MATH454 - Analysis 3

Based on lectures from Fall 2024 by Prof. Linan Chen.
Notes by Louis Meunier

Contents

§1 Sigma Algebras and Measures

§1.1 A Review of Riemann Integration

Let $f : \mathbb{R} \to \mathbb{R}$ and $[a, b] \subset \mathbb{R}$. Define a **partition** of $[a, b]$ as the set

$$\operatorname{part}([a, b]) := \{a =: x_0 < x_1 < \cdots < x_N := b\}.$$

We can then define the upper and lower Riemann integrals of f over the region $[a, b]$ as

$$\text{upper:} \qquad \overline{\int_a^b} f(x)\, \mathrm{d}x := \inf_{\operatorname{part}([a,b])} \left\{ \sum_{\{i=1\}}^{N} \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \right\}$$

$$\text{lower:} \qquad \underline{\int_a^b} f(x)\, \mathrm{d}x := \sup_{\operatorname{part}([a,b])} \left\{ \sum_{\{i=1\}}^{N} \inf_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \right\}.$$

We then say f **Riemann integrable** if these two quantities are equal, and denote this value by $\int_a^b f(x)\, \mathrm{d}x$.

Many "nice-enough" (continuous, monotonic, etc.) functions are Riemann integrable, but many that we would like to be able to "integrate" are simply not, for instance Dirichlet's function $x \mapsto \begin{cases} 1 x \in \mathbb{Q} \setminus [a, b] \\ 0 x \in \mathbb{Q}^c \setminus [a, b] \end{cases}$. Hence, we need a more general notion of integration.

§1.2 Sigma Algebras

↪**Definition 1.1** (Sigma algebra): Let X be a *space* (a nonempty set) and $\mathcal{F}$ a collection of subsets of X. $\mathcal{F}$ a *sigma algebra* or simply σ-algebra of X if the following hold:

1. $X \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ (closed under complement)
3. $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ (closed under countable unions)

↪**Proposition 1.1**:

4. $\varnothing \in \mathcal{F}$
5. $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
6. $A_1, \ldots, A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
7. $A, B \in \mathcal{F} \Rightarrow A \setminus B, B \setminus A \in \mathcal{F}$

⊛ **Example 1.1**: The "largest" sigma algebra of a set X is the power set 2^X, the smallest the trivial $\{\varnothing, X\}$.

Given a set $A \subset X$, the set $\mathcal{F}_A := \{\varnothing, X, A, A^c\}$ is a sigma algebra; given two disjoint sets $A, B \subset X$, then $\mathcal{F}_{A,B} := \{\varnothing, X, A, A^c, B, B^c, A \cup B, A^c \cap B^c\}$ a sigma algebra.

↪**Definition 1.2** (Generating a sigma algebra): Let X be a nonempty set, and $\mathcal{C}$ a collection of subsets of X. Then, the σ-algebra *generated* by $\mathcal{C}$, denoted $\sigma(\mathcal{C})$, is such that
1. $\sigma(\mathcal{C})$ a sigma algebra with $\mathcal{C} \subseteq \sigma(\mathcal{C})$
2. if $\mathcal{F}'$ a sigma algebra with $\mathcal{C} \subseteq \mathcal{F}'$, then $\mathcal{F}' \supseteq \sigma(\mathcal{C})$

Namely, $\sigma(\mathcal{C})$ is the smallest sigma algebra "containing" (as a subset) $\mathcal{C}$.

↪**Proposition 1.2**:
1. $\sigma(\mathcal{C}) = \bigcap\{\mathcal{F} : \mathcal{F} \text{ a sigma algebra containing } \mathcal{C}\}$
2. if $\mathcal{C}$ itself a sigma algebra, then $\sigma(\mathcal{C}) = \mathcal{C}$
3. if $\mathcal{C}_1, \mathcal{C}_2$ are two collections of subsets of X such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_2)$

↪**Definition 1.3** (The Borel sigma-algebra): The *Borel σ-algebra*, denoted $\mathfrak{B}_{\mathbb{R}}$, on the real line is given by

$$\mathfrak{B}_{\mathbb{R}} := \sigma(\{\text{open subsets of } \mathbb{R}\}).$$

We call sets in $\mathfrak{B}_{\mathbb{R}}$ *Borel sets*.

↪**Proposition 1.3**: $\mathfrak{B}_{\mathbb{R}}$ is also generated by the sets
- $\{(a,b) : a < b \in \mathbb{R}\}$
- $\{(a,b] : a < b \in \mathbb{R}\}$
- $\{[a,b] : a < b \in \mathbb{R}\}$
- $\{[a,b) : a < b \in \mathbb{R}\}$ ⊛
- $\{(-\infty,c) : c \in \mathbb{R}\}$
- $\{(-\infty,c] : c \in \mathbb{R}\}$
- etc.

Proof. We prove just ⊛. It suffices to show that the generating sets of each σ-algebra is contained in the other σ-algebra. Let $a < b \in \mathbb{R}$. Then,

$$(a,b) = \bigcup_{n=1}^{\infty} \underbrace{\left[a + \frac{1}{n}, b\right)}_{\in \circledast} \in \sigma(\{[a,b)\}) \Rightarrow \mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[a,b)\}).$$

Conversely,

$$[a,b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right) \in \mathfrak{B}_{\mathbb{R}}.$$

■

↪**Proposition 1.4**: All intervals (open, closed, half open, half closed, finite, etc) are Borel sets; any set obtained from countable set operations of intervals are Borel; all singletons are Borel; any finite and countable sets are Borel.

§1.3 Measures

↪**Definition 1.4** (Measurable Space): Let X be a space and $\mathcal{F}$ a σ-algebra. We call the tuple $(X, \mathcal{F})$ a *measurable space*.

↪**Definition 1.5** (Measure): Let $(X, \mathcal{F})$ be a measurable space. A *measure* is a function $\mu : \mathcal{F} \to [0, \infty]$ satisfying

(i) $\mu(\varnothing) = 0$;

(ii) if $\{A_n\} \subseteq \mathcal{F}$ a sequence of (pairwise) disjoint sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

i.e. μ is *countably additive*. We further call μ

- *finite* if $\mu(X) < \infty$,
- a *probability measure* if $\mu(X) = 1$,
- σ-finite if $\exists\, \{A_n\} \subseteq \mathcal{F}$ such that $X = \bigcup_{n=1}^{\infty} A_n$ with $\mu(A_n) < \infty\ \forall\, n \geq 1$,

and call the triple $(X, \mathcal{F}, \mu)$ a *measure space*.

⊛ **Example 1.2**: The measure on $\mathfrak{B}_{\mathbb{R}}$ given by

$$A \mapsto \begin{cases} |A| \text{ if } A \text{ finite} \\ \infty \quad \text{else} \end{cases}$$

is called the *counting measure*.

Fix $x_0 \in \mathbb{R}$, then the measure on $\mathfrak{B}_{\mathbb{R}}$ given by

$$A \mapsto \begin{cases} 1 \text{ if } x_0 \in A \\ 0 \text{ else} \end{cases}$$

is called the *point mass at x_0*.

↪**Theorem 1.1** (Properties of Measures): Fix a measure space $(X, \mathcal{F}, \mu)$. The following properties hold:

1. (finite additivity) For any sequence $\{A_n\}_{n=1}^N \subseteq \mathcal{F}$ of disjoint sets,
$$\mu\left(\bigcup_{n=1}^N A_n\right) = \sum_{n=1}^N \mu(A_n).$$
2. (monotonicity) For any $A \subseteq B \in \mathcal{F}$, then $\mu(A) \leq \mu(B)$.
3. (countable/finite subadditivity) For any sequence $\{A_n\} \subseteq \mathcal{F}$ (**not** necessarily disjoint),
$$\mu\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty \mu(A_n),$$
an analogous statement holding for a finite collection of sets $A_1, ..., A_N$.
4. (continuity from below) For $\{A_n\} \subseteq \mathcal{F}$ such that $A_n \subseteq A_{n+1}\ \forall\, n \geq 1$ (in which case we say $\{A_n\}$ "increasing" and write $A_n \uparrow$) we have
$$\mu\left(\bigcup_{n=1}^\infty A_n\right) = \lim_{n\to\infty} \mu(A_n).$$
5. (continuity from above) For $\{A_n\} \subseteq \mathcal{F}, A_n \supseteq A_{n+1}\ \forall\, n \geq 1$ (we write $A_n \downarrow$) we have that **if** $\mu(A_1) < \infty$,
$$\mu\left(\bigcap_{n=1}^\infty A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Remark 1.1: In 4., note that since A_n increasing, that the union $\bigcup_{n=1}^\infty A_n \supseteq A_m$ for any arbitrarily large m; indeed, one could logically right $\lim_{n\to\infty} A_n = \bigcup_{n=1}^\infty A_n$. This this notation, then, 4. simply states that we may interchange limit and measure. A similar argument can be viewed for 5. (how?).

Remark 1.2: The finiteness condition in 5. may be slightly modified such as to state that $\mu(A_n) < \infty$ for some n; remark why this would suffice to ensure the entire rest of the sequence has finite measure.

Proof.

1. Extend $A_1, ..., A_N$ to an infinite sequence by $A_n := \varnothing$ for $n > N$. Then this simply follows from countable additivity and $\mu(\varnothing) = 0$.
2. We may write $B = A \cup (B \setminus A)$; this is a disjoint union of sets. By finite additivity, then,
$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A),$$
since the measure is positive.
3. We prove only for a countable union; use the technique from 1. to extend to finite. We first "disjointify" the sequence such that we can use the countable additivity

axiom. Let $B_1 = A_1, B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$ for $n \geq 2$. Remark then that $\{B_n\} \subseteq \mathcal{F}$ is a disjoint sequence of sets, and that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$. By countable additivity and subadditivity,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

4. We again "disjointify" the sequence $\{A_n\}$. Put $B_1 = A_1$, $B_n = A_n \setminus A_{n-1}$ for all $n \geq 2$ (remark that this is equivalent to the construction from the previous proof because the sets are increasing). Then, again, $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$, and in particular, for all $N \geq 1$, $\bigcup_{n=1}^{N} B_n = A_N$. Then

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \sum_{n=1}^{\infty} \mu(B_n) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \mu(B_n) \\ &= \lim_{N\to\infty} \mu\left(\bigcup_{n=1}^{N} B_n\right) = \lim_{N\to\infty} \mu(A_N).\end{aligned}$$

5. We yet again disjointify, backwards (in a way) from the previous case. Put $B_n = A_1 \setminus A_n$ for all $n \geq 1$. Then, $\{B_n\} \subseteq \mathcal{F}$, B_n increasing, and $\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \bigcap_{n=1}^{\infty} A_n$. Then, by continuity from below,

$$\mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \mu(A_1 \setminus A_n)$$

and also

$$\begin{aligned}\mu(A_1) &= \mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) + \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \\ &= \mu(A_1 \setminus A_n) + \mu(A_n),\end{aligned}$$

and combining these two equalities yields the desired result.

■

§1.4 Constructing the Lebesgue Measure on $\mathbb{R}$

↪**Definition 1.6** (Lebesgue outer measure): For all $A \subseteq \mathbb{R}$, define

$$m^*(A) := \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n, I_n \text{ open intervals}\right\},$$

called the *Lebesgue outer measure* of A (where $\ell(I)$ is the length of interval I, i.e. the absolute value of the difference of its endpoints, if finite, or ∞ if not).

