Seminar Report

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Harmonic Analysis on Euclidean Spaces

This report, mainly based on Walter Rudin's article *Trigonometric Series with Gaps* [1], discusses a remarkable result originally presented by Zygmund in *Trigonometrical Series* [2]. The result states that under the condition of a sufficiently "thin" spectrum—such as that of a Sidon set—strong integrability properties hold for functions with frequencies in that set. Specifically, integration bounds hold for all finite $L^p, p \geq 2$ norms and even extend to exponential integrability.

More precisely, if $E \subset \mathbb{Z}$ is a Sidon set, then for every $f \in L^2(\mathbb{T})$ with spectrum in E, we have

$$\int_{\mathbb{T}} \exp\left(|f(x)|^2\right) \, dx < \infty.$$

This shows an exceptional integrability phenomenon associated with thin spectra, and highlights the L^1 anomaly: L^1 is the only space that resists this integrability property.

Historical Context Antoni Zygmund (1900–1992) was a foundational figure in modern harmonic analysis. One of his early successes, which brought him to the attention of mathematicians around the world, was *Trigonometrical Series*, published in Warsaw in 1935. The book quickly became a reference in the field. Jean-Pierre Kahane reportedly referred to Zygmund's *Trigonometrical Series* as a "Bible" for harmonic analysts.

Influenced in the 1930s by G.H. Hardy and J.E. Littlewood during his time in Cambridge, Zygmund would in turn influence and collaborate with his students, most notably Alberto Calderón, Elias M. Stein, and many others.

Walter Rudin (1921–2010) was also influenced by Zygmund and his intellectual successors. He is best known for his widely used textbooks in mathematical analysis: *Functional Analysis*, *Principles of Mathematical Analysis*—nicknamed "Baby Rudin"—and *Real and Complex Analysis*, often referred to as "Big Rudin."

Both Zygmund and Rudin were European-born and immigrated to the United States in response to political instability and anti-Semitic persecution in Europe. Zygmund left Poland, and Rudin, born in Vienna, fled Austria after the Nazi annexation. He spent time in France and England before settling in the United States.

1 Preliminaries

Let f be a Lebesgue integrable function on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Its Fourier series has the form:

$$\sum_{k\in\mathbb{Z}} \hat{f}(k)e^{2\pi ikx}$$

where the Fourier coefficient are given by :

$$\hat{f}(k) = \int_{\mathbb{T}} f(x)e^{-2\pi ikx}dx$$

We define the $L^p(\mathbb{T})$ norm in the usual way:

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p \, dx\right)^{1/p}.$$

When $p = \infty$, this corresponds to the supremum norm:

$$\|f\|_\infty = \sup_{x\in\mathbb{T}} |f(x)|.$$

We now introduce the concept of functions supported on a set $E \subset \mathbb{Z}$.

Definition 1.1. Let $E \subset \mathbb{Z}$. A function f is called an *E-function* if and only if $f \in L^1(\mathbb{T})$ and $\hat{f}(k) = 0$ for all $k \notin E$.

We denote by $C_E(\mathbb{T})$ the space of all continuous E-functions.

Remark 1.1. In the definition above, the fact that $f \in L^1(\mathbb{T})$ guarantees the existence of its Fourier coefficients. The set E can then be interpreted as the *frequency support* of the function f.

Definition 1.2. A trigonometric polynomial whose Fourier frequencies belong to a set $E \subset \mathbb{Z}$ is called an *E-polynomial.*

We now introduce the notion of *Sidon sets*[1], which are special subsets of the integers with useful boundedness properties for Fourier coefficients.

Definition 1.3. A set $E \subset \mathbb{Z}$ is called a *Sidon set* if there exists a constant $S > 0$ such that for every E-polynomial f, one has

$$\sum_{k\in E} |\hat{f}(k)| \leq S\|f\|_\infty.$$

The smallest such constant S is called the *Sidon constant* of the set E.

[1] Simon Sidon was a Hungarian mathematician known for his contributions to harmonic analysis in the early 20th century.

2 Characterization Theorem

We now state and prove a fundamental theorem characterizing Sidon sets.

Theorem 2.1 (Characterization of Sidon Sets)**.** *Let $E \subset \mathbb{Z}$. The following statements are equivalent:*

(i) E is a Sidon set.

(ii) Every bounded E-function has an absolutely convergent Fourier series.

(iii) Every continuous E-function has an absolutely convergent Fourier series.

