(S) = V$. We call such a spanning set *minimal* if no proper subset of S is a spanning set ($\nexists v \in S$ s.t. $S \setminus \{v\}$ spanning).

Remark 1.5. Note that any $S \subseteq V$ is a spanning for $\text{Span}(S)$. But, S may not be minimal; indeed, consider the previous example. We were able to remove a vector from S while having the same span.

⊗ Example 1.7

For \mathbb{F}^n as a vector space over \mathbb{F} , the *standard spanning set*

$$\text{St}_n := \{\underbrace{(1, \dots, 0)}_{:=e_1}, \underbrace{(0, 1, 0, \dots, 0)}_{:=e_2}, \dots, \underbrace{(0, \dots, 1)}_{e_n}\}.$$

Given any $x := (x_1, \dots, x_n) \in \mathbb{F}^n$, we can write

$$x = x_1 \cdot e_1 + \dots x_n \cdot e_n.$$

This is clearly minimal; removing any e_i would then result in a 0 in the i th “coordinate” of a vector, hence $\text{St} \setminus \{e_i\}$ would span only vectors whose i th coordinate is 0.

↪ Definition 1.8: Linear Dependence

Let V be a vector space over a field \mathbb{F} . A set $S \subseteq V$ is said to be *linearly dependent* if there is a nontrivial linear combination of vectors in S that is equal to 0_V .

Conversely, S is called *linearly independent* if there is no nontrivial linear combination of vectors in S that is equal to 0_V ; all linear combinations of vectors in S that equal 0_V are trivial.

⊗ **Example 1.8**

1. The empty set \emptyset is linearly independent; there are no non-trivial linear combinations that equal 0_V (there are no linear combinations at all).
2. For $v \in V$, the set $\{v\}$ is linearly dependent iff $v = 0_V$.
3. $S := \{(1, 0, -1), (0, 1, -1), (1, 1, -2)\} := \{v_1, v_2, v_3\}$; S is linearly dependent ($v_1 + v_2 - v_3 = (0, 0, 0)$).
4. $V := \mathbb{F}^3$; $S := \{(1, 0, -1), (0, 1, -1), (0, 0, 1)\} = \{v_1, v_2, v_3\}$ is linearly independent.

Proof. Suppose

$$\begin{aligned} a_1 v_1 + a_2 v_2 + a_3 v_3 &= 0_V \\ \implies a_1 &= 0 \wedge a_2 = 0 \wedge -a_1 - a_2 + a_3 = 0 \implies a_3 = 0 \\ \implies a_1 &= a_2 = a_3 = 0 \end{aligned}$$

Hence only a trivial linear combination is possible. ■

5. St_n is linearly independent.

Proof.

$$\sum_{i=1}^n a_i e_i = 0_{\mathbb{F}^n} \implies a_i = 0 \forall i$$

■

↪ **Lemma 1.2**

Let V be a vector space over a field \mathbb{F} , and $S \subseteq V$ (possibly infinite).

1. S is linearly dependent \iff there is a finite subset $S_0 \subseteq S$ that is linearly dependent.
2. S is linearly independent \iff all finite subsets of S are linearly independent.

Proof. 2. follows from the negation of 1.

(\Leftarrow) Trivial.

(\Rightarrow) Suppose S linearly dependent. Then, $0_V =$ some nontrivial linear combination of vectors v_1, \dots, v_n in S . Let $S_0 = \{v_1, \dots, v_n\}$, then, S_0 is linearly dependent itself. ■

1.4 Linear Dependence and Span

↪ Proposition 1.4

Let V be a vector space over a field \mathbb{F} and $S \subseteq V$.

1. S linearly dependent $\iff \exists v \in \text{Span}(S \setminus \{v\})$.
2. S linearly independent \iff there is no $v \in \text{Span}(S \setminus \{v\})$.

Proof. 2. follows from the negation of 1.

(\implies) Suppose S linearly dependent. Then, $0_V = \sum_{i=1}^n a_i v_i$ for some nontrivial linear combination of distinct vectors S . At least one of $a_i \neq 0$; we can assume wlog (reindexing) $a_1 \neq 0$. Then,

$$a_1 v_1 = - \sum_{i=2}^n a_i v_i \implies v_1 = (-a_1^{-1}) \sum_{i=2}^n a_i v_i = \sum_{i=2}^n (-a_1^{-1} a_i) v_i,$$

hence, $v_1 \in \text{Span}(\{v_2, \dots, v_n\}) \subseteq \text{Span}(S \setminus \{v\})$

(\impliedby) Suppose $v \in \text{Span}(S \setminus \{v\})$, then $v = a_1 v_1 + \dots + a_n v_n$, with $v_1, \dots, v_n \in S \setminus \{v\}$, thus

$$0_V = a_1 v_1 + \dots + a_n v_n - v,$$

which is not a trivial combination (-1 on the v ; v cannot “merge” with the other vectors), hence S is linearly dependent. ■

↪ Corollary 1.1

$S \subseteq V$ is linearly independent $\iff S$ a minimal spanning set of $\text{Span } S$.

Proof. Follows from proposition 1.4, 2. ■

↪ Definition 1.9: Maximally Independent

Let V be a vector space over a field \mathbb{F} . A set $S \subseteq V$ is called *maximally independent* if S is linearly independent and $\nexists v \in V \setminus S$ s.t. $S \cup \{v\}$ is still linearly independent.

In other words, there is no proper supset $\tilde{S} \supsetneq S$ that is still independent.

↪ Lemma 1.3

If $S \subseteq V$ maximally independent, then S is spanning for V .

Proof. Let $S \subseteq V$ be maximally independent. Let $v \in V$; supposing $v \notin S$ (in the case that $v \in S$, then $v \in \text{Span}(S)$ trivially). By maximality, $S \cup \{v\}$ is linearly dependent, hence there exists a nontrivial linear

combination that equals 0_V . Since S independent, this combination must include v , with a nonzero coefficient. We can write

$$av + \sum_{i=1}^n a_i v_i = 0_V \quad a \neq 0, v_i \in S$$

$$\implies v = \sum_{i=1}^n (-a^{-1}a_i)v_i \in \text{Span } S.$$

■

↪ **Theorem 1.1**

Let V be a vector space over a field \mathbb{F} and let $S \subseteq V$. TFAE:

1. S is a minimal spanning set;
2. S is linearly independent and spanning;
3. S is a maximally linearly independent set;
4. Every vector in V is equal to *unique* linear combination of vectors in S .

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Proof. (1. \implies 2.) Suppose S is spanning for V and is minimal. Then, by corollary 1.1, we have that S is linearly independent, and is thus both linearly independent and spanning.

(2. \implies 3.) Suppose S is linearly independent and spanning. Let $v \in V \setminus S$; S is spanning, hence $v \in \text{Span } S$, that is, there exists a linear combination of vectors in S that is equal to v :

$$v = a_1 v_1 + \cdots + a_n v_n, a_i \in \mathbb{F}, v_i \in S.$$

Thus, $0_V = a_1 v_1 + \cdots + a_n v_n - v$, thus $S \cup \{v\}$ is linearly dependent, and so S is maximally linearly independent.

(3. \implies 1.) Suppose S is maximally linearly independent. By lemma 1.3, S is spanning, and since S is linearly independent, by corollary 1.1, S is minimally spanning for $\text{Span } S$.

(2. \implies 4.) Suppose S is linearly independent and spans V , and let $v \in V$. We have that $v \in \text{Span } S$ and hence is equal to a linear combination of vectors in S . This gives existence; we now need to prove uniqueness.

Suppose there exist two linear combinations that equal v ,

$$v = a_1 v_1 + \cdots + a_n v_n = b_1 u_1 + \cdots + b_m u_m,$$

$a_i, b_j \in \mathbb{F}, v_i, u_j \in S$. With appropriate reindexing/relabelling and allowing certain scalars to equal 0, we can assume that the combinations use the same vectors (with potentially different coefficients), that is,

$$v = a_1 w_1 + \cdots + a_k w_k = b_1 w_1 + \cdots + a_k w_k.$$

This implies, then,

$$(a_1 - b_1)w_1 + \cdots + (a_k - b_k)w_k = 0_V,$$

and by the assumed linear independent of S , each coefficient $(a_i - b_i) = 0 \forall i \implies a_i = b_i \forall i$, hence, these are indeed the same representations, and thus this representation is unique.

(4. \implies 2.) Suppose every vector in V admits a unique linear combination of vectors in S . Clearly, then, S is spanning. It remains to show S is linearly independent. Suppose

$$0_V = a_1v_1 + \cdots + a_nv_n$$

for $v_i \in S$. But we have that every vector has a unique representation, and we know that $a_i = 0 \forall i$ is a (valid) linear combination that gives 0_V ; hence, this must be the unique combination, $a_i = 0 \forall i$, and the linear combination above is trivial. Hence, S is linearly independent and spanning. ■

↪ **Definition 1.10: Basis**

If any (hence all) of the above statements hold, we call S a *basis* for V .

In the words of 4., we call the unique linear combination of vectors in S that is equal to v the *unique representation of v in S* . Its coefficients are called the *Fourier coefficients of v in S* .

⊗ **Example 1.9**

1. $\text{St}_n = \{e_i : 1 \leq i \leq n\}$ is a basis for \mathbb{F}^n .

2. In \mathbb{F}^3 , the set

$$\{(1, 0, -1), (0, 1, -1), (0, 0, 1)\}$$

is a basis; it is linearly independent and spanning.

3. For $\mathbb{F}[t]_n$, the standard basis is

$$\{1, t, t^2, \dots, t^n\}.$$

4. For $\mathbb{F}[t]$, the standard basis is

$$S := \{1, t, t^2, \dots\} = \{t^n : n \in \mathbb{N}\}.$$

5. Let $\mathbb{F}[[t]]$ denote the space of all formal power series $\sum_{n \in \mathbb{N}} a_n t^n$; polynomials are an example, but with only finite nonzero coefficients. Note that, then, the set S defined above is not a basis for this “extended” set. We *can* in fact find a basis for this set; we need more tools first.

↪ **Theorem 1.2**

Every vector space has a basis.

Remark 1.6. This theorem relies on assuming the Axiom of Choice.

Proof (Attempt). (Of theorem 1.2) We will try to “inductively” build a maximally independent set, as follows:

Begin with an empty set $S_0 := \emptyset$, and iteratively add more vectors to it. Let $v_0 \in V$ be a non-zero vector, and let $S_1 := \{v_0\}$.

If S_1 is maximal, then we are done. Otherwise, there exists a new vector $v_1 \in V \setminus S_1$ s.t. $S_2 := \{v_0, v_1\}$ is still independent.