↪**Proposition 1.5**: The following properties of m^* hold:

1. $m^*(A) \geq 0$ for all $A \subseteq \mathbb{R}$, and $m^*(\varnothing) = 0$.
2. (monotonicity) For $A \subseteq B$, $m^*(A) \leq m^*(B)$.
3. (countable subadditivity) For $\{A_n\}, A_n \subseteq \mathbb{R}$, $m^*\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n=1}^\infty m^*(A_n)$.[1]
4. If $I \subseteq \mathbb{R}$ an interval, then $m^*(I) = \ell(I)$.
5. m^* is translation invariant; for any $A \subseteq R, x \in \mathbb{R}$, $m^*(A) = m^*(A+x)$ where $A + x := \{a + x : a \in A\}$.
6. For all $A \subseteq \mathbb{R}$, $m^*(A) = \inf\{m^*(B) : A \subseteq B \subseteq R, B - \text{open}\}$.
7. If $A = A_1 \cup A_2 \subseteq \mathbb{R}$ with $d(A_1, A_2) > 0$,[2] then $m^*(A_1) + m^*(A_2) = m^*(A)$.
8. If $A = \bigcup_{k=1}^\infty J_k$ where J_k's are "almost disjoint intervals" (i.e. share at most endpoints), then $m^*(A) = \sum_{k=1}^\infty m^*(J_k) = \sum_{k=1}^\infty \ell(J_k)$.

Proof.

3. If $m^*(A_n) = \infty$, for any n, we are done, so assume wlog $m^*(A_n) < \infty$ for all n. Then, for each n and $\varepsilon > 0$, one can choose open intervals $\{I_{n,i}\}_{i \geq 1}$ such that $A_n \subseteq \bigcup_{i=1}^\infty I_{n,i}$ and $\sum_{i=1}^\infty \ell(I_{n,i}) \leq m^*(A_n) + \frac{\varepsilon}{2^n}$. Hence

$$\bigcup_{n=1}^\infty A_n \subseteq \bigcup_{n=1,i=1}^\infty I_{n,i}$$

$$\Rightarrow m^*\left(\bigcup_{n=1}^\infty A_n\right) \leq \sum_{n,i=1}^\infty \ell(I_{n,i}) = \sum_{n=1}^\infty \sum_{i=1}^\infty \ell(I_{n,i}) \leq \sum_{n=1}^\infty \left(m^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^\infty m^*(A_n) + \varepsilon,$$

and as ε arbitrary, the statement follows.

4. We prove first for $I = [a, b]$. For any $\varepsilon > 0$, set $I_1 = (a - \varepsilon, b + \varepsilon)$; then $I \subseteq I_1$ so $m^*(I) \leq \ell(I_1) = (b-1) + 2\varepsilon$ hence $m^*(I) \leq b - a = \ell(I)$. Conversely, let $\{I_n\}$ be any open-interval convering of I (wlog, each of finite length; else the statement holds trivially). Since I compact, it can be covered by finitely many of the I_n's, say $\{I_n\}_{n=1}^N$, denoting $I_n = (a_n, b_n)$ (with relabelling, etc). Moreover, we can pick the a_n, b_n's such that $a_1 < a, b_N > b$, and generally $a_n < b_{n-1}$ $\forall\, 2 \leq n \leq N$. Then,

$$\begin{aligned}\sum_{n=1}^\infty \ell(I_n) \geq \sum_{n=1}^N \ell(I_n) &= b_1 - a_1 + \sum_{n=2}^N (b_n - a_n) \\ &\geq b_1 - a_1 + \sum_{n=2}^N (b_n - b_{n-1}) \\ &= b_N - a_1 \geq b - 1 = \ell(I),\end{aligned}$$

hence since the cover was arbitrary, $m^*(A) \geq \ell(I)$, and equality holds.

Now, suppose I finite, with endpoints $a < b$. Then for any $\frac{b-a}{2} > \varepsilon > 0$, then

$$[a + \varepsilon, b - \varepsilon] \subseteq I \subseteq [a - \varepsilon, b + \varepsilon],$$

[1] More generally, any set function on $2^{\mathbb{R}}$ that satisfies 1., 2., and 3. is called an *outer measure*.

[2] Remark: this is a stronger requirement than disjointness!

hence by monotonicity and the previous part of this proof

$$m^*([a+\varepsilon, b-\varepsilon]) = b-a-2\varepsilon \le m^*(I) \le b-a+2\varepsilon = m^*([a-\varepsilon, b+\varepsilon]),$$

from which it follows that $m^*(I) = b-a = \ell(I)$.

Finally, suppose I infinite. Then, $\forall\, M \ge 0, \exists$ closed, finite interval I_M with $I_M \subseteq I$ and $\ell(I_M) \ge M$. Hence, $m^*(I) \ge m^*(I_M) \ge M$ and thus as M arbitrary it must be that $m^*(I) = \infty = \ell(I)$.

6. Denote $\tilde{m}(A) := \inf\{m^*(B) : A \subseteq B \subseteq R, B - \text{open}\}$. For any $A \subseteq B \subseteq \mathbb{R}$ with B open, monotonicity gives that $m^*(A) \le m^*(B)$, hence $m^*(A) \le \tilde{m}(A)$. Conversely, assuming wlog $m^*(A) < \infty$ (else holds trivially), then for all $\varepsilon > 0$, there exists $\{I_n\}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ with $\sum_{n=1}^{\infty} \ell(I_n) \le m^*(A) + \varepsilon$. Setting $B := \bigcup_{n=1}^{\infty} I_n$, we have that $A \subseteq B$ and $m^*(B) = m^*(\bigcup I_n) \le$ (by finite subadditivity) $\sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \ell(I_n) \le m^*(A) + \varepsilon$ hence $m^*(B) \le m^*(A)$ for all B. Thus $m^*(A) \ge \tilde{m}(A)$ and equality holds.

7. Put $\delta := d(A_1, A_2) > 0$. Clearly $m^*(A) \le m^*(A_1) + m^*(A_2)$ by finite subadditivity. wlog, $m^*(A) < \infty$ (and hence $m^*(A_i) < \infty, i = 1, 2$) (else holds trivially). Then $\forall\, \varepsilon > 0, \exists\, \{I_n\} : A \subseteq \bigcup I_n$ and $\sum \ell(I_n) \le m^*(A) + \varepsilon$. Then, for all n, we consider a "refinement" of I_n; namely, let $\{I_{n,i}\}_{i \ge 1}$ such that $I_n \subseteq \bigcup_i I_{n,i}$ and $\ell(I_{n,i}) < \delta$ and $\sum_i \ell(I_{n,i}) \le \ell(I_n) + \frac{\varepsilon}{2^n}$. Relabel $\{I_{n,i} : n, i \ge 1\} \rightsquigarrow \{J_m : m \ge 1\}$ (both are countable). Then, $\{J_m\}$ defines an open-interval cover of A, and since $\ell(J_m) < \delta$ for each m, J_m intersects at most one A_i. For each m and $p = 1, 2$, put

$$M_p := \{m : J_m \cap A_p \neq \varnothing\},$$

noting that $M_1 \cap M_2 = \varnothing$. Then $\{J_m : m \in M_p\}$ is an open covereing of A_p, and so

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\le \sum_{m \in M_1} \ell(J_m) + \sum_{m \in M_2} \ell(J_m) \\ &\le \sum_{m=1}^{\infty} \ell(J_m) = \sum_{n,i=1}^{\infty} \ell(I_n, i) \\ &\le \sum_n \left(\ell(I_n) + \frac{\varepsilon}{2^n} \right) \\ &= \sum_n \ell(I_n) + \varepsilon \\ &\le m^*(A) + 2\varepsilon, \end{aligned}$$

and hence equality follows.

8. If $\ell(J_k) = \infty$ for some k, then since $J_k \subseteq A$, subadditivity gives us that $m^*(J_k) \le m^*(A)$ and so $m^*(A) = \infty = \sum_{k=1}^{\infty} \ell(J_k)$ (since if any J_k infinite, the sum of the lengths of all of them will also be infinite).

Suppose then $\ell(J_k) < \infty$ for all k. Fix $\varepsilon > 0$. Then for all $k \ge 1$, choose $I_k \subseteq J_k$ such that $\ell(J_k) \le \ell(I_k) + \frac{\varepsilon}{2^k}$. For any $N \ge 1$, we can choose a subset $\{I_1, ..., I_N\}$ of intervals such that all are disjoint, with strictly positive distance between them, and so

$$\bigcup_{k=1}^{N} I_k \subseteq \bigcup_{k=1}^{N} I_k \subseteq A$$

$$\Rightarrow m^*(A) \geq m^*\left(\bigcup_{k=1}^{N} I_k\right) \geq \sum_{k=1}^{N} \ell(I_k)$$

$$\geq \sum_{k=1}^{N}\left(\ell(J_k) - \frac{\varepsilon}{2^k}\right)$$

$$\geq \sum_{k=1}^{N} \ell(J_k) - \varepsilon$$

$$\Rightarrow m^*(A) \geq \sum_{k=1}^{\infty} \ell(J_k),$$

the second inequality following from finite subadditivity. The converse of the final inequality holds trivially. ■

§1.5 Lebesgue-Measurable Sets

↪**Definition 1.7**: $A \subseteq \mathbb{R}$ is *m^*-measurable* if $\forall\, B \subseteq \mathbb{R}$,

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c).$$

Remark 1.3: By subadditivity, $\leq$ always holds in the definition above.

↪**Theorem 1.2** (Carathéodary's Theorem): Let

$$\mathcal{M} := \{A \subseteq \mathbb{R} : A \;\; m^* - \text{measurable}\}.$$

Then, $\mathcal{M}$ is a σ-algebra of subsets of $\mathbb{R}$.

Define $m : \mathcal{M} \to [0, \infty]$, $m(A) = m^*(A)$. Then, m is a measure on $\mathcal{M}$, called the *Lebesgue measure* on $\mathbb{R}$. We call sets in $\mathcal{M}$ *Lebesgue-measurable* or simply *measurable* (if clear from context) accordingly. We call $(\mathbb{R}, \mathcal{M}, m)$ the *Lebesgue measure space*.

Proof. The first two σ-algebra axioms are easy. We have for any $B \subseteq \mathbb{R}$ that

$$m^*(B \cap \mathbb{R}) + m^*(B \cap \mathbb{R}^c) = m^*(B) + m^*(B \cap \varnothing) = m^*(B)$$

so $\mathbb{R} \in \mathcal{M}$. Further, $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$ by the symmetry of the requirement for sets to be in $\mathcal{M}$.

The final axiom takes more work. We show first $\mathcal{M}$ closed under finite unions; by induction it suffices to show for 2 sets. Let $A_1, A_2 \in \mathcal{M}$. Then, for all $B \subseteq \mathbb{R}$,

$$\begin{aligned} m^*(B) &= m^*(B \cap A_1) + m^*(B \cap A_1^c) \\ &= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*(B \cap A_1^c \cap A_2^c) \\ &= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*\big(B \cap (A_1 \cup A_2)^c\big) \end{aligned}$$

Note that $(B \cap A_1) \cup (B \cap A_1^c \cap A_2^c) = B \cap (A_1 \cup A_2)$, hence by subadditivity,

$$m^*(B) \geq m^*(B \cap (A_1 \cup A_2)) + m^*\big(B \cap (A_1 \cup A_2)^c\big),$$

and since the other direction of the inequality comes for free, we conclude $A_1 \cup A_2 \in \mathcal{M}$.

