3

FUZZY RELATIONS

3.1 INTRODUCTION

In fuzzy approaches, *relations* possess the computational potency and significance that *functions* possess in conventional approaches. Fuzzy *if/then* rules and their aggregations, known as *fuzzy algorithms*, both of central importance in engineering applications, are fuzzy relations in linguistic disguise. Fuzzy relations may be thought of as fuzzy sets defined over high-dimensional universes of discourse. As the name indicates, a relation implies the presence of an association between elements of different sets. If the degree of association is either 0 or 1, we have *crisp relations*. If the degree of association is between 0 and 1, we have *fuzzy relations*; a number between 0 and 1 is taken to indicate partial absence or presence of association. In this chapter we begin by reviewing crisp relations and various ways for representing them. Next, we look at fuzzy relations and properties used to classify them, and finally we come to *composition of fuzzy relations*, a very important tool for approximate reasoning with applications in the fields of expert systems, control, and diagnosis.

On what basis do we associate various elements in a relation? The association may be due to a common property, a quality, a reference, a condition, or a rule, satisfied by pairs of elements (e.g., objects, numbers, words, variables, etc.). For example, the statements "***is greater than***" or "***is a component of***" indicate an association between two elements. The order of the elements is important. For instance, if the relation "***is a component of***" holds for the pair of elements (*u-tube*, *steam-generator*)—that is, if the statement "*u-tube* ***is a component of*** *steam-generator*" is true—the relation may no longer be true when the elements are interchanged. The relation "*steam* *generator* ***is a component of*** *u-tube*" is not true. Thus, this is an important point to observe: In relations, *order* is important!

A relation such as "***is a component of***" may also be expressed as an *if/then* rule. We can say "*if an object is a u-tube, then it is a component of a steam generator.*" Any ambiguity as to what degree an object is known to be a *u-tube* or a *steam generator*, or any ambiguity as to the degree of truth in such an association, results in a fuzzy relation.

When two elements belong to a relation R, we refer to them as an *ordered pair* denoted as $(a, b) \in R$, or aRb, with element a being distinguished as the first element and b as the second. With two elements in association, we have *binary relations*. With three elements we have *tertiary relations*, and when n elements are in association we have *n-ary relations*. An association of n elements in an *n-ary* relation is called *n-tuple*. A relation is any *set* of ordered *n-tuples*. The keyword here is "*set*." Relations are formed out of sets of elements, and they are sets themselves.

Crisp relations are defined over the *Cartesian product* or *product space* of two or more sets. The *Cartesian product* $X \times Y$ of two sets X and Y is the set of all ordered pairs (x, y) with x in X and y in Y. The product $X \times X$ is often abbreviated as X^2, the product $X^2 \times X$ as X^3, and so on.

We saw that relations are sets where order is important. But relations may also be thought of as *mappings*, with the process of association in mathematics being called a *mapping*. *Functions* are *mappings* as well. Relations, however, are a more general type of mapping. A function performs what is called a *many-to-one mapping*; that is, many elements are associated with one (and only one) element but not vice versa. For example, if the mapping is done between x's and y's in the $X \times Y$ plane, we may have more than one x mapped to the same y but not the other way around. Relations, however, perform *many-to-many* mappings. Many x's can be associated with a single y and vice versa. Many y's can also be associated with a single x. The importance of this abstract-sounding distinction in terms of engineering and computational applications cannot possibly be overstated, as we will see in later chapters. But for the moment let us turn our attention to an example of a crisp relation in order to see some of the ways that relations may be represented.

Example 3.1 A Crisp Relation. Let us consider a *divisibility relation*, R_d, on the set $S = \{1, 2, 3, 4, 6\}$ defined by the statement "***x divides** y*." R_d is a *binary* relation because it involves two elements, x and y, drawn from the Cartesian product of the set S with itself—that is, $S \times S$. Furthermore, it is a *crisp* relation since a number either divides another number or not (assuming integer division only). It is easy to list all the pairs of the relation and to see that the relation itself is a set, namely, the crisp set of all the pairs

$$R_d = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (6,6)\} \qquad \text{(E3.1-1)}$$

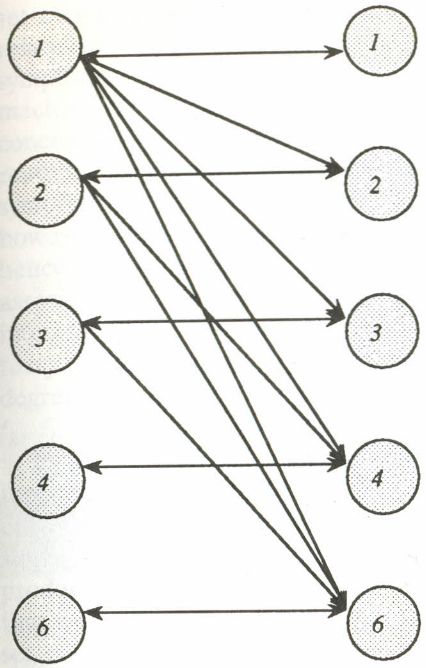

Figure 3.1 The directed graph of the divisibility relation R_d defined on the Cartesian product $S \times S$ of the set $S = \{1, 2, 3, 4, 6\}$.

where the meaning of the elements inside the parentheses is "*1* ***divides*** *1*," and so on. The relation R_d can also be represented through a graph as shown in Figure 3.1. The individual elements are represented by circles, called the *vertices* of the graph. If R_d is true for two elements, we connect them by an arrow, with the direction of the arrow indicating the order of the elements in the relation. For example, given that 3 divides 6, there is an arrow going from 3 to 6; and since 6 does not divide 3, there is no arrow going from 6 to 3. Reflecting the fact that the order of elements or the directions of the arrows is important, we call this a *directed graph*.

The binary relation R_d may also be represented by a table or a matrix. Table 3.1 shows the tabular representation of R_d. When a table entry is 1, it indicates that x (row entry) divides the corresponding y (column entry); for example, in the fourth row and fourth column we simply have that the element 4 divides itself. A 0 indicates the absence of such a relation. Should the divisibility relation have been a fuzzy relation, the table entries would be numbers between 0 and 1 as we will see later on.

R_d can also be represented by a matrix obtained from Table 3.1 by removing the column of x's on the side and the row of y's from the top; that is,

$$R_d = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(E3.1-2)}$$

Table 3.1 A tabular representation of the divisibility relation in Example 3.1

R_d:

x \ y	1	2	3	4	6
1	1	1	1	1	1
2	0	1	0	1	1
3	0	0	1	0	1
4	0	0	0	1	0
6	0	0	0	0	1

Thus we have seen five different ways for representing R_d:

1. *Linguistically*, through the statement "*x* ***divides*** *y*"
2. By listing the *set* of all ordered pairs as in equation (E3.1-1)
3. As a *directed graph* (Figure 3.2)
4. As a *table* (Table 3.1)
5. As a *matrix*, equation (E3.1-2)

It should be noted that the last two ways are generally convenient only for *binary* relations. For *tertiary* relations, for example, we would need a three-dimensional table or matrix (for *n-ary* relations *n*-dimensional tables and matrices), and therefore tables and matrices may be conveniently used only with *binary* relations. □

3.2 FUZZY RELATIONS

In fuzzy relations we consider *pairs* of elements, and more generally *n-tuples*, that are related to *a degree*. Just as the question of whether some element belongs to a set may be considered a matter of degree, whether some elements are associated may also be a matter of degree (Zadeh, 1971; Dubois and Prade, 1980). For example, suppose we have a diagnosis problem involving vibration data with a set of faults $F = \{f_1, \ldots, f_n\}$ associated to a set of symptoms $S = \{s_1, \ldots, s_m\}$. First we need to establish how symptoms relate to faults—that is, establish a relation from F to S. One of these symptoms, let's say s_i, may be "excessive vibration." Knowing whether a machine vibrates depends on the interpretation of vibration data. If the concept of "excessive vibration" has been crisply defined—that is, it can be readily determined whether the machine vibrates and we can associate a symptom s_i with a fault f_j—we have a crisp relation from F to S. In reality, however, it may be rather difficult to crisply define such associations and hence all faults $F = \{f_1, \ldots, f_n\}$ and all symptoms $S = \{s_1, \ldots, s_m\}$ may be associated to a degree, giving us a fuzzy relation from F to S. What is important in such cases is to compute these degrees. Having established the fuzzy relation from F to S, we can subsequently use it to identify the highest degrees of association given a symptom s_i so that it may be linked to faults f_k, f_j, and so forth (Kaufmann, 1975).

