Vision and Image Processing: Linear Algebra

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Plan for today

- Vectors
- Matrices
- Linear mappings
- Traces and determinants

Outline

Vectors and Matrices

Ordered collections of real numbers that represent some quantities

- Position in plane, space, velocity, some geometric transformations, images...
- Series of basic (and less basic operations) defined on them.

Vectors

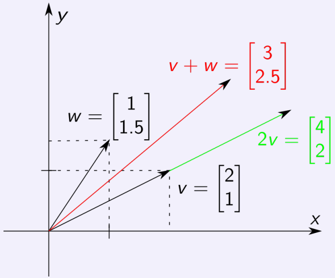
- A *n*-vector is a *n*-uple of real values:

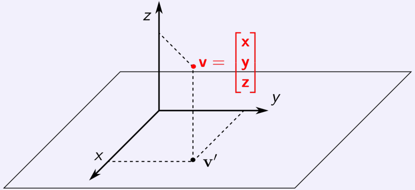
$$v = \left[x_1, \ldots, x_n\right] \text{ (row vector) }, \quad v = \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix} \text{ (column vector, preferred)}$$

- Addition: same length vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$
- Multiplication by a scalar $\lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix}$
- Transposition $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}^\top = \begin{bmatrix} x_1, & \ldots, & x_n \end{bmatrix}, \begin{bmatrix} x_1, & \ldots, & x_n \end{bmatrix}^\top = \begin{bmatrix} x_1, \\ \vdots \\ x_n \end{bmatrix}$
- To save space, I often write a column vector as a transpose of a line vector:

$$\mathbf{x} = \left[x_1, \ldots, x_n\right]^\top$$

Vectors, coordinates, operations – Highschool stuffs!





- Vector space $\mathbb{R}^n$, set of vectors of length n.
- n is the dimension of the vector space.
- Vector subspace: lines (going through origin), planes (going through origin), etc...
- Line: dimension 1, plan dimension 2, etc.

Inner Product, Orthogonality, Norm, Distance

- Two n-D vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$

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- Example: $\mathbf{x} = [1, 2, 2]^\top$: $\|\mathbf{x}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$.

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- Distance between $\mathbf{x}$ and $\mathbf{y}$: $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

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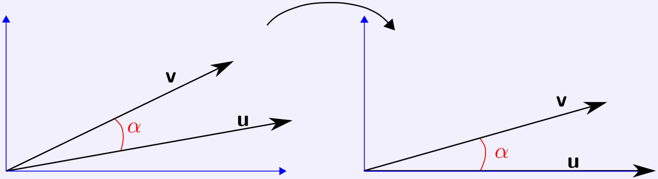
- Exercise: develop the expression $\|\mathbf{x} - \mathbf{y}\|^2$.

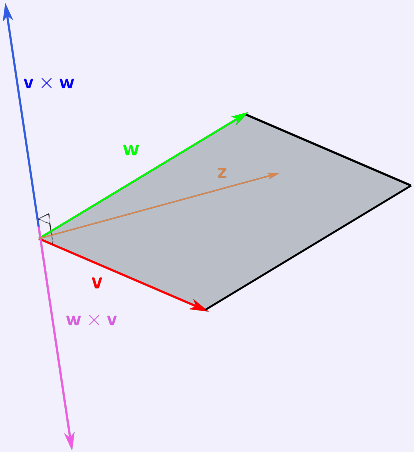
Old and New Inner Product

- Old definition: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\alpha$, with α angle between **u** and **v**
- Definition depends only on norm and angle.
- Rotate so that **u** aligns with the *x*-coordinate



- $\mathbf{u} = [u_1, 0]^T$, $\mathbf{v} = [v_1, v_2]^T$, new inner product: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1$.
- Then express $\cos\alpha$ as function of them, high-school trigonometry – left as exercise.

Cross product of 3D vectors



- **v**,**w** and **v** × **w** form a *direct* basis
- $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \|\mathbf{w}\| |\sin(\mathbf{v}, \mathbf{w})|$
- If **z** is in the vector plane containing **v** and **w** then $\mathbf{z} \perp (\mathbf{v} \times \mathbf{w})$
- Equation for the vector plane containing **v** and **w**:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot (\mathbf{v} \times \mathbf{w}) = 0.$$

Cross product of 3D vectors

In $\mathbb{R}^3$, sort of product of two vectors.

