Optimal Execution and Market Making

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Outline

1. Optimal split of orders across liquidity pools
   - Modelling and mean execution cost
   - Design of the stochastic algorithm
   - Numerical experiments

2. Market Making
   - Fillrates
   - Inventory and Market Making

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Modelling see [6]

The principle of a *Dark pool* is the following:

- It proposes a bid price with no guarantee of executed quantity at the occasion of an OTC transaction.
- Usually this price is lower than the bid price offered on the regular market.

So one can model the impact of the existence of N dark pools ($N \geq 2$) on a given transaction as follows:

- Let $V > 0$ be the random volume to be executed,
- Let $\theta_i \in (0, 1)$ be the *discount factor* proposed by the dark pool i.
- Let r_i denote the percentage of V sent to the dark pool i for execution.
- Let $D_i \geq 0$ be the quantity of securities that can be delivered (or mase available) by the dark pool i at price $\theta_i S$.

Cost of the executed order

The reminder of the order is to be executed on the regular market, at price S.

Then the **cost** C of the whole executed order is given by

$$\begin{aligned} C &= S\sum_{i=1}^{N}\theta_i \min\left(r_i V, D_i\right) + S\left(V - \sum_{i=1}^{N}\min\left(r_i V, D_i\right)\right) \\ &= S\left(V - \sum_{i=1}^{N}\rho_i \min\left(r_i V, D_i\right)\right) \end{aligned}$$

where

$$\rho_i = 1 - \theta_i \in (0,1), i = 1, \dots, N.$$

Mean Execution Cost

Minimizing the mean execution cost, *given the price* S, amounts to solving the following **maximization problem**

$$\max\left\{\sum_{i=1}^{N}\rho_i\mathbb{E}\left(S\min\left(r_iV,D_i\right)\right), r\in\mathcal{P}_N\right\} \tag{1}$$

where $\mathcal{P}_N := \left\{r=(r_i)_{1\le i\le N}\in\mathbb{R}_+^N \mid \sum_{i=1}^{N} r_i = 1\right\}$.

It is then convenient to *include the price* S *into both random variables* V *and* D_i by considering

$$\widetilde{V} := V\,S \quad \text{and} \quad \widetilde{D}_i := D_i S$$

instead of V and D_i.

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Optimal allocation of orders among N dark pools

We set for every $r = (r_1, \ldots, r_N) \in \mathcal{P}_N$,

$$\Phi(r_1, \ldots, r_N) := \sum_{i=1}^{N} \varphi_i(r_i).$$

where for every $i \in I_N = \{1, \ldots, N\}$,

$$\varphi_i(u) := \rho_i \mathbb{E}\left[\min\left(uV, D_i\right)\right], \quad u \in [0, 1].$$

We can formally extend Φ on the whole affine hyperplan spanned by $\mathcal{P}_N$ *i.e.*

$$\mathcal{H}_N := \left\{ r = (r_1, \ldots, r_N) \in \mathbb{R}^N \mid \sum_{i=1}^{N} r_i = 1 \right\}.$$

The Lagrangian Approach

We aim at solving the following maximization problem

$$\max_{r \in \mathcal{P}_N} \Phi(r). \tag{2}$$

The Lagrangian associated to the sole affine constraint is

$$L(r, \lambda) = \Phi(r) - \lambda \left(\sum_{i=1}^{N} r_i - 1 \right) \tag{3}$$

Thus

$$\forall i \in I_N, \quad \frac{\partial L}{\partial r_i} = \varphi_i'(r_i) - \lambda.$$

This suggests that any $r^* \in \arg\max_{\mathcal{P}_N} \Phi$ iff $\varphi_i'(r_i^*)$ is constant when i runs over I_N or equivalently if

$$\forall i \in I_N, \quad \varphi_i'(r_i^*) = \frac{1}{N} \sum_{j=1}^{N} \varphi_j'(r_j^*). \tag{4}$$