Let now $\{A_n\} \subseteq \mathcal{M}$. We "disjointify" $\{A_n\}$; put $B_1 := A_1$, $B_n := \frac{A_n}{} \bigcup_{i=1}^{n-1} A_i, n \geq 2$, noting $\bigcup_n A_n = \bigcup_n B_n$, and each $B_n \in \mathcal{M}$, as each is but a finite number of set operations applied to the A_n's, and thus in $\mathcal{M}$ as demonstrated above. Put $E_n := \bigcup_{i=1}^n B_i$, noting again $E_n \in \mathcal{M}$. Then, for all $B \subseteq \mathbb{R}$,

$$\begin{aligned}
m^*(B) &= m^*\left(\underbrace{B \cap E_n}_{\text{chop up } B_n}\right) + m^*\left(\underbrace{B \cap E_n^c}_{E_n \subseteq \cup B_n \Rightarrow E_n^c \supseteq (\cup B_n)^c}\right) \\
&\geq m^*\left(B \cap \underbrace{E_n \cap B_n}_{=B_n}\right) + m^*\left(B \cap \underbrace{E_n \cap B_n^c}_{=E_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^\infty B_n\right)^c\right) \\
&\geq m^*(B \cap B_n) + m^*\left(\underbrace{B \cap E_{n-1}}_{\text{chop up } B_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^\infty B_n\right)^c\right) \\
&\geq m^*(B \cap B_n) + m^*(B \cap E_{n-1} \cap B_{n-1}) \\
&\qquad + m^*(B \cap E_{n-1} \cap B_{n-1}^c) + m^*\left(B \cap \left(\bigcup_{n=1}^\infty B_n\right)^c\right).
\end{aligned}$$

Notice that the last line is essentially the second applied to B_{n-1}; hence, we have a repeating (essentially, "descending") pattern in this manner, which we repeat until $n \to 1$. We have, thus, that

$$m^*(B) \geq \sum_{i=1}^n [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^\infty B_n\right)^c\right),$$

so taking $n \to \infty$,

$$\begin{aligned}
m^*(B) &\geq \sum_{i=1}^\infty [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^\infty B_n\right)^c\right) \\
&\geq m^*\left(B \cap \left(\bigcup_{n=1}^\infty B_n\right)\right) + m^*\left(B \cap \left(\bigcup_{n=1}^\infty B_n\right)^c\right).
\end{aligned}$$

As usual, the inverse inequality comes for free, and thus we can conclude $\bigcup_{n=1}^\infty B_n$ also m^*-measurable, and thus so is $\bigcup_{n=1}^\infty A_n$. This proves $\mathcal{M}$ a σ-algebra.

We show now m a measure. By previous propositions, we have that $m \geq 0$ and $m(\varnothing) = 0$ (since $m = m^* |_{\mathcal{M}}$), so it remains to prove countable subadditivity.

Let $\{A_n\} \subseteq \mathcal{M}$-disjoint. Following precisely the same argument as above, used to prove that $\mathcal{M}$ closed under countable unions, shows that for any $n \geq 1$

$$m\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(A_i),$$

that is, finite additivity holds, and thus by subadditivity

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq m\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} m(A_i),$$

and so taking the limit of $n \to \infty$, we have

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} m(A_i),$$

with the converse inequality coming for free. Thus, m indeed a measure on $\mathcal{M}$. ■

↪**Proposition 1.6**: $\mathcal{M}, m$ translation invariant; for all $A \in \mathcal{M}, x \in \mathbb{R}, x + A = \{x + a : a \in A\} \in \mathcal{M}$ and $m(A) = m(A + x)$.

Remark 1.4: We would like this to hold, heuristically, since if we shift sets on the real line, we should expect their length to remain constant.

Proof. For all $B \subseteq \mathbb{R}$, we have (since m^* translation invariant)

$$\begin{aligned} m^*(B) = m^*(B - x) = m^*\left(\underbrace{(B - x) \cap A}_{= B \cap (A + x)}\right) + m^*\left(\underbrace{(B - x) \cap A^c}_{= B \cap (A^c + x) = B \cap (A + x)^c}\right) \\ = m^*(B \cap (A + x)) + m^*\left(B \cap (A + x)^c\right), \end{aligned}$$

thus $A + x \in \mathcal{M}$, and since m^* translation invariant, it follows that m is. ■

↪**Theorem 1.3**: $\forall a, b \in \mathbb{R}$ with $a < b$, $(a, b) \in \mathcal{M}$, and $m((a, b)) = b - a$.

Remark 1.5: Again, we'd like this to hold, heuristically, since we would like the measure of an interval to simply be its length; we'd moreover like to be able to measure intervals, i.e. have intervals be contained in $\mathcal{M}$.

↪**Corollary 1.1**: $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M}$

Proof. $\mathfrak{B}_{\mathbb{R}}$ is generated by open intervals of the form (a, b). All such intervals are in $\mathcal{M}$ by the previous theorem, and hence the proof. ■

§1.6 Properties of the Lebesgue Measure

↪**Proposition 1.7** (Regularity Assumptions on m): For all $A \in \mathcal{M}$, the following hold.

- For all $\varepsilon > 0, \exists G$ open such that $A \subseteq G$ and $m(G \setminus A) < \varepsilon$.
- For all $\varepsilon > 0$, $\exists F$-closed such that $F \subseteq A$ and $m(A \setminus F) \leq \varepsilon$.
- $m(A) = \inf\{m(G) : G \text{ open}, G \supseteq A\}$.
- $m(A) = \sup\{m(K) : K \text{ compact}, K \subseteq A\}$.
- If $m(A) < \infty$, then for all $\varepsilon > 0$, $\exists K \subseteq A$ compact, such that $m(A \setminus K) < \varepsilon$.
- If $m(A) < \infty$, then for all $\varepsilon \geq 0$, $\exists$ finite collection of open intervals $I_1, ..., I_N$ such that $m\left(A \triangle \left(\bigcup_{n=1}^{N} I_n\right)\right) \leq \varepsilon$.

↪**Proposition 1.8** (Completeness of m): $(\mathbb{R}, \mathcal{M}, m)$ is *complete*, in the sense that for all $A \subseteq \mathbb{R}$, if $\exists B \in \mathcal{M}$ such that $A \subseteq B$ and $m(B) = 0$, then $A \in \mathcal{M}$ and $m(A) = 0$.

Equivalently, any subset of a null set is again a null set.

Remark 1.6: In general, $A \in \mathcal{F}, B \subseteq A \nRightarrow B \in \mathcal{F}$.

↪**Proposition 1.9**: Up to rescaling, m is the unique, nontrivial measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ that is finite on compact sets and is translation invariant, i.e. if μ another such measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ with $\mu = c \cdot m$ for $c > 0$, then $\mu = m$.

Remark 1.7: Such a c is simply $c = \mu((0, 1))$.

To prove this proposition, we first introduce some helpful tooling:

↪**Theorem 1.4** (Dynkin's π-d): Given a space X, let $\mathcal{C}$ be a collection of subsets of X. $\mathcal{C}$ is called a *π-system* if $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$ (that is, it is closed under finite intersections).

Let $\mathcal{F} = \sigma(\mathcal{C})$, and suppose μ_1, μ_2 are two finite measures on $(X, \mathcal{F})$ such that $\mu_1(X) = \mu_2(X)$ and $\mu_1 = \mu_2$ when restricted to $\mathcal{C}$. Then, $\mu_1 = \mu_2$ on all of $\mathcal{F}$.

↪**Proposition 1.10**: $\{\varnothing\} \cup \{(a, b) : a < b \in \mathbb{R}\}$ a π-system.

↪**Proposition 1.11**: If μ a measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ such that for all intervals I, $\mu(I) = \ell(I)$, then $\mu = m$.

Proof. Consider for all $n \geq 1$ $\mu|_{\mathfrak{B}_{[-n,n]}}$. Clearly, $\mu([-n, n]) = m([-n, n]) = 2n$, and for all $a, b \in \mathbb{R}$, $\mu((a, b) \cap [-n, n]) = \ell((a, b) \cap [-n, n]) = m((a, b) \cap [-n, n])$. Thus, by the previous theorem, μ must match m on all of $\mathfrak{B}_{[-n,n]}$.

Let now $A \in \mathfrak{B}_{\mathbb{R}}$. Let $A_n := A \cap [-n, n] \in \mathfrak{B}_{[-n,n]}$. By continuity of m from below,

$$\begin{aligned}\mu(A) &= \lim_{n\to\infty} \mu(A_n) \\ &= \lim_{n\to\infty} m(A_n) \\ &= m(A),\end{aligned}$$

hence $\mu = m$. ■

↪**Proposition 1.12**: If μ a measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ assigning finite values to compact sets and is translation invariant, then $\mu = cm$ for some $c > 0$.

Remark 1.8: This proposition is also tacitly stating that $\mathfrak{B}_{\mathbb{R}}$ translation invariant; this needs to be shown.

↪**Lemma 1.1**: $\mathfrak{B}_{\mathbb{R}}$ translation invariant; for any $A \in \mathfrak{B}_{\mathbb{R}}, x \in \mathbb{R}, A + x \in \mathfrak{B}_{\mathbb{R}}$.

Proof. We employ the "good set strategy"; fix some $x \in \mathbb{R}$ and let

$$\Sigma := \{B \in \mathfrak{B}_{\mathbb{R}} : B + x \in \mathfrak{B}_{\mathbb{R}}\}.$$

One can check that Σ a σ-algebra, and so $\Sigma \subseteq \mathfrak{B}_{\mathbb{R}}$. But in addition, its easy to see that $\{(a, b) : a < b \in \mathbb{R}\} \subseteq \Sigma$, since a translated interval is just another interval, and since these sets generate $\mathfrak{B}_{\mathbb{R}}$, it must be further that $\mathfrak{B}_{\mathbb{R}} \subseteq \Sigma$, completing the proof. ■

Proof. (of the proposition) Let $c = \mu((0, 1])$, noting that $c > 0$ (why? Consider what would happen if $c = 0$).

This implies that $\forall\, n \geq 1, \mu\left(\left(0, \frac{1}{n}\right]\right) = \frac{c}{n}$ (obtained by "chopping up" $(0, 1]$ into n disjoint intervals); from here we can draw many further conclusions:

$$\begin{aligned}&\forall\, m = 1, \ldots, n-1, \mu\left(\left(0, \frac{m}{n}\right]\right) = \frac{m}{n} c \\ &\Rightarrow \forall\, q \in \mathbb{Q} \cap (0, 1], \mu((0, q]) = qc \\ &\Rightarrow \forall\, q \in \mathbb{Q}^+, \mu((0, q]) = q \cdot c \text{ (translate)} \\ &\Rightarrow \forall\, a \in \mathbb{R}, \mu((a, a + q]) = q \cdot c \\ &\Rightarrow \forall \text{ intervals } I, \mu(I) = c \cdot \ell(I) \text{ (continuity)} \\ &\Rightarrow \forall\, n \geq 1, a, b \in \mathbb{R}, \mu((a, b) \cap [-n, n]) = c \cdot \ell((a, b) \cap [-n, n]) = c \cdot m((a, b) \cap [-n, n]),\end{aligned}$$

but then, $\mu = c \cdot m$ on $\mathfrak{B}_{\mathbb{R}[-n,n]}$, and by appealing again the Dynkin's, $\mu = c \cdot m$ on all of $\mathfrak{B}_{\mathbb{R}}$. ■

↪**Proposition 1.13** (Scaling): m has the *scaling property* that $\forall\, A \in \mathcal{M}, c \in \mathbb{R}, c \cdot A = \{cx : x \in A\} \in \mathcal{M}$, and $m(c \cdot A) = |c|\, m(A)$.

PROOF. Assume $c \neq 0$. Given $A \subseteq \mathbb{R}$, remark that $\{I_n\}$ an open interval cover of A iff $\{cI_n\}$ and open interval cover of cA, and $\ell(cI_n) = |c|\,\ell(I_n)$, and thus $m^*(cA) = |c|\,m^*(A)$.

Now, suppose $A \in \mathcal{M}$. Then, we have for any $B \subseteq \mathbb{R}$,

$$\begin{aligned} m^*(B) = |c|\,m^*\left(\frac{1}{c}B\right) &= |c|\,m^*\left(\frac{1}{c}B \cap A\right) + |c|\,m^*\left(\frac{1}{c}B \cap A^c\right) \\ &= m^*(B \cap cA) + m^*\big(B \cap (cA)^c\big), \end{aligned}$$

so $cA \in \mathcal{M}$. ■

§1.7 Relationship between $\mathfrak{B}_{\mathbb{R}}$ and $\mathcal{M}$

↪**Definition 1.8**: Given $(X, \mathcal{F}, \mu)$, consider the following collection of subsets of X,

$$\mathcal{N} := \{B \subseteq X : \exists\, A \in \mathcal{F} \text{ s.t. } \mu(A) = 0, B \subseteq A\}.$$

Put $\overline{\mathcal{F}} := \sigma(\mathcal{F} \cup \mathcal{N})$; this is called the *completion* of $\mathcal{F}$ with respect to μ.

