Vortex dynamics in the two-fluid model

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We have used two-fluid dynamics to study the discrepancy between the work of Thouless, Ao, and Niu (TAN) and that of Iordanskii. In TAN no transverse force on a vortex due to normal fluid flow was found, whereas the earlier work found a transverse force proportional to normal fluid velocity u_n and normal fluid density ρ_n. We have linearized the time-independent two-fluid equations about the exact solution for a vortex, and find three solutions that are important in the region far from the vortex. Uniform superfluid flow gives rise to the usual superfluid Magnus force. Uniform normal fluid flow gives rise to no forces in the linear region, but does not satisfy reasonable boundary conditions at short distances. A logarithmically increasing normal fluid flow gives a viscous force. As in classical hydrodynamics, and as in the early work of Hall and Vinen, this logarithmic increase must be cut off by nonlinear effects at large distances; this gives a viscous force proportional to $u_n/\ln u_n$, and a transverse contribution that goes like $u_n/(\ln u_n)^2$, even in the absence of an explicit Iordanskii force. In the limit $u_n \to 0$ the TAN result is obtained, but at nonzero u_n there are important corrections that were not found in TAN. We argue that the Magnus force in a superfluid at nonzero temperature is an example of a topological relation for which finite-size corrections may be large.

I. INTRODUCTION

In recent years papers have been published that appear to give an exact expression for the transverse force, including the superfluid Magnus force, acting on a quantized vortex moving in a neutral superfluid.[1,2] These methods were extended to give the transverse force on a flux line in a superconductor in the ultraclean limit,[3] and to give a corresponding (but not easily calculable) expression for the longitudinal force on a vortex in a neutral superfluid.[4] These results have been criticized[5–7] on the grounds that they do not agree with the widely, but by no means universally, accepted results in the literature.[8,9] In particular, a transverse force proportional to the superfluid density, and independent of the normal fluid density and velocity, was found, whereas it is generally supposed that the asymmetrical scattering of phonons or rotons by a vortex, demonstrated many years ago by Lifshitz and Pitaevskii[10] and by Iordanskii,[11] should lead to a contribution to the transverse force proportional to the normal fluid density and to the velocity of the vortex relative to the normal fluid component.

In order to shed new light on this controversy we have decided to examine carefully the predictions of the two-fluid model for a pinned vortex in a moving fluid. The two-fluid equations should be valid under conditions in which the pressure, temperature, and normal and superfluid velocities are varying slowly on the length scale set by the normal fluid mean free path. This cannot be true close to the core of a vortex, but it can be true everywhere if the vortex is pinned by a sufficiently large cylinder, on which either zero velocity or zero rate of shear (slip) boundary conditions are imposed. The equations are also valid at sufficient distances from the pinning center, even if the pinning center has a radius less than the mean free path. This line of thought is not entirely new, since similar arguments were used in the original work

of Hall and Vinen[12] on vortex arrays in helium, and by Vinen[13] on mutual friction. Their arguments, however, did not address the question of a transverse force.

In Sec. II we write down the equations of two-fluid theory for steady-state flow, and examine the simple equilibrium solution associated with a vortex in a stationary fluid.

In Sec. III we set up linearized steady-state equations and study their solutions. We want to find solutions linearized about the equilibrium vortex solution, and examine the behavior of the steady-state solutions at large distances from the vortex. Such an approach should be adequate to derive any transverse force proportional to normal fluid velocity and superfluid circulation, such as is quoted in Refs. 5, 6, and 7, as well as to derive the force proportional to superfluid velocity and superfluid circulation. The partial differential equations are separable in cylindrical polar coordinates, so by a combination of analytical and numerical methods we can get a fairly complete description of those solutions that have the same symmetry as uniform flow past the vortex.

In Sec. IV we examine the influence of boundary conditions on the solutions of the equations of motion. In our initial discussion we look at the case of a cylindrical inner boundary on which the normal fluid is at rest and the temperature is uniform. There is a complication, known long ago for the classical theory of flow of viscous fluid past a cylindrical obstacle, and discussed for superfluids in the works of Hall and Vinen[12] and Vinen,[13] which is that the linearized equations are not valid at very large distances, where the Reynolds number is of order unity even though the speed of flow is small.[14] This is true for any reasonable set of short-distance boundary conditions. This difficulty was circumvented by Oseen, and we can use the same technique. The drag force, in the linear regime, is transmitted by a normal fluid velocity that increases with the logarithm of the dis-

tance from the vortex core, but this must be cut off at some distance R_c at which nonlinear inertial effects become more important than the viscous force. The cutoff may alternatively be set by the spacing between vortices, by the finite size of the system, or by finite viscous diffusion length at nonzero frequency of vortex motion. It was initially surprising to us that for conventional inner boundary conditions there should be a transverse force, but it exists because the coupling between normal fluid flow and superfluid flow at small distances from the inner boundary produces it. We find that the magnitude of this transverse force is proportional to

$$\rho_n \frac{h}{m} \frac{u_n}{[\ln(R_c/r_0)]^2} \frac{L^2}{r_0^2}, \tag{1.1}$$

where u_n is the normal fluid velocity at distances of the order R_c, $r_0 \gg L$ is the distance from the axis at which inner-boundary conditions are applied, and L is the mean free path for excitations. This transverse force due to the normal fluid is small unless the radius of the boundary is comparable with the mean free path. A significant result that we get from our generalization of Lamb's solution[15] of the Oseen equation is that at large distances from the vortex there is a normal fluid circulation, contrary to what was assumed by Thouless, Ao, and Niu (TAN), but this circulation is proportional to $1/\ln^2 U$, where U is the asymptotic normal fluid velocity, so it goes slowly to zero as the normal fluid velocity goes to zero. Our results do agree with TAN's relation between the transverse force and the circulation of momentum at large distances. Some details of the calculation are shown in Appendix A.