Design of the stochastic algorithm

Then using that

$$\forall i \in I_N, \quad \varphi_i'(r) = \rho_i \mathbb{E}\left[\mathbb{1}_{\{r_i V < D_i\}} V\right],$$

this yields that, if $\arg\max_{\mathcal{H}_N} \Phi = \arg\max_{\mathcal{P}_N} \Phi \subset int(\mathcal{P}_N)$, then

$$r^* \in \arg\max_{\mathcal{P}_N} \Phi$$

$$\Updownarrow$$

$$\forall i \in I_N, \quad \mathbb{E}\left[V\left(\rho_i \mathbb{1}_{\{r_i^* V < D_i\}} - \frac{1}{N}\sum_{j=1}^{N} \rho_j \mathbb{1}_{\{r_j^* V < D_j\}}\right)\right] = 0.$$

Consequently, this leads to the following **recursive zero search procedure**

$$r_i^{n+1} = r_i^n + \gamma_{n+1} H_i(r^n, Y^{n+1}), \; r^0 \in \mathcal{P}_N, \; i \in I_N, \tag{5}$$

Design of the stochastic algorithm

where for $i \in I_N$, every $r \in \mathcal{P}_N$, every $V > 0$ and every $D_1, \ldots, D_N \geq 0$,

$$H_i(r, Y) = V \left(\rho_i \mathbb{1}_{\{r_i V < D_i\}} - \frac{1}{N} \sum_{j=1}^{N} \rho_j \mathbb{1}_{\{r_j V < D_j\}} \right)$$

where $(Y^n)_{n \geq 1}$ is a sequence of random vectors with non negative components such that, for every $n \geq 1$ and $i \in I_N$, $(V^n, D_i^n) \stackrel{d}{=} (V, D_i)$.

Comment: The underlying idea of the algorithm is to reward the dark pools which outperform the mean of the N dark pools by increasing the allocated volume sent at the next step (and conversely).

For results on *a.s.* convergence, see [2, 3, 4] for i.i.d. setting and [5] for averaging setting.

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Numerical Tests

Two natural situations of interest can be considered *a priori*:

- *abundance*: $\mathbb{E}\,V \leq \sum_{i=1}^{N} \mathbb{E} D_i$,
- *shortage*: $\mathbb{E}\,V > \sum_{i=1}^{N} \mathbb{E} D_i$.

We define a reference strategy named "**oracle**" devised by an insider who knows all the values V^n and D_i^n before making his/her optimal execution requests to the dark pools. The "oracle" strategy yields a cost reduction of the execution denoted by CR^{oracle}.

We introduce indexes to measure the **performances** of our recursive allocation procedure as follows: the ratios between the relative cost reductions of our allocation algorithm and that of the oracle, *i.e.*

$$\frac{CR^{opti}}{CR^{oracle}}.$$

The pseudo-real data setting

V is the traded volume of a very liquid security – namely the asset BNP – during an 11 day period.

Then we selected the N most correlated assets (in terms of traded volumes) with the original asset, denoted S_i, $i = 1, \dots, N$.

The available volumes of each dark pool i have been modeled as follows

$$\forall 1 \le i \le N, \quad D_i := \beta_i \left((1 - \alpha_i) V + \alpha_i S_i \frac{\mathbb{E}\, V}{\mathbb{E} S_i} \right)$$

where

- $\alpha_i \in (0, 1)$, $i = 1, \dots, N$ are the recombining coefficients,
- β_i, $i = 1, \dots, N$ are some scaling factors,
- $\mathbb{E}\, V$ and $\mathbb{E} S_i$ stand for the empirical mean of the data sets of V and S_i.

Parameters

The shortage situation corresponds to $\sum_{i=1}^{N} \beta_i < 1$ since it implies

$$\mathbb{E}\left[\sum_{i=1}^{N} D_i\right] < \mathbb{E}\, V.$$

We will use the following parameters

$$N = 4, \quad \beta = (0.1, 0.2, 0.3, 0.2)' \quad \text{and} \quad \alpha = (0.4, 0.6, 0.8, 0.2)'$$