Fuzzy relations are fuzzy sets defined on Cartesian products. Whereas the fuzzy sets we encountered in the previous chapter were defined on a single universe of discourse (e.g., X), fuzzy relations are defined on higher-dimensional universes of discourse (e.g., $X \times X$ or $X \times Y \times Z$). A Cartesian product for us is simply a higher-dimensional universe of discourse. Suppose that we have a binary fuzzy relation R defined on $X \times Y$. As with any fuzzy set, we can list all pairs of the relation explicitly as we did in equation (2.2-2); that is,

$$R = \{((x, y), \mu_R(x, y))\} \tag{3.2-1}$$

where every individual pair (x, y) belongs to the Cartesian product $X \times Y$. Alternatively, we can use the notation of equation (2.2-3) to form the union of all $\mu_R(x, y)/(x, y)$ singletons of $X \times Y$. For a discrete Cartesian product we would have

$$R = \sum_{(x_i, y_j) \in X \times Y} \mu_R(x_i, y_j)/(x_i, y_j) \tag{3.2-2}$$

while for a continuous Cartesian product we have

$$R = \int_{X \times Y} \mu_R(x, y)/(x, y) \tag{3.2-3}$$

The same notation is used for any *n-ary fuzzy relation.*

So much for the fuzzy set nature of fuzzy relations and notation. Let us now take a look at alternative ways of representing them. One of them, which is particularly useful for the composition of relations (see Section 3.5), is to form a matrix of grades of membership in a manner analogous to (E3.1-2), only now we have instead of 0's and 1's various numbers between 0 and 1.

The *membership matrix* of an $n \times m$ binary fuzzy relation has the general form

$$R = \begin{bmatrix} \mu_R(x_1, y_1) & \mu_R(x_1, y_2) & \cdots & \mu_R(x_1, y_n) \\ \mu_R(x_2, y_1) & \mu_R(x_2, y_2) & \cdots & \mu_R(x_2, y_n) \\ & \vdots & & \\ \mu_R(x_m, y_1) & \mu_R(x_m, y_2) & \cdots & \mu_R(x_m, y_n) \end{bmatrix} \tag{3.2-4}$$

Let us take a look at some special relations and their membership matrices. The *identity fuzzy relation*, R_I, is a special type of relation which has 1 in all diagonal elements and 0 in all off-diagonal elements—that is,

$$R_I = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \end{bmatrix} \tag{3.2-5}$$

Another special relation is the *universe relation*, R_E, namely a relation with 1 everywhere in its membership matrix—that is,

$$R_E = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & 1 \\ \vdots & & \ddots & \\ 1 & 1 & & 1 \end{bmatrix} \tag{3.2-6}$$

The *null relation*, R_0, has a membership matrix with 0 everywhere—that is,

$$R_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 0 \end{bmatrix} \tag{3.2-7}$$

The transpose of a membership matrix gives the membership matrix of the *inverse relation* of R denoted by R^{-1} and defined by

$$\mu_{R^{-1}}(y, x) \equiv \mu_R(x, y) \tag{3.2-8}$$

Thus the *inverse* of the relation represented by the matrix of equation (3.2-4) has the membership matrix

$$R^{-1} = \begin{bmatrix} \mu_R(x_1, y_1) & \mu_R(x_2, y_1) & \cdots & \mu_R(x_m, y_1) \\ \mu_R(x_1, y_2) & \mu_R(x_2, y_2) & \cdots & \mu_R(x_m, y_2) \\ & \vdots & & \\ \mu_R(x_1, y_n) & \mu_R(x_2, y_n) & \cdots & \mu_R(x_m, y_n) \end{bmatrix} \tag{3.2-9}$$

which is the transpose of the matrix found by interchanging the rows of R to produce the columns of R^{-1}, and the columns of R have become the rows of R^{-1} (Klir and Folger, 1988; Terano et al., 1992). The inverse of an inverse relation is the original relation just as the inverse of the inverse of a matrix is the original matrix—that is,

$$(R^{-1})^{-1} = R \tag{3.2-10}$$

So far we defined fuzzy relations on crisp Cartesian products. However, fuzzy relations can also be defined on fuzzy Cartesian products (Kandel, 1986; Klir and Folger, 1988). Although fuzzy relations defined over fuzzy sets are of interest, particularly in connection with decision making under uncertainty, we will make no actual use of them in this book. Unless otherwise indicated, fuzzy relations in this book are assumed to be defined over crisp Cartesian products.

Example 3.2 Representing a Fuzzy Relation. Let us take two discrete sets $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ and define (subjectively) on their Cartesian product the fuzzy relation R = "*x* ***is similar*** to *y*," shown by the directed graph of Figure 3.2. R may be represented through the five different ways we saw in Example 3.1 with regard to crisp relations:

1. Linguistically, for example by the statement "*x* ***is similar to*** *y*"
2. By listing (or taking the union of) all fuzzy singletons
3. As a *directed graph* (Figure 3.2)
4. In *tabular* form
5. As a *matrix*

Let us represent the relation as a *fuzzy set* by taking the union of all singletons—that is, all ordered pairs and their membership values:

$$R = \int_{X\times Y} \mu_R(x, y)/(x, y) \tag{E3.2-1}$$

Using the data of Figure 3.2, equation (E3.2-1) gives

$$\begin{aligned} R = {} & 1.0/(x_1, y_1) + 0.3/(x_1, y_2) + 0.9/(x_1, y_3) + 0.0/(x_1, y_4) \\ & + 0.3/(x_2, y_1) + 1.0/(x_2, y_2) + 0.8/(x_2, y_3) + 1.0/(x_2, y_4) \\ & + 0.9/(x_3, y_1) + 0.8/(x_3, y_2) + 1.0/(x_3, y_3) + 0.8/(x_3, y_4) \\ & + 0.0/(x_4, y_1) + 1.0/(x_4, y_2) + 0.8/(x_4, y_3) + 1.0/(x_4, y_4) \end{aligned} \tag{E3.2-2}$$

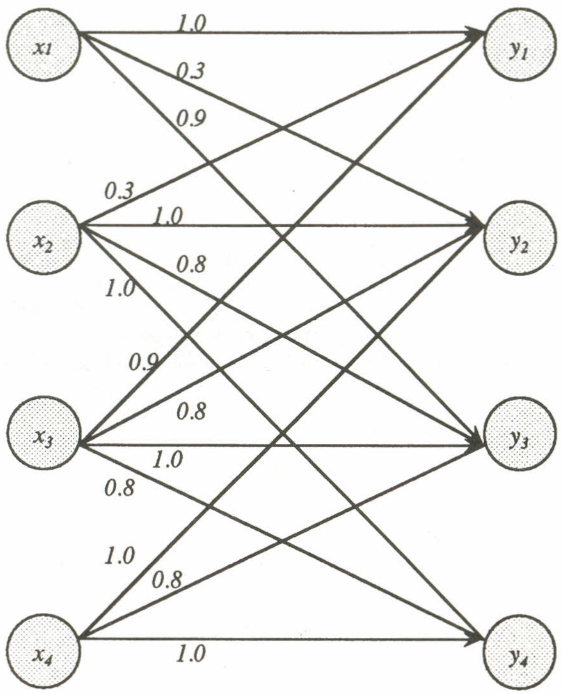

Figure 3.2 The directed graph of the fuzzy relation R in Example 3.2.