If S_2 is maximal, then we are done. Otherwise, there exists a new vector $v_2 \in V \setminus S_2$ s.t. $S_3 := \{v_0, v_1, v_2\}$ is still independent.

Continue in this manner; this would take arbitrarily many finite, or even infinite, steps; we would need some “choice function” that would “allow” us to choose any particular i th vector v_i .

We can make this construction precise via the Axiom of Choice and transfinite induction (on ordinals); alternatively, we will prove a statement equivalent to the Axiom of Choice, Zorn’s Lemma. ■

Remark 1.7. Before stating Zorn’s Lemma, we introduce the following terminology.

↪ **Axiom 1.1: Axiom of Choice**

Let X be a set of nonempty sets. Then, there exists a choice function f defined on X that maps each set of X to an element of that set.

↪ **Definition 1.11: Inclusion-Maximal Element**

A *inclusion-maximal* element of I is a set $S \in I$ s.t. there is no strict super set $S' \supsetneq S$ s.t. $S' \in I$.

↪ **Definition 1.12: Chain**

Let X a set. Call a collection $C \subseteq \mathcal{P}(X)$ a *chain* if any two $A, B \in C$ are comparable, ie, $A \subseteq B$ or $B \subseteq A$.

↪ **Definition 1.13: Upper Bound**

An *upper bound* of a collection $\tau \subseteq \mathcal{P}(X)$ is a set $U \subseteq X$ s.t. $U \supseteq J \forall J \in \tau$; U contains the union of all sets in J .

⊗ **Example 1.10: Of The Previous Definitions**

Let $X := \mathbb{N}, I := \{\emptyset, \{0\}, \{1, 2\}, \{1, 2, 3\}\} \subseteq \mathcal{P}(\mathbb{N})$.

The maximal elements of I would be $\{0\}$ and $\{1, 2, 3\}$.

Chains would include $C_0 := \{\emptyset, \{1, 2\}, \{1, 2, 3\}\}, C_1 := \{\emptyset, \{0\}\}, C_2 := \{\emptyset\}$ (or any set containing a single element).

The sets $\{0, 1, 2, 3\}$ and $\{0, 1, 2, 3, 4, 5\}$ are upper bounds for I , while neither is an element of I . The set $\{1, 2, 3\}$ is an upper bound for C_0 . A chain $\{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}, \dots\}$ has an upper bound of \mathbb{N} .

↪ **Lemma 1.4: Zorn's Lemma**

Let X be an ambient set and $I \subseteq \mathcal{P}(X)$ be a nonempty collection of subsets of X . If every chain $C \subseteq I$ has an upper bound in I , then I has a maximal element.

"Proof". This is equivalent to the Axiom of Choice; proving it is beyond the scope of this course :(.

Proof of theorem 1.2, cnt'd. We obtain a maximal independent set using Zorn's Lemma.

Let I be the collection of all linearly independent subsets of V . I is nonempty; $\emptyset \in I$, as is $\{v\} \in I$ for any nonzero $v \in V$. To apply Zorn's, we need to show that every chain C of sets in I has an upper bound in I ; that is, every linearly independent set has an upper bound that itself is linearly independent.

Let C be a chain in I . Let $S := \bigcup C$ be the union of all sets in C . To show S is linearly independent, it suffices to show that every finite subset $\{v_1, \dots, v_n\} \subseteq S$ is linearly independent. Let $S_i \in C$ be s.t. $v_i \in S_i$ for each i . Because C a chain, for each i, j we have either $S_i \subseteq S_j$ or $S_j \subseteq S_i$, and so we can order S_1, \dots, S_n in increasing order w.r.t \subseteq . This implies, then, there is a maximal S_{i_0} s.t. $S_{i_0} \supseteq S_i \forall i \in \{1, \dots, n\}$. Moreover, we have that $\{v_1, \dots, v_n\} \in S_{i_0}$, and that S_{i_0} is linearly independent and thus $\{v_1, v_2, \dots, v_n\}$ is also linearly independent.

Thus, as we can apply Zorn's Lemma, we conclude that I has a maximal element, ie, there is a maximal independent set, and thus a V indeed has a basis.

↪ Lecture 06; Last Updated: Fri Jan 19 13:36:58 EST 2024

↪ **Theorem 1.3**

For every vector space V over a field \mathbb{F} , any two bases $\mathcal{B}_1, \mathcal{B}_2$ are equinumerous/of equal size/cardinality, ie, there is a bijection between \mathcal{B}_1 and \mathcal{B}_2 .

Remark 1.8. We will only prove this for vector spaces that admit a finite basis.

↪ **Lemma 1.5: Steinitz Substitution**

Let V be a vector space over a field \mathbb{F} . Let $Y \subseteq V$ be a (possibly infinite) linearly independent set and let $Z \subseteq V$ be a finite spanning set. Then:

1. $k := |Y| \leq |Z| =: n$
2. There is $Z' \subseteq Z$ of size $n - k$ s.t. $Y \cup Z'$ is still spanning.

Proof. We prove by induction on k .

$k = 0$ gives that $Y = \emptyset$, and so $Z' = Z$ itself works ($Z' \cup Y = Z$) as a spanning set.

Suppose the statement holds for some $k \geq 0$. Let Y be an independent set such that $|Y| = k + 1$, ie

$$Y := \{y_1, y_2, \dots, y_k, y_{k+1}\}, \quad y \in V.$$

By our inductive assumption, we can consider $Y' := \{y_1, \dots, y_k\} \subseteq Y$ of size k , to obtain a set

$$Z' = \{z_1, z_2, \dots, z_{n-k}\} \subseteq Z, \text{ s.t. } Y' \cup Z' = \{y_1, \dots, y_k, z_1, \dots, z_{n-k}\}$$

is spanning. As this is spanning, we can write y_{k+1} as a linear combination of vectors in $Y' \cup Z'$, ie

$$y_{k+1} = a_1 y_1 + \dots + a_k y_k + b_1 z_1 + \dots + b_{n-k} z_{n-k}, \quad a_i, b_j \in \mathbb{F}.$$

It must be that at least one of b_j 's must be nonzero; if they were all zero, then y_{k+1} would simply be a linear combination of vector y_i giving that y_{k+1} linearly dependent, contradicting our construction of Y linearly independent.

Assume, wlog, $b_{n-k} \neq 0$. Then, we can write

$$z_{n-k} = b_{n-k}^{-1} y_{k+1} - b_{n-k}^{-1} a_1 y_1 - \dots - b_{n-k}^{-1} a_k y_k - b_{n-k}^{-1} b_1 z_1 - \dots - b_{n-k}^{-1} b_{n-k-1} z_{n-k-1},$$

and hence

$$z_{n-k} \in \text{Span}\{y_1, \dots, y_{k+1}, z_1, \dots, z_{n-k-1}\} = \text{Span}\left(\underbrace{\{y_1, \dots, y_{k+1}\}}_Y \cup \underbrace{\{z_1, \dots, z_{n-k-1}\}}_{:=Z''}\right).$$

We had that $Y' \cup Z'$ was spanning, and $(Y' \cup Z') \setminus (Y \cup Z'') = \{z_{n-k}\} \subseteq \text{Span}(Y \cup Z'')$, and we thus have that $Y \cup Z''$ is also spanning. ■

↪ **Corollary 1.2: Finite Basis Case for theorem 1.3**

Let V be a vector space that admits a finite basis. Then, any two bases of V are equinumerous.

Proof. Let Y, Z be two finite bases for V . Then, Y is independent and Z is spanning, so by Steinitz Substitution, $|Y| \leq |Z|$. OTOH, Z is independent, and Y is spanning, so by Steinitz Substitution, $|Z| \leq |Y|$, and we conclude

that $|Y| = |Z|$. Let $n := |Y|$.

It remains to show that there exist no infinite bases for V ; it suffices to show that there is no independent set of size $n + 1$. To this end, let $I \subseteq V$ such that $|I| = n + 1$ be an independent set. Y is still spanning, hence, by the substitution lemma, $n + 1 \leq n$, a contradiction. Hence, I as defined cannot exist and so any basis of V must be of size n . ■

↪ Definition 1.14: Dimension

Let V be a vector space over a field \mathbb{F} . The *dimension* of V , denote

$$\dim(V)$$

as the cardinality/size of any basis for V . We call V *finite dimensional* if $\dim(V)$ is a natural number, i.e. V admits a finite basis. Otherwise, we say V is infinite dimensional.

↪ Corollary 1.3: of Steinitz Substitution

Let V be a finite dimensional vector space over \mathbb{F} and denote $n := \dim(V)$. Then:

1. Every linearly independent subset $I \subseteq V$ has size $\leq n$;
2. Every spanning set $S \subseteq V$ for V has size $\geq n$;
3. Every independent set I can be completed to a basis to V , ie, there exists a basis B for V s.t. $I \subseteq B$.

Proof. Fix a basis B for V , $|B| = n$.

1. If I is a independent set, then because B spanning, Steinitz Substitution gives $|I| \leq |B|$.
2. If S spanning for V , then because B is linearly independent, Steinitz Substitution gives $|B| \leq |S|$.
3. Let I be an independent set. Then, because B is spanning, Steinitz Substitution gives $B' \subseteq B$ of size $n - |I|$ s.t. $I \cup B'$ is spanning. Moreover, $|I \cup B'| \leq n$, and by 2. it must have size $\geq n$, and thus has size precisely n and is thus a minimally spanning set and thus a basis.

↪ Corollary 1.4: Monotonicity of Dimension

Let V be a vector space over a field \mathbb{F} . For any subspace $W \subseteq V$, $\dim W \leq \dim V$, and

$$\dim W = \dim V \iff W = V.$$

Proof. Let $B \subseteq W$ be a basis for W . Because B is independent, $|B| \leq \dim(V)$ by 1. of corollary 1.3, so $\dim(W) = |B| \leq \dim(V)$.

