Parameter Estimation for the McKean-Vlasov Stochastic Differential Equation

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Introduction

Background

- We are interested in a family of McKean-Vlasov SDEs on $\mathbb{R}^d$, parametrised by some unknown $\theta \in \mathbb{R}^p$:

$$\mathrm{d}x_t^\theta = \left[b(\theta, x_t^\theta) + \int_{\mathbb{R}^d} \phi(\theta, x_t^\theta, y)\mu_t^\theta(\mathrm{d}y)\right]\mathrm{d}t + \sigma(x_t^\theta)\mathrm{d}w_t,$$
$$\mu_t^\theta = \mathcal{L}(x_t^\theta).$$

- These processes are non-linear in the sense of McKean: the coefficients depend on the law on the solution, in addition to the solution itself.

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- The Mckean-Vlasov SDE has a natural connection to a non linear, non local PDE: if $u_t^\theta(x)$ denotes the density of x_t^θ, then

$$\partial_t u_t^\theta(x) = -\sum_{k=1}^{d} \partial_{x_k} \left[\left[b_k(\theta, x) + \int_{\mathbb{R}^d} \phi_k(\theta, x, y) u_t^\theta(y) \mathrm{d}y \right] u_t^\theta(x) \right]$$
$$+ \frac{1}{2} \sum_{k,k'=1}^{d} \partial^2_{x_k, x_{k'}} \left[[\sigma(x)\sigma^\top(x)]_{k,k'} u_t^\theta(x) \right].$$

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$$\mathrm{d}x_t^{\theta,i,N} = \left[b(\theta, x_t^{\theta,i,N}) + \frac{1}{N}\sum_{j=1}^{N} \phi(\theta, x_t^{\theta,i}, x_t^{\theta,j}) \right] \mathrm{d}t + \sigma(x_t^{\theta,i,N}) \mathrm{d}w_t^i$$

- Interacting particle systems have many applications, including
  - Mathematical biology (e.g., neuroscience, population dynamics, epidemic dynamics)
  - Social sciences (e.g., opinion dynamics, cooperative behaviours)
  - Financial mathematics
  - Stochastic filtering (e.g., EnKBF, other particle filters)
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- Long history of work on stochastic McKean-Vlasov equations, going back to the work of McKean [McK66].
- Renewed interest in recent years, with a number of new results on:
  - Existence and uniqueness of solutions [BMBP18, MV20].
  - Existence (or non-existence) of a unique invariant measure [BGG13, Tug13, EGZ19]
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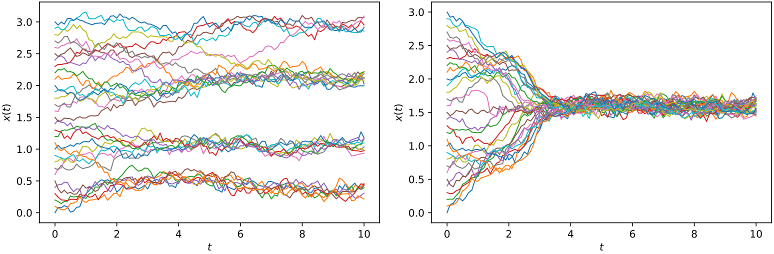
Motivation

- The dynamics of McKean-Vlasov SDEs are often highly dependent on the model parameters.



(a) ‘Weak’ interaction. (b) ‘Strong’ interaction.

Figure: Dynamics of an interacting particle system.

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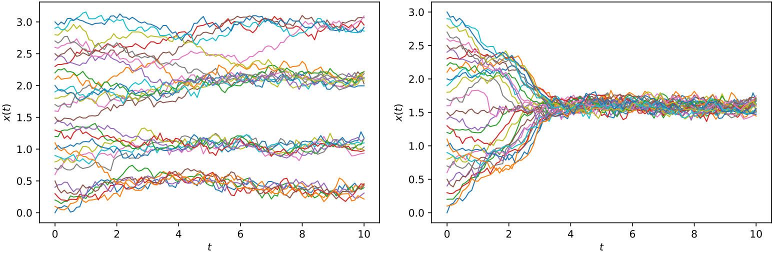

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The Model

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- Let $x = (x_t)_{t\geq 0}$ denote the measured solution of the McKean-Vlasov SDE

$$\mathrm{d}x_t^\theta = \underbrace{\left[b(\theta, x_t^\theta) + \int_{\mathbb{R}^d} \phi(\theta, x_t^\theta, y)\mu_t^\theta(\mathrm{d}y)\right]}_{B(\theta, x_t^\theta, \mu_t^\theta)} \mathrm{d}t + \mathrm{d}w_t,$$

$$\mu_t^\theta = \mathcal{L}(x_t^\theta).$$

- We assume the problem is well specified: that is, there exists a true parameter θ_0 such that $(x_t)_{t\geq 0}$ equals $(x_t^{\theta_0})_{t\geq 0}$.
- We are then interested in estimating θ_0, in either an offline (batch) or online (recursive) fashion.

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- We can relax this assumption. In this case, we will instead let $x = (x_t)_{t \geq 0}$ denote the measured solution of

$$\mathrm{d}x_t = B^*(x_t, \mu_t)\mathrm{d}t + \mathrm{d}w_t,$$
$$\mu_t = \mathcal{L}(x_t),$$

  where, similarly to above, we have defined

$$B^*(x, \mu) = b^*(x) + \int_{\mathbb{R}^d} \phi^*(x, y)\mu(\mathrm{d}y).$$

  for some unknown functions $b^* : \mathbb{R}^d \to \mathbb{R}^d$ and $\phi^* : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$.

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The Likelihood Function

- We can obtain the log-likelihood of the Mckean-Vlasov SDE, via a Girsanov transformation, as (e.g., [WWMX16])

$$\mathcal{L}_t(\theta) = \int_0^t L(\theta, x_s, \mu_s^\theta, \mu_s)\mathrm{d}s + \int_0^t \langle G(\theta, x_s, \mu_s^\theta, \mu_s), \mathrm{d}w_s\rangle,$$

where we have defined

$$G(\theta, x_s, \mu_s^\theta, \mu_s) = B(\theta, x_s, \mu_s^\theta) - B(\theta_0, x_s, \mu_s)$$
$$L(\theta, x_s, \mu_s^\theta, \mu_s) = -\frac{1}{2}|G(\theta, x_s, \mu_s^\theta, \mu_s)|^2$$

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The Likelihood Function

- **Case I**. We observe N independent instances $(x_t^i)_{t\geq 0}^{i=1,\dots,N}$ of the McKean-Vlasov SDE.
- In this case, we can approximate $\mathcal{L}_t(\theta)$ by the following 'Monte Carlo esque' approximation

$$\begin{aligned}\mathcal{L}_t^{[N]}(\theta) &:= \frac{1}{N}\sum_{i=1}^N \mathcal{L}_t^{[i,N]}(\theta)\\ &= \frac{1}{N}\sum_{i=1}^N \left[\int_0^t L(\theta, x_s^i, \mu_s^{[N]}, \mu_s^{[N]})\mathrm{d}s + \int_0^t \langle G(\theta, x_s^i, \mu_s^{[N]}, \mu_s^{[N]}), \mathrm{d}w_s^i\rangle\right],\end{aligned}$$

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- **Case II**. We observe the trajectories of N particles $(x_t^{i,N})_{t\geq 0}^{i=1,\dots,N}$ from the interacting particle system.
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- **Case III**. We observe a single path $(x_t)_{t\geq 0}$ of the McKean-Vlasov SDE, and we can compute the corresponding law $(\mu_t)_{t\geq 0}$.
- In this case, we can use the true log-likelihood for the McKean-Vlasov SDE, that is,

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Offline Parameter Estimation

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- In the offline setting, our objective is to estimate the true parameter θ_0 after a fixed time interval $[0, t]$, using the observations $(x_s)_{s\in[0,t]}$.
- In this case, a standard approach is to compute the maximum likelihood estimator(s), namely

$$\hat{\theta}_t^{[N]} = \arg\sup_{\theta\in\mathbb{R}^p} \mathcal{L}_t^{[N]}(\theta) \quad \textbf{(Case I)}$$

