Introduction of Logistic map and Lyapunov exponent

The mathematical function known as the logistic map is frequently employed as a straightforward model for population increase or other dynamical systems that display chaotic behavior. It appears as follows:

$X_{n+1} = rX_n (1 - X_n)$

Where r is a parameter that governs the growth rate, X_n represents the population at time n, and X_{n+1} represents the population at the subsequent time step.

The logistic map has various fascinating qualities, one of which is that for some values of the growth rate parameter r, it can behave chaotically. This means that over time, even minor modifications to the starting circumstances or the value of r can have drastically different results.

Lyapunov exponents are a way of quantifying the degree of chaos in a dynamical system like the logistic map. They measure how quickly nearby trajectories diverge over time, and can be used to determine whether a system is chaotic, periodic, or stable. Positive Lyapunov exponents indicate chaos, while negative ones indicate stability.

In the case of the logistic map, the Lyapunov exponent varies depending on the value of the growth rate parameter r. When r is below a certain threshold, the Lyapunov exponent is negative and the system is stable. When r is above this threshold, the Lyapunov exponent becomes positive and the system becomes chaotic. This threshold is known as the "chaos threshold" or "Feigenbaum point", and is one of the key features of the logistic map.

Explanation of Chaos by logistic map

Chaos in the logistic map arises due to its sensitivity to initial conditions and the nonlinear dynamics involved.

For certain values of the growth rate parameter r, the logistic map exhibits chaotic behavior. This means that even small changes in the initial population value can lead to drastically different outcomes over time. This happens because the logistic map is a nonlinear function, which means that small changes in the input can lead to large changes in the output.

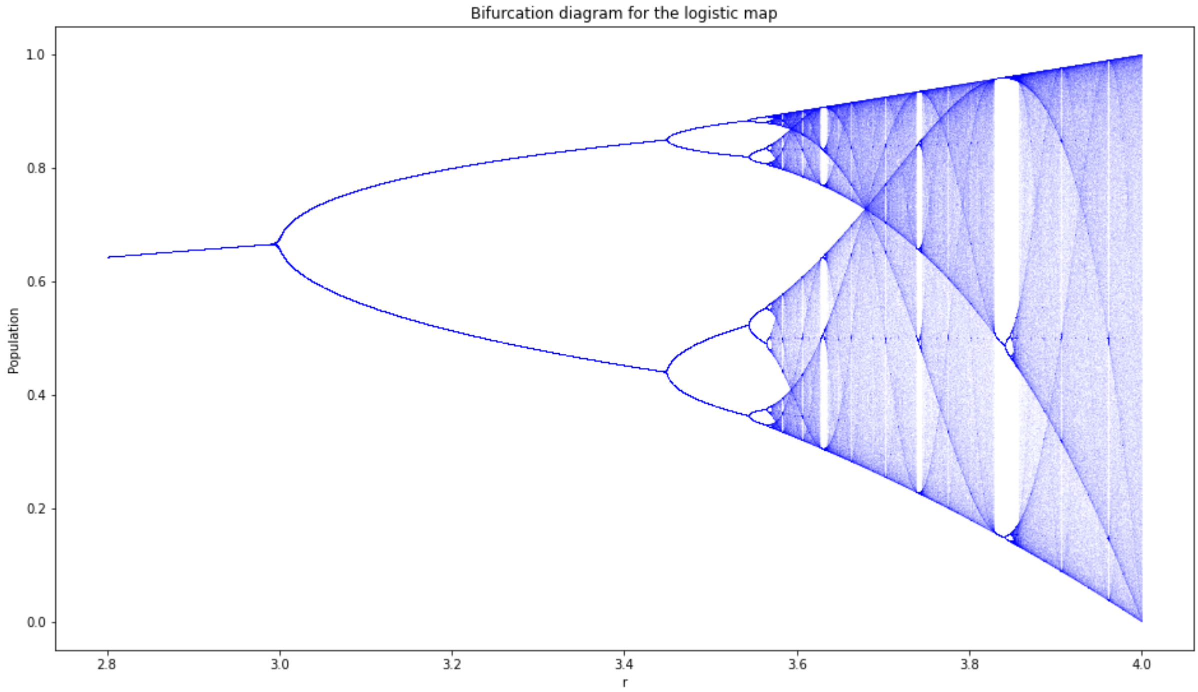
As the population evolves over time, it can exhibit complex patterns that appear random and unpredictable. This is because the logistic map is capable of generating aperiodic (non-repeating) behavior, which is one of the hallmarks of chaotic systems. The chaotic behavior of the logistic map is also characterized by the presence of multiple attractors, or sets of values that the system tends to approach over time.