The relation R may also be represented in tabular form as

R:

	y_1	y_2	y_3	y_4
x_1	1.0	0.3	0.9	0.0
x_2	0.3	1.0	0.8	1.0
x_3	0.9	0.8	1.0	0.8
x_4	0.0	1.0	0.8	1.0

Note that compared to Table 3.1, where we only used 0's and 1's, in the tabular representation of R we find grades of membership between 0 and 1. Consider the pair (x_3, y_4). From the table of R we see that "x_3 ***is similar to*** y_4" is true to a 0.8 degree.

In matrix form, R is given by

$$R = \begin{bmatrix} 1.0 & 0.3 & 0.9 & 0.0 \\ 0.3 & 1.0 & 0.8 & 1.0 \\ 0.9 & 0.8 & 1.0 & 0.8 \\ 0.0 & 1.0 & 0.8 & 1.0 \end{bmatrix} \tag{E3.2-3}$$

The inverse of R, which we denote as R^{-1}, is the transpose of the membership matrix of equation (E3.2-3), given by

$$R^{-1} = \begin{bmatrix} 1.0 & 0.3 & 0.9 & 0.0 \\ 0.3 & 1.0 & 0.8 & 1.0 \\ 0.9 & 0.8 & 1.0 & 0.8 \\ 0.0 & 1.0 & 0.8 & 1.0 \end{bmatrix} \tag{E3.2-4}$$

Of course the inverse fuzzy relation R^{-1} in this case has the same membership matrix due to the fact that R is a symmetric relation (see next section). □

3.3 PROPERTIES OF RELATIONS

Crisp and fuzzy relations alike are classified on the basis of the mathematical properties they possess. We present here a brief introduction to the subject of properties mostly for the sake of reference. We look first at properties of crisp relations and then examine the properties of fuzzy relations. In fuzzy relations, different properties call for different requirements for the membership function of a relation.

Let S be a Cartesian product (e.g., $S = X \times Y$, with x being an element of X and y being an element of Y) and let R be a relation on S. The relation R could have the following properties:

Reflexive. We say that a relation R is *reflexive* if for any arbitrary element x in S we have that xRx is *valid*—that is, the pair (x, x) also belongs to the relation R.

Antireflexive. A relation R is *antireflexive* if there is no x in S for which xRx is valid.

Symmetric. A relation R is *symmetric* if for all x and y in S, the following is true: If xRy holds, then yRx is valid also.

Asymmetric. A relation R is *asymmetric* if there are no elements x and y in S such that both xRy and yRx are valid.

Antisymmetric. A relation R is *antisymmetric* if for all x and y in S when xRy is valid and yRx is also valid, then $x = y$.

Transitive. A relation R is called *transitive* if the following is true for all x, y, z in S: If xRy is valid and yRz is also valid, then xRz is valid as well.

Connected. A relation R is *connected* when for all x, y in S the following is true: If $x \neq y$, then either xRy is valid or yRx is valid.

Left Unique. A relation R is called *left unique* when for all x, y, z in S the following is true: If xRz is valid and yRz is also valid, then we can infer that $x = y$.

Right Unique. A relation R is *right unique* when for all x, y, z in S the following is true: If xRy and xRz hold true, then $y = z$.

Right Biunique. A relation R which is both *left unique* and *right unique* is called *biunique*.

Relations are classified into different groups on the basis of these properties. For example, an important type of crisp relation is the so-called *equivalence relation*. An equivalence relation is a relation that is *reflexive*, *symmetric*, and *transitive* (Klir and Folger, 1988). Equivalence relations are found in every corner of mathematics and are particularly useful in engineering fields such as pattern recognition, measurement, and control. Other important relations are the so-called *order relations*. For example, a relation R is called a *partial ordering* if it is *reflexive*, *transitive*, and *antisymmetric*. If R is also *connected*, then it is called a *total linear ordering*. Order relations are very important in fuzzy arithmetic (Kaufmann and Gupta, 1991).

The properties of fuzzy relations are described in terms of various requirements for their membership function. In a pioneering paper on the subject, Zadeh (1971) showed that most of the important properties of crisp relations stated above are extended to fuzzy relations as well. Let a relation R be a fuzzy relation on the Cartesian product $S = X \times X$. *Reflexivity*, *symmetry*, and *transitivity* are the three most important properties that help us properly categorize fuzzy relations. R is a *reflexive* relation if for all x in X we have that

$$\mu_R(x, x) = 1 \tag{3.3-1}$$

If for at least one x in X but not for all x's, equation (3.3-1) is not true the relation R is called *irreflexive*. If equation (3.3-1) is not satisfied for any x, then R is called *antireflexive*.

A fuzzy relation R is *symmetric* if order is not important—that is, if we can interchange x's and y's. In terms of the membership function of R, this is equivalent to saying that

$$\mu_R(x, y) = \mu_R(y, x) \tag{3.3-2}$$

If equation (3.3-2) is not satisfied for some pairs (x, y), then we say that R is *antisymmetric*; if it is not satisfied for all pairs (x, y), then we say that the relation R is *asymmetric*.

A fuzzy relation R on the Cartesian product $X \times X$ is *max–min transitive* if for two pairs (x, y) and (y, z) both in $X \times X$ we have

$$\mu_R(x, y) \geq \bigvee_z [\mu_R(x, z) \wedge \mu_R(z, y)] \tag{3.3-3}$$

where all the maxima with respect to z are taken for all the mimima inside the brackets in equation (3.3-3). *Transitivity* can be defined for other operations such as *product* $(\cdot)$ instead of min $(\wedge)$ in equation (3.3-3); in such a case we have what is called *max-product transitivity*. A relation that does not satisfy equation (3.3-3) for all pairs is called *nontransitive*, and if it fails to satisfy (3.3-3) for all pairs, then it is called *antitransitive*.

A fuzzy relation that is *reflexive* and *symmetric* is called a *proximity* or *tolerance relation*. A fuzzy relation that is *reflexive*, *symmetric*, and *transitive* is called a *similarity* relation, which is the fuzzy generalization of the *equivalence* property of crisp relations (Zadeh, 1971). Similarity relations are very important in fuzzy logic, and together with proximity relations they are crucially important in the field of fuzzy diagnosis. A *fuzzy ordering* is a fuzzy transitive relation. If a fuzzy relation is *reflexive*, *transitive*, and *antisymmetric*, then we call it a *fuzzy partial ordering*. Fuzzy orderings and similarity relations may be resolved into nonfuzzy partial orderings, in a manner analogous to the way we used the resolution principle in Chapter 2. Let us now look at an example of a fuzzy similarity relation.