and the dark pool trading (rebate) parameters are set to

$$\rho = (0.01, 0.02, 0.04, 0.06)'.$$

Long-term optimization

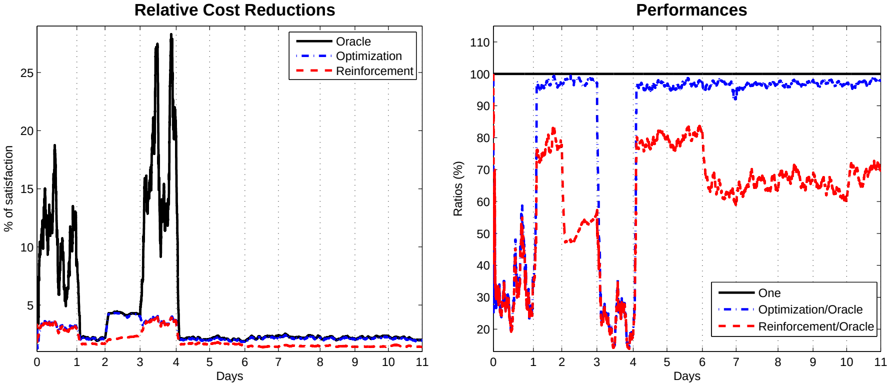

Figure: *Long term optimization*: Case $N = 4$, $\sum_{i=1}^{N} \beta_i < 1$, $0.2 \leq \alpha_i \leq 0.8$ and $r_i^0 = 1/N$, $1 \leq i \leq N$.

Daily resetting of the procedure

We reset the step γ_n at the beginning of each day and the satisfaction parameters and we keep the allocation coefficients of the preceding day.

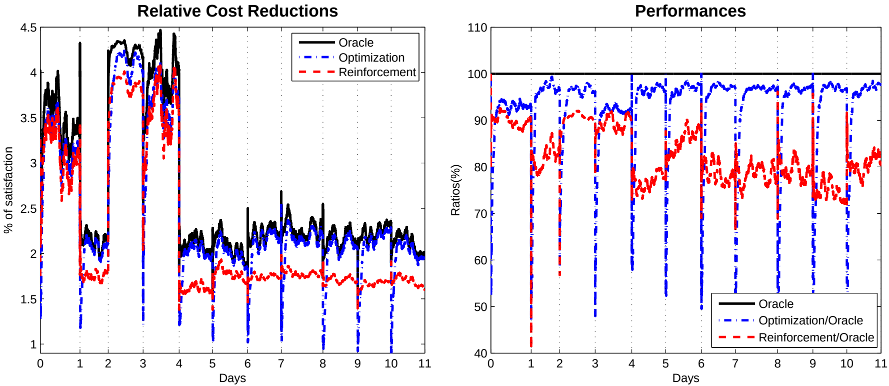

Figure: *Daily resetting of the algorithms parameters*: Case $N = 4$, $\sum_{i=1}^{N} \beta_i < 1$, $0.2 \leq \alpha_i \leq 0.8$ and $r_i^0 = 1/N$ $1 \leq i \leq N$.

Implementation

- Use the datasets on Paris Bourse: you have 4 assets (Bouygues, LVMH, Sanofi and Total). Select two weeks of data.
- Choose one of the asset to model the volume V you want to trade and the other to build the volumes D_i on the dark pools.
- Fix an horizon T (in minutes) and compute the executed quantity on each period of T minutes executed for each assets (namely V and S_i).
- Choose the recombining coefficients α_i and the scaling factors β_i to be in the shortage setting and compute the volumes D_i.
- Choose the ρ_i and implement both the stochastic algorithm and the oracle procedure. Print and comment your results.

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Fillrate

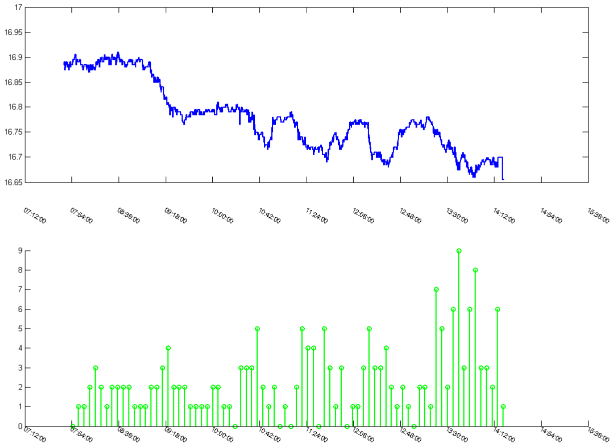
Definition.