If $|B| = \dim(V)$, then B is a basis for V again by 1. of corollary 1.3, so $W = \text{Span}(B) = V$. ■

2 LINEAR TRANSFORMATIONS

2.1 Definitions

↪ **Definition 2.1: Linear Transformation**

Let V, W be vector spaces over a field \mathbb{F} . A function $T : V \rightarrow W$ is called a *linear transformation* if it preserves the vector space structures, that is,

1. $T(v_0 + v_1) = T(v_0) + T(v_1), \forall v_0, v_1 \in V;$
2. $T(\alpha \cdot v) = \alpha \cdot T(v), \forall \alpha \in \mathbb{F}, v \in V;$
3. $T(0_V) = 0_W.$

Remark 2.1. Note that 3. is redundant, implied by 2., but included for emphasis:

$$T(0_V) = T(0_{\mathbb{F}} \cdot 0_V) = 0_{\mathbb{F}} \cdot T(0_V) = 0_W.$$

⊗ **Example 2.1: Linear Transformations**

1. $T : \mathbb{F}^2 \rightarrow \mathbb{F}^2, T(a_1, a_2) := (a_1 + 2a_2, a_1).$
2. Let $\theta \in \mathbb{R}$, and let $T_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the rotation by θ . The linearity of this is perhaps most obvious in polar coordinates, ie $v \in \mathbb{R}^2, v = r(\cos \alpha, \sin \alpha)$ for appropriate r, α , and $T_\theta(v) = r(\cos(\alpha + \theta), \sin(\alpha + \theta)).$
3. $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, a reflection about the x -axis, ie, $T(x, y) = (x, -y).$
4. Projections, $T : \mathbb{F}^n \rightarrow \mathbb{F}^n.$
5. The transpose on $M_n(\mathbb{F})$, ie, $T : M_n(\mathbb{F}) \rightarrow M_n(\mathbb{F})$, where $A \mapsto A^t.$
6. The derivative on space of polynomials of degree leq n , $D : \mathbb{F}[t]_{n+1} \rightarrow \mathbb{F}[t]_n, p(t) \mapsto p'(t).$

↪ **Theorem 2.1**

Linear transformations are completely determined by their values on a basis.

That is, let $\mathcal{B} := \{v_1, \dots, v_n\}$ be a basis for a vector space V over \mathbb{F} . Let W also be a vector space over \mathbb{F} and let $w_1, \dots, w_n \in W$ be arbitrary vectors. Then, there is a unique linear transformation $T : V \rightarrow W$ s.t. $T(v_i) = w_i \forall i = 1, \dots, n.$

Proof. We aim to define $T(v)$ for arbitrary $v \in V$. We can write

$$v = a_1v_1 + \cdots + a_nv_n$$

as the unique representation of v in terms of the basis \mathcal{B} . Then, we simply define

$$T(v) := a_1w_1 + \cdots + a_nw_n,$$

for our given w_i 's. Then, $T(v_i) = 1 \cdot w_i = w_i$, as desired, and T is linear;

1. Let $u, v \in V$; $u := \sum_n a_i v_i, v := \sum_n b_i v_i$. Then,

$$T(u + v) = T\left(\sum_n a_i v_i + \sum_n b_i v_i\right) = T\left(\sum_n (a_i + b_i) v_i\right) = \sum_n (a_i + b_i) w_i = \sum_n a_i w_i + \sum_n b_i w_i = T(u) + T(v).$$

2. Scalar multiplication follows similarly.

To show uniqueness, suppose T_0, T_1 are two linear transformations satisfying $T_0(v_i) = w_i = T_1(v_i)$. Let $v \in V$, and write $v = \sum_n a_i v_i$. By linearity,

$$T_k(v) = T_k\left(\sum_n a_i v_i\right) = \sum_n a_i T(v_i) = \sum_n a_i w_i,$$

for $k = 0, 1$, hence, $T_1(v) = T_0(v)$ for arbitrary v , hence the transformations are equivalent. ■

↪ Definition 2.2: Some Important Transformations

We denote $T_0 : V \rightarrow W$ by $T_0(v) := 0_W \forall v \in V$ the *zero transformation*. We denote $I_V : V \rightarrow V$, $I_V(v) := v \forall v \in V$, as the *identity transformation*.

↪ Lecture 08; Last Updated: Thu Jan 25 12:38:49 EST 2024

2.2 Isomorphisms, Kernel, Image

↪ Definition 2.3: Isomorphism

Let V, W be vector spaces over \mathbb{F} . An *isomorphism* from V to W is a linear transformation $T : V \rightarrow W$ (a homomorphism for vector spaces) which admits an inverse T^{-1} that is also linear.

If such an isomorphism exists, we say V and W are *isomorphic*.

↪ Proposition 2.1

$T : V \rightarrow W$ is an isomorphism $\iff T$ is linear and bijective.

Proof. The direction \implies is trivial. ■

Suppose $T : V \rightarrow W$ is linear and bijective, ie T^{-1} exists. We need to show that T^{-1} is linear. Let $w_1, w_2 \in W, a_1, a_2 \in \mathbb{F}$. Then:

$$\begin{aligned} T^{-1}(a_1 w_1 + a_2 w_2) &= T^{-1}(a_1 T(T^{-1}(w_1)) + a_2 T(T^{-1}(w_2))) \\ (\text{by linearity of } T) \quad &= T^{-1}(T(a_1 T^{-1}(w_1) + a_2 T^{-1}(w_2))) \\ &= a_1 T^{-1}(w_1) + a_2 T^{-1}(w_2). \end{aligned}$$

Remark 2.2. This proposition holds for all structures that only have operations; it does not for those with relations, such as graphs, orders, etc..

↪ **Theorem 2.2**

For $n \in \mathbb{N}$, every n -dimensional vector space V over \mathbb{F} is isomorphic to \mathbb{F}^n . In particular, all n -dim vector spaces over \mathbb{F} are isomorphic.

Proof. Fix a basis $\mathcal{B} := \{v_1, \dots, v_n\}$ for V , and let $T : V \rightarrow \mathbb{F}^n$ be the unique linear transformation determined by \mathcal{B} with $T(v_i) = e_i$, where $\{e_1, \dots, e_n\}$ is the standard basis for \mathbb{F}^n . We show that T is a bijection.

(Injective) Suppose $T(x) = T(y), x, y \in V$. Write $x = a_1 v_1 + \dots + a_n v_n, y = b_1 v_1 + \dots + b_n v_n$, the unique representation of x, y in the basis \mathcal{B} . We have:

$$a_1 e_1 + \dots + a_n e_n = a_1 T(v_1) + \dots + a_n T(v_n) = T(a_1 v_1 + \dots + a_n v_n) = T(x) = T(y) = \dots = b_1 e_1 + \dots + b_n e_n,$$

but by the uniqueness of representation in a basis, it follows that each $a_i = b_i$, hence, $x = y$.

(Surjective) Let $w \in \mathbb{F}^n$. Then, $w = a_1 e_1 + \dots + a_n e_n$ (uniquely). But then,

$$w = a_1 T(v_1) + \dots + a_n T(v_n) = T(a_1 v_1 + \dots + a_n v_n),$$

where $a_1 v_1 + \dots + a_n v_n \in V$, hence T indeed surjective. ■

Remark 2.3. Replacing \mathbb{F}^n with an arbitrary n -dim vector space W over \mathbb{F} yields the following.

↪ **Theorem 2.3: Freeness of Vector Space**

Let W, V be vector spaces over \mathbb{F} and let β, γ be bases for V, W respectively. Every bijection $T : \beta \rightarrow \gamma$ can be extended to an isomorphism $\hat{T} : V \rightarrow W$.

In particular, all vector spaces over \mathbb{F} with equinumerous bases are isomorphic.

Remark 2.4. The proof follows very similarly to the previous theorem, but extended to arbitrary, possible infinite, spaces.

Proof. ■

↪ Definition 2.4: Image/Kernel

For a linear transformation $T : V \rightarrow W$, where V, W are vector spaces over \mathbb{F} , we define the *image*

$$\text{Im}(T) := T(V),$$

and its *kernel*

$$\text{Ker}(T) = T^{-1}(\{0_W\}).$$

↪ Proposition 2.2

$\text{Ker}(T)$ and $\text{Im } T$ are subspaces of V, W resp.

Proof. ($\text{Ker}(T)$) Let $v_0, v_1 \in \text{Ker } T$ and $a_0, a_1 \in \mathbb{F}$, then

$$T(a_0v_0 + a_1v_1) = a_0T(v_0) + a_1T(v_1) = 0_W \implies a_0v_0 + a_1v_1 \in \text{Ker } T.$$

($\text{Im}(T)$) Let $w_0, w_1 \in \text{Im } T$, $a_0, a_1 \in \mathbb{F}$. Then $w_i = T(v_i)$, $v_i \in V$, and so

$$a_0w_0 + a_1w_1 = a_0T(v_0) + a_1T(v_1) = T(a_0v_0 + a_1v_1) \implies a_0w_0 + a_1w_1 \in \text{Im } T.$$

■

↪ Proposition 2.3

Let $T : V \rightarrow W$ be a linear transformation, where V, W vector spaces over \mathbb{F} . Let β be a (possibly infinite) basis for V . Then, $T(\beta)$ spans $\text{Im}(T)$.

In particular, T is surjective iff $T(\beta)$ spans W .

Proof. Let $w \in \text{Im}(T)$, so $w = T(v)$ for some $v \in V$, where we have $v := a_1v_1 + \cdots + a_nv_n$, $v_i \in \beta$. Then,

$$w = T(v) = a_1T(v_1) + \cdots + a_nT(v_n) \in \text{Span}(\{T(v_1), \dots, T(v_n)\}) \subseteq \text{Span}(T(\beta)).$$

■

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↪ Proposition 2.4

Let $T : V \rightarrow W$ be a linear transformation, where V, W vector spaces over \mathbb{F} . TFAE:

1. T is injective.
2. $\text{Ker}(T)$ is the trivial subspace $\{0_V\}$.
3. $T(\beta)$ is independent for each basis β for V .
- 3'. $T(\beta)$ is independent for some basis β for V .

Proof. (1. \implies 2.) Trivial; only 0_V can be mapped to 0_W .

(2. \implies 1.) Suppose $\text{Ker}(T) = \{0_V\}$ and let $T(x) = T(y)$, $x, y \in V$. By linearity,

$$T(x - y) = T(x) - T(y) = 0_W \implies x - y \in \text{Ker}(T) \implies x - y = 0_V \implies x = y.$$

(2. \implies 3.) Fix a basis β for V . To show that $T(\beta)$ linearly independent, take an arbitrary linear combination $a_1w_1 + \cdots + a_nw_n \in T(\beta)$. Suppose $\sum_i a_iw_i = 0_W$. Since $w_i \in T(\beta)$, $w_i = T(v_i)$, $v_i \in \beta$, hence

$$\begin{aligned} 0_W = a_1w_1 + \cdots + a_nw_n &= a_1T(v_1) + \cdots + a_nT(v_n) = T(a_1v_1 + \cdots + a_nv_n) \\ &\implies a_1v_1 + \cdots + a_nv_n \in \text{Ker}(T) \\ &\implies a_1v_1 + \cdots + a_nv_n = 0_V, \end{aligned}$$

but each v_i is linearly independent, hence this must be a trivial linear combination, and thus $a_i = 0 \forall i$.

(3) \implies (3') Trivial; stronger statement implies weaker statement.

