AR and MA models

Time Series for Business

The autoregressive model AR(p)

- The AR(1) model

$$X_t = \phi_0 + \phi_1 X_{t-1} + w_t$$

where $w_t \sim WN(0, \sigma^2)$.

- Properties of AR(1) process:

$$\begin{aligned} E(X_t) &= \phi_0 \\ Var(X_t) &= \frac{\sigma^2}{1-\phi_1^2} \text{ } if \ |\phi_1| < 1 \\ \rho(h) &= \phi_1^h \end{aligned}$$

- ACF examples of AR(1) processes.

```
ts.sim = arima.sim(list(order = c(1,0,0), ar = 0.8),
n = 200);
acf(ts.sim,lag.max=12);
```

AR(1) with positive ϕ_1.

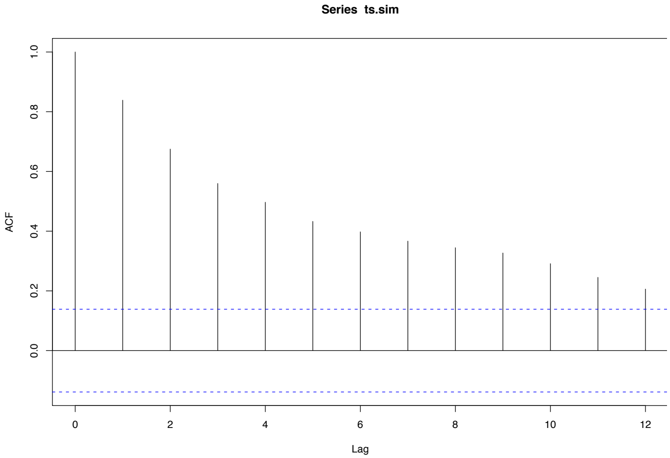
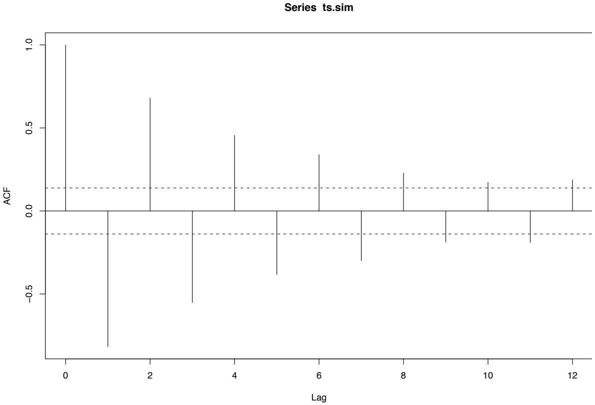


Figure: ACF when ϕ_1=0.8

AR(1) with negative ϕ_1.

```
ts.sim = arima.sim(list(order = c(1,0,0), ar = -0.8),
n = 200);
acf(ts.sim,lag.max=12);
```



AR(1) with small ϕ_1.

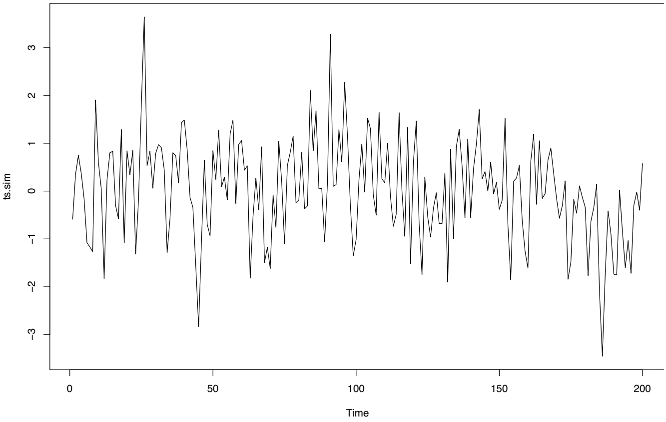


Figure: ACF when ϕ_1=0.2

AR(1) with large ϕ_1. What are the differences?

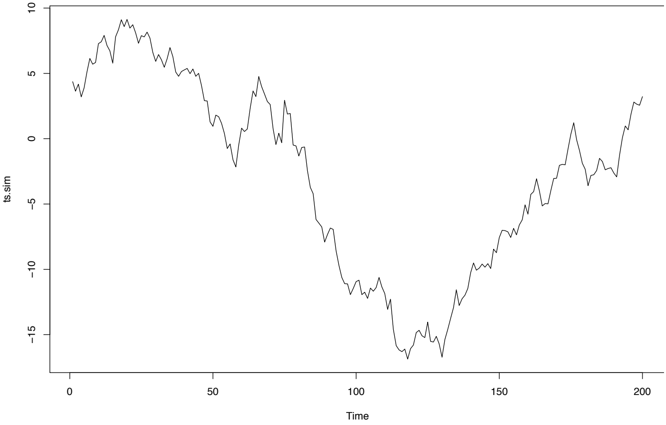


Figure: ACF when ϕ_1=0.99

Properties of AR(1)

