Louis Meunier

Nonlinear Dynamics and Chaos

MATH376

Contents

1 Introduction, Motivations, Overview

Definition 1.1 (*Dynamical Systems*)

Systems which evolve over time. We can categorize them as

- ***continuous,*** *which define ODEs, eg.* $\dot{u}(t) = f(u)$*, where* $u(t)$ *is define over some interval* t*;*
- ***discrete,*** *which are defined by a* map, $S : \mathbb{R}^n \longrightarrow \mathbb{R}^n, u \longmapsto S_u$*, where* $u_1 = S_{u_0}$*,* $u_2 = S_{u_1}$*, etc., where* $n \in \mathbb{Z}$*.*

1.1 Examples of continuous dynamical systems

1.1.1 Exponential growth/decay

Consider $\dot{u} = \lambda u$, where $u \in \mathbb{R}$, $\dot{u} = \frac{\mathrm{d}u}{\mathrm{d}t}$ ($u = f(t)$), and λ is a constant parameter. This is a linear, separable ODE;

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= \lambda u \\ \int \frac{\mathrm{d}u}{u} &= \int \lambda \,\mathrm{d}t \\ \implies u(t) &= u_0 e^{\lambda t}, \text{ where } u(0) = u_0 \end{aligned}$$

Assuming $\lambda \neq 0$ (otherwise $u(t) = u_0$), we can analyze the behavior of u as $t \longrightarrow \infty$ and $t \longrightarrow -\infty$.

- Clearly, if $u_0 = 0$, then $u(t) = 0 \forall t$. This is called a **steady state**.
- Else, (under the assumption $u_0 > 0$), we can consider the cases $\lambda > 0$ and $\lambda < 0$.
  - $\lambda > 0 \implies \lim_{t \to \infty} u(t) = \infty$
  - $\lambda < 0 \implies \lim_{t \to -\infty} u(t) = 0$

Using this fairly simple analysis, we can draw **phase diagrams** describing how the system changes based on initial conditions, for instance, given $\lambda > 0$:

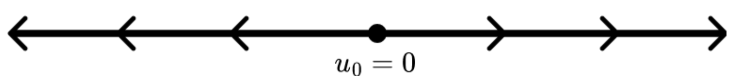


Figure 1: Exponential growth/decay, $\lambda > 0$

Note that the phase diagram is *independent* of the value of λ; naturally, a larger λ will result in a "faster" (so to speak) growth/decay, but the "asymptotic" behavior is identical. We can say that the **dynamics** of the system are independent of the constant λ.

In this case, at $u_0 = 0$, all other u_0 greater than or less than 0 diverge away from u_0; this would be called a *unstable equilibrium*. If $\lambda < 0$, we would see all u_0 converging to 0, which would be an *asymptotically stable equilibrium*.

1.1.2 Logistic ODE

Consider the *logistic* ode $\dot{x} = \lambda x(1-x), x(t) \in \mathbb{R}$. Normally, we would solve this ode (using separation of variables, resulting in a messy fraction decomposition, and lots of algebraic manipulation [1]) This will give the final explicit solution

$$x(t) = \frac{1}{(\frac{1}{x_0} - 1)e^{-\lambda t} + 1},$$

where $x(0) = x_0$. We can then analyze $x(t)$ similar to the previous example. Due to the complexity (the "embedded" exponential, etc), however, this is quite difficult.

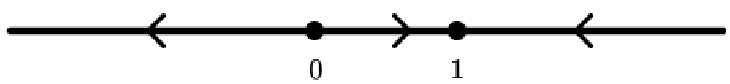
Alternatively, we can consider the original ode

$$\dot{x} = f(x) = \lambda x(1-x),$$

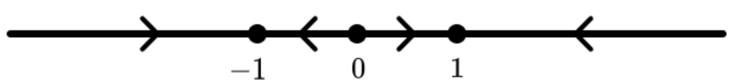
without the exact solutions. Assuming $\lambda > 0$ (similar "methodology" for $\lambda < 0$), we can analyze the behavior:

