

Notes on *Riemannian Geometry*

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Contents

1	Topological Manifolds	2
1.1	Topology	2
1.2	Convergence	4
1.3	Homeomorphisms	4
1.4	Compactness and paracompactness	7
1.5	Connectedness and path-connectedness	8
1.6	Homotopic curves and the fundamental group	9
1.7	Topological manifolds	10
1.7.1	Bundles	10
1.7.2	Atlas	12
2	Differentiable Manifolds	15
2.1	Smooth Atlas and Differentiable Manifolds	15

Chapter 1

Topological Manifolds

1.1 Topology

In order to define the notion of smooth manifolds, we must first begin with some building blocks, such as topology and topological manifolds.

Definition 1.1: Topology

A *topology* on the set M is a family \mathcal{O} of subsets of M satisfying

- (a) the empty set and the set M belong to \mathcal{O} ;
- (b) a finite intersection of elements of \mathcal{O} is a member of \mathcal{O} ; and
- (c) an arbitrary union of members of \mathcal{O} belongs to \mathcal{O} .

The pair (M, \mathcal{O}_M) is named a *topological space*, elements of \mathcal{O} are called *open sets* and elements of $M \setminus \mathcal{O}$ are called *closed sets*. Additionally, given an element $p \in M$ an open set U that contains p is called a *neighborhood* of p .

In [Propositions 1.1](#) to [1.3](#) we show a couple of important examples that illustrate how the axioms of topological spaces given in [Definition 1.1](#) are used.

Proposition 1.1: Standard topology in \mathbb{R}^n

We define the *open ball* $B_n(r, p) \subset \mathbb{R}^n$ of radius $r > 0$ centered in the point $p = (p^1, \dots, p^n)$ as the set

$$B_n(r, p) = \left\{ q = (q^1, \dots, q^n) \in \mathbb{R}^n : \sum_{i=1}^n (q^i - p^i)^2 < r^2 \right\}.$$

Next, we define the *standard topology* $\mathcal{O}_{\text{standard}}$ of \mathbb{R}^n . A subset $U \subset \mathbb{R}^n$ is an open set if for every point $p \in U$ there exists $r > 0$ such that $B_n(r, p) \subset U$. Then, $(\mathbb{R}^n, \mathcal{O}_{\text{standard}})$ is a topological space.

Proof. It is easy to see $\mathbb{R}^n \in \mathcal{O}_{\text{standard}}$ and $\emptyset \in \mathcal{O}_{\text{standard}}$.

Suppose $U, V \in \mathcal{O}_{\text{standard}}$ and let $p \in U \cap V \neq \emptyset$. Then, there exists $r_U > 0$ and $r_V > 0$ such that $B_n(r_U, p) \subset U$ and $B_n(r_V, p) \subset V$. Setting $r = \min\{r_U, r_V\} > 0$ we have $B_n(r, p)$ as subset of both U and V , that is, $B_n(r, p) \subset U \cap V$. It follows that $U \cap V \in \mathcal{O}_{\text{standard}}$.

Let $\{U_\alpha\}_{\alpha \in J}$ be a family of sets in $\mathcal{O}_{\text{standard}}$. Let $p \in \bigcup_{\alpha \in J} U_\alpha$, that is, there exists $\beta \in J$ such that $p \in U_\beta$. Since $U_\beta \in \mathcal{O}_{\text{standard}}$, there exists $r_\beta > 0$ such that $B_n(r_\beta, p) \subset U_\beta \subset \bigcup_{\alpha \in J} U_\alpha$. \square

Proposition 1.2: Subspace topology is a topology

Given a topological space (M, \mathcal{O}_M) and a subset S of M , we define the *subspace topology* $\mathcal{O}_M|_S$ as

$$\mathcal{O}_M|_S = \{U \cap S : U \in \mathcal{O}_M\}.$$

Then $(S, \mathcal{O}_M|_S)$ is a topological space.

Proof. We must show the conditions (a), (b), and (c) of Definition 1.1 are satisfied.

- (a) Since $S = M \cap S$ and $\emptyset = \emptyset \cap S$, we have $S \in \mathcal{O}_M|_S$ and $\emptyset \in \mathcal{O}_M|_S$.
- (b) Let $U, V \in \mathcal{O}_M|_S$. Then, there exists $\tilde{U}, \tilde{V} \in \mathcal{O}_M$ such that $U = \tilde{U} \cap S$ and $V = \tilde{V} \cap S$. Then, $U \cap V = (\tilde{U} \cap S) \cap (\tilde{V} \cap S) = (\tilde{U} \cap \tilde{V}) \cap S$. Since $\tilde{U} \cap \tilde{V} \in \mathcal{O}_M$, we have $U \cap V \in \mathcal{O}_M|_S$.
- (c) Let $\{U_\alpha\}_{\alpha \in J}$ be a family of open sets in $\mathcal{O}_M|_S$. For each $\alpha \in J$, there exists a $\tilde{U}_\alpha \in \mathcal{O}_M$ such that $U_\alpha = \tilde{U}_\alpha \cap S$. Then

$$\begin{aligned} \bigcup_{\alpha \in J} U_\alpha &= \bigcup_{\alpha \in J} \tilde{U}_\alpha \cap S \\ &= \{m \in S : \exists \alpha \in J \text{ such that } m \in \tilde{U}_\alpha\} \\ &= \{m \in M : \exists \alpha \in J \text{ such that } m \in \tilde{U}_\alpha\} \cap S \\ &= S \cap \bigcup_{\alpha \in J} \tilde{U}_\alpha. \end{aligned}$$

Since arbitrary unions of open sets is an open set, it follows that $\bigcup_{\alpha \in J} U_\alpha \in \mathcal{O}_M|_S$. \square

Proposition 1.3: Product topology

Let (M, \mathcal{O}_M) and (N, \mathcal{O}_N) be topological spaces. Define the *product topology* $\mathcal{O}_{M \times N}$ as the collection of subsets $U \subset M \times N$ such that for all $(m, n) \in U$, there exists neighborhoods $S \subset M$ and $T \subset N$ of $m \in M$ and $n \in N$ such that $S \times T \subset U$. Then $(M \times N, \mathcal{O}_{M \times N})$ is a topological space.

