UROPS Report

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Abstract

This report follows the work from Garlappi, Uppal and Wang – “Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach”. Diversification plays an important role in portfolio selection, and it often gives investors stable revenue. However, if the investor is really unfamiliar with the market or that particular assets, should he/she still participate against this ambiguity? Or just concentrate on other familiar assets? The general diversified portfolio may not be the best investment choice for different investors. Specifically, in the following context, we explain in detail what the model is and how it works, reproduce the results of empirical studies, and discuss the improvements in theory and practical performances. The Multi Prior model is based on the classical Markowitz mean-variance portfolio, but takes parameter uncertainty into consideration by using confidence interval on expected returns. For parameter uncertainty case, we implement MSCI international data from 1970 to 2018, to compare the differences in the choice of portfolio weights and out of sample performances. When we also take model uncertainty into consideration, we use Fama-French portfolio and CAPM model as an example, to illustrate how do parameter and model uncertainty effect the portfolio weight allocation in factor and assets. In general, we find out the multi priori approach reduces the fluctuation of changes in portfolio weights and improves the out of sample performances (higher Sharpe ratio).

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ω	$vector$ of portfolio weights
γ	Arrow-Pratt risk aversion parameter
μ	$vector$ of true expected excess returns;
	$\mu = (r_i - r_{i-1})/\ r_{i-1},\ r_i$ is the value/return at time i,
$\sum$	N×N covariance matrix, it is symmetric; (i,j)th entry isσ_{ij}
σ_{ij}	Covariance between returns of assets i and j =$\sigma_i\ \sigma_j\ \rho_{ij}$=$\sigma_{ji}$
1_N	N vector of ones
$\hat{\mu}$	estimate of expected return
T_j	number of observations in the sample for asset j
σ	volatility of return for an asset

1. Introduction to Modern Portfolio Theory

Is there a portfolio that achieves maximum returns and minimum risk at the same time?

In 1952, Harry Markowitz proposed portfolio theory that put return and risk on the same page. The main idea can be summarized to a well-known saying, "Don't put all your eggs in one basket" — diversification.

1.1 Mean Variance Analysis

Assume there are various non-riskfree assets available in market, each with its corresponding expected return and variance. For each level of risk, investor can always find one combination to maximize the total expected return. Similarly, for a given level of expected return, investor find the portfolio with lowest risk since they are assumed to be rational investors who are risk-averse. All these selected portfolios can form an efficient frontier in mean-variance plane. All efficient portfolios, each represented by a point on the efficient frontier, are well-diversified. Note that portfolio with maximum return may not necessarily have minimum risk. Finally, optimal portfolio is the point where$Captical\ Allocation\ line$[1]is tangent to efficient frontier, so that tangency portfolio has highest Sharpe ratio among all rational choices.

[1] CAL on the (σ_p, μ_p) plane is the linear line$E(R_p) = r_0 + \sigma_p \frac{E(R_M)-r_0}{\sigma_M}$, where$r_0$is return of risk free assets,$\frac{E(R_M)-r_0}{\sigma_M}$is the Sharpe ratio (higher is better)

In this paper, variance is considered as risk, standard deviation as volatility, though it is not coherent risk measure. The variance of whole portfolio, Var[R(ω)] =$\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}\,\omega_i\omega_j =$ $\omega^T\sum\omega$. Mean variance portfolio can be considered as the following risk aversion optimization, where$\omega^T\mu$ is investor's expected excess return and$-\frac{\gamma}{2}\omega^T\sum\omega$represents the trade off between return and risk. It can be expressed as maximization over quadratic utility functions of the form:

$$\max_{\omega}\quad \omega^T\mu - \frac{\gamma}{2}\omega^T\sum\omega(1)$$

To solve this maximization problem, simply derive F.O.C:

$$\frac{d}{d\omega}\left(\omega^T\mu - \frac{\gamma}{2}\omega^T\sum\omega\right) = 0$$

$$\mu - \gamma\omega\sum = 0$$

$$\omega = \frac{1}{\gamma}\sum^{-1}\mu(2)$$

Also, it is equivalent to

$a)$ $\min\limits_{\omega}\omega^T\sum\omega$ subject to $\omega^T\mu = \mu^*$ where μ^* is target return;

Or b) $\max\limits_{\omega}\omega^T\mu$ subject to $\omega^T\sum\omega = \sigma_*^2$ where σ_*^2 is target variance.

All the case will give us same efficient portfolio, but a) is a quadratic optimization with linear constraint while b) is vice versa. Note the γ is called the parameter of absolute risk aversion. The greater the γ, the more risk averse the investor is, because the variance becomes more important when γ is greater, resulting a lower utility. The parameter of absolute risk aversion is assumed to be positive, because we assume investors are risk averse. The greater γ will results smaller sum of weights. A negative γ would imply that an investor is risk loving. Several basic constraints will be considered in the paper, including

$$\omega^T\mathbf{1}_N = 1\text{(when absence of riskfree assets)}$$

$$\omega \geq \mathbf{0}_N\text{(short sales not allowed).}$$

2.2 Minimum Variance Portfolio

However, in real world practice, people found mean variance portfolio often has poor performance, due to some extreme holdings in assets. Diversification cannot eliminate all variance. The Global Minimum Variance portfolio is the portfolio on the efficient frontier with lowest variance (most left point). It ignores the expected return, only focus on volatility of the portfolio. A simple case of only two assets (no risk-free assets), the variance minimization problem would be:

$$\text{Min } \{Var[\omega_1 R_1 + (1-\omega_1)R_2] = \omega_1^2\sigma_1^2 + (1-\omega_1)^2\sigma_2^2 + 2\omega_1(1-\omega_1)\sigma_{12}\}$$

F.O.C $$\frac{d}{d\omega_1}Var = 2\omega_1\sigma_1^2 - 2(1-\omega_1)\sigma_2^2 + 2\sigma_{12}(1-2\omega_1) = 0$$

$$\omega_1 \frac{(\sigma_2^2 - \sigma_{12})}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

$$\omega_2 = 1 - \omega_1 = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$

For general case of n assets:

$$\min_{\omega} \frac{1}{2}\omega^T \Sigma \omega \tag{3}$$

Subject to

$$\omega^T 1_N = 1$$

The solution can be easily solved by Lagrange's multiplier

$$\mathcal{L}(\omega, \lambda) = \frac{1}{2}\omega^T \Sigma \omega + \lambda(1 - \omega^T 1_N)$$

$$\omega^* = \frac{\Sigma^{-1}\mathbf{1}_N}{\mathbf{1}_N^T \Sigma^{-1}\mathbf{1}_N} \tag{5}$$

Global minimum variance portfolio only based on estimators for variance-covariance matrix, hence minimum variance optimization should suffer less from estimation error than mean variance portfolio. Recent years, it has been proved that minimum variance portfolio's empirical out of sample performances are better than mean variance portfolio. In addition, the portfolio mixes investments with lower correlations will perform better. Details will be discussed in section 3.

2. Multi-prior Approach with Parameter Uncertainty

However, one of absolutely vital shortcoming of Mean Variance portfolio is we cannot capture the true mean and variance. All the inputs are expected value with estimation error. Investors usually feel uncertain about their estimation because of the turbulence in financial market. There are several new models for handling uncertainty, for example the Bayesian approach. The author Uppal and Wang (2001) think investors' view of model uncertainty cannot be represented by a probability prior, so they introduced a new method - Multi prior approach. Like famous economist Keynes' idea, if an investor is familiar with particular assets, then he/she should consider tilting the portfolio towards those assets first, then consider a diversified portfolio from Markowitz. In this approach, we assume people are rational (averse) to uncertainty, but it clearly divides uncertainty and risk into two parts. The specific proof of this framework is provided in Kogan and Wang (2002).

2.1 what it is

For portfolio management problem, considering only mean and variance is not enough, other parameter should also be added to our constraint. Multi-prior model extends mean variance portfolio and allows parameter uncertainty to take into account of the optimization problem. That is

$$\max_{\omega} \min_{\mu} \quad \omega^T \mu - \frac{\gamma}{2} \omega^T \Sigma \omega \tag{6}$$

Subject to

$$f(\mu, \hat{\mu}, \Sigma) \le \epsilon \tag{7}$$

$$\omega^T 1_N = 1 \tag{8}$$

It imposed an additional constraint, where $f(\mu, \hat{\mu}, \Sigma)$ is a vector valued function. ϵ, selects all the expected return that lie in the confidence interval, and its value is indicated by the percentage size of confidence interval for certain assets distribution.

(6) captures risk of asset returns by Σ (same as mean variance portfolio), but it adds minimization problem before maximization. This step chooses the "worse-case" μ from the results of (7), which gives the minimum portfolio return in certain level of uncertainty. Then, investors and portfolio managers try to allocate optimal portfolio weight to get maximum return in this worse-case scenario. Thus, investors will know portfolio's bottom line when they are unsure about estimated values.

(7) reflects investor's uncertainty about unknown true mean. If investor is more uncertain about asset's estimation, then we choose a higherϵ, a wider confidence interval, so the probability that true mean lie in this interval will be larger (smaller p in following

content). On the other hand, larger ϵ indicates higher level of uncertainty. If we miss this constraint, the minimization process in (6) could pick bad returns in extreme case. To avoid these rare cases, it is necessary to set a confidence interval as the boundary of choosing the worse-case scenario.