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- We are interested in the properties of these estimators in two asymptotic regimes:
  - $t \to \infty$. The properties of $\hat{\theta}_t^N$, $\hat{\theta}_t^{[N]}$, $\hat{\theta}_t$ in this limit are well known (e.g., [WWMX16] and [LSZ94]).
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Main Results

Theorem 1

For all $t > 0$, $\hat{\theta}_t^{[N]}$ and $\hat{\theta}_t^N$ are consistent estimators of θ_0 as $N \to \infty$. That is, as $N \to \infty$,

$$\hat{\theta}_t^{[N]} \longrightarrow \theta_0 \quad \text{and} \quad \hat{\theta}_t^N \longrightarrow \theta_0.$$

Theorem 2

For all $t > 0$, $N^{\frac{1}{2}}(\hat{\theta}_t^{[N]} - \theta_0)$ and $N^{\frac{1}{2}}(\hat{\theta}_t^N - \theta_0)$ are asymptotically normal with mean zero and variance $I_t^{-1}(\theta_0)$, where $I_t(\theta_0) \in \mathbb{R}^{p \times p}$ is the matrix with elements

$$[I_t(\theta_0)]_{kl} = \int_0^t \int_{\mathbb{R}^d} [\nabla_\theta B(\theta_0, x, \mu_s)]_k [\nabla_\theta B(\theta_0, x, \mu_s)]_l \mu_s(\mathrm{d}x)\mathrm{d}s.$$

That is, as $N \to \infty$,

$$N^{\frac{1}{2}}(\hat{\theta}_t^{[N]} - \theta_0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, I_t^{-1}(\theta_0)) \quad \text{and} \quad N^{\frac{1}{2}}(\hat{\theta}_t^N - \theta_0) \xrightarrow{\mathcal{D}} \mathcal{N}(0, I_t^{-1}(\theta_0)).$$

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- In the online setting, the objective is to estimate the true parameter θ_0 in real time, using the continuous stream of observations.
- In this case, a natural objective function is the asymptotic or average log-likelihood function of the McKean-Vlasov SDE,

$$\tilde{\mathcal{L}}(\theta) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}_t(\theta)$$
$$= \int_{\mathbb{R}^d} \underbrace{-\frac{1}{2}||B(\theta, x, \mu_\infty^\theta) - B(\theta_0, x, \mu_\infty)||^2}_{L(\theta,x,\mu_\infty^\theta,\mu_\infty)} \mu_\infty(\mathrm{d}x).$$

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Stochastic Gradient Ascent in Continuous Time

- To find $\arg\max_{\theta\in\Theta} \tilde{\mathcal{L}}(\theta)$, a natural approach is to evolve $(\theta_t)_{t\geq 0}$ according to a gradient ascent scheme, viz

$$\mathrm{d}\theta_t = \gamma_t \nabla_\theta \tilde{\mathcal{L}}(\theta_t)\mathrm{d}t$$

  where $\gamma = (\gamma_t)_{t\geq 0}$ is a non-negative sequence of real numbers known as the learning rate, or learning rate schedule.
- In practice, we can't implement this directly, since the gradient $\nabla_\theta \tilde{\mathcal{L}}(\theta) = \mathbb{E}_{\mu_\infty}\left[\nabla_\theta L(\theta, x, \mu_\infty^\theta, \mu_\infty)\right]$ depends on the (unknown) invariant law of the McKean-Vlasov SDE.

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- Instead, we could instead try a **stochastic** gradient ascent scheme of the form

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- We still can't implement this SDE, since θ_0, and thus $B(\theta_0, x_t, \mu_t)$, is unknown. However, recalling that $\mathrm{d}x_t = B(\theta_0, x_t, \mu_t)\mathrm{d}t + \mathrm{d}w_t$, we can use $\mathrm{d}x_t$ as a noisy estimate of $B(\theta_0, x_t, \mu_t)\mathrm{d}t$.
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- In order to understand these dynamics better, we can consider the following decomposition:

$$\begin{aligned} \mathrm{d}\theta_t &= \gamma_t \Big[\underbrace{\nabla_\theta L(\theta_t, x_t, \mu_t)\mathrm{d}t}_{\text{(noisy) ascent term}} + \underbrace{\nabla_\theta B(\theta_t, x_t, \mu_t)\mathrm{d}w_t}_{\text{noise term}} \Big] \\ &:= \gamma_t \Big[\underbrace{\nabla_\theta \tilde{L}(\theta_t)\mathrm{d}t}_{\text{true ascent term}} + \underbrace{\left(\nabla_\theta L(\theta_t, x_t, \mu_t) - \nabla_\theta \tilde{\mathcal{L}}(\theta_t)\right)\mathrm{d}t}_{\text{fluctuations term}} \\ &\qquad + \underbrace{\nabla_\theta B(\theta_t, x_t, \mu_t)\mathrm{d}w_t}_{\text{noise term}} \Big] \end{aligned}$$

- Of course, this estimator can only be computed under the assumption that we have access to $(x_t)_{t\geq 0}$ and its law $(\mu_t)_{t\geq 0}$ (**Case III**).

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Stochastic Gradient Ascent in Continuous Time

- More realistically, we will observe multiple continuous sample paths: either $(x_t^i)_{t\geq 0}^{i=1,\dots,N}$ (**Case I**) or $(x_t^{i,N})_{t\geq 0}^{i=1,\dots,N}$ (**Case II**).
- In **Case I**, this motivates the 'approximate' update equation

$$\mathrm{d}\theta_t^{[i,N]} = \gamma_t \left[\nabla_\theta L(\theta_t, x_t^i, \mu_t^{[N]})\mathrm{d}t + \nabla_\theta B(\theta_t, x_t^i, \mu_t^{[N]})\mathrm{d}w_t^i \right],$$

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Main Results (Part I)

Theorem 3

Under appropriate conditions, we have

- **Case I**.

$$\lim_{t,N\to\infty}\mathbb{E}\left[||\nabla_\theta\tilde{\underline{\mathcal{L}}}(\theta_t^{[i,N]})||\right]=\lim_{t,N\to\infty}\mathbb{E}\left[||\nabla_\theta\tilde{\underline{\mathcal{L}}}(\theta_t^{[N]})||\right]=0.$$

- **Case II**.

$$\lim_{t,N\to\infty}\mathbb{E}\left[||\nabla_\theta\tilde{\underline{\mathcal{L}}}(\theta_t^{i,N})||\right]=\lim_{t,N\to\infty}\mathbb{E}\left[||\nabla_\theta\tilde{\underline{\mathcal{L}}}(\theta_t^{N})||\right]=0.$$

where

$$\tilde{\underline{\mathcal{L}}}(\theta)=\int_{\mathbb{R}^d}L(\theta,x,\mu_\infty,\mu_\infty)\mu_\infty(\mathrm{d}x)$$

Proof Overview

- We'll need some extra notation to sketch the proof.
- Recall that $(x_t^i)_{t\geq 0}$ denotes a solution of the McKean-Vlasov SDE, and $(x_t^{i,N})_{t\geq 0}$ is a solution of the corresponding IPS.
- In addition, we will write

$$\mathcal{L}_t^i(\theta) = \int_0^t L(\theta, x_s^i, \mu_s^\theta, \mu_s)\mathrm{d}s + \int_0^t \langle G(\theta, x_s^i, \mu_s^\theta, \mu_s)\rangle \mathrm{d}w_s^i$$

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Proof Overview

- Let's focus on Case II. In this case, we can consider the following decomposition

$$
\begin{aligned}
||\nabla_\theta \tilde{\underline{\mathcal{L}}}(\theta_t^{i,N})|| \leq\ & ||\nabla_\theta \tilde{\underline{\mathcal{L}}}(\theta_t^{i,N}) - \frac{1}{t}\nabla_\theta \underline{\mathcal{L}}_t^i(\theta_t^{i,N})|| \\
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where $\tilde{\underline{\mathcal{L}}}(\theta) = \lim_{t\to\infty} \frac{1}{t}\underline{\mathcal{L}}_t^i(\theta)$ and $\tilde{\mathcal{L}}^{i,N}(\theta) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}_t^{i,N}(\theta)$.

