LINMA1510 - Linear feedback systems
General manual for the labs and the tutorials sessions

1 Introduction

The goal of this *general manual* is to summarize all the theoretical concepts, methods and properties needed to students for performing their exercises and practical sessions.

The techniques detailed in this document are for systems with only one single input and one single output (SISO for *Single Input, Single Output*) that can be modelled by linear and invariant equations in time (LTI for *Linear Time Invariant*).

During tutorials and laboratory sessions, you will deal with regulated systems, with at least one feedback loop as shown in the very general block diagram on the figure 1. Indeed, to get an automatically regulated system, it is necessary to compare an output and a setpoint to generate a command signal.

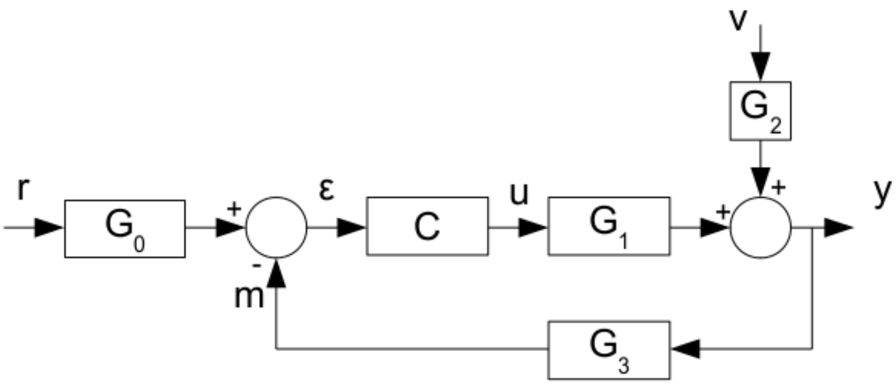


FIGURE 1 – General block diagram for a feedback system.

Based on this block diagram, it is interesting to remind the nomenclature that will be used during this course :

- The signal **r** is named the **Setpoint** ;
- The signal ϵ is named the **Error** ;
- The signal **u** is named the **Command** ;
- The signal **v** is named the **Disturbance** ;
- The signal **y** is named the **Output** ;
- The signal **m** is named the **Measure** ;
- The equivalent system that links **r** to **y** is named the **Closed Loop Setpoint - Output** ;
- The equivalent system that links **v** to **y** is named the **Closed Loop Disturbance - Output** ;
- The equivalent system that links ϵ to **y** is named the **Direct Loop** ;
- The equivalent system that links **y** to **m** is named the **Feedback Loop** ;
- The equivalent system that links **u** and **v** to **m** is named the **Open Loop**.

2 Linearisation and transfer functions

2.1 Linearisation

The tools used for this course allow to deal with linear modelled systems. To use the tools of linear control systems for non-linear modelled systems, the different strategies are :

— In the case of a weakly non linear system, it is sometimes possible to neglect the non linear part of the model.

— Approximate the non linear components with a linear model.

— Use Jacobian linearization technique, that is to develop a model formed by non linear differential equations in Taylor series (of the first order) around an operating point.

To illustrate the Jacobian linearization technique, let us consider a SISO process modelled with a non linear equations system :

$$\begin{aligned} \frac{dx}{dt} &= f(x,u) \mid x \in \mathbb{R}^n, u \in \mathbb{R} \\ y &= h(x,u) \mid y \in \mathbb{R} \end{aligned} \tag{1}$$

where x is the state vector [1], u and y are respectively the command signal and the output. Denote by x_e an equilibrium point of the intermediate state vector and u_e the associate value of the command signal around which we perform a linearization of the process described by (1).