2.2 How to compute

1) Estimate uncertainty Asset by Asset individually

First, we assume all returns follow normal distribution. For each asset, estimate its expected return by Maximum Likelihood Estimation (MLE), which is the sample mean $\hat{\mu}$ from given historic data. A simple case: estimate expected return asset by asset (for j = 1,…., N), then

$$f_j(\mu,\hat{\mu},\Sigma) = \frac{(\hat{\mu_j} - \mu_j)^2}{\sigma_j^2/Tj} \leq \epsilon_j \tag{9}$$

$$-\sqrt{\epsilon_j} \leq \frac{\hat{\mu_j} - \mu_j}{\sigma_j/\sqrt{Tj}} \leq \sqrt{\epsilon_j} \tag{10}$$

$$\hat{\mu_j} - \frac{\sqrt{\epsilon_j}\sigma_j}{\sqrt{Tj}} \leq \mu_j \leq \hat{\mu_j} + \frac{\sqrt{\epsilon_j}\sigma_j}{\sqrt{Tj}} \tag{11}$$

If σ_j is known, $\frac{\hat{\mu_j} - \mu_j}{\sigma_j/\sqrt{Tj}} \sim N(0,1)$, then $\sqrt{\epsilon_j} = z_{1-\frac{p_j}{2}}$

If σ_j is unknown, $\frac{\hat{\mu_j} - \mu_j}{\sigma_j/\sqrt{Tj}} \sim t_{(df=T_j-1)}$, then $\sqrt{\epsilon_j} = t_{1-\frac{p_j}{2}}$

p_j is level of significance. Investor can choose higher ϵ_j , lower p_j to indicate their high level of uncertainty for estimating this particular asset. So, the higher p_j is, the lower 1- p_j, the more certain about this asset.

$$P(\mu_j \in [\hat{\mu_j} - \frac{\sqrt{\epsilon_j}\sigma_j}{\sqrt{Tj}},\ \hat{\mu_j} + \frac{\sqrt{\epsilon_j}\sigma_j}{\sqrt{Tj}}]) = 1 - p_j \tag{12}$$

If all returns are independent identically distributed,

$$1 - p = (1 - p_1)(1 - p_2) \cdots\cdots (1 - p_N) \tag{13}$$

When $1 - p$ tends to 0, $(p \to \infty)$ the portfolio will converge to Mean Variance Portfolio in theory. Because the Confidence interval for true mean will just be a line of the sample mean.

Define $\mu_j's$ lower bound in this specific confidence interval as: $\underline{\mu_j} = \hat{\mu_j} - \frac{\sqrt{\epsilon_j}\sigma_j}{\sqrt{Tj}}$;

Similarly, upper bound $\overline{\mu_j} = \hat{\mu_j} + \frac{\sqrt{\epsilon_j}\sigma_j}{\sqrt{Tj}}$.

Note that in a long position, the worst case is the expected return is actually at the lower bound, so that investor's underlying assets are devaluated. On the other side, if it is in the short position, the worst case is when expected return is higher (highest at the upper bound in this confidence interval), so the investor loses this valuable asset.

Hence, the original (3)(4)(5) optimization problem can be converted into Proposition 1 in the paper, which corresponds to:

$$\max_{\omega}\{ \omega^T(\hat{\mu} - \mu^{adj}) - \frac{\gamma}{2}\omega^T\sum\omega\} \tag{14}$$

$$\mu^{adj} = \{sign(\omega_1)\sigma_1 \frac{\sqrt{\epsilon_1}}{\sqrt{T_1}}, \dots\dots, sign(\omega_N)\sigma_N \frac{\sqrt{\epsilon_N}}{\sqrt{T_N}}\} \tag{15}$$

Short position: $sign(\omega_j)$= -1 long position: $sign(\omega_j)$= +1

2) Estimate Uncertainty jointly for whole assets

The constraint (7) becomes

$$f(\mu, \hat{\mu}, \sum) = \frac{T(T-N)}{(T-1)N}(\hat{\mu} - \mu)^T\sum^{-1}(\hat{\mu} - \mu) \tag{16}$$

$$\text{Or} \quad P\left[\frac{T(T-N)}{(T-1)N}(\hat{\mu} - \mu)^T\sum^{-1}(\hat{\mu} - \mu) \le \epsilon\right] = 1 - p \tag{17}$$

Where N is number of assets being assessed, T is the number of rows of historic data used to estimate expected returns. If true covariance matrix $\sum$ is known, then (16) follows $\mathcal{X}^2$ with (df=N); If $\sum$ is unknown, then we use estimated covariance, and (16) follows F distribution with (df=N, T-N).

As explain in the proposition 2 from Uppal (2004), the max-min problem with constraints can be converted to a max function:

$$\max_{\omega}\{ \omega^T\hat{\mu} - \frac{\gamma}{2}\omega^T\sum\omega - \sqrt{\epsilon \frac{(T-1)N}{T(T-N)}\omega^T\sum\omega} \} \tag{18}$$

St. $\omega^T 1_N = 1$

And the closed-form solution will be

$$w^* = \frac{\sigma_P^*}{\sqrt{\varepsilon} + \gamma\sigma_P^*}\Sigma^{-1}\left(\hat{\mu} - \frac{1}{A}\left(B - \frac{\sqrt{\varepsilon} + \gamma\sigma_P^*}{\sigma_P^*}\right)\mathbf{1}_N\right), \tag{19}$$

where $A = \mathbf{1}_N^\top\Sigma^{-1}\mathbf{1}_N$, $B = \hat{\mu}^\top\Sigma^{-1}\mathbf{1}_N$ and $C = \hat{\mu}^\top\Sigma^{-1}\hat{\mu}$. and $\sigma_p{}^*$is the root of this equation:

$$A\gamma^2\sigma_P^4 + 2A\gamma\sqrt{\varepsilon}\,\sigma_P^3 + (A\varepsilon - AC + B^2 - \gamma^2)\,\sigma_P^2 - 2\gamma\sqrt{\varepsilon}\,\sigma_P - \varepsilon = 0, \tag{20}$$

2.3 Empirical studies

We use 8 countries' MSCI International country equity index for wealth allocation, that is G7 (Canada, France, Germany, Italy, Japan, United Kingdom, and U.S.A.) and Switzerland. Morgan Stanley Capital International Indexes are used as the base for ETFs and are also the benchmarks for mutual funds. It's a good choice for portfolio and fund management. This dataset is the same in the paper section 4.1, but this report illustrates with more details and extensions.

We analysis data by using the similar process, where inputs are every 60 months historic return (sample mean, Covariance matrix), risk aversion parameter (assume to be 1 here) and uncertainty indicatorϵ. We start from the first 60 months data, and then in each simulation, we drop the first row and add the 61st row in the previous dataset of 60, forming the new 60 months data. The higherϵ is, the parameter uncertainty increases. Then, these inputs would provide various portfolio strategies, including their performances like expected return, volatility (standard deviation) and true return in the 61st month. We repeat the whole dataset, moving each month forward at one time, and compute the mean for above variables. Three types of portfolio will be computed for comparison: Mean Variance portfolio (max Sharpe ratio), Minimum Variance portfolio (min standard deviation), and Multi-Prior Portfolio. We omit Bayesian approach since it mixed up risk aversion with uncertainty and it is not our focused research object. In this case, estimate uncertainty about expected return jointly for all assets would be better choice than estimate asset by asset. There are eight asset allocations, which brings the whole portfolio analysis into a higher dimension. From the two assets case in the paper, we conclude that investor will hold less weights on more uncertain asset. However, if we put uncertainty in 8 assets individually, the situation where allocation only put extreme weights on particular one or two assets occurs most of time. For example, the optimal multi prior portfolio suggests investor to put all weights on Japan when investor doesn't have any confident assets (with zero confidence interval or close to). We will not present this result since it is not significant to present the rule of diversification and it does not give us good performances. In addition, high uncertainty in every asset (eg. $90\%, p_i = 0.1$) will result a relatively low uncertainty as whole (eg.$(0.9)^8 = 43\%$, p = 0.57).

Assume returns follow a normal distribution, we will go back to the procedure in 2.2, express uncertainty over the whole set of assets. Since covariance matrix is estimated, we don't know the true$\sum$, it follows F distribution with degree of freedom (N, T-N). Here, N=8, T= 60 because we use 5 years (60 months) excess return to estimate expected returns and optimal weights.

2.3.1

First, I repeat the original dataset from 1970/01 to 2001/07. Table 1 reports out-of-sample Mean (True Return), Standard Deviation, Sharpe Ratio (Mean/SD), and additionally theoretical Expected return (using the historic 60 months data), in order to compare the difference between theoretical expected return and true return. There are 319 observations in total, so we take the average of each performance character among 31 years timeframe.

Indeed, in both cases, Standard Deviation (Volatility) and Expected return are strictly decreasing from 0% to 100% level of uncertainty. The percentage is the percentile of confidence interval for $F_{8,52}$, corresponding to differentϵ. Because of higher level of uncertainty, rational investors would reduce volatility for compensating returns in theory. Minimum Variance Portfolio has highest Sharpe ratio, where Mean Variance has the lowest. Imposing short sale constraint improves the performance and minimizing fluctuations strategies' performances overall. It also narrows the gap between theoretical return and true return, contrasting with short sale allowed case. As mentioned in the paper, Jagannathan and Ma (2003) show that the short sale constraint can be interpreted as adjusting the covariance matrix.