- As we discussed before, the stationary condition for AR(1) process is $|\phi_1| < 1$.

$$X_t = \phi_0 + \phi_1 X_{t-1} + w_t$$

$$Var(X_t) = \phi_1^2 Var(X_{t-1}) + \sigma^2$$

  Due to stationarity,

$$(1 - \phi_1^2) Var(X_t) = \sigma^2,$$

  thus $|\phi_1| < 1$.

- Condition on the past, X_t's dependency on the past can be characterized by X_{t-1}, we have

$$E(X_t|X_{t-1}) = \phi_0 + \phi_1 X_{t-1}, \quad Var(X_t|X_{t-1}) = Var(w_t) = \sigma^2.$$

The AR models

- For AR(1) models, condition on the past, X_t's dependency on the past can be characterized by X_{t-1}, we have

$$E(X_t|X_{t-1}) = \phi_0 + \phi_1 X_{t-1}, \quad Var(X_t|X_{t-1}) = Var(w_t) = \sigma^2.$$

- For general AR(p) models,

$$X_t = \phi_0 + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + w_t,$$

where $\phi_0 = \mu(1 + \phi_1 + \cdots + \phi_p)$. Here μ is the mean of X_t.

AR(2) with complex roots

- Consider an AR(2) $X_t = 1.5X_{t-1} - 0.75X_{t-2} + w_t$,

```
ts.sim = arima.sim(list(ar = c(1.5,-0.75),
 n = 200);
acf(ts.sim);
```

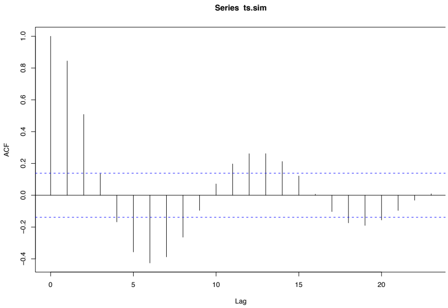


Figure: ACF when for AR(2) with complex root

Operations with autoregressive operators

- Use the backshift operator B,

$$X_t = \phi_1 B X_t + \phi_2 B^2 X_t + \cdots + \phi_p B^p X_t + w_t$$

- We define the AR operator

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$$

- Order p AR model is $\phi(B) X_t = w_t$.

The moving average model

- Let $\{w_t\}$ be $\mathrm{WN}(0, \sigma^2)$ process. The MA(q) model is to represent X_t as a linear combination of q white noise variables.

$$X_t = \theta_0 w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \cdots + \theta_q w_{t-q}$$

- Using the backward shift operator $B^j X_t = X_{t-j}$ we can write

$$X_t = \theta(B) w_t$$

  where

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q$$

- The MA process is stationary for any values of θ.

An MA(2) processes

- An MA(2) with $\theta_1 = 0.45$ and $\theta_2 = -0.45$
  ts.sim = arima.sim(n=200, list(ma=c(0.45, -0.45), sd = 1))

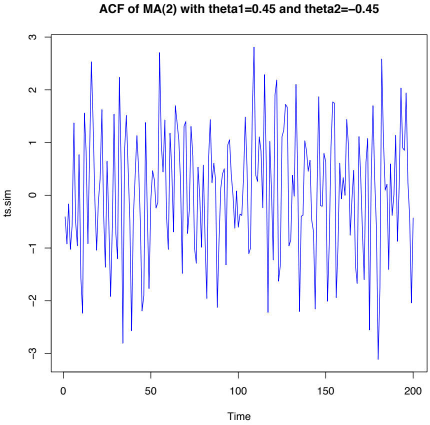


Figure: MA(2) process

The acf of an MA(2) processes

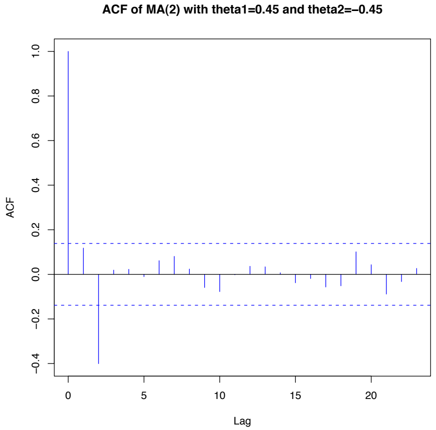


Figure: ACF of the MA(2) process

- Note the sharp cut-off at lag 2.