Math 128A - Programming Assignment #1

Math 128A, Fall 2018

Woo Sik (Lewis) Kim, SID: 25544445

The programs were written in Python.

To run the code for Q1, run `find_extrema(a, b, c, d, e, f)`. `test_q1()` provides solutions to the given test cases. The polynomials were graphed using Desmos.

To run the code for Q2, run `nested_radicals(n)`, where `n` is the nth term of the sequence. `test_q2()` provides solutions to the first 40 terms of the sequence. The code for the graph in Q2 is in `q2plot.py`.

Question 1

The algorithm for question 1 is straightforward; given 6 numbers, a, b, c, d, e, and f, we find the coefficients to the first derivative (a quadratic) of the cubic in the form $f(x) = cx^3 + dx^2 + ex + f$. Using these coefficients, we find the solution to $f'(x) = 0$ through the quadratic formula. If the roots are imaginary, it means there are no turning points in the cubic, and we are done (min/max depend only on a, b and evaluations $f(a), f(b)$). Otherwise, we have 2 solutions, x_1 and x_2. We plug these 2 values into the second derivative $f''(x)$ (determined by the coefficients $6c$ and $2d$) to see if x_1, x_2 are $< 0, = 0$, or > 0. Less than 0 implies local max, greater than 0 implies local min, and equal to 0 implies saddle point.

After determining which of x_1 and x_2 are local minimums and maximums, we check if either or both of x_1 and x_2 lie in $[a, b]$, and determine the local min/max from there (e.g. if both x_1 and x_2 are in $[a, b]$, then the local extrema occur at these 2 points. If neither are in it, then the local extrema depend on evaluations of $f(a), f(b)$ (smaller is min, bigger is max)). A more detailed breakdown of these cases are in `hw1.py`.

The edge cases for this algorithm are when c and/or d and/or e are 0. If $c = 0$ and d, e are not, then we simply have a quadratic, and we run an algorithm for finding the extrema for a quadratic by calculating the vertex $h = \frac{-b}{2a}$ and the concavity of the parabola.

If $c, d = 0$, then we have a simple line, in which case the local min/max depends entirely on whether or not the slope is positive or negative.

Case 1) $f(x) = -x^3 + 2x^2 - x + 1$ on $[-1, 2]$. $(a = -1, b = 2, c = -1, d = 2, e = -1, f = 1)$

- Location of min/max: $x_{min} = 0.333333$, $x_{max} = 1.0$.
- Evaluation at min/max: $p(x_{min}) = 0.851852$, $p(x_{max}) = 1.0$.

Case 2) $f(x) = x^3 - 2x - x + 1$ on $[1, 2]$. $(a = 1, b = 2, c = 1, d = -2, e = -1, f = 1)$

- Location of min/max: $x_{min} = 1.548584$, $x_{max} = 2$.
- Evaluation at min/max: $p(x_{min}) = -1.631130$, $p(x_{max}) = -1$.

Case 3) $f(x) = 4x^3 + 8x^2 - 4x - 2$ on $[-2, 1]$. $(a = -2, b = 1, c = 4, d = 8, e = -4, f = -2)$

- Location of min/max: $x_{min} = 0.215250$, $x_{max} = -1.548584$.
- Evaluation at min/max: $p(x_{min}) = -2.450447$, $p(x_{max}) = 8.524521$.

Case 4) $f(x) = x^3 + x - 3$ on $[-1, 2]$. $(a = -1, b = 2, c = 1, d = 0, e = 1, f = -3)$

- Location of min/max: $x_{min} = -1$, $x_{max} = 4$.
- Evaluation at min/max: $p(x_{min}) = -5$, $p(x_{max}) = 7$.

Case 5) $f(x) = 10^{-14}x^3 + 9x^2 - 3x$ on $[-0.3, 0.6]$. $(a = -0.3, b = 0.6, c = 10^{-14}, d = 9, e = -3, f = 0)$

- Location of min/max: $x_{min} = 0.177635$, $x_{max} = -0.3$.
- Evaluation at min/max: $p(x_{min}) = -0.248918$, $p(x_{max}) = 1.70999$ (most likely a floating point issue; $p(x_{max}) = 1.71$).

