4

FUZZY NUMBERS

4.1 INTRODUCTION

Fuzzy numbers are fuzzy sets used in connection with applications where an explicit representation of the ambiguity and uncertainty found in numerical data is desirable. In an intuitive sense, they are fuzzy sets representing the meaning of statements such as "*about* 3" or "*nearly five and a half*." In other words, fuzzy numbers take into account the "about," "almost," and "not quite" qualities of numerical labels. Fuzzy set operations such as *union* and *intersection*, as well as the notions of *α-cuts*, *resolution*, and the *extension principle* (Chapter 2), are all applicable to fuzzy numbers. In addition, a set of operations very similar to the familiar operations of arithmetic, *addition*, *subtraction*, *multiplication*, and *division* can be defined for fuzzy numbers as well. In this chapter we look at such operations and examples of their use. Fuzzy numbers have been successfully applied in expert systems, fuzzy regression, and fuzzy data analysis methodologies (Kaufmann and Gupta, 1991; Terano et al., 1992). Fuzzy numbers have also been used in connection with fuzzy equations, and alternative operations of fuzzy arithmetic have been introduced for the purpose of reducing fuzziness in successive computations (Sanchez, 1993).

The universe of discourse on which fuzzy numbers are defined is the set of real numbers and its subsets (e.g., integers or natural numbers), and their membership functions ought to be *normal* and *convex*. We recall from Section 2.3 that a fuzzy set is called *normal* if there is at least one point in the universe of discourse where the membership function reaches unity [equation (2.3-11)]. But what is a "convex" fuzzy set? The intuitive meaning

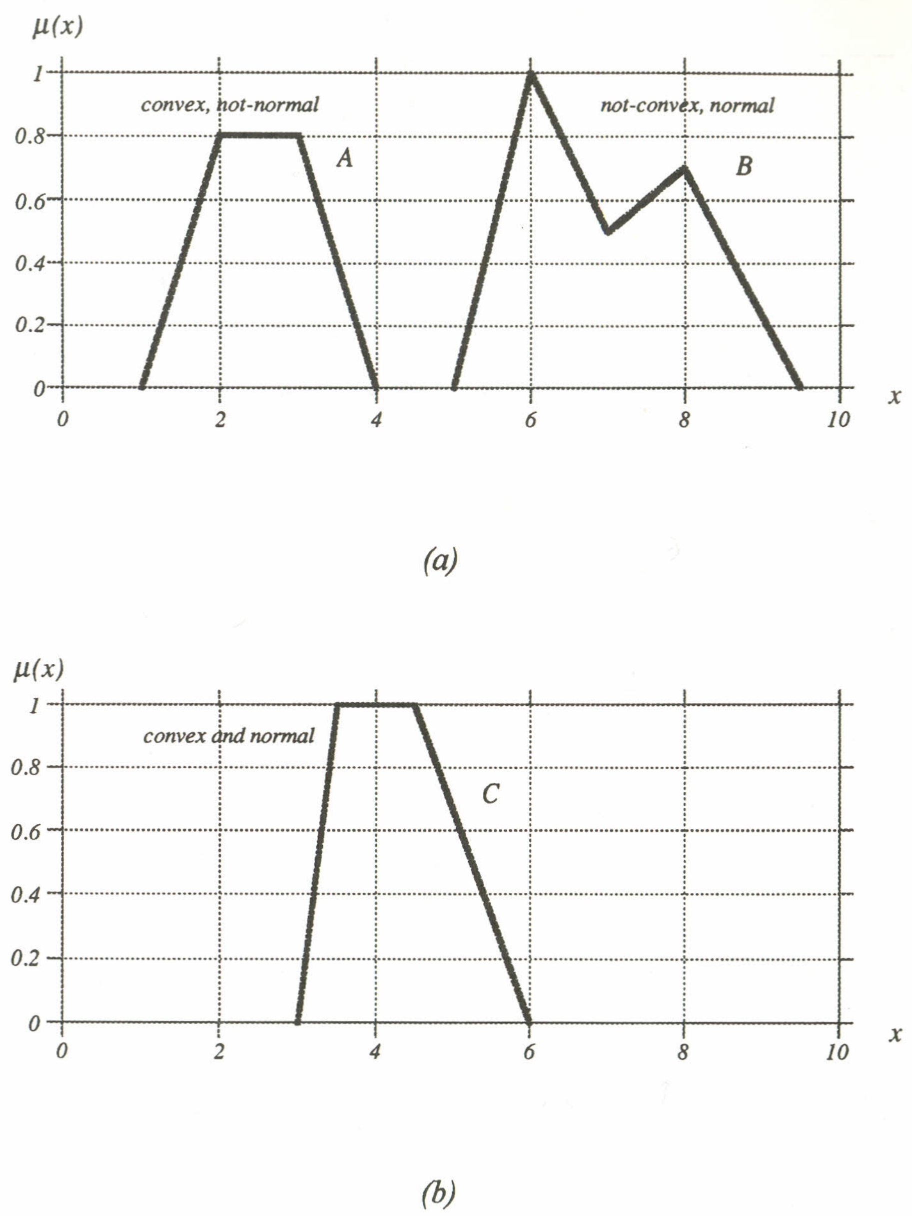


Figure 4.1 (*a*) Two fuzzy sets that *cannot* be used as fuzzy numbers. (*b*) A fuzzy set that may be used as fuzzy number.

of *convexity*[1] is that the membership function of a convex fuzzy set does not go "up-and-down" more than once. Consider, for example, the fuzzy sets A and B shown in Figure 4.1*a*. Fuzzy set A is *convex* but *not normal* since nowhere in the universe of discourse does its membership function reach unity. Therefore it is not a fuzzy number. Fuzzy set B is *normal* but *not convex* since its membership function goes "up-and-down" twice, and hence it is also not a fuzzy number. On the other hand, consider the set C shown in Figure 4.1*b*. It is both *normal* and *convex* and therefore may be considered a fuzzy number. We will see in following sections that changing the shape of a membership function results in a different number. "Shape" is what fuzzy numbers are all about, and fuzzy arithmetic may be thought of as a way of computing with "shapes" (areas) instead of "points" (we consider crisp numbers as "points").

Fuzzy numbers may also be defined on a multidimensional universe of discourse that is a Cartesian product. Such fuzzy numbers are used, for example, in connection with scene analysis and robotics to define the meaning of a region in space, or a domain on the x–y plane, and also to add, subtract, and multiply regions (Pal and Majumder, 1986). In this chapter, however, we consider fuzzy numbers defined on a simple, one-dimensional universe of discourse. A very comprehensive treatment of fuzzy numbers, including multidimensional ones, may be found in the book entitled *Introduction to Fuzzy Arithmetic* by Kaufmann and Gupta (1991).

4.2 REPRESENTING FUZZY NUMBERS

We denote fuzzy numbers by boldfaced italics—for example, ***3*** or ***A***—or by referring to their membership function. As we said earlier, fuzzy numbers are fuzzy sets used to represent the "*about*," "*almost*," or "*nearly*" qualities of numerical data. We observe, however, that there are many possible meanings to a statement such as "*about* 3." Therefore, several different sets may be used to represent "*about* 3." In the context of fuzzy arithmetic operations, however, at any given time we use only one meaning, chosen on the basis of application-specific criteria and needs. Figure 4.2*a* shows a triangular membership function representing the fuzzy number ***3***. Another possible representation is the bell-shaped membership function in Figure 4.2*b*. These are two different ***3***'s. If we start a computation using the triangular ***3***, we cannot halfway through switch to the bell-shaped ***3***. Note that on both instances the shape of the membership function meets our *normal* and *convex* require-

[1]The notion of convexity is derived through references to geometrical objects. A body Ω in Euclidean space is called *convex* if the line segment joining any two points of Ω lies in Ω. Examples of convex bodies in three-dimensional space are the sphere, the ellipsoid, a cylinder, a cube, and a cone.