- Cross (or wedge) product of **v** and **w**

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}, \quad \mathbf{v} \times \mathbf{w} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

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  But *Jacobi Identity* $(\mathbf{v} \times \mathbf{w}) \times \mathbf{z} + (\mathbf{w} \times \mathbf{z}) \times \mathbf{v} + (\mathbf{z} \times \mathbf{v}) \times \mathbf{w} = 0$.

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- Orthogonality relations: $(\mathbf{v} \times \mathbf{w}) \perp \mathbf{v}$, $(\mathbf{v} \times \mathbf{w}) \perp \mathbf{w}$ (compute inner products!)

- As a consequence: a vector $\mathbf{z}$ can be written as a *combination* of vectors $\mathbf{v}$ and $\mathbf{w}$ if and only if $\mathbf{z} \cdot (\mathbf{v} \times \mathbf{w}) = 0$.

Outline

Matrices

- A $m \times n$ matrix is an array of numbers with m rows and n columns
- A $2{\times}3$ matrix F
$$F = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$
- 2 matrices of the same size can be added together: just add the entries:
$$\begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 1 \\ 7 & -3 & 0 \end{bmatrix} = \ ?$$
- a matrix can be multiplied by a scalar: just multiply all entries
$$4 \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \ ?$$
- Null matrix: matrix with all entries = 0: $\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$

- Transposition of a Matrix (Matrix transpose): $(n \times m) \to (m \times n)$

$$A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix}, \quad A^\top = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ \vdots & \vdots & \vdots \\ a_{1m} & \dots & a_{nm} \end{bmatrix}$$

- Example

$$A = \begin{bmatrix} 2 & 3 & 0 & 1 \\ 1 & 8 & 5 & 7 \end{bmatrix}, \quad A^\top = \begin{bmatrix} 2 & 1 \\ 3 & 8 \\ 0 & 5 \\ 1 & 7 \end{bmatrix}$$

- a square matrix A is symmetric if $A = A^T$

$$\underbrace{A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}}_{\text{symmetric}}$$

- a square matrix B is skew-symmetric (or antisymmetric) if $B = -B^T$

$$\underbrace{B = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}, \quad B^\top = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}}_{\text{skew-symmetric}}$$

- A neither symmetric nor skew-symmetric matrix

$$\underbrace{C = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}}_{\text{not symmetric/skew-symmetric}}$$

Product of a Matrix and a Vector

- A matrix of size $m \times n$ and a vector of length n can be multiplied to form a vector of length m.
- Formal rule:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}, v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \quad Av = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \dots a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \dots a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \dots a_{mn}v_n \end{bmatrix}$$

- Each line of A is multiplied in "inner product way" with v: entry i of Av

$$(Av)_i = a_{i1}v_1 + a_{i2}v_2 + \cdots + a_{in}v_n = \underbrace{\begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}}_{\text{transpose of line } i \text{ of } A} \bullet \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Product of Matrices

- Dimension rule: Dimension of A and B must be compatible

$$(m, p).(q, n) \implies \begin{cases} p \neq q : \text{impossible} \\ (m, p).(p, n) \to (m, \cancel{p}).(\cancel{p}, n) \to (m, n) \end{cases}$$

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- Algebraic rule: a_{ij} entry (i, j) of A, b_{jk} entry (j, k) of B

$$A = (a_{ij})_{\substack{i=1\dots m, \\ j=1\dots p}} \quad B = (b_{jk})_{\substack{j=1\dots p \\ k=1\dots n}}$$

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- What does matrix multiplication means? Later!

Special Products

- Row vector $\mathbf{x} = [x_1, \ldots, x_n]$: matrix of size $1 \times n$. Column vector $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$: matrix of size n. The products $\mathbf{x}\,\mathbf{y}$ and $\mathbf{y}\,\mathbf{x}$ well defined.
- $\mathbf{x}\,\mathbf{y}$: dimensions rule says $(1, n)(n, 1) \to (1, 1)$. A $(1, 1)$ dimension matrix? a single number!

$$\mathbf{x}\,\mathbf{y} = [x_1, \ldots, x_n] \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

- $\mathbf{y}\,\mathbf{x}$. What does dimension rule says: $(n, 1).(1, n) \to (n, n)$: A square matrix.

$$\mathbf{y}\,\mathbf{x} = \begin{bmatrix} y_1 x_1 & y_1 x_2 & \ldots & y_1 x_n \\ y_2 x_1 & y_2 x_2 & \ldots & y_2 x_n \\ \vdots & \vdots & \vdots & \vdots \\ y_n x_1 & y_n x_2 & \ldots & y_n x_n \end{bmatrix}$$

- $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$. $\mathbf{x}^T\mathbf{y}$ satisfies the dimensions rule:

  $(1, n)(n, 1) \to (1, 1)$ single number, a *scalar*!

$$\mathbf{x}^T\mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

  This is the inner product!