First, it is necessary to define a new set of deviation variables composed by a state variable χ, a command variable ρ, and an output variable φ like :

$$\begin{aligned} \chi &= x - x_e \\ \rho &= u - u_e \\ \varphi &= y - h(x_e) \end{aligned} \tag{2}$$

The linearised system around the operating point$\{x_e, u_e, y_e\}$ can be written :

$$\begin{cases} \dfrac{d\chi}{dt} &= A\chi + B\rho \\ \varphi &= C\chi + D\rho \end{cases} \tag{3}$$

where the matrix A, B, C, and D are the Jacobian matrix computed for the operating point :

$$\begin{aligned} A &= \tfrac{\partial f(x,u)}{\partial x}|_{(x_e,u_e)} \quad B = \tfrac{\partial f(x,u)}{\partial u}|_{(x_e,u_e)} \\ C &= \tfrac{\partial h(x,u)}{\partial x}|_{(x_e,u_e)} \quad D = \tfrac{\partial h(x,u)}{\partial u}|_{(x_e,u_e)} \end{aligned} \tag{4}$$

MATLAB commands : `syms`, `subs`, `jacobian`, `ss`, `ss2tf`, `tf2ss`, ...

2.2 Transfer functions : generalities

The transfer function, that is the ratio of the Laplace transform of the output signal on the Laplace transform of the input signal, is a common tool in linear control system theory because it easily allows to model the SISO and LTI systems as an Input-Output relation.

1. this method also works for $x \in \Re$

A transfer function is obtained from linear differential equations with constant coefficients. For this tool, rather that dealing with $y(t)$ as a function of $u(t)$, we are looking to get $Y(s) = \mathcal{L}(y(t))$ function of $U(s) = \mathcal{L}(u(t))$. So we get a transfer function of a system defined :

$$\frac{Y(s)}{U(s)} = G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + ... + b_1 s1 + b_0}{a_n s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0} \tag{5}$$

Remember that $G(s)$ is a rational function dependent of the Laplace variable, of numerator $N(s)$ and of denominator $D(s)$. The transfer function $G(s)$ is qualified of order n, and according to the causal rule, we usually get $m \leq n$ (for physical systems). The roots of $N(s)$ and of $D(s)$ are respectively named the *zeros* and *poles* of the transfer function $G(s)$.

If we consider a state space model, as for example the one described in (3) obtained after linearization, it is interesting to show it is possible to convert it as a transfer function. Indeed, consider the Laplace transform of the linear state space model (3) :

$$\begin{cases} sX(s) & = AX(s) + BU(S) \Leftrightarrow X(s) = (sI - A)^{-1}BU(s) \\ Y(s) & = CX(s) + DU(s) \end{cases} \tag{6}$$

And this leads naturally to the expression of the transfer function :

$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D \tag{7}$$

<u>MATLAB commands</u> : `tf`, `zpk`, `residue`, `pole`, `zero`, `damp`, `pzmap`, `rlocus`, `step`, `impulse`, `ltiview`, `bode`, `nyquist` ...

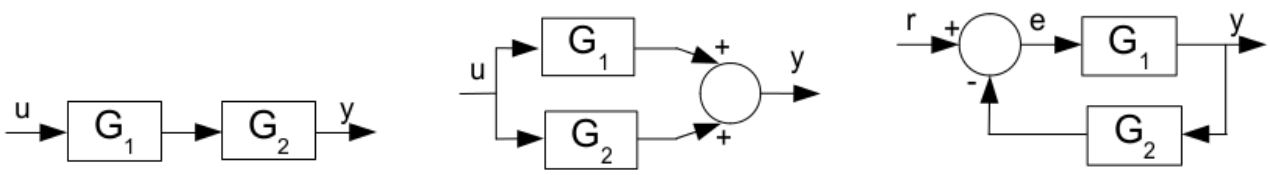
3 Block diagram algebra

The advantage of working with a feedback system modelled with transfer functions is to combine this tool with the block diagram algebra. This coupling of tools allows to easily compute the equivalent transfer function between two signals, by simple algebraic operations on the transfer functions of the blocks connected between these signals.

For example, let us consider a system modelled as an assembly of two transfer functions in serial $G_1(s)$ and $G_2(s)$ connected in series as illustrated on the Figure 2a. For any input signal u, the output signal of the first block will be $G_1 u$ and is also the input of the second block. The output signal y will be :

$$y = G_2(G_1 u) = (G_2 G_1)u \tag{8}$$

The equivalent transfer function of the system will be $G_{uy} = G_1 G_2$. With the same reasoning, it is also possible the two equivalent transfer function of the basic configurations included in the Figure 2 .



