Vision and Image Processing: Very Brief Recall on Derivatives

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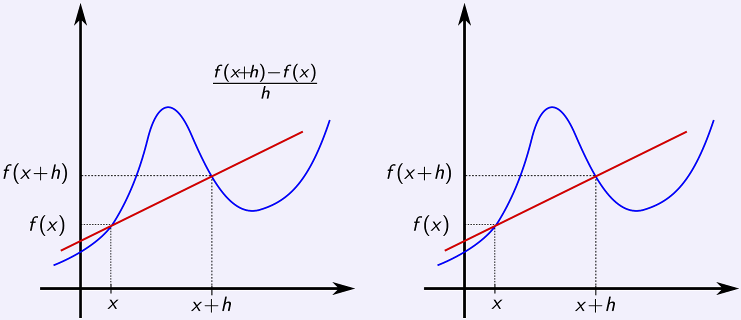
Plan for this lecture

- Derivatives in 1D
- Derivatives in 2D/3D, Gradients

Outline

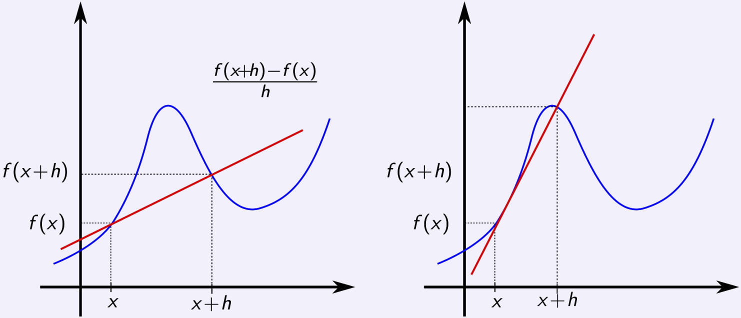
Secants, limits and derivatives



Secant : line joining two point on the graph of f. Slope between $(x, f(x))$ and $(x + h, f(x + h))$:

$$\frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

Secants, limits and derivatives



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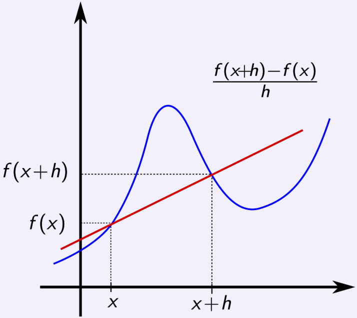

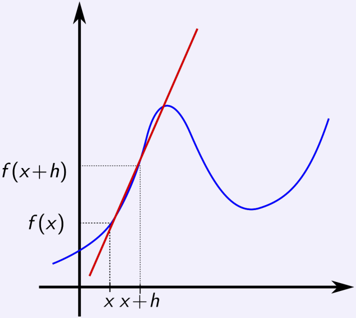

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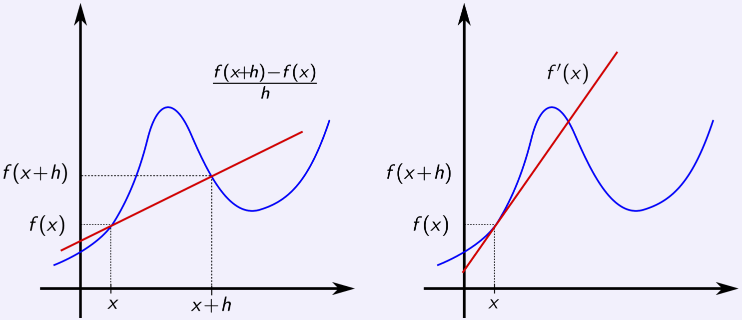

Derivative: limit of the slope when $h \to 0$ (but $h \neq 0$!) Slope of the tangent at x

$$\frac{df}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivative: a limit on the rate of change of a function.

Differentiable function

- f is *differentiable at* x if the limit above exists.
- f is *differentiable* if the limit exists for all x.