Example 3.3 A Similarity Relation. Consider a fuzzy relation R indicating that two points on the $X \times Y$ plane are near the origin. This is a relation we would expect to have a membership function equal to 1 exactly at the origin and to have gradually diminishing membership as we move away from the origin. We can indicate the relation by a statement such as "*x* ***is near the origin with*** *y*" or analytically as a fuzzy set with an appropriately chosen (subjectively) membership function—for example,

$$\mu_R(x, y) = e^{-(x^2+y^2)} \tag{E3.3-1}$$

Thus the relation R is the fuzzy set

$$R = \int_{X\times Y} \mu_R(x, y)/(x, y) \tag{E3.3-2}$$

which using equation (E3.3-1) we can write as

$$R = \int_{X\times Y} e^{-(x^2+y^2)}/(x, y) \tag{E3.3-3}$$

The membership function of R is shown in Figure 3.3. It can be shown that R is a *fuzzy similarity relation*; that is, it is *reflexive*, *symmetric*, and *transitive*.

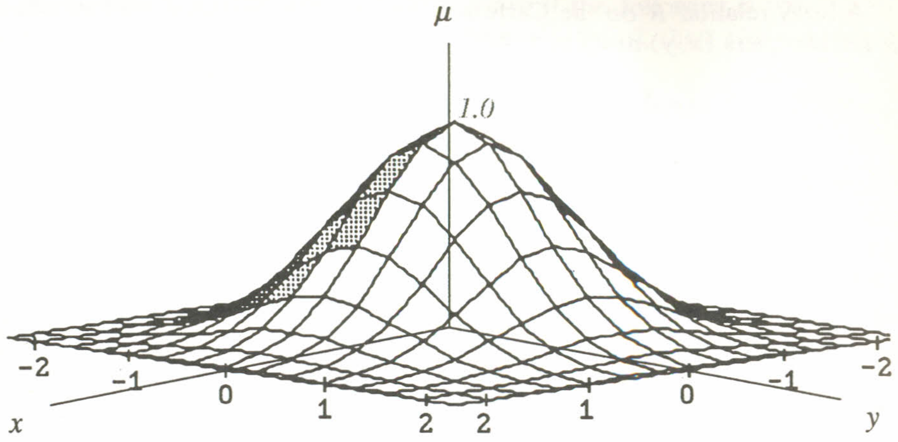

Figure 3.3 The membership function of the relation R indicating that an (x, y) point of the Cartesian plane, $X \times Y$, is close to the origin (0, 0).

Figure 3.3 also illustrates that fuzzy relations are fuzzy sets on high-dimensional universes of discourse. In this case the universe of discourse is the x–y plane—that is, the Cartesian product $X \times Y$. □

3.4 BASIC OPERATIONS WITH FUZZY RELATIONS

Fuzzy relations are fundamentally fuzzy sets defined over higher-dimensional universes of discourse—that is, Cartesian products. All the fuzzy set operations we saw in Chapter 2, such as *union*, *intersection*, *α-cuts*, and so on, are also applicable to fuzzy relations. Here we take a look at the *union*, *intersection*, *inclusion*, *α-cuts*, and *resolution* as well as some operations specific to relations such as *projection* and *cylindrical extension* (Dubois and Prades, 1980; Zimmermann, 1985).

Suppose that we have two fuzzy relations R_1 and R_2. Their *union* is a new relation

$$R_1 \cup R_2 = \int_{X\times Y} \left[\mu_{R_1}(x, y) \vee \mu_{R_2}(x, y) \right] / (x, y) \tag{3.4-1}$$

where the membership function of $R_1 \cup R_2$, as indicated in equation (3.4-1), is

$$\mu_{R_1 \cup R_2}(x, y) \equiv \mu_{R_1}(x, y) \vee \mu_{R_2}(x, y) \tag{3.4-2}$$

for every (x, y) pair of the Cartesian product.

The intersection of fuzzy relations R_1 and R_2 is a new fuzzy relation whose membership function is the minimum of the membership functions of R_1 and R_2 taken at every point (x, y) of the Cartesian product,

$$R_1 \cap R_2 = \int_{X\times Y} \left[\mu_{R_1}(x, y) \wedge \mu_{R_2}(x, y) \right] / (x, y) \tag{3.4-3}$$

where the membership function of $R_1 \cap R_2$ is

$$\mu_{R_1 \cap R_2}(x, y) \equiv \mu_{R_1}(x, y) \wedge \mu_{R_2}(x, y) \tag{3.4-4}$$

We define the α-cut of a fuzzy relation in a manner similar to the way we defined in Section 2.6 the α-cuts of one-dimensional fuzzy sets. The *resolution principle* applied to fuzzy relations offers us an alternative way of representing the membership function of a fuzzy relation. It says that the membership function of a fuzzy relation can be represented through its α-cuts. More specifically, the *resolution principle* asserts that the membership function of a fuzzy relation R is expressed in terms of its α-cuts in a manner analogous to equation (2.7-1) as

$$\mu_A(x) = \bigvee_{0<\alpha\le 1} \left[\alpha \cdot \mu_{R_\alpha}(x, y) \right] \tag{3.4-5}$$

where the maximum is taken over all α's and $\mu_{R_\alpha}(x, y)$ is the α-cut of the membership function of the relation R at level α.

We say that a relation R_1 is *included* in R_2 if both are defined over the same product space and we have everywhere

$$\mu_{R_1}(x, y) \le \mu_{R_2}(x, y) \tag{3.4-6}$$

Note that the *union* and *intersection* of fuzzy relations are meaningful in the context of relations defined over the same Cartesian product. When the product spaces of two relations are different, these operations have no meaning and instead the important and useful operations become the various *composition operations* which we examine later.

Example 3.4 Union and Intersection of Fuzzy Relations. Suppose that we have the following two relations R_1 and R_2 described by the tables below:

R_1 = "*x is larger than y*":

	y_1	y_2	y_3	y_4
x_1	0.0	0.0	0.1	0.8
x_2	0.0	0.8	0.0	0.0
x_3	0.1	0.8	1.0	0.8

$R_2 =$ "*y is much bigger than x*":

	y_1	y_2	y_3	y_4
x_1	0.4	0.4	0.2	0.1
x_2	0.5	0.0	1.0	1.0
x_3	0.5	0.1	0.2	0.6

The union of the two relations, $R_1 \cup R_2$, is formed by taking the maximum of the two grades of membership for the corresponding elements of the two tables. The table of the new relation is as follows:

$R_1 \cup R_2$:

	y_1	y_2	y_3	y_4
x_1	0.4	0.4	0.2	0.8
x_2	0.5	0.8	1.0	1.0
x_3	0.5	0.8	1.0	0.8

For the intersection, $R_1 \cap R_2$, we take the minimum of the two grades of membership in each cell of the tables of the two relations, and the resulting table is as follows:

$R_1 \cap R_2$:

	y_1	y_2	y_3	y_4
x_1	0.0	0.0	0.1	0.1
x_2	0.0	0.0	0.0	0.0
x_3	0.4	0.1	0.2	0.6

Some caution is needed when we interpret the new relations produced by *union* and *intersection*. For example, the union $R_1 \cup R_2$ can be interpreted as a proposition of the form: "*x is quite different than y*." The intersection, however, is not very meaningful, since x cannot be simultaneously larger than y and y cannot be larger than x (Zimmermann, 1985). □

In relations, when it is desired to go to a space of lower dimension we use *projection*. Starting with a fuzzy relation defined on a two-dimensional space,

we can take the *first* and *second projection* and go to one-dimensional universe of discourse, with each projection eliminating the first and second dimension, respectively. The *total projection* takes us to a zero-dimensional singleton, eliminating both dimensions. *Projections* are also called *marginal fuzzy restrictions*. The inverse of projection—that is, going toward higher dimensions—is called *cylindrical extension* (Zadeh, 1971).