(a) $G_{uy} = G_1 G_2$ (b) $G_{uy} = G_1 + G_2$ (c) $G_{ry} = \dfrac{G_1}{1 + G_1 G_2}$

FIGURE 2 – Three basic block diagram configurations.

<u>MATLAB commands</u> : `feedback`, `parallel`, `series`, ...

4 First order systems

4.1 Generalities

Let us consider a R-C circuit as depicted below (Fig. 3). Following electrodynamic theory, $i = r \times v$ and $i = C\dfrac{du}{dt}$ and it is possible to model the circuit as a linear differential equation with constant coefficients :

$$v_{in} - v_{out} = RC\frac{dv_{out}}{dt} \tag{9}$$

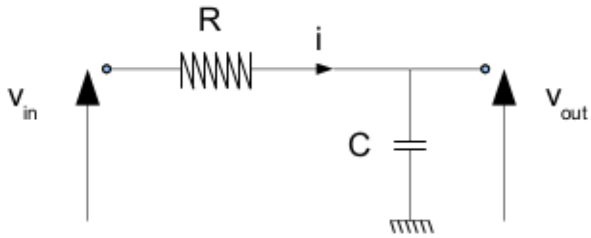


FIGURE 3 – Circuit R-C.

Using Laplace transform (see Appendix A), equation (9) allows to find the transfer function $G(s)$ linking v_{out} to v_{in} :

$$G(s) = \frac{v_{out}(s)}{v_{in}(s)} = \frac{1}{RC\,s + 1} \tag{10}$$

From equation (10), we can show that :
- It is a first order transfer function (the maximum exponent of the Laplace variable in the denominator is 1);
- The term RC is named the time constant. In this example, The units are in seconds;
- The gain ($G(s = 0)$) is equal to 1.

First order transfer functions are usually written in canonical form as :

$$G(s) = \frac{K}{\tau\,s + 1} \tag{11}$$

where K is the gain and τ is the time constant.

4.2 Study the transient of the step response

To test the response of the first order system, one practical method consists in applying a step of amplitude a as input signal u (if $t \geq t_0$, $u = a$, or else $u = 0$). The output signal y can be computed as :

$$y = \frac{K}{\tau\,s + 1}\frac{a}{s} \tag{12}$$

By simplification in rational function and inverse Laplace transform, we find (for $t \geq t_0$) :

$$y(t) = K(1 - e^{-t/\tau})u(t) \tag{13}$$

with $u(t) = a$.

Properties of the step response :

The study of this equation describing the step response in the time domain leads to the followings properties :

- A first order system is always stable ($\Leftrightarrow \tau > 0$) ;
- A first order system never presents any overshoot ;
- The slope in t_0 is a constant equal to τ ;
- Particular points of interest are given as :
  - $y(t = \tau) \approx 0,63\,Ka$;
  - $y(t = 3\tau) \approx 0,95\,Ka$;
  - $y(t = 4\tau) \approx 0,98\,Ka \rightarrow$ response time at 2%.

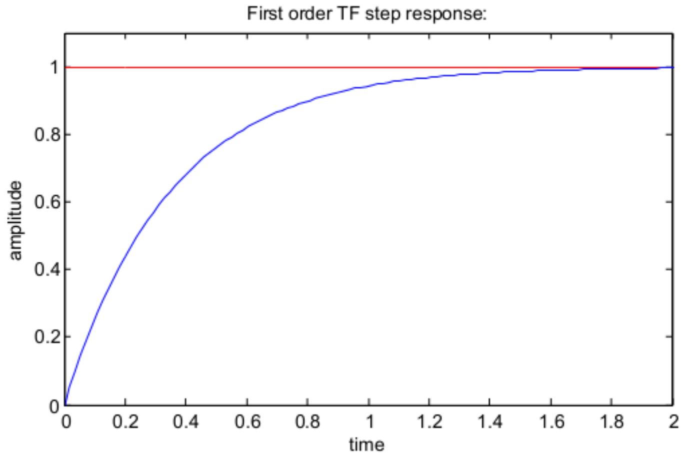


FIGURE 4 – Time plot for the response of a first order system to a step input of amplitude ($a = 1$) describe on the figure 3 with $R = 350\,\Omega$ and $C = 0,001\,F$.