Case 6) $f(x) = 1.7$ on $[-1, 2]$ $(a = -1, b = 2, c = d = e = 0, f = 1.7)$

- Location of min/max: $x_{min} = -1$, $x_{max} = 2$.
- Evaluation at min/max: $p(x_{min}) = 1.7$, $p(x_{max}) = 1.7$.

Case 7) $f(x) = -3x^3 + 9x^2 - 10^{-14}x$ on $[0, 3]$. $(a = 0, b = -3, c = -3, d = 9, e = -10^{-14}, f = 0)$

- Location of min/max: $x_{min} = 5.921189464667501e - 30$, $x_{max} = 1.999999$ (most likely a floating point issue; $x_{max} = 2$.
- Evaluation at min/max: $p(x_{min}) = -2.7657458437834548e - 30$, $p(x_{max}) = 11.99999$ (most likely a floating point issue; $p(x_{max}) = 12$).

Case 8) $f(x) = -2x^2 + 3x - 1$ on $[0, 1]$. $(a = 0, b = 1, c = 0, d = -2, e = 3, f = -1)$

- Location of min/max: $x_{min} = 0$, $x_{max} = 0.75$.
- Evaluation at min/max: $p(x_{min}) = -1$, $p(x_{max}) = 0.125$.

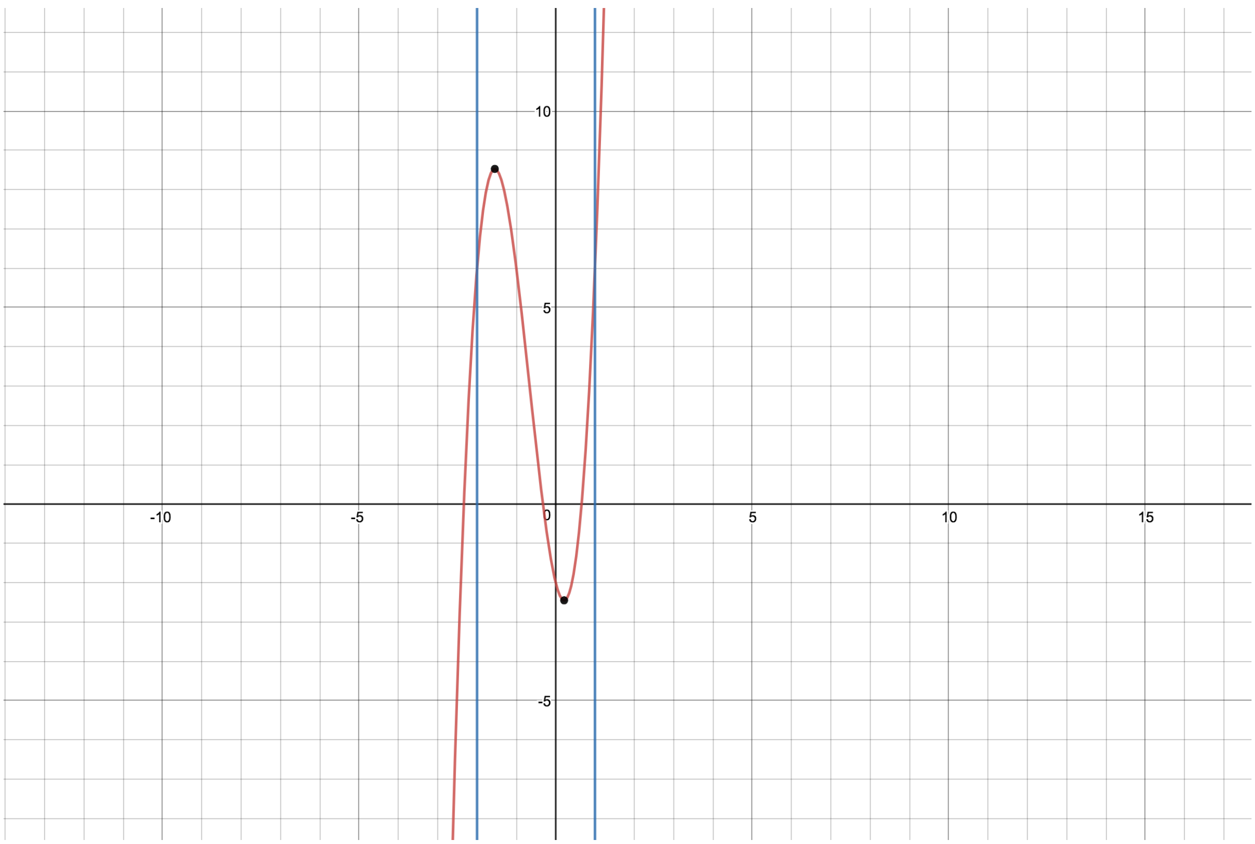

Figure 1: $f(x) = 4x^3 + 8x^2 - 4x - 2$ on $[-2, 1]$. Extrema are marked with a black dot. Blue lines represent the bounds.