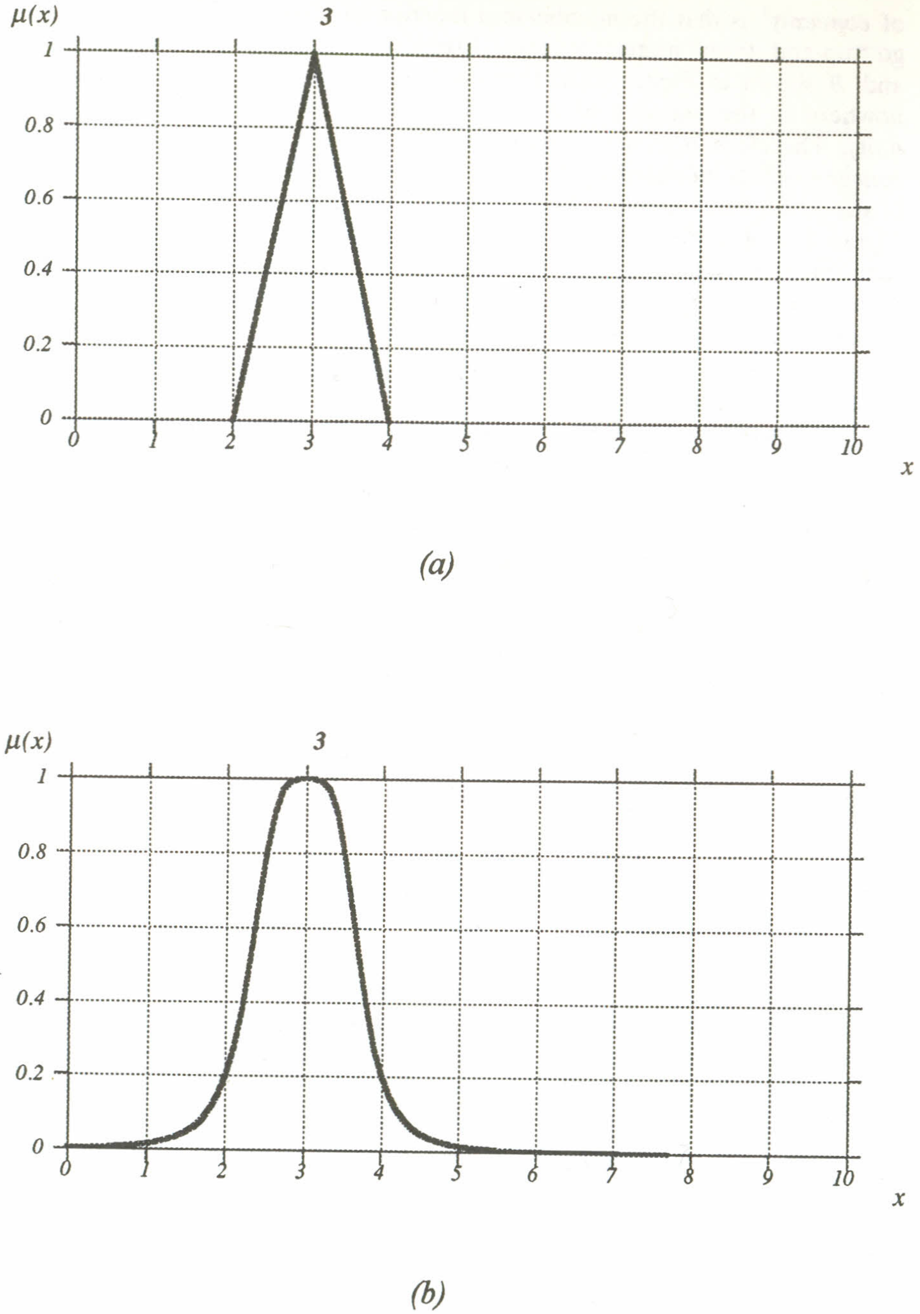


Figure 4.2 Two different fuzzy numbers: (*a*) triangular ***3*** and (*b*) bell-shaped ***3***.

Table 4.1 Tabular representation of a fuzzy number *3*

	0.4	0.7	1	0.7	0.4	0.2	0.1	0	0
$\alpha=1$			1						
$\alpha=0.9$			1						
$\alpha=0.8$			1						
$\alpha=0.7$		1	1	1					
$\alpha=0.6$		1	1	1					
$\alpha=0.5$		1	1	1					
$\alpha=0.4$	1	1	1	1	1				
$\alpha=0.3$	1	1	1	1	1				
$\alpha=0.2$	1	1	1	1	1	1			
$\alpha=0.1$	1	1	1	1	1	1			
$\alpha=0$	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9

ments. Another possible fuzzy number ***3*** is shown in Table 4.1, where the shaded cells, the 1's, indicate the shape of the number. Here ***3*** is defined over the universe of natural numbers shown at the bottom of the table. In the leftmost column we list the values of a parameter, α, ranging between 0 and 1, used to parametrize the shape of the function (Kaufmann and Gupta, 1991). In fact, this is the same α we saw in connection with α-cuts (see Section 2.6). The α-cuts of fuzzy numbers are very useful in fuzzy arithmetic operations. Looking at Table 4.1 we see that the grade of membership of crisp number 4 to the fuzzy number ***3*** is 0.7, and the grade of membership of crisp 3 is 1.0. Although the fuzzy numbers shown in Figure 4.2 and Table 4.1 are all different, we designate them with the same symbol (i.e., ***3***) since they all peak at crisp 3 (Zimmermann, 1985; Kandel, 1986).

Fuzzy numbers, like any fuzzy set, may be represented by its α-cuts. We saw in Chapter 2 that a membership function may be parameterized by a parameter α in a manner similar to the tabular representation of number ***3*** shown in Table 4.1. The parameter α is a number between 0 and 1 (i.e., in the interval [0, 1]). Parameterizing the shape of a fuzzy number by α offers a convenient way for computing with fuzzy numbers because it essentially transforms fuzzy arithmetic operations into operations of interval arithmetic. It is easy to see what we are talking about by looking at Table 4.1. At each level α we have a horizontal "slice," or interval of the membership function, which is its α-cut. For example, at $\alpha = 0.5$, the α-cut is the interval from 2 to 4, and at $\alpha = 0.2$ it is the interval from 1 to 6. The tabular representation exemplifies the length of each α-cut; that is, it shows the number of cells and thus the length of the membership function at level α.

Consider the fuzzy number A shown in Figure 4.3. The membership function of A is parameterized by the parameter α. With each α we identify an interval $[a_1^{(\alpha)}, a_2^{(\alpha)}]$. As may be seen from the figure, we indicate by $a_1^{(\alpha)}$ the left endpoint of the interval ("left" is denoted by the subscript "1") and by $a_2^{(\alpha)}$ the right endpoint of the interval ("right" is denoted by the subscript "2"). Requiring that the membership function of a fuzzy number be *convex* and *normal* is another way of saying that the intervals that comprise the interval representation of A should be nested into one another as we move from the bottom of the membership function to the top (Klir and Folger, 1988; Terano et al., 1992). In other words, when $\alpha_1 < \alpha_2$, as shown in Figure 4.3, we have

$$\left[a_1^{(\alpha_2)}, a_2^{(\alpha_2)}\right] \subset \left[a_1^{(\alpha_1)}, a_2^{(\alpha_1)}\right] \tag{4.2-1}$$

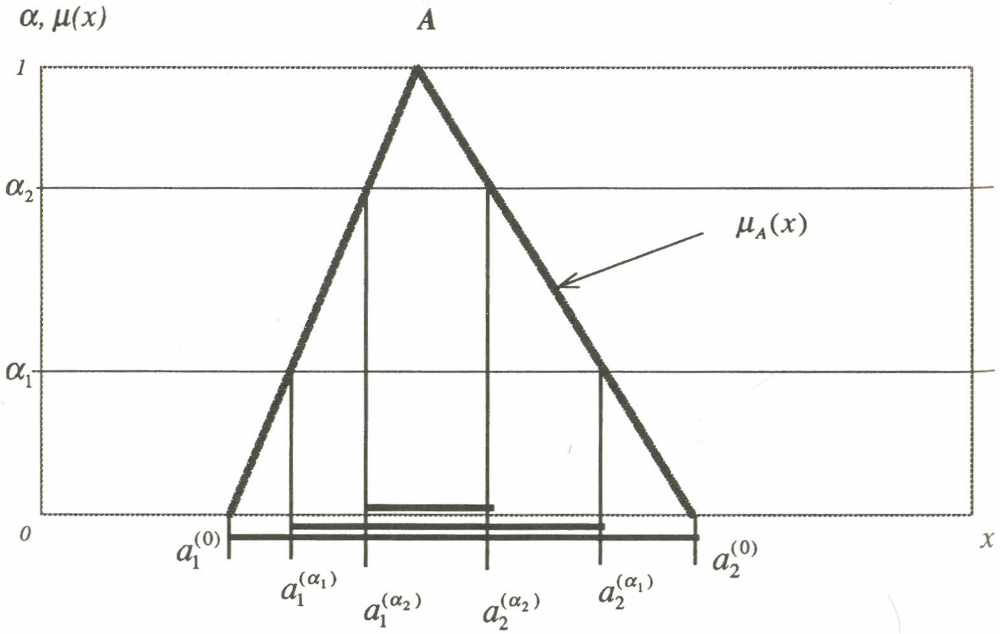


Figure 4.3 Nested intervals (α-cuts) associated with a fuzzy number A.

where the symbol $\subset$ denotes that the interval $[a_1^{(\alpha_2)}, a_2^{(\alpha_2)}]$ is contained within the interval $[a_1^{(\alpha_1)}, a_2^{(\alpha_1)}]$.