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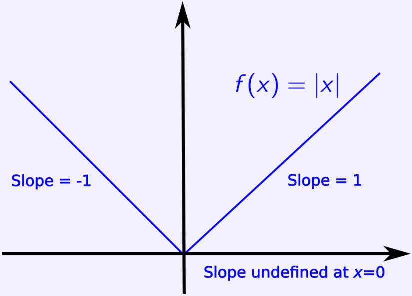

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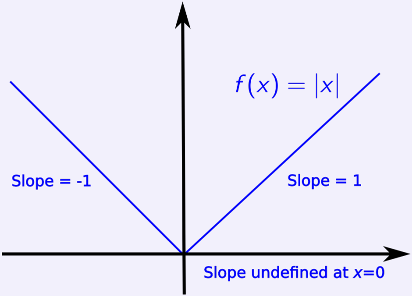
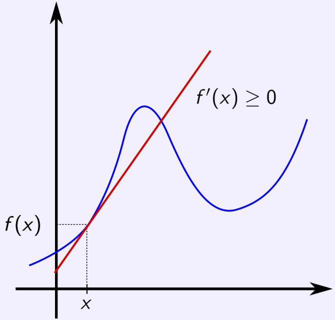


- $f(x) = |x|$ not differentiable at $x = 0$.

Derivatives and variations of a function



$f'(x) \geq 0$: f is increasing around x

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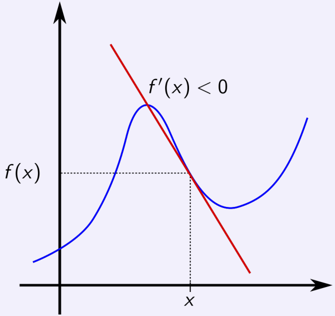
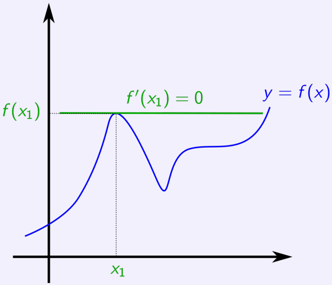


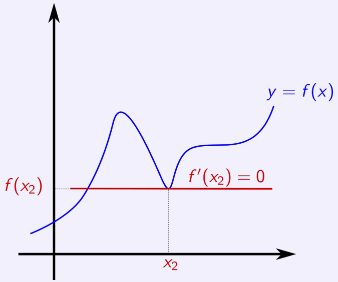
$f'(x) \leq 0$: f is decreasing around x

Null derivative



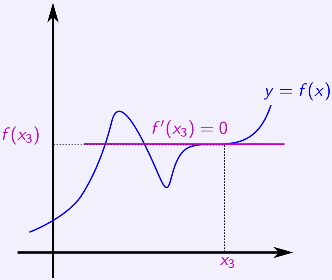
- $f'(x_1) = 0$, f is locally below its tangent: *local maximum*
- $f'(x) > 0$ if $x < x_1$ locally, $f'(x) < 0$ if $x > x_1$ locally.
- x_1 is a *local maximizer* of f.

Null derivative



- $f'(x_2) = 0$, f is locally above its tangent: *local minimum*
- $f'(x) < 0$ if $x < x_2$ locally, $f'(x) < 0$ if $x > x_2$ locally.
- x_2 is a *local minimizer* of f.

Null derivative



- $f'(x_3) = 0$, f is crosses its tangent. neither minimum nor maximum.
- $f'(x) > 0$ if $x < x_3$ locally, $f'(x) > 0$ if $x > x_3$ locally.
- x_3 is an *inflection point*.
- The opposite situation also possible: $f'(x) < 0$ if $x < x_3$ locally, $f'(x) < 0$ if $x > x_3$ locally. Make a picture!

Example: Derivative of x^n

Use the *binomial formula*

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= a^n + na^{n-1}b + \frac{n(n-1)}{2} a^{n-2} b^2 + \dots b^n\end{aligned}$$

Develop

$$\begin{aligned}\frac{(x+h)^n - x^n}{h} &= \frac{x^n + hnx^{n-1} + h^2 \frac{n(n-1)}{2} x^{n-1} + \cdots + h^n - x^n}{h} \\ &= nx^{n-1} + \underbrace{h\left(\frac{n(n-1)}{2} x^{n-1} + \cdots + h^{n-2}\right)}_{\text{goes } \to 0 \text{ when } h \to 0}\end{aligned}$$

The limit when $h \to 0$ is nx^{n-1}.