(iv) For every bounded function $b : E \to \mathbb{C}$, there exists a measure μ on $\mathbb{T}$ such that:

$$\hat{\mu}(n) = b(n), \quad \forall n \in E;$$

(v) If $b(n) \to 0$ as $|n| \to \infty$, then there exists $f \in L^1(\mathbb{T})$ such that $\hat{f}(n) = b(n)$ for all $n \in E$.

Proof. We prove each implication step by step.

(i) $\Rightarrow$ (ii):

Assume that $E \subset \mathbb{Z}$ is a Sidon set with Sidon constant S, and let f be a bounded E-function. Consider the Fejér means $\sigma_N f$ of f, defined by

$$\sigma_N f(x) = \frac{1}{N+1} \sum_{n=0}^{N} S_n f(x),$$

where $S_n f$ denotes the n-th partial sum of the Fourier series of f. Explicitly,

$$\sigma_N f(x) = \sum_{|k| \leq N} \left(1 - \frac{|k|}{N+1}\right) \hat{f}(k) e^{2\pi i k x}.$$

This is a trigonometric polynomial with frequencies in E, i.e., an E-polynomial. Moreover, we can write

$$\sigma_N f = f * F_N,$$

where F_N is the Fejér kernel. Since convolution with the Fejér kernel preserves uniform boundedness, we have

$$\|\sigma_N f\|_\infty \leq \|f\|_\infty.$$

Therefore,

$$\frac{1}{2} \sum_{|k| \leq N/2} |\hat{f}(k)| \leq \sum_{|k| \leq N} \left(1 - \frac{|k|}{N+1}\right) |\hat{f}(k)| \leq S \|f\|_\infty,$$

where the right-hand inequality follows from the definition of the Sidon constant and the fact that $1 - \frac{|k|}{N+1} \leq 1$ for all k.

Letting $N \to \infty$, we conclude that the Fourier series of f converges absolutely.

(ii) ⇒ (iii):

Since every continuous function on $\mathbb{T}$ is bounded, the implication follows directly.

(iii) ⇒ (i):

Assume that every continuous E-function has an absolutely convergent Fourier series. Define the map:

$$\begin{aligned} \mathcal{F}_E \quad : \quad C_E(\mathbb{T}, \|\cdot\|_\infty) &\longrightarrow \ell^1(E, \|\cdot\|_1) \\ f &\mapsto \widehat{f}|_E \end{aligned}$$

This map is linear, continuous, injective, and surjective by assumption. Hence, it is a Banach space isomorphism onto its image.

The continuity implies that there exists a constant S such that $\forall f \in C_E(\mathbb{T})$,

$$\sum_{k\in E} |\hat{f}(k)| \leq S\|f\|_\infty.$$

In particular, this holds for all E-polynomials, proving that E is a Sidon set.

(i), (ii), (iii) are equivalent.

(i) ⇒ (iv):

The key tools here are the Hahn–Banach theorem and the Riesz representation theorem. We recall both:

Theorem 2.2 (Hahn–Banach)**.** *Let X be a complex normed vector space, $M \subset X$ a linear subspace, and $f : M \to \mathbb{C}$ a bounded linear map. Then there exists $F : X \to \mathbb{C}$, linear and bounded, such that $F|_M = f$ and $\|F\| = \|f\|$.*

Theorem 2.3 (Riesz Representation)**.** *Let X be a compact Hausdorff space. Then for every bounded linear functional on $C(X)$, there exists a (complex) Radon measure $\mu \in \mathcal{M}(X)$ such that $f(x) = \int_X f\, d\mu$ for all $f \in C(X)$.*

Let E be a Sidon set with Sidon constant S. Let $b : E \to \mathbb{C}$ be bounded, and define a linear functional Λ on $C_E(\mathbb{T})$ by:

$$\Lambda(f) = \sum_{k\in E} \hat{f}(k)b(k).$$

This functional is bounded on $C_E(\mathbb{T})$ with norm at most $\|b\|_\infty S$, since

$$|\Lambda(f)| \leq \|b\|_\infty \sum_{k\in E} |\hat{f}(k)| \leq S\|b\|_\infty\|f\|_\infty.$$

By the Hahn–Banach theorem, Λ extends to a bounded linear functional on $C(\mathbb{T})$ with the same norm. By the Riesz theorem, there exists a complex Radon measure μ on $\mathbb{T}$, of total variation at most $S\|b\|_\infty$, such that

$$\Lambda(f) = \int_{\mathbb{T}} f(x)\, d\mu(x), \quad \forall f \in C(\mathbb{T}).$$