Lyapunov exponents are used to quantify the degree of chaos in the logistic map. When the Lyapunov exponent is positive, it indicates that nearby trajectories in phase space (i.e. the space of all possible values of the population) diverge exponentially, which is a hallmark of chaotic systems. In contrast, when the Lyapunov exponent is negative, nearby trajectories converge over time and the system is stable.

Overall, the chaotic behavior of the logistic map is an example of how simple mathematical models can exhibit complex and unpredictable dynamics. The logistic map is a classic example of a system that can exhibit chaos, and it has been used to study a wide range of phenomena in fields such as biology, economics, and physics.

Now that that we introduced the subject theoretically, we are going to use the code we wrote to show the followings:

Bifurcation diagram



Critical point

The critical point of the logistic map is the point where the dynamics of the system undergo a bifurcation, resulting in the emergence of chaotic behaviour.

To calculate the critical point, we can use the fact that the critical point occurs when the derivative of the logistic map with respect to the population variable x is equal to 1.

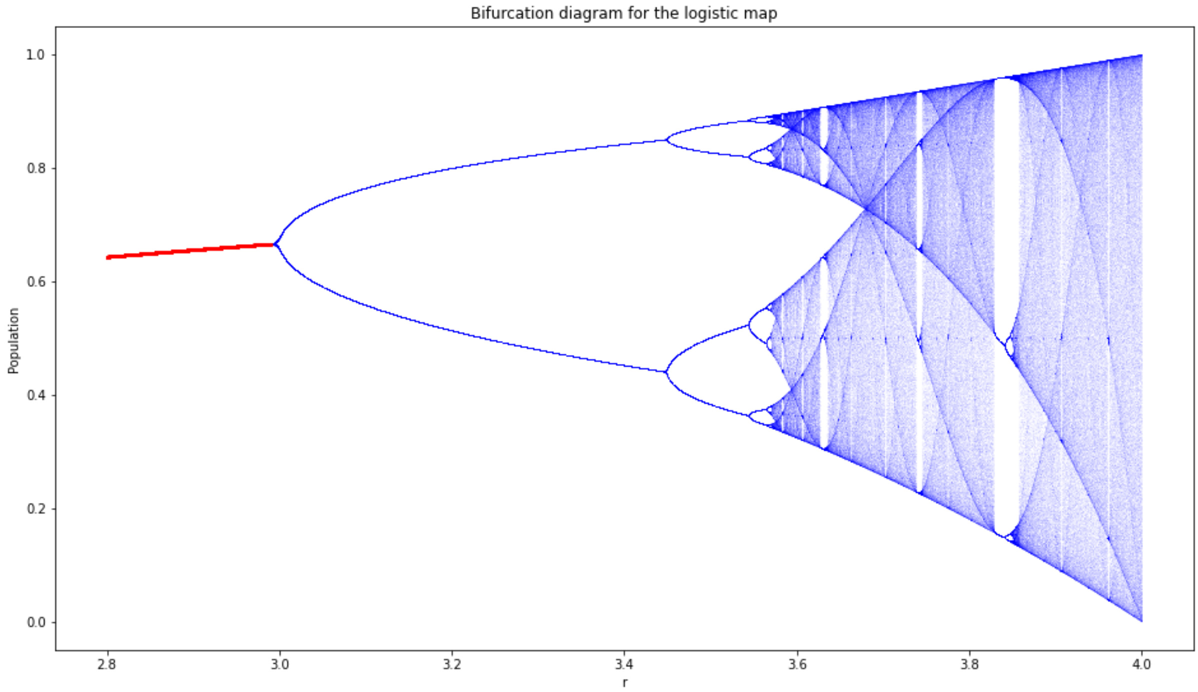
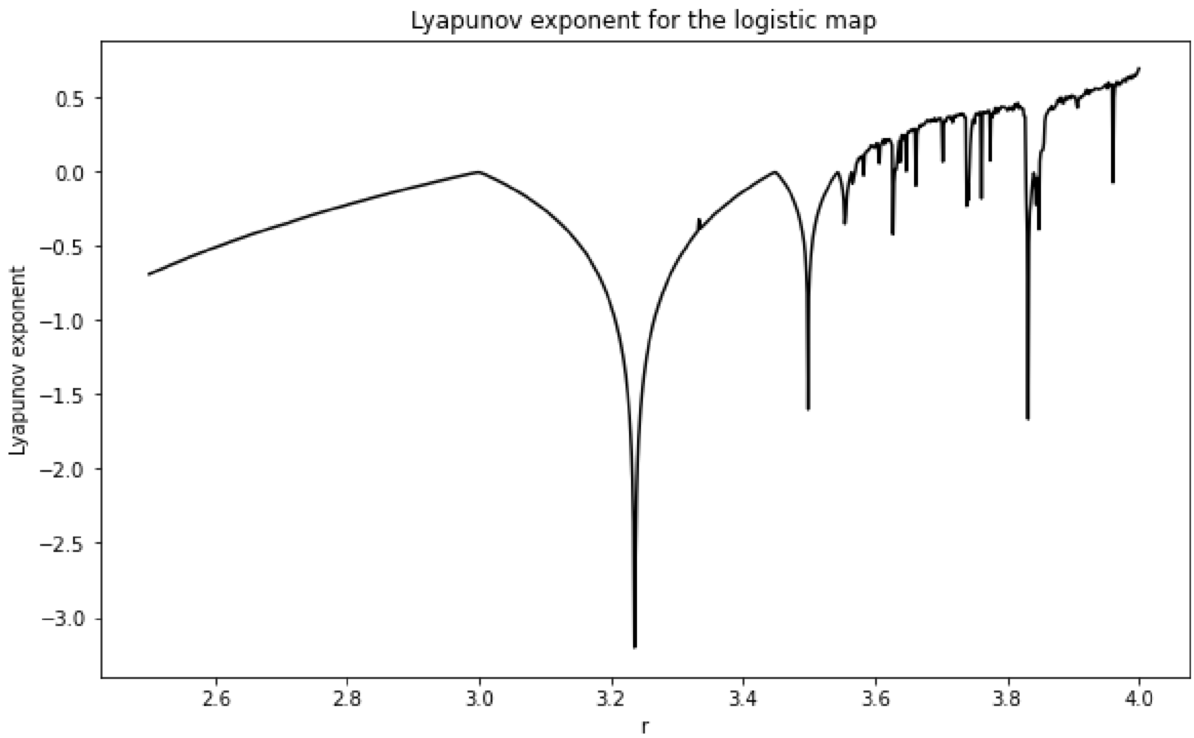


Diagram for Lyapunov exponent



The code generated a plot showing the Lyapunov exponent as a function of the parameter r for the logistic map. The Lyapunov exponent measures the rate of divergence of nearby trajectories in the phase space, and is a key indicator of the chaotic behavior of the system. In the logistic map, the system undergoes a period-doubling bifurcation at a critical value of r, beyond which it exhibits chaotic behavior.