↪**Proposition 1.14**: $\overline{\mathcal{F}} = \{F \subseteq X : \exists\, E, G \in \mathcal{F} \text{ s.t. } \exists\, E \subseteq F \subseteq G \text{ and } m(G \setminus E) = 0\}$.

PROOF. Put $\mathcal{G}$ the set on the right; one can check $\mathcal{G}$ a σ-algebra. Since $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{N} \subseteq \mathcal{G}$, we have $\overline{\mathcal{F}} \subseteq \mathcal{G}$.

Conversely, for any $F \in \mathcal{G}$, we have $E, G \in \mathcal{F}$ such that $E \subseteq F \subseteq G$ with $m(G \setminus E) = 0$. We can rewrite

$$F = \underbrace{E}_{\in \mathcal{F}} \cup \underbrace{(F \setminus E)}_{\substack{\subseteq G \setminus E \\ \Rightarrow \mu(F \setminus E) = 0 \\ \Rightarrow G \setminus E \in \mathcal{N}}},$$

hence $F \in \mathcal{F} \cup \mathcal{N}$ and thus in $\mathcal{F}$, and equality holds. ■

↪**Definition 1.9**: Given $(X, \mathcal{F}, \mu)$, μ can be *extended* to $\overline{\mathcal{F}}$ by, for each $F \in \overline{\mathcal{F}}$ with $E \subseteq F \subseteq G$ s.t. $\mu(G \setminus E) = 0$, put

$$\mu(F) = \mu(E) = \mu(G).$$

We call then $(X, \mathcal{F}, \mu)$ a *complete measure space*.

Remark 1.9: It isn't obvious that this is well defined a priori; in particular, the E, G sets are certainly not guaranteed to be unique in general, so one must check that this definition is valid regardless of choice of "sandwich sets".

↪**Theorem 1.5**: $(\mathbb{R}, \mathcal{M}, m)$ is the completion of $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, m)$.

Proof. Given $A \in \mathcal{M}$, then $\forall\, n \geq 1, \exists\, G_n$-open with $A \subseteq G_n$ s.t. $m^*(G_n \setminus A) \leq \frac{1}{n}$ and $\exists\, F_n$-closed with $F_n \subseteq A$ s.t. $m^*(A \setminus F_n) \leq \frac{1}{n}$.

Put $C := \bigcap_{n=1}^{\infty} G_n$, $B := \bigcap_{n=1}^{\infty} F_n$, remarking that $C, B \in \mathfrak{B}_{\mathbb{R}}$, $B \subseteq A \subseteq C$, and moreover

$$m(C \setminus A) \leq \frac{1}{n}, m(A \setminus B) \leq \frac{1}{n}$$

$$\Rightarrow m(C \setminus B) = m(C \setminus A) + m(A \setminus B) \leq \frac{2}{n},$$

but n can be arbitrarily large, hence $m(C \setminus B) = 0$; in short, given a measurable set, we can "sandwich it" arbitrarily closely with Borel sets. Thus, $A \in \overline{\mathfrak{B}_{\mathbb{R}}} \Rightarrow \mathcal{M} \subseteq \overline{\mathfrak{B}_{\mathbb{R}}}$. But recall that $\mathcal{M}$ complete, so $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M} \Rightarrow \overline{\mathfrak{B}_{\mathbb{R}}} \subseteq \overline{\mathcal{M}} = \mathcal{M}$, and thus $\overline{\mathfrak{B}_{\mathbb{R}}} = \mathcal{M}$ indeed.

Heuristically, this means that any measurable set is "different" from a Borel set by at most a null set. ■

§1.8 Some Special Sets

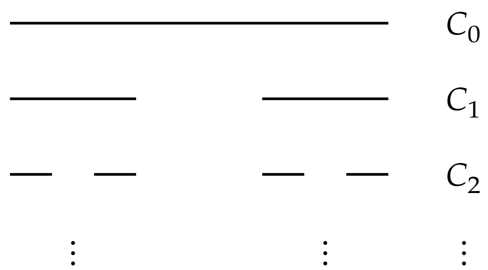
1.8.1 Uncountable Null Set?

Remark that for any countable set $A \in \mathcal{M}$, $m(A) = 0$; indeed, one may write $A = \bigcup_{n=1}^{\infty} \{a_n\}$ for singleton sets $\{a_n\}$, and so

$$m(A) = \sum_{n=1}^{\infty} m(a_n) = 0.$$

One naturally asks the opposite question, does there exist a measurable, *uncountable* set with measure 0? We construct a particular one here, the Cantor set, C.

This requires an "inductive" construction. Define $C_0 = [0, 1]$, and define C_k to be C_{k-1} after removing the middle third from each of its disjoint components. For instance $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, then $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, and so on. This may be clearest graphically:



Remark that the $C_n \downarrow$. Put finally

$$C := \bigcap_{n=1}^{\infty} C_n.$$

↪**Proposition 1.15**: The following hold for the Cantor set C:

1. C is closed (and thus $C \in \mathfrak{B}_{\mathbb{R}}$);
2. $m(C) = 0$;
3. C is uncountable.

Proof.

1. For each n, C_n is the countable (indeed, finite) union of 2^n-many disjoint, closed intervals, hence each C_n closed. C is thus a countable intersection of closed sets, and is thus itself closed.
2. For each n, each of the 2^n disjoint closed intervals in C_n has length $\frac{1}{3^n}$, hence
$$m(C_n) = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$
Since $\{C_n\} \downarrow$, by continuity of m we have
$$m(C) = \lim_{n \Rightarrow \infty} m(C_n) = \lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0.$$
3. This part is a little trickier. Notice that for any $x \in [0, 1]$, we can define a sequence (a_n) where each $a_n \in \{0, 1, 2\}$, and such that
$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n};$$
in particular, this is just the base-3 representation of x, which we denote $(x)_3 = (a_1 a_2 \cdots)$.

I claim now that
$$C = \{x \in [0, 1] : (x)_3 \text{ has no 1's}\}.$$
Indeed, at each stage n of the construction of the Cantor set, we get rid of the segment of the real line that would correspond to the $a_n = 1$. One should note that $(x)_3$ not necessarily unique; for instance $\left(\frac{1}{3}\right)_3 = (1, 0, 0, \ldots) = (0, 2, 2, \ldots)$, but if we specifically consider all x such that there *exists* a base three representation with no 1's, i.e. like $\frac{1}{3}$, then C indeed captures all the desired numbers.

Thus, we have that
$$\text{card}(C) = \text{card}(\{\{a_n\} : a_n = 0, 2\}).$$
Define now the function
$$f : C \to [0, 1], \quad x \mapsto \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot \frac{1}{2^n}, \text{where } (x)_3 = (a_n)$$
i.e., we "squish" the base-3 representation into a base-2 representation of a number. This is surjective; for any $y \in [0, 1]$, $(b_n) := (y)_2$ contains only 0's and 1's, hence $(2b_n)$

contains only 0's and 1's, so let x be the number such that $(x)_3 = (2b_n)$. This necessarily exists, indeed, we simply take our definitions backwards:

$$x := \sum_{n=1}^{\infty} \frac{2b_n}{3^n},$$

which maps to y under f and is contained in C. Hence, $\operatorname{card}(C) \geq \operatorname{card}([0,1])$; but $[0,1]$ uncountable, and thus so is C. ■

We can naturally extend the function f used here to map the entire interval $[0,1] \to [0,1]$ as follows

$$f(x) := \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot \frac{1}{2^n} \text{ if } x \in C, (x)_3 = (a_n) \\ f(a) \text{ if } x \notin C \text{ then } x \in (a,b) \text{ s.t. } (a,b) \text{ removed from } [0,1] \end{cases}.$$

This function is often called the *Devil's Staircase* or *Cantor-Lebesgue function*.

↪**Proposition 1.16**:
1. $f(0) = 0, f(1) = 1, f \equiv \frac{1}{2}$ on $\left(\frac{1}{3}, \frac{2}{3}\right), f \equiv \frac{1}{4}$ on $\left(\frac{1}{9}, \frac{2}{9}\right)$
2. $f : [0,1] \to [0,1]$ a surjection
3. f is nondecreasing
4. f is continuous

Proof. 1., 2., clear from construction.

For 3., let $x_1 < x_2 \in C$, and suppose $(x_1)_3 = (a_n), (x_2)_3 = (b_n)$. Then, since $x_1 < x_2$, it must be that a_n, b_n can only be equal up to some finite N; then the next $0 = a_{N+1} < b_{N+1} = 2$. Hence, it follows that the "modified binary expansion" that arises from f gives directly that $f(x_1) \leq f(x_2)$.

For 4., f is clearly continuous on $[0,1] - C$, since it is piecewise-constant here. Also, f is "one-sided continuous" at each of the "boundary points" $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \ldots$. If $x \in C$, for any $n \geq 1$, there must be x_n, x_n' such that $x_n < x < x_n'$ (if $x = 0$, only need x_n', if $x = 1$, only need x_n) and $f(x_n') - f(x_n) \leq \frac{1}{2^n}$. Then, f is continuous at x by monotonicity of f. ■

1.8.2 Non-Measurable Sets?

We've shown then that there is indeed an uncountable set of measure 0. Another question we may ask ourselves is, is there a $A \subseteq \mathbb{R}$ that is non-measurable? The answer to this turns out to be yes, but the construction requires invoking the axiom of choice:

Axiom 1 (Of Choice): If Σ a collection of nonempty sets, then $\exists$ a function

$$S : \Sigma \to \bigcup_{A \in \Sigma} A,$$

such that $A \in \sigma$, $S(A) \in A$. Such a function is called a *selection function*, and $S(A)$ a *representative* of A.

We construct now a non-measurable set, assuming the above. Consider $[0, 1]$, and define an equivalence relation $\sim$ on $[0, 1]$ by

$$a \sim b \Leftrightarrow a - b \in \mathbb{Q}.$$

Its easy to check that this is indeed an equivalence relation. Denote by E_a the equivalence class containing a, and set $\Sigma = \{E_a : a \in [0, 1]\}$. Note that for any $E_a \in \Sigma, E_a \neq \varnothing$.

Invoking the axiom of choice, we can select exactly one element S_a from E_a for each $E_a \in \Sigma$. Set

$$N := \{S_a : S_a \text{ is a representative of } E_a, E_a \in \Sigma\}.$$

↪**Proposition 1.17**: N, called a *Vitali set*, is non-measurable.

Proof. Assume towards a contradiction that N indeed measurable, $N \in \mathcal{M}$. Consider $[-1, 1] \cap \mathbb{Q}$; this is countable, so we can enumerate it $\{q_k\}$, $k \geq 1$. For each k, put

$$N_k := N + q_k.$$

By the assumption of measurability and translation invariance of m, it must be that each N_k measurable and has the same measure as N.

We claim each N_k disjoint. Assume not, then $\exists k \neq \ell$ (i.e. $q_k \neq q_\ell$) and $S_a, S_b \in N$ such that $S_a + q_k = S_b + q_\ell$. But then $S_a - S_b = q_\ell - q_k \in \mathbb{Q}$, hence $S_a \sim S_b$. But we constructed N to have only one representative from each equivalence class, hence it must be that $S_a = S_b$, and so $S_a + q_k = S_a + q_\ell \Rightarrow q_k = q_\ell$, contradicting the assumed distinctness of the q's; hence, the N_k's indeed disjoint.

We claim next that $[0, 1] \subseteq \bigcup_{n-1}^{\infty} N_k$. Let $x \in [0, 1]$. Then, $x \sim S_a$ for some unique $S_a \in N$ and so $x - S_a \in \mathbb{Q}$. But also, $x, S_a \in [0, 1]$, hence $x - S_a \in [-1, 1]$ (moreover, $x - S_a \in [-1, 1] \cap \mathbb{Q}$) and there must exist a k such that $x - S_a = q_k$, since the q_k's enumerate the entire $[-1, 1] \cap \mathbb{Q}$. Thus, $x \in N_k$ by the construction of the N_k's. Thus, $[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k$ indeed.

