

Vision and Image Processing: Linear Algebra

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TA Sessions this afternoon

- Due to reservation problems, the TA sessions will be held from 15.00 to 17.00, at least today. We have asked the faculty administration to change them to 13.00-15.00. We'll keep you informed as soon as something happens!
- Before that, you're encouraged to work on your assignments.
- There are several places that you can use in the DIKU building: the former library, the very nice new study places in the basement. Of course many others on the campus.



Plan for today

- Vectors
- Matrices
- Traces and determinants
- Linear mappings



Outline

1 Vectors

2 Matrices

3 Square Matrices, Trace, Determinant

4 Linear Mappings



Vectors and Matrices

Ordered collections of real numbers that represent some quantities

- Position in plane, space, velocity, some geometric transformations, images...
- Series of basic (and less basic operations) defined on them.



Vectors

- A n -vector is a n -uple of real values:

$$v = [x_1, \dots, x_n] \text{ (row vector) }, \quad v = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ (column vector, preferred)}$$

- Addition: same length vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$

- Multiplication by a scalar λ $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix}$

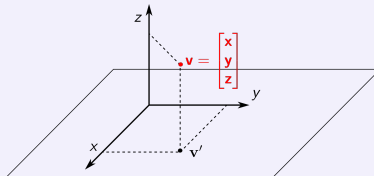
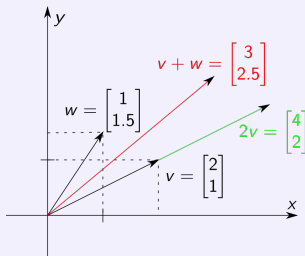
- Transposition $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^T = [x_1, \dots, x_n], [x_1, \dots, x_n]^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

- To save space, I often write a column vector as a transpose of a line vector:

$$\mathbf{x} = [x_1, \dots, x_n]^T$$



Vectors, coordinates, operations – Highschool stuffs!



- Vector space \mathbb{R}^n , set of vectors of length n .
- n is the dimension of the vector space.
- Vector subspace: lines (going through origin), planes (going through origin), etc...
- Line: dimension 1, plan dimension 2, etc.



Inner Product, Orthogonality, Norm, Distance

- Two vector $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^n x_i y_i$

Inner/Dot/Scalar product of \mathbf{x} and \mathbf{y} , also denoted $\mathbf{x}^\top \mathbf{y}$.



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- $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2$. If $\mathbf{x} \perp \mathbf{y}$, Pythagoras Theorem:

$$\mathbf{x} \perp \mathbf{y} \implies \|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$



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- Distance between \mathbf{x} and \mathbf{y} : $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.



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- Distance between \mathbf{x} and \mathbf{y} : $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.
- Exercise: develop the expression $\|\mathbf{x} - \mathbf{y}\|^2$.



Outline

① Vectors

② Matrices

③ Square Matrices, Trace, Determinant

④ Linear Mappings



Matrices

- A $n \times m$ matrix is an array of numbers with n rows and m columns
- A 2×3 matrix F

$$F = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

- 2 matrices **of the same size** can be added together: just add the entries:

$$\begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 1 \\ 7 & -3 & 0 \end{bmatrix} = ?$$

- a matrix can be multiplied by a scalar: just multiply all entries

$$4 \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} = ?$$

- Null matrix: matrix with all entries = 0:

$$\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$



- Transposition of a Matrix (Matrix transpose): $(n \times m) \rightarrow (m \times n)$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix}, \quad A^T = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1m} & \dots & a_{nm} \end{bmatrix}$$

- Example

$$A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 1 & 8 & 5 & 7 \end{bmatrix}, \quad A^T = \begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 0 & 5 \\ 1 & 7 \end{bmatrix}$$

- a square matrix A is **symmetric** if $A = A^T$

$$\underbrace{A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}}_{\text{symmetric}}, \quad \underbrace{B = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}}_{\text{not symmetric}}$$



Product of a Matrix and a Vector

- A matrix of size $m \times n$ and a vector of length n can be multiplied to form a vector of length m .
- Formal rule:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}, v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$Av = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \dots a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \dots a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \dots a_{mn}v_n \end{bmatrix}$$

- Each line of A is multiplied in “inner product way” with v .