(3') \implies (2) Suppose $T(\beta)$ linearly independent for some basis β for V . Suppose $T(v) = 0_W$, $v \in V$. We write

$$v = a_1v_1 + \cdots + a_nv_n, v_i \in \beta.$$

Then,

$$0_W = T(v) = T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n),$$

but $\{T(v_i)\} \subseteq T(\beta)$ is linearly independent, hence, this combination must be trivial and each $a_i = 0$, and thus $v = 0_V$ and so $\text{Ker}(T) = \{0_V\}$ is trivial. ■

↪ **Definition 2.5: Rank, nullity**

Let V, W be vector spaces over \mathbb{F} and $T : V \rightarrow W$ be linear. Define *rank* of T as

$$\text{rank}(T) := \dim(\text{Im}(T)),$$

and *nullity* of T as

$$\text{nullity}(T) := \dim(\text{Ker}(T)).$$

↪ **Theorem 2.4: Rank-Nullity Theorem**

Let V, W be vector spaces over \mathbb{F} , $\dim(V) < \infty$. Let $T : V \rightarrow W$ be a linear transformation. Then,

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

Remark 2.5. *Intuitively: the nullity is the number of vectors we “collapse”; the rank is what is left. Together, we have the entire space.*

Remark 2.6. *This follows directly from the first isomorphism theorem for vector spaces, and the fact that $\dim(V/\text{Ker}(T)) = \dim(V) - \dim(\text{Ker}(T))$; however, we will prove it without this result below.*

Proof. Let $\{v_1, \dots, v_k\}$ be a basis for $\text{Ker}(T)$, and complete it to a basis $\beta := \{v_1, \dots, v_k, u_1, \dots, u_{n-k}\}$ for V , where $n := \dim(V)$. We need to show that $\dim(\text{Im}(T)) = n - k$.

Recall that $\{T(v_1), \dots, T(v_k), T(u_1), \dots, T(u_{n-k})\}$ spans $\text{Im}(T)$. But $v_1, \dots, v_k \in \text{Ker}(T)$, so $T(v_i) = 0_W \forall i = 1, \dots, k$. Hence, letting $\gamma := \{T(u_1), \dots, T(u_{n-k})\}$ spans $\text{Im}(T)$. It remains to show that γ is independent.

Let $a_1 T(u_1) + \dots + a_{n-k} T(u_{n-k}) = 0_W$; by linearity,

$$\begin{aligned} T(a_1 u_1 + \dots + a_{n-k} u_{n-k}) &= 0_W \\ \implies a_1 u_1 + \dots + a_{n-k} u_{n-k} &\in \text{Ker}(T) \\ \implies a_1 u_1 + \dots + a_{n-k} u_{n-k} &= b_1 v_1 + \dots + b_k v_k, \end{aligned}$$

but each of these $u_i, v_j \in \beta$, hence, each coefficient must be identically zero as β linearly independent, and thus $\dim(\text{Im}(T)) = n - k$. This completes the proof. ■

↪ **Corollary 2.1: Pigeonhole Principle for Dimension**

Let $T : V \rightarrow W$ be a linear transformation. If T injective, then $\dim(W) \geq \dim(V)$.

Proof. If $\dim(V) < \infty$, then $\dim(\text{Im}(T)) = \dim(V)$, and we have that $\dim(\text{Im}(T)) \leq \dim(W)$ and conclude $\dim(V) \leq \dim(W)$.

If $\dim(V) = \infty$, then $\dim(\text{Im}(T)) = \infty$ and $\dim(W) \geq \dim(\text{Im}(T)) = \infty$. ■

↪ **Corollary 2.2**

Let $n \in \mathbb{N}$ and V, W be n -dimensional vector spaces over \mathbb{F} . For a linear transformation $T : V \rightarrow W$, TFAE:

1. T injective;
2. T surjective;
3. $\text{rank}(T) = n$.

Proof. (2. \iff 3.) Follows from $\text{rank}(T) = \dim(\text{Im}(T)) = n \iff \text{Im}(T) = W$.

(1. \implies 3.) We have $\text{nullity}(T) = 0$ so $\text{rank}(T) = \dim(V) = n$.

(3. \implies 1.) If $\text{rank}(T) = n$, then $\text{nullity}(T) = 0$. ■

↪ Lecture 10; Last Updated: Mon Feb 5 14:03:23 EST 2024

↪ **Theorem 2.5: First Isomorphism Theorem for Vector Spaces**

Let V, W be vector spaces over \mathbb{F} . Let $T : V \rightarrow W$ be a linear transformation. Then,

$$V/\text{Ker}(T) \cong \text{Im}(T),$$

by the isomorphism given by $v + \text{Ker}(T) \mapsto T(v)$.

Proof. From group theory, we know that $\hat{T} : V/\text{Ker}(T) \rightarrow \text{Im}(T)$, where $\hat{T}(v + \text{Ker}(T)) := T(v)$ is well-defined, and is an isomorphism of abelian groups. We need only to check that \hat{T} is linear, namely, that it respects scalar multiplication. We have

$$\begin{aligned}\hat{T}(a \cdot (v + \text{Ker}(T))) &= \hat{T}((a \cdot v) + \text{Ker}(T)) \\ &= T(av) = a \cdot T(v) \\ &= a\hat{T}(v + \text{Ker}(T)),\end{aligned}$$

as desired. ■

2.3 The Space $\text{Hom}(V, W)$

↪ Definition 2.6: Homomorphism Space

For vector spaces V, W over \mathbb{F} , let $\text{Hom}(V, W)$ (also denoted $\ell(V, W)$) denote the set of all linear transformations from V to W . We can turn this into a vector space over \mathbb{F} as follows:

1. *Addition of linear transformations:* for $T_0, T_1 \in \text{Hom}(V, W)$, define

$$(T_0 + T_1) : V \rightarrow W, \quad v \mapsto T_0(v) + T_1(v).$$

$(T_0 + T_1)$ is clearly a linear transformation, as the linear combination of linear transformations T_0, T_1 .

2. *Scalar multiplication of linear transformations:* for $T \in \text{Hom}(V, W)$, $a \in \mathbb{F}$, define

$$(a \cdot T) : V \rightarrow W, \quad v \mapsto a \cdot T(v),$$

which is again clearly linear in its own right.

↪ Proposition 2.5

Endowed with the operations described above, $\text{Hom}(V, W)$ is a vector space over \mathbb{F} .

Proof. Follows easily from the definitions. ■

↪ Theorem 2.6: Basis for $\text{Hom}(V, W)$

For vector spaces V, W over \mathbb{F} and bases β, γ for V, W resp., the following set

$$\{T_{v,w} = v \in \beta, w \in \gamma\},$$

is a basis for $\text{Hom}(V, W)$, where for each $v \in \beta$ and $w \in \gamma$, $T_{v,w} \in \text{Hom}(V, W)$ defined as the unique linear transformation such that

$$T_{v,w}(v') = \begin{cases} w & v' = v \\ 0_W & v' \neq v \iff v' \in \beta \setminus \{v\} \end{cases}.$$

Proof. Left as a (homework) exercise. ■

↪ Corollary 2.3

If V, W finite dimensional, then $\dim(\text{Hom}(V, W)) = \dim(V) \cdot \dim(W)$.

↪ **Proposition 2.6**

Let $\beta = \{v_1, \dots, v_n\}$, $\gamma = \{w_1, \dots, w_m\}$ be bases for V, W resp. Then, by theorem 2.6,

$$\{T_{v_i, w_j} : i \in \{1, \dots, n\}, j \in \{1, \dots, m\}\}$$

is a basis for $\text{Hom}(V, W)$, and it has $n \cdot m$ vectors by construction.

2.4 Matrix Representation of Linear Transformations, Finite Fields

Consider a linear transformation $T : \mathbb{F}^n \rightarrow \mathbb{F}^m$ between finite fields. We know that T is uniquely determined by its value of basis vectors, so fix the standard bases

$$\beta = \{e_1^{(n)}, \dots, e_n^{(n)}\} = \{v_1, \dots, v_n\},$$

and note that T is determined by $\{T(v_1), \dots, T(v_n)\} \subseteq \mathbb{F}^m$.

Remark 2.7. We denote vectors in \mathbb{F}^n as column vectors, ie $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$.

Each $T(v_i)$ is a column vector in \mathbb{F}^m , and we can put these into a $m \times n$ matrix, namely:⁷

$$[T] := \begin{pmatrix} | & & | \\ T(v_1) & \cdots & T(v_n) \\ | & & | \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_n$$

We call this the *matrix representation* of T in the standard bases. The operation of multiplying an $m \times n$ matrix and a $n \times 1$ vector is precisely defined so that

↪ **Proposition 2.7**

$T(v) = [T] \cdot v$ for all $v \in \mathbb{F}^n$.

⁷Where $[T]$ denotes a matrix named “ T ”.

Proof. Let $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, where $v = x_1v_1 + \cdots + x_nv_n$. Then

$$T(v) = x_1T(v_1) + \cdots + x_nT(v_n)$$

$$T(v_i) = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{pmatrix}$$

so

$$T(v) = \begin{pmatrix} a_{11} \cdot x_1 + \cdots + a_{1n} \cdot x_n \\ \vdots \\ a_{m1} \cdot x_1 + \cdots + a_{mn} \cdot x_n \end{pmatrix} = [T] \cdot v$$

■

↪ Definition 2.7

For a given $m \times n$ matrix A over \mathbb{F} , define $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ by $L_A(v) := A \cdot v$, where v is viewed as an $n \times 1$ column. It follows from definition that the L_A is linear.

In other words, every $T \in \text{Hom}(\mathbb{F}^n, \mathbb{F}^m)$ is equal to L_A for some A .

↪ Lecture 11; Last Updated: Fri Feb 9 14:12:09 EST 2024

↪ Proposition 2.8

The map

$$\begin{aligned} \text{Hom}(\mathbb{F}^n, \mathbb{F}^m) &\rightarrow M_{m \times n}(\mathbb{F}) \\ T &\mapsto [T] \end{aligned}$$

is an isomorphism of vector spaces, with inverse

$$\begin{aligned} M_{m \times n}(\mathbb{F}) &\rightarrow \text{Hom}(\mathbb{F}^n, \mathbb{F}^m) \\ A &\mapsto L_A. \end{aligned}$$

Proof. Linearity: Let $\beta = \{v_1, \dots, v_n\}$ be the standard basis for \mathbb{F}^n . Fix $T_1, T_2 \in \text{Hom}(\mathbb{F}^n, \mathbb{F}^m)$ and $\alpha \in \mathbb{F}$.