We can uniquely describe two fuzzy numbers A and B as two collection of intervals i.e., $[a_1^{(\alpha)}, a_2^{(\alpha)}]$ and $[b_1^{(\alpha)}, b_2^{(\alpha)}]$ respectively. We recall that the α-cuts of A and B (Section 2.6) were defined as the crisp sets

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\} \tag{4.2-2}$$

and

$$B_\alpha = \{x \mid \mu_B(x) \geq \alpha\} \tag{4.2-3}$$

The α-cuts in equations (4.2-2) and (4.2-3) are simply intervals on the x axis, and hence for each α we have

$$A_\alpha = \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] \tag{4.2-4}$$

and

$$B_\alpha = \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] \tag{4.2-5}$$

Thus the fuzzy numbers A and B can be described (using the resolution principle—see Section 2.7) as collections of intervals, that is,

$$A = \bigvee_{0 \leq a \leq 1} \alpha \cdot A_\alpha = \bigvee_{0 \leq a \leq 1} \alpha \cdot \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] \tag{4.2-6}$$

and

$$B = \bigvee_{0 \leq a \leq 1} \alpha \cdot B_\alpha = \bigvee_{0 \leq a \leq 1} \alpha \cdot \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] \tag{4.2-7}$$

To simplify matters, we will not use the rather awkward representation of the two numbers given by equations (4.2-6) and (4.2-7) but will, instead, use equations (4.2-4) and (4.2-5), which we call the *α-cut* or *interval representation* of A and B (with the understanding that the number is the collection of all "slices," all α-cuts as α varies from 0 to 1).

Having two different ways of representing fuzzy numbers, through *membership functions* and through *α-cuts* or *intervals*, gives us the choice of defining arithmetic operations either through the extension principle (i.e., through a fuzzification of arithmetic operations on crisp numbers) or, equivalently, through the operations of interval arithmetic. This last approach is often more practical and straightforward as we will see in several examples.

Let us go next to the definition of *addition*, *subtraction*, *multiplication*, and *division with fuzzy numbers*. Although we will define operations for two numbers A and B, they are generally true for more than two numbers. A word of caution: Some of the properties of crisp numbers—for example,

$(7 \div 3) \times 3 = 7$—may not be valid for arithmetic operations involving fuzzy numbers. We will see that usually when fuzzy numbers are involved we have that $(\mathit{7} \div \mathit{3}) \times \mathit{3}$ may not equal $\mathit{7}$.

4.3 ADDITION

When *adding* two fuzzy numbers $\boldsymbol{A}$ and $\boldsymbol{B}$ we seek to compute a new fuzzy number $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$. The new number $\boldsymbol{C}$ is uniquely described when we obtain its membership function, $\mu_C(z) \equiv \mu_{A+B}(z)$, with z being the crisp sum of x and y, the elements of the universe of discourse of $\boldsymbol{A}$ and $\boldsymbol{B}$. The addition of $\boldsymbol{A}$ and $\boldsymbol{B}$ may be defined in terms of addition of the α-cuts of the two numbers as follows:

$$\boldsymbol{A} + \boldsymbol{B} \equiv \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] + \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] \tag{4.3-1}$$

where $[a_1^{(\alpha)}, a_2^{(\alpha)}]$ is the collection of intervals representing the fuzzy number $\boldsymbol{A}$, and $[b_1^{(\alpha)}, b_2^{(\alpha)}]$ is the collection of intervals representing the fuzzy number $\boldsymbol{B}$. Intervals are added by adding their corresponding left and right endpoints, and therefore equation (4.3-1) becomes

$$\boldsymbol{A} + \boldsymbol{B} = \left[a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}\right] \tag{4.3-2}$$

Equation (4.3-2) indicates that the new number is also a collection of intervals with endpoints obtained from the endpoints of $\boldsymbol{A}$ and $\boldsymbol{B}$.

Another way of defining fuzzy addition is through the *extension principle* (Section 2.5). We give here a cursory description of how this is done; more detailed treatments may be found in Dubois and Prade (1980) and in Terano et al. (1992). Suppose we want to add two crisp numbers x and y. The result is another crisp number $z = x + y$. Now, if x and y are variables, obviously their sum may be thought of as a function of x and y; that is,

$$z(x, y) = x + y \tag{4.3-3}$$

Fuzzifying x and y—that is, defining fuzzy sets on x and y—results in a fuzzified function, $z = f(x, y)$. We saw in Section 2.5 how we can use the extension principle to obtain the fuzzy set $\boldsymbol{C}$ on $z = f(x, y)$. Suppose that we have two fuzzy numbers, $\boldsymbol{A}$ and $\boldsymbol{B}$, defined over x and y (the universe of discourse of real numbers). According to the extension principle, their sum is a fuzzy set on z denoted as $\boldsymbol{C}$, whose membership function is

$$\mu_C(z) \equiv \bigvee_{z=x+y} \left[\mu_A(x) \wedge \mu_B(y)\right] \tag{4.3-4}$$

Equation (4.3-4) tells us that to compute the grade of membership of a certain crisp number z to the fuzzy number $\boldsymbol{C}$, we take the maximum of the minima of the grades of membership of all pairs x and y which add up to z. How equation (4.3-4) works will be seen in Example 4.2, where a rather simple tabular way of carrying out the max–min operations will be presented.

Example 4.1 Addition of Discrete Fuzzy Numbers. Let us compute the sum $\boldsymbol{C}$ of two fuzzy numbers $\boldsymbol{A} = \mathit{3}$ and $\boldsymbol{B} = \mathit{7}$ defined as

$$\boldsymbol{A} = \mathit{3} = 0.3/1 + 0.7/2 + 1.0/3 + 0.7/4 + 0.3/5 + 0/6 \tag{E4.1-1}$$

$$\boldsymbol{B} = \mathit{7} = 0.2/5 + 0.6/6 + 1.0/7 + 0.6/8 + 0.2/9 + 0/10 \tag{E4.1-2}$$

and seen in Table 4.2. We compute $\boldsymbol{C}$ by adding the α-cuts of $\boldsymbol{A}, \boldsymbol{B}$ in accordance with equation (4.3-2). We see from Table 4.2 that when $\alpha = 0.4$, for example, the 0.4-cuts of $\boldsymbol{A}$ and $\boldsymbol{B}$ are

$$\boldsymbol{A}_{0.4} = \left[a_1^{(0.4)}, a_2^{(0.4)}\right] = [2, 4] \tag{E4.1-3}$$

and

$$\boldsymbol{B}_{0.4} = \left[b_1^{(0.4)}, b_2^{(0.4)}\right] = [6, 8] \tag{E4.1-4}$$

The intervals in equations (E4.1-3) and (E4.1-4) are shown as shaded "slices" of cells in Table 4.2. According to equation (4.3-1) the 0.4-cut of $\boldsymbol{C}$ is the sum of the two intervals given by (E4.1-3) and (E4.1-4)—that is,

$$\begin{aligned} \boldsymbol{C}_{0.4} &= \left[a_1^{(0.4)}, a_2^{(0.4)}\right] + \left[b_1^{(0.4)}, b_2^{(0.4)}\right] \\ &= \left[a_1^{(0.4)} + b_1^{(0.4)}, a_2^{(0.4)} + b_2^{(0.4)}\right] \\ &= [2 + 6, 4 + 8] \\ &= [8, 12] \end{aligned} \tag{E4.1-5}$$

We can obtain the same result from Table 4.2 simply by adding the endpoints of the shaded rows. We repeat this for each α to compute the entire sum. We start from the bottom of the table and go up in a *row-by-row* manner identifying the corresponding intervals of the two numbers and adding them up. The result is the number shown in Table 4.3. The 0.4-cut of $\boldsymbol{C}$ is indicated as a shaded group of cells in the table. As seen from the table, the new fuzzy number reaches unity at crisp number 10 (in the universe of discourse shown at the bottom) and therefore we think of it as a fuzzy number $\mathbf{10}$. Thus we see that the sum is $\mathit{7} + \mathit{3} = \mathbf{10}$, as would also be the case with crisp numbers. □

Table 4.2 Fuzzy numbers 3 and 7 in Example 4.1

3:

	0.3	0.7	1	0.7	0.3	0	0	0	0
$\alpha=1.0$			1						
$\alpha=0.9$			1						
$\alpha=0.8$			1						
$\alpha=0.7$		1	1	1					
$\alpha=0.6$		1	1	1					
$\alpha=0.5$		1	1	1					
$\alpha=0.4$		1	1	1					
$\alpha=0.3$	1	1	1	1	1				
$\alpha=0.2$	1	1	1	1	1				
$\alpha=0.1$	1	1	1	1	1				
$\alpha=0.0$	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9

7:

	0	0	0	0	0.2	0.6	1.0	0.6	0.2	0	0
$\alpha=1.0$							1				
$\alpha=0.9$							1				
$\alpha=0.8$							1				
$\alpha=0.7$							1				
$\alpha=0.6$						1	1	1			
$\alpha=0.5$						1	1	1			
$\alpha=0.4$						1	1	1			
$\alpha=0.3$						1	1	1			
$\alpha=0.2$					1	1	1	1	1		
$\alpha=0.1$					1	1	1	1	1		
$\alpha=0.0$	1	1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9	10	11

Table 4.3 Sum of fuzzy numbers 3 and 7 in Example 4.1

10:

	0	0	0	0	0	0.2	0.3	0.6	0.7	1.0	0.7	0.6	0.3	0.2	0	0
$\alpha=1.0$										1						
$\alpha=0.9$										1						
$\alpha=0.8$										1						
$\alpha=0.7$									1	1	1					
$\alpha=0.6$								1	1	1	1	1				
$\alpha=0.5$								1	1	1	1	1				
$\alpha=0.4$								1	1	1	1	1				
$\alpha=0.3$							1	1	1	1	1	1	1			
$\alpha=0.2$						1	1	1	1	1	1	1	1	1		
$\alpha=0.1$						1	1	1	1	1	1	1	1	1		
$\alpha=0.0$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Example 4.2 Addition of Fuzzy Numbers Through the Extension Principle. In this example we compute the sum of the two numbers A and B of Example 4.1 using the alternative definition of addition through the extension principle, namely, equation (4.3-4). At first glance, equation (4.3-4) looks somewhat esoteric. We present here a rather simple technique for using it. The same technique may be used with other fuzzy arithmetic operations as well (Kaufmann and Gupta, 1991). Let's repeat equation (4.3-4) here:

$$\mu_{A+B}(z) \equiv \bigvee_{z=x+y} [\mu_A(x) \wedge \mu_B(y)] \tag{E4.2-1}$$

A convenient way to compute the sum according to equation (E4.2-1) is to create a table as shown in Table 4.4. We take the *support* of B and make as many columns in the table as there are elements in the support; and similarly we take the *support* of A and make as many rows in the table as there are elements in the support of A. We recall that the support is the part of the universe of discourse that has nonzero membership. A and B can be

Table 4.4 Adding fuzzy numbers through the extension principle

Support of **B** (columns) / Support of **A** (rows). In each cell the upper value is $\mu_B(y)$ and the lower value is $\mu_A(x)$.

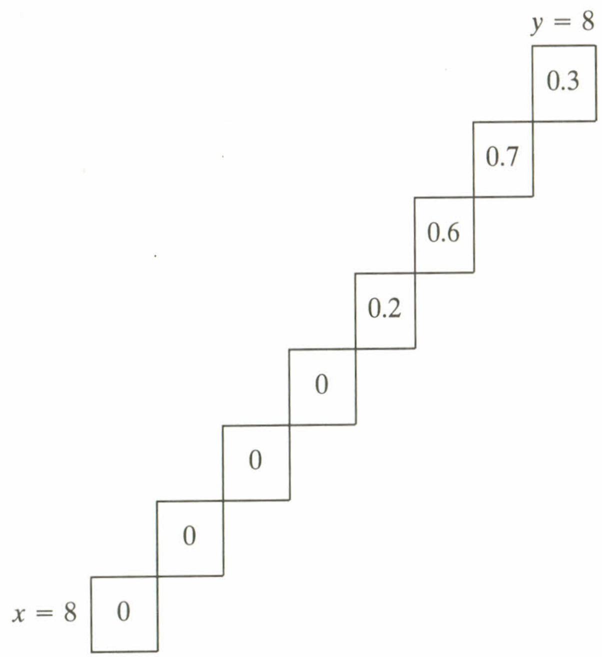
A \ **B**	*y=1*	*y=2*	*y=3*	*y=4*	*y=5*	*y=6*	*y=7*	*y=8*	*y=9*	*y=10*
x=1	0.0 0.3	0.0 0.3	0.0 0.3	0.0 0.3	0.2 0.3	0.6 0.3	1.0 0.3	0.6 0.3	0.2 0.3	0.0 0.3
x=2	0.0 0.7	0.0 0.7	0.0 0.7	0.0 0.7	0.2 0.7	0.6 0.7	1.0 0.7	0.6 0.7	0.2 0.7	0.0 0.7
x=3	0.0 1.0	0.0 1.0	0.0 1.0	0.0 1.0	0.2 1.0	0.6 1.0	1.0 1.0	0.6 1.0	0.2 1.0	0.0 1.0
x=4	0.0 0.7	0.0 0.7	0.0 0.7	0.0 0.7	0.2 0.7	0.6 0.7	1.0 0.7	0.6 0.7	0.2 0.7	0.0 0.7
x=5	0.0 0.3	0.0 0.3	0.0 0.3	0.0 0.3	0.2 0.3	0.6 0.3	1.0 0.3	0.6 0.3	0.2 0.3	0.0 0.3
x=6	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.2 0.0	0.6 0.0	1.0 0.0	0.6 0.0	0.2 0.0	0.0 0.0
x=7	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.2 0.0	0.6 0.0	1.0 0.0	0.6 0.0	0.2 0.0	0.0 0.0
x=8	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.2 0.0	0.6 0.0	1.0 0.0	0.6 0.0	0.2 0.0	0.0 0.0
x=9	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.2 0.0	0.6 0.0	1.0 0.0	0.6 0.0	0.2 0.0	0.0 0.0
x=10	0.0 0.0	0.0 0.0	0.0 0.0	0.0 0.0	0.2 0.0	0.6 0.0	1.0 0.0	0.6 0.0	0.2 0.0	0.0 0.0

interchanged in terms or rows and columns, but for the moment let's make columns from the support of $\boldsymbol{A}$ and make rows from the support of $\boldsymbol{A}$. In every cell of the table we put at the lower left corner the grade of membership of x to $\boldsymbol{A}$ and put in the upper right corner the grade of membership of y to $\boldsymbol{B}$. Thus we have $\mu_A(x)$ in the lower left corner and $\mu_B(y)$ in the upper right corner as shown in Table 4.4. Now, let's take another look in the equation above. It calls for taking the maximum of pairs of singletons that add up to a certain z. For example, suppose that have $z = 9$. There are three different ways to get $z = 9$: adding $y = 8$ and $x = 1$, adding $y = 7$ and

$x = 2$, adding $y = 6$ and $x = 3$ and so on. Both elements for each addition are found inside a cell. These are the shaded cells shown in Table 4.4. Equation (E4.2-1) says that for $z = 9$ we need to take the maximum of the minima of the three pairs of grades of membership inside the shaded cells. First we find the minimum of the grades of membership inside each cell—that is,

$$
\begin{aligned}
\mu_A(1) \wedge \mu_B(8) &= 0.3 \wedge 0.6 = 0.3 \\
\mu_A(2) \wedge \mu_B(7) &= 0.7 \wedge 1.0 = 0.7 \\
\mu_A(3) \wedge \mu_B(6) &= 1.0 \wedge 0.6 = 0.6 \\
\mu_A(4) \wedge \mu_B(5) &= 0.7 \wedge 0.2 = 0.2 \\
\mu_A(5) \wedge \mu_B(4) &= 0.3 \wedge 0 = 0 \\
\mu_A(6) \wedge \mu_B(3) &= 0 \wedge 0 = 0 \\
\mu_A(7) \wedge \mu_B(2) &= 0 \wedge 0 = 0 \\
\mu_A(8) \wedge \mu_B(1) &= 0 \wedge 0 = 0
\end{aligned} \qquad \text{(E4.2-2)}
$$

Now if we look only at the shaded part of the table, we can replace the contents of each cell with the minima found in equations (E4.2-2)—that is,



Next, we take the maximum of these numbers, which in this case is 0.7; this is the maximum with respect to $z = 9$ in equation (E4.2-1). At this point we have completed the entire operation on equation (E4.2-1) for $z = 4$—that is,

$$\begin{aligned}\mu_{A+B}(9) &= [(0.3) \vee (0.7) \vee (0.6) \vee (0.2) \vee (0) \vee (0) \vee (0) \vee (0)] \\ &= 0.7 \qquad \text{(E4.2-3)}\end{aligned}$$

This is the grade of membership of $z = 9$ to the sum $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$. We repeat this procedure for all other cells to obtain the membership function of $\boldsymbol{C}$. The result is

$$\begin{aligned}\boldsymbol{C} = {} & 0/5 + 0.2/6 + 0.3/7 + 0.6/8 + 0.7/9 + 1.0/10 \\ & + 0.7/11 + 0.6/12 + 0.3/13 + 0.2/14 + 0/15\end{aligned}$$

which is the same number as the one we found by the interval approach in Example 4.1—that is, the number shown in Table 4.3. □