On the other hand, $\bigcup_{k=1}^{\infty} N_k \subseteq [-1, 2]$ and so we have the "bound"

$$[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k \subseteq [-1, 2].$$

Taking the measure of all sides then, we have the bound

$$1 \leq \mu\left(\bigcup_{n=1}^{\infty} N_k\right) \leq 3.$$

Invoking the disjointness of the N_k's, we can also use countable additivity to write

$$\mu\left(\bigcup_{n=1}^{\infty} N_k\right) = \sum_{k=1}^{\infty} m(N_k) = \sum_{k=1}^{\infty} m(N),$$

but this final line is a sequence of positive, constant real numbers; hence, it is impossible for it to be within 1 and 3, and we have a contradiction. Hence, N indeed not measurable.

Remark that this proof also shows that $m^*(N_k) > 0$ so $m^*(N) > 0$ (given the interval bound on N we've found). ■

↪**Proposition 1.18**: For every $A \in \mathcal{M}$ such that $m(A) > 0$, there exists $B \subseteq A$ such that B is non-measurable.

Proof. Assume otherwise, that there is a $A \in \mathcal{M}$ with $m(A) > 0$ such that any subset B of A is also measurable.

Remark that $A \subseteq \bigcup_{n \in \mathbb{Z}} A \cap [n, n+1]$. Then, there exists an n such that $m(A \cap [n, n+1]) > 0$ and thus, translating $A' := A \cap [n, n+1] - n)$, $m(A') > 0$, noting that $A' \subseteq [0, 1]$. Now, for any $B' \subseteq A'$, $B' + n \subseteq A$. By assumption, then $B' + n$ must be measurable so B' measurable.

In summary, then, we have $A' \subseteq [0, 1]$ with $m(A') > 0$ such that (by assumption) B' measurable for all $B' \subseteq A'$.

Let $N, \{q_k\}, N_k$ be as in the previous proof. Set

$$A_k' := A' \cap N_k, k \geq 1.$$

Then, A_k' disjoint, and

$$A' = [0, 1] \cap A' \subseteq \bigcup_{k=1}^{\infty} (N_k \cap A') = \bigcup_{k=1}^{\infty} A_{k'}.$$

Since $m(A') > 0$, there exists a k such that $m(A_k') > 0$. Set, for this k,

$$L := \{\ell \geq 1 : q_\ell + q_k \in [-1, 1]\}.$$

This set is again countably infinite. We translate, obtaining a disjoint sequence of sets $\{q_\ell + A_k' : \ell \in L\}$; since $q_\ell + q_k \in [-1, 1] \cap \mathbb{Q}$, then $q_\ell + q_k = q_m$ for some unique m, and so $q_\ell + A_k' = q_\ell + A' \cap (N + q_k) \subseteq N_m$. Hence, we have on the one hand that by countable additivity

$$\bigcup_{\ell \in L} (q_\ell + A_k') \subseteq [-1, 2] \Rightarrow \sum_{\ell \in L} m(q_\ell + A_k') \leq 3,$$

and so it must be that $m(q_\ell + A_k') = m(A_k') = 0$ (else the series couldn't be finite), contradicting the finiteness assumption on $m(A_k')$. ■

1.8.3 Non-Borel Measurable Set?

We may ask, is there $A \in \mathcal{M}$ such that $A \notin \mathfrak{B}_{\mathbb{R}}$?

Let $f : [0,1] \to [0,1]$ be the Cantor-Lebesgue function, and put $g(x) = f(x) + x$; note that g is continuous and strictly increasing, and is defined $g : [0,1] \to [0,2]$. Remark that g bijective; the strictly increasing gives injective, and moreover $g(0) = 0, g(1) = 2$ hence by intermediate value theorem it is surjective. Hence, $g^{-1} : [0,2] \to [0,1]$ exists, and is also continuous, so in short g is a homeomorphism; it maps open to open, closed to closed. In particular, if $A \in \mathfrak{B}_{\mathbb{R}}$, then $g(A) \in \mathfrak{B}_{\mathbb{R}}$.

Recall that if (a, b) an open interval that gets removed from the construction of C, then f is constant and so g will map (a, b) to another open interval of the same length $b - a$. Thus,

$$m(g([0,1] \setminus C)) = m([0,1] \setminus C) = 1.$$

Hence, $m(g(C)) = 2 - 1 = 1 > 0$, since $g(C \cup [0,1] \setminus C) = [0,2]$. Hence, there exists a $B \subseteq G(C)$ such that $B \notin \mathcal{M}$, as per the previous proposition.

Let $A := g^{-1}(B)$; then $A \subseteq g^{-1}(g(C)) = C$. Since $m(C) = 0$, $A \in \mathcal{M}$ and $m(A) = 0$. But, $A \notin \mathfrak{B}_{\mathbb{R}}$; if it were, then $g(A) = B \in \mathfrak{B}_{\mathbb{R}}$, since g "maintains" Borel sets, but B is not even Lebesgue measurable and so this is a contradiction).

§2 Integration Theory

§2.1 Measurable Functions

We will be considering functions f defined on $\mathbb{R}$ or some subset of $\mathbb{R}$ that could take positive or negative infinity as its value i.e.

$$f : \mathbb{R} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\},$$

where $\overline{\mathbb{R}}$ the *extended real line*; we say f is $\overline{\mathbb{R}}$-valued. If f never takes $\infty, -\infty$ for any $x \in \mathbb{R}$, we say f finite-valued, or just $\mathbb{R}$-valued.

For all $a \in \mathbb{R}$, we consider inverse images

$$f^{-1}([-\infty, a)) := \{x \in \mathbb{R} : f(x) \in [-\infty, a)\} = \{f < a\},$$

remarking the inclusion of $-\infty$; similarly

$$f^{-1}((a, \infty]) := \{x \in \mathbb{R} : f(x) \in (a, \infty]\} = \{f > a\},$$

and so on, for any $B \subseteq \mathbb{R}$,

$$f^{-1}(B) := \{x \in \mathbb{R} : f(x) \in B\} = \{f \in B\}.$$

Remark that

$$\begin{aligned} f^{-1}(B^c) &= \left(f^{-1}(B)\right)^c \\ f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B) \\ f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B), \end{aligned}$$

which extend naturally for countable unions/intersections.

↪**Definition 2.1** (Measurable Function): $f : \mathbb{R} \to \overline{\mathbb{R}}$ is *measurable* if $\forall\, a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) \in \mathcal{M}.$$

↪**Proposition 2.1** (Equivalent Definitions of Measurability):

$$\begin{aligned} f \text{ is measurable} &\Leftrightarrow \forall\, a \in \mathbb{R}, f^{-1}([a,\infty]) \in \mathcal{M} \\ &\Leftrightarrow \forall\, a \in \mathbb{R}, f^{-1}((a,\infty]) \in \mathcal{M} \\ &\Leftrightarrow \forall\, a \in \mathbb{R}, f^{-1}([-\infty,a]) \in \mathcal{M} \end{aligned}$$

Proof. We prove just the last equivalence. Notice that $\forall\, a \in \mathbb{R}$, we can use the commuting of inverse images with countable unions, intersections, complement to write

$$f^{-1}([-\infty,a)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a - \frac{1}{n}\right)\right)$$

and

$$f^{-1}([-\infty,a]) = \bigcap_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a + \frac{1}{n}\right)\right).$$

■

↪**Proposition 2.2**: If f finite-valued, Then

$$\begin{aligned} f \text{ is measurable} &\Leftrightarrow \forall\, a < b \in \mathbb{R}, f^{-1}((a,b)) \in \mathcal{M} \\ &\Leftrightarrow \quad \cdots \quad f^{-1}((a,b]) \in \mathcal{M} \\ &\Leftrightarrow \quad \cdots \quad f^{-1}([a,b)) \in \mathcal{M} \\ &\Leftrightarrow \quad \cdots \quad f^{-1}([a,b]) \in \mathcal{M}. \end{aligned}$$

↪**Definition 2.2** (Extended Borel Sigma Algebra): Define the Borel "extended" algebra $\mathfrak{B}_{\overline{\mathbb{R}}}$ of subsets of $\overline{\mathbb{R}}$, defined by

$$\mathfrak{B}_{\overline{\mathbb{R}}} := \sigma(\mathfrak{B}_{\mathbb{R}} \cup \{\{-\infty\},\{\infty\}\}).$$

↪**Proposition 2.3**: $\mathfrak{B}_{\overline{\mathbb{R}}} = \sigma(\{[-\infty,a) : a \in \mathbb{R}\})$.

Proof. For every $a \in \mathbb{R}$, we may write

$$[-\infty, a) = \underbrace{(-\infty, a)}_{\in \mathfrak{B}_{\mathbb{R}}} \cup \{-\infty\} \in \mathfrak{B}_{\overline{\mathbb{R}}},$$

so $\sigma(\{[-\infty, a) : a \in \mathbb{R}\}) \subseteq \mathfrak{B}_{\overline{\mathbb{R}}}$.

Conversely, notice that

$$\{-\infty\} = \bigcap_{n=1}^{\infty} [-\infty, -n),$$

and

$$\{\infty\} = \overline{\mathbb{R}} - \left(\bigcup_{n=1}^{\infty} [-\infty, n) \right),$$

so $\{-\infty\}, \{\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$. Hence, for any $a \in \mathbb{R}$,

$$(-\infty, a) = [-\infty, a) - \{-\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\}),$$

and so $\mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$. $\{-\infty\}, \{\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ already, and thus $\mathfrak{B}_{\overline{\mathbb{R}}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$. ■

↪**Proposition 2.4**: $f : \mathbb{R} \to \overline{\mathbb{R}}$ measurable $\Leftrightarrow$ for all $B \in \mathfrak{B}_{\overline{\mathbb{R}}}, f^{-1}(B) \in \mathcal{M}$.

Proof. $\Leftarrow$ is immediate. For $\Rightarrow$, let $\mathcal{C}$ be a collection of subsets of $\overline{\mathbb{R}}$, then put

$$f^{-1}(\mathcal{C}) := \{f^{-1}(B) : B \in \mathcal{C}\}.$$

By an assignment question (2.6),

$$f^{-1}(\sigma(\mathcal{C})) = \sigma\big(f^{-1}(\mathcal{C})\big).$$

Take $\mathcal{C} = \{[-\infty, a) : a \in \mathbb{R}\}$. Then,

$$f^{-1}(\sigma(\mathcal{C})) = f^{-1}\big(\mathfrak{B}_{\overline{\mathbb{R}}}\big) = \sigma\big(f^{-1}(\{[-\infty, a) : a \in \mathbb{R}\})\big).$$

But f measurable, so $f^{-1}([-\infty, a)) \in \mathcal{M}$ for each $a \in \mathbb{R}$, hence sigma $\big(f^{-1}(\{[-\infty, a) : a \in \mathbb{R}\})\big) \subseteq \mathcal{M}$ and so $f^{-1}(\sigma(\mathcal{C})) \subseteq \mathcal{M}$ completing the proof. ■

↪**Corollary 2.1**: If f finite-valued, then f is measurable $\Leftrightarrow$ for every $B \in \mathfrak{B}_{\mathbb{R}}, f^{-1}(B) \in \mathcal{M}$.

↪**Proposition 2.5**: Given $f : \mathbb{R} \to \overline{\mathbb{R}}$, define the *finite valued component* of f given by

$$f_{\mathbb{R}}(x) := \begin{cases} f(x) : -\infty < f(x) < \infty \\ 0 \text{ otherwise} \end{cases}.$$

Then, f measurable $\Leftrightarrow$ $\forall B \in \mathfrak{B}_{\mathbb{R}}, f_{\mathbb{R}}^{-1}(B) \in \mathcal{M}$ AND $\{f = \infty\}, \{f = -\infty\}$ both in $\mathcal{M}$.