- $\mathbf{x}\,\mathbf{y}^\top$ satisfies the dimensions rule

$$\mathbf{x}\,\mathbf{y}^\top = \begin{bmatrix} x_1 y_1 & x_1 y_2 & \dots & x_1 y_n \\ x_2 y_1 & x_2 y_2 & \dots & x_2 y_n \\ \vdots & \vdots & \vdots & \vdots \\ x_n y_1 & x_n y_2 & \dots & x_n y_n \end{bmatrix}$$

  Outer product. Outer product works in fact for column vectors of different dimensions. Not the case for inner product.

Some afternoon exercises

With Pen and paper.

- $a = [1, 4, -5, 3]$, $b = [3, -1, 0, 2]$. Compute inner product $a^T b$ and outer product ab^T.
- Compute inner product $b^T a$ and outer product ba^T
- $M = ab^T$. Dimensions? Can I compute Ma, Mb. If yes, any observations?

With Python / numpy.

- We can define vectors as 1D numpy arrays

```
>> import numpy as np
>> a = np.array([1,4,-5,3])
>> b = np.array([3,-1,0,2])
>> np.dot(a, b)
>> a @ b # Python 3.x only
```

- `np.dot` can be used for matrix-vector multiplication, matrix-matrix multiplication: try and understand the following and explore!

```
>> a.shape = (4,1)
>> print(a.shape)
>> b.shape = (-1,1)
>> print(a@b.T, a.T@b)
>> print(a@b)
```

Outline

Linear Mapping

- Mapping between vectors with only addition of coordinates, multiplications by scalar and no constant terms.
- Example

$$f\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x+3y\\z-2x\end{bmatrix}$$

- Non linear example

$$g\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x^2+3yz\\z-2x^2+1\end{bmatrix}$$

  There are powers and constant terms.

Linearity

- This means $f(v + \lambda v') = f(v) + \lambda f(v')$

$$
\begin{aligned}
f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \lambda \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}\right) &= f\begin{bmatrix} x + \lambda x' \\ y + \lambda y' \\ z + \lambda z' \end{bmatrix} \\
&= \begin{bmatrix} x + \lambda x' + 3(y + \lambda y') \\ z + \lambda z' - 2(x + \lambda x') \end{bmatrix} \\
&= \begin{bmatrix} x + 3y \\ z - 2x \end{bmatrix} + \lambda \begin{bmatrix} x' + 3y' \\ z' - 2x' \end{bmatrix} \\
&= f\begin{bmatrix} x \\ y \\ z \end{bmatrix} + \lambda f\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}
\end{aligned}
$$

- f is linear.

- Example: Compute the product of

$$A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \text{ and } v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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- We find

$$\begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = f \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 3y \\ -2x + z \end{bmatrix} = \begin{bmatrix} x + 3y \\ z - 2x \end{bmatrix}$$

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- Each linear mapping can be written that way! Often use the same notation for the matrix and the linear mapping.

- We write

$$f \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Rewrite linear mappings as matrices:

- Example 1

$$f\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}3x-2y+z\\ z-3y\end{bmatrix} = \begin{bmatrix}3\times x+(-2)\times y+1\times z\\ 0\times x+(-3)\times y+1\times z\end{bmatrix} = \underbrace{\begin{bmatrix}3 & -2 & 1\\ 0 & -3 & 1\end{bmatrix}}_{\text{matrix of } f}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

  - Dimension of the input vector = number of columns
  - Dimension of the output vector = number of lines

- Example 2

$$f\begin{bmatrix}x\\y\\z\end{bmatrix} = 3z-2x+y = ((-2)x+1y+3z) = \underbrace{\begin{bmatrix}-2 & 1 & 3\end{bmatrix}}_{\text{matrix of } f}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

  - output of f is a single real, the same as a 1D vector.
  - When the output of f is a single real, f is called a *linear form*.