Proof. Clearly, $M \times N$ and \emptyset are open sets in the product topology.

Next, we consider open sets $U, V \in \mathcal{O}_{M \times N}$ and an element $p \in U \cap V$. Let $p = (m, n) \in M \times N$, then there exists neighborhoods $S_U, S_V \subset M$ of m and $T_U, T_V \subset N$ of n such that $S_U \times T_U \subset U$ and $S_V \times T_V \subset V$. Let $S = S_U \cap S_V$ and $T = T_U \cap T_V$, then $S \in \mathcal{O}_M$ and $T \in \mathcal{O}_N$ are neighborhoods of m and n , respectively. Moreover, $S \times T \subset U \cap V$ is a neighborhood of p , from which follows $U \cap V \in \mathcal{O}_{M \times N}$.

Let $\{U_\alpha\}_{\alpha \in J}$ be a family of open sets in the product topology. Let $p \in \bigcup_{\alpha \in J} U_\alpha$, then there exists $\beta \in J$ such that $p \in U_\beta$. By definition, there exists open sets $S \in \mathcal{O}_M$ and $T \in \mathcal{O}_N$ such that $S \times T \subset U_\beta \subset \bigcup_{\alpha \in J} U_\alpha$. Therefore, $\bigcup_{\alpha \in J} U_\alpha$ is an open set. \square

Along with the axioms of topological spaces described in Definition 1.1 one might add further restrictions to specify the space considered. Some common restrictions are called the *separation axioms*. Among these, we will make use of the T2 axiom, namely the Hausdorff property. Historically, Felix Hausdorff used this axiom in his original definition of a

topological space, although the formulation of his other axioms were not exactly those of [Definition 1.1](#), but an equivalent one.

Definition 1.2: Hausdorff space

A topological space (M, \mathcal{O}_M) is called a *Hausdorff space* if for any $p, q \in M$ with $p \neq q$, there exists a neighborhood U of p , i.e. $p \in U \in \mathcal{O}_M$, and a neighborhood V of q such that $U \cap V = \emptyset$.

1.2 Convergence

In analysis on \mathbb{R}^n with the standard topology, we often consider sequences $x : \mathbb{N} \rightarrow \mathbb{R}^n$ and study whether it converges to a value. We say the sequence x converges to $y \in \mathbb{R}^n$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x(i) - y \in B_n(\varepsilon, 0)$ for all $i > N$. We generalize the notion of a convergent sequence to any topological space in [Definition 1.3](#).

Definition 1.3: Convergence of a sequence

A sequence $x : \mathbb{N} \rightarrow M$ on a topological space (M, \mathcal{O}_M) is said to *converge* to a *limit point* $p \in M$ if for every neighborhood $U \in \mathcal{O}_M$ of p there exists $N \in \mathbb{N}$ such that $x(n) \in U$ for all $n > N$.

Theorem 1.1: Unique limit on Hausdorff spaces

Let (M, \mathcal{O}_M) be a Hausdorff space. If a sequence x converges on M , its limit point is unique.

Proof. Let $p, q \in M$ be limit points of the sequence x . Suppose, by contradiction, that $p \neq q$. By the Hausdorff property, there exists neighborhoods $U, V \in \mathcal{O}_M$ of p and q , respectively, such that $U \cap V = \emptyset$. From the definition of convergence, there exists $N_p, N_q \in \mathbb{N}$ such that $x(n) \in U$ for all $n > N_p$ and $x(n) \in V$ for all $n > N_q$. Let $N = \min\{N_p, N_q\}$, then for all $n > N$, $x(n) \in U$ and $x(n) \in V$, that is, $x(n) \in U \cap V = \emptyset$. This contradiction proves the statement. \square

1.3 Homeomorphisms

With the notion of topological spaces, we may ask ourselves whether certain maps between topological spaces can preserve the topology. That is, a map that takes open sets in the domain topology into open sets in the target topology. To define such a map we define *continuity*.

Definition 1.4: Continuous map

Let (M, \mathcal{O}_M) and (N, \mathcal{O}_N) be topological spaces. Then a map $f : M \rightarrow N$ is *continuous* (with respect to \mathcal{O}_M and \mathcal{O}_N) if, for all $V \in \mathcal{O}_N$, the preimage $f^{-1}(V)$ is an open set in \mathcal{O}_M .

In short, a map is continuous if and only the preimages of (all) open sets are open sets. Now a map that preserves the topology is called a *homeomorphism*, which is defined as a continuous bijection with continuous inverse. We now prove such a map satisfies the condition required.

Proposition 1.4: Homeomorphism maps open sets to open sets

Let (M, \mathcal{O}_M) and (N, \mathcal{O}_N) be topological spaces. Suppose a map $f : M \rightarrow N$ is a homeomorphism, then f maps open sets in \mathcal{O}_M into open sets in \mathcal{O}_N .

Proof. Given a subset $U \in \mathcal{O}_M$, we must show the image $V = f(U)$ is open in (N, \mathcal{O}_N) . Taking our attention to the inverse map $g = f^{-1} : N \rightarrow M$, we see the preimage $g^{-1}(U) = V$ must be open in (N, \mathcal{O}_N) , due to continuity. \square

If there exists a homeomorphism between two topological spaces, they are said to be homeomorphic to each other. This begs the question: if (M, \mathcal{O}_M) is homeomorphic to (N, \mathcal{O}_N) and (N, \mathcal{O}_N) is homeomorphic to (P, \mathcal{O}_P) , are (M, \mathcal{O}_M) and (P, \mathcal{O}_P) homeomorphic? To answer this we must show whether the composition of continuous maps is itself continuous.