Applying this to $f(x) = e^{-2\pi ikx}$, we obtain

$$\hat{\mu}(k) = \int_{\mathbb{T}} e^{-2\pi ikx}\, d\mu(x) = b(k), \quad \forall k \in E.$$

(iv) $\Rightarrow$ (v):

Let $b : E \to \mathbb{C}$ satisfy $b(k) \to 0$ as $|k| \to \infty$. Choose an even, positive, convex sequence $(c(k))_{k\in\mathbb{Z}}$ such that $c(k) \to 0$ and $\frac{b(k)}{c(k)}$ is bounded on E. This is possible by ensuring $c(k)$ decays more slowly than $b(k)$.

By (iv), there exists a measure μ on $\mathbb{T}$ such that

$$\hat{\mu}(k) = \frac{b(k)}{c(k)}, \quad \forall k \in E.$$

Now we use the following lemma:

Lemma 2.1. *Let $(a_k)_{k\in\mathbb{Z}}$ be an even, convex, positive sequence with $a_k \to 0$. Then there exists $g \in L^1(\mathbb{T})$ such that $\hat{g}(k) = a_k$ for all $k \in \mathbb{Z}$.*

Let $g \in L^1(\mathbb{T})$ be the function given by the lemma applied to $(c(k))$. Define $f = g * \mu \in L^1(\mathbb{T})$. Then:

$$\hat{f}(k) = \hat{g}(k)\hat{\mu}(k) = c_k \cdot \frac{b(k)}{c_k} = b(k), \quad \forall k \in E.$$

Thus, $f \in L^1(\mathbb{T})$ and has the desired Fourier coefficients.

(v) $\Rightarrow$ (iii):

Let (v) hold and assume $f \in C_E(\mathbb{T})$. We aim to show that for every sequence $(a_k)_{k\in E}$ with $a_k \to 0$ as $|k| \to \infty$, the series

$$\sum_{k\in E} |\hat{f}(k)a_k|$$

is convergent.

Lemma 2.2. *If for all sequences $(a_k) \to 0$, the series $\sum |\hat{f}(k)a_k|$ converges, then $\sum |\hat{f}(k)|$ converges.*

Let $(a_k)_{k\in E}$ be such that $a_k \to 0$ as $|k| \to \infty$. By (v), there exists $g \in L^1(\mathbb{T})$ such that $\hat{g}(k) = a_k$ for all $k \in E$. Since $f \in C_E(\mathbb{T})$, the convolution $f * g \in C_E(\mathbb{T})$.

By Fejér's theorem, the Fejér means of $f * g$ converge pointwise to $f * g(x)$ for every $x \in \mathbb{T}$, and in particular:

$$\sum_{|k|\leq N} \left(1 - \frac{|k|}{N+1}\right) \hat{f}(k)\hat{g}(k) \to f * g(0).$$

Assume $\hat{g}(k) = a_k$ cancels the phase of $\hat{f}(k)$, so $\hat{f}(k)a_k \geq 0$. Then:

$$\frac{1}{2} \sum_{|k|\leq N/2} |\hat{f}(k)a_k| \leq \sum_{|k|\leq N} \left(1 - \frac{|k|}{N+1}\right) \hat{f}(k)\hat{g}(k).$$

As the right-hand side converges, so does $\sum_{k\in E} |\hat{f}(k)a_k|$, which completes the proof.

□

It remains to prove the two lemma:

Proof of Lemma 2.1. Let $(a_k)_{k\in\mathbb{Z}}$ be an even, convex, positive sequence such that $a_k \to 0$. By the convexity of the sequence, for all $k \geq 1$, we have:

$$(a_{k-1} - a_k) - (a_k - a_{k+1}) = a_{k-1} + a_{k+1} - 2a_k \geq 0,$$

so the sequence $(a_k - a_{k+1})_{k\geq 0}$ is monotonically decreasing in k.

Moreover, observe that:

$$\sum_{k=0}^{\infty}(a_k - a_{k+1}) = a_0.$$

From the two previous facts, we deduce that:

$$k(a_k - a_{k+1}) \to 0 \quad \text{as } k \to \infty,$$

otherwise the series $\sum_{k=0}^{\infty}(a_k - a_{k+1})$ would diverge.