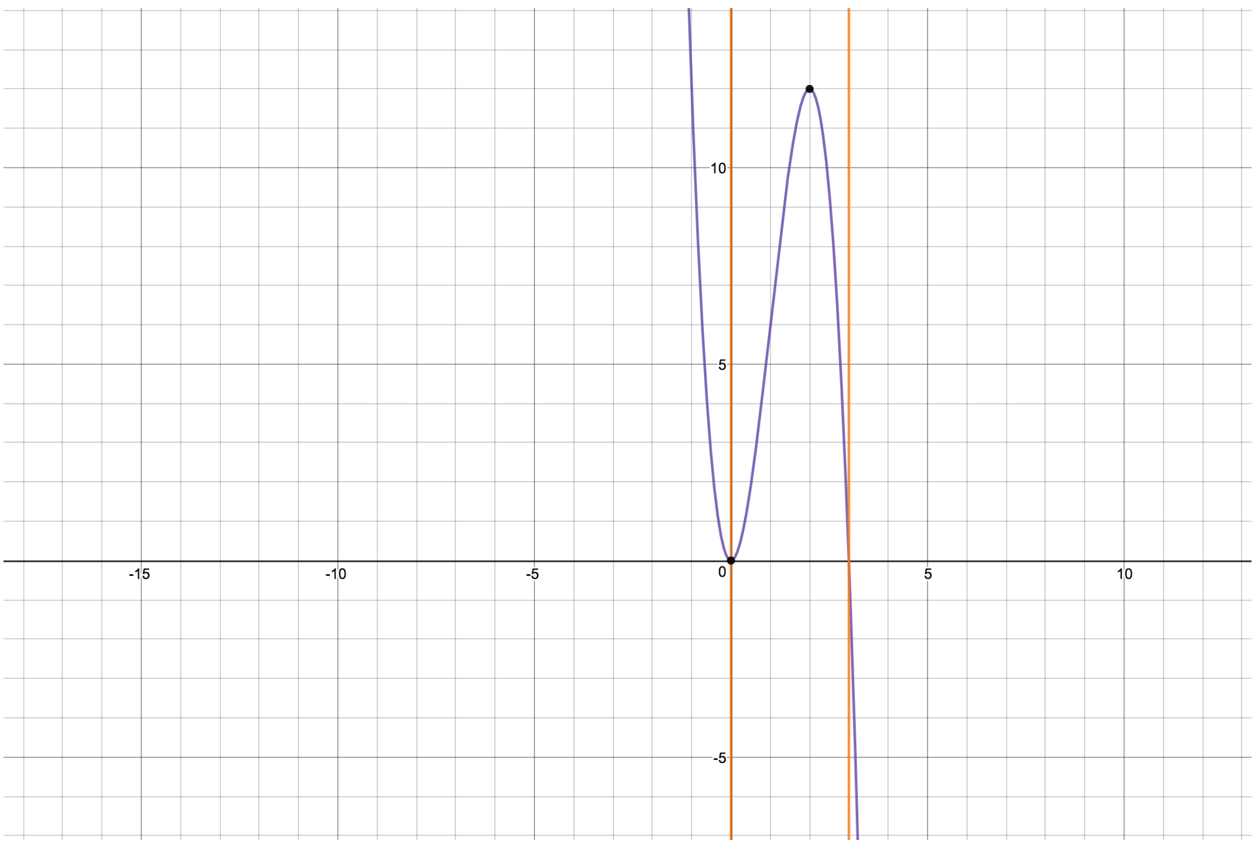

Figure 2: $f(x) = -3x^3 + 9x^2 - 10^{-14}x$ on $[0, 3]$. Extrema are marked with a black dot. Orange lines represent the bounds.

