ARMA models

Time series for business

The Partial ACF

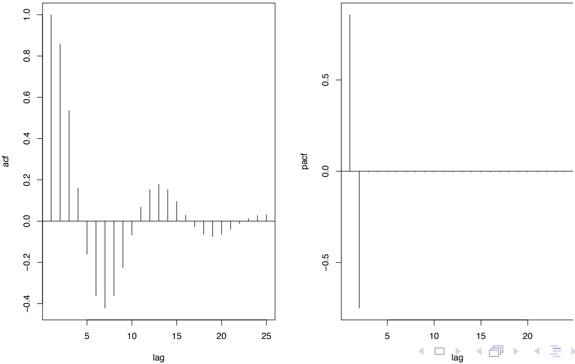
- The MA(q) can be identified from its ACF: cut off at lag q.
- Partial ACF enables similar property for AR(p).
- Motivating example:
  In an AR(1) model $X_t = \phi X_{t-1} + w_t$, where X_t and X_{t-2} are correlated with $\rho_2 = \phi^2$.
  $X_t = \phi X_{t-1} + w_t$ and $X_{t-1} = \phi X_{t-2} + w_{t-1}$. The connection

  $$X_t \leftrightarrow X_{t-1} \leftrightarrow X_{t-2}$$

  Thus if one takes out the common knowledge on X_{t-1}, then X_t and X_{t-2} are uncorrelated.

The Partial ACF

```
acf=ARMAacf(ar=c(1.5,-0.75),ma=0,24);
pacf=ARMAacf(ar=c(1.5,-0.75),ma=0,24,pacf=T);
par(mfrow=c(1,2));
plot(acf,type="h",xlab="lag");  abline(h=0);
plot(pacf,type="h",xlab="lag");abline(h=0);
```



ACF and PACF

- What about the PACF of an invertible ARMA process?

	AR(p)	MA(q)	ARMA(p,q)
ACF	Tails off	Cut off after q	Tails off
PACF	Cut off after p	Tails off	Tails off

Diagnosis-check estimated coefficients

```
> out=arima(x,c(3,0,2))
> out
Call:
arima(x = x, order = c(3, 0, 2))
Coefficients:
         ar1     ar2     ar3     ma1     ma2  intercept
      0.0302  0.8767  0.0296  0.6531  0.4869     0.8687
s.e.  0.1520  0.0373  0.1440  0.1333  0.0722     2.0760
sigma^2 estimated as 0.9767:  log likelihood = -283.91,  a
```

We can roughly check the t-stat = estimate/s.e. for the estimated coefficients. For this case, the 3rd lag coefficient is insignificant. So we should think about fitting a ARMA(2,2) model.

Diagnosis-invertibility and stationarity

Check that the roots of the AR and MA polynomial are outside of the unit circle.

```
> #AR roots
> polyroot(c(1, -0.0302,-0.8767, -0.0296))
[1]   1.033340+0i  -1.106575+0i -29.545008-0i
> a=polyroot(c(1, -0.0302,-0.8767, -0.0296))
> Mod(a)
[1]  1.033340  1.106575 29.545008
```

Diagnosis

- Polynomial factorization: if one of the roots is close to 1, then differencing should be consider. If there are four roots close to $(1, -1, i, -i)$ respectively, then seasonal (quarterly) differencing

$$1 - B^4 = (1 - B)(1 + B)(1 - iB)(1 + iB)$$

should be considered.

- Model redundancy: consider a ARMA(p,q) model $\phi(B)X_t = \theta(B)w_t$, then it also follows ARMA(p+k,q+k) model, since

$$\beta(B)\phi(B)X_t = \beta(B)\theta(B)w_t$$

- Underspecified model: increase model size. Add seasonal terms. If residual shows an isolated significance at lag k, adding X_{t-k} orw_{t-k} usually helps.

Diagnosis- Residual analysis

Check the following things: Time plot of the series, ACF, PACF, Box-Ljung test and outliers.

- If time plot shows non-zero mean, include a constant term in the model.
- If non-constant variance, transformation is needed, try log and cox transformation.
- If ACF PACF are not clean, probably the model is underspecified.
- Outliers: check data, treat as missing and replaced using mean or interpolation ect.