In Sec. V we give a brief discussion of how the inner-boundary conditions can be modified to allow for the transverse force generated by flow of normal fluid past a bare vortex. Our study of the asymptotic flow shows that any force dependent on the normal fluid velocity is modified by a logarithmic dependence on the cut-off distance R_c. In the low-velocity limit any force due to the normal fluid motion, such as the Iordanskii[11] force, is renormalized away by logarithmic denominators; the force on a bare vortex has the form given by Eq. (1.1) in the case $r_0 \approx L$. In practice these terms should not be ignored, and may give a significant normal fluid contribution to the transverse force on a vortex, in addition to the superfluid Magnus force. The relation between vortex velocity and the transverse force on it does have a topological structure similar to the relation between voltage and current in a quantum Hall device,[16] but, unlike the quantum Hall conductance, this relation is subject to large corrections due to the finite size of the system, or the nonzero value of the velocity, decreasing only as the reciprocal of the square of the logarithm of the relevant length parameter.

There is a discussion of the relation of this work to the original work of Hall and Vinen in Appendix B.

In a recent paper by Sonin[17] there is a similar discussion of the relation of the TAN results to two-fluid theory. Sonin argues that there must be an additional contribution to the Berry phase from the flow of phonons or rotons past a vortex, and that this force is transmitted to large distances by the usual hydrodynamic processes.

II. EQUILIBRIUM SOLUTIONS

The equations of two-fluid hydrodynamics are given in the book by Khalatnikov,[18] in Eqs. 9.13, 9.14, 9.15, 9.16, 8.14, 8.19′, and 9.11. We want these equations in steady-state conditions, when all time derivatives are zero. The equation of conservation of matter is then

$$\nabla \cdot (\rho_s \mathbf{v} + \rho_n \mathbf{u}) = 0, \tag{2.1}$$

where we have written $\mathbf{v}$ for the superfluid velocity, and $\mathbf{u}$ for the normal fluid velocity. The momentum balance equation (Navier-Stokes equation) is

$$\nabla_k(\rho_s v_i v_k + \rho_n u_i u_k) + \nabla_i p$$
$$= \nabla_k[\eta(\nabla_k u_i + \nabla_i u_k)]$$
$$- \nabla_i\left[\left(\frac{2}{3}\eta + \zeta_1\rho - \zeta_2\right)\nabla \cdot \mathbf{u} + \zeta_1 \mathbf{u} \cdot \nabla\rho\right], \tag{2.2}$$

where η is the first viscosity, and the ζs are coefficients of second viscosity in the two-fluid model. We have set $\nabla \cdot \mathbf{j}$ equal to zero here, from Eq. (2.1). The equation for superfluid dynamics integrates to give

$$\mu - \mu_0 + \frac{1}{2}v^2 = (\zeta_4 - \rho\zeta_3)\nabla \cdot \mathbf{u} - \zeta_3 \mathbf{u} \cdot \nabla\rho. \tag{2.3}$$

The equation for thermal balance is

$$\nabla \cdot \left(S\mathbf{u} - \kappa\frac{\nabla T}{T}\right) = \frac{R}{T}, \tag{2.4}$$

where S is the entropy density, κ is the thermal conductivity, and R is the rate of heat generation by dissipative processes, a term that is quadratic in the normal fluid velocity and temperature gradient, and will be set equal to zero in the linear approximation. Finally, there is a thermodynamic relation

$$dp = SdT + \rho d\mu + \frac{1}{2}\rho_n d(\mathbf{v} - \mathbf{u})^2. \tag{2.5}$$

The equilibrium vortex, if the pinning center has cylindrical symmetry, is described very simply by the solution

$$\mathbf{u} = 0, \quad \mathbf{v} = \frac{\hbar}{mr}\hat{\theta}, \quad \mu = \mu_0 - \frac{\hbar^2}{2m^2r^2},$$

$$\frac{dp}{dr} = \rho_s(r)\frac{\hbar^2}{m^2r^3}, \quad T = T_0. \tag{2.6}$$

This is compatible with the local thermodynamic relation, Eq. (2.5).

III. LINEARIZED STEADY-STATE EQUATIONS

Now we want to linearize the Eqs. (2.1)–(2.5) about the vortex solution (2.6). We will take full account of first-order terms in $\delta\mathbf{v}, \mathbf{u}, \delta p, \delta T, \delta\mu$, but will neglect the consequent

variations of ρ_s, ρ_n, S, and of the transport coefficients. This is an additional simplifying assumption, and does not affect the asymptotic form of the solutions, since such variations all come into the equations accompanied by an additional factor of r^{-2}. We consider only motion in the xy plane, so all equations will be two dimensional. The equation of conservation of matter becomes

$$\rho_n \nabla \cdot \mathbf{u} = -\rho_s \nabla^2 w, \tag{3.1}$$

where the superfluid velocity has been written as

$$\mathbf{v} = \nabla(W + w), \quad \text{where} \quad W = \hbar\theta/m. \tag{3.2}$$

It is convenient now to write the normal fluid velocity in the form

$$u_i = \epsilon_{ij}\nabla_j\chi - \frac{\rho_s}{\rho_n}\nabla_i w, \tag{3.3}$$

where ϵ_{ij} is the alternating tensor in two dimensions. The linearized Navier-Stokes equation can then be written as

$$\nabla_i[\rho_s(\nabla_j W)\nabla_j w + \delta p] + \rho_s(\nabla_i W)\nabla^2 w$$
$$= \eta\epsilon_{ij}\nabla_j\nabla^2\chi - \frac{\rho_s}{\rho_n}\left(\frac{4}{3}\eta - \zeta_1\rho + \zeta_2\right)\nabla_i(\nabla^2 w). \tag{3.4}$$

This can conveniently be split into two parts by taking the curl and divergence of the equation. The curl gives

$$\rho_s\epsilon_{ij}(\nabla_i W)\nabla_j\nabla^2 w = \eta\nabla^4\chi, \tag{3.5}$$

which is an equation relating the viscous force on the normal fluid to the momentum flow in the superfluid. The divergence gives

$$\nabla^2\delta p + 2\rho_s\nabla_i\nabla_j[(\nabla_i W)\nabla_j w] + \frac{\rho_s}{\rho_n}\left(\frac{4}{3}\eta - \zeta_1\rho + \zeta_2\right)(\nabla^4 w)$$
$$= 0. \tag{3.6}$$

The superfluid equation gives

$$\delta\mu + (\nabla_i W)\nabla_i w = -\frac{\rho_s}{\rho_n}(\zeta_4 - \rho\zeta_3)\nabla^2 w. \tag{3.7}$$

The heat flow equation gives

$$\frac{S\rho_s}{\rho_n}\nabla^2 w + \frac{\kappa}{T}\nabla^2\delta T = 0. \tag{3.8}$$