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&+ \boxed{||\frac{1}{t}\nabla_\theta \mathcal{L}_t^{i}(\theta_t^{i,N}) - \frac{1}{t}\nabla_\theta \mathcal{L}_t^{[i,N]}(\theta_t^{i,N})||} \\
&+ ||\frac{1}{t}\nabla_\theta \mathcal{L}_t^{[i,N]}(\theta_t^{i,N}) - \frac{1}{t}\nabla_\theta \mathcal{L}_t^{i,N}(\theta_t^{i,N})|| \\
&+ ||\frac{1}{t}\nabla_\theta \mathcal{L}_t^{i,N}(\theta_t^{i,N}) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})|| \\
&+ ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})||
\end{aligned}
$$

where $\tilde{\underline{\mathcal{L}}}(\theta) = \lim_{t\to\infty} \frac{1}{t}\underline{\mathcal{L}}_t^{i}(\theta)$ and $\tilde{\mathcal{L}}^{i,N}(\theta) = \lim_{t\to\infty} \frac{1}{t}\mathcal{L}_t^{i,N}(\theta)$.

Proof Overview

- Let's focus on Case II. In this case, we can consider the following decomposition

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\begin{aligned}
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Proof Overview

How do we show that $\boxed{||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})||} \to 0$?

Proof Overview

- Define an arbitrary constant $\kappa > 0$, with $\rho = \rho(\kappa) > 0$ to be determined.
- Set $\sigma = 0$, and define the cycle of random stopping times

$$0 = \sigma_0 \leq \tau_1 \leq \sigma_1 \leq \tau_2 \leq \sigma_2 \leq \dots$$

according to

$$\tau_k = \inf\left\{t > \sigma_{k-1} : ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})|| \geq \kappa\right\}$$

$$\sigma_k = \sup\left\{t > \tau_k : \frac{||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})||}{2} \leq ||\nabla \tilde{\mathcal{L}}^{N}(\theta_s^{i,N})|| \leq 2||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_{\tau_k}^{i,N})|| \quad \forall s \in [\tau_k, t], \quad \int_{\tau_k}^{t} \gamma(s)\mathrm{d}s \leq \rho\right\}$$

- The purpose of these stopping times is to control the periods of time for which $||\nabla \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})||$ is close to zero, and those for which it is away from zero.

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- Let's suppose there are an infinite number of stopping times τ_k.
- For sufficiently large k, we can show that there exist constants $0 < \beta_1 < \beta$ such that

$$\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\sigma_k}) - \tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_k}) \geq \beta \quad , \quad \tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_k}) - \tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\sigma_{k-1}}) \geq -\beta_1.$$

- It follows that

$$\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_{n+1}}) - \tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_{k_0}}) \geq \sum_{k=k_0}^{n} (\beta - \beta_1) = (n + 1 - k_0)(\beta - \beta_1)$$

- This implies that $\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_{n+1}}) \to \infty$ as $n \to \infty$. But $\tilde{\mathcal{L}}^{i,N}(\theta)$ is bounded from above.
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Proof Overview

- This proof relies on two inequalities involving $\tilde{\mathcal{L}}$. We obtain these via Itô's formula, which gives

$$\tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\sigma_k}) - \tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_{\tau_k})$$

$$= \int_{\tau_k}^{\sigma_k} \gamma_s ||\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_s)||^2 \mathrm{d}s$$

$$+ \int_{\tau_k}^{\sigma_k} \gamma_s \langle \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta^{i,N}_s), \nabla_\theta B(\theta^{i,N}_s, x^{i,N}_s, \mu^N_s) \mathrm{d}w^i_s \rangle$$

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Proof Overview

How do we show that the term

$$\boxed{\int_{\tau_k}^{\sigma_k} \gamma_s \langle \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}), \nabla_\theta L(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}) \rangle \mathrm{d}s}$$

goes to zero as $k \to \infty$?

Proof Overview

- The key is to rewrite this in terms of the solution of a related Poisson equation.
- In particular, suppose we define the function

$$T(\theta, x^{i,N}, \mu^N) = \langle \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta), \nabla_\theta L(\theta, x^{i,N}, \mu^N, \mu^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta) \rangle.$$

- Then, extending some classical results [PV01], it is possible to show that the equation

$$\mathcal{A}_{\hat{x}^N} v(\theta, x^{i,N}, \mu^N) = -T(\theta, x^{i,N}, \mu^N)$$

has a unique solution (with some nice properties), which allows us to control this term.

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Main Results (Part II)

Theorem 4

Assume that $\tilde{\mathcal{L}}(\cdot)$ is strongly concave (and some additional technical conditions). Then, for sufficiently large t, we have that

- **Case I.**

$$\mathbb{E}[||\theta_t^{[i,N]} - \theta_0||^2] \leq (K_1 + K_2)\gamma_t + K_3\alpha(N),$$

$$\mathbb{E}[||\theta_t^{[N]} - \theta_0||^2] \leq (K_1 + \frac{K_2}{N})\gamma_t + K_3\alpha(N).$$

- **Case II.**

$$\mathbb{E}[||\theta_t^{i,N} - \theta_0||^2] \leq (K_1^* + K_2^*)\gamma_t + K_3^*\alpha(N) + \frac{K_4^*}{N^{\frac{1}{2}}},$$

$$\mathbb{E}[||\theta_t^{N} - \theta_0||^2] \leq (K_1^* + \frac{K_2^*}{N})\gamma_t + K_3^*\alpha(N) + \frac{K_4^*}{N^{\frac{1}{2}}}.$$

where $\alpha : \mathbb{N} \to \mathbb{R}_+$ is defined according to

$$\alpha(N) = \begin{cases} N^{-\frac{1}{4}} & \text{if} \quad d = 1 \\ N^{-\frac{1}{4}} \log(1+N)^{\frac{1}{2}} & \text{if} \quad d = 2 \\ N^{-\frac{1}{2d}} & \text{if} \quad d \geq 3. \end{cases}$$

Proof Overview

- Once again, we focus on Case II.
- Using the update equation for $\theta_t^{i,N}$, and a Taylor expansion around θ_0, we can write down the following SDE for $Z_t^{i,N} := \theta_t^{i,N} - \theta_0$,

$$
\begin{aligned}
\mathrm{d}Z_t^{i,N} &= \gamma_t \nabla_\theta^2 \tilde{\underline{\mathcal{L}}}(\tilde{\theta}_t^{i,N}) Z_t^{i,N} \mathrm{d}t \\
&\quad + \gamma_t \big(\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) - \nabla_\theta \tilde{\underline{\mathcal{L}}}(\theta_t^{i,N}) \big) \mathrm{d}t \\
&\quad + \gamma_t \big(\nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N, \mu_t^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) \big) \mathrm{d}t \\
&\quad + \gamma_t \nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N) \mathrm{d}w_t^i.
\end{aligned}
$$

where $\tilde{\theta}_t^{i,N}$ is point in the segment connecting $\theta_t^{i,N}$ and θ_0.

Proof Overview

- Once again, we focus on Case II.
- Using the update equation for $\theta_t^{i,N}$, and a Taylor expansion around θ_0, we can write down the following SDE for $Z_t^{i,N} := \theta_t^{i,N} - \theta_0$,

$$
\begin{aligned}
\mathrm{d}Z_t^{i,N} &= \gamma_t \nabla_\theta^2 \underline{\tilde{\mathcal{L}}}(\tilde{\theta}_t^{i,N}) Z_t^{i,N} \mathrm{d}t \\
&+ \gamma_t \big(\nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) - \nabla_\theta \underline{\tilde{\mathcal{L}}}(\theta_t^{i,N}) \big) \mathrm{d}t \\
&+ \gamma_t \big(\nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N, \mu_t^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) \big) \mathrm{d}t \\
&+ \gamma_t \nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N) \mathrm{d}w_t^i.
\end{aligned}
$$

where $\tilde{\theta}_t^{i,N}$ is point in the segment connecting $\theta_t^{i,N}$ and θ_0.