Some Classical Formulas

function	derivative	domain/remark
x^α	$\alpha x^{\alpha-1}$	if α is not an integer, x should be > 0
e^x	e^x	$x \in \mathbb{R}$
$\ln x$	$\frac{1}{x}$	$x > 0$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	special case of first rule with $\alpha = \frac{1}{2}$, $x > 0$
$\cos x$	$-\sin x$	$x \in \mathbb{R}$
$\sin x$	$\cos x$	$x \in \mathbb{R}$
$\tan x$	$\frac{1}{\cos^2 x}$	$x \neq k \pm \frac{\pi}{2}, k \in \mathbb{Z}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$-1 < x < 1$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$-1 < x < 1$
$\arctan x$	$\frac{1}{1+x^2}$	$x \in \mathbb{R}$

Example: Derivative of a Product – Leibniz Rule

Rule for differentiating $x \mapsto f(x)g(x)$. Write secant ratios:

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$$
\begin{aligned}
\frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \frac{f(x+h)(g(x+h) - g(x)) + (f(x+h) - f(x))g(x)}{h} \\
&= f(x+h)\frac{g(x+h) - g(x)}{h} + \frac{f(x+h) - f(x)}{h}g(x)
\end{aligned}
$$

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$$\begin{aligned}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} &= \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \\ &= \frac{f(x+h)(g(x+h)-g(x))+(f(x+h)-f(x))g(x)}{h} \\ &= f(x+h)\frac{g(x+h)-g(x)}{h}+\frac{f(x+h)-f(x)}{h}g(x)\end{aligned}$$

At the limit when $h \to 0$,

- The first term
$$f(x+h)\frac{g(x+h)-g(x)}{h} \to f(x)g'(x)$$
- The second term
$$\frac{f(x+h)-f(x)}{h}g(x) \to f'(x)g(x)$$
- We get Leibniz Rule: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

Classical Rules for Computing Derivatives

function	derivative	rule name
$\lambda f(x)$	$\lambda f'(x)$	scalar multiplication rule
$f(x) + g(x)$	$f'(x) + g'(x)$	sum rule
$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$	Leibniz rule
$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$	quotient rule
$f(g(x))$	$f'(g(x))g'(x)$	chain rule
$e^{f(x)}$	$f'(x)e^{f(x)}$	exponentiation rule (chain rule!)
$\ln \lvert f(x) \rvert$	$\frac{f'(x)}{f(x)}$	logarithm rule
$(f(x)g(x))''$	$f''(x)g(x) + 2f'(x)g'(x) + g''(x)$	iterated Leibniz rule

$f''(x)$ is the derivative of $f'(x)$ at x. Second (order) derivative of f.

A Not That Simple Example: $f(x) = x\cos(e^{\sin(x)})$

- Write as $f(x) = g(x)h(x)$ with $g(x) = x$, $h(x) = \cos(e^{\sin(x)})$

$$\begin{aligned} f'(x) &= g'(x)h(x) + g(x)h'(x) && \text{(Leibniz rule)} \\ &= \cos(e^{\sin(x)}) + xh'(x) && (g'(x) = 1) \end{aligned}$$

- Write $h(x)$ as $l(m(x))$ with $l(x) = \cos(x)$, $m(x) = e^{\sin(x)}$.

$$\begin{aligned} h'(x) &= l'(m(x))m'(x) && \text{(chain rule)} \\ &= -\sin(m(x))m'(x) && (\cos'(x) = -\sin(x)) \end{aligned}$$

- Use Exponential rule $e^{a(x)} = a'(x)e^{a(x)}$ for $m(x) = e^{\sin(x)}$

$$\left(e^{\sin(x)}\right)' = \cos(x)e^{\sin(x)}, \quad (\sin'(x) = \cos(x))$$

- Reassemble the parts

$$f'(x) = \cos(e^{\sin(x)}) - x\sin(e^{\sin(x)})\cos(x)e^{\sin(x)}$$

Example: Arithmetic Mean

- Given n real numbers $x_1, \ldots, x_n$.

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- When is the derivative 0?

$$\begin{aligned} V'(x) &= \frac{1}{2n} \sum_{i=1}^{n} \frac{d(x - x_i)^2}{dx} = \frac{1}{2n} \sum_{i=0}^{n} 2(x - x_i) \\ &= \frac{1}{n} \sum_{i=1}^{n} (x - x_i) = \frac{1}{n} \sum_{i=1}^{n} x - \frac{1}{n} \sum_{i=1}^{n} x_i \\ &= x - \mathbf{x} \end{aligned}$$

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- $V'(x) = 0$ if and only if $x = \mathbf{x}$. Minimum? Maximum? Inflection point?
- Minimum! Why?
- The arithmetic mean is the *unique* point where $V(x)$ is minimum!

