A Derivation of Special Relativity

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***Abstract*—Special Relativity, while giving credit mainly to Albert Einstein, was the work of additional mathematicians and Scientists including: Hendrik Lorentz, Henri Poincaré, and Herman Minkowski. While all of these notable physicists and mathematicians are credited for work (it is the work of Lorentz who originally developed the transforms), Einstein is given credit for actual "concept" of Special Relativity. This paper is a summation of my personal studies (for the past twenty eight years) in the subject.**

I. A Short History

Special Relativity starts in part with the work of Maxwell's Equations of Light. In those equations, Maxwell shows that

1) Light is an electromagnetic wave
2) The Speed of light is a constant based on the wave equation
   a) This was originally predicted in the general wave equation
   b) From the equation we see the speed of light is: $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ where ϵ_0 is the permitivity of free space and μ_0 is the permeability of free space

Hertz would later confirm what Maxwell theorized mathematically. There were still several issues related to the work of Maxwell. Of particular interrest in this subject were the

1) Prediction and verification of the Lumiferous Aether–the theoretical non-viscous medium that allows light (in the form of waves)to travel through free space
2) The Absolute Reference Frame–required to hold that time is absolute for every observer regardless of referece frame as predicted by Newton

A. The Search for the Speed of Light

The search for the Speed of Light proceded in the late 19th-century. In 1887, an experiment was done by Albert Michelson and Edward Morely to determine several factors related to light this included:

1) The verification of the existence the aether
2) The first determination of the speed of light: c
   a) From this, the values of ϵ_0 and μ_0 could be better acertained

1) The Results for the Michelson-Morley Experiment: The Michelson-Morely experiment showed no evidence of the aether.

In addition, the Michelson-Morely experiment showeed the speed of light measurement did not vary based on the speed *relative* to the observer measuring the speed of light.

a) What does that mean?: As stated above the speed of light measured by the Michelson-Morely experiment was a constant and indifferent to the reference frame of the observer. Essentially, that means that if an observer is going $.5c$ towards a light source, they will *not* measure the speed of light from that light source to be $1.5c$, and if an observer is going $.5c$ away from a light source, they will *not* measure the speed of light from that light source to be $.5c$; but rather c in *both* cases.

b) Questions Answered but Left "Unanswered: By nature, the wave equation *shows* the wave's velocity is constant. For light, Maxwell's equation directly verifies the speed of light is a constant (see above), and that velocity of light is *independent of reference frame*. Yet, the Michelson-Morely experiment still had scientists confused and bewildered. The main reason is due to the lack of the verifcation of the aether. Maxwell's Equations were believed to work (or be valid) based on the aether. Afterwards, many scientists proposed the aether to be a non-viscous fluid: that was not verified, regardless. As a result, while there was empirical evidence of the speed of light, one question was answered, this experiment created more questions.

B. The History of the Subject as Presented

As stated in the abstract, there were several scientists and mathematicians involved in the study and development of Special Relativity: Einstein, Poincaré, Lorentz and (later) Minkowski. While all worked on the theory of Special Relativity, Einstein is given credit for the actual theory. This is due to the fact that Einstein presented not only the mathematics of the theory (that was completed by Lorentz and Poincaré; Einstein working independently "discovered" the same equations); rather, Einstein presented a physical description of the Special Theory of Relativity. Mainly, Einstein presented three postulates to the theory that neither Lorentz nor Poincaré presented:

1) The speed of light is a constant regardless of reference frame
2) There is no aether
3) (As a result from the above postulate) There is no absolute reference frame

C. The Mathematics Presented

The Mathematics derived in this paper are based on the work of several of the scientists and mathematicians that worked on the theory of Special Relativity: Mainly Lorentz and Minkowski, as well as Einstein. When credit is known, credit will be given as appropriate.

D. More History to follow

It should be noted even though there is a "history" section, this is not meant to imply there will be no more history will follow in the process of the derivation.

II. Galillean Relativity

This paper starts with Galillean Relativity.
The following are the equations commonly known as Galilean Relativity:

$$x' = x - vt \tag{1}$$
$$y' = y \tag{2}$$
$$z' = z \tag{3}$$
$$t' = t \tag{4}$$

and

$$x = x' + vt \tag{5}$$
$$y = y' \tag{6}$$
$$z = z' \tag{7}$$
$$t = t' \tag{8}$$

Where the prime values (prime frame) are the position values for the observer in the moving frame.
In the history of the study of Special Relativity, plugging the Galillean Transforms (eqns 1-8) into Maxwell's Equations presented an odd anomaly that showed the transforms were non-invariant. From a mathematical perspective, this seems to be the "best place" to start the derivation presented in this paper.

A. Plugging Galillean Relativity into Maxwell's Equations

1) Maxwell's Equations: From Maxwell, it was mathematically shown that light is an electromagnetic wave:

$$\nabla^2 E = \mu_0 \epsilon_0 \ddot{E} \tag{9}$$
$$\nabla^2 B = \mu_0 \epsilon_0 \ddot{B} \tag{10}$$

This can be written in the "Generic Form" of the Wave Equation:

$$\nabla^2 \varphi = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} \tag{11}$$

Where

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \tag{12}$$

and ϵ_0 is the permitivity of free space and μ_0 is the permeability of free space.