The linearized thermodynamic relation gives

$$\delta p = S\,\delta T + \rho\,\delta\mu + \rho(\nabla_i W)\nabla_i w - \rho_n\epsilon_{ij}(\nabla_i W)\nabla_j\chi. \tag{3.9}$$

Substitution of Eqs. (3.9), (3.8), (3.7), in Eq. (3.6) gives

$$2\rho_s\nabla_i\nabla_j[(\nabla_i W)\nabla_j w] - \rho_n\nabla^2[\epsilon_{ij}(\nabla_i W)\nabla_j\chi]$$
$$+ \frac{\rho_s}{\rho_n}\left(\frac{4}{3}\eta - \rho\zeta_1 - \rho\zeta_4 + \zeta_2 + \rho^2\zeta_3\right)(\nabla^4 w)$$
$$- \frac{S^2 T\rho_s}{\kappa\rho_n}\nabla^2 w = 0. \tag{3.10}$$

The last two terms on the left side of this equation describe counterflow between normal and superfluid components, while the first two terms give coupling of this counterflow to the vortex motion. From now on we will ignore the contributions of second viscosity, which greatly simplifies the analysis. Only for ζ_2 do we have experimental values (from the observed attenuation of first sound),[19] although we know from Khalatnikov's[18] Eq. (9.12) and the inequality (9.18) that their contribution to the coefficient of $\nabla^4 w$ is positive. With this simplification we can rewrite Eqs. (3.5) and (3.10) in a dimensionless form. These coupled linear fourth-order equations have eight independent solutions for each angular symmetry.

Since temperature gradient and heat flow may be important both in the interpretation of these equations and in the boundary conditions, we take the gradient of Eq. (3.9), and substitute Eqs. (3.4) and (3.7) to get

$$S\nabla_i T = -\rho_s(\nabla_i W)\nabla^2 w - \rho_s\nabla_i[(\nabla_j W)\nabla_j w]$$
$$- \frac{\rho_s}{\rho_n}\frac{4}{3}\eta\nabla_i\nabla^2 w + \eta\epsilon_{ij}\nabla_j\nabla^2\chi$$
$$+ \rho_n\nabla_i[\epsilon_{jk}(\nabla_j W)\nabla_k\chi]. \tag{3.11}$$

We introduce the length scale L defined by

$$L^2 = \frac{4\eta\kappa}{3S^2 T}. \tag{3.12}$$

From kinetic theory L should be of the order of the mean free path for the excitations. We rewrite the equations in terms of the dimensionless complex variables $z = (x + iy)/L$ and its conjugate complex $\bar{z}$, using the symbols ∂ and $\bar{\partial}$ to denote the partial derivatives with respect to these two variables. In these variables we have

$$W = \frac{\hbar}{2mi}\ln\frac{z}{\bar{z}} \tag{3.13}$$

for the vortex potential. In these variables there is one dimensionless parameter in the equations of motion, which is

$$\alpha = \frac{3\hbar\rho_n}{4m\,\eta}; \tag{3.14}$$

this is of order unity at high temperatures, but falls off at lower temperatures in ${}^4\text{He}$ in a similar manner to ρ_n/ρ, since $\hbar\rho/m\eta$ remains of order unity. We look for solutions of the form

$$\rho_s w = \text{Re}\, zf(z\bar{z}), \quad \rho_n\chi = \text{Im}\, zg(z\bar{z}) \tag{3.15}$$

since these have the symmetry that corresponds to uniform flow. We use s to denote the variable $z\bar{z} = r^2/L^2$, and then the components of velocity are given by

$$L\rho_s(v_x - iv_y) = f + sf' + s\bar{f}'e^{-2i\theta},$$

$$L\rho_n(u_x - iu_y) = g - f + s(g' - f') - s(\bar{g}' + \bar{f}')e^{-2i\theta}. \tag{3.16}$$

In these new variables Eq. (3.5) gives

$$3s\left(s\frac{d^2}{ds^2} + 2\frac{d}{ds}\right)^2 g = -i\alpha(2s^2f''' + 7sf'' + 2f'), \tag{3.17}$$

and Eq. (3.10) gives

$$4s\left(s\frac{d^2}{ds^2} + 2\frac{d}{ds} - \frac{1}{4}\right)\left(s\frac{d^2}{ds^2} + 2\frac{d}{ds}\right)f$$

$$= -2i\alpha(sf'' + f') + i\alpha(2s^2g''' + 5sg''). \tag{3.18}$$

It is helpful to consider the case $\alpha = 0$, where the two Eqs. (3.17) and (3.18) decouple. In this case Eq. (3.17) has four solutions, namely,

$$g \propto s, \quad g \propto \ln s, \quad g \text{ constant}, \quad g \propto 1/s. \tag{3.19}$$

The first of these gives a solution of the Navier-Stokes equation with flow that is quadratic at large distances, the second is the well-known solution that is logarithmic at large distances and gives rise to a constant viscous force on any surface surrounding the axis, the third gives uniform flow, and the fourth dipolar flow falling off like $1/r^2$ at large distances. Equation (3.18), in the limit $\alpha \to 0$, has solutions

$$f \text{ constant}, \quad f \propto 1/s,$$

$$f \propto s^{-1/2}I_1(\sqrt{s}), \quad f \propto s^{-1/2}K_1(\sqrt{s}), \tag{3.20}$$

where I_1, K_1 are modified Bessel functions. The first two of these, combined with the corresponding solutions of Eq. (3.19), give uniform and dipolar superfluid flow, while the third and fourth give counterflow of the normal and superfluid components, growing and decreasing exponentially with distance from the axis.

