$$g_{11} = 1+u^2 \qquad g_{12} = 2v-u \qquad g_{21} = 2v-u \qquad g_{22} = 1+\kappa v^2$$

$$g\,g^{11} = 1+\kappa v^2 \qquad g\,g^{12} = u-2v \qquad g\,g^{21} = u-2v \qquad g\,g^{22} = 1+u^2$$

$$\Gamma^1{}_{11} = \frac{\kappa u v^2 + 2v}{g} \qquad \partial_2 \Gamma^1{}_{11} = \frac{2(\kappa u v+1)g - (\kappa u v^2+2v)\partial_v g}{g^2}$$

$$\Gamma^1{}_{22} = \frac{2+\kappa u v}{g} \qquad \Gamma^2{}_{11} = -\frac{(1+2uv)}{g} \Rightarrow \Gamma^1{}_{22}\Gamma^2{}_{11} = -\frac{(2+4uv+\kappa u v + 2\kappa u^2 v^2)}{g^2}$$

$$g = 1+4uv+\kappa u^2 v^2 + (\kappa-4)v^2 \quad \rightsquigarrow \quad \partial_v g = 4u + 2\kappa u^2 v + 2(\kappa-4)v$$

$$\partial_2 \Gamma^2{}_{11} = \frac{-2ug+(1+2uv)\partial_v g}{g^2} = \frac{-2u-8u^2v-2\kappa u^3v^2-2(\kappa-4)uv^2+4u+2\kappa u^2 v+2(\kappa-4)v+8u^2v+4\kappa u^3v^2+4(\kappa-4)uv^2}{g^2}$$

$$= \frac{2u+2\kappa u^2 v+2\kappa u^3 v^2+2(\kappa-4)uv^2+2(\kappa-4)v}{g^2}$$

$$\Gamma^2{}_{22} = \frac{2u+(\kappa-4)v+\kappa u^2 v}{g} \Rightarrow \Gamma^2{}_{22}\Gamma^2{}_{11} = -\frac{2u+4u^2v+(\kappa-4)v+2(\kappa-4)uv^2+\kappa u^2 v+2\kappa u^3 v^2}{g^2}$$

$$= -\frac{\cancel{2u}+(\kappa+4)u^2 v+\cancel{(\kappa-4)v}+\cancel{2(\kappa-4)uv^2}+\cancel{2\kappa u^3 v^2}}{g^2}$$

$$g^2\left(\partial_2\Gamma^2{}_{11}+\Gamma^2{}_{22}\Gamma^2{}_{11}\right) = (\kappa-4)u^2 v$$

$$g^2\partial_2\Gamma^1{}_{11} = 2(\kappa u v+1)g-(\kappa u v^2+2v)\partial_v g = 2\kappa u v + 8\kappa u^2 v^2 + 2\kappa^2 u^3 v^3 + 2\kappa(\kappa-4)uv^3 + 2 + 8uv + 2\kappa u^2 v^2 + 2(\kappa-4)v^2 - (\kappa u v + 2v)\partial_v g$$

$$= 2 + (2\kappa+8)uv + 10\kappa u^2 v^2 + 2\kappa^2 u^3 v^3 + 2\kappa(\kappa-4)uv^3 + 2(\kappa-4)v^2 - 8uv - 4\kappa u^2 v^2 - 4(\kappa-4)v^2 - \kappa u v\,\partial_v g$$

$$= 2 + 2\kappa u v + 6\kappa u^2 v^2 + 2\kappa^2 u^3 v^3 + 2\kappa(\kappa-4)uv^3 - 2(\kappa-4)v^2 - 4\kappa u^2 v^2 - 2\kappa^2 u^3 v^3 - 2\kappa(\kappa-4)uv^3$$

$$= 2 + 2\kappa u v + 2\kappa u^2 v^2 - 2(\kappa-4)v^2$$

$$\Gamma^1{}_{22}\Gamma^2{}_{11} = -\frac{(\cancel{2}+4uv+\kappa u v+\cancel{2\kappa u^2 v^2})}{g^2}$$

$$g^2\left(\partial_2\Gamma^1{}_{11}+\Gamma^1{}_{22}\Gamma^2{}_{11}\right) = (\kappa-4)uv - 2(\kappa-4)v^2 = (\kappa-4)(u-2v)v$$

$$g_{21}(u-2v)+g_{22}u^2 = -(\cancel{u^2}-4uv+4v^2)+\cancel{u^2}+\kappa u^2 v^2 = \kappa u^2 v^2 + 4uv - 4v^2$$