- Example 3

$$f(t) = \begin{bmatrix}3t\\-11t\end{bmatrix} = \begin{bmatrix}(3)t\\(-11)t\end{bmatrix} = \begin{bmatrix}3\\-11\end{bmatrix}[t] = t\begin{bmatrix}3\\-11\end{bmatrix}$$

scalar-vector multiplication is linear, its matrix is the vector itself.

Cross Product in $\mathbb{R}^3$

- Remember: $\mathbf{v}, \mathbf{w} \mapsto \mathbf{v} \times \mathbf{w}$
- The cross - product is linear for each term:
  - $(\lambda \mathbf{v} + \mu \mathbf{w}) \times \mathbf{z} = \lambda \mathbf{v} \times \mathbf{z} + \mu \mathbf{w} \times \mathbf{z}$
  - $\mathbf{v} \times (\lambda \mathbf{w} + \mu \mathbf{z}) = \lambda \mathbf{v} \times \mathbf{w} + \mu \mathbf{v} \times \mathbf{z}$
- Fix the first factor $\mathbf{v} = [v_1, v_2, v_3]^T$. The mapping $\mathbf{x} = [x, y, z]^T \mapsto \mathbf{v} \times \mathbf{x}$ is linear in $\mathbf{x}$.
- Then it can be written in matrix form. Which one? Assigment!
- The corresponding matrix is usually denoted by $\hat{\mathbf{v}}$ or $[\mathbf{v}]_\times$ ($\hat{\mathbf{v}}$ in the assignment).

Matrices /linear mappings as geometric transformations

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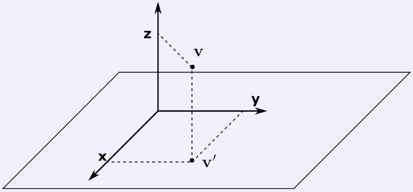
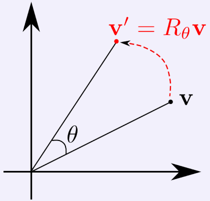
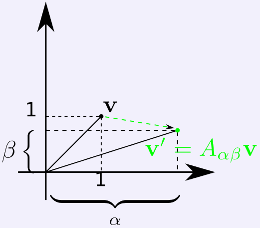
Projection on $x - y$ plane

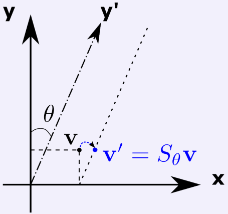




Rotation of angle θ



anisotropic scaling



shear

- projection $\mathbb{R}^3 \to \mathbb{R}^2$

$$F \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- Rotation of angle θ from $\mathbb{R}^2 \to \mathbb{R}^2$:

$$R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}, \quad R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Scaling by a factor α in x and β in y:

$$S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x \\ \beta y \end{bmatrix}, \quad S = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

- Shear of the y-axis with angle θ:

$$\mathcal{S} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \sin\theta y \\ y \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} 1 & \sin\theta \\ 0 & 1 \end{bmatrix}$$

Meaning of the Product

- M and N the linear mappings $\begin{bmatrix} 2 & 2 \\ 1 & 3 \\ 1 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}$
- Apply N to $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and M to the result:

$$N\mathbf{v} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 3y \\ -2x + z \end{bmatrix}$$

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- and M to Nv

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x + 3y \\ -2x + z \end{bmatrix} = \begin{bmatrix} -2x + 6y + 2z \\ -5x + 3y + 3z \\ 3x + 3y - z \end{bmatrix} =$$

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- Compute the product of M and N

$$\begin{bmatrix} 2 & 2 \\ 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 6 & 2 \\ -5 & 3 & 3 \\ 3 & 3 & -1 \end{bmatrix}$$

Matrix Product as Chain Application of Linear Mappings

- We found that

$$M\,(N\mathbf{v}) = \underbrace{MN}_{\text{Matrix product}}\,\mathbf{v}$$

- Very Important Property: Matrix product corresponds to chain application (composition) of linear mappings!