4.4 SUBTRACTION

The difference $\boldsymbol{C}$ of two fuzzy numbers $\boldsymbol{A}, \boldsymbol{B}$ may be defined either through interval subtraction utilizing the *α-cut representation* of the two numbers or through the *extension principle*. Using α-cuts we subtract them as follows

$$\boldsymbol{A} - \boldsymbol{B} \equiv \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] - \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] \qquad (4.4\text{-}1)$$

where $[a_1^{(\alpha)}, a_2^{(\alpha)}]$ is the collection of closed intervals representing $\boldsymbol{A}$, and $[b_1^{(\alpha)}, b_2^{(\alpha)}]$ is the collection of closed intervals representing $\boldsymbol{B}$. Two intervals are subtracted by subtracting their left and right endpoints, and thus equation (4.4-1) becomes

$$\boldsymbol{A} - \boldsymbol{B} = \left[a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}\right] \qquad (4.4\text{-}2)$$

The alternative way to define the difference of fuzzy numbers $\boldsymbol{A}$ and $\boldsymbol{B}$ is through the extension principle—that is, by fuzzifying a function $z = x - y$. Fuzzification means that we define fuzzy sets on the universes of discourse where the crisp elements x and y are found. As a result, z gets fuzzified as well; that is, there is a fuzzy set $\boldsymbol{C}$ over the universe of discourse of the z's, which is the result of fuzzifying the function $z = f(x, y) = x - y$. The membership function of $\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$ can be computed from

$$\mu_{A-B}(z) \equiv \bigvee_{z=x-y} [\mu_A(x) \wedge \mu_B(y)] \qquad (4.4\text{-}3)$$

Equation (4.4-3) gives, of course, the same number $\boldsymbol{C}$ obtained through (4.4-2).

Example 4.3 Subtracting Fuzzy Numbers as Intervals. Let us compute a fuzzy number $\boldsymbol{C} = \boldsymbol{7} - \boldsymbol{3}$, where the fuzzy numbers $\boldsymbol{7}$ and $\boldsymbol{3}$ are as defined in Table 4.2 (Example 4.1):

$$\boldsymbol{A} = \boldsymbol{3} = 0.3/1 + 0.7/2 + 1.0/3 + 0.7/4 + 0.3/5 + 0/6 \qquad \text{(E4.3-1)}$$

$$\boldsymbol{B} = \boldsymbol{7} = 0.2/5 + 0.6/6 + 1.0/7 + 0.6/8 + 0.2/9 + 0/10 \qquad \text{(E4.3-2)}$$

Subtracting the two numbers is the same as interval subtraction at each α. From Table 4.2 we see that when $\alpha = 0.3$, for example, the 0.3-cuts of the two numbers are

$$\boldsymbol{A}_{0.3} = \left[a_1^{(0.3)}, a_2^{(0.3)}\right] = [1, 5] \qquad \text{(E4.3-3)}$$

and

$$\boldsymbol{B}_{0.3} = \left[b_1^{(0.3)}, b_2^{(0.3)}\right] = [6, 8] \qquad \text{(E4.3-4)}$$

The α-cut of $\boldsymbol{C}$ at $\alpha = 0.3$ is the difference of the α-cuts in by (E4.3-3) and (E4.3-4)

$$\begin{aligned}\boldsymbol{C}_{0.3} &= \left[b_1^{(0.3)}, b_2^{(0.3)}\right] - \left[a_1^{(0.3)}, a_2^{(0.3)}\right] \\ &= \left[b_1^{(0.3)} - a_2^{(0.3)}, b_2^{(0.3)} - a_1^{(0.3)}\right] \\ &= [6 - 5, 8 - 1] \\ &= [1, 7] \qquad \text{(E4.3-5)}\end{aligned}$$

shown as a "slice" of shaded cells in Table 4.5. In a similar manner we compute the α-cuts of $\boldsymbol{C}$ at the other levels of α and obtain the fuzzy number

$$\begin{aligned}\boldsymbol{C} = {} & 0.2/0 + 0.3/1 + 0.6/2 + 0.7/3 + 1.0/4 \\ & + 0.7/5 + 0.6/6 + 0.3/7 + 0.2/8\end{aligned}$$

which is also shown in Table 4.5. As may be seen from Table 4.5, $\boldsymbol{C}$ can be considered a fuzzy $\boldsymbol{4}$. □

Example 4.4 Subtracting Fuzzy Numbers with Continuous Membership Functions. Consider the two triangular fuzzy numbers $\boldsymbol{A}$ and $\boldsymbol{B}$ shown in Figure 4.4. We want to compute their difference—that is, find a fuzzy number $\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$. When continuous (or piecewise continuous) membership functions are used, we subtract them by parameterizing their membership functions by α and subtracting their α-cuts. The membership functions of

Table 4.5 Difference of fuzzy numbers 7 and 3 in Example 4.3

4:

	0.2	0.3	0.6	0.7	1	0.7	0.6	0.3	0.2	0
α=1.0					1					
α=0.9					1					
α=0.8					1					
α=0.7				1	1	1				
α=0.6			1	1	1	1	1			
α=0.5			1	1	1	1	1			
α=0.4			1	1	1	1	1			
α=0.3		1	1	1	1	1	1	1		
α=0.2	1	1	1	1	1	1	1	1	1	
α=0.1	1	1	1	1	1	1	1	1	1	
α=0.0	1	1	1	1	1	1	1	1	1	1
	0	1	2	3	4	5	6	7	8	9

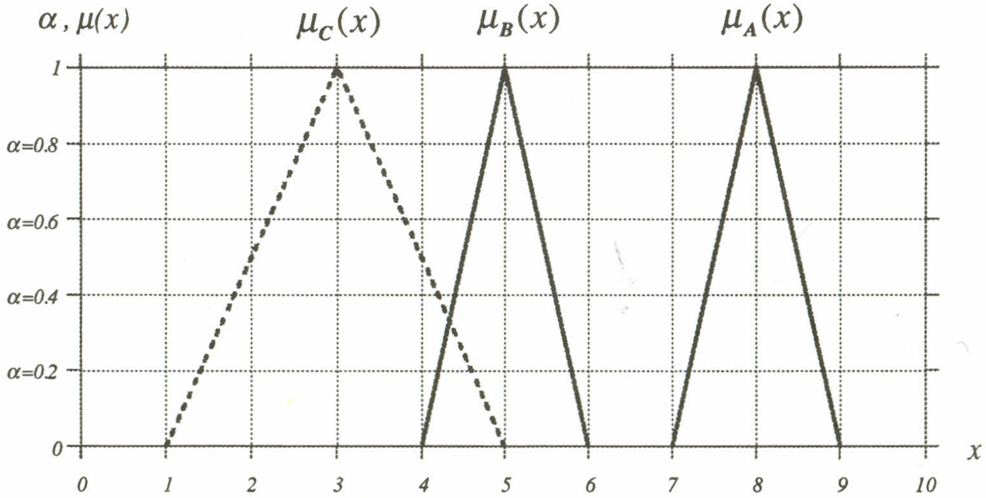


Figure 4.4 Subtracting two fuzzy numbers, $C = A - B$.

A, B are

$$\begin{aligned} \mu_A(x) &= 0, && x \le 7 \\ &= x - 7, && 7 \le x \le 8 \\ &= -x + 9, && 8 \le x \le 9 \\ &= 0, && x \ge 9 \end{aligned} \tag{E4.4-1}$$

and

$$\begin{aligned} \mu_B(x) &= 0, && x \le 4 \\ &= x - 4, && 4 \le x \le 5 \\ &= -x + 6, && 5 \le x \le 6 \\ &= 0, && x \ge 6 \end{aligned} \tag{E4.4-2}$$

Let us parameterize them by α. To simplify matters, consider the left and right side of each membership function separately. There is one equation for the left side and another for the right side of the membership function of A, and likewise for B. Thus, we have a total of four equations to parameterize. From equations (E4.4-1) we take the part that describes the left side of A, $\mu_A^L(x) = x - 7$, and write it in terms of a. We note that the value of α is the same as the value of the membership function at the left endpoint $a_1^{(\alpha)}$ of an α-cut, and $a_1^{(\alpha)}$ is the value of x at that point. Thus we have for the left side of A,

$$\alpha = a_1^{(\alpha)} - 7 \Rightarrow a_1^{(\alpha)} = \alpha + 7 \tag{E4.4-3}$$

where $a_1^{(\alpha)}$ is the left endpoint of the "slice" of A at level α.