Now, define:

$$f = \sum_{k=1}^{\infty} k(a_{k-1} + a_{k+1} - 2a_k)F_{k-1},$$

where F_k denotes the Fejér kernel of order k.

First, note that:

$$\sum_{k=1}^{N} k(a_{k-1} + a_{k+1} - 2a_k) = a_0 - a_N - N(a_N - a_{N+1}) \to a_0 \quad \text{as } N \to \infty,$$

and since $\|F_{k-1}\|_1 = 1$, the series defining f converges in $L^1(\mathbb{T})$.

Moreover, the Fourier coefficients of f are given by:

$$\widehat{f}(n) = \sum_{k=|n|+1}^{\infty} k(a_{k-1} + a_{k+1} - 2a_k)\left(1 - \frac{|n|}{k}\right) = a_{|n|}.$$

Remark: The last equality can be proved by induction on $m \geq 1$, showing that:

$$\sum_{k=|n|+1}^{|n|+m} k(a_{k-1} + a_{k+1} - 2a_k)\left(1 - \frac{|n|}{k}\right) = a_{|n|} + m a_{|n|+m+1} - (m+1)a_{|n|+m}.$$

□

Proof of Lemma 2.2. Suppose that $\sum_{k\in\mathbb{Z}} |\hat{f}(k)| = \infty$.

We construct a sequence $(a_k) \to 0$ such that $\sum_k |\hat{f}(k)a_k| = \infty$.

Let $\{I_n\}_{n\in\mathbb{N}}$ be a partition of $\mathbb{Z}$ into finite, symmetric, disjoint subsets such that

$$\sum_{k\in I_n} |\hat{f}(k)| \geq 1 \quad \text{for all } n.$$

Define

$$a_k = \begin{cases} \frac{1}{n+1} & \text{if } k \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $a_k \to 0$, but

$$\sum_{k \in \mathbb{Z}} |\hat{f}(k) a_k| = \sum_{n \geq 0} \frac{1}{n+1} \sum_{k \in I_n} |\hat{f}(k)| \geq \sum_{n \geq 1} \frac{1}{n} = \infty.$$

This proves the contrapositive. Hence, if $a_k \to 0$, then $\sum |\hat{f}(k) a_k| < \infty$.

$\square$

3 An example: Hadamard Sets

We have just proven a characterization theorem for Sidon sets. Before going further, we may want to examine some examples of what a Sidon set can be. For this, we will use one of the characterizations of Sidon sets.

Let us introduce a category of sets named after Hadamard[2].

Definition 3.1. A set of positive integers $H := \{h_n\}_{n\geq 0}$ is called Hadamard lacunary (or simply lacunary) if there exists $r > 1$ such that

$$\frac{h_{n+1}}{h_n} > r \quad \forall n \geq 0.$$

Remark: A trigonometric series is called lacunary if all the frequencies appearing in it are of the form $\pm h_n$.

Example 3.1. A classical example of a Hadamard set is the sequence of powers of an integer $r > 1$:

$$H = \{r^n : n \geq 0\}.$$

We are going to show that every Hadamard lacunary set is a Sidon set. The general idea is that the sparsity of the exponents in the series imposes a certain uniformity in the behavior of the functions built from them.

Let us admit the following lemma for now, and see where it leads us:

Lemma 3.1. *Let f be an H-polynomial of the form*

$$f(x) = \sum_{k=0}^{N} c_k e^{2\pi i h_k x}.$$

Then we have

$$\sum |c_k| \lesssim_r {}^3 \|f\|_\infty.$$

Now let $f = \sum c_k e^{2\pi i h_k x}$ be a continuous H-function.

Observe that the Fejér means of f converge uniformly to f:

$$\sigma_N f(x) \to f(x) \quad \text{as } N \to \infty, \quad \forall x \in \mathbb{T},$$

and in particular,

$$\|\sigma_N f\|_\infty \to \|f\|_\infty \quad \text{as } N \to \infty.$$

Moreover, $\sigma_N f$ is an H-polynomial, so by the lemma:

$$\sum_{k=1}^{N} |\widehat{\sigma_N f}(k)| \lesssim_r \|\sigma_N f\|_\infty.$$

[2] Jacques Hadamard (1865–1963) was a French mathematician known for his work in complex analysis, number theory, and functional analysis. These sets are named after him due to his theorem on Hadamard gaps in trigonometric series.