Product of Matrices

- Dimension rule: Dimension of A and B must be compatible

$$(m, p) \cdot (q, n) \implies \begin{cases} p \neq q : \text{impossible} \\ (m, p) \cdot (p, n) \rightarrow (m, \cancel{p}) \cdot (\cancel{p}, n) \rightarrow (m, n) \end{cases}$$



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- Algebraic rule: a_{ij} entry (i, j) of A , b_{jk} entry (j, k) of B

$$A = (a_{ij})_{\substack{i=1 \dots m, \\ j=1 \dots p}}, \quad B = (b_{jk})_{\substack{j=1 \dots p, \\ k=1 \dots n}}$$

Denote entry (i, k) of product $C = AB$ by c_{ik} :



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- Example

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 2 \\ -5 & 3 & 3 \\ 3 & 3 & -1 \end{bmatrix}$$



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- What does matrix multiplication means? Later!



Special Products

- Row vector $\mathbf{x} = [x_1, \dots, x_n]$: matrix of size $1 \times n$. Column vector

$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$: matrix of size n . The products $\mathbf{x}\mathbf{y}$ and $\mathbf{y}\mathbf{x}$ well defined.

- $\mathbf{x}\mathbf{y}$: dimensions rule says $(1, n)(n, 1) \rightarrow (1, 1)$. A $(1, 1)$ dimension matrix? a single number!

$$\mathbf{x}\mathbf{y} = [x_1, \dots, x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n.$$

- $\mathbf{y}\mathbf{x}$. What does dimension rule says: $(n, 1).(1, n) \rightarrow (n, n)$: A square matrix.

$$\mathbf{y}\mathbf{x} = \begin{bmatrix} y_1 x_1 & y_1 x_2 & \dots & y_1 x_n \\ y_2 x_1 & y_2 x_2 & \dots & y_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ y_n x_1 & y_n x_2 & \dots & y_n x_n \end{bmatrix}$$



- $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. $\mathbf{x}^T \mathbf{y}$ satisfies the dimensions rule:

$(1, n)(n, 1) \rightarrow (1, 1)$ single number, a *scalar*!

$$\mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^n x_i y_i.$$

This is the **inner** product!

- $\mathbf{x} \mathbf{y}^T$ satisfies the dimensions rule

$$\mathbf{x} \mathbf{y}^T = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_n \\ \vdots & \vdots & \vdots & \vdots \\ x_n y_1 & x_n y_2 & \cdots & x_n y_n \end{bmatrix}$$

Outer product. Outer product works in fact for column vectors of different dimensions. **Not** the case for inner product.



Some afternoon exercises

With Pen and paper.

- $a = [1, 4, -5, 3]$, $b = [3, -1, 0, 2]$. Compute inner product $a^T b$ and outer product ab^T .
- Compute inner product $b^T a$ and outer product ba^T
- $M = ab^T$. Dimensions? Can I compute Ma , Mb . If yes, any observations?

With Python / numpy.

- We can define vectors as 1D numpy arrays

```
>> import numpy as np
>> a = np.array([1, 4, -5, 3])
>> b = np.array([3, -1, 0, 2])
>> np.dot(a, b)
>> a @ b # Python 3.x only
```

- `np.dot` can be used for matrix-vector multiplication, matrix-matrix multiplication: try and understand the following and explore!

```
>> a.shape = (4, 1)
>> print(a.shape)
>> b.shape = (-1, 1)
>> print(a@b.T, a.T@b)
>> print(a@b)
```



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Square Matrices

- The product of two $n \times n$ square matrices has the same size.

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}, AB = \begin{bmatrix} -2 & 5 \\ -24 & 5 \end{bmatrix}$$

- Beware that $AB \neq BA$ in general! $BA = \begin{bmatrix} 1 & 12 \\ -9 & 2 \end{bmatrix}$

- I can have $A \neq 0$, $B \neq 0$, $AB = 0$!

- $AA = A^2$: powers of a matrix. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

- **Identity** matrix: 1 on the diagonal, 0 elsewhere:

$$I_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Identity because $AI = IA = A$



Trace of a Square Matrix

- $\text{Tr}(A)$: **Trace** of A = sum of the diagonal elements of A :

$$\text{Tr} \left(\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 1 \\ -1 & 4 & 5 \end{bmatrix} \right) = 1 + 3 + 5 = 9.$$

- Invariant to a lot of transformations, used massively in linear algebra.
- Linear:

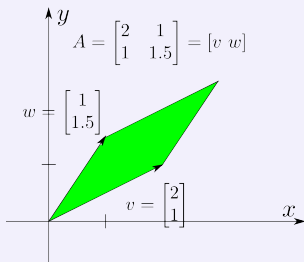
$$\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B) \quad \text{Tr}(\lambda A) = \lambda \text{Tr}(A).$$

- Product: $\text{Tr}(AB) = \text{Tr}(BA) \neq \text{Tr}(A) \text{Tr}(B)$.