Solutions of the actual Eqs. (3.17) and (3.18) have the same character at large distances, but some of them are modified at small distances from the axis. The four solutions that are unchanged are $f' = 0$, $g' = 0$, g' constant, and the solution $f = 1/s, g = -1/s$. Combinations of the first two give uniform superfluid or normal fluid flow, the third gives quadratically growing normal fluid velocity, and is not of interest to us, while the fourth gives dipolar superfluid flow. Equation (3.16) shows that the solution

$$f = \rho_s v_0, \quad g = \rho_s v_0, \tag{3.21}$$

gives uniform superfluid flow, no normal fluid flow. The temperature is a constant T_0 everywhere, and the other thermodynamic variables are given by

$$\delta p = \rho_s v_0 \frac{\hbar}{mr}\sin\theta, \quad \delta\mu = v_0\frac{\hbar}{mr}\sin\theta. \tag{3.22}$$

This is an exact solution of the equations as we have approximated them, but would be perturbed by terms from the inhomogeneity in the equilibrium value of ρ_s. These perturbations lead to corrections of w that are of order $1/r$, and to corrections of the pressure that are of order $1/r^3$. The net force per unit length on a large cylindrical volume of fluid of radius R due to pressure is

$$-\int_0^{2\pi}\delta p\mathbf{r}d\theta = -\frac{\pi\hbar}{m}\rho_s\hat{z}\times\mathbf{v}_0, \tag{3.23}$$

while the momentum flowing into the cylinder per unit length is

$$-\int\frac{\hbar}{m}\rho_s\hat{\theta}\mathbf{v}_0\cdot\hat{r}d\theta = -\frac{\pi\hbar}{m}\rho_s\hat{z}\times\mathbf{v}_0. \tag{3.24}$$

This gives the usual expression for the Magnus force on the vortex pinning center as the sum of these two contributions, equal to

$$\mathbf{F}_M = \frac{h}{m}\rho_s\hat{z}\times\mathbf{v}_0. \tag{3.25}$$

This derivation is exactly the same as the derivation for the Magnus force in a classical compressible fluid.

Uniform normal fluid flow is given by

$$f = 0, \quad g = \rho_n u_0. \tag{3.26}$$

This gives uniform pressure, no momentum flow, and no viscous forces. The dipolar superfluid flow is obtained by taking the derivative of the vortex solution with respect to the position of the vortex core, which is, in our notation,

$$f(s) \propto \frac{1}{s}, \quad g(s) = -f(s). \tag{3.27}$$

Various methods are available to find the remaining four solutions of the equation, or, rather, the three solutions that do not grow exponentially at large distances. We have solved the equations numerically, we have studied the power series solutions in positive or negative powers of s, and we have looked at the systematic expansion of the solutions in increasing powers of α using a standard Green function technique for solving the inhomogeneous ordinary differential equations that arise. Since α is indeed a small parameter for much of the temperature range of superfluid ^{4}He, and the expansion in powers of α is more transparent than the other methods, we will outline this method, and mention some of the results from the other methods.

We start with a zero-order approximation that is a linear superposition of the solutions in the limit $\alpha = 0$, excluding those that increase faster than logarithmically at large distances, and excluding the uniform superfluid flow, which we know about already, so that we have

$$g_0(s) = c_1 + c_2\frac{a^2}{s} + c_3\ln(s/a^2),$$

$$f_0(s) = d_2\frac{a^2}{s} + d_3\frac{1}{\sqrt{s}}K_1(\sqrt{s}). \tag{3.28}$$

Substitution of this in the right-hand sides of Eqs. (3.17) and (3.18) gives two functions $i\alpha G_0, i\alpha F_0$ that have the form

$$G_0 = -d_3\frac{1}{4}K_1'(\sqrt{s}),$$

$$F_0 = -2(c_2+d_2)\frac{a^2}{s^2} - c_3\frac{1}{s} + d_3\frac{1}{2\sqrt{s}}K_2'(\sqrt{s}), \tag{3.29}$$

where $aL = r_0$ is the distance at which short-distance boundary conditions will be imposed. This leads to the inhomogeneous equations

$$3s\left(s\frac{d^2}{ds^2} + 2\frac{d}{ds}\right)^2 g_1 = i\alpha G_0,$$

$$4s\left(s\frac{d^2}{ds^2} + 2\frac{d}{ds} - \frac{1}{4}\right)\left(s\frac{d^2}{ds^2} + 2\frac{d}{ds}\right)f_1 = i\alpha F_0. \tag{3.30}$$

Particular integrals of these inhomogeneous equations can be obtained by a standard Green function technique, in the form

$$g_1(s) = i\alpha\int_{a^2}^{\infty}K_g(s,t)G_0(t)dt,$$

$$f_1(s) = i\alpha\int_{a^2}^{\infty}K_f(s,t)F_0(t)dt, \tag{3.31}$$

and solutions $c_1' + c_2'a^2/s, d_2'a^2/s + d_3'K_1(\sqrt{s})/\sqrt{s}$, should be added to the particular integral so that the boundary conditions at $s = a^2$ are satisfied. We keep the coefficient of $\ln(s/a^2)$, the dominant term at large distances, constant during the iterative procedure. The Green functions K_f, K_g can be written as

$$K_f(s,t) = \begin{cases} 1/s - (2/\sqrt{st})K_1(\sqrt{s})I_1(\sqrt{t}) & \text{if } t < s \\ 1/t - (2/\sqrt{st})I_1(\sqrt{s})K_1(\sqrt{t}) & \text{if } t > s, \end{cases}$$

$$K_g(s,t) = \begin{cases} (2\ln t - t/s)/6 & \text{if } t < s \\ (2\ln s - s/t)/6 & \text{if } t > s, \end{cases} \tag{3.32}$$

but it is actually convenient sometimes to use other forms that differ from these by a solution of the homogeneous equation.

The process can be repeated by substituting f_1, g_1, etc., in the right-hand sides of Eqs. (3.17) and (3.18) to generate F_1, G_1, and so to find the second-order corrections in f, g.

All the solutions generated in this way have series solutions in ascending powers of s, with the leading power n_0 given by the solutions of the complex indicial equation

$$12n_0^2(n_0^2 - 1) + 6i\alpha n_0^2 - \alpha^2(4n_0^2 - 1) = 0. \tag{3.33}$$

The solution generated by a combination of c_2, d_2 has a power series solution in descending powers of s, with $1/s$ as the leading term. The solution generated by c_3 is $\ln s$ times a descending power series, and is the solution that leads to a net viscous force at large distances. The solution generated by d_3 has a singular point at infinity.

IV. BOUNDARY CONDITIONS AND SOLUTIONS

In the previous section we have shown how linearized solutions of the two-fluid equations in the presence of a straight superfluid vortex can be found. We must combine these solutions in a way that satisfies appropriate boundary conditions. We already know that superfluid flow gives rise to a Magnus force whose magnitude depends on the superfluid velocity at large distances. It remains to find the corresponding solution for the normal fluid component. As with a classical fluid, we cannot satisfy boundary conditions simply with a flow that is asymptotically uniform, but we need the logarithmically growing term. Four relations between the other five coefficients must be determined by boundary conditions on some inner boundary.

