Notes on *Riemannian Geometry*

Louis Bergamo Radial

January 2, 2024

Contents

1 Differentiable Manifolds **2**
1.1 Topology 2
1.2 Homeomorphisms 3
1.3 Compactness and paracompactness 6

Chapter 1

Differentiable Manifolds

The theory of smooth manifolds is a very useful generalization of the differential calculus on $\mathbb{R}^n$. Namely, a smooth manifold is a topological space endowed with a differentiable structure such that it locally resembles Euclidean space.

1.1 Topology

In order to define the notion of smooth manifolds, we must first begin with some building blocks, such as topology and topological manifolds.

Definition 1.1: Topology

A *topology* on the set M is a family $\mathcal{O}$ of subsets of M satisfying

(a) the empty set and the set M belong to $\mathcal{O}$;

(b) a finite intersection of elements of $\mathcal{O}$ is a member of $\mathcal{O}$; and

(c) an arbitrary union of members of $\mathcal{O}$ belongs to $\mathcal{O}$.

The pair $(M, \mathcal{O}_M)$ is named a *topological space*, elements of $\mathcal{O}$ are called *open sets* and elements of $M \setminus \mathcal{O}$ are called *closed sets*.

In Propositions 1.1 and 1.2 we show a couple of important examples that illustrate how the axioms of topological spaces given in Definition 1.1 are used.

Proposition 1.1: Standard topology in $\mathbb{R}^n$

We define the *open ball* $B_n(r,p) \subset \mathbb{R}^n$ of radius $r > 0$ centered in the point $p = (p^1, \ldots, p^n)$ as the set

$$B_n(r,p) = \left\{ q = (q^1, \ldots, q^n) \in \mathbb{R}^n : \sum_{i=1}^{n} (q^i - p^i)^2 < r^2 \right\}.$$

Next, we define the *standard topology* $\mathcal{O}_{\text{standard}}$ of $\mathbb{R}^n$. A subset $U \subset \mathbb{R}^n$ is an open set if for every point $p \in U$ there exists $r > 0$ such that $B_n(r,p) \subset U$. Then, $(\mathbb{R}^n, \mathcal{O}_{\text{standard}})$ is a topological space.

Proof. It is easy to see $\mathbb{R}^n \in \mathcal{O}_{\text{standard}}$ and $\emptyset \in \mathcal{O}_{\text{standard}}$.

Suppose $U, V \in \mathcal{O}_{\text{standard}}$ and let $p \in U \cap V \neq \emptyset$.Then, there exists $r_U > 0$ and $r_V > 0$ such that $B_n(r_U, p) \subset U$ and $B_n(r_V, p) \subset V$. Setting $r = \min\{r_U, r_V\} > 0$ we have $B_n(r, p)$ as subset of both U and V, that is, $B_n(r, p) \subset U \cap V$. It follows that $U \cap V \in \mathcal{O}_{\text{standard}}$.

Let $\{U_\alpha\}_{\alpha \in J}$ be a family of sets in $\mathcal{O}_{\text{standard}}$. Let $p \in \bigcup_{\alpha \in J} U_\alpha$, that is, there exists $\beta \in J$ such that $p \in U_\beta$. Since $U_\beta \in \mathcal{O}_{\text{standard}}$, there exists $r_\beta > 0$ such that $B_n(r, p) \subset U_\beta \subset \bigcup_{\alpha \in J} U_\alpha$. □

Proposition 1.2: Subspace topology is a topology

Given a topological space $(M, \mathcal{O}_M)$ and a subset S of M, we define the *subspace topology* $\mathcal{O}_M|_S$ as

$$\mathcal{O}_M|_S = \{U \cap S : U \in \mathcal{O}_M\}.$$

Then $(S, \mathcal{O}_M|_S)$ is a topological space.

Proof. We must show the conditions (a), (b), and (c) of Definition 1.1 are satisfied.

(a) Since $S = M \cap S$ and $\emptyset = \emptyset \cap S$, we have $S \in \mathcal{O}_M|_S$ and $\emptyset \in \mathcal{O}_M|_S$.