Notice that the results are not exactly the same, the errors of true return might come from using different risk aversion parametersγsince it doesn't define clearly in the paper. Meanwhile, an interesting phenomenon is that the highest average true return on 61st month does not necessary present in Minimum Variance portfolio. I used more specific scale of uncertainty levels, we found that in between these special extreme cases of Mean Variance or Minimum Variance Portfolio, the performances of Multi-prior strategy are not strictly increasing/ decreasing as uncertainty increases. Take a closer look, there are actually lots of fluctuations in the result of true return. Sometimes, the best true return performance might be hidden in the middle part of uncertainty level. Here, we conclude that Minimum variance portfolio is optimal in this case, but that is not always true.

Table 1(1970-2001)

Out-of-sample Performance of Various Portfolio Using Original International Dataset

F(8,52)	0%	0.10%	1%	10%	55%	75%	90%	95%	99%	~99.99%	~99.9999%	
ϵ =	0	0.102	0.1977	0.426	0.9963	1.3296	1.79	2.122	2.87	7.676	26.33095	INF

Panel A

Allow Short Sale

	Allow Short Sale			
	Expected Return	**True Return**	**SD**	**Sharpe ratio**
Mean Var	11.68364	-3.281798	33.23481	-0.0987458
Multiprior				
0%	11.68364	-3.281732	33.23481	-0.0987438
~0%	11.39146	-2.759878	32.35698	-0.0852947
0.10%	7.64737	5.257778	21.18808	0.24814792
1%	6.143645	1.662063	16.80814	0.09888441
10%	3.9868352	-4.035653	10.77891	-0.3744027
50%	2.059289	0.4289098	6.107059	0.07023181
55%	1.940491	0.522735	5.850667	0.08934622
75%	1.531547	0.7449699	5.015608	0.14853033
90%	1.238543	0.863696	4.474033	0.19304641
95%	1.118478	0.9024936	4.271448	0.21128517
99%	0.9393419	0.9393419	4.026989	0.2332616
~99.99%	0.582033	0.9659384	3.614988	0.26720376
~99.9999%	0.5002685	0.964746	3.564513	0.27065296
MIN VAR	0.2906421	0.9574651	3.509438	0.27282576

Panel B

No Short Sale

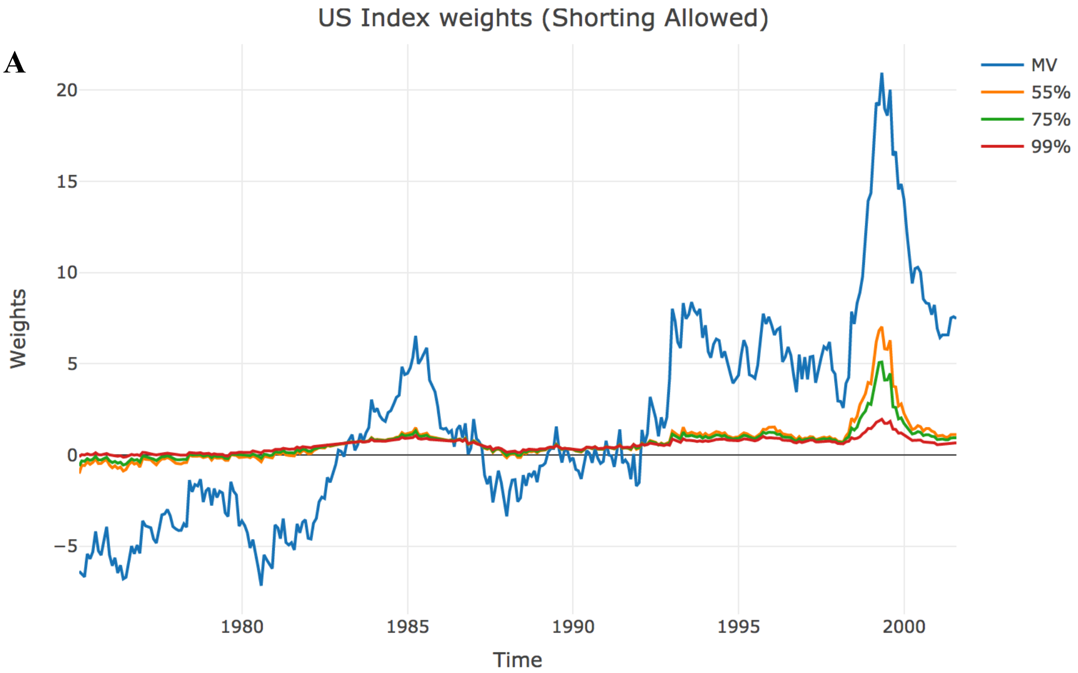
	No Short Sale			
	Expected Return	**True Return**	**SD**	**Sharpe ratio**
Mean Var	1.169246	0.9049886	5.824435	0.155377921
Multiprior				
0%	1.169246	0.9049886	5.824311	0.155381229
~0%	0.876438	0.9846971	5.824023	0.169075071
0.10%	0.79966	0.9414593	5.734044	0.164187666
1%	0.664548	0.9846893	5.56015	0.177097614
10%	0.6375533	1.012917	5.2699	0.192208012
50%	0.5968529	1.129133	4.967947	0.227283624
55%	0.5906646	1.127891	4.957195	0.227526051
75%	0.5634589	1.087178	4.861969	0.223608583
90%	0.5403702	1.055459	4.766976	0.221410597
95%	0.524338	1.047088	4.706645	0.222470146
99%	0.4984819	1.005833	4.630337	0.217226737
~99.99%	0.469149	0.971175	4.47576	0.216985495
~99.9999%	0.431919	0.9744322	4.451374	0.218905938
MIN VAR	0.3253721	0.9520361	3.643126	0.261323956

Note: In this application, Risk-free assets are not available, so constraint (8) $\omega^T 1_N = 1$ is considered.

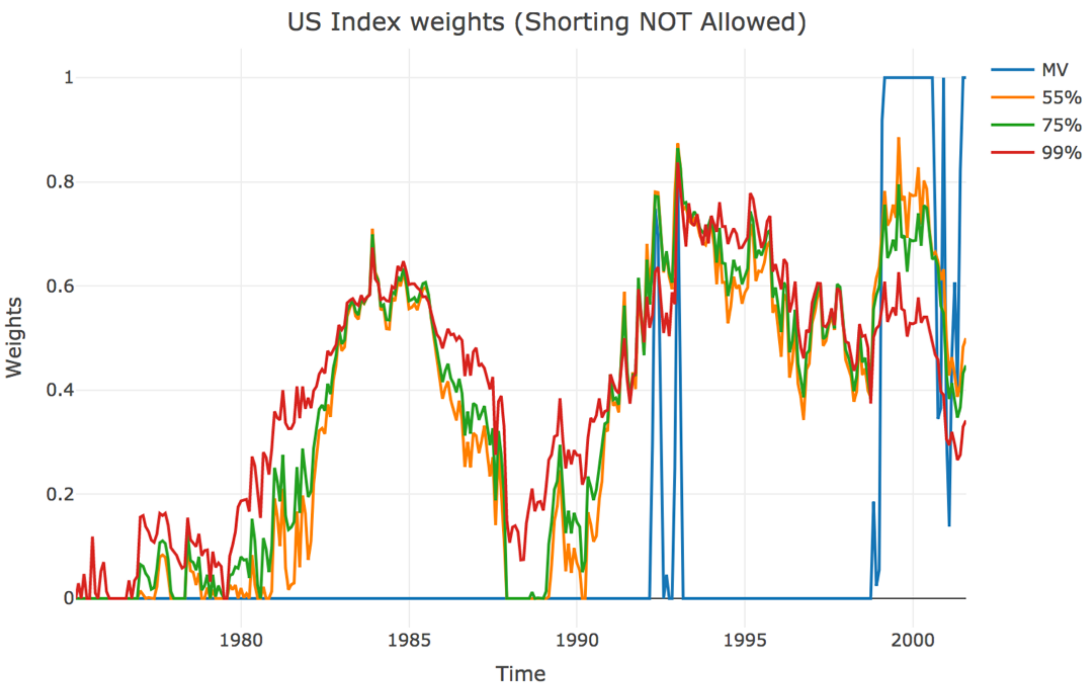
Figure 1:

Portfolio weights in US index over time (presenting exactly same results)

Panel A



Panel B



2.3.2

Second, I use recent years data that haven't been assessed in the paper, from 2001/8 to 2018/6 (total 203 rows of observation). Same procedure as before, using previous 5 years (60 months window) to estimate, and check next month return. The true return (out of sample performance) starts from August 2006 (Total 143 out of sample performances).

Figure 2 provides a general situation of optimal portfolio's theoretical structure based on the whole dataset. The pie chart shows the proportion of weights will be invested in Mean Variance Portfolio, US index is the main portion of it, so we will investigate more in US weights and how it changed overtime by using different historic data.

Table 2 uses the same procedure and structure as Table 1 and presents similar results. With $\epsilon = 0$, Multi prior portfolio converges to mean variance portfolio; and converges to minimum variance portfolio when $\epsilon \to \infty$, so that investor has zero confidence about expected return, only uses variance as significant data for computing portfolio weights. Expected return and standard deviation are strictly declining. There are less fluctuations in "No short sale" situation, and smaller gap between expected and true mean. It also confirms the conclusion that portfolio strategies with parameter uncertainty have better out of sample performance (higher Sharpe Ratio) than Mean Variance portfolio.