MATLAB commands : `tf`, `step`, `ltiview`, ...

5 Second order system

5.1 Generalities

Let us consider a *Mass - Spring - Damper* system as shown on the Figure 5 where a mass M is fixed on a wall (y its position from the wall) with a damper of viscous friction coefficient f and a spring with a constant k, and that we pull with a force U to the right.

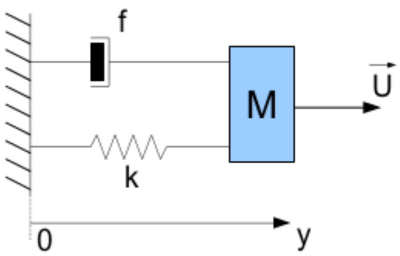


FIGURE 5 – *Mass - Spring - Damper* system.

Following the laws of mechanics, it is possible to model the relationship between the position y of the mass M and the force U according the value of the components by :

$$M\frac{d^2y}{dt^2} = U - ky - f\frac{dy}{dt} \tag{14}$$

that is a linear differential equation with constant coefficients.

Using Laplace transform, we can find the transfer function that links the position of the mass and the force :

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{Ms^2 + fs + k} \tag{15}$$

From equation (15), we can show that :

— It is a second order transfer function (the maximum exponent of the Laplace variable in the denominator is 2) ;
— The term $\sqrt{k/M}$ is named *natural frequency* ;
— The term $\frac{f}{2\sqrt{kM}}$ is named *damping ratio* ;
— The system gain is equal to $1/k$.

Second order transfer functions are usually written in canonical form as :

$$G(s) = K\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{16}$$

where K is the system gain, ζ the damping ratio and ω_n the natural frequency.

5.2 Study the transient of the step response

The study of the transient response of a system modelled with a second order transfer function needs a rational function simplification :

$$s^2 + 2\zeta\omega_n s + w_n^2 \tag{17}$$

The discriminant is given by :

$$\rho = 4w_n^2\sqrt{\zeta^2 - 1} \tag{18}$$

Equation (18) and its 2 poles leads to three cases that will be treated separately :

- $\zeta > 1$: 2 real poles ;
- $\zeta = 1$: 1 double pole ;
- $\zeta < 1$: 2 complex conjugate poles.

5.2.1 Case of $\zeta > 1$: 2 real poles

If we apply a step of amplitude a on the input u, the output y of a second order system can be written :

$$Y(s) = \frac{a}{s}\frac{K\omega_n^2}{\left(\frac{-1}{p_1}s + 1\right)\left(\frac{-1}{p_2}s + 1\right)\omega_n^2} \tag{19}$$

with p_1 and p_2 the poles of the characteristic equation (17). By simplification in rational function and inverse Laplace transform, we can find the temporal equation of the output signal :

$$y(t) = K + \frac{K}{\frac{-1}{p_1} + \frac{-1}{p_2}}\left(\frac{-1}{p_1}e^{tp_1} - \frac{-1}{p_2}e^{tp_2}\right) \tag{20}$$

Properties of the step response :

The study of equation (20), leads to the followings properties :