(b) Let $U, V \in \mathcal{O}_M|_S$. Then, there exists $\tilde{U}, \tilde{V} \in \mathcal{O}_M$ such that $U = \tilde{U} \cap S$ and $V = \tilde{V} \cap S$.Then, $U \cap V = (\tilde{U} \cap S) \cap (\tilde{V} \cap S) = (\tilde{U} \cap \tilde{V}) \cap S$. Since $\tilde{U} \cap \tilde{V} \in \mathcal{O}_M$, we have $U \cap V \in \mathcal{O}_M|_S$.

(c) Let $\{U_\alpha\}_{\alpha \in J}$ be a family of open sets in $\mathcal{O}_M|_S$. For each $\alpha \in J$, there exists a $\tilde{U}_\alpha \in \mathcal{O}_M$ such that $U_\alpha = \tilde{U}_\alpha \cap S$. Then

$$\begin{aligned}
\bigcup_{\alpha \in J} U_\alpha &= \bigcup_{\alpha \in J} \tilde{U}_\alpha \cap S \\
&= \{m \in S : \exists \alpha \in J \text{ such that } m \in \tilde{U}_\alpha\} \\
&= \{m \in M : \exists \alpha \in J \text{ such that } m \in \tilde{U}_\alpha\} \cap S \\
&= S \cap \bigcup_{\alpha \in J} \tilde{U}_\alpha.
\end{aligned}$$

Since arbitrary unions of open sets is an open set, it follows that $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{O}_M|_S$.

□

1.2 Homeomorphisms

With the notion of topological spaces, we may ask ourselves whether certain maps between topological spaces can preserve the topology. That is, a map that takes open sets in the domain topology into open sets in the target topology. To define such a map we define *continuity*.

Definition 1.2: Continuous map

Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces. Then a map $f : M \to N$ is *continuous* (with respect to $\mathcal{O}_M$ and $\mathcal{O}_N$) if, for all $V \in \mathcal{O}_N$, the preimage $f^{-1}(V)$ is an open set in $\mathcal{O}_M$.

In short, a map is continuous if and only the preimages of (all) open sets are open sets. Now a map that preserves the topology is called a *homeomorphism*, which is defined as a continuous bijection with continuous inverse. We now prove such a map satisfies the condition required.

Proposition 1.3: Homeomorphism maps open sets to open sets

Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces. Suppose a map $f : M \to N$ is a homeomorphism, then f maps open sets in $\mathcal{O}_M$ into open sets in $\mathcal{O}_N$.

Proof. Given a subset $U \in \mathcal{O}_M$, we must show the image $V = f(U)$ is open in $(N, \mathcal{O}_N)$. Taking our attention to the inverse map $g = f^{-1} : N \to M$, we see the preimage $g^{-1}(U) = V$ must be open in $(N, \mathcal{O}_N)$, due to continuity. □

If there exists a homeomorphism between two topological spaces, they are said to be homeomorphic to each other. This begs the question: if $(M, \mathcal{O}_M)$ is homeomorphic to $(N, \mathcal{O}_N)$ and $(N, \mathcal{O}_N)$ is homeomorphic to $(P, \mathcal{O}_P)$, are $(M, \mathcal{O}_M)$ and $(P, \mathcal{O}_P)$ homeomorphic? To answer this we must show whether the composition of continuous maps is itself continuous.

Theorem 1.1: Composition of continuous maps

Let $(M, \mathcal{O}_M)$, $(N, \mathcal{O}_N)$, and $(P, \mathcal{O}_P)$ be topological spaces. If the maps $f : M \to N$ and $g : N \to P$ are continuous (with respect to the appropriate topologies), then the map $g \circ f : M \to P$ is continuous with respect to $\mathcal{O}_M$ and $\mathcal{O}_{\mathcal{P}}$.

Proof. Let V be an open set of $(P, \mathcal{O}_P)$. We must show the preimage $(g \circ f)^{-1}(V)$ is an open set of $(M, \mathcal{O}_M)$. We have

$$\begin{aligned} (g \circ f)^{-1}(V) &= \{m \in M : g \circ f(m) \in V\} \\ &= \{m \in M : f(m) \in g^{-1}(V)\} \\ &= f^{-1}\left(g^{-1}(V)\right). \end{aligned}$$

Since the map g is continuous and V is an open set in $(P, \mathcal{O}_P)$, it follows that $g^{-1}(V)$ is open in $(N, \mathcal{O}_N)$. By the same argument, $f^{-1}\left(g^{-1}(V)\right)$ is an open set in $(M, \mathcal{O}_M)$. □