Figure 2

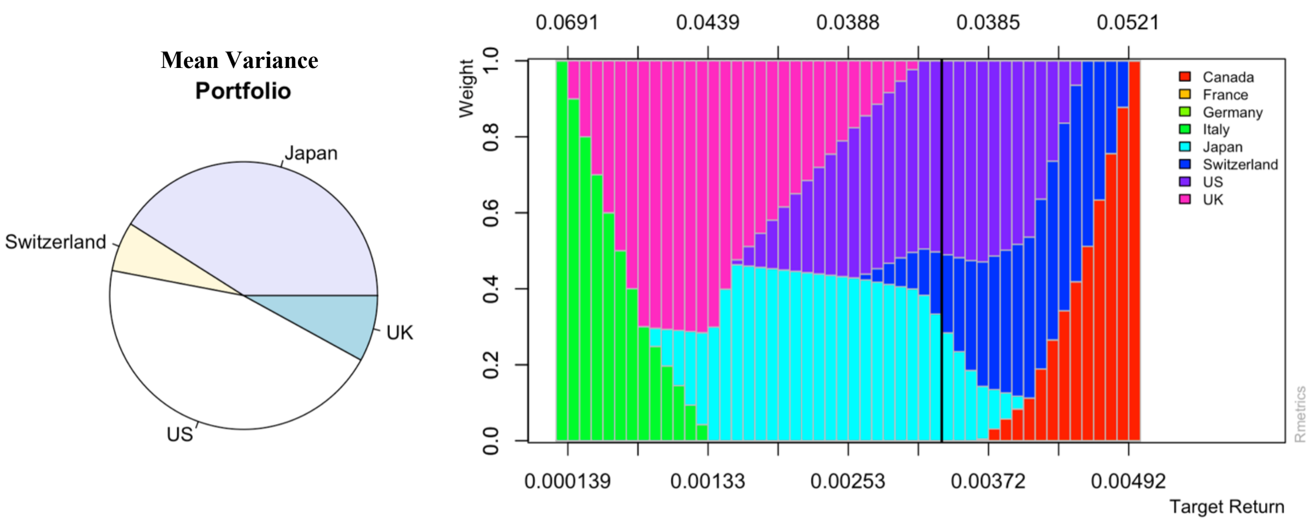


Table 2(2001-2018)

Out-of-sample Performance of Various Portfolio Using Following Years International Data

F(8,52)	0%	0.10%	1%	10%	55%	75%	90%	95%	99%	~99.99%	~99.9999%	
ϵ =	0	0.102	0.1977	0.426	0.9963	1.3296	1.79	2.122	2.87	7.676	26.33095	INF

Panel A

Allow Short Sale				
	Expected Return	**True Return**	**SD**	**Sharpe ratio**
Mean Var	21.518	0.9034899	45.44157	
Multiprior				
0%	21.518	2.574536	45.44157	0.056655965
~0%	21.11535	2.54702	44.55605	0.057164403
0.10%	15.908	2.30368	33.1247	0.069545686
1%	13.73285	2.785701	28.37496	0.098174623
10%	10.19472	6.785358	20.74412	0.327097896
50%	5.529809	-0.3935656	11.08641	-0.03549982
55%	5.113918	12.84246	10.2631	1.251323674
75%	3.557571	4.300689	7.279136	0.590824103
90%	2.424772	1.307357	5.251591	0.248944939
95%	2.0064	1.116448	4.5549	0.245109223
99%	1.552066	0.9258985	3.85777	0.240008736
~99.99%	0.8649396	0.6172	3.070467	0.201011768
~99.9999%	0.6795144	0.5245087	2.95291	0.177624343
Min Var	0.3770825	0.3436449	2.880189	0.119313316

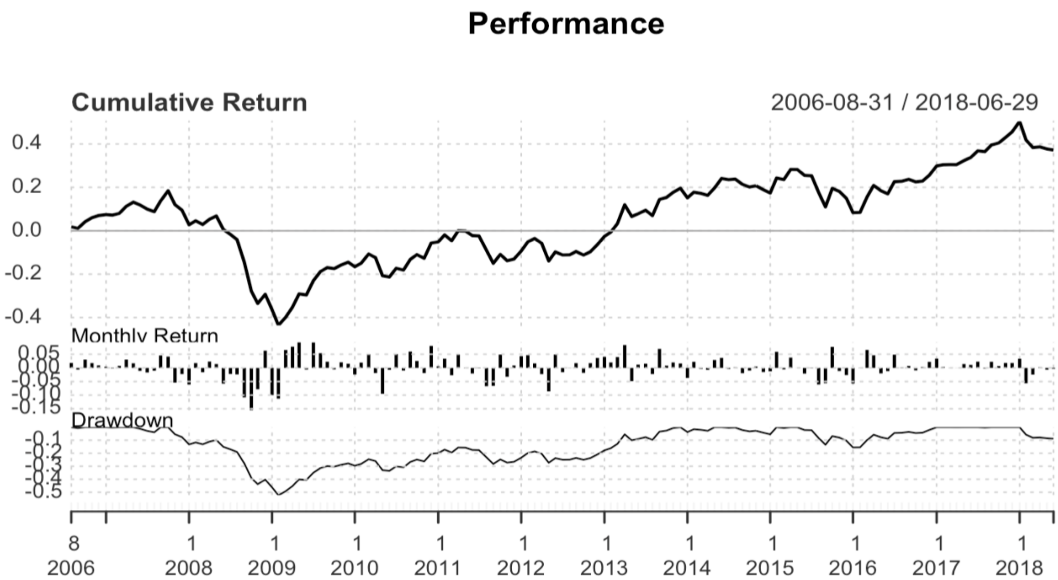
Panel B

No Short Sale				
	Expected Return	**True Return**	**SD**	**Sharpe ratio**
Mean Var	0.9168193	0.5653596	5.049412	0.11196543
Multiprior				
0%	0.9178422	0.5652352	5.049189	0.11194574
~0%	0.9158168	0.5280925	5.01943	0.10520966
0.10%	0.8531677	0.4798525	4.539081	0.10571578
1%	0.83085	0.5425915	4.429176	0.12250394
10%	0.8019757	0.5993835	4.317553	0.13882482
50%	0.7517687	0.5262419	4.175964	0.12601687
55%	0.746299	0.528303	4.1631118	0.12690099
75%	0.7194684	0.5107346	4.105074	0.12441544
90%	0.6856548	0.4866298	4.040967	0.1204241
95%	0.6662843	0.4778941	4.0079	0.11923803
99%	0.63417	0.459888	3.958767	0.1161695
~99.99%	0.5504742	0.404049	3.858751	0.10470979
~99.9999%	0.457631	0.3439796	3.787476	0.09082027
Min Var	0.3413218	0.2539	3.75851	0.06755337

Parameter uncertainty reduces extreme positions, short sale constraint has the same effect. Mean Variance portfolio shows higher Sharpe ratio when short sale is not allowed. This dataset verifies again that "parameter uncertainty is less sensitive to the introduction of a short sale constraint". Note that fromϵgreater than 0.9963 (55-percentile of$F_{8,52}$), the Sharpe ratio for parameter uncertainty portfolios with shorting allowed are larger than the constrained mean-variance portfolio.

However, minimum variance portfolio is not the optimal choice in this dataset. It is inside the multi prior approach with relatively low uncertainty level. When short sale is not allowed, the performance is relatively stable. It seems the true return has a regular pattern, it climbs up to the peak and then goes down, similar to Sharpe ratio. When shorting is allowed, the higher standard deviation is, the huger variation between expected and true return. One of special notice is when $\epsilon = 1(55\%)$, the true return is extremely high. Based on different dataset, the optimal portfolio occurs with random uncertainty level, hence we cannot induce a general conclusion from this point. But overall, multi prior model performs better than other portfolios that do not incorporate with uncertainty. The result may not fully support for the performances in 3.1, there are many reasons. For instance, serious economy crisis in 2008, larger standard deviation.

Figure 3

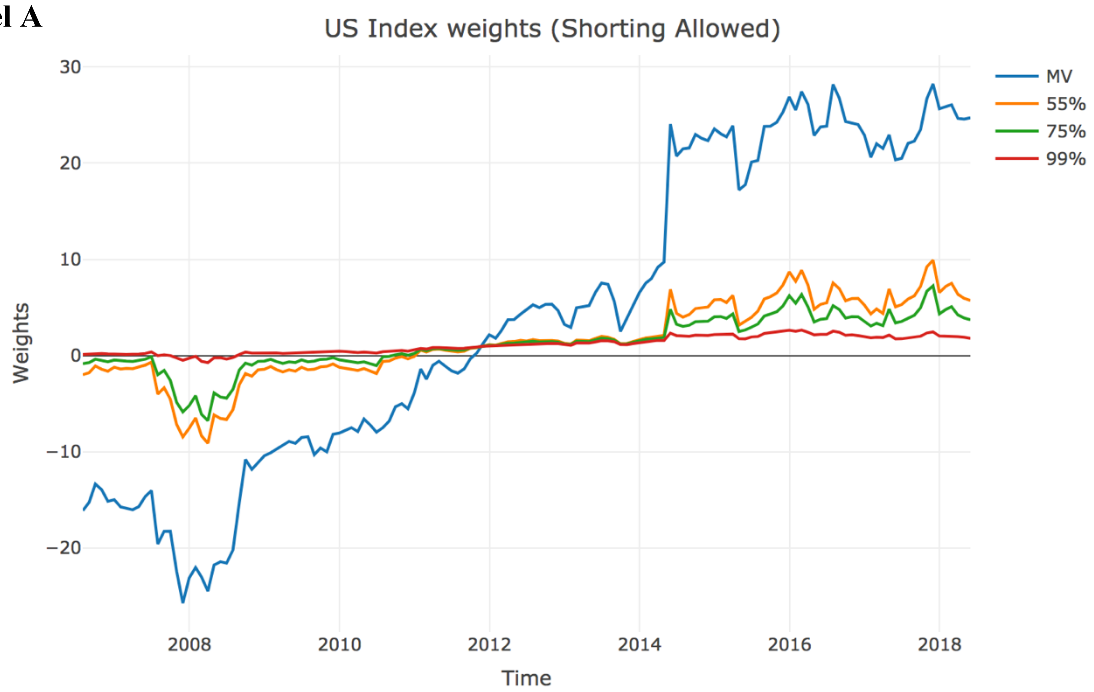


This is a big picture of true returns on the 61st month, staring from August 2006. There is a notable slip in 2008 when financial crisis occurred.