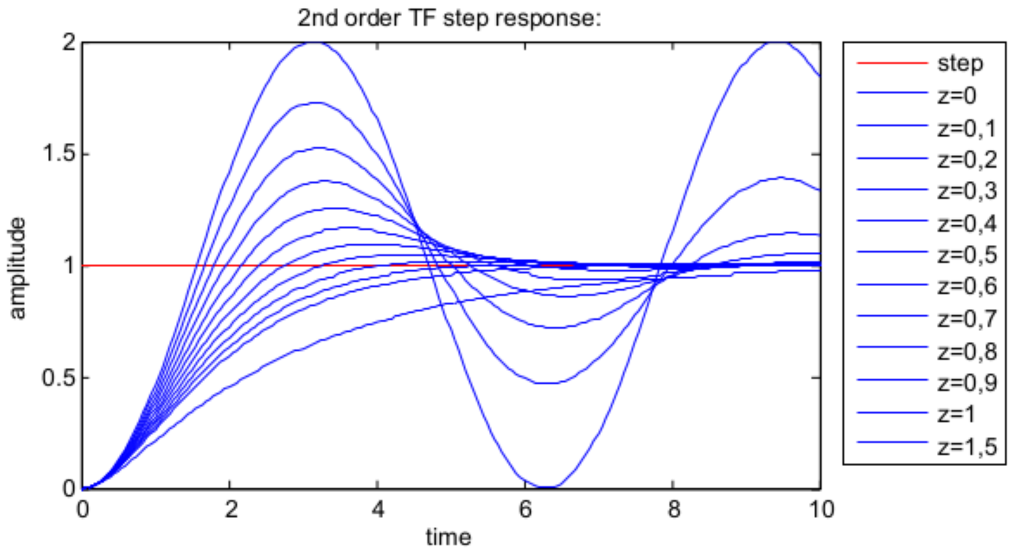


FIGURE 6 – Transient response of a second order system to an unit step for several values of ζ.

- The system is stable :
- The overshoot is zero ;
- The response time is even greater than ζ is large ;
- The slope of the signal plot in $t = t_0$ (first instant of the step) is zero ;
- The response time is given by $t_R = \dfrac{4}{\omega_n(\zeta - \sqrt{\zeta^2 - 1})}$.

5.2.2 Case of $\zeta = 1$: 1 double pole

In this case, the characteristic equation (17) shows a double pole equal to ω_n.

Properties of the step response :
The curve presents the same characteristics as cited in the item 5.2.2, with a smaller response time.

5.2.3 Case of $\zeta < 1$: 2 complex conjugate poles

In this case where the characteristic equation (17) has two complex conjugate poles and when a step of amplitude a is applied on the input u, the simplification in rational functions of the output y of the system gives :

$$Y(s) = \frac{K}{s} + \frac{Cs + D}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} \tag{21}$$

with C and D some constants.

After inverse Laplace transform, we obtain :

$$y(t) = K + e^{-\zeta\omega_n t} cos\left(\omega_n \sqrt{1 - \zeta^2 t}\right) \tag{22}$$

Properties of the step response :
Equation (22) allows to show that :

- The response is oscillatory, with natural frequency $\omega = \omega_n\sqrt{1-\zeta^2}$;
- The response is damped oscillatory if $\zeta > 0$, and purely oscillatory if $\zeta = 0$;
- By computing $\dfrac{dy(t)}{dt} = 0$, we can find that the first overshoot (and thus the largest) occurs when $t_D = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ and its value is $D = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$;
- The value of the overshoot only depends of ζ ;
- The response time is given by $t_R = \dfrac{4}{\zeta\omega_n}$.

5.3 Practical approximation of a non-oscillating second order

To simplify computations, it is common to approximate a second order transfer function with $\zeta \geq 1$ as a delayed first order (see figure 7).

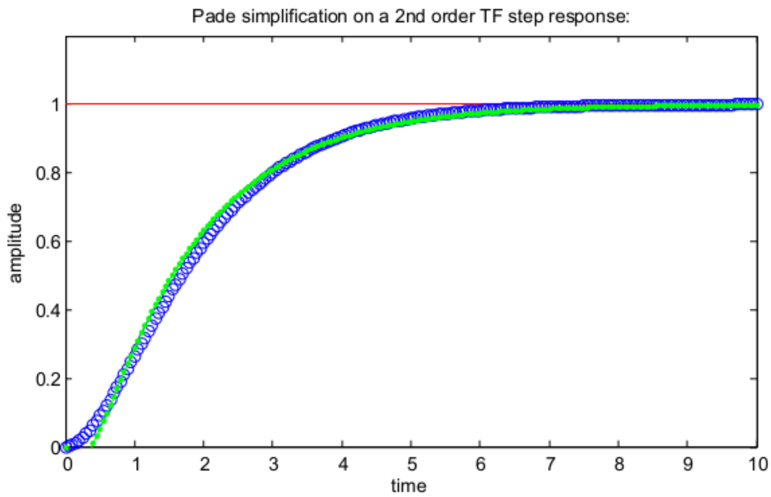


FIGURE 7 – Comparison of a second order transfer function (blue) ($\zeta = 1$) approximated by a delayed first order (green) (Padé approximation with $t_r = 0.5\,s$)

Furthermore, the delay $\mathcal{L}(y(t - t_R)) = e^{-t_R s}$ of a duration t_R can easily be evaluated with two different methods :

- If it is small $e^{-t_R s} \approx 1 - st_R$;
- Else with the simplified approximation of Padé $e^{-t_R s} \approx \dfrac{1 - \frac{t_R s}{2}}{1 + \frac{t_R s}{2}}$.