[3] The notation $\lesssim_r$ means the inequality holds up to a constant depending on r.

As seen in the proof of the implication (i) $\Rightarrow$ (ii) in the previous theorem:

$$\frac{1}{2}\sum_{k=1}^{N}|c_k| \leq \sum_{k=1}^{N}|\widehat{\sigma_N f}(k)|$$

Taking the limit as $N \to \infty$, we get:

$$\sum |c_k| \lesssim_r \|f\|_\infty.$$

Hence, the Fourier series of f is absolutely convergent, so H is a Sidon set.

Let us now prove the lemma.

Proof. Let f be an H-polynomial:

$$f(x) = \sum_{k=0}^{N} c_k e^{2\pi i h_k x}.$$

By the triangle inequality, it suffices to prove the result for $\Re f$ and $\Im f$.

These functions have Fourier series of the form:

$$\sum_{|k|\leq N} \tilde{c}_k e^{2\pi i h_k x}, \quad \text{with } h_{-k} = -h_k.$$

We define the Riesz product:

$$P_N(x) = \prod_{j=1}^{N}\left(1 + \cos(2\pi h_j x + \phi_j)\right),$$

where the phases $\phi_j \in \mathbb{T}$ are arbitrary.

Case $r > 3$: In this case, every integer $n \in \mathbb{Z}$ admits a unique representation of the form:

$$n = \sum \epsilon_j h_j, \quad \epsilon_j \in \{-1, 0, 1\}.$$

This uniqueness ensures that in the expansion of $P_N(x)$ given below, the second sum does not introduce any of the frequencies h_j already present in the first sum:

$$P_N(x) = 1 + \sum_{j=1}^{N}\cos(2\pi h_j x + \phi_j) + \sum A_n \cos(2\pi n x + \phi_n),$$

where the second sum runs over those integers n expressible as above with at least two nonzero ϵ_j, and the coefficients A_n are constants.

Thus, for each $k \in \mathbb{Z}$:

$$\widehat{P_N}(k) = \delta_{k=0} + \sum_{j=1}^{N}\frac{1}{2}\left(e^{i\phi_j}\delta_{k=h_j} + e^{-i\phi_j}\delta_{k=-h_j}\right) + \sum A_n\left(e^{i\phi_n}\delta_{k=n} + e^{-i\phi_n}\delta_{k=-n}\right).$$

In particular,

$$\|P_N\|_1 = \widehat{P_N}(0) = 1, \quad \widehat{P_N}(h_j) = \frac{1}{2}e^{i\phi_j}, \quad \widehat{P_N}(-h_j) = \frac{1}{2}e^{-i\phi_j}.$$

We choose the phases ϕ_j so that

$$\overline{\tilde{c}_j}e^{i\phi_j} = |\tilde{c}_j|.$$

Since P_N, $\Re f$, and $\Im f$ are continuous (and hence in $L^2(\mathbb{T})$), we apply Parseval's identity:

$$\frac{1}{2}\sum |\tilde{c}_k| = \frac{1}{2}\sum \overline{\tilde{c}_k}e^{i\phi_k} \leq \int_{\mathbb{T}} P_N(x) \cdot \overline{\Re f(x)}\, dx \leq \|\Re f\|_\infty.$$

The same estimate holds for $\Im f$, hence we conclude:

$$\sum |c_k| \leq 4\|f\|_\infty.$$

Case $1 < r < 3$:

In this case, there may exist nontrivial relations of the form:

$$h_k = \sum \epsilon_j h_j.$$

To reduce to the previous case, choose $M \in \mathbb{N}$ large enough such that:

$$r^M > 3, \quad 1 - \frac{1}{r^M - 1} > \frac{1}{r}, \quad 1 + \frac{1}{r^M - 1} < r.$$

Define the subsequence:

$$h_j^{(m)} := h_{m+jM}.$$

Then,

$$h_{j+1}^{(m)} > r^M h_j^{(m)}.$$

We construct the corresponding Riesz product:

$$P_N^{(m)}(x) = \prod_{j=1}^{N} \left(1 + \cos(2\pi h_j^{(m)}x + \phi_{m+jM})\right),$$

with $\phi_{m+jM} \in \mathbb{T}$ as before.

Its expansion takes the same form:

$$P_N^{(m)}(x) = 1 + \sum_{j=1}^{N} \cos(2\pi h_j^{(m)}x + \phi_{m+jM}) + \sum A_n \cos(2\pi nx + \phi_n),$$

where $n = \sum \epsilon_j h_j^{(m)}$ and the A_n are constants.