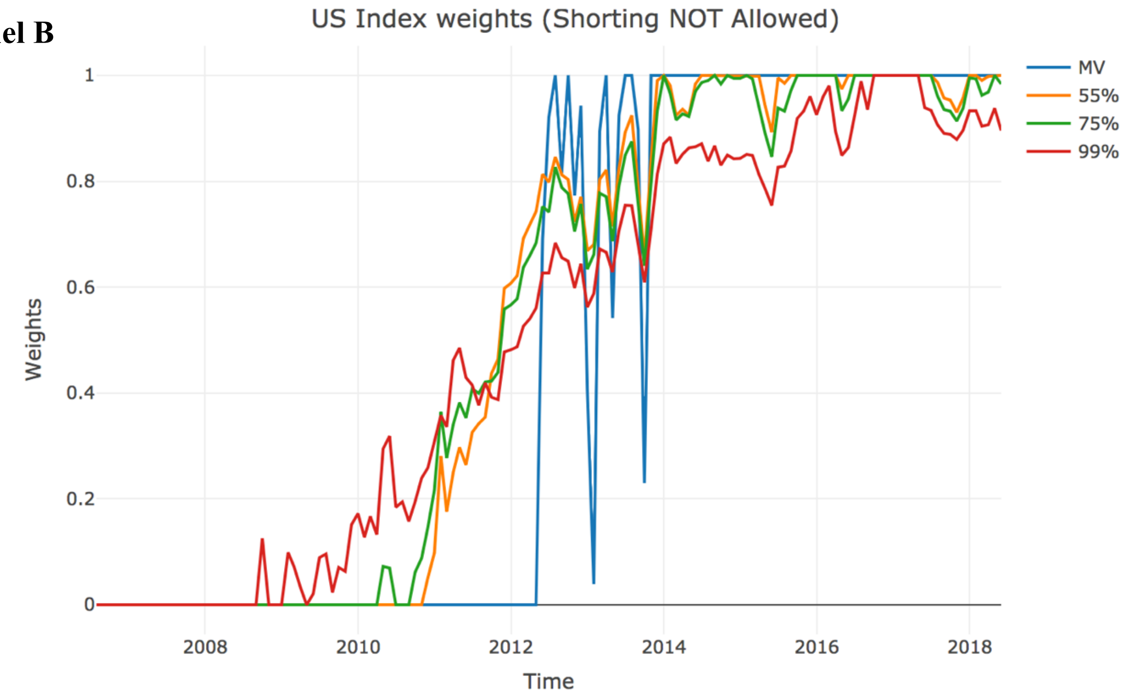
Meanwhile, the US index weight allocation graph follows same result in the original paper. The weights in portfolios that considered with parameter uncertainty has fewer extreme positions and fluctuate less over time. Weights allocation are more stabilize in the case of higher level of uncertainty aversion. The Higher aversion to uncertainty, less extreme weights on assets.

Figure 4
Portfolio weights in US index over time

Panel A



Panel B



3. Multi-prior Approach with Model and Parameter Uncertainty

Uncertainty about return generating model and expected returns

There are many Return generating models, for instance, MLE, CAPM, Bayesian. All of them are not the true model, they just predict and estimate what's future return might like. And in reality, assets returns are not necessarily multivariate normally distributed. Hence, model uncertainty exists.

Previously, we use historic returns and estimation method MLE to get our expected excess returns. The only uncertainty is about estimation, investors are not sure if MLE get them precise estimation of return and risk. However, if we use other return-generating models which predict return based on other financial factors. Then, investors should also consider model uncertainty, whether this model is the true (best) model to generate returns.

There are benchmark and non-benchmark assets in financial market. Benchmarks are usually represented by board market index, like S&P 500 and MSCI. It gives investors and managers a good reference of the stock performances. However, for those non-benchmark assets, investors should use factors and pricing model to estimate their "index", here we only consider up to second moment as before.

3.1 what it is

Consider the situation with N risky assets with K factors:

$$\mu = \begin{pmatrix} \mu_a \\ \mu_f \end{pmatrix} = \begin{pmatrix} \alpha + \beta\mu_f \\ \mu_f \end{pmatrix}$$
$$\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{af} \\ \Sigma_{fa} & \Sigma_{ff} \end{pmatrix} = \begin{pmatrix} \beta\Sigma_{ff}\beta^\top + \Omega & \beta\Sigma_{ff} \\ \Sigma_{ff}\beta^\top & \Sigma_{ff} \end{pmatrix}. \tag{21}$$

$\alpha(N \times 1)$, $\beta(N \times K)$ is coefficient of linear regression model of asset returns about factor returns, and Ω is the covariance matrix of residuals/errors. And uncertainty averse investor will solve similar MaxMin problem for the same reason, but adding one more constraint for model uncertainty:

$$\max_{w} \min_{\mu_a,\mu_f} w^\top \mu - \frac{\gamma}{2} w^\top \Sigma w, \quad (22)$$

St.

$$(\hat{\mu}_a - \mu_a)^\top \Sigma_{aa}^{-1} (\hat{\mu}_a - \mu_a) \leq \epsilon_a, \quad (23)$$

$$(\hat{\mu}_f - \mu_f)^\top \Sigma_{ff}^{-1} (\hat{\mu}_f - \mu_f) \leq \epsilon_f. \quad (24)$$

ϵ_a represents uncertainty level of the model, since assets' expected return now depends on factor parameter and model type. For factor expected returns, we will continue using MLE and take sample mean under normal assumption, hence the effect of ϵ_f still represents uncertainty level about estimation, separated from model uncertainty. So, in the case of $\epsilon_f = 0,\ \epsilon_a \neq 0$, investor is very certain about the factor, but leave uncertainty to the model, he/she will put most of money into factor portfolio.

One example we will present in the following empirical study is CAPM (Capital Asset Pricing Model) derived from Mean Variance Analysis, says if all investors have same market information (same input), they will hold same optimal portfolio, and the *expected return of portfolio* can be estimated from market factors:

$E(R_P) = r_0 + \beta[E(r_m) - r_0]$; where $\beta = \frac{Cov(R_p,\ r_m)}{\sigma_m^2}$, r_m is return of market portfolio.

3.2 How to compute

To solve optimization system (22)(23)(24), we could use Lagrange multiplier to solve the inner minimization related with constraints of μ_a, μ_f first, then optimize outer maximization problem by choosing optimal weights. The detail of the proof is showed in Uppal (2004) Proposition 3, it gives max function (25)

$$\max_{w_a,w_f} w_a^\top \hat{\mu}_a + w_f^\top \hat{\mu}_f - \frac{\gamma}{2}\left[w_a^\top \hat{\Sigma}_{aa}(w_a,\epsilon_\alpha) w_a - w_a^\top \Sigma_{af} w_f - w_f^\top \Sigma_{fa} w_a - w_f^\top \hat{\Sigma}_{ff}(w_f,\epsilon_f) w_f\right] \quad (25)$$

And finally gets the following system of solutions:

$$w_a = \max\left[1 - \frac{\sqrt{\epsilon_a}}{\sqrt{g(w_f)^\top \Sigma_{aa}^{-1} g(w_f)}}, 0\right] \frac{1}{\gamma} \Sigma_{aa}^{-1} g(w_f), \quad (26)$$

$$w_f = \max\left[1 - \frac{\sqrt{\epsilon_f}}{\sqrt{h(w_a)^\top \Sigma_{ff}^{-1} h(w_a)}}, 0\right] \frac{1}{\gamma} \Sigma_{ff}^{-1} h(w_a), \quad (27)$$

Where $g(w_f) = \hat{\mu}_a - \gamma \Sigma_{af} w_f,\ \ h(w_a) = \hat{\mu}_f - \gamma \Sigma_{fa} w_a.$

Additionally, in Uppal (2004) version's Appendix B, it provides the proof for model and parameter uncertainty for the case of individually estimated moments, similar to 2.2.1) introduced above but add model uncertainty in μ_f.

3.3 Empirical Study

In the second application, we use Domestic Fama-French portfolio from http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html, the 3 Factors (monthly) dataset. It includes monthly excess returns on SMB(Small Minus Big), HML(High Minus Low), Market portfolio and risk-free rate, using value-weighted return on all NYSE, AMEX and NASDAQ stocks (from CRSP) minus the one-month Treasury bill rate (from Ibbotson Associates). We use the similar method before, use windows of 120 months of data to estimate expect returns and compute the weights, but add model uncertainty into account this time. SMB and HML are considered as two assets, and Market is considered as a factor. In return generating models, factors are usually excess return over benchmark assets (r_f), as a parameter to estimate excess returns for non-benchmark assets (r_a).