Outline

Square Matrices

- The product of two $n \times n$ square matrices has the same size.
$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}, \quad AB = \begin{bmatrix} -2 & 5 \\ -24 & 5 \end{bmatrix}$

- Beware that $AB \neq BA$ in general! $BA = \begin{bmatrix} 1 & 12 \\ -9 & 2 \end{bmatrix}$

- we can have $A \neq 0$, $B \neq 0$, $AB = 0$. Example

$$A = \begin{bmatrix} -5 & 9 & -3 \\ -4 & 7 & -2 \\ -2 & 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -9 & 3 \\ 4 & -6 & 2 \\ 2 & -3 & 1 \end{bmatrix}, \quad A.B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- This can also happen for product of non square matrices.

Identity matrices

- Identity matrix: 1 on the diagonal, 0 elsewhere:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \dots$$

- Identity: Check that $AI_n = I_nA$ for any square n-matrix A.

- To which linear mapping does I_n correspond to?

Inverse Matrices

- $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

- A and B are inverse of each other: $A = B^{-1}$, $B = A^{-1}$.

- A is invertible if and only if its *determinant* $\det(A) \neq 0$ (defined in a later slide).

Linear Systems of Equations

A 2×2 system:

$$\begin{cases} 3x + 2y & = 5 \\ 2x + y & = -1 \end{cases} \quad \text{in matrix form:} \quad \underbrace{\overbrace{\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}}^{C}}_{\det = 3\times 1 - 2\times 2 = -1 \neq 0} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

Solution:

$$C^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}, \quad C^{-1} C \begin{bmatrix} x \\ y \end{bmatrix} = I \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} = C^{-1} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 13 \end{bmatrix}$$

A 3x3 system

$$\begin{cases} 2x + 3y - z &= 5 \\ x + 2y - 2z &= 3 \\ 2x - y + 4z &= -2 \end{cases} \text{ in matrix form: } \overbrace{\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & -2 \\ 2 & -1 & 4 \end{bmatrix}}^{D} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

Solution: with `numpy.linalg.inv()`, $\det(D) = -7 \neq 0$, inverse of D:

$$D^{-1} = \begin{bmatrix} -0.85714286 & 1.57142857 & 0.57142857 \\ 1.14285714 & -1.42857143 & -0.42857143 \\ 0.71428571 & -1.14285714 & -0.14285714 \end{bmatrix}$$

$$D^{-1}[5, 3, -1]^T = [-0.14285714, 1.85714286, 0.28571429]^T$$

Symbollically

$$D^{-1} = \begin{bmatrix} -\frac{6}{7} & \frac{11}{7} & \frac{4}{7} \\ \frac{8}{7} & -\frac{10}{7} & -\frac{3}{7} \\ \frac{5}{7} & -\frac{8}{7} & -\frac{1}{7} \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -6 & 11 & 4 \\ 8 & -10 & -3 \\ 5 & -8 & -1 \end{bmatrix}, D^{-1} \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -1 \\ 13 \\ 2 \end{bmatrix}$$

Check that they match and solve indeed the system!

Exercise

- *Pen and paper. Let C be the matrix*

$$\begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$$

- *Compute its inverse by writing*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = C \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a + 3c & b + 3d \\ 2a + 8c & 3b + 8d \end{pmatrix}$$

- *Explain why. There is a linear system to solve. Can you?*
- *Try with Python and the* `numpy.linalg.inv()` *function.*

Trace of a Square Matrix

- $\mathrm{Tr}(A)$: Trace of A = sum of the diagonal elements of A:

$$\mathrm{Tr}\left(\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 1 \\ -1 & 4 & 5 \end{bmatrix}\right) = 1 + 3 + 5 = 9.$$

- Invariant to a lot of transformations, used massively in linear algebra.

- Trace is linear:

$$\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B) \quad \mathrm{Tr}(\lambda A) = \lambda\,\mathrm{Tr}(A).$$

- Product: $\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \neq \mathrm{Tr}(A)\,\mathrm{Tr}(B)$. The trace does not depend on product order, but *does not transform a product into a product*.

About Linearity of the Trace

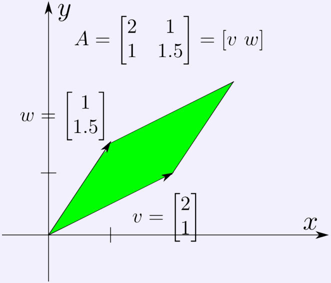
- A bit tricky question: A matrix for the trace operation?
- A (n, n) matrix can be represented by an ordered list of n^2 numbers.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ h & i & j \end{bmatrix} \mapsto \mathbf{vec}(A) = [a, b, c, d, e, f, g, h, i, j]^T.$$

- **vec**: vectorization. We could have used an other ordering!
- In Python-`numpy`, it can be done by *changing shape* or *flattening*.
- So what is the matrix for the Trace operation?