Theorem 1.2: Composition of continuous maps

Let (M, \mathcal{O}_M) , (N, \mathcal{O}_N) , and (P, \mathcal{O}_P) be topological spaces. If the maps $f : M \rightarrow N$ and $g : N \rightarrow P$ are continuous (with respect to the appropriate topologies), then the map $g \circ f : M \rightarrow P$ is continuous with respect to \mathcal{O}_M and \mathcal{O}_P .

Proof. Let V be an open set of (P, \mathcal{O}_P) . We must show the preimage $(g \circ f)^{-1}(V)$ is an open set of (M, \mathcal{O}_M) . We have

$$\begin{aligned} (g \circ f)^{-1}(V) &= \{m \in M : g \circ f(m) \in V\} \\ &= \{m \in M : f(m) \in g^{-1}(V)\} \\ &= f^{-1}(g^{-1}(V)). \end{aligned}$$

Since the map g is continuous and V is an open set in (P, \mathcal{O}_P) , it follows that $g^{-1}(V)$ is open in (N, \mathcal{O}_N) . By the same argument, $f^{-1}(g^{-1}(V))$ is an open set in (M, \mathcal{O}_M) . \square

Corollary 1.1. If (M, \mathcal{O}_M) is homeomorphic to (N, \mathcal{O}_N) and (N, \mathcal{O}_N) is homeomorphic to (P, \mathcal{O}_P) , then (M, \mathcal{O}_M) is homeomorphic to (P, \mathcal{O}_P) .

Proof. Let $f : M \rightarrow N$ and $g : N \rightarrow P$ be homeomorphisms from (M, \mathcal{O}_M) to (N, \mathcal{O}_N) and (N, \mathcal{O}_N) to (P, \mathcal{O}_P) , respectively. Consider the composition $g \circ f : M \rightarrow P$.

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & \searrow g \circ f & \downarrow g \\ & & P \end{array}$$

By [Theorem 1.2](#), the map $g \circ f$ is a homeomorphism from (M, \mathcal{O}_M) to (P, \mathcal{O}_P) . \square

As was done for the subspace topology, we prove a similar result for continuous maps.

Proposition 1.5: Restriction of a continuous map

Let (M, \mathcal{O}_M) and (N, \mathcal{O}_N) be topological spaces and let $f : M \rightarrow N$ be a continuous map. Let S be a subset of M and let (S, \mathcal{O}_S) be the subspace topology, then $f|_S : S \rightarrow N$ is a continuous map with respect to \mathcal{O}_S and \mathcal{O}_N .

Proof. Let $V \in \mathcal{O}_N$. Then, by the definition of preimage, we have

$$\begin{aligned} f|_S^{-1}(V) &= \{s \in S : f|_S(s) \in V\} \\ &= \{s \in S : f(s) \in V\} \\ &= f^{-1}(V) \cap S. \end{aligned}$$

By hypothesis, the preimage $f^{-1}(V)$ is an open set in (M, \mathcal{O}_M) , so $f|_S^{-1}(V)$ is an open set in the subspace topology. \square

We can now define the notion of a topological space locally resembling Euclidean space.

Definition 1.5: Locally Euclidean topological space

A topological space (M, \mathcal{O}_M) is *locally Euclidean* of dimension n if for all $m \in M$ there exists an open subset $U \in \mathcal{O}_M$ about m that is homeomorphic to \mathbb{R}^n with respect to the subspace topology and the standard topology of \mathbb{R}^n .

It is sufficient to show the subspace topology (U, \mathcal{O}_U) is homeomorphic to an open ball in \mathbb{R}^n , due to [Proposition 1.6](#).

Proposition 1.6: Open ball is homeomorphic to the Euclidean space

Let $r > 0$, then the map $f : B_n(r, 0) \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by

$$f(x) = \frac{x}{r - \|x\|}$$

is a homeomorphism with respect to the standard topology.

Proof. We begin by checking f is one-to-one and onto.

Suppose there exists $x_1, x_2 \in B_n(r, 0)$ such that $f(x_1) = f(x_2)$. It follows from

$$\begin{aligned} f(x_2) - f(x_1) &= \frac{x_2}{r - \|x_2\|} - \frac{x_1}{r - \|x_1\|} \\ &= \frac{(r - \|x_1\|)x_2 - (r - \|x_2\|)x_1}{(r - \|x_2\|)(r - \|x_1\|)} \end{aligned}$$

that $(r - \|x_1\|)x_2 = (r - \|x_2\|)x_1$. Taking the norm on both sides, we have $\|x_1\| = \|x_2\|$. Substituting back, we have $x_1 = x_2$, proving f is injective.

Suppose $y \in \mathbb{R}^n$ and consider $\xi = \frac{ry}{1 + \|y\|}$. Clearly, $\xi \in B_n(r, 0)$. We have

$$\begin{aligned} f(\xi) &= f\left(\frac{ry}{1 + \|y\|}\right) \\ &= \frac{ry}{1 + \|y\|} \frac{1}{r - \left\|\frac{ry}{1 + \|y\|}\right\|} \\ &= \frac{1}{(1 + \|y\|)\left(1 - \frac{\|y\|}{1 + \|y\|}\right)} y \\ &= y, \end{aligned}$$

so f is onto.

We have shown f is a bijection with inverse $f^{-1} : \mathbb{R}^n \rightarrow B_n(r, 0)$ defined by

$$f^{-1}(x) = \frac{rx}{1 + \|x\|}. \quad (1.3.1)$$

With the standard topology, continuity of f and f^{-1} follows from techniques of elementary calculus, and we conclude f is a homeomorphism. \square

1.4 Compactness and paracompactness

Definition 1.6: Compactness

A topological space (M, \mathcal{O}_M) is *compact* if every *open cover* of M has a finite subcover. That is, the topological space is compact if for every family of open sets C that covers M , i.e. $\bigcup_{U \in C} U = M$ with $U \in \mathcal{O}_M$, there exists a finite family of open sets $F \subset C$ such that $\bigcup_{U \in F} U = M$.