5.4 Particular case : nonminimum phase system

A rational transfer function $G(s) = \frac{N(s)}{D(s)}$ (where $N(s)$ and $D(s)$ are two relatively prime polynomials, i.e. there is no any common root in $\mathbb{C}$) is named *nonminimum phase* if one of its zero has a stictly positive real part. This is called an unstable zero.

Properties of a step response :
This type of system is deemed difficult to control. These system are associated with an inverse response (see figure 8) for a step on the input.

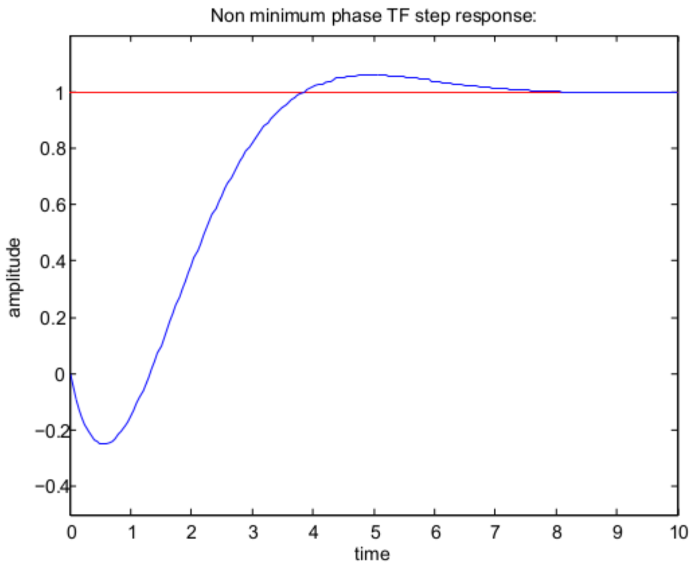


FIGURE 8 – Unit step response of a transfer function $G(s) = \dfrac{1 - sT_\alpha}{1 + 2\zeta T_n s + T_n^2 s^2}$ with $T_\alpha = T_n = 1$ et $\zeta = 0,7$.

6 System of order > 2

In general, any transfer function described by equation (5) can just be reformulated as :

$$G(s) = \frac{N(s)}{D(s)} = K_0 \frac{\prod_{i=1}^{m} (s - a_i)}{\prod_{j=1}^{n} (s - b_i)} \tag{23}$$

$$= \frac{K}{s^a} \frac{\prod_i (\tau_i s + 1)^{n_i}}{\prod_j (\tau_j s + 1)^{n_j}} \frac{\prod_k \left[s^2 + 2\zeta_k \omega_{nk} s + {\omega_{nk}}^2\right]^{n_k}}{\prod_l \left[s^2 + 2\zeta_l \omega_{nl} s + {\omega_{nl}}^2\right]^{n_l}} \tag{24}$$

Equation (24) shows that a system of order > 2 can be formulated as a system composed of pure integrators (first part of the equation), of real poles and zeros (second part of the equation), and of complex conjugated poles and zeros (third part of the equation).

The transient response of a system characterized by an high order transfer function is mainly influenced by some poles and zeros called *dominant*. On the pole and zeros diagram, they correspond to points near the imaginary axis (with a slightly negative real part), they corresponds to longer time constants and lower damping factors.

In the case of real poles, if the time constants are of the same order of magnitude, the effect of each poles in the transient response will be in proportion. However, if the time constants present a important ratio, the smallest of them can be neglected vis-a-vis the dominant ones.

In the case of an high order transfer function has real and complex conjugates poles, we must examine the real parts of each of them. As mentioned above, if there is an important ratio between the real parts, the system can usually be reduced to an oscillatory second order system (if the complex conjugate poles present the less negative part), and to a system of order 1 or 2 otherwise.