If the fluid is rotating around a wire of circular cross section, radius $r_0 = aL$, as in the Vinen experiment,[22] it is natural to take both components of normal fluid velocity, and the radial component of superfluid velocity, to be zero at r_0, or, in the notation of Eq. (3.15),

$$2a^2f'(a^2) + f(a^2) = 0, \quad g(a^2) = 0,$$

$$2a^2g'(a^2) - f(a^2) = 0. \tag{4.1}$$

An additional condition must be imposed, such as that the temperature is uniform on the boundary (alternatively, it might have been more realistic to take the heat flux to be zero at the boundary). From Eq. (3.11) the condition for constant temperature at $r = aL$ is

$$\frac{\rho_s\hbar}{maL}\left(\nabla^2 w + \frac{1}{a^2L^2}\partial_\theta^2 w\right) + \frac{4\rho_s\eta}{3\rho_n aL}\nabla^2\partial_\theta w$$

$$+ \left(\eta\nabla^2 + \frac{\rho_n\hbar}{ma^2L^2}\partial_\theta\right)\partial_r\chi = 0, \tag{4.2}$$

or, in terms of f and g, and making use of Eq. (4.1),

$$3(2a^4g''' + 7a^2g'' + 2g') = 4(1 - i\alpha)(a^2f'' + 2f'). \tag{4.3}$$

Only the terms for which $\nabla^2\chi$ and $\nabla^2 w$ are nonzero contribute to this.

Our main concern in this work is to what extent the normal fluid flow past a vortex can generate a transverse force. The direction of the viscous force is determined by the phase of the coefficient of the term in g that is logarithmic in s for large s, while the direction of flow is determined also by the constant term. The existence of a transverse force depends on the complexity of the solutions of Eqs. (3.17) and (3.18), imposed either by the form of the equations or by the boundary conditions. Numerical solution shows that in this case

(no tangential normal fluid flow or temperature gradient at the boundary) there is a transverse force due to normal fluid flow, and that this transverse force is proportional to ρ_n and more or less independent of η. This transverse force is small when the radius of the inner boundary is significantly greater than L, but rises as a approaches unity.

In Appendix A some of the details of the solution of the equations with these boundary conditions to first order in α and lowest nonvanishing order in $1/a$ are shown. To zero order in α and for $a \gg 1$, Eq. (A3) gives

$$c_1 = \left[-1 + \frac{3}{2a} \right] c_3, \tag{4.4}$$

which shows that the solution is similar to the solution of the classical problem, and the coupling between normal and superfluid components simply reduces the effective radius of the cylinder by an amount close to $3L/4$.

There are imaginary corrections to this solution to first order in α, coming both from the combined effect of the Green functions in Eq. (3.31), and from the boundary conditions. The detailed work in Appendix A shows that the most important terms for $a \gg 1$ are of order $i\alpha/a^2$. When the coefficient c_3 of the logarithmic term in $\mathbf{u}$ is kept equal to a constant C_3 in the iterative procedure, the coefficient of the constant term is, from Eq. (A10),

$$C_1 \approx c_1 + c_1' = \left[-1 + \frac{3K_0(a)}{2aK_1(a)} \right] C_3 - \frac{2i\alpha}{a^2} C_3. \tag{4.5}$$

In general the normal fluid velocity at large distances should have the form

$$L\rho_n(u_x - iu_y) = C_3[\ln(s/a^2) + 1] - \bar{C}_3 e^{-2i\theta} + C_1. \tag{4.6}$$

Because of the logarithmic term, this does not tend to uniform flow at large distances, which is the basis of the well-known Stokes paradox for slow viscous flow past a cylinder. The resolution of the Stokes paradox was provided by Oseen in 1910 (see Refs. 14 and 15), who pointed out that at sufficiently large distances the nonlinear inertial term $\rho_n \mathbf{u} \cdot \nabla \mathbf{u}$ in the equation of motion, overwhelms the viscous term $\eta \nabla^2 \mathbf{u}$ however small the fluid velocity $\mathbf{u}$ is. This was discussed in the context of the drag on superfluid vortices by Hall and Vinen[12] and by Vinen[13], and Appendix B shows how our discussion relates to this early work. In the region where these nonlinear terms are important there is negligible coupling between the superfluid and normal components, so the classical theory can be taken over with little modification.

Oseen proposed replacing the nonlinear term $\rho \mathbf{u} \cdot \nabla \mathbf{u}$ in the Navier-Stokes equation by $\rho \mathbf{U} \cdot \nabla \mathbf{u}$, where $\mathbf{U}$ is a constant vector equal to the asymptotic velocity of the fluid. This gives the leading term in a small Reynolds number asymptotic expansion, uniform in r, and is a good approximation in the regions where the nonlinear term is more important than the viscous term. The vorticity for the normal fluid then satisfies the equation

$$\rho_n \mathbf{U} \cdot \text{curl} \, \mathbf{u} = \eta \nabla^2 \text{curl} \, \mathbf{u}, \tag{4.7}$$

which follows from the curl of Eq. (2.2) by dropping all terms involving the divergence of $\mathbf{u}$. The general solution of this equation with the right asymptotic behavior, with $\mathbf{U}$ in the x direction, is

$$\text{curl} \, \mathbf{u} = e^{kx} \sum_n p_n K_n(kr) e^{in\theta}, \tag{4.8}$$

where the p_n are arbitrary coefficients, K_n is a modified Bessel function, and

$$k = \frac{\rho_n U}{2\eta}. \tag{4.9}$$

Lamb[15] gave an explicit solution for $\mathbf{u}$, and a simple generalization of this solution to allow for a transverse component of the force is

$$u_x - iu_y = U - A e^{kx} K_0(kr) - \bar{A} e^{-i\theta}[e^{kx}K_1(kr) - 1/kr] = U$$
$$- A e^{kL(z+\bar{z})/2} K_0(kL\sqrt{z\bar{z}})$$
$$- \bar{A} \sqrt{\frac{z}{\bar{z}}} e^{kL(z+\bar{z})/2} K_1(kL\sqrt{z\bar{z}}) + \frac{\bar{A}}{kLz}. \tag{4.10}$$