Here, we present the simple one factor CAPM model and MLE estimation in comparison. In CAPM estimation, $\hat{\mu}_a = \beta_a \hat{\mu}_f$ where β_a is the vector of betas in the model. As mentioned above, uncertainty about μ_a is considered as model uncertainty represented by ϵ_a. When $\epsilon_a = 0$, this means investor firmly believes CAPM model. In addition, if $\epsilon_f = 0$ also, the multi prior portfolio will be mean variance portfolio because investor assumes there is no uncertainty. In MLE estimation, $\hat{\mu}_a$ is taken by the sample mean since we keep the assumption of normal distribution, and we also allowed uncertainty by indicating in ϵ_a. Since there is only one factor (MKT), and we don't know the true variance for its excess return, we assume it follows t distribution with degree 119 (=120-1). For the same normality assumption, assets SMB and HML as a whole will follow $F_{2,118}$. And ϵ_f, ϵ_a are the value of the confidence interval for its corresponding distribution.

3.3.1 MLE Model

First, we revise the process discussed in Uppal (2004) section 4.2, using the same dataset from July 1926 to December 2002. Using 120 months of data, the first portfolio is formed at July 1936, and there are total 918 simulations. The following results are comparison between our simulations (bottom graph) and paper work (top graph).

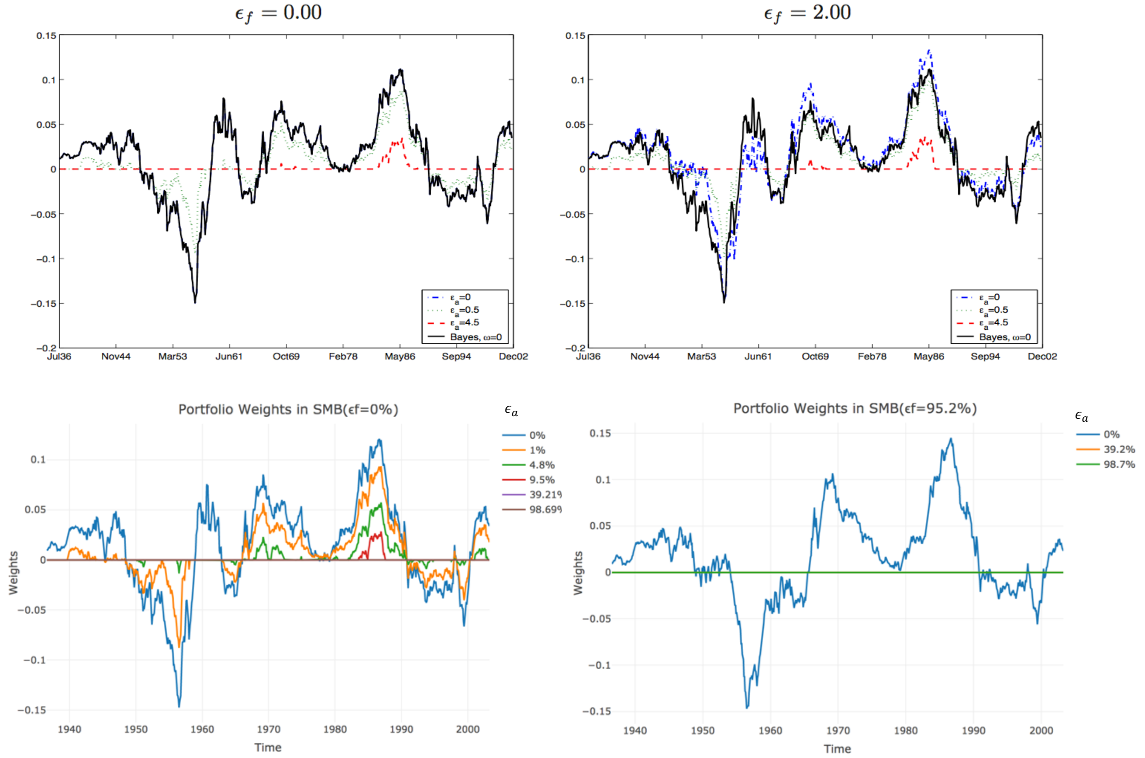
In each plot, we present the portfolio holdings for different levels of model and parameter uncertainty in corresponding time period. The left figure is when there is no parameter uncertainty on factor MKT, so $\epsilon_f = 0$. The right side is $\epsilon_f = 2$, corresponds to 95.2% significant confidence interval which stands for a relative high uncertainty. Special note that the confidence interval in t distribution can be negative infinity to infinity, however, ϵ_f is supposed to be positive in multi prior model because there is root process in between. Therefore, we will use the "absolute value" instead, that is, $|\widehat{\mu_f} - \mu_f| \leq \epsilon_f * S.D.$

For example, $\epsilon_f = 0.5$, originally it is the confidence level of 69.1% in t (df=119), now it means the confidence level between (-0.5, 0.5), which corresponds to (0.691-0.5)*2= 38.2%. For assets/model uncertainty($F_{2,118}$), ϵ_a= 0(0%); ϵ_a= 0.01(1%); ϵ_a= 0.05(4.8%); ϵ_a= 0.1 (9.5%); ϵ_a= 0.5(39.21%); ϵ_a= 1(62.9%);ϵ_a= 4.5 (98.69%), listed as reference.

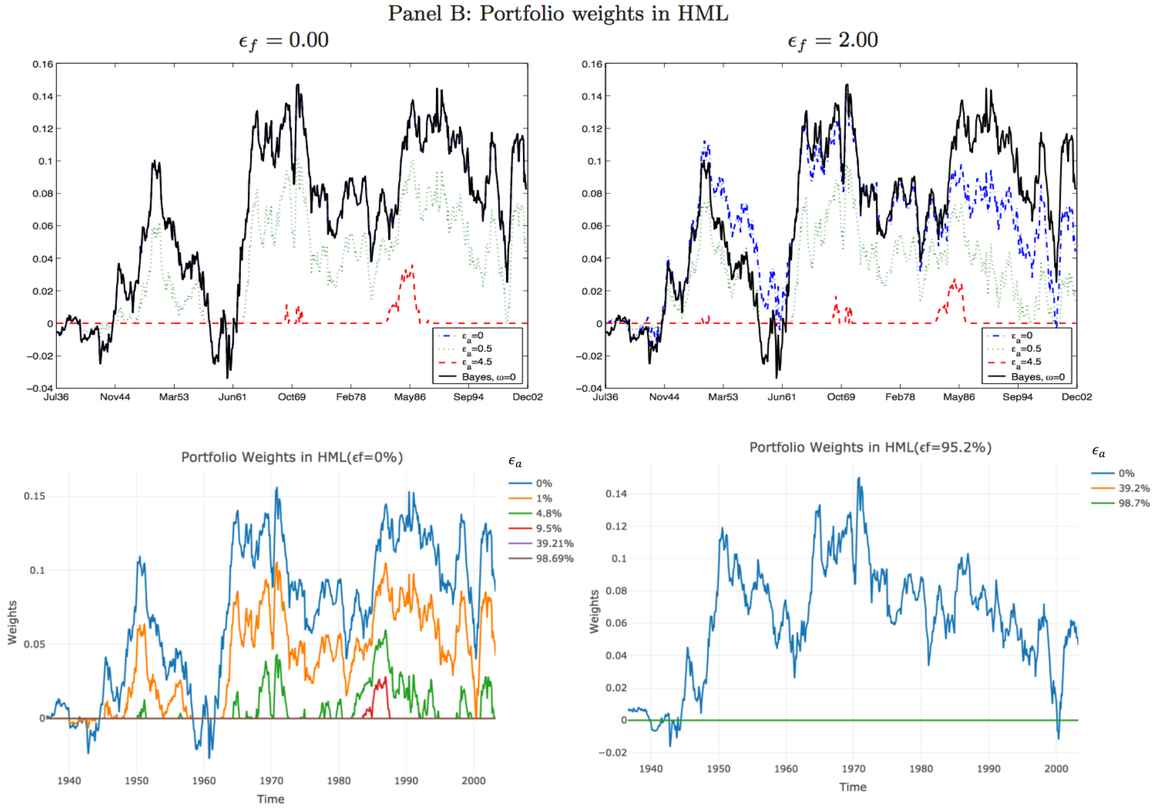
For portfolio solving method, I use the system of solutions (26) (27) as mentioned in 3.2, where we can treat (26) ω_a as a function of ω_f, where covariance matrix and sample means come from data. Then, insert ω_a into (27), so this is a single variable equation for ω_f. Solve the root for ω_f, then plug back in (26), get ω_a. But this approach might result some extreme case. When$\epsilon_f \neq 0$, the max function in (27) may return 0 due to big value ofand $\frac{\sqrt{\epsilon_f}}{\sqrt{h(w_a)^\top \Sigma_{ff}^{-1} h(w_a)}}$ result zero weight in MKT. This might be the reason for the following result of zero weights. On the other hand, in the paper, it inverts the original Maxmin saddle point problem to solve a standard maximization problem. However, I found out this max function (25) can go to infinity as the weight increases in my algorithm. Hence, I didn't use it for computing results.