7 Design of a control law by pole placement

To get the desired performances of a feedback system, one way to compute the parameters of a controller assigning the poles of the closed loop transfer function on the appropriate locations. Given the block diagram, the poles of the closed loop transfer function depend of the poles of the controlled system (fixed) and of the controller transfer function (tunable).

7.1 Example of pole placement design

Let us consider for example a system $G(s)$ of order 1 controlled by state feedback and integral action as shown on figure 9 below :

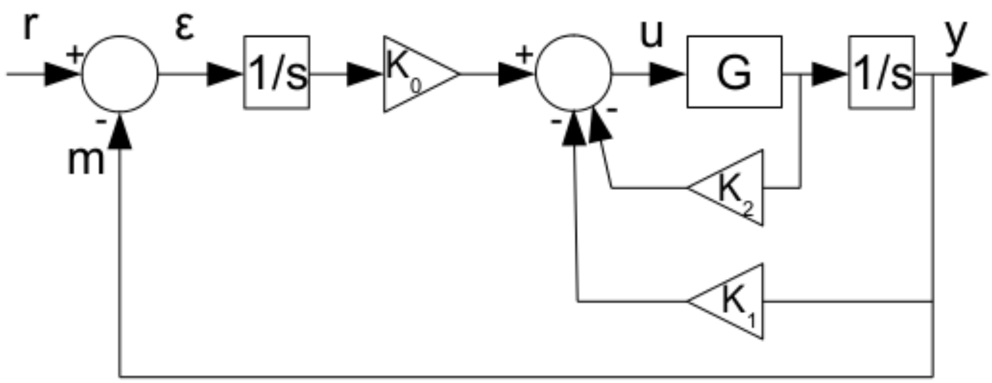


FIGURE 9 – System controlled by state feedback and integral action. $G(s) = \dfrac{K}{\tau s + 1}$

By block diagram algebra, we compute the closed loop transfer function :

$$T_r(s) = \frac{k_0 K}{\tau s^3 + (1 + k_2 K)s^2 + k_1 K s + k_0 K} \tag{25}$$

The desired performances for this system are :
- A time response of $T_{R_{T_r(s)}} \approx 3$ seconds ;
- A maximal overshoot of $D = 5$ %.

In this case, it is a third order transfer function. We will place the poles to get a dominant second order system (associated with a first order system with a shorter time constant, as it may be neglected) :

$$\begin{aligned} Den_{T_r(s)} &= s^3 + \frac{1 + k_2 K}{\tau} s^2 + \frac{k_1 K}{\tau} s + \frac{k_0 K}{\tau} && (26) \\ &= (s^2 + 2\zeta\omega_n s + \omega_n^2)(s + a) && (27) \end{aligned}$$

The maximum overshoot and time response formulas for the step response of a second order transfer function (or the tables in the appendix B) allow to compute the damping ratio ζ and the natural frequency ω_n of the first term of the equation (27). The parameter a of equation (27) has to be fixed as the pole it form can be neglected. A good practice is to fix it as $a \geq 10\omega_n$.

To know the 3 parameters k_0, k_1 et k_2 of equation (25), there is no more than to equal term-to-term the denominator of $T_r(s)$ (equation (25)) with he right side of equation (27).

<u>Commandes MATLAB</u> : `tf`, `zpk`, `pole`, `zero`, `damp`, `pzmap`, `rlocus`, `place`, `acker` ...

7.2 Particular case : pole-zero cancellation

The pole-zero cancellation is a particular case of the pole/zero placement, where in the direct loop, a parameter of the controller $C(s)$ is fixed as a zero/pole of $C(s)$ is equal to a pole/zero of $G(s)$. This cancellation in the direct loop causes an interesting cancellation when computing the closed loop transfer function (reduction of order).

For example, let us consider a simple feedback loop where a first order system $G(s) = \dfrac{K}{\tau s + 1}$ is controlled by an output feedback controller $C(s) = k_p\left(1 + \dfrac{k_i}{s}\right)$ and leads to $T_r(s) = \dfrac{C(s).G(s)}{1 + C(s).G(s)}$. It is possible to make a pole-zero cancellation in the direct chain $C(s).G(s)$ by setting $k_i = \tau^{-1}$:

$$C(s).G(s) = \frac{k_p\,(s + k_i)}{s}\frac{K}{\tau s + 1} \tag{28}$$

$$= \frac{k_p K}{s} \tag{29}$$

Given this cancellation, the closed-loop transfer function $T_r(s)$ will be reduced to a first order system.