- Instantaneous volume obtained when orders are posted inside the orderbook which can be estimated by modeling from empirical data.
- Dependent on order parameters (size, distance, duration, type), also dependent on market factors (volatility, spread, volume, etc).

We consider an agent who catches liquidity (at the bid or at the ask) on a market by only posting passive limit orders. Her placement strategy consists in

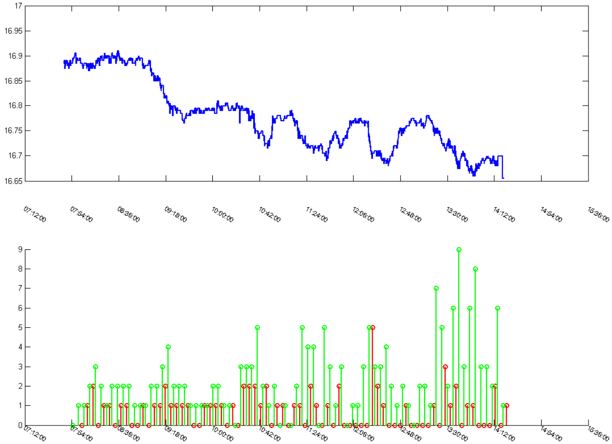
- posting a limit order of size 1 at a distance d (in ticks) from the best price (bid or ask) ; then waiting t minutes ;
- if the order is executed before the end of the period, she instantenously posts the same order (side, price, volume) ;
- counting at the end the number of executions of her type of orders on the period ;
- canceling the last unexecuted order and replacing the same quantity at the same distance d for the same duration t.

Buy Order : $d = 0$ (best bid), $t = 15$ trades



Order Size = 1, executed volume on 15 trades slice, strong correlation to market movements – therefore to volatility.

Buy Order : $d = 0$ (best bid) vs $d = 1$ (second best bid), $t = 15$ trades



Legend : $d = 0$ in green, $d = 1$ in red.

Implementation

- Plot the price dynamics and the associated fillrate of this posting strategy over a day of a stock you choose.
- Compare with different durations t (in minutes) : 1 minute, 5 minutes, 10 minutes and 15 minutes.
- Compare with different fixed distances ($0 \leq d \leq 4$).
- Comment.

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Inventory

Definition.

- Volume obtained during the day using a liquidity seeking tactic.
- Can be regarded as the integral of the fillrate over a period of time.
- Thus it is dependent on the same parameters.

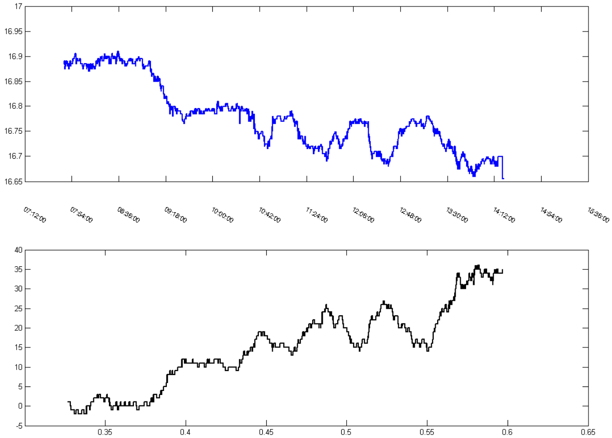
Implementation.

- Plot the inventory evolution V of the previous strategy depending on d and t (cumulative fillrates).
- Study the dependence of this inventory V on volatility σ and spread S.
- Comment.