Consider the fuzzy relation R defined over the Cartesian product $X \times Y$—that is,

$$R = \int_{X\times Y} \mu_R(x, y)/(x, y) \tag{3.4-7}$$

The first projection is a fuzzy set that results by eliminating the second set Y of the Cartesian product, $X \times Y$, hence projecting the relation on the universe of discourse of the first set X. We write the first projection as

$$R^1 = \int_X \mu_{R^1}(x)/x \tag{3.4-8}$$

The membership function of the first projection is defined as

$$\mu_{R^1}(x) \equiv \bigvee_y [\mu_R(x, y)] \tag{3.4-9}$$

To obtain $\mu_{R^1}(x)$, equation (3.4-9) indicates that we take the maximum of $\mu_R(x, y)$ with respect to y. Similarly the second projection (projecting on the Y universe of discourse) is a fuzzy set:

$$R^2 = \int_Y \mu_{R^2}(y)/y \tag{3.4-10}$$

with membership function defined as

$$\mu_{R^2}(y) \equiv \bigvee_x [\mu_R(x, y)] \tag{3.4-11}$$

where we take the maximum of $\mu_R(x, y)$ with respect to x. The *total projection* of R simply identifies the peak point of the relation—that is, a singleton (x_0, y_0) where the membership function of the original relation reaches its highest value.

$$R^T = \bigvee_x \bigvee_y \mu_R(x_0, y_0)/(x_0, y_0) \tag{3.4-12}$$

The opposite of projection is called the *cylindrical extension*. Through cylindrical extension we go from a fuzzy relation defined over a lower-dimensional space to a fuzzy relation on a higher-dimensional space. If a relation R is defined on a subsequence of a product space $X = X_1 \times X_2 \times X_3 \times \cdots \times X_n$, call it $X_{i1} \times X_{i2} \times X_{i3} \times \cdots \times X_{ik}$, then the cylindrical extension of R, denoted as CE(R), is defined as

$$\mathrm{CE}(R) \equiv \int_{X_1 \times \cdots \times X_n} \mu_R(x_{i1}, \ldots, x_{ik}) / (x_1, \ldots, x_n) \qquad (3.4\text{-}13)$$

Let us look at an example of projection and cylindrical extension.

Example 3.5 Projection and Cylindrical Extension. Consider the relation R defined over the Cartesian product $X \times Y$ of the sets $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5, y_6\}$ as shown in Table 3.2. The membership functions for the first and second projection are indicated by the column to the right of the table and the row below the table, respectively. The first projection is what the relation would look like if seen from the direction of the arrow on the left side of the table. Imagine that we look in the direction that the arrow on the left indicates. We see in front of us three rows of the relation and select the highest value in each row. As a result, we obtain the first projection, namely,

$$R^1 = \sum_X \mu_{R^1}(x_i)/x_i = 1.0/x_1 + 0.9/x_2 + 1.0/x_3 \qquad (\text{E}3.5\text{-}1)$$

Equation (E3.5-1) indicates that the first projection of the binary fuzzy relation R is simply a fuzzy set on a one-dimensional universe of discourse.

Table 3.2 Fuzzy relation and projections

$\Downarrow$

		y_1	y_2	y_3	y_4	y_5	y_6	$\mu_{R^1}(x)$	
	x_1	0.1	0.2	0.4	0.8	1.0	0.6	1.0	
$\Rightarrow$	x_2	0.2	0.4	0.8	0.9	0.8	0.6	0.9	
	x_3	0.5	0.9	1.0	0.8	0.4	0.2	1.0	
$\mu_{R^2}(y)$		0.5	0.9	1.0	0.9	1.0	0.6	1.0	μ_{R^T}

The second projection is what the relation would look like if seen from the direction of the arrow on top of the table.

$$R^2 = \sum_Y \mu_{R^2}(y_j)/y_j = 0.5/y_1 + 0.9/y_2 + 1.0/y_3 + 0.9/y_4 + 0.6/y_5 + 0.8/y_6 \qquad (\text{E}3.5\text{-}2)$$

The total projection is the single cell in the corner and represents the highest grade of membership that the relation has, namely, 1.

Let us next take a look at the cylindrical extension of the second projection. In a way the cylindrical extension is the *opposite* of projection. We expect therefore to obtain a relation on $X \times Y$ somewhat similar to the original relation R. As equation (E3.5-2) indicates, the second projection is defined on the Y universe of discourse. The generalization of this to the $X \times Y$ two-dimensional space is given by the *cylindrical extension.* Using equation (3.4-7) we obtain that the cylindrical extension of the second projection of the relation R^2 is simply the fuzzy set of the second projection extended in one more dimension, namely,

CE(R^2):

	y_1	y_2	y_3	y_4	y_5	y_6
x_1	0.5	0.9	1.0	0.9	1.0	0.6
x_2	0.5	0.9	1.0	0.9	1.0	0.6
x_3	0.5	0.9	1.0	0.9	1.0	0.6

Note that although the cylindrical extension of the second projection R^2 results in a relation of higher dimensionality, it did not recover the original relation R. Some information was lost through the operation of the cylindrical extension. □

3.5 COMPOSITION OF FUZZY RELATIONS

Fuzzy relations defined on different Cartesian products can be combined with each other in a number of different ways through *composition.* Composition may be thought of metaphorically as a bridge that allows us to connect one product space to another, provided that there is a common boundary. Figure 3.4 illustrates the notion. Given two fuzzy relations—one in $X \times Y$ and another on $Y \times Z$—we want to associate directly elements of X with elements of Z. The set Y is the common boundary. Composition results in a