- Steady states will occur when $\dot{x} = 0 \implies \lambda x(1-x) = 0 \implies x = 0$ or $x = 1$.
- Next, we can consider the behavior of x in the intervals $(-\infty, 0)$, $(0, 1)$, and $(1, \infty)$.
  - $(-\infty, 0)$: as $x \to -\infty$, $\dot{x} \to -\infty$ $((+) \times (-) \times (+) \sim (-))$
  - $(0, 1)$: as $x \to 1$, $\dot{x}$ increases.
  - $(1, \infty)$: as $x \to \infty$, $\dot{x} \to -\infty$ $((+) \times (+) \times (-) \sim (-))$

This means that $x_0 = 0$ and $x_0 = 1$ are unstable and stable, respectively. We can then draw the phase diagram:



We can compare the logistic ODE to the seemingly unrelated[2] $\dot{x} = x - x^3$. Factoring, we write $\dot{x} = x(1-x)(1+x)$, indicating steady states at $x = -1, 0, 1$. This results in a very similar phase diagram:



As is clear from the diagram, the two equations have very similar dynamics; however, at no initial condition does the second ODE tend to positive/negative infinity.

[1] Note that this is also a *Bernoulli ODE*, ie one of the form $y' + p(x)y = q(x)y^n$; you can solve it by dividing by y^n and making the substitution $u = y^{1-n}$ to get a far nicer (though obviously equivalent) linear equation which you can solve using an integrating factor.

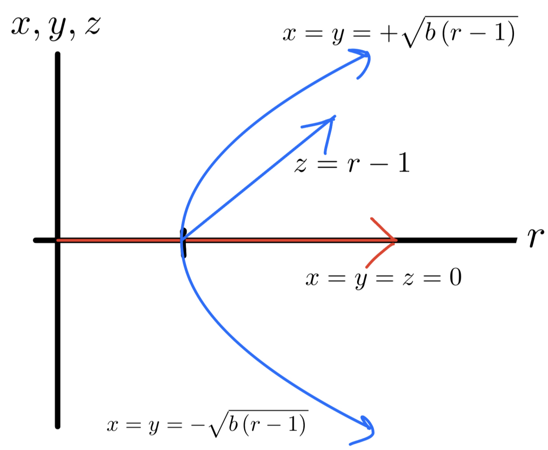
[2] Also a Bernoulli

1.2 Analyzing the Lorenz Equations

The Lorenz equation is defined by

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases},$$

where solutions $u(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \in \mathbb{R}^3$. A trivial steady state exists at $(x, y, z) = (0, 0, 0), \forall r > 0$, and two more exist[3] $(x, y, z) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1), \forall r > 1$. Notice that when $r = 1$, the two non-trivial steady states collapse into the trivial steady state. This is what we call a **bifurcation**, or in this case specifically, a **pitchfork bifurcation**. This can make sense if we plot [4] (x, y, z) of the steady points as a function of r:



Further analyzing the dynamics of the system is a little trickier - we can't exactly use the same approaches as before in the $\mathbb{R}^2$ space. The system is clearly not linear because of the xz and xy terms. However, if we assume that x, y, z are small and remain small, then we can approximate the system as linear by dropping these terms [5]. Thus, we can approximate the system as $\dot{x} = \sigma(y - x), \dot{y} = rx - y, \dot{z} = -bz$, or, equivalently,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Now, if $x(0) = y(0) = 0$, then $\dot{x} = \dot{y} = 0 \implies x(t) = y(t) = 0 \forall t$, and thus the solution evolves solely on the z-axis; ie $\dot{z} = -bz \implies z = z(0)e^{-bt}$, which $\to 0$ as $t \to \infty$, supporting our assumption that x, y, z remain small.

[3] These are fairly easy to find by considering different possible cases that would cause each of $\dot{x}, \dot{y}, \dot{z} = 0$.

[4] This is an odd way to look at the system (as the parameter r is suddenly becoming the independent variable), but it is helpful to analyze how exactly the steady states behave due to the parameters.

[5] xy and xz would be "very small" if x, y, z are small, as they are funtionally quadratic terms. Intuitively, $\alpha \times \beta \ll 1$ given $\alpha, \beta < 1$.

On the other hand, let's assume[6] $|x(t), y(t), z(t)| \ll 1$; again, this results in x, y, z remaining small, and thus "allows" us to study the dynamics approximately near $(x, y, z) = (0, 0, 0)$. Solving the matrix formulation of the system, we get

$$u(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = c_1 e^{\lambda_1 t}\underline{\mathbf{v_1}} + c_2 e^{\lambda_2 t}\underline{\mathbf{v_2}} + c_3 e^{\lambda_3 t}\underline{\mathbf{v_3}},$$

where λ_i are eigenvalues and $\underline{\mathbf{v_i}}$ are corresponding eigenvectors.