Proof. ($\Leftarrow$) For any $a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) = \{f = -\infty\} \cup f^{-1}((-\infty, a)) = \{f = -\infty\} \cup f_{\mathbb{R}}^{-1}((-\infty, a)),$$

a union of measurable sets and hence is itself measurable.

($\Rightarrow$) Remark that $\{f=\infty\},\{f=-\infty\}\in\mathcal{M}$ automatically. For any $B\in\mathfrak{B}_{\mathbb{R}}$, we have

$$f_{\mathbb{R}}^{-1}(B)=\{x\in\mathbb{R}:f_{\mathbb{R}}(x)\in B\}=\{x\in\mathbb{R}:f(x)\in B,-\infty<f<\infty\}\cup\{x\in\mathbb{R}:0\in B,f(x)=\pm\infty\}\in\mathcal{M}.$$

■

↪**Definition 2.3**: If a statement is true for every $x\in A$ where $A\in\mathcal{M}$ s.t. $m(A^c)=0$, then we say the statement is true a.e. (almost everywhere).

↪**Proposition 2.6**: If $f:\mathbb{R}\to\overline{\mathbb{R}}$ is measurable and $f=g$ a.e. then g is measurable.

↪**Corollary 2.2**: If f is finite-valued a.e., then f is measurable $\Leftrightarrow f_{\mathbb{R}}$ is measurable $\Leftrightarrow\ \forall\, a<b\in\mathbb{R}, f^{-1}((a,b))\in\mathcal{M}$.

↪**Proposition 2.7**: If $f\equiv c$ then f measurable.

If $f=\mathbb{1}_A$ for some $A\subseteq\mathbb{R}$, then f is measurable $\Leftrightarrow A\in\mathcal{M}$.

Proof. Assume $f\equiv c$. Then

$$f^{-1}([-\infty,a))=\begin{cases}\mathbb{R}\text{ if }c<a\\ \varnothing\text{ if }c\geq a\end{cases}\in\mathcal{M}.$$

Assume now $f=\mathbb{1}_A$. For all $a\in\mathbb{R}$,

$$f^{-1}([-\infty,a))=\begin{cases}\mathbb{R}\text{ if }a>1\\ A^c\text{ if }0<a\leq 1\\ \varnothing\text{ if }a\leq 0\end{cases}\in\mathcal{M}\Leftrightarrow A\in\mathcal{M}.$$

■

↪**Proposition 2.8**: If f is (finite-valued) continuous, then f is measurable.

Proof. $f:\mathbb{R}\to\mathbb{R}$ continuous $\Leftrightarrow$ for all $G\subseteq\mathbb{R}$ open, $f^{-1}(G)$ open. For all $a<b\in\mathbb{R}$, then $f^{-1}((a,b))$ open so $f^{-1}((a,b))\in\mathcal{M}$ so f measurable.

In fact, if $f:\mathbb{R}\to\mathbb{R}$ continuous, then for all $B\in\mathfrak{B}_{\mathbb{R}}, f^{-1}(B)\in\mathfrak{B}_{\mathbb{R}}$;

$$f^{-1}(\mathfrak{B}_{\mathbb{R}})=f^{-1}(\sigma(\{\text{open sets}\}))=\sigma\left(\underbrace{f^{-1}(\{\text{open sets}\})}_{\text{all open}}\right)\subseteq\sigma(\{\text{open sets}\})=\mathfrak{B}_{\mathbb{R}}.$$

Moreover, if f^{-1} (inverse) exists and is continuous, then for any $B\in\mathfrak{B}_{\mathbb{R}}, f(B)\in\mathfrak{B}_{\mathbb{R}}$. ■

↪**Proposition 2.9**: If $f : \mathbb{R} \to \mathbb{R}$ is measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g \circ f$ is measurable.

Remark 2.1: The order matters! The converse doesn't hold in general.

Proof. For all $a \in \mathbb{R}$,

$$\begin{aligned}(g \circ f)^{-1}((-\infty, a)) &= \{x \in \mathbb{R} : g(f(x)) < a\} \\ &= \left\{x \in \mathbb{R} : f(x) \in g^{-1}([-\infty, a))\right\} \\ &= f^{-1}\left(g^{-1}([-\infty, a))\right) \in \mathcal{M}.\end{aligned}$$

■

↪**Proposition 2.10**: If $f : \mathbb{R} \to \overline{\mathbb{R}}$ is measurable, then:
1. for every $c \in \mathbb{R}$, cf is measurable (in particular $-f$ measurable);
2. $|f|$ is measurable;
3. for every $k \in \mathbb{N}$, f^k is a measurable.

Proof. We prove just 3. If $k = 0$ this is trivial. For any $a \in \mathbb{R}$,

$$\left(f^k\right)^{-1}([-\infty, a)) = \begin{cases} f^{-1}\left([-\infty, a^{\frac{1}{k}})\right) & \text{if } k \text{ is odd} \\ \varnothing & \text{if } k \text{ is even and } a \leq 0 \\ f^{-1}\left([-a^{\frac{1}{k}}, a^{\frac{1}{k}})\right) & \text{if } k \text{ is even and } a > 0 \end{cases} \in \mathcal{M}.$$

■

↪**Proposition 2.11**: If f, g are two finite-valued measurable functions, then $f + g, f \cdot g, f \vee g := \max\{f, g\}, f \wedge g := \min\{f, g\}$ are measurable functions, where

$$(f \vee g)(x) = \max\{f(x), g(x)\}.$$

Proof. For all $a \in \mathbb{R}$,

$$\begin{aligned}(f + g)^{-1}([-\infty, a) &= \{x \in \mathbb{R} : f(x) + g(x) < a\} \\ &= \{x \in \mathbb{R} : f(x) < a - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \{x \in \mathbb{R} : f(x) < q < a - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \underbrace{\{x \in \mathbb{R} : f(x) < q\}}_{\in \mathcal{M}} \cap \underbrace{\{x \in \mathbb{R} : g(x) < a - q\}}_{\in \mathcal{M}} \in \mathcal{M}.\end{aligned}$$

This implies, then, that $f - g$ measurable, as are $(f + g)^2$ and $(f - g)^2$, and thus

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$$

is measurable.

We have too that

$$f \vee g = \frac{1}{2}(|f-g| + (f+g))$$

and so is measurable, and so

$$f \wedge g = -\max\{-f, -g\} = -(-f \vee -g)$$

is measurable. ■

↪**Corollary 2.3**: If f is measurable, then $f^+ := f \vee 0 = \max\{f, 0\}$ and $f^- := -(f \wedge 0) = \max\{-f, 0\}$ are measurable, as is $f \wedge k$ for any $k \in \mathbb{R}$.

Remark 2.2: Notice that $f = f^+ - f^-$, even with "infinities", and $|f| = f^+ + f^-$.

↪**Proposition 2.12**: Let $\{f_n\}$ be a sequence of measurable functions. Then, $\sup_n f_n$, $\inf_n f_n$, $\limsup_{n\to\infty} f_n$, and $\liminf_{n\to\infty} f_n$ are all measurable (where $(\limsup_{n\to\infty} f_n)(x) := \limsup_{n\to\infty} f_n(x) = \inf_{m\geq 1} \sup_{n\geq m} f_n(x) = \lim_{m\to\infty} \sup_{n\geq m} f_n(x)$).

Proof. To show $\sup_n f_n$ measurable, we will show for all $a \in \mathbb{R}$ $\{\sup_n f_n \leq a\} \in \mathcal{M}$.

$$x \in \left\{\sup_n f_n \leq a\right\} \Leftrightarrow \sup_n f_n(x) \leq a \Leftrightarrow f_n(x) \leq a \,\forall\, n \geq 1 \Leftrightarrow x \in \bigcap_{n=1}^{\infty} \{f_n \leq a\},$$

hence $\{\sup_n f_n \leq a\} = \bigcap_{n=1}^{\infty} \underbrace{\{f_n \leq a\}}_{\in \mathcal{M}} \in \mathcal{M}$ and hence $\sup_n f_n$ is measurable. Note that using $\leq$ was important; $\{\sup_n f_n < a\} \subsetneq \bigcap_{n=1}^{\infty} \{f_n < a\}$, since the $\sup_n f_n$ could equal a. We could say the following, however:

$$\left\{\sup_n f_n < a\right\} = \bigcup_{k=1}^{\infty} \left\{\sup_n f_n \leq a - \frac{1}{k}\right\} = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{f_n \leq a - \frac{1}{k}\right\} \in \mathcal{M}.$$

Next, we have $\inf_n f_n = -\sup_n(-f_n)$ so we are done.

For lim sup, lim inf, we have

$$\limsup_n f_n = \inf_{m\geq 1} \underbrace{\sup_{n\geq m} f_n}_{:= g_m}.$$

g_m is measurable for each $m \geq 1$, hence $\inf_m g_m$ is measurable, hence $\limsup_n f_n$ is measurable. Similar logic follows for lim inf.

We could have show, more directly, that

$$
\begin{aligned}
\left\{\limsup_n f_n < a\right\} &= \left\{\inf_{m\geq 1} \sup(n \geq m) f_n < a\right\} \\
&= \bigcup_{m=1}^{\infty} \left\{\sup_{n\geq m} f_n < a\right\} \\
&= \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \left\{\sup_{n\geq m} f_n \leq a - \frac{1}{k}\right\} \\
&= \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{f_n \leq a - \frac{1}{k}\right\}.
\end{aligned}
$$

■

↪**Proposition 2.13**: Let $\{f_n\}$ be a sequence of measurable functions. Then, all of the following sets are also measurable:

$$
\begin{aligned}
&\left\{x \in \mathbb{R} : \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\right\} =: \left\{\lim_{n\to\infty} f_n \text{ exists in } \mathbb{R}\right\}, \\
&\{\lim f_n = \infty\}, \{\lim f_n = -\infty\}, \{\lim f_n = c \in \mathbb{R}\}.
\end{aligned}
$$

Moreover, if $\lim_{n\to\infty} f_n$ exists (in $\mathbb{R}$ or as $\pm\infty$) a.e. with $f = \lim_{n\to\infty} f_n$ a.e. then f is measurable.

Proof. We have

$$
\begin{aligned}
\{\lim f_n \text{ exists in } \mathbb{R}\} &= \{\limsup f_n = \liminf f_n \text{ and } -\infty < \limsup f_n < \infty\} \\
&= \{-\infty < \liminf f_n < \infty\} \cap \{-\infty < \limsup f_n < \infty\} \cap \{\limsup f_n - \liminf f_n = 0\} \in \mathcal{M}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\{\lim f_n = c\} &= \left\{x \in \mathbb{R} : \forall\, k \geq 1, \exists\, n \geq 1 \;\text{ s.t.} \forall\, m \geq n, |f_n(x) - c| \leq \frac{1}{k}\right\} \\
&= \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \left\{|f_n(x) - c| \leq \frac{1}{k}\right\}.
\end{aligned}
$$

■

§2.2 Approximation by Simple Functions

Given a function $f : \mathbb{R} \to \overline{\mathbb{R}}$, measurable, we may write

$$
f = f^+ - f^-,
$$

where f^+, f^- are non-negative measurable functions; so, it suffices to study non-negative measurable functions. For any $n \geq 1$, we have

$$
f_n^+ := (f^+ \wedge n) \cdot \mathbb{1}_{[-n,n]},
$$

i.e., we cap f^+ at n, and disregard values of f^+ outside of $[-n, n]$; hence we limit our view to a $2n \times n$ "box". Then, f_n^+ is non-negative, measurable, bounded (by n), compactly supported (zero outside a bounded set), and in particular $f_n^+ \uparrow$, with limit

$$
\lim_{n\to\infty} f_n^+ = f^+.
$$

An identical construction follows for f^- with

$$f_n^- := (f^- \wedge n)\mathbb{1}_{[-n,n]},$$

with $f_n^- \uparrow$ and

$$\lim_{n\to\infty} f_n^- = f^-.$$

Fix some n and consider f_n^+. For $k = 0, 1, 2, ..., 2^n n$, define

$$A_{n,k} := \left\{x \in [-n,n] : \frac{k}{2^n} \leq f_n^+(x) < \frac{k+1}{2^n}\right\} = \left\{\frac{k}{2^n} \leq f_n^+ < \frac{k+1}{2^n}\right\} \cap [-n,n] \in \mathcal{M},$$

noting that $A_{n,k} \cap A_{n,\ell} = \emptyset$ if $k \neq \ell$. Set now

$$\varphi_n := \sum_{k=0}^{n\cdot 2^n} \mathbb{1}_{A_{n,k}} \frac{k}{2^n} = \sum_{k=0}^{n\cdot 2^n} \begin{cases} \frac{k}{2^n} \text{ if in } A_{n,k} \\ 0 \text{ else} \end{cases}.$$

We call φ_n a "simple function"; more generally:

↪**Definition 2.4**: φ is a *simple function* if $\varphi = \sum_{k=1}^{L} \mathbb{1}_{E_k} \cdot a_k$ where L a positive integer, a_k's are constant, E_k's are measurable sets of finite measure.