1.

$$\begin{aligned}
[T_1 + T_2] &= \begin{pmatrix} \cdots & \begin{array}{c} | \\ (T_1 + T_2)(v_i) \\ | \end{array} & \cdots \end{pmatrix} = \begin{pmatrix} \cdots & \begin{array}{c} | \\ T_1(v_i) + T_2(v_i) \\ | \end{array} & \cdots \end{pmatrix} \\
&= \begin{pmatrix} \cdots & \begin{array}{c} | \\ T_1(v_i) \\ | \end{array} & \cdots \end{pmatrix} + \begin{pmatrix} \cdots & \begin{array}{c} | \\ T_2(v_i) \\ | \end{array} & \cdots \end{pmatrix} \\
&= [T_1] + [T_2]
\end{aligned}$$

2. It remains to show that $\alpha \cdot [T] = [\alpha \cdot T]$; the proof follows similarly to 1.

Inverse: We need to show that 1. $A \mapsto L_A \mapsto [L_A]$ is the identity on $M_{m \times n}(\mathbb{F})$, and conversely, that 2. $T \mapsto [T] \mapsto L_{[T]}$ is the identity on $\text{Hom}(\mathbb{F}^n, \mathbb{F}^m)$.

1. We need to show that $[L_A] = A$. The j th column of $[L_A]$ is $L_A(v_j) = A \cdot v_j = j$ th column of $A =: A^{(j)}$. Hence, the j th column of $[L_A]$ is equal to the j th column of A , and thus they are equal.
2. We showed this in proposition 2.7.

■

↪ Corollary 2.4

$$\dim(\text{Hom}(\mathbb{F}^n, \mathbb{F}^m)) = \dim(M_{m \times n}(\mathbb{F})) = m \cdot n.$$

Remark 2.8. This was stated previously in proposition 2.6 by constructing an explicit basis. Indeed, this basis is precisely the image of the standard basis for $M_{m \times n}(\mathbb{F})$ under the map $A \mapsto L_A$.

2.5 Matrix Representation of Linear Transformations, General Spaces

Remark 2.9. The previous section was concerned with representing transformations between finite fields $\mathbb{F}^n, \mathbb{F}^m$; this section aims to make the same construction for any finite dimensional V, W .

↪ Definition 2.8: Coordinate Vector

Let V be a finite dimensional space over \mathbb{F} and let $\beta := \{v_1, \dots, v_n\}$ be a basis for V . Let $v \in V$, with (unique) representation $v = a_1 v_1 + \dots + a_n v_n$. We denote

$$[v]_\beta := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{F}^n$$

the coordinate vector of v in base β .

Remark 2.10. Recall that $V \cong \mathbb{F}^n$ where $\dim(V) = n$, by the unique linear transformation $v_i \mapsto e_i$, where $\{e_1, \dots, e_n\}$ the standard basis for \mathbb{F}^n . We denote this transformation

$$I_\beta : V \rightarrow \mathbb{F}^n.$$

For an arbitrary $v \in V$, $I_\beta(v)$ maps v to its coordinate vector:

$$\begin{aligned} I_\beta(v) &= I_\beta(a_1v_1 + \dots + a_nv_n) = a_1I_\beta(v_1) + \dots + a_nI_\beta(v_n) \\ &= a_1e_1 + \dots + a_ne_n = [v]_\beta. \end{aligned}$$

↪ Proposition 2.9

The map

$$I_\beta : V \rightarrow \mathbb{F}^n, \quad v \mapsto [v]_\beta$$

is an isomorphism.

Suppose we are given a linear transformation $T : V \rightarrow W$, where V, W finite dimensional spaces over \mathbb{F} . Fix $\beta := \{v_1, \dots, v_n\}$ and $\gamma := \{w_1, \dots, w_m\}$ as bases for V, W resp. We can denote $[T(v_i)]_\gamma$ as $T(v_i)$ in base γ (in the field m), and construct a matrix for T :⁸

$$[T]_\beta^\gamma := \begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix}$$

We call this the *matrix representation* of T from β to γ .

↪ Theorem 2.7

Let $T : V \rightarrow W, \beta, \gamma$ as above.

1. The following diagram commutes:

$$\begin{array}{ccc} \bullet V & \xrightarrow{T} & \bullet W \\ I_\beta \downarrow & & \downarrow I_\gamma \\ \bullet \mathbb{F}^n & \xrightarrow{L_{[T]_\beta^\gamma}} & \bullet \mathbb{F}^m \end{array}$$

Namely, $I_\gamma \circ T = L_{[T]_\beta^\gamma} \circ I_\beta$, or equivalently, given $v \in V$, $[T(v)]_\gamma = [T]_\beta^\gamma \cdot [v]_\beta$.

2. The map $\text{Hom}(V, W) \rightarrow M_{m \times n}(\mathbb{F}), T \mapsto [T]_\beta^\gamma$ is a vector space isomorphism with inverse begin the map $M_{m \times n}(\mathbb{F}) \rightarrow \text{Hom}(V, W), A \mapsto I_\gamma^{-1} \circ L_A \circ I_\beta$

⁸Where we denote $[T]_\beta^\gamma$ as the matrix representation of the transform $T : V \rightarrow W$, with basis β, γ for V, W respectively.

Proof. 2. is left as a (homework) exercise; it follows directly from 1.

Fix $v \in V$. We need to show that $I_\gamma \circ T(v) = L_{[T]_\beta^\gamma} \circ I_\beta(v)$. We have

$$I_\gamma \circ T(v) = [T(v)]_\gamma.$$

OTOH,

$$L_{[T]_\beta^\gamma} \circ I_\beta(v) = L_{[T]_\beta^\gamma}([v]_\beta) = [T]_\beta^\gamma \cdot [v]_\beta.$$

We need to show, then, that $[T(v)]_\gamma = [T]_\beta^\gamma \cdot [v]_\beta$. Let $v = a_1v_1 + \cdots + a_nv_n$, so $[v]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$. Recall that

$$[T]_\beta^\gamma = \begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix}. \text{ Thus, we have}$$

$$\begin{aligned} [T]_\beta^\gamma \cdot [v]_\beta &= a_1[T(v_1)]_\gamma + \cdots + a_n[T(v_n)]_\gamma = [a_1T(v_1) + \cdots + a_nT(v_n)]_\gamma \quad (\text{by linearity of } I_\gamma) \\ &= [T(a_1v_1 + \cdots + a_nv_n)]_\gamma \quad (\text{by linearity of } T) \\ &= [T(v)]_\gamma, \end{aligned}$$

which is precisely what we wanted to show. ■

Remark 2.11. For $A \in M_{m \times n}(\mathbb{F})$ and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$, we have

$$A \cdot x = x_1 \cdot A^{(1)} + x_2 \cdot A^{(2)} + \cdots + x_n \cdot A^{(n)},$$

where $A^{(j)}$ is the j th column of A ; thus $A \cdot x$ is a linear combination of A , with coefficients given by the vector x ; this interpretation can make it easier to make sense of computations.

↔ Lecture 12; Last Updated: Fri Feb 9 11:12:11 EST 2024

2.6 Composition of Linear Transformations, Matrix Multiplication

↪ Proposition 2.10

Composition is associative; given $T : V \rightarrow W$, $S : W \rightarrow U$, and $R : U \rightarrow X$, then

$$(R \circ S) \circ T = R \circ (S \circ T).$$

Proof. Fix $v \in V$. Then

$$(R \circ S) \circ T(v) = (R \circ S)(T(v)) = R(S(T(v)))$$

OTOH:

$$R \circ (S \circ T)(v) = R((S \circ T)(v)) = R(S(T(v))).$$

■

Let $A \in M_{m \times n}(\mathbb{F})$ and $B \in M_{l \times m}(\mathbb{F})$. Then, $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^m$ and $L_B : \mathbb{F}^m \rightarrow \mathbb{F}^l$, and have composition $L_B \circ L_A : \mathbb{F}^n \rightarrow \mathbb{F}^l$. We know that $L_B \circ L_A$ is a linear transformation, and thus must be equal to L_C for some matrix $C \in M_{l \times n}(\mathbb{F})$. Indeed, C is the matrix representation of the transformation $[L_B \circ L_A]$, as proven previously.

Let $\beta = \{e_1, \dots, e_n\}$ for \mathbb{F}^n , then

$$[L_B \circ L_A] = \begin{pmatrix} | & & | \\ L_B \circ L_A(e_1) & \cdots & L_B \circ L_A(e_n) \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ B \cdot (A \cdot e_1) & \cdots & B \cdot (A \cdot e_n) \\ | & & | \end{pmatrix}$$

↪ Definition 2.9: Matrix Multiplication

For matrices $A \in M_{m \times n}(\mathbb{F})$ and $B \in M_{l \times m}(\mathbb{F})$, define their product $B \cdot A$ to be the matrix

$$[L_B \circ L_A] = \begin{pmatrix} | & & | \\ B \cdot (A \cdot e_1) & \cdots & B \cdot (A \cdot e_n) \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ B \cdot A^{(1)} & \cdots & B \cdot A^{(n)} \\ | & & | \end{pmatrix} = (c_{ij})_{1 \leq i \leq l, 1 \leq j \leq n}$$

where $A^{(j)}$ is the j th column of A , $c_{ij} := \begin{pmatrix} - & B_{(i)} & - \end{pmatrix} \cdot \begin{pmatrix} | \\ A^{(j)} \\ | \end{pmatrix}$.

↪ Proposition 2.11

$[L_B \circ L_A] = B \cdot A$, ie $L_B \circ L_A = L_{B \cdot A}$.

Proof. Follows from our definition. ■

↪ Corollary 2.5

Matrix multiplication is association; $C \cdot (B \cdot A) = (C \cdot B) \cdot A$ for $A \in M_{m \times n}(\mathbb{F})$, $B \in M_{l \times m}(\mathbb{F})$, $C \in M_{k \times l}(\mathbb{F})$.

Proof. $C \cdot (B \cdot A) = [L_C \circ (L_B \circ L_A)] = [(L_C \circ L_B) \circ L_A] = (C \cdot B) \cdot A$. ■

Remark 2.12. This is proven by the linear transformation representation of matrices; try proving this directly from our definition.