It can be verified that this indeed gives incompressible flow whose curl satisfies the Oseen equation, since we have

$$\frac{2i}{L} \frac{\partial}{\partial \bar{z}} (u_x - iu_y) = \text{curl} \, \mathbf{u} + i \, \text{div} \, \mathbf{u}$$
$$= 2k e^{kx} [\text{Im} \, A K_0 - \text{Im}(A e^{i\theta}) K_1]. \tag{4.11}$$

The vorticity given by this expression vanishes for $kr \gg 1$, except in a parabolic wake bounded by $\theta^2 \approx 2/kr$; outside the wake the flow is a combination of dipolar and vortex motion. A general discussion of this method has been given by Proudman and Pearson.[23]

The leading term in the expansion of Eq. (4.10) in powers of k is

$$u_x - iu_y \approx U + A \ln(kL|z|/2) + A\gamma - \frac{\bar{A}}{2} - \frac{\bar{A}z}{2z}, \tag{4.12}$$

where γ is Euler's constant. In the range $L \ll r \ll 1/k$ this should match the form given in Eq. (4.6), and this requires

$$A = 2C_3/L\rho_n, \tag{4.13}$$

and

$$L\rho_n U = C_3[2 \ln(2/kLa) - 2\gamma] + C_3 + \bar{C}_3 + C_1. \tag{4.14}$$

The complex ratio C_1/C_3 is determined by the solution of the differential Eqs. (3.17) and (3.18), and by the boundary conditions, so the phases of C_1, C_3 are determined by the imaginary part of Eq. (4.14), and the amplitudes are related to the asymptotic normal fluid velocity U by the real part. In the case that we have studied in detail this is given in Eq. (4.5).

Substitution of Eq. (4.6) in Eq. (2.2) gives the change of pressure and the viscous stress tensor in the region $L \ll r \ll 1/k$ as

$$\delta p = -\frac{4\eta}{rL\rho_n}(\text{Re}\,C_3\cos\theta - \text{Im}\,C_3\sin\theta),$$

$$S_\eta = \frac{4\eta}{rL\rho_n}(\hat{r}\hat{r} - \hat{\theta}\hat{\theta})(\text{Re}\,C_3\cos\theta - \text{Im}\,C_3\sin\theta). \quad (4.15)$$

Pressure and viscous stress each contribute half the net force, so that the total force per unit length due to normal fluid flow is

$$\mathbf{F}_n = \frac{8\pi\eta}{L\rho_n}(\text{Re}\,C_3\hat{x} - \text{Im}\,C_3\hat{y}), \quad (4.16)$$

in agreement with the results for an ordinary fluid quoted in hydrodynamics text books.[14] Equations (4.10) and (4.13) show that the normal fluid has circulation

$$\kappa_n = \frac{4\pi\,\text{Im}\,C_3}{kL\rho_n} \quad (4.17)$$

in the region $r \gg 1/k$. Comparison of this with Eqs. (4.16) and (4.9) shows that the transverse component of the force is given by

$$F_t = -\rho_n\kappa_n U. \quad (4.18)$$

This is in full agreement with the equations derived in TAN,[1] but not with the discussion, since it was assumed that the normal fluid circulation is zero. In fact, as we shall show, κ_n goes to zero as U goes to zero, but only as fast as $1/|\ln U|^2$.

The real parts of Eqs. (4.14) and (4.16) combined with the approximation (4.4) give

$$\frac{\mathbf{F}_n\cdot\hat{x}}{U} \approx \frac{8\pi\eta}{2\ln(4\eta/\rho_n Ua) - 2\gamma + 1 + 3/2a}, \quad (4.19)$$

which is the standard result from classical hydrodynamics, apart from a small correction of order $1/a$ in the denominator. Combination of the imaginary parts of Eqs. (4.5) and (4.14) gives

$$\text{Im}\,C_3 \approx \frac{4\alpha/a^2}{2\ln(4\eta/\rho_n Ua) - 1 + 3/2a}\,\text{Re}\,C_3, \quad (4.20)$$

so that, after substitution of the value of α from Eq. (3.14), the transverse force is given by

$$\mathbf{F}_n\cdot\hat{y} \approx -\frac{3\rho_n(h/m)}{a^2[\ln(4\eta/\rho_n Ua) - \gamma/2]^2}U. \quad (4.21)$$

which agrees with the result written down in Eq. (1.1). As the cutoff $R_c = 4\eta/\rho_n Ua$ goes to infinity both these components go to zero, the transverse force faster than the longitudinal force. For any finite value of the cutoff, these corrections of order $1/\ln R_c$ and $1/(\ln R_c)^2$ will be needed. Our numerical results agree with these expressions (4.19) and (4.21).

This calculation shows that even a solid core can give rise to a transverse force that is proportional to the normal fluid density, and its ratio to the normal fluid velocity close to the core can be quite large if the core radius is comparable with

L (a of order unity), however, its ratio to the asymptotic value of the normal fluid velocity is reduced by a factor proportional to $1/(\ln R_c)^2$.

As was observed by Hall and Vinen, the actual cutoff may be reduced to lower values, if a nonzero angular velocity produces a finite spacing of vortices, producing a situation analogous to the array of cylindrical obstacles considered for a viscous fluid by Sewell,[24] or if there is a nonzero frequency ν of oscillation of the normal fluid relative to the vortex, in which case a cutoff of magnitude

$$R_c \approx \sqrt{\frac{\eta}{\rho_n\nu}}, \quad (4.22)$$

is appropriate.