Clearly[7], $\underline{\mathbf{v_3}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ with a corresponding $\lambda_3 = -b$. Thus, $\underline{\mathbf{v_1}}$ and $\underline{\mathbf{v_2}}$ must lie in the xy-plane, and similarly, are eigenvectors of the top left, $\begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$ (with 0 z-component, of course). Solving for these by standard methods, we write

$$\begin{aligned} 0 &= \det\left(\begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} - \lambda I\right) \\ &= \begin{vmatrix} -\sigma - \lambda & \sigma \\ r & -1 - \lambda \end{vmatrix} = (\sigma + \lambda)(1 + \lambda) - r\sigma \\ &= \lambda^2 + (1 + \sigma)\lambda - (r - 1) \\ \implies \lambda_{1,2} &= \frac{-(1 + \sigma) \pm \sqrt{(1 + \sigma)^2 + 4(r - 1)\sigma}}{2} \end{aligned}$$

Notice that if $r \in (0, 1)$, then[8] $(1 + \sigma)^2 + 4(r - 1)\sigma < (1 + \sigma)^2$. Assuming [9] the lhs of this inequality is greater than 0, we can further say that $\left|\sqrt{(1 + \sigma)^2 + 4(r - 1)\sigma}\right| < |1 + \sigma|$. Thus, both λ_1 and λ_2 are < 0, as taking either the positive or negative sign in the quadratic necessarily yields a negative[10]. We could work out a full solution, but this is unncessary; clearly, as all $\lambda_i < 0$ when $r < 1$, $(x, y, z) \to 0$ as $t \to \infty$, which supports

[6] In this context, $\ll$ means "much less".

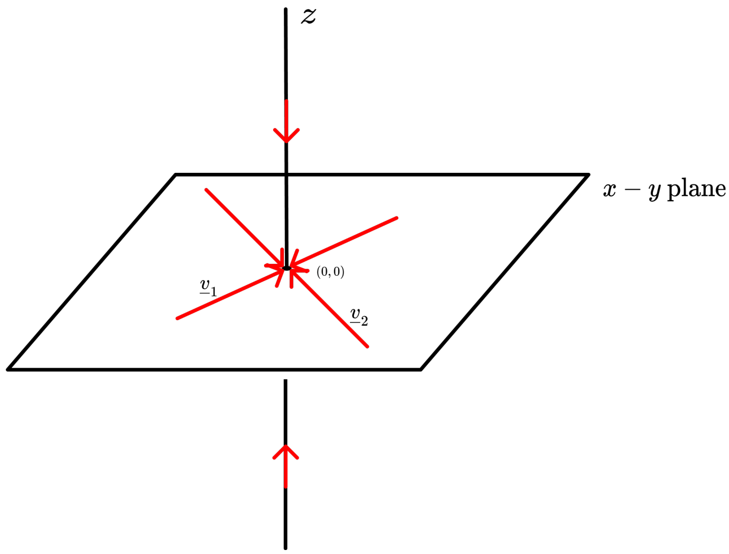
[7] Note the $-b$ in the bottom right of the matrix, surround by 0's; the only component of the "position" vector that will multiply to that $-b$ is z, thus, any vector with only a z component will remain unchanged, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ being the "unit" of these.

[8] As the (r-1) term is thus negative.

[9] Allowing us to operate in $\mathbb{R}^n$, as this is the part under the radical.

[10] Based on the reasoning "above", we are essentially saying (in "pseudomath") $-\alpha + (\alpha - \epsilon) < 0$, as does $-\alpha - (\alpha - \epsilon)$, taking α to represent the "terms" of the quadratic and ϵ the undetermined-but-clearly-there difference.

our original assumption in simpifying the system that (x, y, z) remain "small".[11] Thus, in all, $\underline{\mathbf{u}}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \sum_{j=1}^{3} \left[c_j e^{\lambda_j t} \, \underline{\mathbf{v_j}} \right] \to 0$ as $t \to \infty$, $\implies$ $(|x(t)|, |y(t)|, |z(t)|)$ "small" $\forall t > 0$.[12]



Again, this is all under $r > 1$; when $r > 1$, $4(r-1)\sigma > 0 \implies \sqrt{(1+\sigma)^2 + 4(r-1)\sigma} > 0 \implies -(1+\sigma) + \sqrt{(1+\sigma)^2 + 4(r-1)\sigma} > 0$, ie the root of the characteristic polynomial when we take the positive is now greater than one. In practical terms, this indicates a positive eigenvalue (we will take this to be λ_1 and $\underline{v}_1$), and thus one of our terms will grow as $t \to \infty$; however, the other two eigenvalues remain < 1 and will continue to shrink with time.