Moreover, note that $\varphi_n \uparrow$; at each new stage $n \to n+1$, the regions are cut in two, $A_{n,k} = A_{n+1,2k} \cup A_{n+1,2k+1}$. In addition, we have $\varphi_n \leq f_n^+ \leq f^+$ for all n. Moreover, we have the following:

↪**Proposition 2.14**:

$$\lim_{n\to\infty} \varphi_n(x) = f^+(x)$$

for all $x \in \mathbb{R}$.

Proof. For all $x \in \mathbb{R}$, for sufficiently large n we have that $x \in [-n,n]$ and so $f^+(x) = f^+(x)\mathbb{1}_{[-n,n]}(x)$. Assume for now $f^+ < \infty$. Then, for sufficiently large(r?) n, we can ensure $f^+(x) < n$ and so $f^+(x) = f_n^+(x)$ for such an x. Further, we have that $x \in A_{n,k}$ for some k so $\varphi_n(x) = \frac{k}{2^n}$ and $f_n^+(x) < \frac{k+1}{2^n}$ and thus

$$0 \leq f_n^+(x) - \varphi_n(x) < \frac{k+1}{2^n} - \frac{k}{2^n} = 2^{-n}$$

by construction and so $0 \leq f^+(x) - \varphi_n(x) \leq 2^{-n}$ and thus $\lim_{n\to\infty} \varphi_n(x) = f^+(x)$.

In the case that $f^+(x) = \infty$, then $\varphi_n(x) = n$ for all sufficiently large n hence

$$\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} n = \infty = f^+(x).$$

■

↪**Theorem 2.1**: If g is measurable and non-negative, there exists a sequence of simply functions $\{\varphi_n\}$ such that $\varphi_n \uparrow$ and $\lim_{n\to\infty} \varphi_n(x) = g(x)$ for every $x \in \mathbb{R}$.

We can repeat this same construction and proof for f^- with a sequence $\widetilde{\varphi_n}$. Even better:

↪**Theorem 2.2**: If f is measurable, then $\exists$ a sequence of simple functions $\{\psi_n\}$ such that $|\psi_n| \uparrow$ and $|\psi_n| \leq |f|$ for all n and for all $x \in \mathbb{R}$, $\lim_{n\to\infty} \psi_n(x) = f(x)$.

PROOF. Take $\psi_n = \varphi_n - \widetilde{\varphi_n}$ as above; then for all $x \in \mathbb{R}$, at least one of $\varphi_n(x), \widetilde{\varphi_n}(x)$ equals zero. Then

$$|\psi_n| = \varphi_n + \widetilde{\varphi_n} < f^+ + f^- = |f|,$$

and

$$\lim_{n\to\infty} \psi_n(x) = \lim_{n\to\infty} \varphi_n(x) - \lim_{n\to\infty} \widetilde{\varphi_n}(x) = f^+ - f^- = f.$$

■

↪**Definition 2.5** (Step Function): θ a *step function* if it takes the form

$$\theta(x) = \sum_{k=1}^{L} a_k \mathbb{1}_{I_k}(x),$$

where $L \in \mathbb{N}, a_k$'s constant, and I_k finite, open intervals.

↪**Theorem 2.3**: If f is measurable, then there exists a sequence of step functions $\{\theta_n\}$ such that

$$\lim_{n\to\infty} \theta_n(x) = f(x) \text{ for } \textbf{almost every } x \in \mathbb{R}.$$

In particular, we do not have pointwise convergence as for general simple functions, but we have convergence outside a zero-measure set.

PROOF. Assume, wlog, that f non-negative (by the previous construction, we can "split" f if not and approximate its positive, negative parts). Given $A \in \mathcal{M}$ with finite measure, recall that for every $\varepsilon > 0$, there exists finitely many finite open intervals $I_1, \ldots, I_N$ such that

$$m\left(A \,\triangle \left(\bigcup_{i=1}^{N} I_i\right)\right) < \varepsilon.$$

By renaming/rearranging I_i's if necessary, we may assume that I_i's are disjoint; hence

$$\mathbb{1}_{\bigcup_{i=1}^{N} I_i} = \sum_{i=1}^{N} \mathbb{1}_{I_i}.$$

Put

$$\theta_A := \sum_{i=1}^{N} \mathbb{1}_{I_i},$$

noting this is indeed a step function as the name suggests. Then, remark that

$$\underbrace{m(\{x \in \mathbb{R} : \mathbb{1}_A(x) \neq \theta_A(x)\})}_{=A\triangle\left(\bigcup_{n=1}^{N} I_i\right)} < \varepsilon.$$

Since f measurable and non-negative, $\exists \{\varphi_n\}$ sequence of simple functions with limit f. In particular,

$$\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}}.$$

Applying our above analysis to each $A_{n,k}$, then, we have that for any $n \geq 1$ and $k = 0, 1, ..., n2^n$ we can find a step function $\theta_{n,k}$ such that

$$m\Big(\Big\{x \in \mathbb{R} : \mathbb{1}_{A_{n,k}} \neq \theta_{n,k}(x)\Big\}\Big) < \frac{1}{2^n(n2^n + 1)} \quad ('' = \varepsilon'').$$

Put then

$$\theta_n := \sum_{k=0}^{n2^n} \frac{k}{2^n} \theta_{n,k},$$

which is itself a step function. Put

$$E_n := \{x \in \mathbb{R} : \theta_n(x) \neq \varphi_n(x)\}.$$

Then,

$$m(E_n) \leq m\left(\bigcup_{k=0}^{n2^n} \Big\{\theta_{n,k} \neq \mathbb{1}_{A_{n,k}}\Big\}\right) \leq \sum_{k=0}^{n2^n} m\Big(\Big\{\theta_{n,k} \neq \mathbb{1}_{A_{n,k}}\Big\}\Big) \leq 2^{-n}.$$

The φ_n's are chosen such that $\forall\, x \in \mathbb{R}, |\varphi_n(x) - f_n(x)| \leq \frac{1}{2^n}$. Putting

$$F_n := \{x \in \mathbb{R} : |\theta_n(x) - f_n(x)\,| > 2^{-n}\},$$

then remark that $F_n \subseteq E_n$ so $m(F_n) \leq \frac{1}{2^n}$.

We claim now that for a.e. $x \in \mathbb{R}$, $\exists\, m \geq 1$ such that $\forall\, n \geq m$, $|\theta_n(x) - f_n(x)| \leq \frac{1}{2^n}$, remarking that such an m is *dependent* on x. Consider the complement of this statement; if this set has measure 0, we are done. The logical negation would be "for every $m \geq 1$, exist $n \geq m$ such that $|\theta_n(x) - f_n(x)\,| > 2^{-n}$", which is equivalent to the set

$$\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in \mathbb{R} : |\theta_n(x) - f_n(x)| > 2^{-n}\} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n.$$

Let $B_m := \bigcup_{n=m}^{\infty} F_n$; notice $B_m \downarrow$. Then, by continuity from above ****

$$m\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}F_n\right) = \lim_{m\to\infty} m(B_m) \le \lim_{m\to\infty}\sum_{n=m}^{\infty} m(F_n) \le \lim_{m\to\infty}\sum_{n=m}^{\infty}\frac{1}{2^n} = 0,$$

since the tail of a convergent series must converge to zero. Hence, the set has measure 0 as desired so for almost every $x \in \mathbb{R}$ there exists $m \ge 1$ such that for all $n \ge m$, $|\theta_n - f_n| \le \frac{1}{2^n}$, hence almost every where $\lim_{n\to\infty}(\theta_n - f_n) = 0$. Therefore, almost everywhere,

$$\theta_n = (\theta_n - f_n) + f_n \overset{n\to\infty}{\longrightarrow} f.$$

■

In this proof, we have proven (and then used) more generally:

↪**Lemma 2.1** (Borel-Cantelli Lemma): If $\{F_n\} \subseteq \mathcal{M}$ such that $\sum_{n=1}^{\infty} m(F_n) < \infty$, then

$$m\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}F_n\right) = 0.$$

§2.3 Convergence Almost Everywhere vs Convergence in Measure

↪**Definition 2.6** (Convergence Almost Everywhere): For measurable functions $\{f_n\}, f$ we say f_n converges to f a.e. and write $f_n \to f$ a.e. if for almost every $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Similarly, we say $f_n \to f$ a.e. on A if $\exists\, B \subseteq A$ with $m(B) = 0$ such that $\forall\, x \in A - B$, $\lim_{n\to\infty} f_n(x) = f(x)$.

↪**Definition 2.7** (Convergence in Measure): For measurable, finite-valued functions $\{f_n\}, f$ we say f_n converges to f in measure and write $f_n \to f$ in measure if for every $\delta > 0$,

$$\lim_{n\to\infty} m(\{x \in \mathbb{R} : |f_n(x) - f(x)| \ge \delta\}) = 0.$$

Similarly, we say $f_n \to f$ in measure on A if $\forall\, \delta > 0$, $\lim_{n\to\infty} m(\{x \in A : |f_n(x) - f(0)| \ge \delta\}) = 0$.

↪**Proposition 2.15**: Given finite-valued measurable functions $\{f_n\}, f$ and $A \in \mathcal{M}$ with finite measure, then if $f_n \to f$ a.e. on A, then $f_n \to f$ in measure on A.

Proof. For all $\delta > 0$,

$$\bigcap_{m=1}^{\infty}\bigcup_{n=m}\{x \in A : |f_n(x) - f(x)| > \delta\} \subseteq \left\{x \in A : \lim_{n\to\infty} f_n(x) \ne f(x)\right\}.$$

The set on the RHS has measure zero and thus so does the left one. Then,

$$\lim_{m\to\infty} m\left(\bigcup_{n=m}\{x \in A : |f_n(x) - f(x)| > \delta\}\right) = 0$$

by continuity, and

$$\{|f_m - f| > \delta\} \subseteq \bigcup_{n=m}^{\infty} \{|f_n - f| > \delta\}$$

hence $m(\{|f_m - f| > \delta\}) \le m\left(\bigcup_{n=m}^{\infty}\{|f_n - f| > \delta\}\right) \overset{m\to\infty}{\longrightarrow} 0$. ■

⊛ **Example 2.1**: We give an example of why the assumption that $m(A) < \infty$ is necessary. Let, $f_n = \mathbb{1}_{[n,\infty)}$ and $f \equiv 0$. Then, $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in \mathbb{R}$. But $m(\{x \in \mathbb{R} : |f_n(x) - f(x)| = 1\}) = m([n,\infty)) = \infty$.