↪ Corollary 2.6

Let V, W, U be finite-dimensional vector spaces over \mathbb{F} , $T : V \rightarrow W, S : W \rightarrow U$ be linear transformations and α, β, γ be bases for V, W, U resp. Then,

$$[S \circ T]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta}.$$

Proof. Follows from the commutativity of the diagrams:

$$\begin{array}{ccccc} V & \xrightarrow{T} & W & \xrightarrow{S} & U \\ \wr \downarrow & & \wr \downarrow & & \wr \downarrow \\ \mathbb{F}^n & \xrightarrow{[T]_{\alpha}^{\beta}} & \mathbb{F}^m & \xrightarrow{[S]_{\beta}^{\gamma}} & \mathbb{F}^l \end{array} \iff \begin{array}{ccc} V & \xrightarrow{T \circ S} & U \\ \wr \downarrow & & \wr \downarrow \\ \mathbb{F}^n & \xrightarrow{[S \circ T]_{\alpha}^{\gamma}} & \mathbb{F}^l \end{array}$$

In “words”, for $v \in V$,

$$[S \circ T]_{\alpha}^{\gamma} \cdot [v]_{\alpha} = [(S \circ T)(v)]_{\alpha}^{\gamma} = [S(T(v))]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma} \cdot [T(v)]_{\beta} = [S]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta} \cdot [v]_{\alpha},$$

ie we have shown that $L_{[S \circ T]_{\alpha}^{\gamma}} = L_{[S]_{\beta}^{\gamma}} \cdot L_{[T]_{\alpha}^{\beta}}$. Because $A \mapsto L_A$ is an isomorphism, it follows that $[S \circ T]_{\alpha}^{\gamma} = [S]_{\beta}^{\gamma} \cdot [T]_{\alpha}^{\beta}$. ■

↪ Lecture 13; Last Updated: Sat Feb 3 22:20:36 EST 2024

2.7 Inverses of Transformations and Matrices

Remark 2.13. Recall that, given a function $f : X \rightarrow Y$, a function $g : Y \rightarrow X$ is called

1. a left inverse of f if $g \circ f = \text{Id}_X$;
2. a right inverse of f if $f \circ g = \text{Id}_Y$;
3. a (two-sided) inverse of f if g both a left and right inverse of f .

If an inverse exists, it is unique; let g_0, g_1 be inverse of f , then, $g_0 = g_0 \circ (f \circ g_1) = (g_0 \circ f) \circ g_1 = g_1$.

↪ Proposition 2.12

Let $f : X \rightarrow Y$. Then,

1. f has a left-inverse $\iff f$ injective;
2. f has a right-inverse $\iff f$ surjective;
3. f has an inverse $\iff f$ bijective.

Proof. ((a), \implies) Suppose $g : Y \rightarrow X$ is a left-inverse of f and $f(x_1) = f(x_2)$. Then, $g \circ f(x_1) = g \circ f(x_2) \implies x_1 = x_2$ and so f injective.

((b), \implies) Suppose $g : Y \rightarrow X$ is a right-inverse of f and let $y \in Y$. Then, $f(g(y)) = y \implies y \in f(X)$.

The remainder of the cases and directions are left as an exercise. ■

Remark 2.14. *Proof of (b), \Leftarrow uses Axiom of Choice.*

⊗ Example 2.2

1. The differentiation transform $\delta : \mathbb{F}[t]_{n+1} \rightarrow \mathbb{F}[t]_n, p(t) \mapsto p'(t)$ has a right inverse, the integration transform, $\iota : \mathbb{F}[t]_n \rightarrow \mathbb{F}[t]_{n+1}, p(t) \mapsto \text{antiderivative of } p(t)$; conversely, ι has left inverse δ ; they do not admit inverses.
2. Let $f : \mathbb{F}[[t]] \rightarrow \mathbb{F}[[t]]$ be the left-shift map, where $\sum_{n=0}^{\infty} a_n t^n \mapsto \sum_{n=1}^{\infty} a_n t^{n-1}$. Then, $g : \mathbb{F}[[t]] \rightarrow \mathbb{F}[[t]]$ with $\sum_{n=0}^{\infty} a_n t^n \mapsto \sum_{n=0}^{\infty} a_n t^{n+1}$, the right-shift map, is a right inverse of f , but f has no left inverse (it is not injective).

Remark 2.15. *The existence of only one-sided inverses existing happens only when in infinite-dimensional vectors spaces, or when the dimension of the domain is not the same as the dimension of the codomain.*

↪ Corollary 2.7: Of Rank-Nullity Theorem

Let $T : V \rightarrow W$ s.t. $\dim(V) = \dim(W) < \infty$. TFAE:

1. T has a left-inverse;
2. T has a right-inverse;
3. T is invertible (has an inverse).

Proof. We have already that T injective $\iff T$ surjective $\iff T$ bijective. ■

↪ Definition 2.10: Matrix Inverse

We call a $n \times n$ matrix B over \mathbb{F} the *inverse* of an $n \times n$ matrix A over \mathbb{F} if $A \cdot B = B \cdot A = I_n$. We denote $B = A^{-1}$.

↪ Proposition 2.13

Let $A \in M_n(\mathbb{F})$. Then,

1. L_A is invertible $\iff A$ is invertible, in which case $L_A^{-1} = L_{A^{-1}}$;
2. A is invertible \iff it has a left-inverse, ie $B \cdot A = I_n \iff$ it has a right-inverse, ie $A \cdot B = I_n$.

Proof. 1. L_A invertible $\iff \exists T : \mathbb{F}^n \rightarrow \mathbb{F}^n$ -linear s.t. $L_A \circ T = T \circ L_A = I_{\mathbb{F}^n} \iff \exists$ a matrix $B \in M_n(\mathbb{F})$ such that $L_A \circ L_B = L_B \circ L_A = I_{\mathbb{F}^n} \iff$ there is a matrix $B \in M_n(\mathbb{F})$ s.t. $L_{AB} = L_{BA} = I_{\mathbb{F}^n} \iff$ there is a $B \in M_n(\mathbb{F})$ s.t. $A \cdot B = B \cdot A = I_n$.

2. Follows directly from corollary 2.7 and part 1. ■

2.8 Invariant Subspaces and Nilpotent Transformations

↪ Definition 2.11: T -Invariant

Let $T : V \rightarrow V$ be a linear transformation.⁹ We call a subspace $W \subseteq V$ T -invariant if $T(W) \subseteq W$.

⊗ Example 2.3: Examples of Invariant Subspaces

1. For any $T : V \rightarrow V$, $\text{Im}(T)$ is T -invariant.
2. For any $T : V \rightarrow V$, $\text{Ker}(T)$ is T -invariant, since $T(v) = 0_V \in \text{Ker}(T) \forall v \in \text{Ker}(T)$. Moreover, for any $n \in \mathbb{N}$, the space $\text{Ker}(T^n)$ is T -invariant.¹⁰

↪ Lecture 14; Last Updated: Mon Feb 12 08:34:27 EST 2024

↪ Proposition 2.14

For a linear operator $T : V \rightarrow V$, the following hold:

1. $V \supseteq \text{Im}(T) \supseteq \text{Im}(T^2) \supseteq \dots \supseteq \text{Im}(T^n) \supseteq \dots$. Moreover, $\text{Im}(T^n)$ is T -invariant for any $n \in \mathbb{N}$.
2. $\{0_V\} \subseteq \text{Ker}(T) \subseteq \text{Ker}(T^2) \subseteq \dots \subseteq \text{Ker}(T^n) \subseteq \dots$. Moreover, $\text{Ker}(T^n)$ is T -invariant for any $n \in \mathbb{N}$.

Proof. 1. If $x \in \text{Im}(T^{n+1})$, then $x = T^{n+1}(y) = T^n(T(y)) \in \text{Im}(T^n)$ for some $y \in V$, hence $\text{Im}(T^{n+1}) \subseteq \text{Im}(T^n)$.
If $x \in \text{Im}(T^n)$, then $x = T^n(y)$ so $T(x) = T(T^n(y)) = T^n(T(y)) \in \text{Im}(T^n)$, so $T(\text{Im}(T^n)) \subseteq \text{Im}(T^n)$.

2. If $x \in \text{Ker}(T^n)$, then $T^{n+1}(x) = T(T^n(x)) = T(0_V) = 0_V$ hence $x \in \text{Ker}(T^{n+1})$ so $\text{Ker}(T^n) \subseteq \text{Ker}(T^{n+1})$.
Moreover, $T(x) \in \text{Ker}(T^n)$ since $T(x) \in \text{Ker}(T^{n-1}) \subseteq \text{Ker}(T^n)$, since $T^{n-1}(T(x)) = T^n(x) = 0_V$ so $T(\text{Ker}(T^n)) \subseteq \text{Ker}(T^n)$. ■

⁹Because the domain and codomain are the same, we often call T a “linear operator”.

¹⁰ $T^n := T \circ T \circ \dots \circ T$, n times; $T^0 := I_V$.

⊗ **Example 2.4: More Examples of Invariant Subspaces**

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T(x, y, z) := (2x + y, 3x - y, 7z)$. Then, the $x - y$ plane, $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ is T -invariant, as is the z axis, $\{(x, y, z) \in \mathbb{R}^3 : x = y = 0\}$. Hence, we can decompose \mathbb{R}^3 into two T -invariant subspaces, namely $x - y$ plane \oplus z -axis.

↪ **Definition 2.12: Nilpotent**

In a ring R , an element $r \in R$ is called *nilpotent* if $r^n = 0$ for some $n \in \mathbb{N}^+$.

A linear transformation $T : V \rightarrow V$ is called nilpotent if $T^n = 0$ for some $n \in \mathbb{N}^+$.¹¹

For a matrix $A \in M_n(\mathbb{F})$, A is called nilpotent if $A^n = 0_n$ for some $n \in \mathbb{N}^+$.

¹¹One can verify that all linear transformations $T : V \rightarrow V$ from a vector space to itself form a ring with $(\circ, +)$, ie composition and (“standard”) addition of transformations. The same holds for linear operators defined over an abelian group (where the same $+$ operation is endowed by the ring).

⊗ **Example 2.5: Examples of Nilpotent Transformations**

1. Let V , n -dimensional vector space over \mathbb{F} with basis $\beta := \{v_1, \dots, v_n\}$. Let $T : V \rightarrow V$ be the unique linear transformation that “shifts” β : ie, $T(v_1) := 0_V$, $T(v_2) := v_1, \dots, T(v_n) = v_{n-1}$.
2. The differentiation operation, $\delta : \mathbb{F}[t]_n \rightarrow \mathbb{F}[t]_n$ is nilpotent, since $\delta^{n+1} = 0$ for any polynomial.
3. For any matrix $A \in M_n(\mathbb{F})$, A is nilpotent iff $L_A : \mathbb{F}^n \rightarrow \mathbb{F}^n$ is nilpotent.