Proof Overview

- Applying Itô's formula to the function $f(Z) = ||Z||^2$, we obtain

$$\begin{aligned}
\mathrm{d}||Z_t^{i,N}||^2 = {} & 2\gamma_t \big\langle Z_t^{i,N}, \nabla_\theta^2 \underline{\tilde{\mathcal{L}}}(\tilde{\theta}_t^{i,N}) Z_t^{i,N} \big\rangle \mathrm{d}t \\
& + \gamma_t \big\langle Z_t^{i,N}, \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) - \nabla_\theta \underline{\tilde{\mathcal{L}}}(\theta_t^{i,N}) \big\rangle \mathrm{d}t \\
& + \gamma_t \big\langle Z_t^{i,N}, \nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N, \mu_t^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) \big\rangle \mathrm{d}t \\
& + \gamma_t \big\langle Z_t^{i,N}, \nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N) \mathrm{d}w_t^i \big\rangle \\
& + \gamma_t^2 \big|\big|\nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\big|\big|_F^2 \mathrm{d}t
\end{aligned}$$

Proof Overview

- Due to the strong concavity of $\tilde{\underline{\mathcal{L}}}(\theta)$, it then follows that

$$
\begin{aligned}
\mathrm{d}||Z_t^{i,N}||^2 + 2\eta\gamma_t||Z_t^{i,N}||^2\mathrm{d}t
&\leq \gamma_t\big\langle Z_t^{i,N}, \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N}) - \nabla_\theta\tilde{\underline{\mathcal{L}}}(\theta_t^{i,N})\big\rangle\mathrm{d}t \\
&+ \gamma_t\big\langle Z_t^{i,N}, \nabla_\theta L(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N, \mu_t^N) - \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_t^{i,N})\big\rangle\mathrm{d}t \\
&+ \gamma_t\big\langle Z_t^{i,N}, \nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\mathrm{d}w_t^i\big\rangle \\
&+ \gamma_t^2\big|\big|\nabla_\theta B(\theta_t^{i,N}, x_t^{i,N}, \mu_t^N)\big|\big|_F^2\mathrm{d}t
\end{aligned}
$$

Proof Overview

- Let $\Phi_{s,t} = \exp\left[-2\eta\int_s^t \gamma_u \mathrm{d}u\right]$. Then, after some algebra, and taking expectations, the previous expression implies

$$
\begin{aligned}
\mathbb{E}\left[||Z_t^{i,N}||^2\right] \leq\ & \mathbb{E}\left[\Phi_{1,t}||Z_1||^2\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s \Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}) - \nabla_\theta \underline{\tilde{\mathcal{L}}}(\theta_s^{i,N})\rangle \mathrm{d}s\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s \Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta L(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s)\rangle \mathrm{d}s\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s \Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N)\mathrm{d}w_s^i\rangle\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s^2 \Phi_{s,t}||\nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N)||_F^2 \mathrm{d}s\right].
\end{aligned}
$$

Proof Overview

- Let $\Phi_{s,t} = \exp\left[-2\eta\int_s^t \gamma_u \mathrm{d}u\right]$. Then, after some algebra, and taking expectations, the previous expression implies

$$
\begin{aligned}
\mathbb{E}\left[||Z_t^{i,N}||^2\right] \leq\ & \boxed{\mathbb{E}\left[\Phi_{1,t}||Z_1||^2\right]} \\
&+ \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}) - \nabla_\theta\tilde{\underline{\mathcal{L}}}(\theta_s^{i,N})\rangle \mathrm{d}s\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta L(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N) - \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s)\rangle \mathrm{d}s\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N)\mathrm{d}w_s^i\rangle\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s^2\Phi_{s,t}||\nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N)||_F^2 \mathrm{d}s\right].
\end{aligned}
$$

Proof Overview

- Let $\Phi_{s,t} = \exp\left[-2\eta\int_s^t \gamma_u \mathrm{d}u\right]$. Then, after some algebra, and taking expectations, the previous expression implies

$$\mathbb{E}\left[||Z_t^{i,N}||^2\right] \leq \mathbb{E}\left[\Phi_{1,t}||Z_1||^2\right]$$

$$+\ \boxed{\mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}) - \nabla_\theta\underline{\tilde{\mathcal{L}}}(\theta_s^{i,N})\rangle \mathrm{d}s\right]}$$

$$+\ \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta L(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N) - \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s)\rangle \mathrm{d}s\right]$$

$$+\ \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N)\mathrm{d}w_s^i\rangle\right]$$

$$+\ \mathbb{E}\left[\int_1^t \gamma_s^2\Phi_{s,t}||\nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N)||_F^2 \mathrm{d}s\right].$$

Proof Overview

- Let $\Phi_{s,t} = \exp\left[-2\eta \int_s^t \gamma_u \mathrm{d}u\right]$. Then, after some algebra, and taking expectations, the previous expression implies

$$
\begin{aligned}
\mathbb{E}\left[||Z_t^{i,N}||^2\right] \leq\ & \mathbb{E}\left[\Phi_{1,t}||Z_1||^2\right] \\
& + \mathbb{E}\left[\int_1^t \gamma_s \Phi_{s,t} \langle Z_s^{i,N}, \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}) - \nabla_\theta \underline{\tilde{\mathcal{L}}}(\theta_s^{i,N})\rangle \mathrm{d}s\right] \\
& + \boxed{\mathbb{E}\left[\int_1^t \gamma_s \Phi_{s,t} \langle Z_s^{i,N}, \nabla_\theta L(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s)\rangle \mathrm{d}s\right]} \\
& + \mathbb{E}\left[\int_1^t \gamma_s \Phi_{s,t} \langle Z_s^{i,N}, \nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N) \mathrm{d}w_s^i\rangle\right] \\
& + \mathbb{E}\left[\int_1^t \gamma_s^2 \Phi_{s,t} ||\nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N)||_F^2 \mathrm{d}s\right].
\end{aligned}
$$

Proof Overview

- Let $\Phi_{s,t} = \exp\left[-2\eta \int_s^t \gamma_u \mathrm{d}u\right]$. Then, after some algebra, and taking expectations, the previous expression implies

$$
\begin{aligned}
\mathbb{E}\left[||Z_t^{i,N}||^2\right] \leq\ & \mathbb{E}\left[\Phi_{1,t}||Z_1||^2\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s \Phi_{s,t} \langle Z_s^{i,N}, \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}) - \nabla_\theta \underline{\tilde{\mathcal{L}}}(\theta_s^{i,N})\rangle \mathrm{d}s\right] \\
&+ \mathbb{E}\left[\int_1^t \gamma_s \Phi_{s,t} \langle Z_s^{i,N}, \nabla_\theta L(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N) - \nabla_\theta \tilde{\mathcal{L}}^{i,N}(\theta_s)\rangle \mathrm{d}s\right] \\
&+ \boxed{\mathbb{E}\left[\int_1^t \gamma_s \Phi_{s,t} \langle Z_s^{i,N}, \nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N) \mathrm{d}w_s^i\rangle\right]} \\
&+ \mathbb{E}\left[\int_1^t \gamma_s^2 \Phi_{s,t} ||\nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N)||_F^2 \mathrm{d}s\right].
\end{aligned}
$$

Proof Overview

- Let $\Phi_{s,t} = \exp\left[-2\eta\int_s^t \gamma_u \mathrm{d}u\right]$. Then, after some algebra, and taking expectations, the previous expression implies

$$\begin{aligned}\mathbb{E}\left[||Z_t^{i,N}||^2\right] \leq\ & \mathbb{E}\left[\Phi_{1,t}||Z_1||^2\right] \\ &+ \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s^{i,N}) - \nabla_\theta\underline{\tilde{\mathcal{L}}}(\theta_s^{i,N})\rangle\mathrm{d}s\right] \\ &+ \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta L(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N) - \nabla_\theta\tilde{\mathcal{L}}^{i,N}(\theta_s)\rangle\mathrm{d}s\right] \\ &+ \mathbb{E}\left[\int_1^t \gamma_s\Phi_{s,t}\langle Z_s^{i,N}, \nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N)\mathrm{d}w_s^i\rangle\right] \\ &+ \boxed{\mathbb{E}\left[\int_1^t \gamma_s^2\Phi_{s,t}||\nabla_\theta B(\theta_s^{i,N}, x_s^{i,N}, \mu_s^N)||_F^2\mathrm{d}s\right]}.\end{aligned}$$