Additionally, in a topological space (N, \mathcal{O}_N) , a subset $S \subset N$ is called compact if the subspace topology is compact.

Theorem 1.3: Heine-Borel theorem

A subset $S \subset \mathbb{R}^n$ with the standard topology is compact if it is closed and bounded.

Proof. Refer to [3]. \square

Definition 1.7: Locally finite cover

A cover C of a topological space (M, \mathcal{O}_M) is called *locally finite* if each point in the space has a neighborhood that intersects only finitely many sets in C . More precisely, for all $p \in M$ there exists a neighborhood $U \in \mathcal{O}_M$ about p such that $U \cap V \neq \emptyset$ only for finitely many $V \in C$.

Definition 1.8: Refinement

A *refinement* of a cover C of a topological space (M, \mathcal{O}_M) is a cover D such that every set in D is contained in some set in C . Precisely, let $C = \{U_\alpha\}_{\alpha \in A}$ and $D = \{V_\beta\}_{\beta \in B}$ such that $\bigcup_{\alpha \in A} U_\alpha = M$ and $\bigcup_{\beta \in B} V_\beta = M$, then D is a refinement of C if for all $\beta \in B$ there exists $\alpha \in A$ such that $V_\beta \subset U_\alpha$.

Definition 1.9: Paracompactness

A topological space (M, \mathcal{O}_M) is called *paracompact* if every open cover C has an *open refinement* \tilde{C} that is *locally finite*.

Theorem 1.4: Stone's theorem

Any *metrizable space* is paracompact.

Proof. Refer to [2]. \square

Definition 1.10: Partition of unity subordinate to a cover

Let $C = \{U_\alpha\}_{\alpha \in J}$ be an open cover of the topological space (M, \mathcal{O}_M) . The family of functions $\mathcal{F}_C = \{f_\alpha\}_{\alpha \in J}$ from M to $[0, 1] \subset \mathbb{R}$ is a (continuous) *partition of unity subordinate to U_α* if

- (a) each $f_\alpha : X \rightarrow [0, 1]$ is continuous;
- (b) for each $\alpha \in J$, the support of f_α is contained in U_α , that is, for every $f \in \mathcal{F}_C$ there exists $U \in C$ such that $f(p) \neq 0 \implies p \in U$;
- (c) the family $\{\text{supp } f_\alpha\}_{\alpha \in J}$ is locally finite, i.e. for any $p \in M$ there exists a neighborhood $U \in \mathcal{O}_M$ about p where all but finitely many functions f_α vanish on U ;
- (d) for any point $p \in M$, $\sum_{\alpha \in J} f_\alpha = 1$.

Theorem 1.5: Paracompactness and partitions of unity

Let (M, \mathcal{O}_M) be a Hausdorff space. Then it is paracompact if and only if every open cover C admits a partition of unity subordinate to that cover.

Proof. Refer to [2]. □

1.5 Connectedness and path-connectedness

Definition 1.11: Connectedness

A topological space (M, \mathcal{O}_M) is *connected* unless there exists two non-empty, non-intersecting open sets $A, B \in \mathcal{O}_M$ such that $M = A \cup B$.

Theorem 1.6: Interval is connected

Every interval $I \subset \mathbb{R}$ is connected with respect to the standard topology.

Proof. Suppose I is not connected, then $I = A \cup B$, where $A, B \subset I$ are non-empty, non-intersecting open sets. Let $a \in A$ and $b \in B$. Without loss of generality, we assume $a < b$.

Consider $\alpha = \sup\{x \in \mathbb{R} : [a, x) \cap I \subset A\}$. Then $a \leq \alpha \leq b$, so $\alpha \in I$. Since $B = I \setminus A$ is open, we have A closed, hence $\alpha \in A$. Since A is open, there exists $r > 0$ such that $(\alpha - r, \alpha + r) \cap I \subset A$. We conclude $(a, \alpha + r) \cap I \subset A$, which is a contradiction. □

Theorem 1.7: Open and closed subsets in a connected space

A topological space (M, \mathcal{O}_M) is connected if and only if \emptyset and M are the only subsets that are both open and closed.

Proof. Suppose the topological space is connected. Suppose, by contradiction, the non-empty subset $U \subsetneq M$ is open and closed. It follows from $M = U \cup (M \setminus U)$ that the topological space is not connected, a contradiction.

Suppose the empty set and M are the only subsets that are both open and closed. Suppose, by contradiction, that the topological space is not connected. Then there exists two

non-empty non-intersecting open sets $A, B \in \mathcal{O}_M$ such that $M = A \cup B$. Clearly, $B = M \setminus A$ is closed, and likewise, A is closed. By hypothesis, $A = B = M$ since they are both open and closed and are non-empty. We have thus arrived at a contradiction, since $A \cap B = M$ is non-empty, which proves the topological space is connected. \square

Definition 1.12: Path-connectedness

A topological space (M, \mathcal{O}_M) is called *path-connected* if for every pair of points $p, q \in M$ there exists a continuous curve $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

Theorem 1.8: Path-connectedness implies connectedness

A path-connected topological space is connected.