Proof. $L_{A^k} = L_A^k \implies A^k = 0 \iff L_{A^k} = 0 \iff L_A^k = 0$ ■

4. $n \times n$ matrices that are strictly upper triangular¹² are nilpotent. For instance, for 3×3 , we need to show¹³

$$\begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}^3 = 0 \iff \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}^2 \cdot \begin{pmatrix} * \\ * \\ * \end{pmatrix} = 0$$

We have:

$$\begin{aligned} \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} * \\ * \\ * \end{pmatrix} &= \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned}$$

↪ **Proposition 2.15**

If V is n -dimensional and $T : V \rightarrow V$ is a linear nilpotent transformation, then $T^n = 0$.

Proof. Left as a (homework) exercise. ■

¹²ie zeros everywhere except cells strictly above diagonal.

¹³Where we denote arbitrary elements $*$; different $*$ s are not necessarily equal.

↪ Definition 2.13: Domain Restriction

For a function $f : X \rightarrow Y$ and $A \subseteq X$, we define the *restriction* of f to A as the function $f|_A : A \rightarrow Y$ given by $a \mapsto f(a)$.

↪ Definition 2.14: Direct Sum

Let V be a vector space over \mathbb{F} , and let $W_0, W_1 \subseteq V$ be subspaces of V . If

1. $W_0 \cap W_1 = \{0_V\}$ (the subspaces are *linearly independent*), and
2. $W_0 + W_1 = \{w_0 + w_1 : w_0 \in W_0, w_1 \in W_1\} = V$,

we write $V = W_0 \oplus W_1$, and say V is the *direct sum* of W_0, W_1 .

↪ Theorem 2.8: Fitting's Lemma

For finite dimensional vector space V over \mathbb{F} and a linear transformation $T : V \rightarrow V$, there is a decomposition

$$V = U \oplus W$$

as a direct sum of T -invariant subspaces U, W such that $T|_U : U \rightarrow U$ is nilpotent and $T|_W : W \rightarrow W$ is an isomorphism.

Proof. Recall that $\text{Im}(T) \supseteq \cdots \supseteq \text{Im}(T^n)$ and $\text{Ker}(T) \subseteq \cdots \subseteq \text{Ker}(T^n)$. Both of these become constant eventually, ie the inequalities become strict equalities, hence $\exists N \in \mathbb{N}^+$ such that $\forall k \in \mathbb{N}$, $\text{Im}(T^{N+k}) = \text{Im}(T^N)$ and $\text{Ker}(T^{N+k}) = \text{Ker}(T^N)$.

Let $U := \text{Ker}(T^N)$ and $W := \text{Im}(T^N)$. These are clearly T -invariant.

$T^N(\text{Ker}(T^N)) = \{0_V\}$, and $T(\text{Im}(T^N)) = \text{Im}(T^{N+1}) = \text{Im}(T^N) = W$ and thus $T|_W : W \rightarrow W$ is surjective and hence $T|_W$ must be injective and thus an isomorphism.

It remains to show that $V = U \oplus W$. If $v \in U \cap W$, $T^N(v) = 0_V$ but $T|_W$ an isomorphism so $T^N(v) = 0 \iff v = 0_V$, hence $U \cap W = \{0_V\}$.

Thus, we have $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W) = \dim(U) + \dim(W) = \dim(V)$; moreover, it must be that $U + W = V$.¹⁴ ■

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2.9 Dual Spaces

¹⁴It is precisely here that we use finiteness of V .

↪ Definition 2.15: Dual Space

For a vector space V over a field \mathbb{F} , linear transformations from $V \rightarrow \mathbb{F}$ (where we view \mathbb{F} as a one-dimensional vector space over \mathbb{F}) are called *linear functionals*. The space of linear functionals (namely, $\text{Hom}(V, \mathbb{F})$) is denoted V^* , and called the *dual space* of V .

↪ Proposition 2.16

If V is finite dimensional, $\dim(V^*) = \dim(V)$.¹⁵

Proof. For finite dimensional V , we know that $\dim(\text{Hom}(V, \mathbb{F})) = \dim(V) \cdot \dim(\mathbb{F}) = \dim(V)$, hence $\dim(V^*) = \dim(V)$. In the same notation with which we proved this originally in proposition 2.6; fix a basis $\beta := \{v_1, \dots, v_n\}$ for V and the standard basis $\gamma := \{1\}$ for \mathbb{F} , and defined $\beta^* := \{f_1, \dots, f_n\}$, where $f_i := T_{v_i, 1} : V \rightarrow \mathbb{F}$ maps $v_i \mapsto 1$ and every other basis vector to 0. ■

Remark 2.16. The basis β^* for V^* is called the *dual basis*. Explicitly, we have:

↪ Corollary 2.8

Let V be a finite dimensional vector space over \mathbb{F} and let $\beta := \{v_1, \dots, v_n\}$ be a basis for V . Then,

$$\beta^* := \{f_1, \dots, f_n\}$$

is a basis for V^* . Moreover, for each linear functional $f \in V^*$,

$$f = \sum_{i=1}^n f(v_i) \cdot f_i.$$

Proof. Linear independence: let $a_1 f_1 + \dots + a_n f_n = 0_{V^*} =: 0$. Then,

$$(a_1 f_1 + \dots + a_n f_n)(v_i) = a_i f_i(v_i) = a_i \cdot 1 = a_i \implies a_i = 0,$$

hence β^* indeed linearly independent.

Spanning: let $f \in V^*$. We claim that $f = \sum_{i=1}^n f(v_i) f_i$. It suffices to show these two sides are equal on the basis vectors, as linear transformations are determined by their effect on basis vectors. We have:

$$\left(\sum_{i=1}^n f(v_i) f_i \right)(v_j) = \sum_{i=1}^n f(v_i) f_i(v_j) = \sum_{i=1}^n f(v_i) \cdot \delta_{ij} = f(v_j),$$

as desired.¹⁶ ■

¹⁵This does *not* hold for infinite dimensional spaces.

¹⁶Where $\delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ is the Kronecker delta.

⊗ **Example 2.6**

1. Let $V := \mathbb{F}^n$ and $\beta := \{v_1, \dots, v_n\}$ be a basis for \mathbb{F}^n , viewed as column vectors, and let $\beta^* := \{f_1, \dots, f_n\}$ be the dual basis for V^* . Recall that $f_i : \mathbb{F}^n \rightarrow \mathbb{F}$, hence $f_i := L_{A_i}$ for some matrix $A_i \in M_{1 \times n}(\mathbb{F}) := \text{space of } 1 \times n \text{ row vectors}$. Hence, $A_i = e_i^t$.
2. Consider V^{**} , the dual of the dual. If V is finite-dimensional, then as $\dim(V) = \dim(V^*)$, we have $\dim(V) = \dim(V^*) = \dim(V^{**})$, ie, they are (abstractly) isomorphic.
We have that $T : V \rightarrow V^*$, $v_i \mapsto f_i$ is an isomorphism; we define an explicit isomorphism to V^{**} below.

↪ **Definition 2.16**

Let V be an arbitrary vector space over \mathbb{F} . For each $x \in V$, define $\hat{x} \in V^{**}$ by $\hat{x} : V^* \rightarrow \mathbb{F}$, where $\hat{x}(f) := f(x)$.

Remark 2.17. Note that \hat{x} is linear.

↪ **Theorem 2.9**

The map $x \mapsto \hat{x} : V \rightarrow V^{**}$ is a linear injection. In particular, if V is finite dimensional, it is an isomorphism.

Proof. Let $x \in V$ and suppose $\hat{x} = 0_{V^{**}}$. Let β be a basis for V and β^* its dual basis. Let $x = a_1v_1 + \dots + a_nv_n$ for $v_i \in \beta, a_i \in \mathbb{F}$. Let f_i such that $f_i(v_j) = \delta_{ij}$. Then,

$$\hat{x}f_i = f_i(x) = f_i(a_1v_1 + \dots + a_nv_n) = a_i = 0,$$

hence, $a_i = 0 \forall i$. Hence, $x = 0$, and thus \hat{x} has a trivial kernel and is thus injective. ■

↪ Lecture 16; Last Updated: Mon Feb 12 13:34:17 EST 2024

Remark 2.18. Notice that to get an isomorphism $V \cong V^*$, we fixed a basis for V to define it. However, for $V \cong V^{**}$, we had a canonical isomorphism independent of choice of basis. Writing $S \subseteq V$, $\hat{S} := \{\hat{x} : x \in S\} \subseteq V^{**}$, our theorem says that $\hat{V} = V^{**}$ for finite-dimensional V .

↪ **Definition 2.17: Annihilator**

Let V be a vector space over \mathbb{F} and $S \subseteq V$. We call

$$S^\perp := \{f \in V^* : f|_S = 0\} = \{f \in V^* : f(u) = 0 \forall u \in S\}$$

the *annihilator* of S .

↪ **Proposition 2.17**

Let V be a vector space over \mathbb{F} and $S \subseteq V$.

1. S^\perp is a subspace of V^{*17}
2. $S_1 \subseteq S_2 \subseteq V \implies S_1^\perp \supseteq S_2^\perp$
3. $S^\perp = (\text{Span}(S))^\perp$

Proof. 1. If $f_1, f_2 \in S^\perp, a \in \mathbb{F}$, then $\forall u \in S$,

$$(af_1 + f_2)(u) = af_1(u) + f_2(u) = a \cdot 0 + 0,$$

so $af_1 + f_2 \in S^\perp$.

2. Clear.

3. If $f \in V^*$ takes all vectors in S to 0, then it does the same for linear combinations. ■

↪ **Proposition 2.18**

If V is finite dimensional and $U \subseteq V$ a subspace, then $(U^\perp)^\perp = \hat{U}$.

Proof. We know that $V^{**} = \hat{V}$, so we fix $\hat{x} \in \hat{V}$ and show that

$$\hat{x} \in (U^\perp)^\perp \iff \hat{x} \in \hat{U} \iff x \in U.$$

We have

$$\hat{x} \in (U^\perp)^\perp : \iff \forall f \in U^\perp, \hat{x}(f) = f(x) = 0$$

hence if $x \in U$, then $\hat{x} \in (U^\perp)^\perp$, so $\hat{U} \subseteq (U^\perp)^\perp$.

Conversely, let $\hat{x} \in (U^\perp)^\perp$. Then, $\forall f \in U^\perp, f(x) = 0$.

Suppose towards a contradiction that $x \notin U$. We aim to define $f \in U^\perp$ s.t. $f(x) = 1$, obtaining a contradiction. Let $\{u_1, \dots, u_k\}$ be a basis for U , noting that $\{u_1, \dots, u_k, x\}$ still linearly independent by assumption of $x \notin U = \text{Span}(\{u_1, \dots, u_k\})$. Thus, we can extend this to a basis $\beta = \{u_1, \dots, u_k, x, v_1, \dots, v_m\}$ for V . Define $f : V \rightarrow \mathbb{F} \in V^*$ as the unique linear transformation such that $f(u_i) = f(v_j) = 0$ and $f(x) = 1$. Then, $f \in U^\perp$ by definition, and $f(x) = 1$ by definition. This is a contradiction that $x \notin U$. ■

¹⁷Even if S is not a subspace itself.