V. FREE VORTICES AND THE IORDANSKII FORCE

For a free vortex core, or one bound by a core whose size is much less than the mean free path of excitations, then we should take account of the effect of the superfluid flow on noninteracting excitations, in accordance with the discussions of Lifshitz and Pitaevskii,[10] Iordanskii,[11] and others.[20,21] According to these works the flow of phonons or rotons past a stationary vortex produces a transverse force equal to

$$\mathbf{F}_t = -\frac{\rho_n h}{m}\hat{z}\times\mathbf{u}_c, \quad (5.1)$$

where $\mathbf{u}_c$ is the normal fluid velocity at a distance of the order of a mean free path from the vortex core. In the phonon-dominated regime there is a longitudinal force

$$\mathbf{F}_{lp} = \frac{\rho_n h}{m}\frac{k_B T}{mc^2}\frac{15\pi\zeta(5)}{4\zeta(4)}\mathbf{u}_c, \quad (5.2)$$

where c is sound velocity and ζ denotes the Riemann ζ-function. In the roton-dominated region the longitudinal force is

$$\mathbf{F}_{lr} \approx \frac{\rho_n h}{m}3\pi\sqrt{\frac{2\pi\mu}{m}\frac{p_0^2}{mk_B T}}\ln\left(\sqrt{2\mu k_B T}\frac{2L}{h}\right)\mathbf{u}_c, \quad (5.3)$$

where μ is roton mass and p_0 is the magnitude of roton momentum. For helium in the phonon-dominated region, the longitudinal force due to normal fluid flow is about 40% of the transverse force, while in the roton-dominated region the longitudinal force is considerably larger than the transverse force.

A detailed calculation of what happens in this case is not easy to undertake, since we have no satisfactory theory of the transition region between the collisionless region of the order of a mean free path in radius and the hydrodynamic region beyond a few mean free paths from the vortex core. The best we have been able to do is to join one region abruptly to the other. Qualitatively the same thing will happen that happens for a solid core, which is that the growth of the logarithmic term in the normal fluid velocity will lead to an increasing alignment of the direction of normal fluid flow with the direction of the force at larger distances from the core, until the

Oseen or other cutoff radius R_c is reached. Indeed, with a solid core whose radius r_0 is comparable with the mean free path L, two-fluid hydrodynamics, although it is not strictly applicable, predicts a transverse force comparable with the Iordanskii force. The phonon-dominated regime is a little different, because the transverse force is initially somewhat larger than the longitudinal force, so, even after the logarithmic growth of the normal fluid comes into effect, there is still a significant angle between the direction of the force and the direction of flow.

VI. DISCUSSION

In this paper we have investigated how to combine the TAN expression relating the transverse force on a vortex to the circulation of momentum round the vortex to the usual results based on a combination of two-fluid hydrodynamics with the Lifshitz-Pitaevskii-Iordanskii expressions for the force on a free vortex in the presence of superfluid and normal flow. We have found that for a single stationary vortex in an infinite superfluid the TAN relation between transverse force and circulation of momentum is correct—this is not surprising, since the relation depends on little more than conservation of momentum and the independence of normal and superfluid flows at large distances from the vortex. The assumption that is wrong in Ref. 1 is that there is no normal fluid circulation, which is only literally true when the normal fluid is at rest. Application of the methods of Oseen and Lamb to the two-fluid situation shows that normal fluid flow with speed U round a vortex induces a circulation proportional to $\rho_n hU/m[\ln(4\pi\eta U/\rho_n a)]^2$.

For an isolated vortex this factor proportional to the inverse square of a logarithm will be rather small, and so the transverse force due to normal fluid flow will be small compared with the transverse force due to superfluid flow. There is a possible modification of this conclusion when the normal fluid component is primarily phonons, but then the normal fluid density is very small relative to the superfluid density, so the transverse force due to the normal fluid flow is still going to be small.

Similar conclusions have been reached in a recent paper by Sonin.[17]

APPENDIX A: FIRST-ORDER CORRECTIONS

We work out the first-order corrections to the flow, taking the specific case of the boundary conditions in which there is no normal fluid flow and constant temperature at $r=aL$, where a is large, as given in Eqs. (4.1) and (4.3).

To zero order the functions g and f are given by

$$g=c_1+c_2\frac{a^2}{s}+c_3\ln\frac{s}{a^2}, \quad f=d_2\frac{a^2}{s}+d_3\frac{1}{\sqrt{s}}K_1(\sqrt{s}). \quad (A1)$$

The four boundary conditions give

$$d_2=d_3K_1'(a), \quad c_2+c_1=0, \quad -2c_2+2c_3=d_2$$

$$+d_3\frac{1}{a}K_1(a), \quad 3c_3=-(1-i\alpha)d_3aK_1(a). \quad (A2)$$

These equations have solutions

$$c_1=\left[-1+\frac{3K_0(a)}{2aK_1(a)(1-i\alpha)}\right]c_3=-c_2,$$

$$d_2=-\frac{3K_1'(a)}{aK_1(a)(1-i\alpha)}c_3,$$

$$d_3=-\frac{3}{aK_1(a)(1-i\alpha)}c_3. \quad (A3)$$

Substitution of Eq. (3.29) in Eqs. (3.31) and (3.32) gives

$$g_1(s)=d_3\frac{i\alpha}{24}\int_s^\infty\left[2\ln\frac{t}{s}-\frac{t^2-s^2}{st}\right]K_1'(\sqrt{t})dt$$

$$=d_3\frac{i\alpha}{3}\int_s^\infty\frac{t+s}{t^{5/2}}K_1(\sqrt{t})dt, \quad (A4)$$

where three integrations by parts have been performed, and

$$f_1(s)=i\alpha(c_2+d_2)\left[\int_s^\infty\frac{2a^2(t-s)}{st^3}dt\right.$$

$$+\int_{a^2}^s\frac{4a^2}{s^{1/2}t^{5/2}}K_1(\sqrt{s})I_1(\sqrt{t})dt$$

$$+\int_s^\infty\frac{4a^2}{s^{1/2}t^{5/2}}I_1(\sqrt{s})K_1(\sqrt{t})dt+\frac{8}{s^{1/2}a^2}K_1(\sqrt{s})I_1(a)\right]$$

$$+i\alpha c_3\left[-\int_{a^2}^s\frac{1}{st}dt+\int_{a^2}^s\frac{2}{s^{1/2}t^{3/2}}K_1(\sqrt{s})I_1(\sqrt{t})dt\right.$$

$$+\int_s^\infty\frac{2}{s^{1/2}t^{3/2}}I_1(\sqrt{s})K_1(\sqrt{t})dt+\frac{4}{s^{1/2}a^2}K_1(\sqrt{s})I_1(a)\right]$$

$$+i\alpha d_3\left\{-\int_s^\infty\frac{t-s}{2st^{3/2}}K_2'(\sqrt{t})dt\right.$$

