Applying Markov Chain Monte-Carlo Techniques for an Inverse Problem Regarding Fish Larval Settlement

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Fish Larvae Settle Among Coral Reefs

- Life cycles of oceanic organisms are **complex**:
  - **External** fertilization
  - Larvae drift among currents
  - **Settle** among coral reef structures
  - Grow, reproduce, repeat...
- Observing/modeling settlers tells us about the overall **health** of the reef

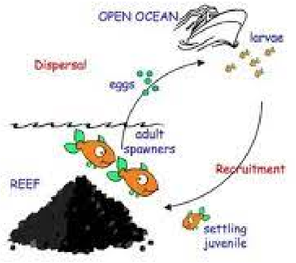


Figure: Cartoon of oceanic fish life cycle

Toy Example of Fish Settlement

- Hill function, $\phi(x)$ models **settlers** at **locations**, x_i, away from the shore
  - "Settlers from the Black Lagoon"

$$\phi(x) = \lambda \left(1 - \frac{x^\beta}{\theta^\beta + x^\beta}\right)$$

- Where:
  - λ: maximum **magnitude**
  - θ: **half-way** point
  - β: **steepness** of slope

Toy Example of Fish Settlement

- Suppose number settlers at location x_i are **Poisson** dist. w/ mean ϕ
  - $Y_i \sim \mathcal{P}(\phi(x_i))$
- From **observations** of settlers, what can we learn about λ, θ, β?
  - Inverse problem, **estimate** parameters
- Particularly, what **configuration** of x_i's gives us most **confidence** in our parameter estimations?

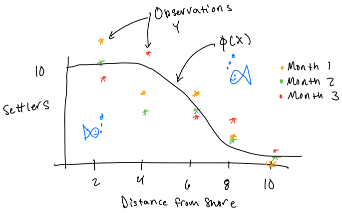


Figure: Hill function as the **mean** of settlers away from shoreline

Utilize MCMC Techniques to Simulate Posterior Distributions

Problem:

- What is the best configuration of location markers x_i that gives us best estimate of model parameters?

Methods:

- **Simulate** observations with 'true parameters' at various configurations of x_i
- Use **Bayes' Theorem** to yield distributions of parameters
- Use Markov Chain Monte-Carlo (**MCMC**) algorithm to simulate posteriors and compare

Bayes' Theorem

- Statistical analysis is an **inversion** process:
  - Aim to retrieve **causes**, parameters θ, from **effects**, observations X
- **Bayesian statistical model** is made of a parametric statistical model, **likelihood** $f(X|\theta)$, and a **prior** distribution on the parameters $\pi(\theta)$ [2]
- Our model, **posterior** $\pi(\theta|X)$ is defined using Bayes' Theorem:

$$\pi(\theta|X) = \frac{f(X|\theta)\pi(\theta)}{\int_{\Theta} f(X|\theta)\pi(\theta)d\theta}$$

- How do we **choose** on a prior?

Prior Selection

- One of the **most criticized** aspects of Bayesian statistics
- Choice of prior can drastically alter inference
- **Informed** priors (some prior knowledge of parameters is known):
  - Maximum entropy priors
  - Conjugate priors
- **Non-informed** priors (no prior knowledge of parameters is known):
  - Uniform priors $\pi(\theta) = c$
  - Scale-invariant priors (for $\theta > 0$) $\pi(\theta) = \frac{1}{\theta}$
  - **Jeffery's prior**[2]

Jeffery's Prior

- Here we derive $\pi(\theta)$ from $f(X|\theta)$
- Define **Fischer Information**:

$$I(\theta) = -\mathbb{E}_\theta \left[\frac{\partial^2 \log f(X|\theta)}{\partial \theta^2} \right]$$

- Jeffery's prior:

$$\pi(\theta) \propto I^{1/2}(\theta)$$

(**Our problem**, assume $\lambda \perp\!\!\!\perp \theta \perp\!\!\!\perp \beta$ and compute each Jeffery's prior)

- Now we wish to **simulate** our posteriors

$$\pi(\theta|X) \propto f(X|\theta)\pi(\theta)$$

MCMC Algorithms

- Generate **Markov Chain** $(\theta^{(m)})_{m>0}$ with limiting distribution $\pi(\theta|X)$
  - Chain is **irreducible** (for any set $A \subseteq \Theta$ then $\pi(A|X) > 0)$)
  - Chain is **ergodic** (distribution of $\theta^{(m)}$ converges to $\pi(\cdot|X)$ for almost any $\theta^{(0)}$) [2]
- Examples of MCMC algorithms:
  - Gibb's sampler
  - **Metropolis-Hastings** algorithm
  - Hamiltonian Monte Carlo (HMC)

Metropolis-Hastings Algorithm

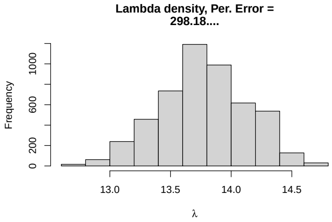
- Need a **target distribution** π up to a normalizing factor
  - $\pi(\theta|X) \propto f(X|\theta)\pi(\theta)$
- Need a conditional density $q(\theta'|\theta)$
  - **Random-walk** proposal $\theta' = \theta^{(m)} + \epsilon$ where ϵ is a symmetric RV (ex. $\epsilon \sim \mathcal{N}(0, \sigma^2)$)
- Generally aim for **acceptance ratio** about 0.44 [1]
  - **Adjust** σ^2 to influence acceptance ratio