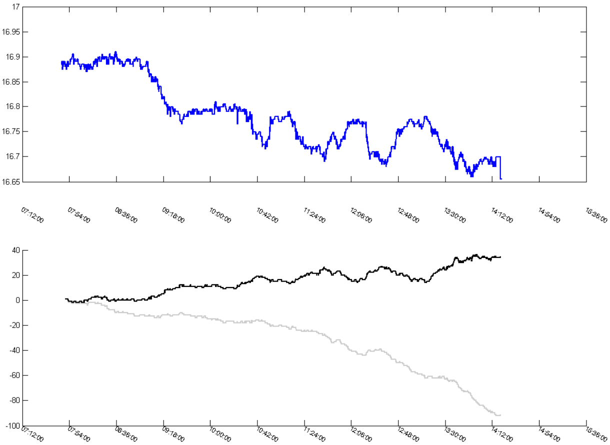
Static Market Making

- We consider now a market-maker using the same strategy as the agent before to buy and sell simultaneously stocks on the same market (then the distance is fixed over the day for both bid and ask sides, so it's a static strategy).
- We assume that at the beginning of the day the market-maker has an intial inventory $V_0 = 0$ stocks.
- Let d_b and d_a the posting distances for buying and selling respectively.

Inventory for $d_b = 0$ and $t_b = 15$ trades, $d_a = 0$ and $t_a = 15$ trades



Inventory comparison with different posting parameters



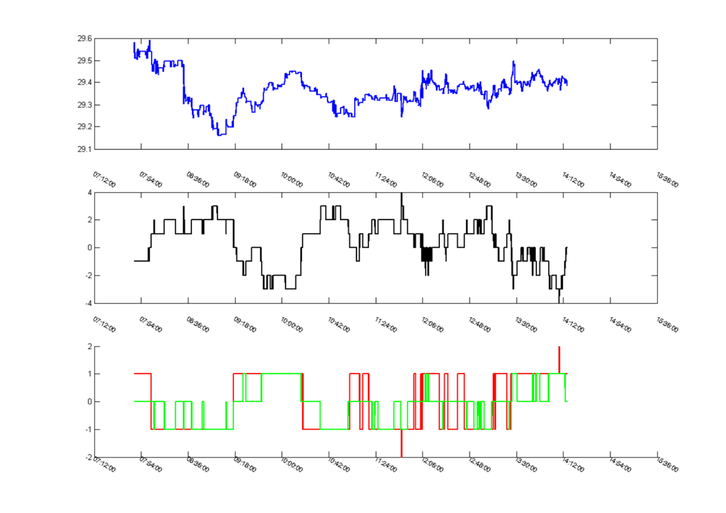
Legend : $d_b = 0$ and $t_b = 15$ trades, $d_a = 0$ and $t_a = 15$ trades (black) ; $d_b = 1$ and $t_b = 15$ trades, $d_a = 0$ and $t_b = 15$ trades (gray).

Implementation

- Let fix the duration t to a few minutes (1, 5, 10 or 15 minutes).
- Study the inventory evolution regarding d_b and d_a.
- What are the optimal d_b and d_a to have a "stationary" inventory for each duration ? (combination of the fillrates/inventory computed before).

Dynamical Market Making

Realtime reassessments, dynamical inventory, use market indicators to compute the posting distance.



Implementation

Let fix the duration t to a few minutes (1, 5, 10 or 15 minutes).

- Set the posting distances d_b and d_a as functions of the intraday volatility : $d_{b,t} = c_b \sigma_t$ and $d_{a,t} = c_a \sigma_t$.
- Determine the optimal constants c_b and c_a to have a "stationary" inventory with $V_0 = 0$.
- Plot on the same graph the evolutions of inventory, price and volatility for the optimal values of c_b and c_a. Comment.
- Propose other intraday parameters which could be used to optimize the posting price. Justify your choice.

P&L : Implementation

For the same market maker, we consider that the remaining stocks at the end of the day have to be traded at the close (the last price of the day) to get a zero final inventory.

- Plot and compare the P&L of the optimal strategies obtained in the previous section (for fixed d_b and d_a and as functions of intraday volatility). Add on the graphs the price, the inventory and the volatility.
- Do you think that it is always possible to liquidate an inventory at the closing price ?
- Study the dependence of the P&L to the inventory. If the posting distance depends on both inventory and volatility $d_t = f(V_t, \sigma_t)$, what could be the shape of f to optimize the closing P&L ?

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