Numerical Results

Linear Mean-Field Dynamics

- We first consider a one-dimensional linear mean field model, parametrised by $\theta = (\theta_1, \theta_2)^T \in \mathbb{R}^2$, of the form

$$\mathrm{d}x_t = -\left[\theta_1 x_t + \theta_2 \int_{\mathbb{R}} (x_t - y)\mu_t(\mathrm{d}y)\right]\mathrm{d}t + \sigma \mathrm{d}w_t,$$
$$\mu_t = \mathcal{L}(x_t).$$

- The corresponding system of interacting particles is given by

$$\mathrm{d}x_t^{i,N} = -\left[\theta_1 x_t^{i,N} + \theta_2 \frac{1}{N}\sum_{j=1}^{N}(x_t^{i,N} - x_t^{j,N})\right]\mathrm{d}t + \sigma \mathrm{d}w_t^{i,N}.$$

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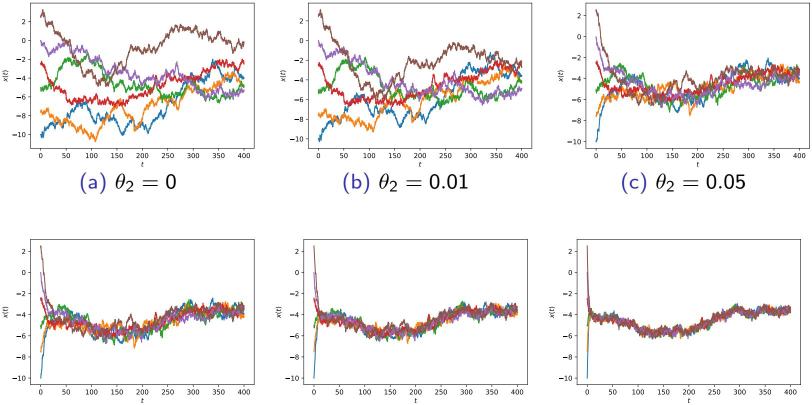
Linear Mean-Field Dynamics



(a) $\theta_2 = 0$ (b) $\theta_2 = 0.01$ (c) $\theta_2 = 0.05$

(d) $\theta_2 = 0.1$ (e) $\theta_2 = 0.2$ (f) $\theta_2 = 0.5$

Figure: Sample paths of the linear mean field interacting particle system for different values of the interacting parameter θ_2.

Linear Mean-Field Dynamics: Offline MLE

- Since this model is linear in both of the parameters, in this case it is possible to obtain the MLE in closed form as

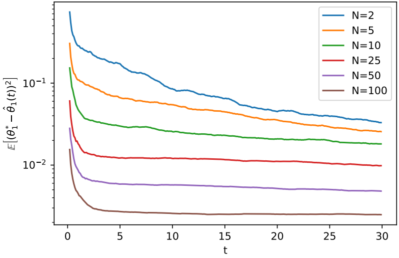
$$\hat{\theta}_{1,t}^N = \frac{A_t^N - B_t^N}{C_t^N - D_t^N} \quad , \quad \hat{\theta}_{2,t}^N = \frac{D_t^N A_t^N - C_t^N B_t^N}{(C_t^N)^2 - C_t^N D_t^N}$$

where we have defined, writing $\bar{x}_s^N = \frac{1}{N}\sum_{j=1}^N x_s^{j,N}$,

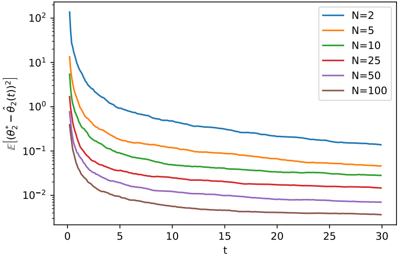
$$A_t^N = \int_0^t \sum_{i=1}^N (x_s^{i,N} - \bar{x}_s^N)\mathrm{d}x_s^{i,N} \ , \quad B_t^N = \int_0^t \sum_{i=1}^N x_s^{i,N}\mathrm{d}x_s^{i,N}$$

$$C_t^N = \int_0^t \sum_{i=1}^N (x_s^{i,N} - \bar{x}_s^N)^2\mathrm{d}s \quad , \quad D_t^N = \int_0^t \sum_{i=1}^N (x_s^{i,N})^2\mathrm{d}s.$$

Linear Mean-Field Dynamics: Offline MLE



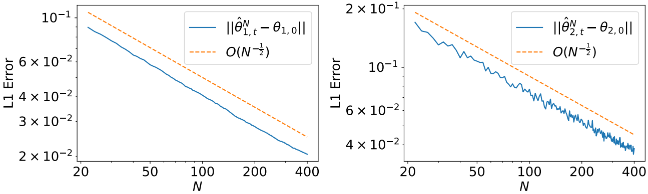
(a) $\hat{\theta}_1^N(t)$



(b) $\hat{\theta}_2^N(t)$

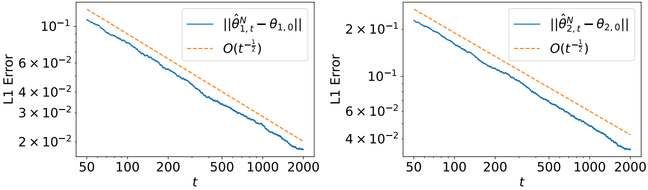
Figure: $\mathbb{L}^2$ error of the offline MLE for $t \in [0, 30]$ and $N = \{2, 5, 10, 25, 50, 100\}$.

Linear Mean-Field Dynamics: Offline MLE



(a) $\hat{\theta}_1^N(t)$ (b) $\hat{\theta}_2^N(t)$

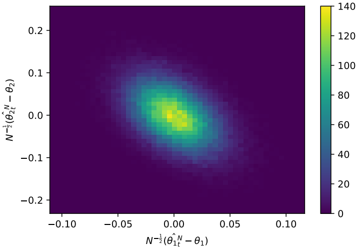
Figure: Log-log plot of the $\mathbb{L}^1$ error of the MLE for $T = 5$, $N = \{20, \ldots, 400\}$.



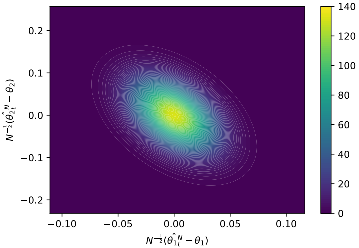
(a) $\hat{\theta}_1^N(t)$ (b) $\hat{\theta}_2^N(t)$

Figure: Log-log plot of the $\mathbb{L}^1$ error of the MLE for $T \in [50, 2000]$, $N = 2$.

Linear Mean-Field Dynamics: Offline MLE



(a) The approximate bivariate density.



(b) The asymptotic bivariate density.

Figure: A comparison between the asymptotic normal distribution and the approximate normal distribution of the MLE for $N = 500$ particles.

Linear Mean-Field Dynamics: Online MLE

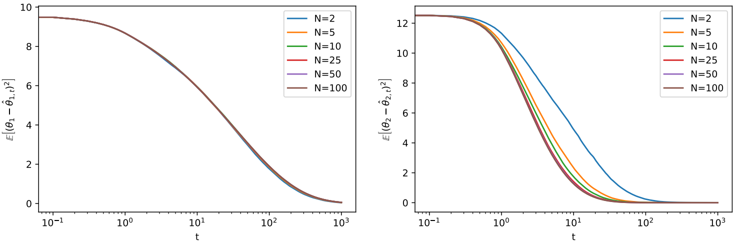
- For this model, the online parameter updates are given by

$$\mathrm{d}\theta_{1,t}^N = \frac{\gamma_{1,t}}{N\sigma^2}\sum_{i=1}^N \left[-x_t^{i,N}\mathrm{d}x_t^{i,N} - x_t^{i,N}(\theta_{1,t}^N x_t^{i,N} + \theta_{2,t}^N(x_t^{i,N} - \bar{x}_t^N))\mathrm{d}t\right],$$

$$\mathrm{d}\theta_{2,t}^N = \frac{\gamma_{2,t}}{N\sigma^2}\sum_{i=1}^N \Big[-(x_t^{i,N} - \bar{x}_t^N)\mathrm{d}x_t^{i,N} - (x_t^{i,N} - \bar{x}_t^N)$$

$$(\theta_{1,t}^N x_t^{i,N} + \theta_{2,t}^N(x_t^{i,N} - \bar{x}_t^N))\mathrm{d}t\Big].$$

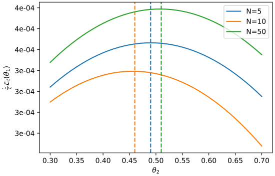
Linear Mean-Field Dynamics: Online MLE



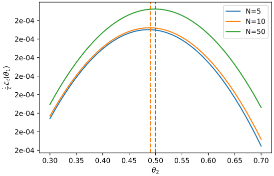
(a) $\theta^N_{1,t}$. (b) $\theta^N_{2,t}$.