Remember that, this whole time, we are working with a linearized version of the original Lorenz equations, and thus these diagrams are not fully reflective of reality. As shown, for instance, the Lorenz equations have two other steady states (unless bifurcation…), which influence trajectories in the original system. However, this linearization is still useful in analyzing the dynamics of the system near the steady state $(0, 0, 0)$.

The other two steady states influence the dynamics of the system such that, at certain initial conditions, the trajectories will tend to spiral towards one of the two steady states, as well as (in chaotic systems) jump "randomly" from one steady state to the other, hence the "attractor" name.

1.3 Motivations of Maps

While aforementioned dynamical systems were defined via ordinary differential equations, we can also define them via **maps**, which are discrete dynamical systems. These can be defined:

1. Taking the maxima of a function in a system plotted against the previous maxima (for instance, in the Lorenz map, taking the maxima of $z(t)$ as z_n and plotting z_n against z_{n+1} for natural n).

[11]NB: just because the assumption "held" sts does not mean that it is *always true*; it simply validates the approximation we made in the particular scenario where $r < 1$.

[12]Note that we did not need to find the eigenvectors, or even the values (albeit, the matrix was in a quite nice form to allow this).

2. "Redefining" ODE's as maps. For instance, the **logistic map** defined

$$x_{n+1} = f(x_n) = \lambda x_n(1 - x_n)$$

2 Chapter 2

2.1 Flows on the Line

Consider $\dot{x} = f(x)$ or $\dot{x} = f(x, \mu)$ where μ some parameter. Solutions will be $x(t) \in \mathbb{R}$, and will either have $f : \mathbb{R} \to \mathbb{R}(f(x))$ or $f : \mathbb{R}^2 \to \mathbb{R}(f(x, \mu))$.

We may consider initial conditions $x(t = 0) = x_0 \in \mathbb{R}$; different x_0 lead to different solutions. Typically, we do not plot $x(t)$ against t, rather, we show the dynamics in $\mathbb{R}$ on a phase plot.

We could, in principle solve ODEs exactly; eg $\dot{x} = f(x), t \geq 0, x(0) = x_0$. We have

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= f(x) \\
\int \frac{dx}{f(x)} &= \int dt \\
\int_{x(0)}^{x(t)} \frac{dx}{f(x)} &= \int_0^t dt \\
\int_{x(0)}^{x(t)} \frac{dx}{f(x)} &= t
\end{aligned}$$

From here, we would have to solve the integral on the left and solve for $x(t)$.

However, we will approach this by determining the dynamics graphically. We do the following:

1. Graph $f(x)$
2. Draw steady states when $\dot{x} = f(x) = 0$
3. For $f(x) \neq 0$ we have either $\dot{x} = f(x) > 0, x(t)$ increasing, or $\dot{x} = f(x) < 0, x(t)$ decreasing

Remark 2.1

*If $f(x) = 0$ and $f'(x) < 0$, then x is a **stable steady state**. If $f(x) = 0$ and $f'(x) > 0$, then x is an **unstable steady state**.*

If $f(x) = f'(x) = 0$, we have a steady state which is "half-stable", eg $\dot{x} = f(x) = x^2$.

Example 2.1. *$\dot{x} = f(x, \mu) = x^2 + \mu$. If $\mu = 0$, we have a "half-stable" point.*

If $\mu > 0$, we have $f(x, \mu) \geq \mu > 0 \forall x$, and we have no steady state.

If $\mu < 0$, we have two stable steady states (on stable, one unstable).

*There is a **bifurcation** at $x = 0$ when $\mu = 0$; the number of steady states changes.*

In short; sign$(f'(x))$ determines the stability of the steady state, given $f'(x) \neq 0$, in which case we need to study further.

In particular, cases where $f(x) = f'(x) = 0$ are "delicate", and small parameter changes can cause large changes in the dynamics.