Proof. Suppose, by contradiction, a topological space (M, \mathcal{O}_M) is path-connected but not connected. Then, there exists non-empty, non-intersecting open sets $A, B \in \mathcal{O}_M$ such that $M = A \cup B$. Choose $a \in A$ and $b \in B$. Since (M, \mathcal{O}_M) is path-connected, there exists a continuous curve $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = a$ and $\gamma(1) = b$. Consider the preimage $[0, 1] = \gamma^{-1}(M)$. It follows from

$$\begin{aligned}\gamma^{-1}(M) &= \gamma^{-1}(A \cup B) \\ &= \{x \in [0, 1] : \gamma(x) \in A \cup B\} \\ &= \gamma^{-1}(A) \cup \gamma^{-1}(B)\end{aligned}$$

that $[0, 1]$ is the union of two non-empty sets, since $0 \in \gamma^{-1}(A)$ and $1 \in \gamma^{-1}(B)$. Suppose $\gamma^{-1}(A) \cap \gamma^{-1}(B)$ is non-empty. Then there exists $t \in [0, 1]$ such that $\gamma(t) \in A$ and $\gamma(t) \in B$, that is, $\gamma(t) \in A \cap B$. This contradiction shows $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are non-intersecting. By continuity, these sets are both open and closed. Therefore, $[0, 1]$ is not connected. By Theorem 1.6, this is a contradiction, and the theorem follows. \square

1.6 Homotopic curves and the fundamental group

Definition 1.13: Homotopic curves

Let (M, \mathcal{O}_M) be a topological space and let $p, q \in M$. Two curves $\gamma, \eta : [0, 1] \rightarrow M$ from p to q , i.e. $\gamma(0) = \eta(0) = p$ and $\gamma(1) = \eta(1) = q$, are *homotopic* if there exists a continuous map $h : [0, 1] \times [0, 1] \rightarrow M$ such that $h(0, \lambda) = \gamma(\lambda)$ and $h(1, \lambda) = \eta(\lambda)$, for all $\lambda \in [0, 1]$.

It is easy to check that homotopic curves form an equivalence relation. (do that)

Definition 1.14: Loops at a point

Let (M, \mathcal{O}_M) be a topological space and let $p \in M$. We define the *space of loops at p* as the set \mathcal{L}_p of continuous curves $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = \gamma(1) = p$.

A *concatenation* is a binary operation $*_p : \mathcal{L}_p \times \mathcal{L}_p \rightarrow \mathcal{L}_p$ defined by

$$(\gamma *_p \eta)(\lambda) = \begin{cases} \gamma(2\lambda), & \text{for } \lambda \in [0, \frac{1}{2}] \\ \eta(2\lambda - 1), & \text{for } \lambda \in (\frac{1}{2}, 1] \end{cases}$$

for all $\gamma, \eta \in \mathcal{L}_p$.

Definition 1.15: Fundamental group

The *fundamental group* $(\pi_{1,p}, \cdot)$ of a topological space is the set

$$\pi_{1,p} = \mathcal{L}_p / \text{homotopy} = \{[\gamma]_{\text{homotopy}} : \gamma \in \mathcal{L}_p\}$$

together with the product $\cdot : \pi_{1,p} \times \pi_{1,p} \rightarrow \pi_{1,p}$ defined by

$$[\gamma] \cdot [\eta] = [\gamma *_p \eta].$$

Example 1.1. (a) On the two-sphere S^2 , the fundamental group has a single element, represented by the constant loop.

(b) For the cylinder $C = \mathbb{R} \times S^1$, the fundamental group is homomorphic to the group $(\mathbb{Z}, +)$. There exists a bijection $f : \pi_1 \rightarrow \mathbb{Z}$ with the property $f(\alpha \cdot \beta) = f(\alpha) + f(\beta)$.

(c) On the torus $T^2 = S^1 \times S^1$, we have π_1 isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Definition 1.16: Simply connected

A topological space (M, \mathcal{O}_M) is *simply connected* if it is path-connected and if for every point $p \in M$ the fundamental group $(\pi_{1,p}, \cdot)$ is the trivial group.

Remark 1.1. Building up on the previous examples, we see that the two-sphere is simply connected, while the cylinder and the torus are not, although path-connected.

1.7 Topological manifolds

We now finally define the notion of a manifold, which is a “well-behaving” topological space that *locally* looks like Euclidean space.

Definition 1.17: Topological manifold

A paracompact Hausdorff space (M, \mathcal{O}_M) locally Euclidean of dimension n is called a *n -dimensional topological manifold*.

Example 1.2. We list a few examples of constructions of new manifolds from other manifolds.

(a) As with topological spaces, it is clear that the subspace topology (N, \mathcal{O}_N) is in its own right a manifold. This construction takes the name of submanifold.

(b) Let (M, \mathcal{O}_M) be an m -dimensional manifold and (N, \mathcal{O}_N) be an n -dimensional manifold. Then the product topology $(M \times N, \mathcal{O}_{M \times N})$ is an $(n + m)$ -dimensional manifold. As an example, the circle S^1 is a 1-dimensional manifold, so the torus $T^2 = S^1 \times S^1$ is a 2-dimensional manifold.

1.7.1 Bundles

It is clear we may not always construct a manifold as a product manifold. To see this, take the Möbius strip. We generalize this concept with *bundles*.

Definition 1.18: Bundles of topological manifolds

A *bundle of topological manifolds* is a triple (E, π, M) , where (E, \mathcal{O}_E) is a topological manifold called the *total space*, (M, \mathcal{O}_M) is a topological manifold called the *base space*, and $\pi : E \rightarrow M$ is a continuous surjective map called the *projection*. Let $p \in M$, then $\pi^{-1}(\{p\}) = F_p$ is the *fiber at p* .

- Example 1.3.** (a) The first example is when the total space $E = M \times F$ is a product manifold of the base space M and a fiber F . We take the projection $\pi : M \times F \rightarrow M$ as the continuous map $(p, f) \mapsto p$.
- (b) We now take the Möbius strip as the total space and the base space as S^1 . For any point in S^1 , the preimage of the projection is some interval on the real line.
- (c) We consider $M = \mathbb{R}$ as the base space. We begin construct the total space by attaching to each point of M a circle. Then, we continuously deform the circles such that they collapse to a point on the positive numbers of the real line. Finally, we continuously stretch the point on that half of the line to increasing intervals. The fiber at any given point may be homeomorphic to a circle or to a point, or to an interval.

In the last example, a bundle was constructed with a fiber space that is not the same at different points. This motivates the following definition.