Outline

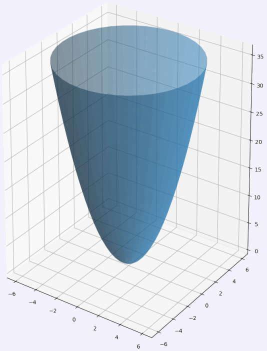
Partial derivatives

- Function $(x, y) \in \mathbb{R}^2 \mapsto f(x, y) \in \mathbb{R}$: two variables x and y.
- partial derivative $\frac{\partial f}{\partial x}(x, y)$ in x-direction: derivative of $x \mapsto f(x, y)$.

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

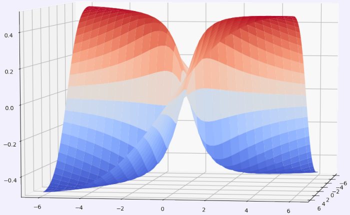
- partial derivative $\frac{\partial f}{\partial x}(x, y)$ in y-direction: derivative of $y \mapsto f(x, y)$

$$\frac{\partial f}{\partial y}(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$



- if partial derivatives exist at (x_0, y_0) f is *differentiable* at (x_0, y_0).
- if partial derivatives exist everywhere, f is *differentiable*.
- Example $f(x, y) = x^2 + y^2$ (squared Euclidean distance).
- Partial derivatives $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$
- f is differentiable.
- Euclidean distance: Is $f(x, y) = \sqrt{x^2 + y^2}$ differentiable?

Example $f(x,y) = \frac{xy}{1+x^2+y^2}$



- Partial derivatives
- in x

$$\frac{\partial f}{\partial x} = \frac{y(1-x^2+y^2)}{(1+x^2+y^2)^2}$$

- in y

$$\frac{\partial f}{\partial x} = \frac{x(1+x^2-y^2)}{(1+x^2+y^2)^2}$$

- Computation in x. y supposed to be fixed: Using quotient rule, power rule,

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= \frac{\frac{\partial xy}{\partial x}(1+x^2+y^2) - xy\frac{\partial(1+x^2+y^2)}{\partial x}}{(1+x^2+y^2)^2} \\ &= \frac{y(1+x^2+y^2) - xy(2x)}{(1+x^2+y^2)^2} = \frac{y(1-x^2+y^2)}{(1+x^2+y^2)^2}\end{aligned}$$

Differential, Gradient

- Differential of $f(x, y)$: the *line vector* made of partial derivatives.

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- Gradient of $f(x, y)$: the *column vector* made of partial derivatives.

$$\nabla f(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = Df(x, y)^T$$

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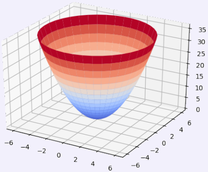
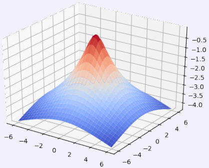
- $\nabla f(x,y)^T \nabla f(x,y) = \|\nabla f(x,y)\|^2$: length of gradient vector.

Gradients and Critical Points

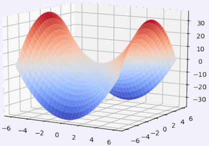
- A point (x, y) is *critical* for f if $\nabla f(x, y) = \vec{0}$. Minima, Maxima, Saddle points...



$f(x, y) = x^2 + y^2$



$g(x, y) = -\log(1 + x^2 + y^2)$



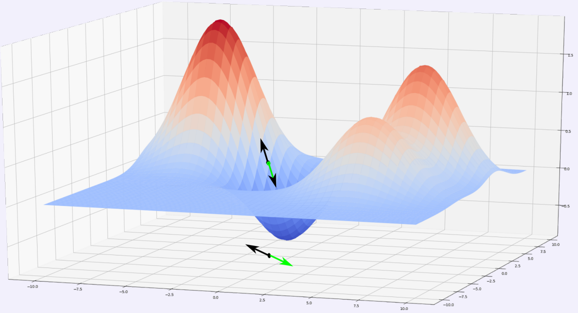
$h(x, y) = x^2 - y^2$

- All the three functions have only one critical point, at $(x, y) = (0, 0)$.
- For f: minimum
- For g: maximum
- For h: saddle point.