Determinant of a square matrix

- $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$



- Area of the green parallelogram spanned by v and w

$$\det A = 2 \times 1.5 - 1 \times 1 = 2$$

- Order of vectors matters

$$det \begin{bmatrix} b & a \\ d & c \end{bmatrix} = -\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- $\det[w \ v] = -\det[v \ w]$. Reversing the orientation changes the sign
- if $v = \lambda w$: parallelogram is flat, area is 0.

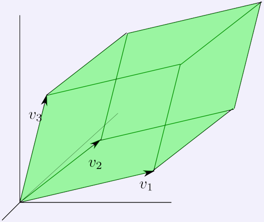
$$\det \begin{bmatrix} a & \lambda a \\ b & \lambda b \end{bmatrix} = \lambda ab - \lambda ab = 0.$$

- A matrix with null determinant is singular.
- Important rule

$$\boxed{det(AB) = \det(BA) = \det(A)\det(B)}$$

Note that
$\det(A + B) \neq \det(A) + \det(B)$

In 3D



$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} =$$

$a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 -$
$a_2 b_1 c_3 - a_1 b_3 c_2 - a_3 b_2 c_1$

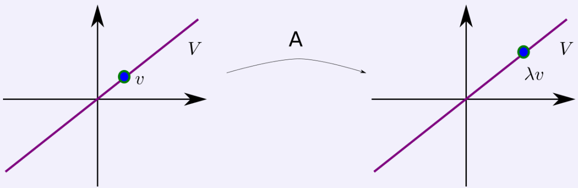
- $v_1 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, v_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, v_3 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$
- $\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \det[v_1, v_2, v_3]$
- Signed volume of the parallelepiped spanned by v_1, v_2 and v_3.
- The sign depends on the *orientation* of (v_1, v_2, v_3): switch two fo them, the sign will change.
- If one vector is combination of the others: parallepiped box is flat, volume is 0.
- Python-`numpy`: `numpy.linalg.det()`. Matlab `det()`. Not limited to 3×3 matrices.

What is an eigenvalue and its eigenvector?

- An *eigenvector* for a matrix A is a vector $\mathbf{e}$ such that

$$A\mathbf{e} = \lambda\mathbf{e}$$

for some real number λ, called the *eigenvalue* of $\mathbf{e}$. In this course, we further ask that $\|\mathbf{e}\| = 1$.



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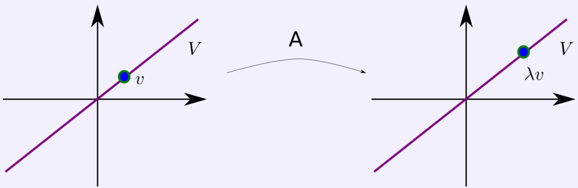

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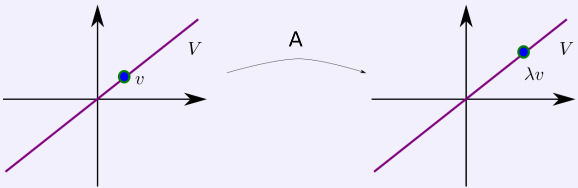

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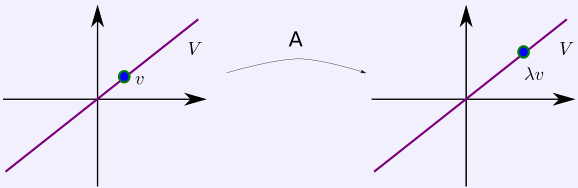
- The eigenvector $\mathbf{e}$ spans a subspace V which is left invariant by A – that is, for any $\mathbf{v} \in V$, we also have $A\mathbf{v} \in V$.



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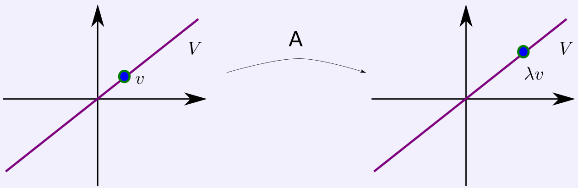
- The eigenvector $\mathbf{e}$ spans a subspace V which is left invariant by A – that is, for any $\mathbf{v} \in V$, we also have $A\mathbf{v} \in V$.
- The eigenvalue λ tells you how much A *stretches* V.