Use model selection criteria for order specification

- Akaike Information Criterion (AIC)

$$AIC = ln\hat{\sigma}^2 + \frac{2(p+q)}{T}.$$

- Bayesian Information Criterion (BIC)

$$BIC = ln\hat{\sigma}^2 + \frac{(p+q)ln(T)}{T}.$$

- AICc, DIC, MDL ect.

ARMA model forecast

- The underlying governing rule does not change and future behavior can be induced from the past
- Assume we have the observed series $x_1, \cdots, x_n$, where x_n is the current state. Our **Goal** is to forecast h steps ahead. Namely, estimate the random variable X_{n+h}.
  Denote $X_n(h)$ as forecast of X_{n+h} made at time n.
  Examples: $X_n(1)$, $X_n(5)$, $X_{n+2}(2)$.
- Our forecast: $X_n(h) = E(X_{n+h}|\mathcal{F}_n)$. Namely, onditional expectation given past events.

Prediction for AR(1)

- Consider an AR(1) model with mean $E(X_t) = \mu$

$$X_t = c + \phi_1 X_{t-1} + w_t,$$

where $w_t \sim N(0, \sigma^2)$ and $c = (1 - \phi_1)\mu$.
Since $X_{n+1} = c + \phi_1 X_n + w_{n+1}$, the one step ahead forecast is

$$\begin{aligned} X_n(1) &= E(X_{n+1}|x_1, \cdots, x_n) = E(c + \phi_1 X_n + w_{n+1}|x_1, \cdots, x_n) \\ &= E(c + \phi_1 x_n + w_{n+1}) = c + \phi_1 x_n. \end{aligned}$$

- You can also compute the prediction error.

Prediction for AR(1)

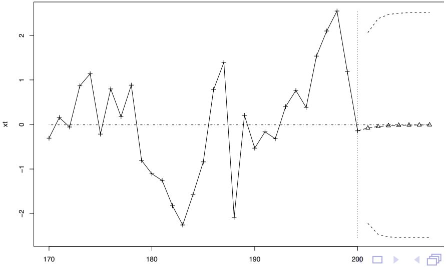
- Two steps forecast.
  $X_{n+2} = c + \phi_1 X_{n+1} + w_{n+2}$.

$$\begin{aligned} X_n(2) &= E(X_{n+2}|x_1, \cdots, x_n) = E(c + \phi_1 X_{n+1} + w_{n+2}|x_1, \cdots, x_n) \\ &= c + \phi_1 E(X_{n+1}|x_1, \cdots, x_n) \\ &= c + \phi_1 X_n(1). \end{aligned}$$

In general $X_n(h) = c + \phi_1 X_n(h-1)$.

Example

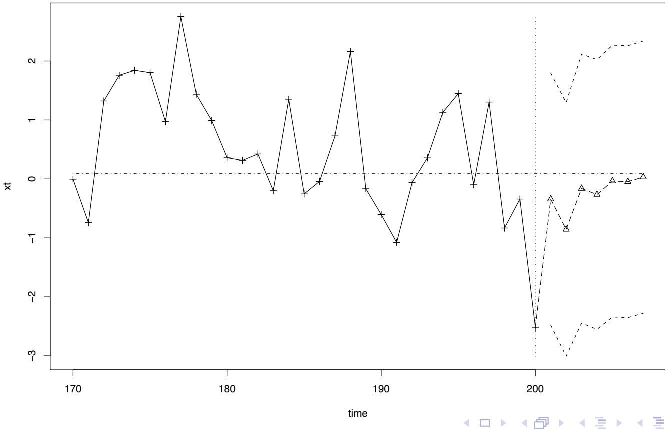
```
x=arima.sim(n = 200, list(ar = c(0.5)), sd = 1)
out=arima(x,c(1,0,0))
pp=predict(out,7)
print(t(cbind(pp$pred,pp$se)))
              [,1]       [,2]       [,3]       [,4]       [,5]
pp$pred -0.847106 -0.3918024 -0.2172091 -0.1502585 -0.1245852
pp$se    1.016656  1.0888409  1.0990556  1.1005497  1.1007692
               [,6]       [,7]
pp$pred -0.1147404 -0.1109652
pp$se    1.1008015  1.1008062
```