↪ Corollary 2.9

For a finite dimensional V and subspace $U \subseteq V$,

$$U = \{x \in V : \forall f \in U^\perp, f(x) = 0\}.$$

↪ Definition 2.18: Dual/Transpose of T

Let V, W be vector spaces over a field \mathbb{F} and $T : V \rightarrow W$. We define the *dual/transpose* of T is the map $T^t : W^* \rightarrow V^*$, given by $g \mapsto g \circ T$. Ie, $T^t(g)(v) := g \circ T(v) = g(T(v))$.

↪ Proposition 2.19

Let V, W be vector spaces over a field \mathbb{F} and $T : V \rightarrow W$.

1. T^t is linear.
2. $\text{Ker}(T^t) = (\text{Im}(T))^\perp$.
3. $\text{Im}(T^t) \subseteq (\text{Ker}(T))^\perp$ and is equal if V, W are finite dimensional.
4. If V, W are finite dimensional and β, γ are bases resp., then

$$[T^t]_{\gamma^*}^{\beta^*} = ([T]_{\beta}^{\gamma})^t.$$

Proof. 1. $T^t(ag_1 + g_2) = (ag_1 + g_2) \circ T = a \cdot g_1 \circ T + g_2 \circ T = a \cdot T^t(g_1) + T^t(g_2), \forall g_1, g_2 \in W^*, a \in \mathbb{F}$.

2. For $g \in W^*$,

$$\begin{aligned} g \in \text{Ker}(T^*) : &\iff T^t(g) = 0_{V^*} \iff T^t(g)(v) = 0 \forall v \in V \\ &\iff g(T(v)) = 0 \forall v \in V \\ &\iff g(w) = 0 \forall w \in \text{Im}(T) \\ &\iff g \in (\text{Im}(T))^\perp \end{aligned}$$

3. Fix $f = T^t(g) \in \text{Im}(T^t)$, $g \in W^*$, and $u \in \text{Ker}(T)$, noting that $f(u) = T^t(g)(u) = g(T(u)) = g(0_W) = 0$ so $f \in (\text{Ker}(T))^\perp$.

Suppose now V, W are finite dimensional; we've shown an inclusion, so it suffices now to show that

$\dim(\text{Im}(T^t)) = \dim(\text{Ker}(T))^\perp$. We have:

$$\begin{aligned}\dim(\text{Im}(T^t)) &= \dim(W^*) - \dim(\text{Ker}(T^t)) \\ &= \dim(W) - \dim(\text{Im}(T)^\perp) \\ &= \dim(W) - \dim(W) + \dim(\text{Im}(T)) \\ &= \dim(\text{Im}(T))\end{aligned}$$

OTOH:

$$\dim(\text{Ker}(T)^\perp) = \dim(V) - \dim(\text{Ker}(T)) = \dim(\text{Im}(T)),$$

and thus $\dim(\text{Im}(T^t)) = \dim(\text{Ker}(T))^\perp$ (remarking that the first equality follows from 1. of the following theorem, and 2. from the dimension theorem).

4. Let $\beta := \{v_1, \dots, v_n\}, \gamma := \{w_1, \dots, w_m\}$ be finite bases for V, W resp. Recall that

$$A := [T]_\beta^\gamma := \begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix},$$

ie $A^{(j)} = [T(v_j)]_\gamma$ hence $T(v_j) = \sum_{k=1}^m A_{kj} w_k$.

Similarly, write $\gamma^* := \{g_1, \dots, g_m\}$ and $\beta := \{f_1, \dots, f_n\}$, then

$$B := [T^t]_{\gamma^*}^{\beta^*} := \begin{pmatrix} | & & | \\ [T^t(g_1)]_{\beta^*} & \cdots & [T^t(g_m)]_{\beta^*} \\ | & & | \end{pmatrix},$$

so $T^t(g_i) = \sum_{\ell=1}^n B_{\ell i} f_\ell = \sum_{\ell=1}^n T^t(g_i)(v_\ell) f_\ell$, so $B_{\ell i} = T^t(g_i)(v_\ell)$. To complete the proof, we must show that $A_{ij} = B_{ji}$ for all i, j :

$$B_{ji} = T^t(g_i)(v_j) = g_i(T(v_j)) = g_i\left(\sum_{k=1}^m A_{kj} w_k\right) = \sum_{k=1}^m A_{kj} g_i(w_k) = A_{ij},$$

where the last equality $g_i(w_k) = \delta_{ik}$, by construction. ■

↪ **Theorem 2.10**

Let V be a finite-dimensional vector space over \mathbb{F} and $U \subseteq V$ be a subspace.

1. $\dim(U^\perp) = \dim(V) - \dim(U)$. In fact, if $\{v_1, \dots, v_k\}$ is a basis for U and $\beta := \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ is a basis for V with the dual basis $\beta^* = \{f_1, \dots, f_n\}$, then $\{f_{k+1}, \dots, f_n\}$ is a basis for U^\perp .
2. $(V/U)^* \cong U^\perp$ by the map $f \mapsto f_U$, where $f_U : V \rightarrow \mathbb{F}$ given by $f_U(v) := f(v + U)$.

Proof. Left as a (homework) exercise. ■

↪ **Corollary 2.10: of proposition 2.19**

Let V, W be vector spaces over \mathbb{F} and $T : V \rightarrow W$ be a linear transformation.

1. T^t injective $\iff T$ surjective.
2. If V, W finite dimensional, then T^t surjective $\iff T$ injective.

Proof. 1. T^t injective $\iff \text{Ker}(T^t) = \{0_{W^*}\} \iff \text{Im}(T)^\perp = \{0_{W^*}\} \implies {}^\circ \text{Im}(T) = W \iff T$ surjective.
Conversely, if $\text{Im}(T) = W \implies (\text{Im}(T))^t = \{0_{W^*}\}$ (and the rest follows identically).

2. $\text{Im}(T^t) = \text{Ker}(T)^\perp \implies \text{Im}(T^\perp) = V^* \iff \text{Ker}(T) = \{0_V\}$, following similar logic to above. ■

Remark 2.19. Part 4. of proposition 2.19 establishes a dependency between the columns and rows of a matrix; precisely:

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2.9.1 Application to Matrix Rank

↪ **Definition 2.19: Matrix Rank/C-Rank, R-Rank**

For a matrix $A \in M_{m \times n}(\mathbb{F})$, we define

$$\text{rank}(A) := \text{rank}(L_A)$$

and the *column rank* of

$$\text{c-rank}(A) := \text{size of maximal indep. subset of columns } \{A^{(1)}, \dots, A^{(n)}\}$$

and *row rank* of

$$\text{r-rank}(A) := \text{size of maximal indep. subset of rows } \{A_{(1)}, \dots, A_{(m)}\}.$$

Remark 2.20. Notice that $\text{rank}(A) = \text{c-rank}(A)$.

↪ **Corollary 2.11**

$$\text{rank}(A) = \text{rank}(A^t) = \text{r-rank}(A)$$

Proof. We know already that $\text{rank}(A^t) = \text{c-rank}(A^t) = \text{r-rank}(A)$, as remarked previously, hence we need only to show that $\text{rank}(A^t) = \text{rank}(A)$. But $A = [L_A]$ and $A^t = [L_{A^t}] = [L_A]^t = [L_A^t]$. Thus, $\text{rank}(A) = \text{rank}(L_A) = \text{rank}(L_A^t) = \text{rank}(A^t)$. ■

↪ **Corollary 2.12**

$$\text{rank}(A) = \text{c-rank}(A) = \text{r-rank}(A), \quad \forall A \in M_{m \times n}(\mathbb{F})$$

2.10 Systems of Linear Equations

We can write a system of m equations of n unknowns x_i

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \quad \quad \quad \ddots \quad \quad \quad \ddots \quad \quad \quad \ddots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{cases}$$

succinctly as a matrix equation

$$A \cdot \vec{x} = \vec{b},$$

where $A := (a_{ij}) \in M_{m \times n}(\mathbb{F})$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, and $\vec{b} := \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{F}^m$. Hence, \vec{x} solves $A\vec{x} = \vec{b} \iff L_A(\vec{x}) = \vec{b} \iff$

$\vec{x} \in L_A^{-1}(\vec{b})$. In other words, a solution exists iff $\vec{b} \in \text{Im}(L_A) = \text{Span}(A^{(1)}, \dots, A^{(n)})$. In particular, when $\vec{b} = \vec{0}$, a solution always exists, $\vec{x} = \vec{0}$. We call $A \cdot \vec{x} = \vec{0}$ the *homogeneous system of equations* of A .

It follows that $A \cdot \vec{x} = \vec{0}$ has nonzero solutions $\iff \text{Ker}(L_A)$ non-trivial. Moreover, if $A \cdot \vec{x} = \vec{b}$ and $A \cdot \vec{y} = \vec{0}$, then $A \cdot (\vec{x} + \vec{y}) = \vec{b}$ as well by linearity.

↪ **Proposition 2.20**

For $A \in M_{m \times n}(\mathbb{F})$ and $b \in \text{Im}(L_A)$ the set of solutions to $A\vec{x} = \vec{b}$ is precisely the coset $\vec{v} + \text{Ker}(L_A)$ where $\vec{v} \in \mathbb{F}^n$ is a particular solution to $A\vec{x} = \vec{b}$; $A\vec{v} = \vec{b}$.

Proof. $\vec{v} +$ an element of $\text{Ker}(L_A)$ is a solution to $A\vec{x} = \vec{b}$. Conversely, if \vec{v}, \vec{w} are solutions to $A\vec{x} = \vec{b}$, then $A \cdot (\vec{v} - \vec{w}) = \vec{b} - \vec{b} = \vec{0}$ so $\vec{v} - \vec{w} \in \text{Ker}(L_A)$, thus $\vec{w} = \vec{v} + (\vec{v} - \vec{w}) \in \vec{v} + \text{Ker}(L_A)$. ■

↪ Corollary 2.13

If $m < n$ and $A \in M_{m \times n}(\mathbb{F})$, then there is always a nonzero solution to the homogeneous equation $A\vec{x} = \vec{0}$

Proof. nullity $(L_A) = n - \text{rank}(L_A) = n - \dim(\text{Im}(L_A)) \geq n - m > 0$ hence $\text{Ker}(L_A)$ nontrivial. ■

↪ Lecture 19; Last Updated: Fri Feb 16 14:23:00 EST 2024