Now observe that any $n = \sum \epsilon_j h_j^{(m)}$ is at a distance less than $\frac{h_{j_0}^{(m)}}{r^M-1}$ from some $h_{j_0}^{(m)}$. Indeed, if $h_{j_0}^{(m)}$ is the dominant term, then by lacunarity:

$$|n - h_{j_0}^{(m)}| \leq \sum_{j<j_0} h_j^{(m)} < h_{j_0}^{(m)} \sum_{l=1}^{\infty} \frac{1}{r^{Ml}} = \frac{h_{j_0}^{(m)}}{r^M - 1}.$$

The only potential issue is if a linear combination of the $h_j^{(m)}$ coincides exactly with some h_k outside the subsequence. We now prove this never happens — that is, for $k \not\equiv m \mod M$, the frequencies $\pm h_k$ do not appear in the Riesz product.

Indeed, if $|h_k - h_{j_0}^{(m)}| < \frac{h_{j_0}^{(m)}}{r^M-1}$, then:

$$\frac{1}{r} < 1 - \frac{1}{r^M - 1} < \frac{h_k}{h_{j_0}^{(m)}} < 1 + \frac{1}{r^M - 1} < r.$$

Moreover, by lacunarity, if $k < m + j_0 M$, then

$$\frac{h_k}{h_{m+j_0M}} < \frac{1}{r},$$

and if $k > m + j_0 M$, then

$$\frac{h_k}{h_{m+j_0M}} > r,$$

In both cases, this contradicts the above inequality, hence such h_k cannot appear.

Thus, the only frequencies h_k that appear in the Riesz product satisfy:

$$k \equiv m \mod M.$$

Consequently, by repeating the argument from the $r > 3$ case, we deduce that

$$\sum_j |c_{m+jM}| \leq 4\|f\|_\infty.$$

Summing over $m = 1, \ldots, M$, the result follows. □

Remark. A more conceptual understanding of the role of the Riesz product arises from duality: according to characterization (iv) of Sidon sets, the Riesz product provides an explicit construction of the interpolating measures involved.

4 Norm Inequalities for Sidon Sets

The characterization theorem proved earlier is not only conceptually important but also technically useful. In particular, the constructions of interpolating measures and the absolute convergence of Fourier series for E-functions (established in points (iv) and (v)) are key tools in proving norm inequalities. The inequality we are about to prove directly leads us to the desired result of exponential integration.

Theorem 4.1. *Let $E \subset \mathbb{Z}$ be a Sidon set with Sidon constant S. Then, for any E-polynomial f, if $2 < q < \infty$, we have*

$$\|f\|_q \leq S\sqrt{q}\|f\|_2.$$

Proof. Let f be an E-polynomial, where E is a Sidon set with constant S. For $x \in \mathbb{T}$, define the random sum

$$s_n(x) = \sum_{|k| \leq n} \epsilon_k \hat{f}(k) e^{2\pi i k x},$$

where ϵ_k are independent Rademacher variables. Then $s_n(x)$ is a random trigonometric polynomial.

Since f is an E-polynomial, for n large enough the sum becomes stationary, and we denote this finite sum by $s(x)$. Also, define

$$\sigma = \left(\sum_k |\hat{f}(k)|^2 \right)^{1/2} = \|f\|_2$$

by Parseval's identity.

We state the following lemma, which will be proven later:

Lemma 3.1. For every $x \in \mathbb{T}$, we have

$$\mathbb{E}(|s(x)|^{2m}) \leq m^m \sigma^{2m}.$$

As shown in the implication (iii) $\Rightarrow$ (iv) of the previous theorem, there exists a measure μ on $\mathbb{T}$ such that

$$\|\mu\| \leq S \quad \text{and} \quad \forall k \in E, \quad \hat{\mu}(k) = \epsilon_k.$$

Since $\hat{s} \cdot \hat{\mu} = \hat{f}$, we conclude $f = s * \mu$.