AR(2) Example

- Example for predicting AR(2) model. Data generated using

```
x=arima.sim(n = 200, list(ar = c(0.2,0.3)), sd = 1)
```



Prediction for MA(1) models

- MA(1) model with $E(X_t) = \mu$ and $w_t \sim N(0, \sigma^2)$.

$$X_t = c + w_t + \theta_1 w_{t-1}, \quad \text{where} \quad c = \mu.$$

$$X_n(1) = E(c + w_{n+1} + \theta_1 w_n | x_1, \cdots, x_n) = c + \theta_1 E(w_n | x_1, \cdots, x_n).$$

- What is $X_n(2)$?

Prediction for MA models

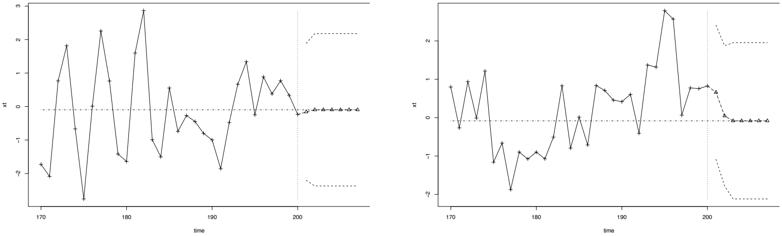


Figure: Prediction and prediction interval for MA(1) and MA(2)

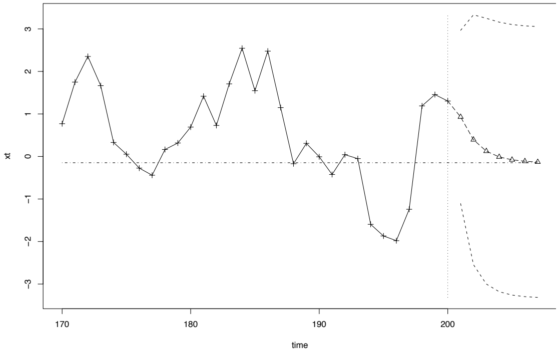
Prediction for ARMA(1,1) models

- The ARMA(1,1)

$$X_t = c + \phi_1 X_{t-1} + w_t + \theta_1 w_{t-1}, \quad c = (1-\phi_1)\mu.$$

In general we have

$$X_n(h) = c + \phi_1 X_n(h-1).$$



Details of arima in R

```
arima(x, order = c(0, 0, 0), seasonal = list(order = c(0, 0, 0),
period = NA), xreg = NULL, include.mean = TRUE,
method = c(CSS-ML, ML, CSS))
```

- x: a univariate time series
- order: A specification of the non-seasonal part of the ARIMA model: the three components (p, d, q) are the AR order, the degree of differencing, and the MA order.
- seasonal: A specification of the seasonal part of the ARIMA model, plus the period (which defaults to frequency(x)). This should be a list with components order and period, but a specification of just a numeric vector of length 3 will be turned into a suitable list with the specification as the order.
- xreg: Optionally, a vector or matrix of external regressors, which must have the same number of rows as x.

- include.mean: Should the ARMA model include a mean/intercept term? The default is TRUE for undifferenced series, and it is ignored for ARIMA models with differencing.
- method: Fitting method: maximum likelihood or minimize conditional sum-of-squares. The default (unless there are missing values) is to use conditional-sum-of-squares to find starting values, then maximum likelihood.

```
> out=arima(x,c(2,0,2))
> out
Call:
arima(x = x, order = c(2, 0, 2))
Coefficients:
         ar1     ar2     ma1     ma2  intercept
      0.0604  0.8760  0.6296  0.4787     0.8578
s.e.  0.0372  0.0374  0.0662  0.0627     2.0488
sigma^2 estimated as 0.9769:  log likelihood = -283.93,  a
```