Definition 1.19: Fiber bundle

A bundle (E, π, M) is a *fiber bundle with typical fiber F* if for all $p \in M$, the preimage $\pi^{-1}(\{p\})$ is homeomorphic to the manifold F . The diagram

$$F \longrightarrow E \xrightarrow{\pi} M$$

denotes a fiber bundle.

As an example, we consider the \mathbb{C} -line bundle over M as the bundle (E, π, M) with typical fiber \mathbb{C} .

Definition 1.20: Section of the bundle

Let (E, π, M) be a bundle. A map $\sigma : M \rightarrow E$ is called a *section of the bundle* if $\pi \circ \sigma = \text{id}_M$.

As a special case we consider the "product bundle"

$$F \longrightarrow M \times F \xrightarrow{\pi} M,$$

where $\pi : M \times F \rightarrow M$ is the projection $(p, f) \mapsto p$. Given any function $s : M \rightarrow F$, we may construct a section $\sigma : M \rightarrow M \times F$ defined by $p \mapsto (p, s(p))$.

Given a bundle (E, π, M) , we may consider the submanifolds $E' \subset E$ and $M' \subset M$ and the restriction $\pi' = \pi|_{E'}$, then we call (E', π', M') a *subbundle*. Similarly, we consider another submanifold $N \subset M$ and the preimage $G = \pi^{-1}(N)$, then $(G, \pi|_G, N)$ is called the *restricted bundle*.

Given two bundles (E, π, M) and (E', π', M') and a pair of maps $u : E \rightarrow E'$ and $f : M \rightarrow M'$. Then the pair (u, f) is called a *bundle morphism* if the diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & E' \\ \downarrow \pi & & \downarrow \pi' \\ M & \xrightarrow{f} & M' \end{array}$$

commutes. The bundles are called *isomorphic as bundles* if there exists bundle morphisms (u, f) and (u^{-1}, f^{-1}) . In this case (u, f) are called *bundle isomorphisms* and are the structure-preserving maps for bundles.

We may weaken this condition by not requiring it globally. Two bundles (E, π, M) and (E', π', M') are *locally isomorphic as bundles* if for every $p \in M$ there exists a neighborhood $U \in \mathcal{O}_M$ such that the restricted bundle $(\pi^{-1}(U), \pi|_{\pi^{-1}(U)}, U)$ is isomorphic to (E', π', M') .

A bundle is called *(locally) trivial* if it is (locally) isomorphic to a product bundle. As an example, the cylinder is trivial and thus locally trivial and a Möbius strip is not trivial, but it is locally trivial. From now on we will disregard bundles that are not locally trivial. Then, locally, any section can be represented as a map from the base space to fiber.

Definition 1.21: Pullback bundle

Let (E, π, M) be a bundle, M' be a manifold and $f : M' \rightarrow M$ be a continuous map.

$$\begin{array}{ccc} f^*E & \xrightarrow{u} & E \\ \downarrow \pi' & & \downarrow \pi \\ M' & \xrightarrow{f} & M \end{array}$$

Let $f^*E = \{(m', e) \in M' \times E : \pi(e) = f(m')\} \subset M' \times E$ equipped with the subspace topology, define the map $u : f^*E \rightarrow E$ by $(m', e) \mapsto e$ and the projection $\pi' : f^*E \rightarrow M'$ by $(m', e) \mapsto m'$ such that the diagram above commutes. The bundle (f^*E, π', M') is called the *pullback bundle by f* or the *bundle induced by f* .

Given a bundle (E, π, M) with section $\sigma : M \rightarrow E$ and a continuous map $f : M' \rightarrow M$, where M' is a manifold, we may define the pullback section $f^*\sigma : M' \rightarrow f^*E$ on the bundle induced by f by the map $m' \mapsto (m', \sigma \circ f(m'))$. Since $\pi \circ \sigma = \text{id}_M$, it is clear the image of the pullback section is contained in f^*E , so the map is well-defined.

1.7.2 Atlas

Since a topological manifold is locally Euclidean, for any point there is a neighborhood homeomorphic to Euclidean space. That is, there exists a homeomorphism from one such neighborhood to Euclidean space. Moreover, the collection of all such neighborhoods must cover the manifold. These remarks motivate the [Definitions 1.22](#) and [1.23](#).

Definition 1.22: Chart of a manifold

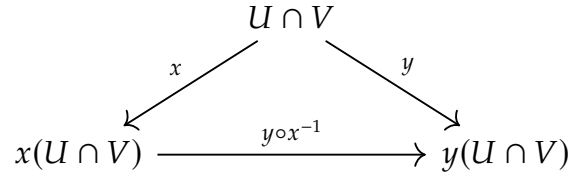
Let (M, \mathcal{O}_M) be a topological manifold of dimension n . Then a pair (U, x) where $U \in \mathcal{O}_M$ and $x : U \rightarrow x(U) \subset \mathbb{R}^n$ is called a *chart* of the manifold.

The component functions of x , the maps $x^i : U \rightarrow \mathbb{R}$ defined by $p \mapsto \text{proj}_i(x(p))$, are called the *coordinates of the point $p \in U$ with respect to the chart (U, x)* .

Definition 1.23: Atlas of a manifold

Let (M, \mathcal{O}_M) be a topological manifold. The *atlas* $\mathcal{A} = \{(U_\alpha, x_\alpha)\}_{\alpha \in J}$ is a family of charts of the manifold such that $\bigcup_{\alpha \in J} U_\alpha = M$.

It is easy to see that the domains of different charts may overlap. Let (U, x) and (V, y) be charts of a manifold (M, \mathcal{O}_M) with $U \cap V \neq \emptyset$. Since $U \cap V \in \mathcal{O}_M$ is a non-empty open set, the pairs $(U \cap V, x)$ and $(U \cap V, y)$ are charts on the manifold.



As the maps x and y are bijections, the map $y \circ x^{-1} : x(U \cap V) \rightarrow y(U \cap V)$ is well defined and it is called the *chart transition map*. By [Theorem 1.2](#), the transition map is a homeomorphism.