Similarly for the right side of A we parameterize the right endpoint $a_2^{(\alpha)}$ of each α-cut in terms of α as

$$\alpha = -a_2^{(\alpha)} + 9 \Rightarrow a_2^{(\alpha)} = -\alpha + 9 \tag{E4.4-4}$$

Using equations (E4.4-3) and (E4.4-4) the α-cut representation of A is written as

$$A = \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] = [\alpha + 7, -\alpha + 9] \tag{E4.4-5}$$

The membership function of the number B is parameterized in terms of α in a similar fashion. We express the left endpoint $b_1^{(\alpha)}$ in terms of α by

$$\alpha = b_1^{(\alpha)} - 4 \Rightarrow b_1^{(\alpha)} = \alpha + 4 \tag{E4.4-6}$$

The right endpoint $b_2^{(\alpha)}$ is given as a function of α by

$$\alpha = -b_2^{(\alpha)} + 6 \Rightarrow b_2^{(\alpha)} = -\alpha + 6 \tag{E4.4-7}$$

From equations (E4.4-6) and (E4.4-7) the interval representation of $\boldsymbol{B}$ is

$$\boldsymbol{B} = \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] = [\alpha + 4, -\alpha + 6] \tag{E4.4-8}$$

From the α-cut representations of $\boldsymbol{A}$ and $\boldsymbol{B}$ (equations (E4.4-5) and (E4.4-8)), we find their difference by subtracting their corresponding intervals at each α, that is,

$$\begin{aligned} \boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B} &= \left[a_1^{(\alpha)} - b_2^{(\alpha)}, a_2^{(\alpha)} - b_1^{(\alpha)}\right] \\ &= [(\alpha + 7) - (-\alpha + 6), (-\alpha + 9) - (\alpha + 4)] \\ &= [2\alpha + 1, -2\alpha + 5] \end{aligned} \tag{E4.4-9}$$

Therefore, $\boldsymbol{C}$ is

$$\boldsymbol{C} = \left[c_1^{(\alpha)}, c_2^{(\alpha)}\right] = [2\alpha + 1, -2\alpha + 5] \tag{E4.4-10}$$

We note that the left and right endpoints of $\boldsymbol{C}$ are functions of α. To express the fuzzy number $\boldsymbol{C}$ in terms of a membership function, we derive equations for the left and right side of $\boldsymbol{C}$. The left endpoint $c_1^{(\alpha)}$ in equation (E4.4-10) is equal to the value of x when the left-side membership function's value is α. Similarly, the right endpoint $c_2^{(\alpha)}$ is equal to the value of x when the right-side membership function is α. Thus the equation of the left side is obtained by setting $c_1^{(\alpha)} = x$ and recalling that $\alpha = \mu_C^L(x)$, where $\mu_C^L(x)$ is the left-side membership function for $\boldsymbol{C}$. We have

$$x = 2\mu_C^L(x) + 1 \Rightarrow \mu_C^L(x) = \tfrac{1}{2}(x - 1) \tag{E4.4-11}$$

In a similar manner we obtain an equation for $\mu_C^R(x)$, the right side of the membership function of $\boldsymbol{C}$, and solve it to obtain the membership function of the right side—that is,

$$x = -2\mu_C^R(x) + 5 \Rightarrow \mu_C^R(x) = -\tfrac{1}{2}(x - 5) \tag{E4.4-12}$$

From equations (E4.4-11) and (E4.4-12) we obtain

$$\begin{aligned} \mu_C(x) &= 0, & x &\le 1 \\ &= \tfrac{1}{2}(x - 1), & 1 &\le x \le 3 \\ &= -\tfrac{1}{2}(x - 5), & 3 &\le x \le 5 \\ &= 0, & x &\ge 5 \end{aligned} \tag{E4.4-13}$$

The number $\boldsymbol{C}$ described by equations (E4.4-11) is shown in Figure 4.4. Note that $\boldsymbol{C}$ has its peak at crisp 3, and therefore it can be considered as a fuzzy number $\boldsymbol{3}$ (as expected since, $\boldsymbol{8} - \boldsymbol{5} = \boldsymbol{3}$). □

4.5 MULTIPLICATION

As in the case of addition and subtraction, fuzzy number multiplication may be defined either as α-cut multiplication or through the extension principle. Using the α-cut representation of two numbers $\boldsymbol{A}$, $\boldsymbol{B}$, their product is defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} \equiv \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] \cdot \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] \tag{4.5-1}$$

In general, the product of two intervals is a new interval whose left endpoint is the product of the left endpoints of the two intervals and the right endpoint is the product of the right endpoints of the two intervals. Thus, equation (4.5-1) is

$$\boldsymbol{A} \cdot \boldsymbol{B} = \left[a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}\right] \tag{4.5-2}$$

Alternatively, we define the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ through the extension principle by fuzzifying the function $z(x, y) = x \cdot y$. The extension principle tells us that their product is a fuzzy set on z, denoted as $\boldsymbol{A} \cdot \boldsymbol{B}$, whose membership function is

$$\mu_{A \cdot B}(z) \equiv \bigvee_{z = x \cdot y} [\mu_A(x) \wedge \mu_B(y)] \tag{4.5-3}$$

Of course, equations (4.5-2) and (4.5-3) are equivalent in that they give us the same number $\boldsymbol{C} = \boldsymbol{A} \cdot \boldsymbol{B}$.

A special case of fuzzy multiplication is the product of fuzzy number by crisp number. Let k be a crisp positive real number and $\boldsymbol{A}$ a fuzzy number defined over the universe of discourse of positive real numbers also. We define the product of k with $\boldsymbol{A}$ either as interval multiplication or through the extension principle. Crisp number k may be viewed as an interval also, a trivial interval whose left and right endpoints are the same—that is, $k = [k, k]$. We use equations (4.5-1) and (4.5-2) to obtain the product of k with $\boldsymbol{A}$ as

$$\begin{aligned} k \cdot \boldsymbol{A} &\equiv [k, k] \cdot \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] \\ &= \left[k a_1^{(\alpha)}, k a_2^{(\alpha)}\right] \end{aligned} \tag{4.5-4}$$

Alternatively, we define the product of fuzzy number $\boldsymbol{A}$ with a crisp number k, $k \cdot \boldsymbol{A}$, through the extension principle. It may be shown using equation

(4.5-3) that the membership function of $k \cdot A$ is

$$\mu_{k\cdot A}(x) = \mu_A\left(\frac{x}{k}\right) \tag{4-5-5}$$

where equations (4.5-4) and (4.5-5) give the same result.

Example 4.5 Multiplication of Two Fuzzy Numbers. Consider the triangular fuzzy numbers $A = 8$ and $B = 2$ defined over the positive real numbers as shown in Figure 4.5 (since both numbers are defined over the same universe of discourse we simply use x to indicate an element of the universe of discourse, instead of x, y, etc.). We want to compute a fuzzy number C which is the product of A and B—that is, $C = A \cdot B$. Let us do this through α-cut multiplication—that is, by parameterizing their membership functions and multiplying their α-cuts in the manner indicated by equation (4.5-2).

First, we write the analytical expressions for the membership functions of A and B:

$$\begin{aligned}
\mu_A(x) &= 0, && x \leq 4 \\
&= \tfrac{1}{4}x - 1, && 4 \leq x \leq 8 \\
&= -\tfrac{1}{4}x + 3, && 8 \leq x \leq 12 \\
&= 0, && x \geq 12
\end{aligned} \tag{E4.5-1}$$

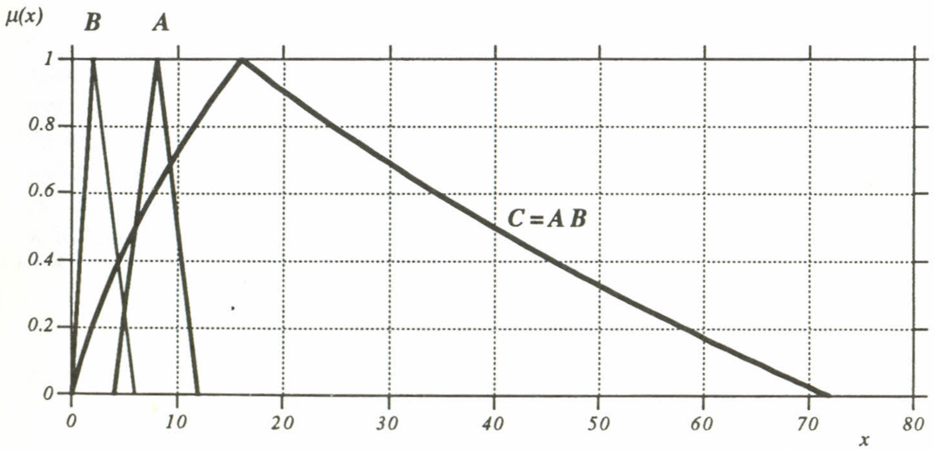


Figure 4.5 The product $C = A \cdot B$ of numbers $A = 8$ and $B = 2$ in Example 4.5.