Figure: $\mathbb{L}^2$ error of the online maximum likelihood estimates for $T \in [0, 1000]$ and $N = \{2, 5, 10, 25, 50, 100\}$. The time is plotted on a log-scale.

Linear Mean-Field Dynamics: Online MLE

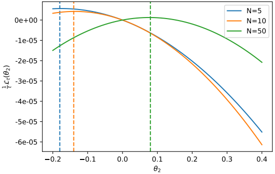


(a) $T = 1.0$.

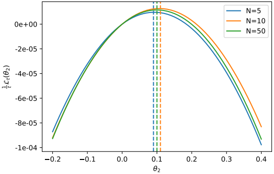


(b) $T = 7.5$.

Figure: The log-likelihood w.r.t. $\theta_1 = 0.5$ for $T = \{1, 7.5\}$, $N = \{5, 10, 50\}$.



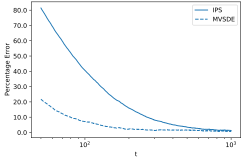
(a) $T = 1.0$.



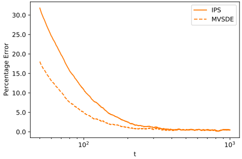
(b) $T = 7.5$.

Figure: The log-likelihood w.r.t. $\theta_2 = 0.1$ for $T = \{1, 7.5\}$, $N = \{5, 10, 50\}$.

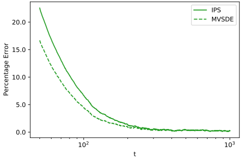
Linear Mean-Field Dynamics: Online MLE



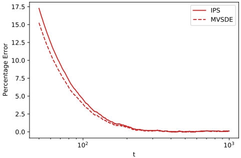
(a) $N = 2$.



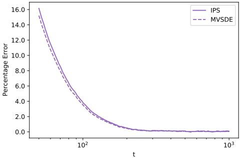
(b) $N = 5$.



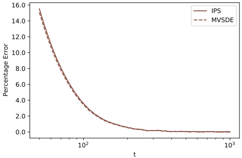
(c) $N = 10$.



(d) $N = 25$.



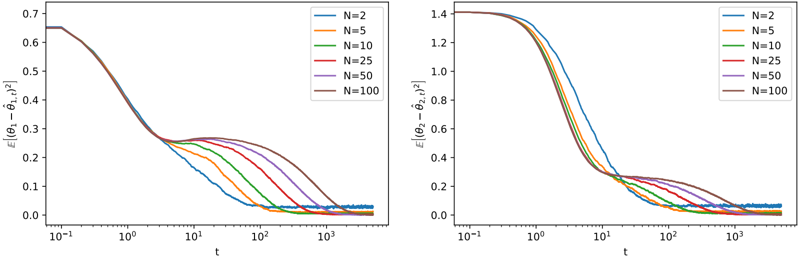
(e) $N = 50$.



(f) $N = 100$.

Figure: Percentage error of the online maximum likelihood estimates of the interaction parameter $\hat{\theta}^N_{2,t}$ for $T \in [0, 1000]$ and $N = \{2, 5, 10, 25, 50, 100\}$, generated using the IPS and the McKean-Vlasov SDE.

Linear Mean-Field Dynamics: Online MLE

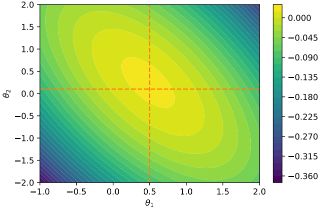


(a) $\theta^N_{1,t}$.

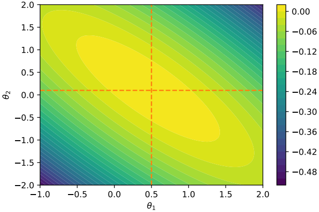
(b) $\theta^N_{2,t}$.

Figure: $\mathbb{L}^2$ error of the online MLEs for $T \in [0, 5000]$, $N = \{2, 5, 10, 25, 50, 100\}$. The time is plotted on a log-scale.

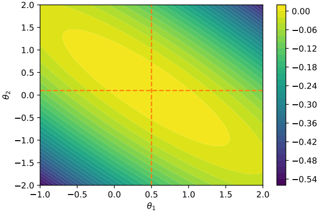
Linear Mean-Field Dynamics: Online MLE



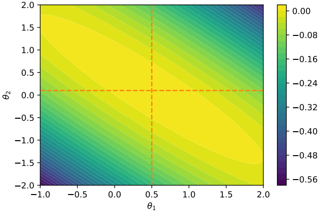
(a) $N = 2$.



(b) $N = 5$.



(c) $N = 10$.



(d) $N = 100$.

Figure: Contour plots of $\tilde{\mathcal{L}}^N(\theta)$ for $N = \{2, 5, 10, 100\}$.

Bistable Potential

- We next consider a one-dimensional mean-field model, parametrised by $\theta \in \mathbb{R}$, of the form

$$\mathrm{d}x_t = -\left[\nabla_x V(x) + \int_{\mathbb{R}} \nabla_x W(\theta, x_t - y)\mu_t(\mathrm{d}y)\right]\mathrm{d}t + \sqrt{2\beta^{-1}}\mathrm{d}w_t,$$
$$\mu_t = \mathcal{L}(x_t).$$

where $\beta > 0$ is inverse temperature, $w = (w_t)_{t\geq 0}$ is a standard Brownian motion, $V : \mathbb{R} \to \mathbb{R}$ is the bistable confinement potential

$$V(x) = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]$$

and $W(\theta, \cdot) : \mathbb{R} \to \mathbb{R}$ is the Curie-Weiss (i.e., quadratic) interaction potential

$$W(\theta, x-y) = \frac{1}{2}\theta\left[x-y\right]^2.$$

Bistable Potential

- An interesting feature of this model is that it may admit multiple invariant measures (dependent on θ).
  - The one-parameter family of invariant densities for this model is given by

$$p_{\infty}(x;\theta,\beta,m)=\frac{e^{-\beta\left(\left[\frac{1}{4}x^4-\frac{1}{2}x^2\right]+\theta\left(\frac{1}{2}x^2-xm\right)\right)}}{\int_{\mathbb{R}}e^{-\beta\left(\left[\frac{1}{4}x^4-\frac{1}{2}x^2\right]+\theta\left(\frac{1}{2}x^2-xm\right)\right)}\mathrm{d}x}$$

  - These solutions are subject to the constraint that they provide the correct formula for the first moment, viz

$$m=\int_{\mathbb{R}}xp_{\infty}(x;\theta,\beta,m)\mathrm{d}x=R(m;\theta,\beta).$$

  - The number of solutions to this equation (the 'self-consistency' equation) determines the number of invariant measures.