Determinant of a square matrix

- $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$



- Area of the green parallelogram spanned by v and w

$$\det A = 2 \times 1.5 - 1 \times 1 = 2$$

- Order of vectors matters

$$\det \begin{bmatrix} b & a \\ d & c \end{bmatrix} = -\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- $\det[w \ v] = -\det[v \ w]$. Reversing the orientation changes the sign
- if $v = \lambda w$: parallelogram is flat, area is 0.

$$\det \begin{bmatrix} a & \lambda a \\ b & \lambda b \end{bmatrix} = \lambda ab - \lambda ab = 0.$$

- A matrix with null determinant is **singular**.
- Important rule

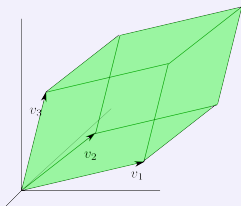
$$\det(AB) = \det(BA) = \det(A) \det(B)$$

Note that

$$\det(A + B) \neq \det(A) + \det(B)$$



In 3D



$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} =$$

$$a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - \\ a_2 b_1 c_3 - a_1 b_3 c_2 - a_3 b_2 c_1$$

- $v_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $v_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, $v_3 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$
- $\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \det[v_1, v_2, v_3]$
- Volume of the parallelepiped spanned by v_1 , v_2 and v_3 .
- Order of the vectors counts!
- If one vector is combination of the others: parallelepiped flat, volume is 0.
- Matlab `det` command! - Python Numpy has a similar one (in `linalg` module). Not limited to 3x3 matrices.



Exercises

Pen and paper.

- Compute the products AB and BA for the following matrices

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

- Compute their determinant and traces.

With numpy.

- `numpy.linalg` subpackage has functions for trace and determinants.
- Find them and play with the two matrices above.



Inverse Matrices

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}, B = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

A and B are **inverse** of each other: $A = B^{-1}, B = A^{-1}$.

- A is **invertible** iff $\det(A) \neq 0$.
- Example: system of equations

$$\begin{cases} 3x + 2y = 5 \\ 2x + y = -1 \end{cases} \quad \text{in matrix form:} \quad \underbrace{\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}}_C \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$\det = 3 - 4 = -1 \neq 0$

- Solution

$$C^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}, \quad C^{-1}C \begin{bmatrix} x \\ y \end{bmatrix} = I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = C^{-1} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 13 \end{bmatrix}$$

- Matlab and Python-numpy have functions to invert matrices and solve linear systems.



Exercise

- *Pen and paper. Let C be the matrix*

$$\begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$$

- *Compute its inverse by writing*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = C \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + 3c & b + 3d \\ 2a + 8c & 3b + 8d \end{pmatrix}$$

- *Explain why. There is a linear system to solve. Can you?*
- *Try with Python and the `numpy.linalg.inv()` function.*



For Non-Square-Matrices

- Notions of right-inverse or left inverses.
- General construction of the Moore-Penrose Pseudo-Inverse. Works both with matrices with more lines than columns – **overdetermined linear systems** and the opposite: less lines than columns – **underdetermined linear systems**.
- `pinv` function in Matlab, `pinv` function in `numpy.linalg` python package.
- Intimately connected to **linear least-squares problems** and the **Singular Value Decomposition**.
- In turn intimately connected to eigenvalues and eigenvectors problems (spectral theory) for square matrices.



Outline

① Vectors

② Matrices

③ Square Matrices, Trace, Determinant

④ Linear Mappings



Linear Mapping

- Mapping between vectors with only addition of coordinates, multiplications by scalar and no constant terms.
- Example

$$f \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 3y \\ z - 2x \end{bmatrix}$$

- Non linear example

$$g \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x^2 + 3yz \\ z - 2x^2 + 1 \end{bmatrix}$$

There are powers and constant terms.



Linearity

- This means $f(v + \lambda v') = f(v) + \lambda f(v')$

$$\begin{aligned} f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \lambda \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) &= f\begin{bmatrix} x + \lambda x' \\ y + \lambda y' \\ z + \lambda z' \end{bmatrix} \\ &= \begin{bmatrix} x + \lambda x' + 3(y + \lambda y') \\ z + \lambda z' - 2(x + \lambda x') \end{bmatrix} \\ &= \begin{bmatrix} x + 3y \\ z - 2x \end{bmatrix} + \lambda \begin{bmatrix} x' + 3y' \\ z' - 2x' \end{bmatrix} \\ &= f\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \lambda f\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \end{aligned}$$

- f is linear.