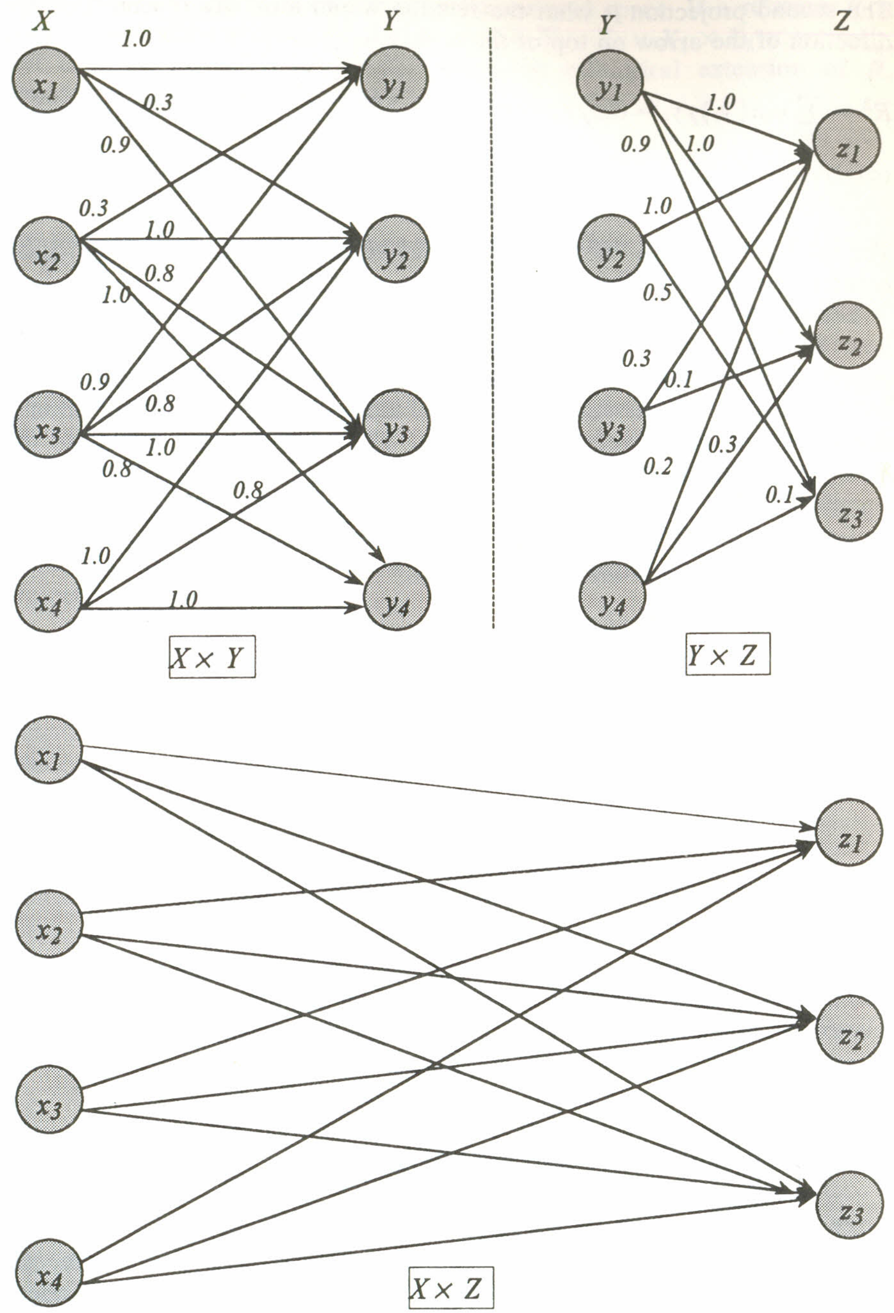

Figure 3.4 The composition of two fuzzy relations is a new relation directly associating elements from X and Z.

new relation shown at the bottom of Figure 3.4 that directly relates X to Z. Our main task in composition is to compute the grades of membership of the pairs (x, z) in the composed relation, namely, $\mu(x, z)$ (not shown in Figure 3.4).

Composition is very important for inferencing procedures used in linguistic descriptions of systems and is particularly useful in fuzzy controllers and expert systems (Klir and Folger, 1988). As we shall see in Chapters 5 and 6, collections of fuzzy *if/then* rules or *fuzzy algorithms* are mathematically equivalent to fuzzy relations, and the problem of *inferencing* or (*evaluating* them with specific inputs) is mathematically equivalent to *composition*. There are several types of composition. By far the most common in engineering applications is *max–min composition*, but we will also look at *max-star*, *max-product*, and *max-average*. In general, different types of composition result in different composed relations.

Max–Min Composition

The max–min composition of two fuzzy relations uses the familiar operators of fuzzy sets, max ($\vee$) and min ($\wedge$) (see Section 2.3). Suppose that we have two fuzzy relations $R_1(x, y)$ and $R_2(y, z)$ defined over the Cartesian products $X \times Y$ and $Y \times Z$, respectively. The max–min composition of R_1 and R_2 is a new relation $R_1 \circ R_2$ defined on $X \times Z$ as

$$R_1 \circ R_2 \equiv \int_{X \times Z} \bigvee_y \left[\mu_{R_1}(x, y) \wedge \mu_{R_2}(y, z) \right] \big/ (x, z) \qquad (3.5\text{-}1)$$

where the symbol "$\circ$" stands for max–min composition of relations R_1 and R_2. When the Cartesian product $X \times Y$ is discrete, then the integral (union) sign in (3.5-1) is replaced by summation. From equation (3.5-1) we see that the grade of membership of each (x, z) pair in the new relation is

$$\mu_{R_1 \circ R_2}(x, z) = \bigvee_y \left[\mu_{R_1}(x, y) \wedge \mu_{R_2}(y, z) \right] \qquad (3.5\text{-}2)$$

where the outer maximum is taken with respect to the elements y of the common boundary. The operation on the right-hand side of equation (3.5-2) is actually very similar to matrix multiplication, with max ($\vee$) being analogous to *summation* ($+$) and min ($\wedge$) being analogous to *multiplication* ($\cdot$), as we will see in the examples that follow. Interchanging min and max in (3.5-1) is known as the *min–max composition*. In this book, however, we will mostly use max–min composition and compositions where the final (outer) operand is max ($\vee$). Max–min composition is used extensively in diagnostic and control applications of fuzzy logic.

Max–Star Composition

We can use *multiplication, summation*, or some other *binary operation* $(*)$ in place of min $(\wedge)$ in equations (3.5-1) and (3.5-2) while still performing maximization with respect to y. This type of composition of two fuzzy relations is generally known as the "*max-star*" or "*max-* $*$ *composition*."[1]

Suppose that we have two fuzzy relations R_1 and R_2 defined over the Cartesian products $X \times Y$ and $Y \times Z$, respectively. The max-$*$ composition of R_1 and R_2 is the new relation

$$R_1 * R_2 \equiv \int_{X \times Z} \bigvee_y \left[\mu_{R_1}(x, y) * \mu_{R_2}(y, z) \right] \big/ (x, z) \qquad (3.5\text{-}3)$$

We see from equation (3.5-3) that the membership function of the new relation is

$$\mu_{R_1 * R_2}(x, z) = \bigvee_y \left[\mu_{R_1}(x, y) * \mu_{R_2}(y, z) \right] \qquad (3.5\text{-}4)$$

When the Cartesian product is discrete the integral sign in equation (3.5-3) is replaced by summation. Again as we shall see in the examples that follow this is essentially a computational procedure very similar to matrix multiplication. Two special cases of the max-star composition are the *max-product* (or *max-prod*) and the *max-average* composition.

Max-Product Composition

In max-product composition we use product $(\cdot)$ in place of $(*)$ in equations (3.5-3) and (3.5-4). Thus the max-product composition of two relations R_1 and R_2 is

$$R_1 \cdot R_2 \equiv \int_{X \times Z} \bigvee_y \left[\mu_{R_1}(x, y) \cdot \mu_{R_2}(y, z) \right] \big/ (x, z) \qquad (3.5\text{-}5)$$

For discrete product spaces we use the summation sign in equation (3.5-5). The membership function of the composed relation is given by

$$\mu_{R_1 \cdot R_2}(x, z) = \bigvee_y \left[\mu_{R_1}(x, y) \cdot \mu_{R_2}(y, z) \right] \qquad (3.5\text{-}6)$$

[1]The name "star" refers to the star symbol that stands for a number of operations such as *average* and *product*.

Max–Average Composition

In the max-average composition of fuzzy relations we use the arithmetic sum $(+)$ divided by 2 in place of $(*)$ in equations (3.5-3) and (3.5-4). Thus the max-average composition of R_1 with R_2 is a new relation $R_1\langle + \rangle R_2$ given by

$$R_1\langle + \rangle R_2 \equiv \int_{X \times Z} \bigvee_y \left[\tfrac{1}{2}\left(\mu_{R_1}(x, y) + \mu_{R_2}(y, z) \right) \right] \big/ (x, z) \qquad (3.5\text{-}7)$$

with membership function

$$\mu_{R_1\langle + \rangle R_2}(x, z) = \bigvee_y \left[\tfrac{1}{2}\left(\mu_{R_1}(x, y) + \mu_{R_2}(y, z) \right) \right] \qquad (3.5\text{-}8)$$

Let us take a look at a few examples of composition.