and

$$\begin{aligned}
\mu_B(x) &= 0, && x \leq 0 \\
&= \tfrac{1}{2}x, && 0 \leq x \leq 2 \\
&= -\tfrac{1}{4}x + \tfrac{3}{2}, && 2 \leq x \leq 6 \\
&= 0 && x \geq 6
\end{aligned} \tag{E4.5-2}$$

Next, we parameterize the membership functions in equations (E4.5-1) and (E4.5-2) in terms α (a procedure of renaming the left and right side of the membership functions and thus the endpoints of all intervals in terms of α). Let us take the left and right side of each membership function separately and rewrite it in terms of α. It should be noted that a given value of α is the same as the value of the membership function at that level. From equation (E4.5-1) we have that the left and right endpoints of A are

$$\alpha = \tfrac{1}{4}a_1^{(\alpha)} - 1 \Rightarrow a_1^{(\alpha)} = 4(\alpha + 1) \tag{E4.5-3}$$

and

$$\alpha = -\tfrac{1}{4}a_2^{(\alpha)} + 3 \Rightarrow a_2^{(\alpha)} = -4(\alpha - 3) \tag{E4.5-4}$$

Using equations (E4.5-3) and (E4.5-4) we obtain A as

$$A = \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] = \left[4(\alpha + 1), -4(\alpha - 3)\right] \tag{E4.5-5}$$

Similarly, we parameterize the membership function of B and write its left and right endpoints at each α as

$$\alpha = \tfrac{1}{2}b_1^{(\alpha)} \Rightarrow b_1^{(\alpha)} = 2\alpha \tag{E4.5-6}$$

and

$$\alpha = -\tfrac{1}{4}b_2^{(\alpha)} + \tfrac{3}{2} \Rightarrow b_2^{(\alpha)} = -4\left(\alpha - \tfrac{3}{2}\right) \tag{E4.5-7}$$

Thus, from equations (E4.5-6) and (E4.5-7) the interval representation of B is

$$B = \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] = \left[2\alpha, -4\left(\alpha - \tfrac{3}{2}\right)\right] \tag{E4.5-8}$$

Having the endpoints of A and B in terms of α, we multiply the two numbers using equation (4.5-2) and obtain

$$\begin{aligned}
C = A \cdot B &= \left[a_1^{(\alpha)} \cdot b_1^{(\alpha)}, a_2^{(\alpha)} \cdot b_2^{(\alpha)}\right] \\
&= \left[4(\alpha + 1) \cdot 2\alpha, -4(\alpha - 3) \cdot \left(-4\left(\alpha - \tfrac{3}{2}\right)\right)\right] \\
&= \left[8\alpha^2 + 8\alpha, 16\alpha^2 - 72\alpha + 72\right]
\end{aligned} \tag{E4.5-9}$$

The interval representation of C is

$$C = \left[c_1^{(\alpha)}, c_2^{(\alpha)}\right] = [8\alpha^2 + 8\alpha, 16\alpha^2 - 72\alpha + 72] \quad \text{(E4.5-10)}$$

where the left and right endpoints in equation (E4.5-10) are functions for α. We can obtain the membership function of C as well. Equation (E4.5-10) provides us with left and right endpoints of each α-cut. The equation for the left-side membership function $\mu_C^L(x)$ is obtained by setting $c_1^{(\alpha)} = x$ and recalling that $\alpha = \mu_C^L(x)$. Thus, we obtain an equation involving $\mu_C^L(x)$, which is

$$8\left(\mu_C^L(x)\right)^2 + 8\mu_C^L(x) - x = 0 \quad \text{(E4.5-11)}$$

Solving quadratic equation (E4.5-11) for $\mu_C^L(x)$, we obtain two solutions and accept only the value of $\mu_C^L(x)$ in [0, 1], ignoring the other one. The result is

$$\mu_C^L(x) = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \tfrac{1}{2}x} \quad \text{(E4.5-12)}$$

Similarly we obtain an equation for $\mu_C^R(x)$, the right side of the membership function of C, and solve it, keeping the solution which is within [0, 1]. The result is

$$\mu_C^R(x) = \tfrac{1}{2}\left(4.5 - \sqrt{(4.5)^2 - 4\left(4.5 - \tfrac{1}{16}x\right)}\right) \quad \text{(E4.5-13)}$$

The membership function of C is

$$\begin{aligned}
\mu_C(x) &= 0, && x \le 0 \\
&= -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \tfrac{1}{2}x}, && 0 \le x \le 16 \\
&= \tfrac{1}{2}\left(4.5 - \sqrt{(4.5)^2 - 4\left(4.5 - \tfrac{1}{16}x\right)}\right), && 16 \le x \le 72 \\
&= 0, && x \ge 72 \quad \text{(E4.5-14)}
\end{aligned}$$

as shown in Figure 4.5. It should be noted that C has its peak point at crisp 16 and therefore may be considered a fuzzy number ***16***. It should also be noted that multiplying two fuzzy numbers results in a new number whose shape has been considerably changed, no longer having a triangular membership function with linear sides but in this case parabolic sides. Multiplication in general has the effect of "fattening" the lower part of the membership functions involved. □

4.6 DIVISION

We can find the *quotient* of two fuzzy numbers A and B either through interval division or by the extension principle. In terms of their α-cut representation, we write the *quotient* of the two numbers as

$$A \div B \equiv \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] \div \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] \quad \text{(4.6-1)}$$

In general, the *quotient* of two intervals is a new interval given by

$$\left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] \div \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] \equiv \left[\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right]$$

Hence, provided that $b_2^{(\alpha)} \neq 0$ and $b_1^{(\alpha)} \neq 0$, the *quotient* of A, B is

$$A \div B = \left[\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right] \quad \text{(4.6-2)}$$

Alternatively, we find the *quotient* of A and B through the extension principle by fuzzifying the function $z(x, y) = x \div y$, where x and y are crisp elements of the universe of discourse of A and B. The extension principle tells us that $A \div B$ is a fuzzy set with membership function

$$\mu_{A \div B}(z) \equiv \bigvee_{z = x \div y} [\mu_A(x) \wedge \mu_B(y)] \quad \text{(4.6-3)}$$

The results obtained through equations (4.6-4) and (4.6-2) are of course the same. Equation (4.6-3) may be used in the manner shown in Example 4.2. We construct a table such as Table 4.4 and proceed as outlined in the example. A word of caution: Fuzzy number division is not the reverse of multiplication; that is, generally it is not true that $(A \div B) \times B = A$.

Example 4.6 Division of Fuzzy Numbers. Consider the triangular fuzzy numbers A = ***8*** and B = ***2*** used in Example 4.5. Let us find $C = A \div B$ using interval division. The analytical expressions for the membership functions of A and B are given in Example 4.5 [equations (E4.5-1) and (E4.5-2)], and their parameterized interval representation is found in equations (E4.5-5) and (E4.5-8), which for convenience we repeat here:

$$A = \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] = [4(\alpha + 1), -4(\alpha - 3)] \quad \text{(E4.6-1)}$$

$$B = \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] = \left[2\alpha, -4\left(\alpha - \tfrac{3}{2}\right)\right] \quad \text{(E4.6-2)}$$

Thus their *quotient* $C = A \div B$ is obtained using equation (4.6-2):

$$C = A \div B = \left[\frac{a_1^{(\alpha)}}{b_2^{(\alpha)}}, \frac{a_2^{(\alpha)}}{b_1^{(\alpha)}}\right]$$

$$= \left[\frac{4(\alpha+1)}{(-4(\alpha-\frac{3}{2}))}, \frac{-4(\alpha-3)}{2\alpha}\right] \qquad \text{(E4.6-3)}$$