2) Putting Galillean Relativity Equations in Maxwell's equations: Using the equations:

$$x' = x - vt \tag{13}$$
$$y' = y \tag{14}$$
$$z' = z \tag{15}$$
$$t' = t \tag{16}$$

Given:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{17}$$

From the chain rule of partial derivatives, it can be shown that:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial x'} \cancelto{1}{\frac{\partial x'}{\partial x}} + \cancelto{0}{\frac{\partial \varphi}{\partial y'} \frac{\partial y'}{\partial x}} + \cancelto{0}{\frac{\partial \varphi}{\partial z'} \frac{\partial z'}{\partial x}} + \cancelto{0}{\frac{\partial \varphi}{\partial t'} \frac{\partial t'}{\partial x}} \tag{18}$$

Therefore:

$$\boxed{\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial x'}} \tag{19}$$

$$\boxed{Let \quad \alpha \equiv \frac{\partial \varphi}{\partial x'}} \tag{20}$$

Continuing, the same thing can be done for y, z and t.

$$\frac{\partial \varphi}{\partial y} = \cancelto{0}{\frac{\partial \varphi}{\partial x'} \frac{\partial x'}{\partial y}} + \frac{\partial \varphi}{\partial y'} \cancelto{1}{\frac{\partial y'}{\partial y}} + \cancelto{0}{\frac{\partial \varphi}{\partial z'} \frac{\partial z'}{\partial y}} + \cancelto{0}{\frac{\partial \varphi}{\partial t'} \frac{\partial t'}{\partial y}} \tag{21}$$

Therefore:

$$\boxed{\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial y'}} \tag{22}$$

$$\boxed{Let \quad \beta \equiv \frac{\partial \varphi}{\partial y'}} \tag{23}$$

$$\frac{\partial \varphi}{\partial z} = \cancelto{0}{\frac{\partial \varphi}{\partial x'} \frac{\partial x'}{\partial z}} + \cancelto{0}{\frac{\partial \varphi}{\partial y'} \frac{\partial y'}{\partial z}} + \frac{\partial \varphi}{\partial z'} \cancelto{1}{\frac{\partial z'}{\partial z}} + \cancelto{0}{\frac{\partial \varphi}{\partial t'} \frac{\partial t'}{\partial z}} \tag{24}$$

Therefore:

$$\boxed{\frac{\partial \varphi}{\partial z} = \frac{\partial \varphi}{\partial z'}} \tag{25}$$

$$\boxed{Let \quad \gamma \equiv \frac{\partial \varphi}{\partial z'}} \tag{26}$$

The partial with respect to t is more tricky:

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial x'} \cancelto{-v}{\frac{\partial x'}{\partial t}} + \cancelto{0}{\frac{\partial \varphi}{\partial y'} \frac{\partial y'}{\partial t}} + \cancelto{0}{\frac{\partial \varphi}{\partial z'} \frac{\partial z'}{\partial t}} + \frac{\partial \varphi}{\partial t'} \cancelto{1}{\frac{\partial t'}{\partial t}} \tag{27}$$

$$\boxed{\frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial t'} - v \frac{\partial \varphi}{\partial x'}} \tag{28}$$

$$\boxed{Let \quad \Delta \equiv \frac{\partial \varphi}{\partial t'} - v \frac{\partial \varphi}{\partial x'}} \tag{29}$$

Using $\underline{\alpha}$ substitution from above:

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial \varphi}{\partial x} \right] = \frac{\partial}{\partial x} \left[\frac{\partial \varphi}{\partial x'} \right] = \frac{\partial \alpha}{\partial x} \tag{30}$$

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \alpha}{\partial x'} \cancelto{1}{\frac{\partial x'}{\partial x}} + \cancelto{0}{\frac{\partial \alpha}{\partial y'} \frac{\partial y'}{\partial x}} + \cancelto{0}{\frac{\partial \alpha}{\partial z'} \frac{\partial z'}{\partial x}} + \cancelto{0}{\frac{\partial \alpha}{\partial t'} \frac{\partial t'}{\partial x}} \tag{31}$$

$$\frac{\partial \alpha}{\partial x} = \frac{\partial}{\partial x'} \left[\frac{\partial \varphi}{\partial x'} \right] = \frac{\partial^2 \varphi}{\partial x'^2} \tag{32}$$

Therefore:

$$\boxed{\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \varphi}{\partial x'^2}} \tag{33}$$