Gradients and Optimization

Gradients and Optimization

Fact: The gradient points in the direction of the steepest ascent of the function.



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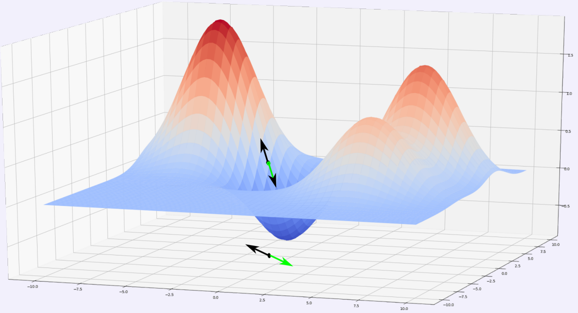

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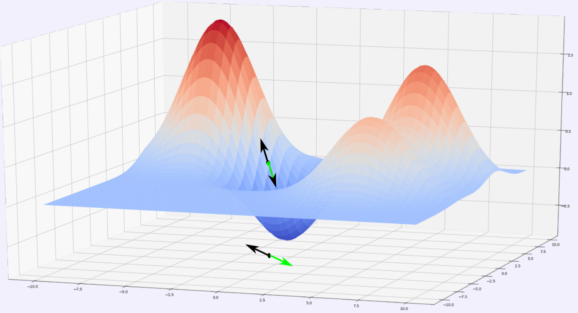

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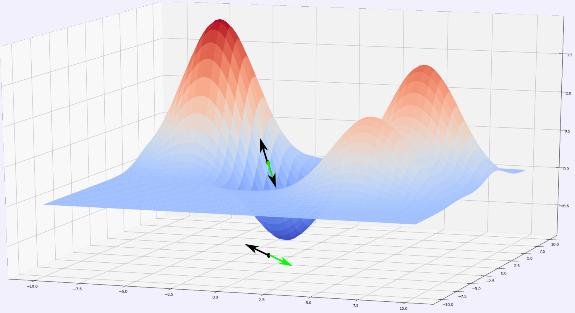

What can this be useful for? **Optimization – gradient descent!**

Second order derivatives

- Second order derivatives in x: Different combinations

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x}, \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y}$$

- *Mixed* Partial Derivatives: (equality from Schwarz' Theorem)

$$\frac{\partial^2 f}{\partial xy} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial yx}$$

- *Hessian matrix* of f: *symmetric* matrix (still function of x and y)

$$\operatorname{Hess} f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial xy} \\ \frac{\partial^2 f}{\partial xy} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

- *Laplacian* of f:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \operatorname{Trace} \operatorname{Hess} f$$

Some Examples

- Hessian of $f(x, y) = x^2 + y^2$

$$\text{Hess} f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_2, \quad I_2 = 2 \times 2\text{-identity matrix}$$

- $\Delta f = 4$.

- Hessian of $f(x, y) = \frac{xy}{1+x^2+y^2}$: not as nice as previous one!

$$\text{Hess} f = \frac{1}{(x^2 + y^2 + 1)^3} \begin{bmatrix} 2xy\left(x^2 - 3y^2 - 3\right) & 6x^2y^2 - x^4 - y^4 + 1 \\ 6x^2y^2 - x^4 - y^4 + 1 & -2xy\left(3x^2 - y^2 + 3\right) \end{bmatrix}$$

- Laplacian of f

$$\Delta f = -\frac{4xy\left(x^2 + y^2 + 3\right)}{\left(x^2 + y^2 + 1\right)^3}$$

In More Variables

Function $f(x_1, x_2, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$.

- Partial Derivative w.r.t x_i

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots x_n) - f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots x_n)}{h}$$

- The same, but we need more letters!
- Differentials, Gradients:

$$Df = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right), \quad \nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)^T$$

- Hessian: $n \times n$ symmetric matrices

$$\operatorname{Hess} f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 x_n} \\ \frac{\partial^2 f}{\partial x_1 x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 x_n} & \frac{\partial^2 f}{\partial x_2 x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- Laplacian

$$\nabla f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^i}$$

That's all Folk!