a) MLE **Figure 5**

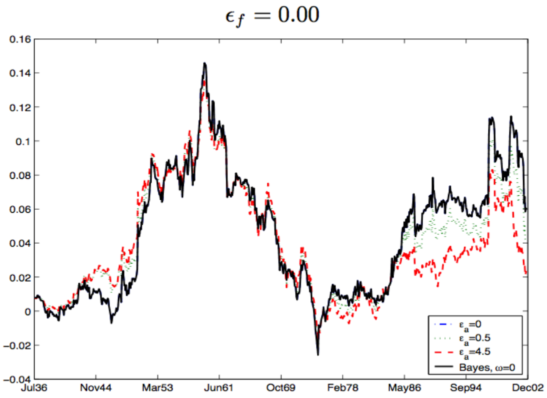
Panel A: Portfolio weights in SMB

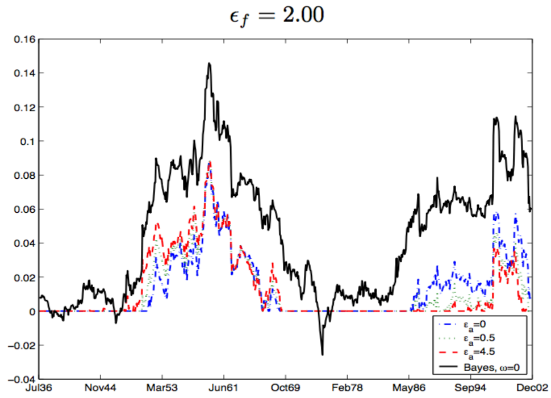


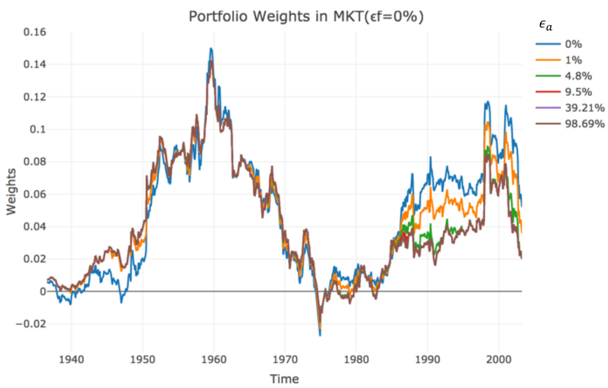
Panel B: Portfolio weights in HML

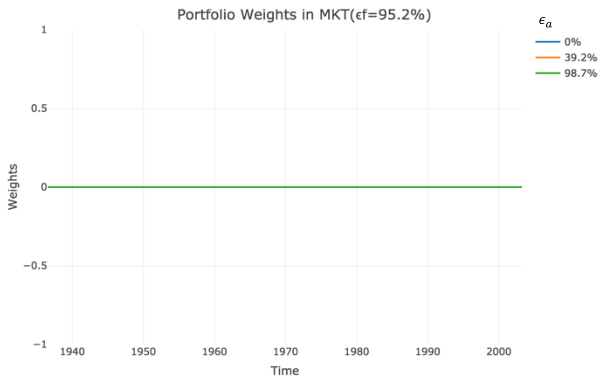


Panel C: Portfolio weights in MKT

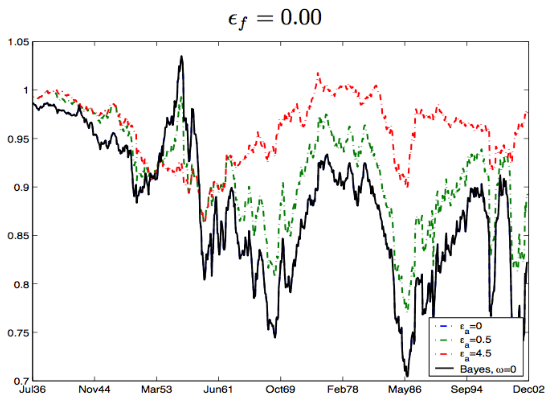


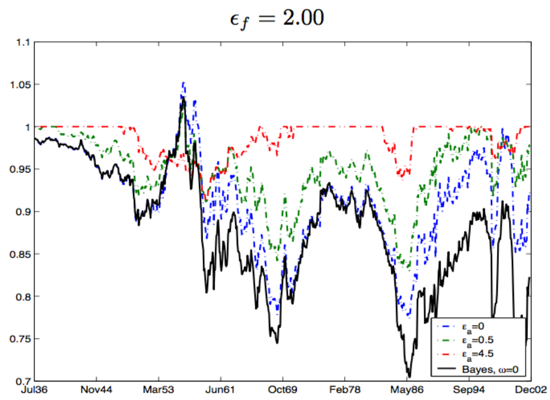


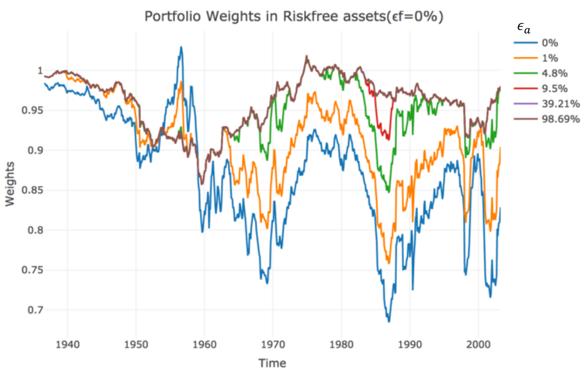


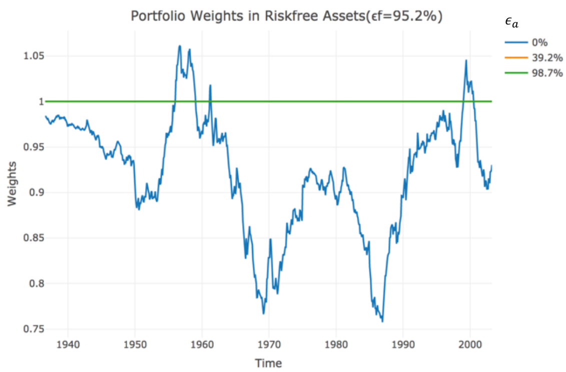


Panel D: Portfolio weights in the riskfree asset









In figure 5, my results are almost same which verifies my algorithm. There are more sensitive changes when $\epsilon_f = 0$,so I add several cases ofϵ_a. Discuss the left side case first, when $\epsilon_f = 0$, $\epsilon_a = 0$,all the weights are the same in portfolio. However, notice that the track of ϵ_a= 0.01(1%) in my result coincides likely with ϵ_a= 0.5(39.21%) in the paper, and ϵ_a= 0.1(9.5%) is about ϵ_a= 4.5 (98.69%) in the paper. Instead, when ϵ_a= 4.5 (98.69%), assets weights SMB and HML are all zero in the case, but factor weight MKT doesn't have such strong effect. This scale mistake should be confirmed in later studies. On the right side, the differences are bigger. We notice a strange phenomenon when$\epsilon_f = 2$,the only case for non-zero asset weights are $\epsilon_a = 0$,and MKT weights are more extreme, because of large factor uncertainty, it seems that optimal portfolio will not put market portfolio into account.

In my result overall, it indeed confirms the theory that the optimal holding in assets (SMB,HML) decreases as investors' uncertainty in assets increases, as we keep a fixed level of factor uncertainty. A higher level of market uncertainty(ϵ_f) makes investors to hold more assets portfolio. And only in the case $\epsilon_f = 0$, it shows why higher assets uncertainty(ϵ_a) induces more holding in the market.