By Young's inequality (with $1/p + 1/q = 1 + 1/r$ for $r = q$, $p = 1$), we get:

$$\|f\|_q \leq \|s\|_q \|\mu\| \leq S\|s\|_q.$$

Assume $q = 2m$. Raising the inequality to the $2m$-th power and taking expectation yields:

$$\|f\|_{2m}^{2m} = \mathbb{E}(\|f\|_{2m}^{2m}) \leq S^{2m}\mathbb{E}(\|s\|_{2m}^{2m}) = S^{2m} \int_{\mathbb{T}} \mathbb{E}(|s(x)|^{2m})\, dx \leq S^{2m} m^m \sigma^{2m}.$$

Thus,

$$\|f\|_{2m} \leq S\sqrt{m}\|f\|_2.$$

To complete the proof for $2 < q < \infty$, observe that if $q \in [2m-2, 2m]$, then

$$\|f\|_q \leq \|f\|_{2m} \leq S\sqrt{m}\|f\|_2 \leq S\sqrt{q}\|f\|_2,$$

since $m \leq q$, and the left inequality follows from Hölder's inequality.

This proves the result for all $2 < q < \infty$. $\square$

We now prove the lemma.

Proof of Lemma 3.1. Using Chernoff bounds, one can derive a version of the Khintchine inequality:

$$\mathbb{E}\left[|s(x)|^r\right] \leq 2^{r+1} r\, \Gamma\left(\frac{r}{2}\right) \sigma^r,$$

where Γ denotes the Gamma function and σ is the standard deviation.

For even integers $r = 2m$, Stirling's approximation gives

$$\Gamma(m) \sim \sqrt{2\pi m}\, m^m e^{-m},$$

so the bound becomes

$$\mathbb{E}\left[|s(x)|^{2m}\right] \lesssim m^{3/2} \left(\frac{4}{e}\right)^m \sigma^{2m}.$$

Since $\frac{4}{e} > 1$, the previous estimate is not sufficient, and we need to improve the Khintchine bound.

It can actually be shown that a stronger version of the Khintchine inequality holds in the case $p > 2$:

$$\mathbb{E}\left[|s(x)|^r\right] \leq \frac{2^{r/2}}{\sqrt{\pi}}\, \Gamma\left(\frac{r+1}{2}\right) \sigma^r,$$

which allows us to replace $\frac{4}{e} > 1$ with $\frac{2}{e} < 1$, and this yields the desired conclusion. However, this result is far from trivial; the interested reader may refer to [4], p. 265 for a complete proof.

An alternative argument based on Zygmund's book is also possible and will be explained here.

As previously observed, for sufficiently large N, we have $s(x) = s_N(x)$. Let us introduce the notation

$$c_k^f(x) := \left|\widehat{f}(k)e^{2\pi i k x}\right|.$$

Then:

$$\mathbb{E}\left(|s_N(x)|^{2m}\right) = \mathbb{E}\left(\left|\sum_{k \leq N} \epsilon_k \widehat{f}(k) e^{2\pi i k x}\right|^{2m}\right)$$

$$\leq \mathbb{E}\left(\sum \frac{(\alpha_1 + \cdots + \alpha_\ell)!}{\alpha_1! \cdots \alpha_\ell!} \left(c_{k_1}^f(x)\right)^{\alpha_1} \cdots \left(c_{k_\ell}^f(x)\right)^{\alpha_\ell} \epsilon_{k_1}^{\alpha_1} \cdots \epsilon_{k_\ell}^{\alpha_\ell}\right),$$

where the sum is taken over indices $k_1, \ldots, k_\ell$ and exponents $\alpha_1, \ldots, \alpha_\ell$ satisfying:

- $0 \leq k_i \leq N$,

- $0 \leq \alpha_i \leq 2m$, for $i = 1, \ldots, \ell$,
- $1 \leq \ell \leq 2m$,
- $\alpha_1 + \cdots + \alpha_\ell = 2m$.

Though the expression may seem complicated, it is just the multinomial expansion.

If any of the α_i are odd, then $\mathbb{E}\left(\epsilon_{k_1}^{\alpha_1} \cdots \epsilon_{k_\ell}^{\alpha_\ell}\right) = 0$, due to the symmetry of the Rademacher variables.