Question 2

The algorithm for question 2 is an iterative evaluation (iterating backwards from n). We reassign a value `nested_sum` to an inner square root each step, i.e.:

```
nested_sum = 1
for k in range(1, n, -1):
  nested_sum = math.sqrt(1 + k*nested_sum)
```

We evaluate a_n for $1 \leq n \leq 40$:

$a_1 = 1$

$a_2 = 1.7320508075688772$

$a_3 = 2.23606797749979$

$a_4 = 2.559830165300118$

$a_5 = 2.755053261329896$

...

$a_{38} = 2.9999999999454037$

$a_{39} = 2.9999999999725286$

$a_{40} = 2.99999999998618$

From the first 40 terms of the sequence a_n, it appears that $\lim_{n\to\infty} a_n = 3$.

Graph of $\ln(|a_n - a|)$ *vs.* n *and* $y = 3 - (\ln(3))n$ *is on the next page.*

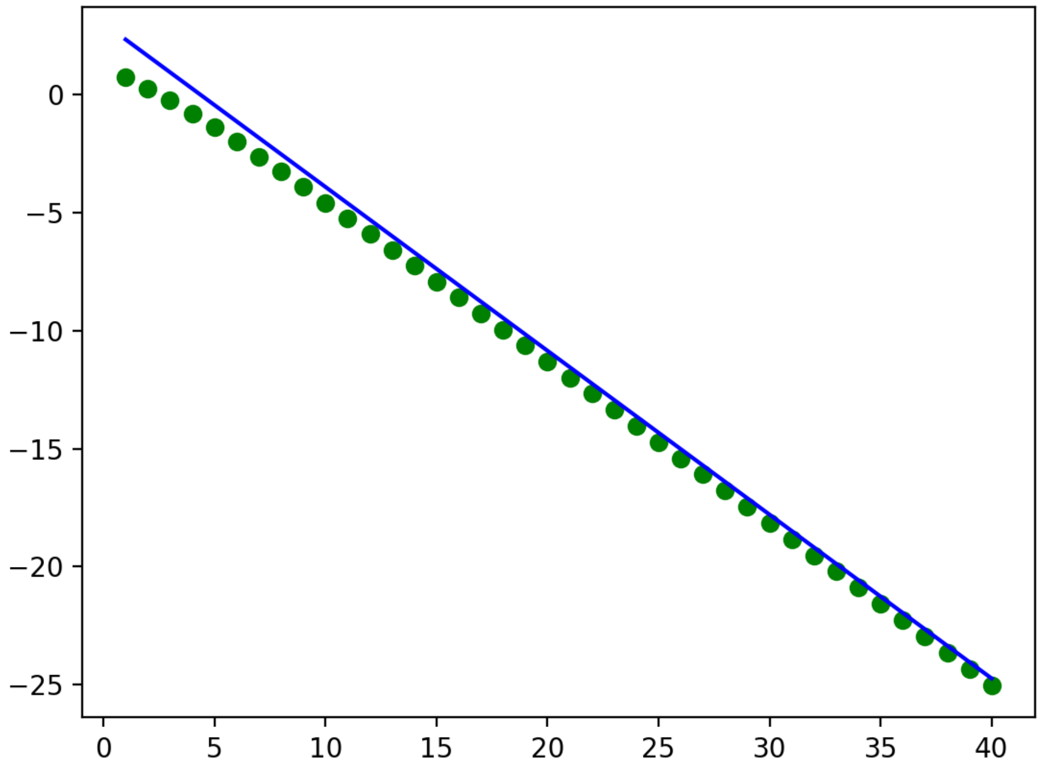

Figure 3: Green: Graph of $\ln(|a_n - a|)$ vs. n. Blue: $y = 3 - (\ln(3))n$.

From the graph, it appears that $\beta_n = 3 - (\ln(3))n$, i.e. $\beta_n = y$ is appropriate for the upper bound $O(\beta_n) = a_n - a$.

We can verify this by representing $\ln(|a_n - a|)$ as a continuous function f, letting $g(x) = 3 - (\ln(3))x$, and calculating $\lim_{n\to\infty} \frac{f(x)}{g(x)}$. Doing so gives us a limit of 0, which, by asymptotic analysis, shows that $f = O(g)$.