2.2 Linear Stability Analysis

$\dot{x} = f(x)$, let $x^* \in \mathbb{R}$ be a s.s., ie $f(x^*) = 0$; what does the dynamics look like near x^*?

First, change variables such that the s.s. is at the origin; let $v(t) = x(t) - x^*$, then

$$\begin{aligned}\dot{v} &= \frac{\mathrm{d}}{\mathrm{d}t}(x(t) - x^*) = \frac{\mathrm{d}x}{\mathrm{d}t} - 0 \\ &= f(x(t)) = f(x^* + v(t)) = g(v(t))\end{aligned}$$

Note that this "new" system has a steady state at $v = 0$; $g(0) = f(x^*) = 0$. Here, we can Taylor expand $\dot{v}$:

$$\begin{aligned}\dot{v} &= \sum_{j=0}^{\infty} \frac{v^j f^j(x^*)}{j!} \\ &= f(x^*) + vf'(x^*) + \frac{v^2}{2}f''(x^*) + \dots \\ f(x^*) = 0 \implies \dot{v} &= f'(x^*)v + \frac{1}{2}f''(x^*)v^2 + O(v^3)\end{aligned}$$

For $x(t) \approx x^*$, we have $|v(t)| = |x(t) - x^*| \ll 1$, then $1 \gg |v(t)| \gg |v(t)|^2 \gg |v(t)|^3 \gg \cdots > 0$. Provided $f'(x^*) \neq 0$ for $|v|$ sufficiently small the $f'(x^*)v$ term will dominate others and we can write

$$\dot{v} \approx f(x^*)v.$$

Let $\lambda = f(x^*)$ (just a constant), then we say

$$\begin{aligned}\dot{v} &= \lambda v \\ \implies \frac{\mathrm{d}v}{\mathrm{d}t} &= \lambda v \\ \implies \int_{v_0}^{v(t)} \frac{dv}{v} &= \int_0^t \lambda dt \\ \implies [\ln|v|]_{v_0}^{v(t)} &= \lambda t \\ \implies \ln\frac{v(t)}{v_0} &= \lambda t\end{aligned}$$

We can drop the absolute value bars as $v(t)$ and v_0 must have the same sign.

$$v(t) = v_0 e^{\lambda t}$$

Thus, if $\lambda < 0$, $v(t) \to 0$ as $t \to \infty$, then $v(t) = x - x^* \implies x(t) \to x^*$ as $t \to \infty$. Else, if $\lambda > 0$, then

$|v_0 e^{\lambda t}|$ grows with t and $|v(t)| \to \infty$ as $t \to \infty$. Importantly, $|v(t)|$ becomes large before it becomes unbounded, meaning that our initial assumption doesn't work as the $O(v^2)$ term becomes significant. However, we can still make conclusions about the (in)stability local to x^*, and we can't draw conclusions for dynanmics $t \to \infty$.

Example 2.2. $\dot{x} = f(x) = x - x^3 = x(1 - x^2) = x(1 - x)(1 + x)$. *We have steady states at* $x = -1, 0, 1$.

$f(x) = x - x^3 \implies f'(x) = 1 - 3x^2$, *so* $f(0) = 1 \implies 0$ *unstable,* $f'(\pm 1) = 1 - 3 = -2 \implies \pm 1$ *stable.*

Alternatively, graph $f(x)$ *and the steady states are visually obvious.*

This is called **linear stability analysis**, as are reducing the nonlinear differential equation $\dot{x} = f(x)$ to a linear ODE $\dot{v} = \lambda v$ (we'll see in higher dimensions that we replace λ with some Jacobian).

2.3 Existence & Uniqueness

We are studying the *qualitative* behavior of solutions, which only makes sense if these solutions exist. Usually we require that the IVP $\dot{x} = f(x), x(0) = x_0$ to have a unique solution $x(t) \forall t \geq 0$, for every $x_0 \in \mathbb{R}$. If they were not unique, they have multiple solutions starting at some point or worse, at points x_0 where multiple solutions are possible orbits will cross each other. The very aspect that this doesn't happen is what makes phase plots useful.

This is also why only autonomous ODEs are considereed; in nonautonomous ODES $\dot{x} = f(t, x)$, the value of $\dot{x}(t)$ at a particular value of x will depend of t.