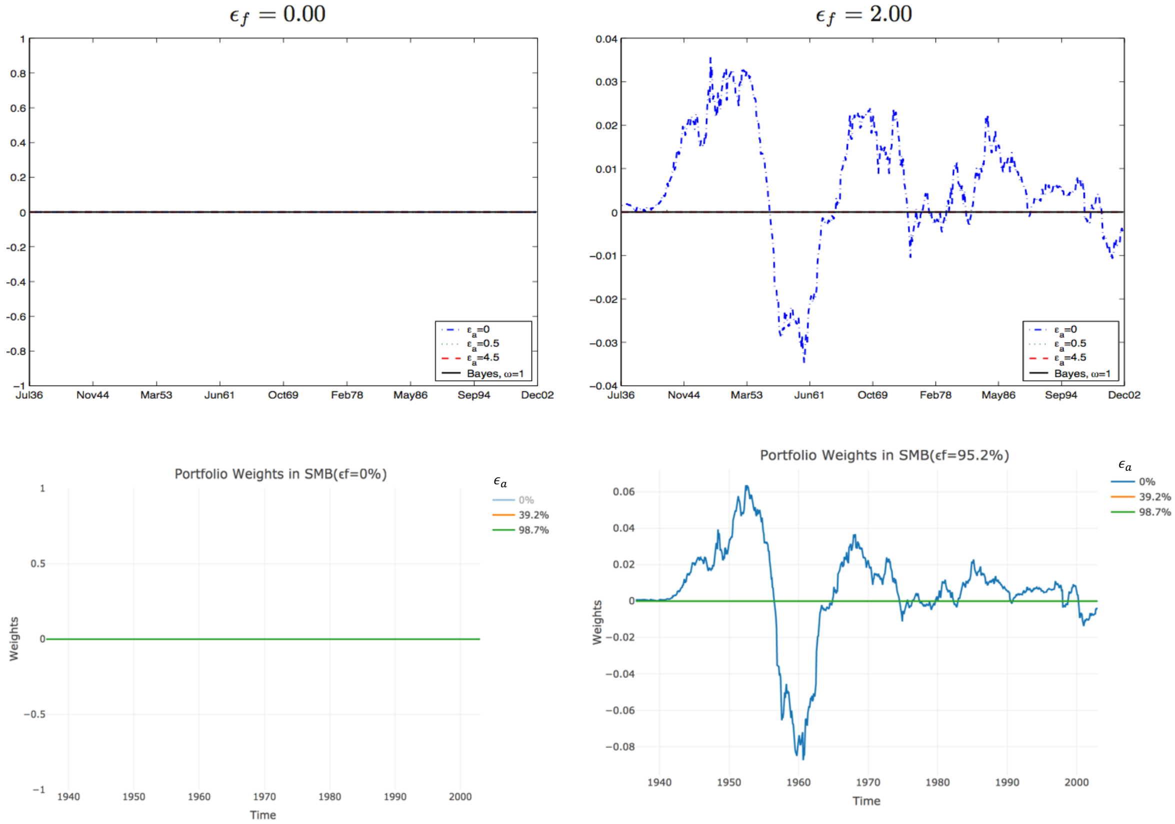
3.3.2 CAPM Model

In figure 6, it again repeats the same process, but using CAPM model and $\hat{\mu}_f$ to estimate $\hat{\mu}_a$, connecting by β_a which is determined from every historic 120 months data. $\epsilon_a$is model uncertainty now, intuitively thinking, asset and factor's estimated returns are related now, and so asset uncertainty might be more sensitive compared to MLE. Hence, in the case of no factor uncertainty, investors will put almost entire weights on MKT. This is showed on the left side, the optimal multi-prior portfolio put all weight on MKT in all cases of ϵ_a.
Similar situation happens on the right side, like in the previous MLE case, with high factor uncertainty and zero model uncertainty, uncertainty aversion investors tend to put weights on assets they are more familiar with. And the same strange phenomenon happens on MKT weight.

b) CAPM

Figure 6

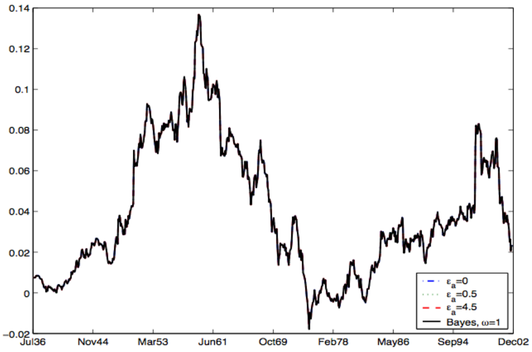
Panel A: Portfolio weights in SMB

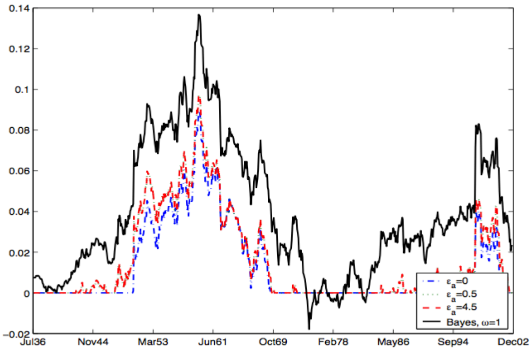


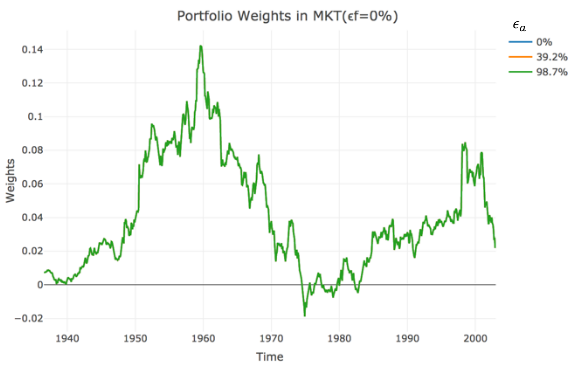
Panel B: Portfolio weights in HML

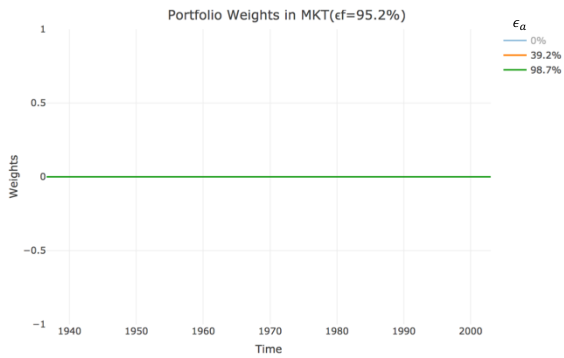
Panel C: Portfolio weights in MKT

$\epsilon_f = 0.00$ | $\epsilon_f = 2.00$



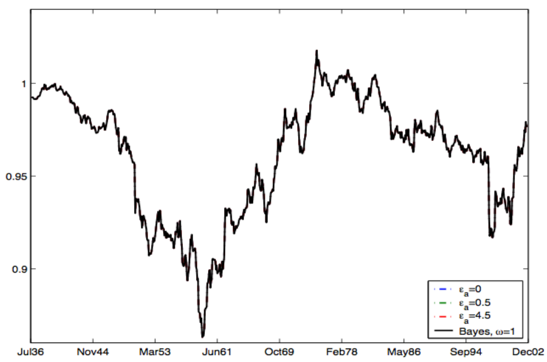


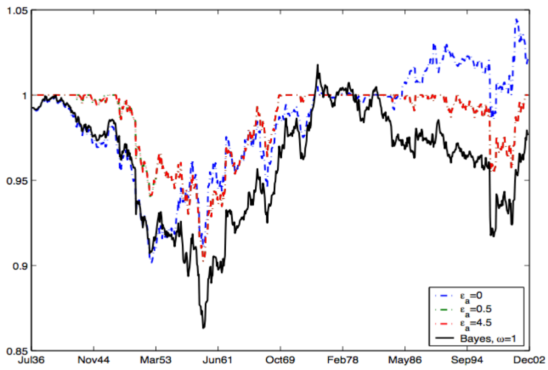


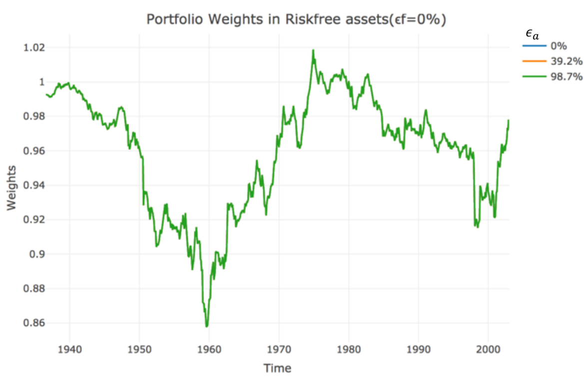


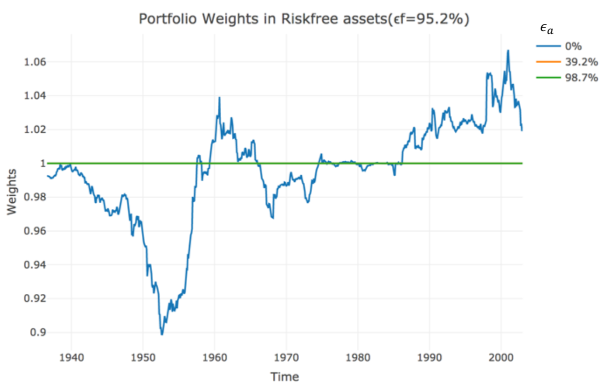
Panel D: Portfolio weights in the riskfree asset

$\epsilon_f = 0.00$ | $\epsilon_f = 2.00$









This case reminds me of one of Keynes' idea, if investors are sufficiently ambiguous about all risky asset, then they should not participate in the market. Here, investors both faces high factor and model uncertainty, this might be the reason why they quit the market and put all their money to risk-free asset.

Details of my algorithm should be further examined. Note that, if we repeat this in the following year as we did in 2.3.2, the weights allocation gives us the same regularity.

4. Conclusion

This model partially answered the question: should investors still put their money into the general diversification process when they have large ambiguity about the assets?

Overall, we used both international and domestic dataset to test Multi-prior model. In general, when there is parameter uncertainty, the changes of asset allocation weights will be less fluctuated, there are less extreme weights on particular assets, and as uncertainty increases, the weight on that asset decreases. Meanwhile, Multi prior model can give investor better performances, higher Sharpe ratio at some point. It confirms the results in Wang (2005) that an investor who is averse to uncertainty does not hold a mean-variance "efficient" portfolio, because that does not fit for every investor. Kan and Zhou showed that with parameter uncertainty, holding the sample tangency portfolio and the riskless asset is never optimal. Further, when we also consider model uncertainty, investor will allocate more assets in the relative smaller uncertainty part.

At last, some of my thoughts for improvement are:

1. Previously, we estimate model uncertainty with assets jointly, what if we try estimate each asset individually, when there are different uncertainty level between factor?

2. What's the diversified level of multi-prior model? Check diversification ratio, cooperate with uncertainty, think about our beginning question, Will the most diversified multi-prior portfolio give us better performance?

3. In a multi-period setting, investors might be more and more familiar with risky assets, How should we change the uncertainty level? How to measure? Will they eventually go back to mean variance portfolio with zero uncertainty?

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