Corollary 1.1. *If $(M, \mathcal{O}_M)$ is homeomorphic to $(N, \mathcal{O}_N)$ and $(N, \mathcal{O}_N)$ is homeomorphic to $(P, \mathcal{O}_P)$, then $(M, \mathcal{O}_M)$ is homeomorphic to $(P, \mathcal{O}_P)$.*

Proof. Let $f : M \to N$ and $g : N \to P$ be homeomorphisms from $(M, \mathcal{O}_M)$ to $(N, \mathcal{O}_N)$ and $(N, \mathcal{O}_N)$ to $(P, \mathcal{O}_P)$, respectively. Consider the composition $g \circ f : M \to P$.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[dr, "g\circ f"'] & N \arrow[d, "g"] \\ & P \end{tikzcd}$$

By Theorem 1.1, the map $g \circ f$ is a homeomorphism from $(M, \mathcal{O}_M)$ to $(P, \mathcal{O}_P)$. □

As was done for the subspace topology, we prove a similar result for continuous maps.

Proposition 1.4: Restriction of a continuous map

Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces and let $f : M \to N$ be a continuous map. Let S be a subset of M and let $(S, \mathcal{O}_S)$ be the subspace topology, then $f|_S : S \to N$ is a continuous map with respect to $\mathcal{O}_S$ and $\mathcal{O}_N$.

Proof. Let $V \in \mathcal{O}_N$. Then, by the definition of preimage, we have

$$\begin{aligned} f|_S^{-1}(V) &= \{s \in S : f|_S(s) \in V\} \\ &= \{s \in S : f(s) \in V\} \\ &= f^{-1}(V) \cap S. \end{aligned}$$

By hypothesis, the preimage $f^{-1}(V)$ is an open set in $(M, \mathcal{O}_M)$, so $f|_S^{-1}(V)$ is an open set in the subspace topology. □

We can now define the notion of a topological space locally resembling Euclidean space.

Definition 1.3: Locally Euclidean topological space

A topological space $(M, \mathcal{O}_M)$ is *locally Euclidean* of dimension n if for all $m \in M$ there exists an open subset $U \in \mathcal{O}_M$ about m that is homeomorphic to $\mathbb{R}^n$ with respect to the subspace topology and the standard topology of $\mathbb{R}^n$.

It is sufficient to show the subspace topology $(U, \mathcal{O}_U)$ is homeomorphic to an open ball in $\mathbb{R}^n$, due to Proposition 1.5.

Proposition 1.5: Open ball is homeomorphic to the Euclidean space

Let $r > 0$, then the map $f : B_n(r, 0) \subset \mathbb{R}^n \to \mathbb{R}^n$ given by

$$f(x) = \frac{x}{r - \|x\|}$$

is a homeomorphism with respect to the standard topology.

Proof. We begin by checking f is one-to-one and onto.

Suppose there exists $x_1, x_2 \in B_n(r, 0)$ such that $f(x_1) = f(x_2)$. It follows from

$$\begin{aligned} f(x_2) - f(x_1) &= \frac{x_2}{r - \|x_2\|} - \frac{x_1}{r - \|x_1\|} \\ &= \frac{(r - \|x_1\|)\, x_2 - (r - \|x_2\|)\, x_1}{(r - \|x_2\|)(r - \|x_1\|)} \end{aligned}$$

that $(r - \|x_1\|)\, x_2 = (r - \|x_2\|)\, x_1$. Applying the norm to both sides, we have $\|x_1\| = \|x_2\|$. Substituting back, we have $x_1 = x_2$, proving f is injective.

Suppose $y \in \mathbb{R}^n$ and consider $\xi = \frac{ry}{1+\|y\|}$. Clearly, $\xi \in B_n(r, 0)$. We have

$$\begin{aligned} f(\xi) &= f\left(\frac{ry}{1 + \|y\|}\right) \\ &= \frac{ry}{1 + \|y\|} \frac{1}{r - \left\|\frac{ry}{1+\|y\|}\right\|} \\ &= \frac{1}{(1 + \|y\|)\left(1 - \frac{\|y\|}{1+\|y\|}\right)} y \\ &= y, \end{aligned}$$

so f is onto.