The following definitions will seem redundant, but will serve as a framework of definitions as we require more and more structure on manifolds. Namely, later on we will replace the continuity requirement with a differentiability class.

Definition 1.24: C^0 -compatible charts

Two charts (U, x) and (V, y) of a n -dimensional manifold (M, \mathcal{O}_M) are C^0 -compatible if either

- (a) $U \cap V = \emptyset$; or
- (b) $U \cap V \neq \emptyset$ and the transition map $y \circ x^{-1}$ is continuous as a map $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

As discussed above, it is clear that any two charts on a manifold are C^0 -compatible. However, as we can study, for example, the differentiability class of the transition map as a map $\mathbb{R}^n \rightarrow \mathbb{R}^n$, we may not conclude any such thing from the chart maps before adding structure to the manifold.

Definition 1.25: C^0 -atlas

A C^0 -atlas \mathcal{A} is an atlas whose charts are pairwise C^0 -compatible.

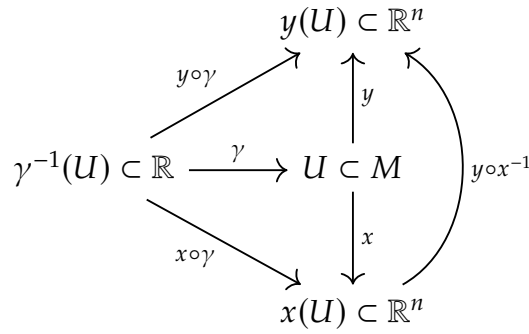
Definition 1.26: Maximal C^0 -atlas

A C^0 -atlas \mathcal{A} is *maximal* if any chart (U, x) that is C^0 -compatible with any $(V, y) \in \mathcal{A}$ is already contained in \mathcal{A} .

Even though every atlas is a C^0 -atlas, not every atlas is maximal. As an example we consider (M, \mathcal{O}_M) as the real line equipped with the standard topology. Then, the family $\{(\mathbb{R}, \text{id}_{\mathbb{R}})\}$ is an atlas, but the chart $((-\infty, 0), \text{id}_{\mathbb{R}})$ is not contained in it.

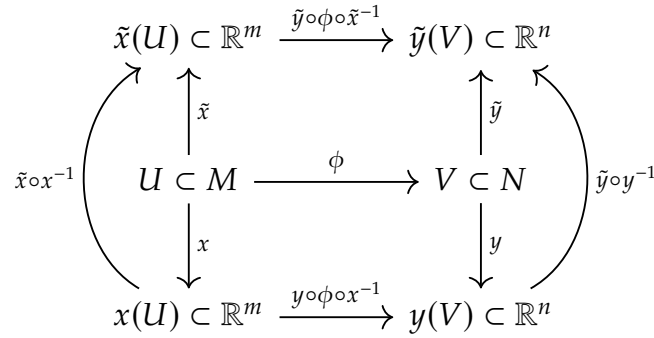
With an atlas and charts, one may study objects on an n -dimensional topological manifold (M, \mathcal{O}_M) from different points of view. As an example, we consider a curve $\gamma : \mathbb{R} \rightarrow M$ and ask ourselves whether the curve is continuous. Clearly, by definition, we can check whether

the preimage of open sets in M are open. From another perspective, we may employ charts.



Let $U \subset M$ be an open set that contains the image of the curve, and let (U, x) be a chart. The *expression* of γ in this chart is the map $x \circ \gamma : \gamma^{-1}(U) \subset \mathbb{R} \rightarrow x(U) \subset \mathbb{R}^n$. Then, the curve is continuous if its expression is continuous as a function from $\mathbb{R} \rightarrow \mathbb{R}^n$, that is, if its components are continuous real-valued functions of a single variable. Additionally, if (U, y) is another chart, then the expression of the curve in this chart $y \circ \gamma : \gamma^{-1}(U) \rightarrow y(U)$ is continuous if and only if the expression in the other chart $x \circ \gamma$ is continuous, because the chart transition map $y \circ x^{-1} : x(U) \rightarrow y(U)$ is continuous. In this perspective, it is possible to ignore altogether the inner workings of the manifolds and use only the coordinate systems given by the charts.

Analogously, a map $\phi : M \rightarrow N$, where (M, \mathcal{O}_M) is an m -dimensional topological manifold and (N, \mathcal{O}_N) is an n -dimensional topological manifold, is continuous if preimages of open sets in N are open sets in M . By employing charts (U, x) on M and (V, y) on N , the expression of the map is $y \circ \phi \circ x^{-1} : x(U) \subset \mathbb{R}^m \rightarrow y(V) \subset \mathbb{R}^n$, and its continuity can be determined as a function of $\mathbb{R}^m \rightarrow \mathbb{R}^n$.



Taking another pair of charts (U, \tilde{x}) and (V, \tilde{y}) , it follows from the continuity of the chart transition maps that the expression of ϕ in these charts $\tilde{y} \circ \phi \circ \tilde{x}^{-1} : \tilde{x}(U) \subset \mathbb{R}^m \rightarrow \tilde{y}(V) \subset \mathbb{R}^n$ is a continuous function if and only if the expression $y \circ \phi \circ x^{-1}$ is continuous.

Chapter 2

Differentiable Manifolds

2.1 Smooth Atlas and Differentiable Manifolds

Even though for topological manifolds the atlas was mostly redundant, although useful, we may add a differentiable structure to a topological manifold by refining the maximal C^0 -atlas. The following definitions are almost the same as [Definitions 1.24 to 1.26](#), substituting the continuity trivial requirement to the differentiability class C^k , with $k \geq 1$. Analogously, we may change the differentiability class C^k to the smoothness condition C^∞ or to real C^ω or complex analytic functions.