$$+\int_s^\infty\frac{1}{s^{1/2}t}[K_1(\sqrt{s})I_1(\sqrt{t})$$

$$-I_1(\sqrt{s})K_1(\sqrt{t})]K_2'(\sqrt{t})dt\right\}. \quad (A5)$$

To leading orders in a,s, these give

$$g_1(s) \approx d_3 i\alpha \sqrt{\frac{\pi}{2}} \left(\frac{4}{3s^{5/4}} - \frac{25}{6s^{7/4}} \right) e^{-\sqrt{s}},$$

$$f_1(s) \approx i\alpha \left[(c_2 + d_2) \frac{a^2}{s^2} - c_3 \frac{1}{s} \ln(s/a^2) - d_3 \sqrt{\frac{\pi}{2}} \frac{1}{s^{5/4}} e^{-\sqrt{s}} \right].$$

$$(A6)$$

Now we substitute the expressions

$$g_1(s) + c_1' + c_2' \frac{a^2}{s}, \quad f_1(s) + d_2' \frac{a^2}{s} + d_3' \frac{1}{\sqrt{s}} K_1(\sqrt{s}), \quad (A7)$$

in the boundary conditions (4.1) and (4.3) to get

$$c_1' = \frac{K_0(a)}{aK_1(a)}$$

$$\times \left[-\frac{3}{2}(2a^6 g_1''' + 7a^4 g_1'' + 2a^2 g_1') + 2(a^4 f_1'' + 2a^2 f_1') \right]$$

$$+ a^2(f_1' - g_1') + f_1 - g_1,$$

$$c_2' = -c_1' - g_1,$$

$$d_3' K_0(a) = 2c_2' + 2a^2(f_1' - g_1') + 2f_1,$$

$$d_2' = d_3' K_1'(a) + 2a^2 f_1' + f_1. \quad (A8)$$

The coefficient c_1' is the most important one, since, with c_1, c_3 it dominates the behavior for large s. In the expression for c_1' it can be seen that the only terms giving contributions of order $i\alpha/a^2$ are the terms proportional to $a^3 g_1'', a^2 g_1', a^3 f_1'', a^2 f_1', f_1$, while the term proportional to $a^5 g_1'''$ gives a contribution of order $i\alpha/a$ that cancels with a similar term in c_1. There is no contribution to this order from the terms in f_1 proportional to d_2 or d_3. With the substitution of Eqs. (A3) and (A6) in this we get

$$\frac{c_1'}{i\alpha} \approx d_3 \frac{K_0(a)}{2} - (c_2 + c_3) \frac{1}{a^2}$$

$$+ d_3 \sqrt{\frac{\pi}{2}} \frac{2}{3a^{3/2}} e^{-a}$$

$$\approx -c_3 \left[\frac{3K_0(a)}{2aK_1(a)} + \frac{4}{a^2} \right]. \quad (A9)$$

Combined with Eq. (A3) this gives

$$\frac{c_1 + c_1'}{c_3} \approx \left[-1 + \frac{3K_0(a)}{2aK_1(a)} \right] - 4\frac{i\alpha}{a^2}. \quad (A10)$$

The last term in Eq. (A10) gives the imaginary part of the coefficient, that leads to a velocity at large distances that is not in the same direction as the force.

APPENDIX B: RELATION TO THE HALL-VINEN THEORY

As is clear from earlier sections of this paper, some of the problems that we have addressed were foreseen by Hall and Vinen[12] (HV) and by Vinen,[13] and it may help the reader if we examine the extent to which the results derived in these earlier works accord with those derived here. The earlier works referred to a vortex with a core of microscopic size (much smaller than the excitation mean free path); furthermore, as we have noted, it did not include any transverse force due to flow of the normal fluid, such a force having been ruled out, incorrectly, by a supposed symmetry of the scattering of excitations by the vortex. However, it is easy to modify the earlier work by the addition of a transverse force, as we now show.

In the presence of a transverse component of the force, Eq. (2) of HV must be generalized to

$$f_\parallel = Dv_{R\parallel} - D'v_{R\perp}, \quad f_\perp = Dv_{R\perp} + D'v_{R\parallel}; \quad (B1)$$

The symbols $\parallel$ and $\perp$ refer to directions parallel and perpendicular to the asymptotic normal fluid velocity, U (i.e., $f_\parallel = \mathbf{F}_n \cdot \hat{x}$ and $f_\perp = \mathbf{F}_n \cdot \hat{y}$ in the notation of Sec. IV), and $\mathbf{v}_R$ is the velocity of the normal fluid close to the (stationary) vortex. Equation (4) of HV still holds and gives the relationship between U and $\mathbf{v}_R$

$$U - v_{R\parallel} = \frac{f_\parallel}{E}, \quad -v_{R\perp} = \frac{f_\perp}{E}, \quad (B2)$$

where E is given by Eq. (5) of HV for the case of finite frequencies. The parameter L in E is roughly the mean free path of the excitations constituting the normal fluid, as in earlier parts of this paper. In the case of zero frequency, with which this paper is concerned, the parameter λ in Eq. (5) of HV must be replaced by[13] $\rho_n U/2\eta$, so that the logarithmic term in Eq. (5) of HV becomes $\ln R_e$, where R_e is the Reynolds number $L\rho_n U/4\eta$. When U is sufficiently small the magnitude of this (negative) logarithmic term is much larger than unity, and then E can be written in the approximate form

$$E \approx \frac{4\pi\eta}{\ln R_e}. \quad (B3)$$

From these equations we can derive the relations between the components of the force on the vortex and U, analogous to Eqs. (4.19) and (4.21) in Sec. IV.

$$\frac{f_\parallel}{U} = E - \frac{(D+E)E^2}{(D+E)^2 + D'^2};$$

$$(B4)$$

$$\frac{f_\perp}{U} = \frac{E^2 D'}{(D+E)^2 + D'^2}.$$

For sufficiently small values of U we must have $D \gg E$. It follows that the leading term in $f_\parallel/U$ is proportional to $[\ln R_e]^{-1}$, while the leading term in $f_\perp/U$ is proportional to

$[\ln R_e]^{-2}$. Equations (4.19) and (4.21) in Sec. IV exhibit the same feature. According to HV the feature has its origin in the form of Eqs. (B2), which give the response of the normal fluid to the line force **f**. The cutoff in the solution of the linearized Navier-Stokes equation that is required at large distances (Sec. I) forces this response to decrease logarithmically as U decreases and the corresponding cutoff distance increases.

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