We have shown f is a bijection with inverse $f^{-1} : \mathbb{R}^n \to B_n(r, 0)$ defined by

$$f^{-1}(x) = \frac{rx}{1 + \|x\|}. \tag{1.2.1}$$

With the standard topology, continuity of f and f^{-1} follows from techniques of elementary calculus, and we conclude f is a homeomorphism. □

1.3 Compactness and paracompactness

Definition 1.4: Hausdorff space

A topological space $(M, \mathcal{O}_M)$ is called a *Hausdorff space* if for any $p, q \in M$ with $p \neq q$, there exists a neighborhood U of p, i.e. $p \in U \in \mathcal{O}_M$, and a neighborhood V of q such that $U \cap V = \emptyset$.

Remark 1.1. *The Hausdorff property is one of the* separation axioms *of topological spaces. Namely, a Hausdorff space is also called a* T2 space.

Definition 1.5: Compactness

A topological space $(M, \mathcal{O}_M)$ is *compact* if every *open cover* of M has a finite subcover. That is, the topological space is compact if for every family of open sets C that covers M, i.e. $\bigcup_{U \in C} U = M$ with $U \in \mathcal{O}_M$, there exists a finite family of open sets $F \subset C$ such that $\bigcup_{U \in F} U = M$.
Additionally, in a topological space $(N, \mathcal{O}_N)$, a subset $S \subset N$ is called compact if the subspace topology is compact.

Theorem 1.2: Heine-Borel theorem

A subset $S \subset \mathbb{R}^n$ with the standard topology is compact if it is closed and bounded.

Proof. Refer to [2]. □

Definition 1.6: Locally finite collection

A collection of subsets C of a topological space $(M, \mathcal{O}_M)$ is called *locally finite* if each point in the space has a neighborhood that intersects only finitely many sets in C. More precisely, for all $p \in M$ there exists a neighborhood $U \in \mathcal{O}_M$ about p such that $U \cap V \neq \emptyset$ only for finitely many $V \in C$.

Definition 1.7: Refinement

A *refinement* of a cover C of a topological space $(M, \mathcal{O}_M)$ is a cover D such that every set in D is contained in some set in C. Precisely, let $C = \{U_\alpha\}_{\alpha \in A}$ and $D = \{V_\beta\}_{\beta \in B}$ such that $\bigcup_{\alpha \in A} U_\alpha = M$ and $\bigcup_{\beta \in B} V_\beta = M$, then D is a refinement of C if for all $\beta \in B$ there exists $\alpha \in A$ such that $V_\beta \subset U_\alpha$.

Definition 1.8: Paracompactness

A topological space $(M, \mathcal{O}_M)$ is called *paracompact* if every open cover C has an *open refinement* $\tilde{C}$ that is *locally finite*.

Definition 1.9: Partition of unity

A *partition of unity* of a topological space $(M, \mathcal{O}_M)$ is a set $\mathcal{F}$ of continuous functions from M to $[0, 1] \subset \mathbb{R}$ such that for every point $p \in M$

(a) there exists a neighborhood $U \in \mathcal{O}_M$ about p where all but finitely many functions of $\mathcal{F}$ vanish on U;

(b) the sum of all function values at p is 1, that is, $\sum_{f\in\mathcal{F}} f(p) = 1$.

Moreover, let $C = \{U_\alpha\}_{\alpha\in J}$ be an open cover of M. A *partition of unity subordinate to the open cover* C is a family $\mathcal{F}_C$ of continuous maps $f_\alpha : p \to [0, 1] \subset \mathbb{R}$ indexed over the same set J such that the support of f_α is contained in U_α, for all $\alpha \in J$. That is, for every $f \in \mathcal{F}_C$ there exists an open set $U \in C$ such that $f(p) \neq 0 \implies p \in U$.

Theorem 1.3: Paracompactness and partitions of unity

Let $(M, \mathcal{O}_M)$ be a Hausdorff space. Then it is paracompact if and only if every open cover C admits a partition of unity subordinate to that cover.

Proof. Refer to [1]. □

Bibliography

[1] J.R. Munkres. *Topology*. 2000.

[2] Walter Rudin. *Principles of Mathematical Analysis*. International series in pure and applied mathematics. 1964.