- Example: Compute the product of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \text{ and } v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



- Example: Compute the product of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \text{ and } v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- We find precisely the value of

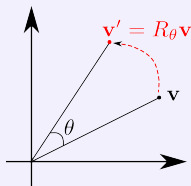
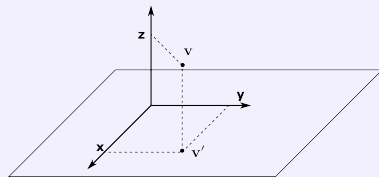
$$f \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 3y \\ -2x + z \end{bmatrix} = \begin{bmatrix} x + 3y \\ z - 2x \end{bmatrix}$$

- Each linear mapping can be written that way. Often use the same notation for the matrix and the linear mapping.

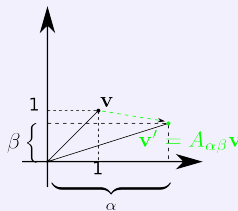


Matrices /linear mappings as geometric transformations

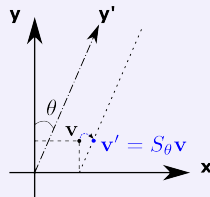
Projection on $x - y$ plane



Rotation of angle θ



anisotropic scaling



shear



- projection $\mathbb{R}^3 \rightarrow \mathbb{R}^2$

$$F \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Rotation of angle θ from $\mathbb{R}^2 \rightarrow \mathbb{R}^2$:

$$R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix}, \quad R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

- Scaling by a factor α in x and β in y :

$$S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \beta y \end{bmatrix}, \quad S = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

- Shear of the y -axis with angle θ :

$$S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \sin \theta y \\ y \end{bmatrix}, \quad S = \begin{bmatrix} 1 & \sin \theta \\ 0 & 1 \end{bmatrix}$$



Eigenvalues/vectors of a Square Matrix

- $A = \begin{bmatrix} 11 & 27 \\ -4 & -10 \end{bmatrix}$. $A \begin{bmatrix} 9 \\ -4 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$. $A \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$



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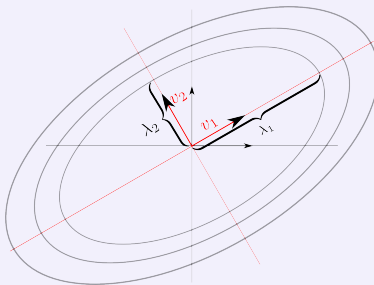
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- A acts as scaling by -1 in the direction of vector $[9, -4]^\top$ and by scaling by 2 in the direction of vector $[-3, 1]^\top$.
- Any vector of \mathbb{R}^2 can be written as $\alpha[9, -4]^\top + \beta[-3, 1]^\top$. Action of A :

$$A \left(\alpha[9, -4]^\top + \beta[-3, 1]^\top \right) = -\alpha[9, -4]^\top + 2\beta[-3, 1]^\top.$$



Case of Symmetric Matrices

- Eigenvectors for different eigenvalues are orthogonal. Can be chosen with norm 1.
- Eigenvectors + eigenvalues: Linear “elliptic-like” scaling.



- Can be interpreted as “Rotate, scale in each axis direction, then rotate back”.

Matrix Rank

- Linear mapping $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, A matrix associated. **Rank** of A = dimension of the image of f , i.e. dimension of the set made of the $f(x_1, \dots, x_n)$.
- Rank of projection F above: 2: all vectors of \mathbb{R}^3 are projected on the plan of vectors $[x, y, 0]^T$.
- Take $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. A has rank 1.

$$f(x, y) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + 4y \end{bmatrix}$$

All the values of f belong to the line $y = 2x$, dimension 1 subspace of \mathbb{R}^2 .

- The square matrix A , says size $n \times n$, is invertible if its rank is n .
- For a square matrix, its rank is the number of non-zeros eigenvalues.



Meaning of the Product

- M and N the linear mappings $\begin{bmatrix} 2 & 2 \\ 1 & 3 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$
- Apply N to $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and M to the result:

$$N\mathbf{v} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 3y \\ -2x + z \end{bmatrix}$$

- and

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Matrix Product as Chain Application of Linear Mappings

- We found that

$$M(N\mathbf{v}) = \underbrace{MN}_{\text{Matrix product}} \mathbf{v}$$

- Very Important Property: Matrix product corresponds to chain application (composition) of linear mappings!



So far

- We talked of vectors, vector spaces, dimension, inner products
- Matrices, operations on them, transposition, product of matrices,
- Square matrices and their algebra, symmetric matrices, Traces, determinants.
- linear mappings, ranks, eigenvalues/vectors.
- matrix products and composition of linear mappings.

Read the Linear Algebra Tutorial and Reference on Absalon!
Next time: pencil and paper for manual computations of
convolutions!

This will also be useful for other courses!