Metropolis-Hastings Algorithm

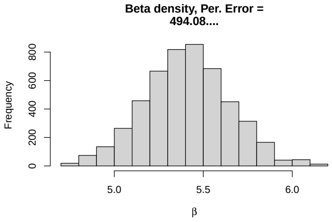
MH algorithm is as follows:
(for $\pi(\theta|X)$ target and $\theta' = \theta^{(m)} + \epsilon$ with ϵ symmetric RV)

1. Start with arbitrary initial guess $\theta^{(0)}$
2. Update $\theta^{(m)}$ to $\theta^{(m+1)}$ (for $m = 1, 2, ...$) by:
   - Generate $\theta' \sim q(\theta'|\theta^{(m)})$
   - Define
   $$\alpha = \frac{f(X|\theta')\pi(\theta')}{f(X|\theta^{(m)})\pi(\theta^{(m)})} \wedge 1$$
   - Take
   $$\theta^{(m+1)} = \begin{cases} \theta' \text{ with probability } \alpha \\ \theta^{(m)} \end{cases}$$
3. Repeat 2

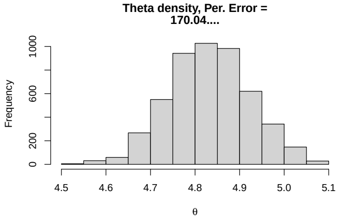
Results– Sample Distributions of Parameters



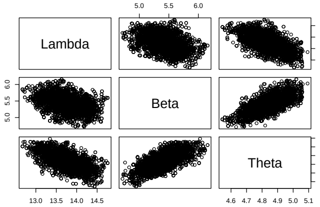
(a) True $\lambda = 13$



(b) True $\beta = 6$



(c) True $\theta = 5$



(d) Parameter clouds

Results– Evenly spaced locations

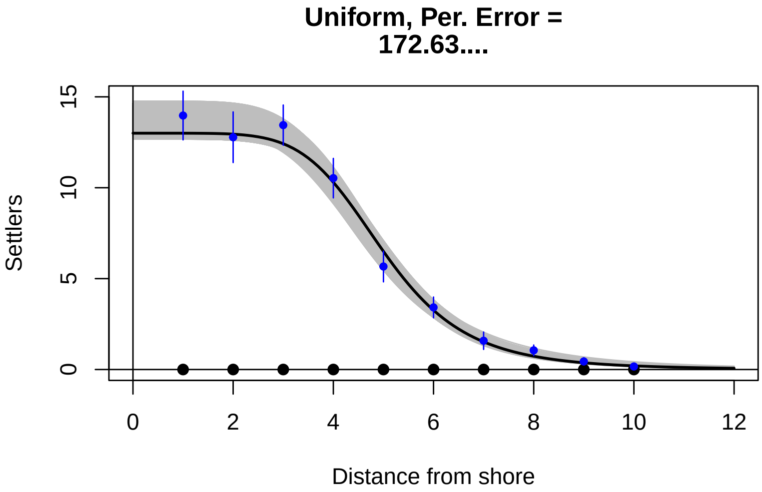
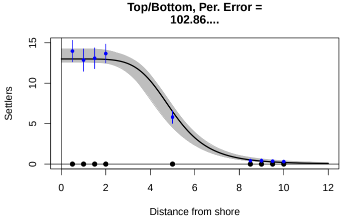
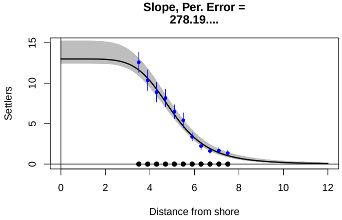


Figure: Markers evenly spaced along the shore

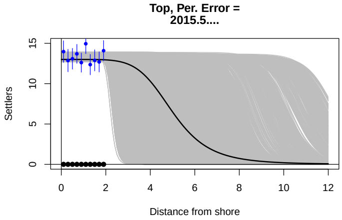
Results– Changing marker locations



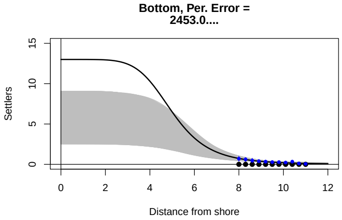
(a) Top/Bottom



(b) Slope



(c) Only Top



(d) Only Bottom

Conclusions

- Observe best results from markers that identify the **range** of the Hill function
  - Uniform, Top/Bottom, Slope
- Scattered results from markers that are confined to a **small section**
  - Only Top, Only Bottom
- Require **max** and **min** settlement locations in order to make accurate estimates
  - Allows for better **initial guesses** of parameters
- Improvements:
  - Reconsider independent parameter assumption, clear **correlation** among β and θ
  - Implement HMC with **Stan** in R

Resources

A. Gelman, G. O. Roberts, and W. R. Gilks.
Efficient metropolis jumping rules.
In J. M. Bernardo, J. O. Berger, A. P. Dawid, and A. F. M. Smith, editors, *Bayesian Statistics*, pages 599–608. Oxford University Press, Oxford, 1996.

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Springer-Verlag, 2007.