Example 3.6 Max–Min Composition of Fuzzy Relations. Let's use max–min composition with the two relations shown in the upper part of Figure 3.4. The membership matrices of the relations R_1 on $X \times Y$ and R_2 on $Y \times Z$ are

$$R_1 = \begin{bmatrix} \mu_{R_1}(x_1, y_1) & \mu_{R_1}(x_1, y_2) & \mu_{R_1}(x_1, y_3) & \mu_{R_1}(x_1, y_4) \\ \mu_{R_1}(x_2, y_1) & \mu_{R_1}(x_2, y_2) & \mu_{R_1}(x_2, y_3) & \mu_{R_1}(x_2, y_4) \\ \mu_{R_1}(x_3, y_1) & \mu_{R_1}(x_3, y_2) & \mu_{R_1}(x_3, y_3) & \mu_{R_1}(x_3, y_4) \\ \mu_{R_1}(x_4, y_1) & \mu_{R_1}(x_4, y_2) & \mu_{R_1}(x_4, y_3) & \mu_{R_1}(x_4, y_4) \end{bmatrix}$$

$$= \begin{bmatrix} 1.0 & 0.3 & 0.9 & 0.0 \\ 0.3 & 1.0 & 0.8 & 1.0 \\ 0.9 & 0.8 & 1.0 & 0.8 \\ 0.0 & 1.0 & 0.8 & 1.0 \end{bmatrix} \qquad (E3.6\text{-}1)$$

and

$$R_2 = \begin{bmatrix} \mu_{R_2}(y_1, z_1) & \mu_{R_2}(y_1, z_2) & \mu_{R_2}(y_1, z_3) \\ \mu_{R_2}(y_2, z_1) & \mu_{R_2}(y_2, z_2) & \mu_{R_2}(y_2, z_3) \\ \mu_{R_2}(y_3, z_1) & \mu_{R_2}(y_3, z_2) & \mu_{R_2}(y_3, z_3) \\ \mu_{R_2}(y_4, z_1) & \mu_{R_2}(y_4, z_2) & \mu_{R_2}(y_4, z_3) \end{bmatrix}$$

$$= \begin{bmatrix} 1.0 & 1.0 & 0.9 \\ 1.0 & 0.0 & 0.5 \\ 0.3 & 0.1 & 0.0 \\ 0.2 & 0.3 & 0.1 \end{bmatrix} \qquad (E3.6\text{-}2)$$

We want to compute the membership matrix of the max–min composition of R_1 and R_2. We can use equation (3.5-2) to obtain the membership function of the composed relation. The operations in (3.5-2) are similar to matrix multiplication, with $(\vee)$ being treated like *summation* $(+)$ and $(\wedge)$ being treated like *multiplication* $(\cdot)$. With this in mind, instead of using

$$\mu_{R_1 \circ R_2}(x, z) = \bigvee_y \left[\mu_{R_1}(x, y) \wedge \mu_{R_2}(y, z) \right]$$

we can use the matrix form of max–min composition, namely,

$$R_1 \circ R_2 = \begin{bmatrix} 1.0 & 0.3 & 0.9 & 0.0 \\ 0.3 & 1.0 & 0.8 & 1.0 \\ 0.9 & 0.8 & 1.0 & 0.8 \\ 0.0 & 1.0 & 0.8 & 1.0 \end{bmatrix} \circ \begin{bmatrix} 1.0 & 1.0 & 0.9 \\ 1.0 & 0.0 & 0.5 \\ 0.3 & 0.1 & 0.0 \\ 0.2 & 0.3 & 0.1 \end{bmatrix} \qquad \text{(E3.6-3)}$$

To evaluate equation (E3.6-3) we proceed, like in matrix multiplication, by forming the pairs of minima of each element in the first row of membership matrix R_1 with every element in the first column of membership matrix R_2. For example, to obtain the first element, (x_1, z_1), of the composition we perform the following operations:

$$[1.0 \quad 0.3 \quad 0.9 \quad 0.0] \circ \begin{bmatrix} 1.0 \\ 1.0 \\ 0.3 \\ 0.2 \end{bmatrix}$$

$$= [1.0 \wedge 1.0] \vee [0.3 \wedge 1.0] \vee [0.9 \wedge 0.3] \vee [0.0 \wedge 0.2]$$
$$= 1.0 \vee 0.3 \vee 0.3 \vee 0.0$$
$$= 1.0$$

We repeat this procedure for all rows and columns and the result is the membership matrix of the composed relation $R_1 \circ R_2$ given by

$$R_1 \circ R_2 = \begin{bmatrix} 1.0 & 1.0 & 0.9 \\ 1.0 & 0.3 & 0.5 \\ 0.9 & 0.9 & 0.9 \\ 1.0 & 0.3 & 0.5 \end{bmatrix} \qquad \text{(E3.6-4)}$$

The new relation is a fuzzy set over the Cartesian product $X \times Z$ which may also be written as

$$\begin{aligned} R_1 \circ R_2 = {} & 1.0/(x_1, z_1) + 1.0/(x_1, z_2) + 0.9/(x_1, z_3) \\ & + 1.0/(x_2, z_1) + 0.3/(x_2, z_2) + 0.5/(x_2, z_3) \\ & + 0.9/(x_3, z_1) + 0.9/(x_3, z_2) + 0.9/(x_3, z_3) \\ & + 1.0/(x_4, z_1) + 0.3/(x_4, z_2) + 0.5/(x_4, z_3) \qquad \text{(E3.6-5)} \end{aligned}$$

□

Example 3.7 Max–Min, Max-Product, and Max-Average Composition of Fuzzy Relations. Suppose we have the two relations R_1 and R_2, shown below, and we want to compute a new relation which is the max–min composition of the two, $R = R_1 \circ R_2$. We will also find the max-product and max-average compositions. We perform max–min composition using the tabular representation of the relations and the definition of max–composition given in equations (3.5-1) or (3.5-2). The relations to be composed are described by the following membership tables:

R_1:

	y_1	y_2	y_3	y_4	y_5
x_1	0.1	0.2	0.0	1.0	0.7
x_2	0.3	0.5	0.0	0.2	1.0
x_3	0.8	0.0	1.0	0.4	0.3

R_2:

	z_1	z_2	z_3	z_4
y_1	0.9	0.0	0.3	0.4
y_2	0.2	1.0	0.8	0.0
y_3	0.8	0.0	0.7	1.0
y_4	0.4	0.2	0.3	0.0
y_5	0.0	1.0	0.0	0.8

Questão 2

To find the new relation $R = R_1 \circ R_2$ we use equation (3.5-2), the definition of max–min composition, namely,

$$\mu_{R_1 \circ R_2}(x, z) = \bigvee_y \left[\mu_{R_1}(x, y) \wedge \mu_{R_2}(y, z) \right] \qquad \text{(E3.7-1)}$$

To use (E3.7-1) we proceed in the following manner. First, we fix x and z—for example, $x = x_1$ and $z = z_1$—and vary y. Next, we evaluate the following pairs of minima, using the numbers from the shaded cells in the tables of the two relations:

$$
\begin{aligned}
\mu_{R_1}(x_1, y_1) \wedge \mu_{R_2}(y_1, z_1) &= 0.1 \wedge 0.9 = 0.1 \\
\mu_{R_1}(x_1, y_2) \wedge \mu_{R_2}(y_2, z_1) &= 0.2 \wedge 0.2 = 0.2 \\
\mu_{R_1}(x_1, y_3) \wedge \mu_{R_2}(y_3, z_1) &= 0.0 \wedge 0.8 = 0.0 \\
\mu_{R_1}(x_1, y_4) \wedge \mu_{R_2}(y_4, z_1) &= 1.0 \wedge 0.4 = 0.4 \\
\mu_{R_1}(x_1, y_5) \wedge \mu_{R_2}(y_5, z_1) &= 0.7 \wedge 0.0 = 0.0
\end{aligned}
\tag{E3.7-2}
$$