In general, the converse statement $f_n \to f$ in measure does *not* imply that $f_n \to f$ almost everywhere, even on finite measure sets. Put $\varphi_{1,1} = \mathbb{1}_{[0,1)}$, $\varphi_{2,1} = \mathbb{1}_{\left[0,\frac{1}{2}\right)}$, $\varphi_{2,2} = \mathbb{1}_{\left[\frac{1}{2},1\right)}$, $\varphi_{3,1} = \mathbb{1}_{\left[0,\frac{1}{3}\right)}$, $\varphi_{3,2} = \mathbb{1}_{\left[\frac{1}{3},\frac{2}{3}\right)}$, $\varphi_{3,3} = \mathbb{1}_{\left[\frac{2}{3},1\right)}$, or in general $\varphi_{k,j} = \mathbb{1}_{\left[\frac{j-1}{k},\frac{j}{k}\right)}$ for $j = 1, \ldots, k$. Reorder $\varphi_{k,j}$ "lexicographically" into $\{f_n\}$. Then, we claim $f_n \to 0$ in measure on $[0,1)$; for any $\delta \in (0,1)$,

$$m(\{|f_n - 0| > \delta\}) = \frac{1}{k(n)} \to 0,$$

where $k(n)$ the "row" that f_n comes from. Hence, f_n converges in measure. However, f_n does not converge almost eveywhere on $[0,1)$. Indeed, for each $x \in \mathbb{R}$ and $k \ge 1$, there exists a *unique* j such that $x \in \left[\frac{j-1}{k}, \frac{j}{k}\right]$ hence $\varphi_{k,j}(x) = 1$, so in other notation there always exists an n such that $f_n(x) = 1$, and so precisely $f_n(x) = 1$ for infinitely many n. Hence, we do not have convergence everywhere (in fact, anywhere).

↪**Proposition 2.16**: Given $\{f_n\}, f$ measurable, finite-valued functions, if $f_n \to f$ in measure, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. as $k \to \infty$.

PROOF. Assume $f_n \to f$ in measure, that is for every $\delta > 0$, $m(\{|f_n - f| > \delta\}) \to 0$. Hence, for all $k \ge 1$, with $\delta = \frac{1}{k}$, we have that for some sufficiently large n_k, we have that $m\left(\underbrace{\left\{|f_{n_k} - f| > \frac{1}{k}\right\}}_{:=A_k}\right) \le \frac{1}{k^2}$, hence $\sum_{k=1}^{\infty} m(A_k) < \infty$. Hence,

$$m\left(\bigcap_{\ell=1}^{\infty}\bigcup_{k=\ell}^{\infty} A_k\right) = \lim_{\ell\to\infty} m\left(\bigcup_{k=\ell}^{\infty} A_k\right) \le \lim_{\ell\to\infty}\sum_{k=\ell}^{\infty} m(A_k) = 0,$$

since $\sum_{k=\ell}^{\infty} m(A_k)$ the tail of a converging series. Hence, complementing the above, a.e. there $\exists\, \ell$ such that for every $k \ge \ell$, $|f_{n_k} - f| \le \frac{1}{k}$ and so $\lim_{k\to\infty}|f_{n_k} - f| = 0$ almost everywhere, and so $f_{n_k} \to f$ a.e. (as $k \to \infty$). ■

↪**Proposition 2.17** (Subsequence Test): Given $\{f_n\}, f$ measurable, finite-valued functions, $f_n \to f$ in measure $\Leftrightarrow$ for every subsequence $\{n_k\}$, there exists a subsubsequence $\{n_{k_\ell}\} \subset \{n_k\}$ such that $f_{n_{k_\ell}} \to f$ in measure as $\ell \to \infty$.

PROOF. $\Rightarrow$ is clear. For $\Leftarrow$, suppose towards a contradiction that $f_n \not\to f$ in measure. Then, $\exists\,\delta > 0$ and subsequence $\{n_k\}$ $m\left(\left\{|f_{n_k} - f| > \delta\right\}\right) > \delta$ for every k. By the assumption of the RHS, there exists a further subsequence $\left\{n_{k_\ell}\right\}$ such that $f_{n_{k_\ell}} \to f$ in measure. This is a contradiction. ■

⊛ **Example 2.2** (Assignment Exercise): Prove that if $f_n \to f$ in measure and $g_n \to g$ in measure, $f_n g_n \to fg$ in measure (everything finite valued, measurable).

§2.4 Egorov's Theorem and Lusin's Theorem

Recall that if f is measurable, then $\exists\,\{\theta_n\}$ sequence of step functions such that $\theta_n \to f$ almost everywhere.

↪**Theorem 2.4** (Egorov's): Given $\{f_n\}, f$ measurable functions and $A \in \mathcal{M}$ with $m(A) < \infty$, if $f_n \to f$ a.e. on A, then $\forall\,\varepsilon > 0$, there exists a closed subset $A_\varepsilon \subseteq A$ with $m(A \setminus A_\varepsilon) \le \varepsilon$ such that $f_n \to f$ uniformly on A_ε.

PROOF. We assume first f is finite-valued on A (otherwise, replace A with $A \cap \{-\infty < f < \infty\}$; we'll deal with $\{f = \pm\infty\}$ later). We want to show that $\forall\,\varepsilon > 0, \exists$ closed $A_\varepsilon \subseteq A$ s.t. $m(A \setminus A_\varepsilon) < \varepsilon$ *and* $\sup_{x \in A_\varepsilon} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

For each $k \ge 1$ and $n \ge 1$, put

$$E_n^{(k)} := \left\{x \in A : |f_j(x) - f(x)| \le \frac{1}{k}\ \forall\, j \ge n\right\}.$$

For fixed k, remark that $E_n^{(k)} \subseteq E_{n+1}^{(k)}$, i.e. $E_n^{(k)}$ increasing (wrt n), so we may consider

$$\bigcup_{n=1}^{\infty} E_n^{(k)} = \left\{x \in A : \exists\, n \ge 1 \text{ s.t.} \forall\, j \ge n, |f_j(x) - f(x)\,| \le \frac{1}{k}\right\} \supseteq \left\{x \in A : \lim_{n\to\infty} f_n(x) = f(x)\right\} =: A'.$$

By assumption, $m(A') = m(A)$, so by continuity and the superset relation above, $m(A) = m(A') \le m\left(\bigcup_{n=1}^{\infty} E_n^{(k)}\right) = \lim_{n\to\infty} m\left(E_n^{(k)}\right) \le m(A)$, and thus $\lim_{n\to\infty} m\left(E_n^{(k)}\right) = m(A)$ for every $k \ge 1$.

Given, then, any $\varepsilon > 0$, there exists a n_k such that $m\left(A \setminus E_{n_k}^{(k)}\right) = m(A) - m\left(E_{n_k}^{(k)}\right) < \frac{1}{2^k}\frac{\varepsilon}{2}$. Set

$$B := A \setminus \left(\bigcap_{k=1}^{\infty} E_{n_k}^{(k)}\right),$$

then

$$m(B) = m\left(\bigcup_{k=1}^{\infty} A \setminus E_{n_k}^{(k)}\right) \le \sum_{k=1}^{\infty} m\left(A \setminus E_{n_k}^{(k)}\right) \le \frac{\varepsilon}{2}.$$

Put

$$\tilde{A} := A \setminus B = \bigcap_{k=1}^{\infty} E_{n_k}^{(k)}.$$

Then, if $x \in \tilde{A}$, then $x \in E_{n_k}^{(k)}$ for every k, and hence for every $k \geq 1$ and $j \geq n_k$, $|f_j(x) - f(x)| \leq \frac{1}{k}$. This shows then that $f_n \to f$ uniformly on $\tilde{A}$. By regularity of m, there exists a closed $A_\varepsilon \subseteq \tilde{A}$ such that $m(\tilde{A} \setminus A_\varepsilon) \leq \frac{\varepsilon}{2}$. Then, $f_n \to f$ uniformly on A_ε, and $m(A \setminus A_\varepsilon) = m(A \setminus \tilde{A}) + m(\tilde{A} \setminus A_\varepsilon) < \varepsilon$.

Now, if $f = \infty / -\infty$ on A, then $A = A^{\infty} \cup A^{-\infty} \cup A^{\mathbb{R}}$ (with $A^{\bullet} := \{f = \bullet\} \cap A$). The last case is done. For A^{∞} (similar construction for $A^{-\infty}$), define for every $k, n \geq 1$,

$$E_n^{(k)} := \left\{x \in A : f_j(x) > k \; \forall \, j \geq n\right\}.$$

Then, the remainder of the proof follows precisely the same for the sequence of sets $E_n^{(k)}$. ■

Remark 2.3:

1. The assumption $m(A) < \infty$ is necessary. For instance $f_n = \mathbb{1}_{[n,\infty)} \to 0$ pointwise, but for any $a \in \mathbb{R}$, f_n does not converge to 0 uniformly on (a, ∞).
2. In general, Egorov's $\not\Rightarrow f_n \to f$ uniformly a.e.. For instance, on $[0,1]$, let $f_n(x) = x^n$ and $f(x) \equiv 0$. For every $x \in [0,1)$, $f_n(x) \to f(x)$ as $n \to \infty$. Hence, $f_n \to f$ a.e. on $[0,1]$ (the only point that doesn't converge, indeed, is at 1). If $A \subseteq [0,1]$ is closed such that $1 \in A$, then $f_n \nrightarrow f$ uniformly on A. To see this, let $\{x_m\} \subseteq A$ such that $x_m \uparrow$ and $\lim_{m\to\infty} x_m = 1$. Then, for any fixed n,

$$\sup_{x \in A} |f_n(x) - f(x)| \geq \sup_m |f_n(x_m) - f(x_m)| = \sup_m x_m^n = 1,$$

hence f_n does not converge uniformly on A.

↪**Theorem 2.5** (Lusin's Theorem): Given f measurable and finite-valued and $A \in \mathcal{M}$ with $m(A) < \infty$, for all $\varepsilon > 0$, there exists a closed $A_\varepsilon \subseteq A$ with $m(A \setminus A_\varepsilon) < \varepsilon$ such that $f|_{A_\varepsilon}$ is continuous.

Remark 2.4: Lusin's Theorem states that $f|_{A_\varepsilon}$ is continuous as a function on ε, which is *not* the same as saying f as a function of A is continuous at points in A_ε.

For instance, $f = \mathbb{1}_{\mathbb{Q} \cap [0,1]}$ is not continuous anywhere on $[0,1]$. However, $f|_{\mathbb{Q}\cap[0,1]}$ is constant and therefore continuous *on* $\mathbb{Q} \cap [0,1]$.

Proof. Let $\{\theta_n\}$ be a sequence of step functions such that $\theta_n \to f$ a.e. on A. Note that θ_n piecewise constant and hence piecewise continuous. Given $\varepsilon > 0$ and $n \geq 1$, we can find an open set E_n such that $\theta_n|_{E_n^c}$ is continuous and $m(E_n) \leq \frac{\varepsilon}{2}\frac{1}{2^n}$. Meanwhile, Egorov's implies that there exists a closed $B \subseteq A$ such that $m(A \setminus B) \leq \frac{\varepsilon}{2}$ such that $\theta_n \to f$ uniformly on B. Set

$$A_\varepsilon = B \setminus \bigcup_{n=1}^{\infty} E_n,$$

noting that $A_\varepsilon \subset A$ closed and

$$m(A \setminus A_\varepsilon) = m(A \setminus B) + m\left(\bigcup_{n=1}^{\infty} E_n\right) = \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} m(E_m) \leq \varepsilon.$$

Finally, on A_ε, $\theta_n \to f$ uniformly and $\theta_n|_{A_\varepsilon}$ continuous, and hence $f|_{A_\varepsilon}$ continuous (uniform limit of continuous functions is continuous). ■

Remark 2.5:
1. Lusin's Theorem $\nRightarrow f$ is continuous almost everywhere in general.
2. (Exercise) The $\{\theta_n\}$'s are not continuous on $\mathbb{R}$, but you can choose a sequence $\left\{\widetilde{\theta_n}\right\}$ to be continuous on $\mathbb{R}$ such that $\widetilde{\theta_n} \to f$ a.e..