It is right to pay attention to the stability of the transfer functions $T_r(s)$ and $T_v(s)$. In a general way, it is very strongly advised not to cancel any unstable pole/zero of $G(s)$ with any unstable zero/pole of $C(s)$!

7.3 Further : link with the Nyquist plot

Further, the Nyquist plot and the associated stability criterion are tools that allow to link poles of the open loop with the performance and stability of the closed loop. See *Feedback Systems, Astrom and Murray* page 267...

A Laplace Transform table

Function $f(t)$, $t \geq 0$	$F(s)$	Function $f(t)$, $t \geq 0$	$F(s)$
$K.\delta(t)$	K	K	$\frac{K}{s}$
$K.t$	$\frac{K}{s^2}$	t^n	$\frac{n!}{s^{(n+1)}}$
$e^{-a.t}$	$\frac{1}{s+a}$	$\frac{1}{a}(1-e^{-a.t})$	$\frac{1}{s(s+a)}$
$t^n e^{-a.t}$	$\frac{n!}{(s+a)^{n+1}}$	$\frac{1}{b-a}(e^{-a.t}-e^{-b.t})$	$\frac{1}{(s+b)(s+a)}$
$\frac{1}{b-a}(b.e^{-a.t}-a.e^{-b.t})$	$\frac{s}{(s+b)(s+a)}$	$1+\frac{1}{a-b}(b.e^{-a.t}-a.e^{-b.t})$	$\frac{a.b}{s(s+b)(s+a)}$
$\frac{1}{a^2}(1-e^{-a.t}-a.t.e^{-a.t})$	$\frac{1}{s(s+a)^2}$	$\frac{1}{a^2}(a.t-1+e^{-a.t})$	$\frac{1}{s^2(s+a)}$
$sin(\omega.t)$	$\frac{\omega}{s^2+\omega^2}$	$cos(\omega.t)$	$\frac{s}{s^2+\omega^2}$
$e^{-a.t}.sin(\omega.t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-a.t}.cos(\omega.t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$1-cos(\omega.t)$	$\frac{\omega^2}{s(s^2+\omega^2)}$	$\omega.t-sin(\omega.t)$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
$t.cos(\omega.t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega}.t.sin(\omega.t)$	$\frac{s}{(s^2+\omega^2)^2}$
$\frac{1}{2\omega}(sin(\omega.t)+\omega.t.cos(\omega.t))$	$\frac{s^2}{(s^2+\omega^2)^2}$	$sin(\omega.t)-\omega.t.cos(\omega.t)$	$\frac{2.\omega^3}{(s^2+\omega^2)^2}$
$sinh(\omega.t)$	$\frac{\omega}{s^2-\omega^2}$	$cosh(\omega.t)$	$\frac{s}{s^2-\omega^2}$

B Step response of a 2nd order system

ζ	$t_R.\omega_n$	$t_{O_{s_{Max}}}.\omega_n$	$O_{s_{Max}}\%$
$0,1$	30	$3,16$	73
$0,15$	20	$3,18$	62
$0,2$	14	$3,21$	53
$0,25$	11	$3,24$	44
$0,3$	$10,1$	$3,29$	37
$0,35$	$7,9$	$3,35$	31
$0,4$	$7,7$	$3,43$	25
$0,45$	$5,4$	$3,52$	21
$0,5$	$5,3$	$3,63$	16
$0,55$	$5,3$	$3,76$	$12,6$
$0,6$	$5,2$	$3,93$	$9,5$
$0,65$	$5,0$	$4,13$	$6,8$
$0,7$	$3,0$	$4,40$	$4,6$
$0,75$	$3,1$	$4,75$	$2,84$
$0,8$	$3,4$	$5,24$	$1,52$
$0,85$	$3,7$	$5,96$	$0,63$
$0,9$	$4,0$	$7,21$	$0,15$
$0,95$	$4,1$	$10,06$	$0,01$
1	$4,6$		$0,00$