Ergodicity of the solution of Lorenz Equation

Lokit Khemka

August 11, 2020

For the given set of ODEs, which are called Lorenz equations:

$$\frac{dx}{dt} = 10(x - y) \tag{1a}$$

$$\frac{dy}{dt} = x(28 - y) - y \tag{1b}$$

$$\frac{dz}{dt} = xy - \frac{8}{3}z \tag{1c}$$

and the initial conditions of $(x, y, z) = (1, 1, 1)$, the trajectory of the solution is as given below:

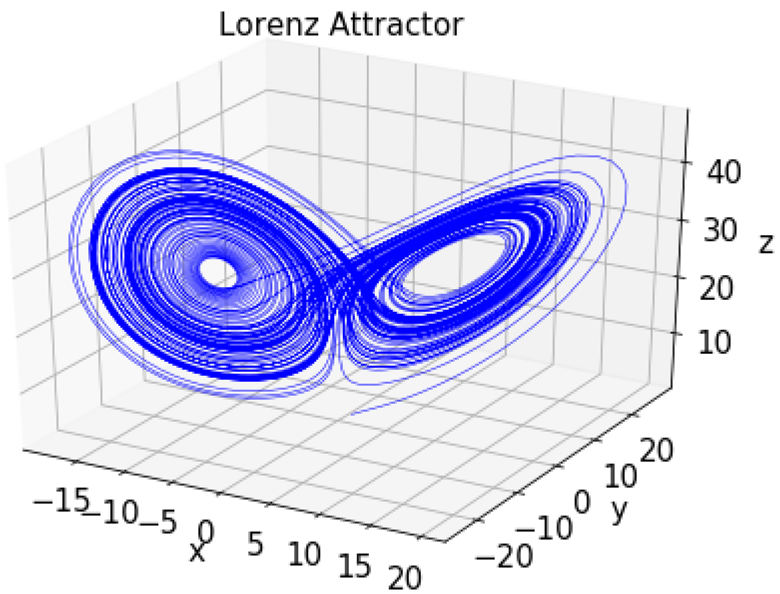

Figure 1: Solution Trajectory of the Lorenz Equations with Initial Conditions $(x, y, z) = (1, 1, 1)$

This structure is called 'Lorenz Attractor'. For the second part of the question, the joint probability density functions of (x, y, z) for a one particular time-step with randomly sampled initial conditions are as follows:

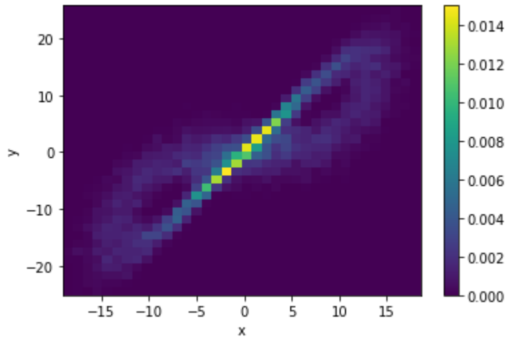

Figure 2: Joint PDF of (x, y) at one particular time step

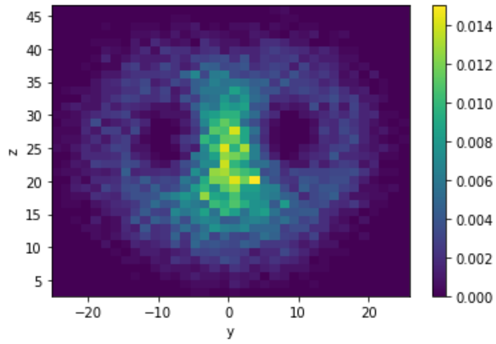

Figure 3: Joint PDF of (y, z) at one particular time step

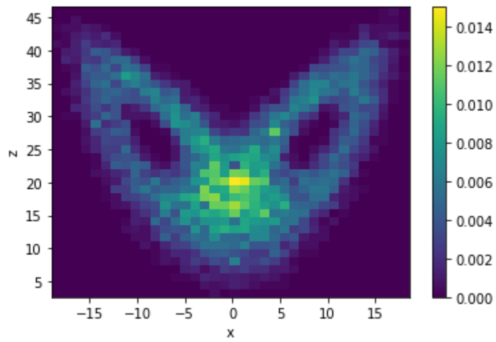

Figure 4: Joint PDF of (x, z) at one particular time step

The joint PDF of (x, y, z) are same for all time steps. Further, it should be noted that the joint PDF of (x, y, z) at any particular time step looks like the projection of the Lorenz attractor at the corresponding planes. Thus, the solution of the Lorenz equations is ergodic.