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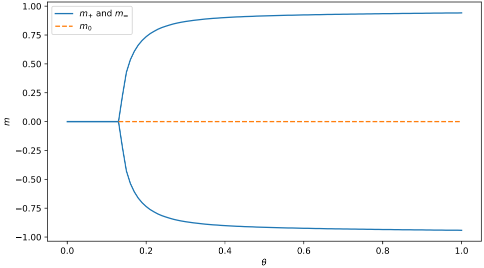
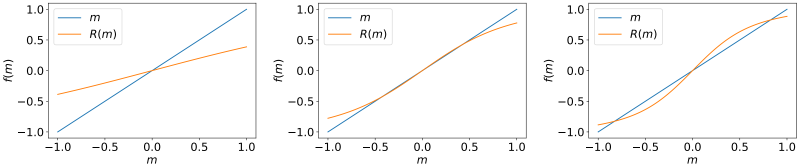


Figure: Bifurcation diagram of m as a function of $\theta \in [0, 1]$ for fixed $\beta = 10$.

Bistable Potential



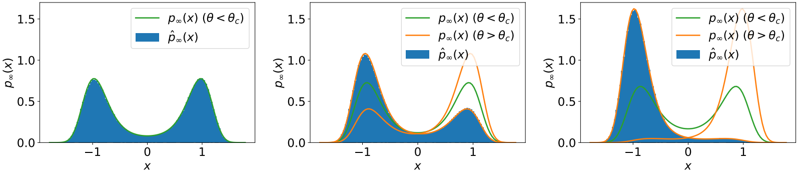
(a) $\theta = 0.05$.

(b) $\theta = 0.15$.

(c) $\theta = 0.25$.

Figure: Plots of $f(m) = m$ and $f(m) = R(m; \theta, \beta)$ for several values of θ, and fixed $\beta = 10$. The intersection points correspond to solutions of the self-consistency equation.

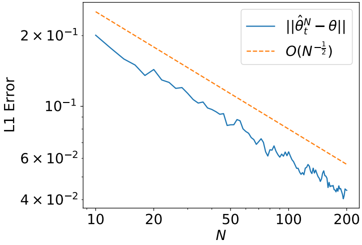
Bistable Potential



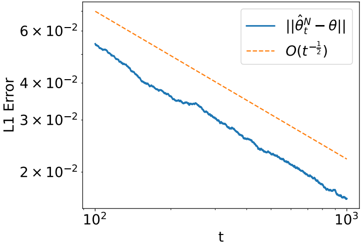
(a) $\theta = 0.05$. (b) $\theta = 0.15$. (c) $\theta = 0.25$.

Figure: Plots of the empirical invariant density $\hat{p}_\infty(x;\theta,\beta,m)$ (blue), and the true invariant density (densities) $p_\infty(x;\theta,\beta,m)$ (green, orange) for several values of θ, and fixed $\beta = 10$. We distinguish between the invariant density which exists for $\theta < \theta_c$ (green) and the two invariant densities which only exist for $\theta > \theta_c$ (orange).

Bistable Potential: Offline MLE



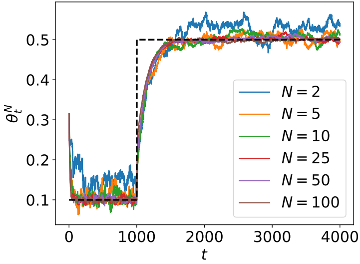
(a) $\hat{\theta}^N_{1,t=5}$.



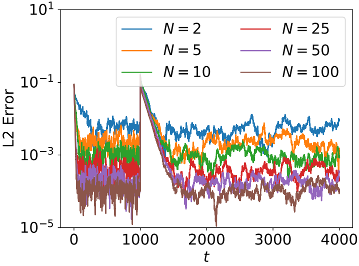
(b) $\hat{\theta}^{N=2}_{2,t}$.

Figure: Log-log plot of the $\mathbb{L}^1$ error of the offline MLE for $t = 0.5$ and $N \in \{20, \ldots, 200\}$ (left hand panel), and for $t \in [100, 1000]$ and $N = 2$ (right hand panel).

Bistable Potential: Online MLE



(a) Sequence of online parameter estimates.



(b) $\mathbb{L}^2$ error.

Figure: Performance of the recursive maximum likelihood estimator for values of $N \in \{2, 5, 10, 25, 50, 100\}$. The true value(s) of the time varying parameter are shown in black (dashed).

Stochastic Kuramoto Model

- Our next example is a non-linear SDE on the one-dimensional torus $\mathbb{T}$, parametrised by $\theta \in \mathbb{R}$, of the form

$$\mathrm{d}x_t = \left[\theta \int_{\mathbb{R}} \sin(x_t - y)\mu_t(\mathrm{d}y)\right] \mathrm{d}t + \sqrt{2\beta^{-1}}\mathrm{d}w_t,$$
$$\mu_t = \mathcal{L}(x_t).$$

  where $\beta > 0$ and $w = (w_t)_{t \geq 0}$ is a $\mathbb{T}$-valued Brownian motion, and all other terms are as defined previously.
- This equation represents the mean-field limit of the stochastic Kuramoto model, which models the synchronisation of noisy oscillators through their phases.

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\mu_t &= \mathcal{L}(x_t).
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$$

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- This model, similarly to the last, may admit multiple invariant measures (depending on θ).
  - In this case, every stationary solution can be written as $p_\infty(x + x_0)$ for some $x_0 \in [0, 2\pi)$, where

$$p_\infty(x; \theta, \beta, r) = \frac{e^{\beta\theta r \cos x}}{2\pi \int_{\mathbb{S}} e^{\beta\theta r \cos x} \mathrm{d}x}.$$

with r being a non-negative solution to the equation

$$r := \Psi(\beta\theta r) = \frac{\int_{\mathbb{S}} \cos x \exp(\beta\theta r \cos x) \mathrm{d}x}{\int_{\mathbb{S}} \exp(\beta\theta r \cos x) \mathrm{d}x}$$

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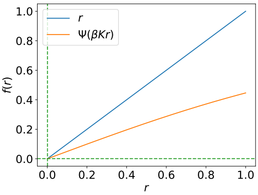
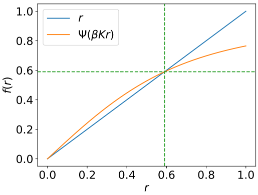
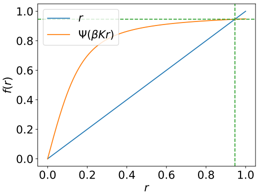


(a) $\theta = 0.05$.



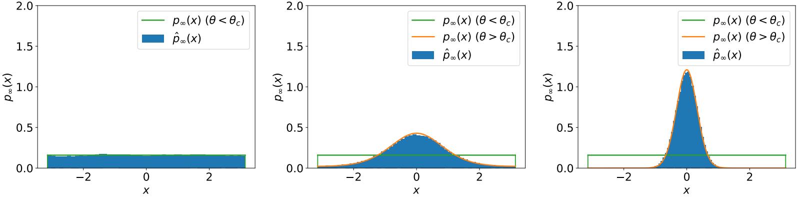
(b) $\theta = 0.15$.



(c) $\theta = 0.25$.

Figure: Plots of $f(r) = r$ and $f(r) = \Psi(\beta\theta r)$ for several values of θ, and fixed $\beta = 10$. The intersection points correspond to solutions of the fixed point equation.

Stochastic Kuramoto Model



(a) $\theta = 0.05$. (b) $\theta = 0.05$. (c) $\theta = 0.05$.

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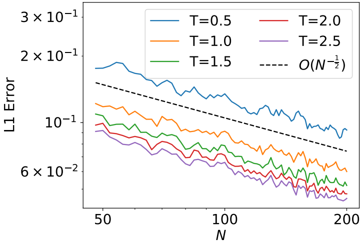
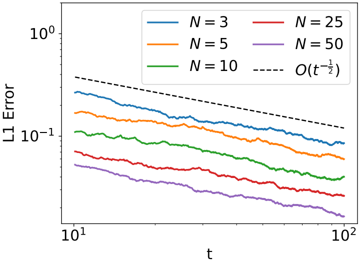
Stochastic Kuramoto Model: Offline MLE



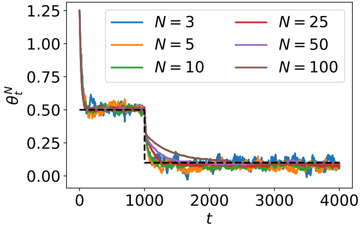
(a) $\mathbb{L}^1$ error as a function of N.