The α-cut representation of C is

$$C = \left[c_1^{(\alpha)}, c_2^{(\alpha)}\right] = \left[-\frac{(\alpha+1)}{(\alpha-\frac{3}{2})}, -\frac{2(\alpha-3)}{\alpha}\right] \qquad \text{(E4.6-4)}$$

where the left and right endpoints are functions of α. We may also express C in terms of a membership function by deriving equations for the left and right sides of the membership function as we did in Example 4.5. Equation (E4.6-4) gives us the endpoints of the interval of each α-cut. The equation of the left side is obtained by setting $c_1^{(\alpha)} = x$ and recalling that $\alpha = \mu_C^L(x)$, where, $\mu_C^L(x)$ is the left side membership function for C. The result is

$$\mu_C^L(x) = \frac{\frac{3}{2}x - 1}{x+1} \qquad \text{(E4.6-5)}$$

Similarly we obtain an equation for $\mu_C^R(x)$, the right side of the membership function of C, and solve it to obtain

$$\mu_C^R(x) = \frac{6}{x+2} \qquad \text{(E4.6-6)}$$

The quotient is shown in Figure 4.6, and the analytical description of the

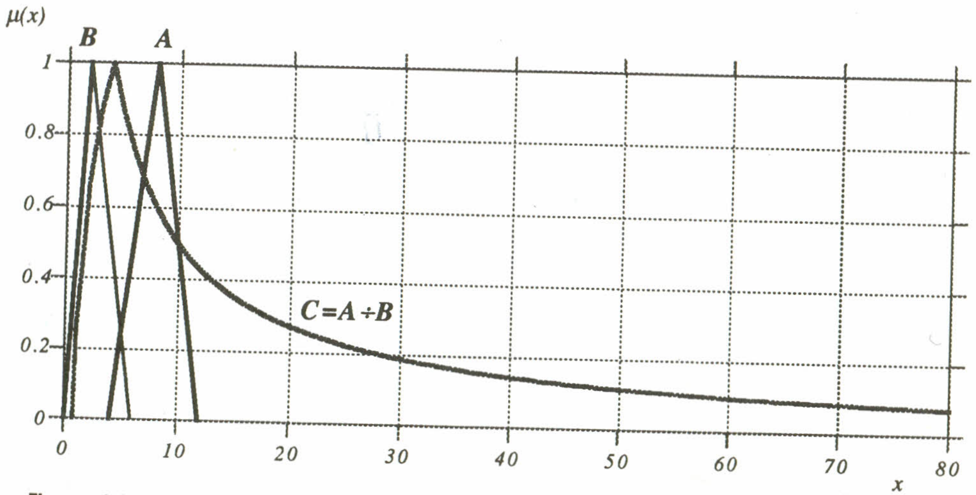


Figure 4.6 The quotient $C = A \div B$ of the fuzzy numbers $A = 8$ and $B = 2$ in Example 4.6.

membership function of C is

$$\mu_C(x) = \begin{cases} 0, & x \le 0 \\ \dfrac{\frac{3}{2}x - 1}{x+1}, & 0 \le x \le 4 \\ \dfrac{6}{x+2}, & 4 \le x \le 72 \\ 0, & x \ge 72 \end{cases} \qquad \text{(E4.6-7)}$$

It should be noted from equation (E4.6-7) that the quotient is a new fuzzy number that no longer has a triangular shape with linear sides. As may be seen from the figure, the fuzzy number C only asymptotically reaches zero and hence we may consider the use of a *level fuzzy set* (Chapter 2) in order to limit and exclude trivially small grades of membership—for example, less than 0.2. □

4.7 MINIMUM AND MAXIMUM

The minimum and maximum of two fuzzy numbers A, B result in finding the smallest and the biggest one, respectively, and may be defined either through their interval representation or by the extension principle. In interval arithmetic the minimum of two intervals is a new interval whose left endpoint is the minimum of the left endpoints of the original intervals and whose right endpoint is the minimum of the right endpoints of the two intervals. Thus the minimum of A, B is a new number, $A \wedge B$, given by

$$A \wedge B \equiv \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] \wedge \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right]$$

$$= \left[a_1^{(\alpha)} \wedge b_1^{(\alpha)}, a_2^{(\alpha)} \wedge b_2^{(\alpha)}\right] \qquad \text{(4.7-1)}$$

Alternatively, the minimum of two fuzzy numbers may be obtained through the extension principle. The membership function of $A \wedge B$ is

$$\mu_{A \wedge B}(z) \equiv \bigvee_{z = x \wedge y} \left[\mu_A(x) \wedge \mu_B(y)\right] \qquad \text{(4.7-2)}$$

In an analogous manner we define the maximum of two fuzzy numbers A and B, recalling that in interval arithmetic the maximum of two intervals is a new interval whose left endpoint is the maximum of the left endpoints of the original intervals and whose right endpoint is the maximum of the right endpoints of the two intervals. Thus the maximum $A \vee B$ is given by

$$A \vee B \equiv \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right] \vee \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right]$$

$$= \left[a_1^{(\alpha)} \vee b_1^{(\alpha)}, a_2^{(\alpha)} \vee b_2^{(\alpha)}\right] \qquad \text{(4.7-3)}$$

Alternatively, by the extension principle the membership function of the

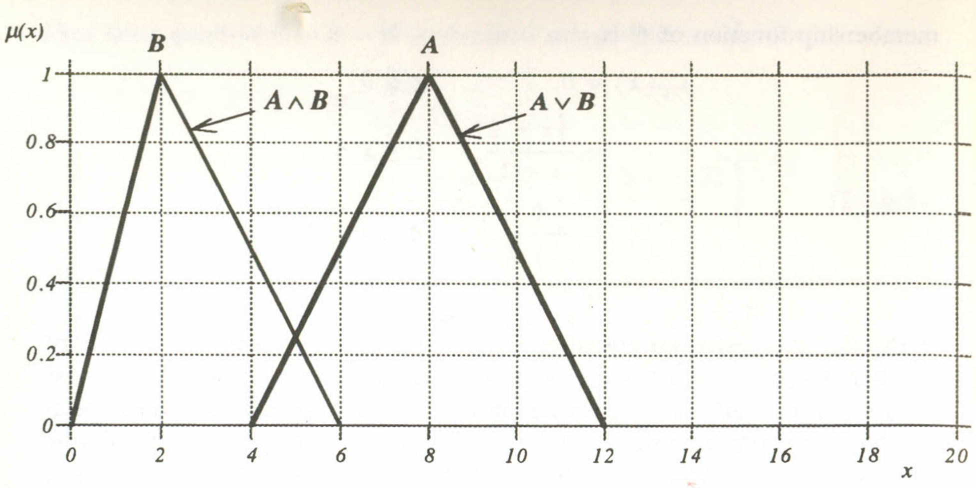


Figure 4.7 The minimum and maximum of the two numbers $A = 8$ and $B = 2$ used in Example 4.6.

maximum of the two numbers A and B is

$$\mu_{A \vee B}(z) \equiv \bigvee_{z = x \vee y} [\mu_A(x) \wedge \mu_B(y)] \tag{4.7-4}$$

It should be noted that the *maximum* and *minimum* of two fuzzy numbers are different than the maximum and minimum of membership functions used in connection with the *union* and *intersection* of two fuzzy sets. Let us illustrate this by finding the minimum of the numbers $A = 8$ and $B = 2$ used in Examples 4.5 and 4.6 and redrawn in Figure 4.7. Equations (4.7-1) or (4.7-2) do not give us the little wedge between A and B, which is the *intersection* of A and B. They will simply give us the number $B = 2$ itself, which is the smallest of the two fuzzy numbers. Similarly the largest of the numbers is found by using the maximum operation of either equation (4.7-3) or (4.7-4), which is simply the number $A = 8$, as shown in Figure 4.7. For more intricately overlapping membership functions the maximum or minimum may not simply be a number with the membership function of either A or B, but may have a totally new shape (Kaufmann and Gupta, 1991).

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PROBLEMS

1. The fuzzy numbers A and B are given by

$$A = 0.33/6 + 0.67/7 + 1.00/8 + 0.67/9 + 0.33/10$$

$$B = 0.33/1 + 0.67/2 + 1.00/3 + 0.67/4 + 0.33/5$$

   Subtract B from A to give fuzzy number C. Draw a sketch of C.

2. Multiply fuzzy numbers A and B of Problem 1. Draw a sketch of C.

3. Divide fuzzy number A by fuzzy number B where the fuzzy numbers are defined in Problem 1. Draw a sketch of C.

4. Modify Example 4.2 to subtract the two fuzzy numbers using the extension principle.

5. Consider the fuzzy numbers A and B described by the membership functions:

$$\begin{aligned}
\mu_A(x) &= 0, && x \le 8,\\
&= \frac{1}{10}x - \frac{8}{10}, && 8 \le x \le 18,\\
&= -\frac{1}{14}x + \frac{32}{14}, && 18 \le x \le 32,\\
&= 0, && x > 32,\\
\mu_B(x) &= 0, && x \le -3,\\
&= \frac{1}{9}x + \frac{1}{3}, && -3 \le x \le 6,\\
&= -\frac{1}{18}x + \frac{4}{3}, && 6 \le x \le 24,\\
&= 0, && x > 24
\end{aligned}$$

Compute:

(a) $A\ (+)\ B$,

(b) $A\ (-)\ B$,

(c) $A\ (\div)\ B$.

6. Repeat the computations in Problem 5 for the fuzzy numbers A and B given below, and using C state and show the distributivity property (with respect to addition and multiplication)

$$A = 0.6/1 + 0.8/2 + 1.0/3 + 0.6/4$$

$$B = 0.5/0 + 0.7/1 + 0.9/2 + 1.0/3 + 0.4/4$$

$$C = 0.7/1 + 0.8/2 + 1.0/3 + 0.3/4$$