An example

$$A = \begin{bmatrix} 62 & 164 & -167 \\ 72 & 180 & -186 \\ 96 & 236 & -245 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 8 \\ 9 \\ 12 \end{bmatrix}, \quad f_2 = \begin{bmatrix} -11 \\ -12 \\ -16 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 9 \\ 10 \\ 13 \end{bmatrix}$$

$$Af_1 = \begin{bmatrix} -32 \\ -36 \\ -48 \end{bmatrix} = -4f_1, \quad Af_2 = \begin{bmatrix} -22 \\ -24 \\ -32 \end{bmatrix} = -2f_2, \quad Af_3 = ?$$

Eigenvalues/vectors of a Square Matrix

- $A = \begin{bmatrix} 11 & 27 \\ -4 & -10 \end{bmatrix}$. $\quad A \begin{bmatrix} 9 \\ -4 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$. $\quad A \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix}$

Eigenvalues/vectors of a Square Matrix

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- A multiplies the vector $[-9, 4]^\top$ by -1 and multiplies the vector $[-3, 1]^\top$ by 2. If I take $v = [9\alpha, -4\alpha]^\top$, $Av = -v$. If I take $w = [-3\beta, \beta]^\top$, $Aw = 2w$.

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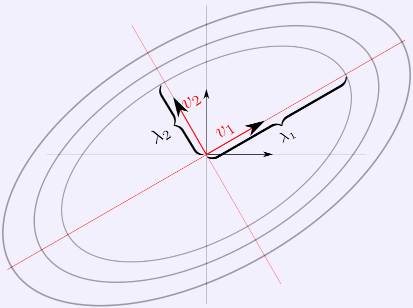
- Any vector of $\mathbb{R}^2$ can be written as $\alpha[9, -4]^\top + \beta[-3, 1]^\top$. Action of A:

$$A\left(\alpha[9, -4]^\top + \beta[-3, 1]^\top\right) = -\alpha[9, -4]^\top + 2\beta[-3, 1]^\top.$$

Case of Symmetric Matrices

- Eigenvectors for different eigenvalues are orthogonal. Can be chosen with norm 1.
- Eigenvectors + eigenvalues: Linear“elliptic–like” scaling.



- Can be interpreted as “Rotate, scale in each axis direction, then rotate back”.

Matrix Rank

- Linear mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, A matrix associated. Rank of A = dimension of the image of f, i.e. dimension of the set made of the $f(x_1, \ldots, x_n)$.
- Rank of projection F above: 2: all vectors of $\mathbb{R}^3$ are projected on the plan of vectors $[x, y, 0]^T$.
- Take $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. A has rank 1.

$$f(x, y) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 2x + 4y \end{bmatrix}$$

  All the values of f belong to the line $y = 2x$, dimension 1 subspace of $\mathbb{R}^2$.
- The square matrix A, says size $n \times n$, is invertible is its rank is n.
- For a square matrix, its rank is the number of non-zeros eigenvalues.

Exercises

Pen and paper.

- Compute the products AB and BA for the following matrices

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

- Compute their determinant and traces.

With numpy.

- *numpy.linalg* subpackage has functions for trace and determinants.
- Find them and play with the two matrices above.

For Non-Square-Matrices

- Notions of right-inverse or left inverses.
- General construction of the Moore-Penrose Pseudo-Inverse. Works both with matrices with more lines than columns – overdetermined linear systems and the opposite: less lines than columns – underdetermined linear systems.
- `pinv` function in Matlab, `pinv` function in `numpy.linalg` python package.
- Intimately connected to linear least-squares problems and the Singular Value Decomposition.
- In turn intimately connected to eigenvalues and eigenvectors problems (spectral theory) for square matrices.

So far

- We talked of vectors, vector spaces, dimension, inner products
- Matrices, operations on them, transposition, product of matrices,
- Square matrices and their algebra, symmetric matrices, Traces, determinants.
- linear mappings, ranks, eigenvalues/vectors.
- matrix products and composition of linear mappings.

Read the Linear Algebra Tutorial and Reference on Absalon!
Next time: pen and paper for manual computations of convolutions!

This will also be useful for other courses!