Using the $\underline{\beta}$ substitution from above:

$$\frac{\partial^2\varphi}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial\varphi}{\partial y}\right] = \frac{\partial}{\partial y}\left[\frac{\partial\varphi}{\partial y'}\right] = \frac{\partial\beta}{\partial y} \tag{34}$$

$$\frac{\partial\beta}{\partial y} = \cancelto{0}{\frac{\partial\beta}{\partial x'}\frac{\partial x'}{\partial y}} + \frac{\partial\beta}{\partial y'}\cancelto{1}{\frac{\partial y'}{\partial y}} + \cancelto{0}{\frac{\partial\beta}{\partial z'}\frac{\partial z'}{\partial y}} + \cancelto{0}{\frac{\partial\beta}{\partial t'}\frac{\partial t'}{\partial y}} \tag{35}$$

$$\frac{\partial\beta}{\partial y} = \frac{\partial}{\partial y'}\left[\frac{\partial\varphi}{\partial y'}\right] = \frac{\partial^2\varphi}{\partial y'^2} \tag{36}$$

Therefore:

$$\boxed{\frac{\partial^2\varphi}{\partial y^2} = \frac{\partial^2\varphi}{\partial y'^2}} \tag{37}$$

By the similarites above (using the $\underline{\gamma}$ substitution):

$$\boxed{\frac{\partial^2\varphi}{\partial z^2} = \frac{\partial^2\varphi}{\partial z'^2}} \tag{38}$$

Using Δ substitution:

$$\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{\partial\varphi}{\partial t}\right] = \frac{\partial}{\partial t}\left[\frac{\partial\varphi}{\partial t'} - v\frac{\partial\varphi}{\partial x'}\right] = \frac{\partial\Delta}{\partial t} \tag{39}$$

$$\frac{\partial\Delta}{\partial t} = \frac{\partial\Delta}{\partial x'}\cancelto{-v}{\frac{\partial x'}{\partial t}} + \cancelto{0}{\frac{\partial\Delta}{\partial y'}\frac{\partial y'}{\partial t}} + \cancelto{0}{\frac{\partial\Delta}{\partial z'}\frac{\partial z'}{\partial t}} + \frac{\partial\Delta}{\partial t'}\cancelto{1}{\frac{\partial t'}{\partial t}} \tag{40}$$

$$\frac{\partial\Delta}{\partial t} = \frac{\partial\Delta}{\partial t'} - v\frac{\partial\Delta}{\partial x'} \tag{41}$$

$$\frac{\partial\Delta}{\partial t} = \frac{\partial}{\partial t'}\left[\frac{\partial\varphi}{\partial t'} - v\frac{\partial\varphi}{\partial x'}\right] - v\frac{\partial}{\partial x'}\left[\frac{\partial\varphi}{\partial t'} - v\frac{\partial\varphi}{\partial x'}\right] \tag{42}$$

$$\frac{\partial\Delta}{\partial t} = v^2\frac{\partial^2\varphi}{\partial x'^2} - 2v\frac{\partial^2\varphi}{\partial x'\partial t'} + \frac{\partial^2\varphi}{\partial t'^2} \tag{43}$$

Therefore:

$$\frac{\partial^2\varphi}{\partial t^2} = v^2\frac{\partial^2\varphi}{\partial x'^2} - 2v\frac{\partial^2\varphi}{\partial x'\partial t'} + \frac{\partial^2\varphi}{\partial t'^2} \tag{44}$$

and:

$$\boxed{\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{v^2}{c^2}\frac{\partial^2\varphi}{\partial x'^2} - \frac{2v}{c^2}\frac{\partial^2\varphi}{\partial x'\partial t'} + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t'^2}} \tag{45}$$

Combining equations 33,37,38 and 45, equation 11 becomes:

$$\boxed{\frac{\partial^2\varphi}{\partial x'^2} + \frac{\partial^2\varphi}{\partial y'^2} + \frac{\partial^2\varphi}{\partial z'^2} = \frac{v^2}{c^2}\frac{\partial^2\varphi}{\partial x'^2} - \frac{2v}{c^2}\frac{\partial^2\varphi}{\partial x'\partial t'} + \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t'^2}} \tag{46}$$

or:

$$\boxed{\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2\varphi}{\partial x'^2} + \frac{\partial^2\varphi}{\partial y'^2} + \frac{\partial^2\varphi}{\partial z'^2} + \frac{2v}{c^2}\frac{\partial^2\varphi}{\partial x'\partial t'} = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t'^2}} \tag{47}$$

This violates the principle of invariance. Another way of putting this: the equations are non-invariant.

The demonstration of non-invariace continues with the related derivation:

$$\frac{\partial\varphi}{\partial x'} = \frac{\partial\varphi}{\partial x}\cancelto{1}{\frac{\partial x}{\partial x'}} + \cancelto{0}{\frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial x'}} + \cancelto{0}{\frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial x'}} + \cancelto{0}{\frac{\partial\varphi}{\partial t}\frac{\partial t}{\partial x'}} \tag{48}$$

$$\boxed{Let \quad \alpha \equiv \frac{\partial\varphi}{\partial x}} \tag{49}$$

Continuing with y',z' and t':

$$\frac{\partial\varphi}{\partial y'} = \cancelto{0}{\frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial y'}} \tag{50}$$