We take the maximum of all these terms and obtain the value of the (x_1, z_1) element of the relation, namely,

$$\mu_{R_1 \circ R_2}(x_1, z_1) = 0.1 \vee 0.2 \vee 0.0 \vee 0.4 = 0.4 \tag{E3.7-3}$$

This is the value in the shaded cell in the table of the composed relation shown below. In a similar manner, we determine the grades of membership for all other pairs and finally we have

$R = R_1 \circ R_2$:

	z_1	z_2	z_3	z_4
x_1	0.4	0.7	0.3	0.7
x_2	0.3	1.0	0.5	0.8
x_3	0.8	0.3	0.7	1.0

Let us now compose these two relations using max-product composition as defined by equation (3.5-6)—that is,

$$\mu_{R_1 \cdot R_2}(x, z) = \bigvee_y \left[\mu_{R_1}(x, y) \cdot \mu_{R_2}(y, z) \right] \tag{E3.7-4}$$

Again we fix x and z and vary y—for example, $x = x_1$, $z = z_1$, and $y = y_i$ for $i = 1, \ldots, 5$. We form and evaluate the products of the shaded cells in the relation tables—that is,

$$
\begin{aligned}
\mu_{R_1}(x_1, y_1) \cdot \mu_{R_2}(y_1, z_1) &= 0.1 \times 0.9 = 0.09 \\
\mu_{R_1}(x_1, y_2) \cdot \mu_{R_2}(y_2, z_1) &= 0.2 \times 0.2 = 0.04 \\
\mu_{R_1}(x_1, y_3) \cdot \mu_{R_2}(y_3, z_1) &= 0.0 \times 0.8 = 0.0 \\
\mu_{R_1}(x_1, y_4) \cdot \mu_{R_2}(y_4, z_1) &= 1.0 \times 0.4 = 0.4 \\
\mu_{R_1}(x_1, y_5) \cdot \mu_{R_2}(y_5, z_1) &= 0.7 \times 0.0 = 0.0
\end{aligned}
\tag{E3.7-5}
$$

Taking the maximum of these terms, we obtain the grade of membership of the (x_1, z_1) pair in the composed relation, namely,

$$\mu_{R_1 \cdot R_2}(x_1, z_1) = 0.09 \vee 0.04 \vee 0.0 \vee 0.4 \vee 0.0 \tag{E3.7-6}$$

which coincidentally evaluates also to $\mu_{R_1 \cdot R_2}(x_1, z_1) = 0.4$. This is the number in the shaded cell of the table below. Similarly, we obtain the membership of all other pairs and finally we get the membership table of the composition as

$R_1 \cdot R_2$:

	z_1	z_2	z_3	z_4
x_1	0.4	0.7	0.3	0.56
x_2	0.27	1.0	0.4	0.8
x_3	0.8	0.3	0.7	1.0

For the max-average composition of the two relations, again we fix x and z and vary y in order to find the max with respect to y in equation (3.5-8) for each (x, z) pair. Thus first we form and evaluate the sums of the shaded cells as before:

$$
\begin{aligned}
\mu_{R_1}(x_1, y_1) + \mu_{R_2}(y_1, z_1) &= 0.1 + 0.9 = 1.0 \\
\mu_{R_1}(x_1, y_2) + \mu_{R_2}(y_2, z_1) &= 0.2 + 0.2 = 0.4 \\
\mu_{R_1}(x_1, y_3) + \mu_{R_2}(y_3, z_1) &= 0.0 + 0.8 = 0.8 \\
\mu_{R_1}(x_1, y_4) + \mu_{R_2}(y_4, z_1) &= 1.0 + 0.4 = 1.4 \\
\mu_{R_1}(x_1, y_5) + \mu_{R_2}(y_5, z_1) &= 0.7 + 0.0 = 0.7
\end{aligned}
\tag{E3.7-7}
$$

Thus, using equation (3.5-8), the grade of membership of the (x_1, z_1) pair is

$$\mu_{R_1\langle + \rangle R_2}(x_1, z_1) = \tfrac{1}{2}[1.0 \vee 0.4 \vee 0.8 \vee 1.4 \vee 0.7] = 0.7 \quad \text{(E3.7-8)}$$

This is the grade of membership of the shaded cell in the table shown below. In a similar manner the membership function for each pair is computed, and finally we get the max-average composition of the two relations in the table:

$R_1\langle + \rangle R_2$:

	z_1	z_2	z_3	z_4
x_1	0.7	0.85	0.65	0.75
x_2	0.6	1.0	0.65	0.9
x_3	0.9	0.65	0.85	1.0

We observe from the tables of the composed relations that max–min, max-product, and max-average compositions of R_1 and R_2 may result in different relations. □

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PROBLEMS 03/08/2005

1. A fuzzy "diagnostic relation" $\boldsymbol{R}_d$ for an automobile relates the system set $\boldsymbol{S}$ to the fault set $\boldsymbol{F}$. These sets are given below.

$$S = [x_1 \text{ (low gas mileage)}, x_2 \text{ (excessive vibration)}, x_3 \text{ (loud noise)}, x_4 \text{ (high collant temperature)}, x_5 \text{ (steering instability)}]$$

$$F = [y_1 \text{ (bad spark plugs)}, y_2 \text{ (wheel imbalance)}, y_3 \text{ (bad muffler)}, y_4 \text{ (thermostat stuck closed)}]$$

Assume reasonable numerical values $(0 \to 1)$ for membership values relating members of sets $\boldsymbol{S}$ and $\boldsymbol{F}$ and use them. Give all five representations of this fuzzy diagnostic relationship $\boldsymbol{R}_d$ in terms of x_i and y_j.

2. Give the max–min composition, max-star composition, and the max-average composition of the relation fuzzy "diagnostic relation" of Problem 1.

3. Repeat Example 3.3 for a fuzzy relation R indicating that "x is near the perimeter of a circle having a radius 1 with y".

4. In Example 3.4, give a table for $[R_1 \cap R_2] \cup [R_1 \cap R_2]$.

5. Find the first, second, and total projection as well as the cylindrical extension of the fuzzy relation R given by Equation (E3.2-2).

6. Find the max-product and max-average composition of relations R_1 and R_2 given by Equations (E3.6-1) and (E3.6-2), respectively.

7. Find the max–min composition of relations R_1 and R_2 given in Example 3.7.

8. a. Show that the max–min composition of fuzzy relations is *associative*. Illustrate with an example of your own.

Verificar **b.** Consider the max-min composition and a relation R which is *reflexive*. e transitiva
Show that:

$$R \circ R = R.$$

9. Suppose that we have three relations involved in max–min composition

$$P \circ Q = R$$

When two of the components in the above equation are given and the other is unknown we have a set of equations known as *fuzzy relation equations*. Solve the following fuzzy relation equations:

(a) $P \circ \begin{bmatrix} .9 & .6 & 1 \\ .8 & .8 & .5 \\ .6 & .4 & .6 \end{bmatrix} = [.6 \quad .6 \quad .5]$

Verificar (b) $P \circ \begin{bmatrix} .2 & .4 & .5 & .7 \\ .3 & .1 & .6 & .8 \\ .1 & .4 & .6 & .7 \\ 0 & .3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} .2 & .4 & .6 & .7 \\ .1 & .1 & .2 & .2 \end{bmatrix}$

10. Consider two probability distributions that are independent and described by

$$dP(x_1) = e^{-x_1}\, dx_1 \text{ and } dP(x_2) = x_2 e^{-x_2}\, dx_2,\ x_1, x_2 \geq 0$$

How can we model the *similarity* of x_1, x_2 through a fuzzy set and what would be the probability of occurrence of such a set?