Therefore, we may assume that each α_i is even, and we write $\alpha_i = 2\beta_i$. Now, let us observe that:

$$\sum \frac{(\beta_1 + \cdots + \beta_\ell)!}{\beta_1! \cdots \beta_\ell!} \left(c_{k_1}^f(x)\right)^{2\beta_1} \cdots \left(c_{k_\ell}^f(x)\right)^{2\beta_\ell} = \left(\sum_{|k| \leq N} \left(c_k^f(x)\right)^2\right)^m = \sigma^{2m}$$

Now, consider the ratio of the multinomial coefficients:

$$\frac{\dfrac{(2\beta_1 + \cdots + 2\beta_\ell)!}{(2\beta_1)! \cdots (2\beta_\ell)!}}{\dfrac{(\beta_1 + \cdots + \beta_\ell)!}{\beta_1! \cdots \beta_\ell!}} = \frac{(2m)!}{m!} \cdot \frac{\beta_1! \cdots \beta_\ell!}{(2\beta_1)! \cdots (2\beta_\ell)!}.$$

Let us focus on the following term:

$$\frac{(2\beta_1)! \cdots (2\beta_\ell)!}{\beta_1! \cdots \beta_\ell!}$$

We claim this term is bounded below by 2^m. Indeed, when $\beta_1 = \cdots = \beta_\ell = 1$ (i.e., $\ell = m$), the ratio is exactly 2^m.

Now, assume that for some index i, $\beta_i > 1$, so $\ell < m$. We decompose β_i into two integers $\beta_{i0} = 1$ and $\beta_{i1} = \beta_i - 1 \geq 1$. Then:

$$\frac{(2\beta_{i0})!(2\beta_{i1})!}{\beta_{i0}!\beta_{i1}!} = 2\frac{(2\beta_i - 2)!}{(\beta_i - 1)!} \leq \frac{(2\beta_i)!}{\beta_i!},$$

which implies that this decomposition decreases the ratio. By iterating this process, we show that the minimal value of the full ratio is 2^m.

Finally, observe that:

$$\frac{(2m)!}{m! \cdot 2^m} = \frac{(m+1)(m+2)\cdots(2m)}{2^m} \leq m^m,$$

which completes the argument.

□

Remark 4.1. This estimate is sharp: there exists an E-polynomial f such that

$$\|f\|_q \geq \frac{1}{4}\sqrt{q}\|f\|_2,$$

as shown in [1, Section 3.3, p.212].

5 Application to Exponential Integrability

Corollary 5.1. *Let $E \subset \mathbb{Z}$ be a Sidon set. Then for every E-function $f \in L^2(\mathbb{T})$, we have*

$$\int_{\mathbb{T}} \exp\left(|f(x)|^2\right) dx < \infty.$$

Proof. Let s_n denote the n-th Fourier partial sum of f:

$$s_n(x) = \sum_{|k| \leq n} \hat{f}(k) e^{2\pi i k x}.$$

By convexity of $y \mapsto y^2$, we have:

$$|f|^2 \leq 2|f - s_n|^2 + 2|s_n|^2,$$

so that:

$$\exp(|f(x)|^2) \leq \exp(2|f(x) - s_n(x)|^2) \cdot \exp(2|s_n(x)|^2).$$

Fix $\delta > 0$. Since $f \in L^2(\mathbb{T})$, we can choose N such that

$$\|f - s_N\|_2 < \delta,$$

where s_N denotes the Nth partial sum of the Fourier series of f. Since s_N is a trigonometric polynomial, it is bounded. Therefore:

$$\int_{\mathbb{T}} \exp(|f(x)|^2)\, dx \lesssim_N \int_{\mathbb{T}} \exp(2|f(x) - s_N(x)|^2)\, dx,$$

and we may expand the exponential in power series:

$$\int_{\mathbb{T}} \exp(2|f - s_N|^2)\, dx = \sum_{q \geq 0} \frac{2^q}{q!} \int_{\mathbb{T}} |f - s_N|^{2q}\, dx.$$

By the previous theorem, for all $q \geq 2$:

$$\|f - s_N\|_{2q}^{2q} \leq (S\|f - s_N\|_2)^{2q} q^q \leq (S\delta)^{2q} q^q.$$

So the full series is bounded (up to a multiplicative constant) by:

$$\sum_{q \geq 2} \frac{(S\delta)^{2q} q^q}{q!}.$$

Using Stirling's formula $q! \sim e^{-q} q^q \sqrt{2\pi q}$, we estimate:

$$\frac{(S\delta)^{2q} q^q}{q!} \sim \frac{(eS^2\delta^2)^q}{\sqrt{q}}.$$

Choosing $\delta > 0$ such that $eS^2\delta^2 < 1$, the series converges and the integral is finite.

□

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[4] Uffe Haagerup, *The best constants in the Khintchine inequality*, Studia Mathematica, 70 (1981), no. 3, 231–283 (1982).