Interaction Diversity in Mean-Field Settings

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IPAM: ECH2025

Sept 23, 2025

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Objects

particles (z^N)

empirical measures

$$f_n(\bar{z}^N) = \prod_{i=1}^N \delta_{z_i}(\bar{z}_i)$$

N-distribution (f_N)

mean-field limit

$$f_N(\bar{z}^N) = f^{\otimes N}(\bar{z}^N) = \prod_{i=1}^N f(\bar{z}_i)$$

1-distribution (f)

Equations

Investigations of mean-field limits are often broken down into various systems:

Flow Dynamics (NODE)	$\{\dot{z}_i = \frac{1}{N}\sum_{j\neq i}^{N} K(z_i, z_j)$	high dimension
Liouville	$\partial_t f_N + \frac{1}{N}\sum_{i,j=1}^{N} \nabla_{z_i} \cdot (f_N K(z_i, z_j)) = 0$	high dimension
Vlasov (forward-Kolmogorov)	$\partial_t f + \nabla_z \cdot (f\mathcal{K}[f]) = 0$	low dimension

where $\mathcal{K}[f](x) = \int K(x, y)f(y)\, dy$ (sometimes $\mathcal{K}[f] = K * f$).

Classical Mechanics [1]

Some physical systems, such as
1) Vlasov-Poisson 2) Vlasov-Maxwell and 3) incompressible fluids
are adjacent to the following result:

Theorem

Assume K

1) is skew-symmetric 2) has continuous bounded derivatives and 3) is divergence-free

then, given an initial distribution f_0,

$$\begin{array}{c}\text{Solutions of NODE}\\ \text{(initial conditions sampled from } f_0\text{)}\end{array} \xrightarrow{\textit{Wasserstein}} \begin{array}{c}\text{Solutions of the Vlasov}\\ \text{(with initial conditions } f_0\text{)}\end{array}$$

Optimization [2]

To search a surface F for minima, one may apply a "swarm-based" approach by coupling
1) gradient descent and 2) simulated annealing
in the following system:

$$\begin{cases} \dot{x}_j = -\nabla F(x_j) + \sqrt{2\sigma(m_j)}\ W_j \\ \dot{m}_j = -(F(x_j) - \mathcal{F}^N)m_j \\ \mathcal{F}^n = \frac{\sum_{j=1}^N m_j F(x_j)}{\sum_{j=1}^N m_j} \end{cases} \rightarrow \begin{cases} \partial_t f = \nabla_x \cdot (f\nabla F) + (F(x) - \mathcal{F})\partial_m(mf) + \sigma(m)\Delta_x f \\ \mathcal{F} = \frac{\int\int mF(x)f(x,m)\ dx\ dm}{\int\int mf(x,m)\ dx\ dm} \end{cases}$$

Optimization [2]

The previous NODE to mean-field system admits the following result:

Theorem

Under reasonable (physical) constraints of σ, F, and initial conditions f_0,

$$\begin{array}{c}\text{Solutions of NODE}\\ \text{(initial conditions sampled from } f_0\text{)}\end{array} \xrightarrow{\textit{Wasserstein}} \begin{array}{c}\text{Solutions of the forward-Kolomogorov}\\ \text{(with initial conditions } f_0\text{)}\end{array}$$

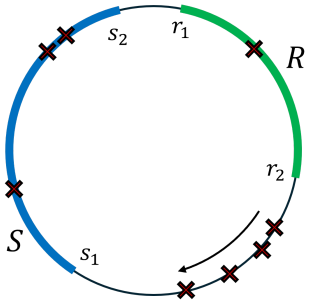
Moreover, $\liminf_t \lim_N \mathcal{F}^N = \min_x F$.

Modeling Yeast Metabolism [3]

Yeast cells have been observed synchronizing their metabolic cycles.

To explain the the pattern formation:

Cells in a (S)ignaling region release metabolites that inhibit cells in a (R)esponsive region.



Schematics of Model for Yeast Cell Metabolism with signaling region (blue), responsive region (green), and yeast cells (red)

Equations

The previous model to translates to a NODE of

$$\left\{\dot{z}_i = 1 + \chi_R(z_i) f\left(\frac{1}{N}\sum_{j=1}^{N} \chi_S(z_j)\right)\right. \quad \xrightarrow{f(x)=-\alpha x} \left\{\dot{z}_i = \frac{1}{N}\sum_{j=1}^{N} \alpha\left(1 - \chi_R(z_i)\chi_S(z_j)\right)\right.$$

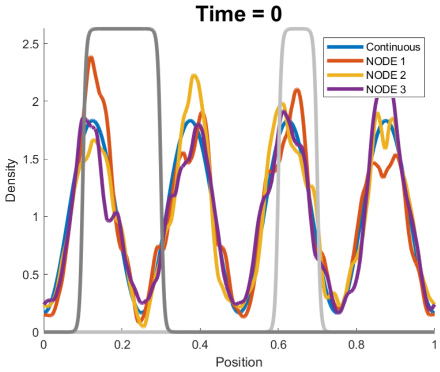
which may be pushed to a Vlasov of

$$\partial_t u + \partial_x\left(u\left(1 + \chi_R(x) f\left(\int_S u\right)\right)\right) = 0 \xrightarrow{f(x)=-\alpha x} \partial_t u + \partial_x u - \alpha\langle \chi_S, u\rangle \partial_x\left(u\chi_R(x)\right) = 0$$

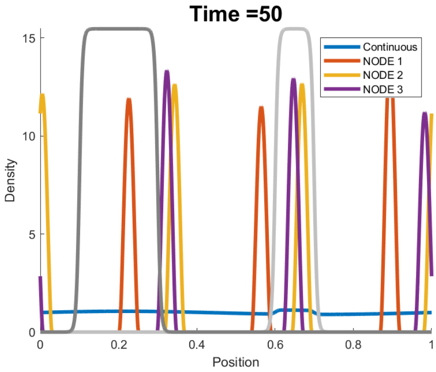
or Vlasov-McKean of

$$\cdots \xrightarrow{f(x)=-\alpha x} \partial_t u + \epsilon^2 \Delta u + \partial_x u - \alpha\langle \chi_S, u\rangle \partial_x\left(u\chi_R(x)\right) = 0$$

Simulation



Initial Conditions for Vlasov-McKean (blue) and three 1000-particle samplings of initial conditions (red, yellow, purple).



Numerical simulation of Vlasov-McKean (blue) and three 1000-particle simulations of NODE.

References

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Thank You!
Any Questions?

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