Definition 2.1: C^k -compatible charts

Let (M, O_M) be an n -dimensional topological manifold. Two charts (U, x) and (V, y) are C^k -compatible if

- (a) $U \cap V = \emptyset$; or
- (b) $U \cap V \neq \emptyset$ and the transition map $y \circ x^{-1}$ is of class C^k as a map $\mathbb{R}^n \rightarrow \mathbb{R}^n$.

Definition 2.2: C^k -atlas

A C^k -compatible atlas \mathcal{A} is an atlas whose charts are pairwise C^k -compatible.

Definition 2.3: Maximal C^k -atlas

A C^k -atlas \mathcal{A} is *maximal* if any chart (U, x) that is C^k -compatible with any $(V, y) \in \mathcal{A}$ is already contained in \mathcal{A} .

We now state a theorem [\[1\]](#) that allows us to consider either maximal C^0 -atlas or a maximal smooth atlas. That is, the construction of a maximal C^1 -atlas is not weaker than the construction of a maximal C^∞ -atlas.

Theorem 2.1: Whitney

Any maximal C^k -atlas with $k \geq 1$ contains a smooth atlas. And two maximal C^k -atlases that contain the same smooth atlas are already identical.

Definition 2.4: C^k -manifold

A C^k -manifold is a triple $(M, \mathcal{O}_M, \mathcal{A}_M)$ when (M, \mathcal{O}_M) is a topological manifold and \mathcal{A}_M is a maximal C^k -atlas.

Remark 2.1. A given topological manifold can carry different incompatible atlases. As an example, take the manifold as the real line equipped with the standard topology. We consider two atlases $\mathcal{A}_1 = \{(\mathbb{R}, \text{id}_{\mathbb{R}})\}$ and $\mathcal{A}_2 = \{(\mathbb{R}, x)\}$ where $x : \mathbb{R} \rightarrow \mathbb{R}$ is the map $p \mapsto p^{\frac{1}{3}}$. Clearly, \mathcal{A}_1 can be completed to be a maximal smooth atlas. The other atlas is a smooth atlas, as there is only one chart, so the chart transition map is the identity map, which is smooth. As such, it is also possible to complete this atlas to a maximal smooth atlas. We observe however that the atlas $\mathcal{A}_1 \cup \mathcal{A}_2$ is not even C^1 -compatible, as the transition map is not differentiable at $p = 0$.

Definition 2.5: Differentiable map between manifolds

Let $\phi : M \rightarrow N$ be a map, where $(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(N, \mathcal{O}_N, \mathcal{A}_N)$ are C^k -manifolds of dimensions m and n respectively.

$$\begin{array}{ccc} U \subset M & \xrightarrow{\phi} & V \subset N \\ \downarrow x & & \downarrow y \\ x(U) \subset \mathbb{R}^m & \xrightarrow{y \circ \phi \circ x^{-1}} & y(V) \subset \mathbb{R}^n \end{array}$$

The map ϕ is *differentiable* at $p \in M$ if there exists charts $(U, x) \in \mathcal{A}_M$ and $(V, y) \in \mathcal{A}_N$, where U and V are neighborhoods of p and $\phi(p)$, such that the expression of ϕ in these charts, that is the map $y \circ \phi \circ x^{-1} : x(U) \subset \mathbb{R}^m \rightarrow y(V) \subset \mathbb{R}^n$ is a C^k map from $\mathbb{R}^m \rightarrow \mathbb{R}^n$.

Although the definition relies on the existence of charts, we must show the differentiability of a map between manifolds is independent of the choice of charts. Without loss of generality, we suppose there are another pair of charts $(U, \tilde{x}) \in \mathcal{A}_M$ and $(V, \tilde{y}) \in \mathcal{A}_N$.

$$\begin{array}{ccc} \tilde{x}(U) \subset \mathbb{R}^m & \xrightarrow{\tilde{y} \circ \phi \circ \tilde{x}^{-1}} & \tilde{y}(V) \subset \mathbb{R}^n \\ \uparrow \tilde{x} & & \uparrow \tilde{y} \\ U \subset M & \xrightarrow{\phi} & V \subset N \\ \downarrow x & & \downarrow y \\ x(U) \subset \mathbb{R}^m & \xrightarrow{y \circ \phi \circ x^{-1}} & y(V) \subset \mathbb{R}^n \end{array}$$

$\tilde{x} \circ x^{-1}$ (left curved arrow) $\tilde{y} \circ y^{-1}$ (right curved arrow)

Because the atlases are C^k -compatible, the chart transition maps $\tilde{x} \circ x^{-1}$ and $\tilde{y} \circ y^{-1}$ are C^k maps. Therefore, the expression $\tilde{y} \circ \phi \circ \tilde{x}^{-1} : \tilde{x}(U) \rightarrow \tilde{y}(V)$ is a C^k map if and only if $y \circ \phi \circ x^{-1} : x(U) \rightarrow y(V)$ is a C^k . This shows the definition is independent of the choice of charts.

We now define the maps that preserve the differentiable structure on manifolds.

Definition 2.6: Diffeomorphism

The map $\phi : M \rightarrow N$ is a *diffeomorphism* if it is bijective and the maps ϕ and ϕ^{-1} are smooth.

Definition 2.7: Diffeomorphic manifolds

Two manifolds $(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(N, \mathcal{O}_N, \mathcal{A}_N)$ are *diffeomorphic* if there exists a diffeomorphism between them.

It is custom to regard diffeomorphic manifolds to be the same up to diffeomorphism. A natural question arises: how many different differentiable structures can one add on a given n -dimensional topological manifold up to diffeomorphism? The answer differs for the dimension of the manifold.

- (a) The case $1 \leq n \leq 3$. Radó-Moise theorems. There is a unique smooth manifold one can construct of a given topological manifold.
- (b) The case $n = 4$. Depending on the structure of the topological manifold, there are possibly uncountably many different smooth manifolds.
- (c) The case $n > 4$. Surgery theory (1960s). For compact manifolds, there are finitely many smooth manifolds one can make from a given topological manifold.

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