MATH457 - Algebra 4

Representation Theory; Galois Theory

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§1 Representation Theory

§1.1 Introduction

↪**Definition 1.1** (Linear Representation): A *linear representation* of a group G is a vector space V over a field $\mathbb{F}$ equipped with a map $G \times V \to V$ that makes V a G-set in such a way that for each $g \in G$, the map $v \mapsto gv$ is a linear homomorphism of V.

This induces a homomorphism

$$\rho : G \to \mathrm{Aut}_{\mathbb{F}}(V),$$

or, in particular, when $n = \dim_{\mathbb{F}} V < \infty$, a homomorphism

$$\rho : G \to \mathrm{GL}_n(\mathbb{F}).$$

Alternatively, a linear representation V can be viewed as a module over the group ring $\mathbb{F}[G] = \left\{\sum_{g \in G} : \lambda_g g : \lambda_g \in \mathbb{F}\right\}$ (where we require all but finitely many scalars λ_g to be zero).

↪**Definition 1.2** (Irreducible Representation): A linear representation V of a group G is called *irreducible* if there exists no proper, nontrivial *subspace* $W \subsetneq V$ such that W is G-stable.

⊛ **Example 1.1**:

1. Consider $G = \mathbb{Z}/2 = \{1, \tau\}$. If V a linear representation of G and $\rho : G \to \mathrm{Aut}(V)$. Then, V uniquely determined by $\rho(\tau)$. Let $p(x)$ be the minimal polynomial of $\rho(\tau)$. Then, $p(x) \mid x^2 - 1$. Suppose $\mathbb{F}$ is a field in which $2 \neq 0$. Then, $p(x) \mid (x-1)(x+1)$ and so $p(x)$ has either $1, -1$, or both as eigenvalues and thus we may write

$$V = V_+ \oplus V_-,$$

   where $V_\pm := \{v \mid \tau v = \pm v\}$. Hence, V is irreducible only if one of V_+, V_- all of V and the other is trivial, or in other words τ acts only as multiplication by 1 or -1.

2. Let $G = \{g_1, ..., g_N\}$ be a finite abelian group, and suppose $\mathbb{F}$ an algebraically closed field of characeristic 0 (such as $\mathbb{C}$). Let $\rho : G \to \mathrm{Aut}\ (V)$ and denote $T_j := \rho\left(g_j\right)$ for $j = 1, ..., N$. Then, $\{T_1, ..., T_N\}$ is a set of mutually commuting linear transformations. Then, there exists a simultaneous eigenvector, say v, for $\{T_1, ..., T_N\}$, and so $\mathrm{span}\ (v)$ a G-stable subspace of V. Thus, if V irreducible, it must be that $\dim_{\mathbb{F}} V = 1$.

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↪**Theorem 1.1**: If G a finite abelian group and V an irreducible finite dimensional representation over an algebraically closed field of characeristic 0, then $\dim V = 1$.

PROOF. Let $\rho : G \to \mathrm{Aut}(V)$, label $G = \{g_1, ..., g_N\}$ and put $T_j := \rho\left(g_j\right)$ for $j = 1, ..., N$. Then, $\{T_1, ..., T_N\}$ a family of mutually commuting linear transformations on V. Then, there is a simultaneous eigenvector v for $\{T_1, ..., T_N\}$ and thus $\mathrm{span}(v)$ is $T_1, ..., T_N$-stable and so $V = \mathrm{span}(v)$. ■

↪**Lemma 1.1**: Let V be a finite dimensional vector space over $\mathbb{C}$ and let $T_1, ..., T_N : V \to V$ be a family of mutually commuting linear automorphisms on V. Then, there is a simultaneous eigenvector for $T_1, ..., T_N$.

↪**Proposition 1.1**: Let $\mathbb{F}$ a field where $2 \neq 0$ and V an irreducible representation of S_3. Then, there are three distinct (i.e., up to homomorphism) possibilities for V.

PROOF. Let $\rho : G \to \mathrm{Aut}(V)$ and let $T = \rho((23))$. Then, notice that $p_T(x) \mid \left(x^2 - 1\right)$ so T has eigenvalues in $\{-1, 1\}$.

If the only eigenvalue of T is -1, we claim that V one-dimensional.

If T has 1 as an eigenvalue. ■

↪**Proposition 1.2**: D_8 has a unique faithful irreducible representation, of dimension 2 over a field F in which $0 \neq 2$.

PROOF. Write $G = D_8 = \left\{1, r, r^2, r^3, v, h, d_1, d_2\right\}$ as standard. Let ρ be our irreducible, faithful representation and let $T = \rho\left(r^2\right)$. Then, $p_T(x) \mid x^2 - 1 = (x - 1)(x + 1)$ and so $V = V_+ \oplus V_-$, the respective eigenspaces for $\lambda = +1, -1$ respectively for T. Then, notice that since r^2 in the center of G, both V_+ and V_- are preserved by the action of G, hence one must be trivial and the other the entirety of V. V can't equal V_+, else $T = I$ on all of V hence ρ not faithful so $V = V_-$.

Next, it must be that $\rho(h)$ has both eigenvalues 1 and -1. Let $v_1 \in V$ be such that $hv_1 = v_1$ and $v_2 = rv_1$. We claim that $W := \mathrm{span}\,\{v_1, v_2\}$, namely $V = W$ 2-dimensional.

We simply check each element. $rv_1 = v_2$ and $rv_2 = r^2 v_1 = -v_1$ which are both in W hence r and thus $\langle r \rangle$ fixes W. Next, $hv_1 = v_1$ and $vv_2 = vrv_1 = rhv_1 = rv_1 = v_2$ (since $rhr^{-1} = v$) and so $hv_2 = -v_2$ and $vv_1 = -v_1$ and so W G-stable. Finally, d_1 and d_2 are just products of these elements and so W G-stable. ■