(b) $\mathbb{L}^1$ error as a function of t.

Figure: Log-log plot of the $\mathbb{L}^1$ error of the offline MLE as a function of N, for several values of t (left hand panel), and as a function of t, for several values of N (right hand panel).

Stochastic Kuramoto Model: Online MLE



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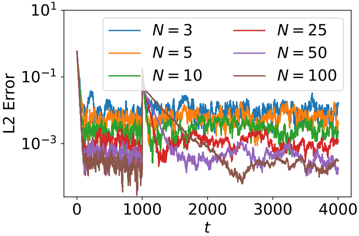
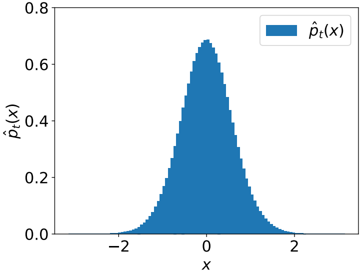


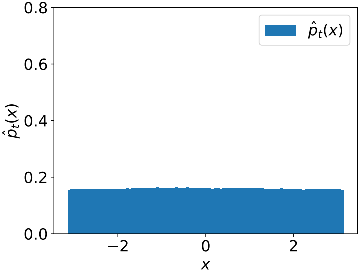
(b) $\mathbb{L}^2$ error.

Figure: Performance of the recursive maximum likelihood estimator for values of $N \in \{3, 5, 10, 25, 50, 100\}$. The true value(s) of the time varying parameter are shown in black (dashed).

Stochastic Kuramoto Model: Online MLE



(a) $t = 1000$.



(b) $t = 4000$.

Figure: Plots of the empirical density $\hat{p}_t(x)$ at two times. The two plots approximate the invariant densities for the two values of the true (time varying) parameter from the previous slide.

Stochastic Opinion Dynamics

- We now consider a model, parametrised by $\theta = (\theta_1, \theta_2)^T \in \mathbb{R}^2$, of the form

$$\mathrm{d}x_t^{i,N} = -\frac{1}{N}\sum_{j=1}^{N} \varphi_\theta(||x_t^{i,N} - x_t^{j,N}||)(x_t^{i,N} - x_t^{j,N})\mathrm{d}t + \sigma \mathrm{d}w_t.$$

where the 'interaction kernel' $\varphi_\theta : \mathbb{R}^+ \to \mathbb{R}$ is given by[1]

$$\varphi_\theta(r) = \theta_1 \mathbb{1}_{r\in[0,\theta_2]}\ , \quad r \in \mathbb{R},$$

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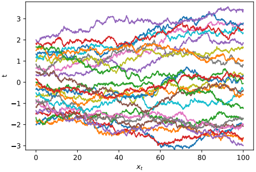
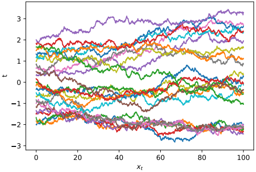
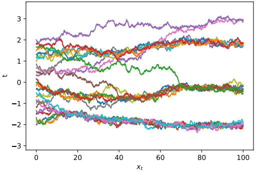


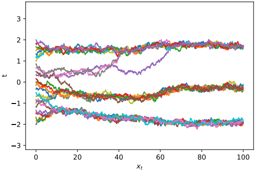
(a) $\theta_2 = 0.0$



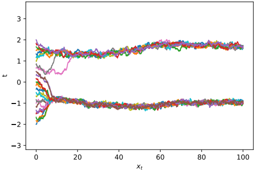
(b) $\theta_2 = 0.2$



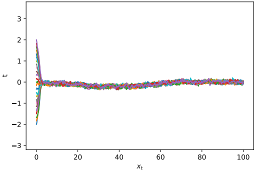
(c) $\theta_2 = 0.3$



(d) $\theta_2 = 0.5$



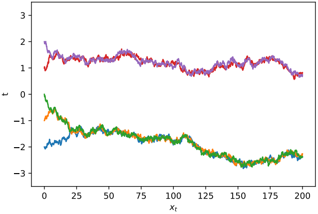
(e) $\theta_2 = 0.7$



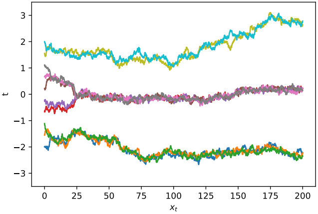
(f) $\theta_2 = 1.0$

Figure: Simulations from the IPS for different values of θ_2.

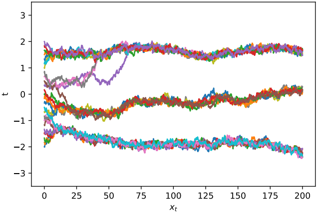
Stochastic Opinion Dynamics



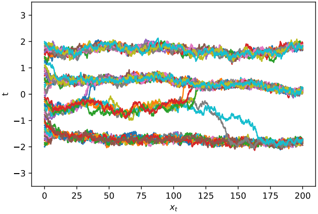
(a) $N = 5$



(b) $N = 10$



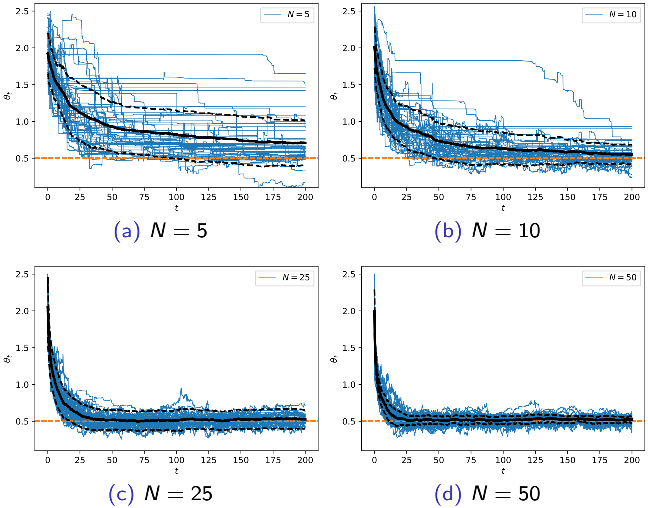
(c) $N = 25$



(d) $N = 50$

Figure: Simulations from the IPS for $\theta_2 = 0.5$, for $N \in \{5, 10, 25, 50\}$.

Stochastic Opinion Dynamics



(a) $N = 5$ (b) $N = 10$ (c) $N = 25$ (d) $N = 50$

Figure: Sequence of online parameter estimates for θ_2, for $N \in \{5, 10, 25, 50\}$.

Stochastic Opinion Dynamics

- One can also imagine even more interesting dynamics, whereby the interaction kernel is given by

$$\tilde{\varphi}_{\theta}(r) = \sum_{i=1}^{p} \theta_{1,i} \mathbb{1}_{r \in [0, \theta_{2,i}]}$$

- This allows more granular control over how the interaction strengths (controlled by $\theta_{1,i}$) in different ranges (controlled by $\theta_{2,i}$)

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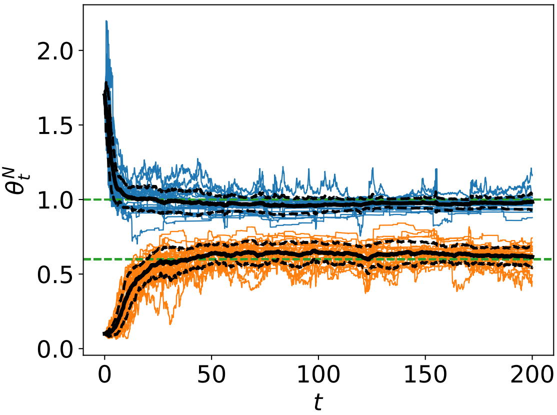

Figure: Sequence on online parameter estimates for two range parameters $\theta_{2,1}$ (blue) and $\theta_{2,2}$ (orange).

Summary

Conclusions

- We provide a theoretical analysis of parameter estimation for a McKean-Vlasov SDE, and the associated IPS.
  - Offline parameter estimation: strong consistency and asymptotic normality as $N \to \infty$.
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Any Questions?

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